Preprint typeset in JHEP style - HYPER VERSION

MIT-CTP-4228

Constraints on 6D Supergravity Theories with Abelian Gauge Symmetry

Daniel S. Park and Washington Taylor

Center for Theoretical Physics
Department of Physics
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

dspark81 at mit.edu, wati at mit.edu

Abstract: We study six-dimensional $\mathcal{N} = (1, 0)$ supergravity theories with abelian, as well as non-abelian, gauge group factors. We show that for theories with fewer than nine tensor multiplets, the number of possible combinations of gauge groups—including abelian factors—and non-abelian matter representations is finite. We also identify infinite families of theories with distinct $U(1)$ charges that cannot be ruled out using known quantum consistency conditions, though only a finite subset of these can arise from known string constructions.

Contents

1. Introduction

Using quantum consistency conditions to identify interesting theories among a general class of theories has proven to be a fruitful practice historically. This is epitomized by the sequence of developments that initiated with the classic works by Alvarez-Gaumé and Witten [1] and by Green and Schwarz [2] on anomalies, and that culminated in the discovery of the heterotic string by Gross, Harvey, Martinec and Rohm [3]. Gauge, gravitational

and mixed anomalies [1, 4] constrain the gauge symmetry of ten-dimensional supergravity theories. The anomalies cannot cancel unless the gauge group is one of the groups $SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$ or $U(1)^{496}$[2, 5]. The $SO(32)$ type I string theory was known at the time that these constraints were discovered, but this result motivated the discovery of the $E_8 \times E_8$ and $Spin(32)/\mathbb{Z}_2$ heterotic string theories. Recent results show that the $E_8 \times U(1)^{248}$ and $U(1)^{496}$ theories cannot be made simultaneously anomaly-free and consistent with supersymmetry [6]. Together, these results show that the spectra of all anomaly-free gravity theories with minimal supersymmetry in 10D are realized in string theory.

Gravitational anomaly constraints were also used to identify the unique massless spectrum that a six-dimensional supergravity theory with $\mathcal{N} = (2, 0)$ symmetry can have [7]. This spectrum precisely agrees with the massless spectrum of type IIB string theory compactified on a K3 manifold. These results suggest that quantum consistency conditions—especially anomaly constraints—are an effective tool for obtaining clues about the microscopic theory of gravity from the structure of the macroscopic theory.

An interesting class of theories to approach from this perspective is the set of six-dimensional $\mathcal{N} = (1, 0)$ supergravity theories. These theories have strong anomaly constraints, but at the same time admit diverse consistent string vacua with a wide range of gauge groups and matter representations. A sampling of the substantial literature on six-dimensional $\mathcal{N} = (1, 0)$ supergravity theories and their string realizations is given in references [8]-[34]; reviews of this work and further references appear in [35], [36]. Constraints on the set of nonabelian gauge groups and matter that can appear in such theories were analyzed in [32, 37, 38, 39, 40, 41, 42]. We are interested here in obtaining constraints on these theories in the case when there are abelian vector fields in the massless spectrum.

Many six-dimensional $(1, 0)$ supergravity theories with abelian vector fields have been identified as arising from string theory vacuum constructions. Erler has analyzed the $U(1)$ sector of heterotic orbifold constructions in [15] while Honecker and Trapletti have extended the analysis to more general heterotic backgrounds [33]. Abelian gauge symmetry of six-dimensional F-theory backgrounds has also been studied by various authors [27, 29, 31]. Determining the abelian gauge symmetry of a given string model is often subtle because vector bosons in the spectrum that are naively massless can be lifted at the linear level by coupling to Stückelberg fields. This is a generic phenomenon in many types of string models including heterotic, orbifold, intersecting brane, fractional brane and F-theory models[1]. Therefore, a detailed analysis of all the vector-matter couplings in play is necessary in order to ensure that a vector boson is indeed in the massless spectrum in a given string model.

The anomaly cancellation structure for specific six-dimensional $(1, 0)$ string vacua with abelian gauge symmetry has been worked out in the literature [15, 27, 29, 31, 33]. General analyses on the full landscape of six-dimensional $(1, 0)$ supergravity theories, however, have not fully incorporated the structure of abelian gauge symmetries. In this paper we treat the abelian anomaly cancellation conditions carefully and show how they place bounds on

[1]We have given an incomplete list of references that address this in the bibliography [9, 10, 15, 21, 25, 27, 29, 31, 33, 43, 44, 45, 46, 47].

the massless spectrum of general six-dimensional supergravity theories with abelian factors in the gauge group.

It was shown in [38, 40] that the number of non-abelian theories with $T < 9$ tensor multiplets is finite. In light of this result it is natural to ask whether this bound continues to hold when we allow abelian gauge group factors. It is useful to divide this question into two parts. That is, we ask;

I. Whether the number of different gauge/matter structures is finite when we ignore the charges of the matter under the $U(1)$'s.

II. Whether given the gauge/matter structure, the number of distinct combinations of $U(1)$ charges each matter multiplet can have is finite.

In this paper we show that the answer to the first question is "yes" when $T < 9$. As is the case with non-abelian theories, when $T \geq 9$ we can generate an infinite class of theories in which the bounds that hold for $T < 9$ theories are violated. An example of such an infinite class is given in section 3.3.

In addressing the second question, it is important to note that theories with multiple $U(1)$ gauge symmetries (say $U(1)^n$) are defined up to arbitrary linear redefinitions of the gauge symmetry. If we assume that all the $U(1)$'s are compact and normalize the unit charge to be 1 for each $U(1)$ factor, the theories are defined up to $SL(n, \mathbb{Z})$.

From this fact, we may deduce that there are an infinite number of distinct $U(1)$ charge assignments possible for certain non-anomalous gauge/matter structures. This is because there are many known examples of theories with two $U(1)$ factors and at least one uncharged scalar, so that the non-anomalous gauge group can be written in the form $U(1)^2 \times \mathcal{G}_0$. Since any linear combination of the two $U(1)$'s is a non-anomalous $U(1)$ gauge symmetry, it is possible to construct an infinite class of apparently consistent 6D supergravity theories with gauge group $U(1) \times \mathcal{G}_0$ by simply removing the other $U(1)$ along with a neutral scalar from the spectrum.

Hence we see that the answer to the second question is negative. We may now ask, however,

III. Whether all infinite families of $U(1)$'s could be generated in the trivial manner presented above.

IV. Whether additional quantum consistency conditions that are unknown to us at the present could be employed to constrain the set of $U(1)$ charges in a given theory.

Regarding question III, we find that there are non-trivially generated infinite families of $U(1)$ charge solutions. We have not addressed the last question here, though some speculation in this regard is included at the end of the conclusions in Section 5.

This paper is organized as follows: In section 2 we review the massless spectrum and anomaly structure of six-dimensional $(1, 0)$ theories. In section 3 we address the first question. In section 4 we address the second and third questions. In particular, we present examples of infinite classes of $T = 1$ theories with $U(1)$'s that are trivially/nontrivially

generated. We also discuss subtleties arising in the case $T = 0$, where there are no tensor multiplets. We conclude our discussion in section 5.

The results in this paper are based on consistency conditions on low-energy supergravity theories, and do not depend upon a specific UV completion such as string theory. In some cases, however, examples are drawn from string theory and F-theory to illuminate the structure of the set of allowed models as determined from macroscopic considerations.

2. 6D $(1,0)$ Theories and Anomaly Cancellation

In this section we review six-dimensional theories with $\mathcal{N} = (1,0)$ supersymmetry and anomaly cancellation in these theories. In section 2.1 we present an overview of the field content of these theories. We compute the anomaly polynomial in section 2.2 and review anomaly cancellation and factorization in section 2.3. We give explicit formulae for the anomaly factorization condition in the presence of $U(1)$'s in section 2.4 and discuss some salient features of these equations.

In sections 2.5 and 2.6 we summarize the aforementioned aspects of anomaly cancellation specializing to the cases of $T = 1$ and $T = 0$ respectively. In section 2.7 we discuss aspects of the generalized Green-Schwarz mechanism that come into play when the theory has abelian gauge symmetry, and also explain why this issue can be safely ignored when discussing the massless spectrum.

2.1 The Massless Spectrum

The massless spectrum of the models we consider can contain four different multiplets of the supersymmetry algebra: the gravity and tensor multiplet, vector multiplet, and hypermultiplet. The contents of these multiplets are summarized in Table 1.

We consider theories with one gravity multiplet. There can in general be multiple tensor multiplets; we denote the number of tensor multiplets by T. When $T = 1$ it is possible to write a Lagrangian for the theory; the self-dual and anti-self-dual tensors can combine into a single antisymmetric tensor. Theories with T tensor multiplets have a moduli space with $SO(1,T)$ symmetry; the T scalars in each multiplet combine into a $SO(1,T)$ vector j that can be taken to have unit norm. We consider theories with arbitrary gauge group and matter content.

Note that a theory with a general number of tensor multiplets can still be defined despite the lack of a covariant Lagrangian. The partition function can be defined by coupling the three-form field strength to a 3-form gauge potential as in [48]. Classical equations of motion can be formulated as in [11, 14]. Supersymmetry and anomaly cancellation may be discussed at the operator level of a theory obtained by quantizing the classical theory defined by these equations.

We write the gauge group for a given theory as[2]

$$\mathcal{G} = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa \times \prod_{i=1}^{V_A} U(1)_i \,. \tag{2.1}$$

[2]The gauge group generally can have a quotient by a discrete subgroup, but this does not affect the gauge algebra, which underlies the anomaly structure analyzed in this paper.

Multiplet	Field Content
Gravity	$(g_{\mu\nu}, \psi_\mu^+, B_{\mu\nu}^+)$
Tensor	$(\phi, \chi^-, B_{\mu\nu}^-)$
Vector	(A_μ, λ^+)
Hyper	$(4\varphi, \psi^-)$

Table 1: Six-dimensional (1,0) supersymmetry multiplets. The signs on the fermions indicate the chirality. The signs on antisymmetric tensors indicate self-duality/anti-self-duality.

Lowercase greek letters $\kappa, \lambda, \cdots$ are used to denote the simple non-abelian gauge group factors; lowercase roman letters $i, j, k, \cdots$ are used to denote $U(1)$ factors. ν and V_A denote the numbers of nonabelian and abelian gauge group factors of the theory.

We denote by N the number of irreducible representations of the non-abelian gauge group under which the matter hypermultiplets transform (including trivial representations); we use uppercase roman letters to index these representations. We say that hypermultiplets transform in the representation R_κ^I under $\mathcal{G}_\kappa$ and have $U(1)_i$ charge $q_{I,i}$.

We characterize theories by their massless spectrum. There is a slight subtlety we must consider when dealing with $U(1)$ gauge symmetries. It is possible to break $U(1)$ at the linearized level by certain hypermultiplets, called "linear hypermultiplets" in the literature [49]. We will refer to these multiplets simply as "linear multiplets" throughout this paper. When a linear multiplet couples to a vector multiplet the two merge into a long (or non-BPS) multiplet and are lifted from the massless spectrum. Once lifted from the massless spectrum, these long multiplets can be safely ignored. This issue is discussed in more detail in section 2.7.

2.2 The Anomaly Polynomial for Theories with $U(1)$'s

In six-dimensional chiral theories there can be gravitational, gauge and mixed anomalies [4]. The sign with which each chiral field contributes to the anomaly is determined by their chirality.

The 6D anomaly can be described by the method of descent from an 8D anomaly polynomial. The anomaly polynomial is obtained by adding up the contributions of all the chiral fields present in the theory [1]. For the $T = 1$ case this is given in [15, 33]. In general

we obtain

$$\begin{aligned}
I_8 = &-\frac{1}{5760}(H-V+29T-273)[\mathrm{tr}R^4+\frac{5}{4}(\mathrm{tr}R^2)^2] \\
&-\frac{1}{128}(9-T)(\mathrm{tr}R^2)^2 \\
&-\frac{1}{96}\mathrm{tr}R^2[\sum_\kappa \mathrm{Tr}F_\kappa^2-\sum_{I,\kappa}\mathcal{M}_I^\kappa \mathrm{tr}_{R_\kappa^I}F_\kappa^2] \\
&+\frac{1}{24}[\sum_\kappa \mathrm{Tr}F_\kappa^4-\sum_{I,\kappa}\mathcal{M}_I^\kappa \mathrm{tr}_{R_\kappa^I}F_\kappa^4-6\sum_{I,\kappa,\lambda}\mathcal{M}_I^{\kappa\lambda}(\mathrm{tr}_{R_\kappa^I}F_\kappa^2)(\mathrm{tr}_{R_\lambda^I}F_\lambda^2)] \\
&+\frac{1}{96}\mathrm{tr}R^2\sum_{I,i,j}\mathcal{M}_I q_{I,i}q_{I,j}F_iF_j \\
&-\frac{1}{6}\sum_{I,\kappa,i}\mathcal{M}_I^\kappa q_{I,i}(\mathrm{tr}_{R_\kappa^I}F_\kappa^3)F_i-\frac{1}{4}\sum_{I,\kappa,i,j}\mathcal{M}_I^\kappa q_{I,i}q_{I,j}(\mathrm{tr}_{R_\kappa^I}F_\kappa^2)F_iF_j \\
&-\frac{1}{24}\sum_{I,i,j,k,l}\mathcal{M}_I q_{I,i}q_{I,j}q_{I,k}q_{I,l}F_iF_jF_kF_l\,.
\end{aligned} \tag{2.2}$$

$\mathcal{M}_I$ is the size of the representation I that is given by

$$\mathcal{M}_I=\prod_\kappa d_{R_\kappa^I}\,, \tag{2.3}$$

where d_{R_κ} is the dimension of the representation R_κ of $\mathcal{G}_\kappa$. Similarly, $\mathcal{M}_I^\kappa$ ($\mathcal{M}_I^{\kappa\lambda}$) is the number of $\mathcal{G}_\kappa$ ($\mathcal{G}_\kappa\times\mathcal{G}_\lambda$) representations in I, which is given by

$$\mathcal{M}_I^\kappa=\prod_{\mu\neq\kappa}d_{R_\mu^I}\qquad(\mathcal{M}_I^{\kappa\lambda}=\prod_{\mu\neq\kappa,\lambda}d_{R_\mu^I}) \tag{2.4}$$

respectively. V and H are the number of massless vector multiplets and hypermultiplets in the theory. They are given by

$$V\equiv V_{NA}+V_A\equiv\sum_\kappa d_{\mathrm{Adj}_\kappa}+V_A,\qquad H\equiv\sum_I\mathcal{M}_I \tag{2.5}$$

where d_{Adj_κ} is the dimension of the adjoint representation of gauge group $\mathcal{G}_\kappa$. V_{NA} is the number of non-abelian vector multiplets in the theory. The integer N, which is the number of irreducible representations of the non-abelian gauge group, plays an important role in bounding the number of $U(1)$'s. We use 'tr' to denote the trace in the fundamental representation, and 'Tr' to denote the trace in the adjoint. Multiplication of forms should be interpreted as wedge products throughout this paper unless stated otherwise.

2.3 Anomaly Cancellation and Factorization

The Green-Schwarz mechanism [2] can be generalized to theories with more than one tensor multiplet when the anomaly polynomial is factorizes in the following form [14, 26]:

$$I_8=-\frac{1}{32}\Omega_{\alpha\beta}X_4^\alpha X_4^\beta\,, \tag{2.6}$$

where Ω is a symmetric bilinear form (or metric) in $SO(1,T)$ and X_4 is a four form that is an $SO(1,T)$ vector. X_4 can be written as

$$X_4^\alpha = \frac{1}{2}a^\alpha \mathrm{tr}R^2 + \sum_\kappa (\frac{2b_\kappa^\alpha}{\lambda_\kappa})\mathrm{tr}F_\kappa^2 + \sum_{ij} 2b_{ij}^\alpha F_i F_j \,, \tag{2.7}$$

where we define b_{ij} to be symmetric in i,j. The a and b's are $SO(1,T)$ vectors and α are $SO(1,T)$ indices. Note that the anomaly coefficients for the $U(1)$'s can be written in this way due to the fact that the field strength is gauge invariant on its own [30]. The λ_κ's are normalization factors that are fixed by demanding that the smallest topological charge of an embedded SU(2) instanton is 1. These factors, which are equal to the Dynkin indices of the fundamental representation of each gauge group, are listed in table 2 for all the simple groups. The b_κ's form an integral $SO(1,T)$ lattice when we include these normalization factors [40].

The gauge-invariant three-form field strengths are given by

$$H^\alpha = dB^\alpha + \frac{1}{2}a^\alpha \omega_{3L} + 2\sum_\kappa \frac{b_\kappa^\alpha}{\lambda_\kappa}\omega_{3Y}^\kappa + 2\sum_{ij} b_{ij}^\alpha \omega_{3Y}^{ij}, \tag{2.8}$$

where ω_{3L} and ω_{3Y} are Chern-Simons 3-forms of the spin connection and gauge fields respectively. If the factorization condition (2.6) is satisfied, anomaly cancellation can be achieved by adding the local counterterm

$$\delta\mathcal{L}_{GS} \propto -\Omega_{\alpha\beta}B^\alpha \wedge X_4^\beta \,. \tag{2.9}$$

Meanwhile, supersymmetry determines the kinetic term for the gauge fields to be (up to an overall factor) [14, 30]

$$-\sum_\kappa (\frac{j\cdot b_\kappa}{\lambda_\kappa})\mathrm{tr}(F_\kappa \wedge *F_\kappa) - \sum_{ij}(j\cdot b_{ij})(F_i \wedge *F_j) \,, \tag{2.10}$$

where j is the unit $SO(1,T)$ vector that parametrizes the T scalars in the tensor multiplets. The inner product of j and the b vectors are defined with respect to the metric Ω. There must be a value of j such that all the gauge fields have positive definite kinetic terms. This means that there should be some value of j such that all $j\cdot b_\kappa$ are positive and such that $j\cdot b_{ij}$ is a positive definite matrix with respect to i,j.

If we did not have any $U(1)$'s, (2.6) would be the only way in which the anomaly can be cancelled. When we have abelian vector multiplets, however, a generalized version of the Green-Schwarz mechanism is available [25]. In this case, it is possible to cancel terms in the 8-form anomaly polynomial that are proportional to

$$F \wedge X_6 \,, \tag{2.11}$$

	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
λ	1	2	1	2	6	12	60	6	2

Table 2: Normalization factors for the simple groups.

where X_6 is a six form, by a counter-term in the action of the form

$$-C \wedge X_6 \,, \tag{2.12}$$

where C is a Stückelberg 0-form that belongs to a linear multiplet. The coupling of C to the vector boson V is given by

$$\frac{1}{2}(\partial_\mu C - V_\mu)^2 \,, \tag{2.13}$$

which is what we mean by C being a Stückelberg 0-form. The anomalous gauge boson V recieves a mass, hence rendering the $U(1)$ broken; the abelian vector multiplet is lifted from the massless spectrum by coupling to the linear multiplet by the Stückelberg mechanism. When all the anomalous $U(1)$'s are lifted and we look at the pure massless spectrum of the theory, all the gravitational anomalies and gauge/mixed anomalies induced by the massless fields are cancelled completely by two forms through the conventional Green-Schwarz mechanism. The lesson is that when we are discussing the massless spectrum, this generalized version of the Green-Schwarz mechanism does not come into play and can be safely ignored.[3] We elaborate further on this issue in section 2.7.

2.4 The Factorization Equations

We are now ready to write down the factorization equations in the presence of $U(1)$'s. The factorization equations come from demanding that the anomaly polynomial (2.2) factorize in the form (2.6). Comparing the terms with no abelian field strength factors gives the conditions

$$R^4 \ : \qquad 273 = H - V + 29T \tag{2.14}$$

$$(R^2)^2 \ : \qquad a \cdot a = 9 - T \tag{2.15}$$

$$F^2R^2 \ : \qquad a \cdot b_\kappa = \frac{1}{6}\lambda_\kappa(A_{\mathrm{Adj}_\kappa} - \sum_I \mathcal{M}_I^\kappa A_\kappa^I) \tag{2.16}$$

$$F^4 \ : \qquad 0 = B_{\mathrm{Adj}_\kappa} - \sum_I \mathcal{M}_I^\kappa B_\kappa^I \tag{2.17}$$

$$(F^2)^2 \ : \qquad b_\kappa \cdot b_\kappa = \frac{1}{3}\lambda_\kappa^2(\sum_I \mathcal{M}_I^\kappa C_\kappa^I - C_{\mathrm{Adj}_\kappa}) \tag{2.18}$$

$$F_\kappa^2 F_\mu^2 \ : \qquad b_\kappa \cdot b_\mu = \lambda_\kappa \lambda_\mu \sum_I \mathcal{M}_I^{\kappa\mu} A_\kappa^I A_\mu^I \tag{2.19}$$

The inner products on the left-hand-side of the equations are taken with respect to the $SO(1,T)$ metric Ω. For each representation R of a given group, the group theory coefficients A_R, B_R, C_R are defined by

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2, \qquad \mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2 \,, \tag{2.20}$$

[3] The situation is quite the opposite when we are taking the top-down approach, for example when we are constructing theories from string compactifications. Since we are working downward from the high-energy end, it is then important to figure out which $U(1)$ vector bosons that naively seem to be massless are lifted by this mechanism.

where tr denotes the trace with respect to the fundamental representation. For each hypermultiplet I we use the shorthand notation

$$A^I_\kappa = A_{R^I_\kappa}, \quad B^I_\kappa = B_{R^I_\kappa}, \quad C^I_\kappa = C_{R^I_\kappa}\,. \tag{2.21}$$

We refer to the first anomaly equation (2.14) as the gravitational anomaly constraint.

The $U(1)$ anomaly equations obtained by comparing terms with abelian field strength factors are given by

$$F_iF_jR^2 \;:\quad a\cdot b_{ij} = -\frac{1}{6}\sum_I \mathcal{M}_I q_{I,i}q_{I,j} \tag{2.22}$$

$$F_i^3F_\kappa \;:\quad 0 = \sum_I \mathcal{M}_I^\kappa E^I_\kappa q_{I,i} \tag{2.23}$$

$$F_iF_jF_\kappa^2 \;:\quad (\frac{b_\kappa}{\lambda_\kappa})\cdot b_{ij} = \sum_I \mathcal{M}_I^\kappa A^I_\kappa q_{I,i}q_{I,j} \tag{2.24}$$

$$F_iF_jF_kF_l \;:\quad b_{ij}\cdot b_{kl} + b_{ik}\cdot b_{jl} + b_{il}\cdot b_{jk} = \sum_I \mathcal{M}_I q_{I,i}q_{I,j}q_{I,k}q_{I,l} \tag{2.25}$$

for all i,j,k,l. The group theory coefficient E is defined to be

$$\mathrm{tr}_R F^3 = E_R \mathrm{tr} F^3 \tag{2.26}$$

and $E^I_\kappa = E_{R^I_\kappa}$.

It is useful to summarize these anomaly constraints by the following polynomial identities;

$$a\cdot P(x_i) = -\frac{1}{6}\sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.27}$$

$$0 = \sum_I \mathcal{M}_I^\kappa E^I_\kappa f_I(x_i) \tag{2.28}$$

$$b_\kappa \cdot P(x_i) = \lambda_\kappa \sum_I \mathcal{M}_I^\kappa A^I_\kappa f_I(x_i)^2 \tag{2.29}$$

$$P(x_i)\cdot P(x_i) = \frac{1}{3}\sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.30}$$

Here we have defined the $SO(1,T)$ scalar and vector polynomials

$$f_I(x_i) \equiv \sum_i q_{I,i}x_i \tag{2.31}$$

$$P^\alpha(x_i) \equiv \sum_i b^\alpha_{ij}x_ix_j\,. \tag{2.32}$$

The reason that $U(1)$ factorization conditions can be written as polynomial identities is because the field strengths of the $U(1)$'s behave like numbers rather than matrices in the

anomaly polynomial. We note that the x_i are auxiliary variables and do not have any physical significance.

A theory with charges $q_{I,i}$ assigned to the hypermultiplets is only consistent if there exist b_{ij} satisfying these equations that give a positive-definite kinetic matrix $j \cdot b_{ij}$ for the $U(1)$ gauge fields. It is useful to define the charge vector with respect to $U(1)_i$ whose components are the charges of the N nonabelian representations:

$$\vec{q_i} \equiv (q_{1,i}, q_{2,i}, \cdots, q_{N,i}) \tag{2.33}$$

There is a $GL(V_A, \mathbb{R})$ symmetry of the $U(1)$ anomaly equations that originates from the fact that there is a freedom of redefining $U(1)$'s. If there are multiple $U(1)$'s one could take some new linear combination of them to define a new set of non-anomalous $U(1)$'s. The equations are invariant under

$$\begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix} \to M \begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix}, \quad (b^\alpha)_{ij} \to \left(M^t (b^\alpha) M\right)_{ij}, \tag{2.34}$$

for $M \in GL(V_A, \mathbb{R})$. We have denoted (b^α) to be the matrix whose (i,j) element is b^α_{ij}. When we are discussing properly quantized charges of compact $U(1)$'s the linear redefinitions of the $U(1)$'s must be given by elements of $SL(V_A, \mathbb{Z}) \subset GL(V_A, \mathbb{R})$. Here, however, we merely use the fact that the anomaly equations are invariant under $GL(V_A, \mathbb{R})$ as a tool for obtaining bounds on the number of $U(1)$'s we can add to a given theory. Therefore, we do not need to be concerned with the issue of integrality of charges.

The factorization equations, combined with the positive-definite condition on b_{ij}, impose stronger constraints on the theory when $T < 9$. This is because a is timelike when $T < 9$:

$$a \cdot a = 9 - T > 0 \tag{2.35}$$

When a is timelike,

$$a \cdot y = 0, \quad y \cdot y \geq 0 \qquad \Rightarrow \qquad y = 0 \tag{2.36}$$

for any arbitrary $SO(1,T)$ vector y. This fact is used in [40] to bound the number of theories with nonabelian gauge groups, and is also crucial in bounding the space of theories with abelian factors. In particular, this fact implies that the charge vectors $\vec{q_i}$ must be linearly independent in order to get a positive definite kinetic term for the $U(1)$'s when $T < 9$. If they are not, there exists non-zero (x_i) such that $f_I(x_i) = 0$ for all I since

$$\vec{f}(x_i) \equiv (f_1(x_i), \cdots, f_N(x_i)) = \sum_i x_i \vec{q_i}\,. \tag{2.37}$$

For such x_i, we see that

$$a \cdot P(x_i) = 0, \qquad P(x_i) \cdot P(x_i) = 0\,. \tag{2.38}$$

This implies that $P(x_i) = 0$, which in turn implies that $j \cdot P(x_i) = 0$, *i.e.,*

$$\sum_{ij} (j \cdot b)_{ij} x_i x_j = 0\,. \tag{2.39}$$

This would mean that the kinetic term is not positive-definite. Hence we have proven that in order for the kinetic term to be positive-definite, $\vec{q}_i$ must be linearly independent when $T < 9$. This in particular means that we cannot have a massless $U(1)$ vector under which nothing is charged, *i.e.,* that when $T < 9$, the trivial solutions to the $U(1)$ factorization equations where all the charges are set to $q_{I,i} = 0$ are not acceptable. The analogous connection in 10D between $U(1)$ charges and the BF^2 term, which is related by supersymmetry to the gauge kinetic term [14], also played a key role in the analysis in [6] showing that the ten-dimensional supergravity theories with gauge group $U(1)^{496}$ and $E_8 \times U(1)^{248}$ are inconsistent.

The fact that $\vec{q}_i$ are all linearly independent for $T < 9$ also implies that

$$P(x_i) \cdot P(x_i) = \sum_I \mathcal{M}_I f_I(x_i)^4 > 0 \tag{2.40}$$

for all non-zero x_i as $f_I(x_i)$ cannot be made simultaneously zero for all I. We make use of (2.40) in bounding the set of abelian theories in section 3.

2.5 $T = 1$

In this section we discuss anomaly cancellation and the factorization equations in the special case of one tensor multiplet. As discussed earlier, these theories have a Lagrangian description, unlike theories with other T values. $T = 1$ string models are the most thoroughly studied string vacua in the six-dimensional string landscape. The most widely studied string constructions that give $T = 1$ vacua are K3 manifold/orbifold compactifications of the heterotic string [9, 10, 12, 14, 15, 17, 18, 19, 21, 22, 25, 33], and Calabi-Yau threefold compactifications of F-theory [21, 24, 26, 27, 28, 29, 30, 31]. These were intensively investigated as they played an essential role in understanding various string dualities.

The basis most commonly used for $T = 1$ theories in the literature is

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad a = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \qquad b = \frac{1}{2}\lambda_\kappa \begin{pmatrix} \alpha_\kappa \\ \tilde{\alpha}_\kappa \end{pmatrix}, \qquad j = \frac{1}{\sqrt{2}} \begin{pmatrix} e^\phi \\ e^{-\phi} \end{pmatrix} \tag{2.41}$$

for which the factorization condition becomes

$$I_8 = -\frac{1}{16}(\mathrm{tr}R^2 - \sum_\kappa \alpha_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \alpha_{ij} F_i F_j) \wedge (\mathrm{tr}R^2 - \sum_\kappa \tilde{\alpha}_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \tilde{\alpha}_{ij} F_i F_j)\,. \tag{2.42}$$

For the abelian factors we have

$$b_{ij} = \frac{1}{2} \begin{pmatrix} \alpha_{ij} \\ \tilde{\alpha}_{ij} \end{pmatrix}, \tag{2.43}$$

and the gravitational anomaly constraint becomes

$$H - V = 244\,. \tag{2.44}$$

The kinetic term for the antisymmetric tensor is given by

$$\mathcal{L} = -\frac{1}{2}e^{-2\phi}(dB-\omega)\cdot(dB-\omega)\,, \tag{2.45}$$

where ϕ is the dilaton. We define the Chern-Simons forms ω and $\tilde{\omega}$ as

$$d\omega = \frac{1}{16\pi^2}(\mathrm{tr}R^2 - \sum_\kappa \alpha_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij}\alpha_{ij}F_iF_j) \tag{2.46}$$

$$d\tilde{\omega} = \frac{1}{16\pi^2}(\mathrm{tr}R^2 - \sum_\kappa \tilde{\alpha}_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij}\tilde{\alpha}_{ij}F_iF_j) \tag{2.47}$$

The variation of the two form under gauge transformations becomes

$$\delta B = -\frac{1}{16\pi^2}(\sum_\kappa \alpha_\kappa \mathrm{tr}\Lambda_\kappa F_\kappa + \sum_{ij}\alpha_{ij}\Lambda_i F_j) \tag{2.48}$$

and the anomaly can be gotten rid of by adding the term

$$-B\wedge d\tilde{\omega} \tag{2.49}$$

to the Lagrangian. Supersymmetry determines the kinetic term for the gauge fields to be

$$-\sum_\kappa(\alpha_\kappa e^\phi + \tilde{\alpha}_\kappa e^{-\phi})\mathrm{tr}F_\kappa\wedge *F_\kappa - \sum_{ij}(\alpha_{ij}e^\phi + \tilde{\alpha}_{ij}e^{-\phi})F_i\wedge *F_j\,. \tag{2.50}$$

For a consistent theory without instabilities there must be a value of the dilaton such that all the gauge fields have positive kinetic terms. This means that the matrix

$$\gamma_{ij} \equiv \alpha_{ij}e^\phi + \tilde{\alpha}_{ij}e^{-\phi} = 2\sqrt{2}j\cdot b_{ij} \tag{2.51}$$

must be positive definite for some value of ϕ. Also, in order for the distinct $U(1)_i$ vector multiplets to be independent degrees of freedom, γ_{ij} must be non-degenerate.

In order to discuss the factorization equations coming from terms with abelian gauge field factors, in addition to $f_I(x_i) = q_{I,i}x_i$ it is convenient to define the quadratic forms

$$F(x_i) = \sum_{ij}\alpha_{ij}x_ix_j \qquad \tilde{F}(x_i) = \sum_{ij}\tilde{\alpha}_{ij}x_ix_j. \tag{2.52}$$

These are the components of $P^\alpha(x)$ defined through (2.32).

The factorization condition can then be summarized by the polynomial identities

$$0 = \sum_I(\mathcal{M}_I^\kappa E_\kappa^I)f_I(x_i) \qquad \text{for all } \kappa \tag{2.53}$$

$$F(x_i) + \tilde{F}(x_i) = \frac{1}{6}\sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.54}$$

$$\tilde{\alpha}_\kappa F(x_i) + \alpha_\kappa\tilde{F}(x_i) = 4\sum_I(\mathcal{M}_I^\kappa A_\kappa^I)f_I(x_i)^2 \qquad \text{for all } \kappa \tag{2.55}$$

$$F(x_i)\tilde{F}(x_i) = \frac{2}{3}\sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.56}$$

The basis chosen for the $U(1)$ factors is defined up to $GL(V_A, \mathbb{R})$:

$$\begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix} \to M \begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix}, \quad (\alpha)_{ij} \to \left(M^t(\alpha)M\right)_{ij}, \quad (\tilde{\alpha})_{ij} \to \left(M^t(\tilde{\alpha})M\right)_{ij}. \tag{2.57}$$

(α) and $(\tilde{\alpha})$ denote the matrices whose (i,j) element is α_{ij} and $\tilde{\alpha}_{ij}$, respectively.

As proven in the last section, since $T = 1 < 9$, the charge vectors $\{\vec{q_i}\}$ are linearly independent for solutions of the factorization equations that give a non-degenerate kinetic term for some value of the dilaton. Linear independence of $\vec{q_i}$ imposes positive-definiteness on both α_{ij} and $\tilde{\alpha}_{ij}$. The reason is that the r.h.s.'s of (2.54) and (2.56) are both positive for any real x_i if $\vec{q_i}$ are linearly independent. This is because $\vec{f}(x_i)$ cannot be zero for any real x_i. Hence $F(x_i)$ and $\tilde{F}(x_i)$ are positive for all real x_i. Therefore, α_{ij} and $\tilde{\alpha}_{ij}$ both have to be positive definite.

We now work out two explicit examples where we can see the equations at play. The first example is given by orbifold compactifications of the $E_8 \times E_8$ heterotic string theory [15]. This theory has gauge group $E_7 \times E_8 \times U(1)$ with 10 **56**'s and 66 singlets with respect to E_7. Nothing is charged under the E_8. This matter structure solves the non-abelian factorization equations. The non-abelian part of the anomaly polynomial factorizes to

$$-\frac{1}{16}(\mathrm{tr}R^2 - \frac{1}{6}\mathrm{tr}F_{E_7}^2 - \frac{1}{30}\mathrm{tr}F_{E_8}^2) \wedge (\mathrm{tr}R^2 - \mathrm{tr}F_{E_7}^2 + \frac{1}{5}\mathrm{tr}F_{E_8}^2). \tag{2.58}$$

We index the hypermultiplet representations **56** by $I = 1, \cdots, 10$ and the singlets by $I = 11, \cdots, 76$. Since there is only one $U(1)$, there is only a single $\alpha = \alpha_{11}$ and a single $\tilde{\alpha} = \tilde{\alpha}_{11}$. Also, $f_I(x) = q_I x$.

Therefore, the anomaly equations can be obtained by plugging in

$$F(x) = \alpha x, \quad \tilde{F}(x) = \tilde{\alpha} x, \quad f_I(x) = q_I x \tag{2.59}$$

to equations (2.53)-(2.56). Since E_7 and E_8 do not have third order invariants, and no matter is charged under E_8, we obtain

$$-\frac{1}{5}\alpha + \frac{1}{30}\tilde{\alpha} = 0 \tag{2.60}$$

$$\alpha + \tilde{\alpha} = \frac{1}{6}(56\sum_{I=1}^{10} q_I^2 + \sum_{I=11}^{76} q_I^2) \tag{2.61}$$

$$\alpha + \frac{1}{6}\tilde{\alpha} = 4\sum_{I=1}^{10} q_I^2 \tag{2.62}$$

$$\alpha\tilde{\alpha} = \frac{2}{3}(56\sum_{I=1}^{10} q_I^4 + \sum_{I=11}^{76} q_I^4) \tag{2.63}$$

This can be re-written as

$$56\sum_{I=1}^{10} q_I^4 + \sum_{I=11}^{76} q_I^4 = 36\left(\sum_{I=1}^{10} q_I^2\right)^2 = \frac{9}{196}\left(\sum_{I=11}^{76} q_I^2\right)^2. \tag{2.64}$$

Five distinct charge assignments that give solutions to these equations can be obtained by different abelian orbifold—by which we mean an orbifold whose orbifold group is abelian—compactifications. For example, there is a Z_8 orbifold compactification that assigns the charges

$$q_1 = q_2 = -3/8, \quad q_3 = q_4 = q_5 = -1/4, \quad q_8 = q_9 = q_{10} = 0, \quad q_6 = q_7 = -1/8,$$
$$q_{11} = \cdots = q_{30} = 1/8, \quad q_{31} = \cdots = q_{34} = -7/8, \quad q_{35} = q_{44} = 1/4,$$
$$q_{45} = \cdots = q_{50} = -3/4, \quad q_{51} = \cdots = q_{54} = 3/8, \quad q_{55} = \cdots = q_{58} = -5/8,$$
$$q_{59} = \cdots = q_{77} = 1/2, \quad q_{78} = \cdots = q_{66} = -1/2$$

to the hypermultiplets. The anomaly coefficients for these charge assignments are

$$\alpha = 1, \qquad \tilde{\alpha} = 6. \tag{2.65}$$

All five solutions from abelian orbifolds are given in table 1 of [15].

We present one more example that will prove to be useful later in this paper. Consider the gauge group $SU(13) \times U(1)$ with 4 two-index anti-symmetric, 6 fundamental and 23 singlet representations of $SU(13)$. These solve the anomaly equations that do not concern the $U(1)$ field strengths. The non-abelian part factorizes to

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)})\,. \tag{2.66}$$

Denoting the charges of hypermultiplets in the antisymmetric/fundamental/singlet representations as $a_x (x = 1, \cdots, 4)/f_y (y = 1, \cdots, 6)/s_z (z = 1, \cdots, 23)$ the anomaly equations become

$$0 = \sum_x 9a_x + \sum_y f_y \tag{2.67}$$
$$\alpha + \tilde{\alpha} = \frac{1}{6}(\sum_x 78a_x^2 + \sum_y 13f_y^2 + \sum_z s_z^2) \tag{2.68}$$
$$2\alpha + 2\tilde{\alpha} = 4(\sum_x 11a_x^2 + \sum_y f_y^2) \tag{2.69}$$
$$\alpha\tilde{\alpha} = \frac{2}{3}(\sum_x 78a_x^4 + \sum_y 13f_y^4 + \sum_z s_z^4) \tag{2.70}$$

If there exist for given a_x, f_y, s_z a solution $\alpha, \tilde{\alpha}$ to these equations, the anomaly polynomial factorizes into

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)} - \alpha F^2_{U(1)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)} - \tilde{\alpha} F^2_{U(1)}) \tag{2.71}$$

We identify infinite classes of charge assignments and $\alpha, \tilde{\alpha}$ values that solve these equations in section 4.

2.6 $T = 0$

We now discuss anomaly cancellation and the factorization equations in the case of $T = 0$. Theories without tensor multiplets are in some ways the simplest type of 6D supergravity theory. Although such vacua do not arise directly from geometric heterotic or type II compactifications, $T = 0$ vacua are easily constructed in F-theory by compactifications on the base $\mathbb{P}^2$, and can be reached from $T = 1$ vacua by tensionless string transitions [21, 22]. F-theory models for $T = 0$ vacua based on toric Calabi-Yau threefolds that are elliptic fibrations over $\mathbb{P}^2$ have recently been systematically studied in [50]. Analysis of 6D supergravity theories with $T = 0$ are given in [42], and F-theory vacua giving $T = 0$ vacua are systematically studied in [34]. Further references for string constructions of $T = 0$ vacua can be found in these papers.

In the case $T = 0$ all the $SO(1, T)$ vectors a, b, j reduce to numbers. $j^2 = 1$ leaves us with a sign ambiguity. Without loss of generality we can set $j = 1$. Positivity of the kinetic term imposes that the b_κ's be positive and that b_{ij} be a positive definite matrix. This and equation (2.27) sets $a < 0$. The relation $a^2 = 9 - T$ fixes $a = -3$. To summarize,

$$\Omega = 1, \qquad a = -3, \qquad j = 1\,. \tag{2.72}$$

In this case the factorization condition becomes

$$I_8 = -\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + \sum_\kappa \frac{2b_\kappa}{\lambda_\kappa}\mathrm{tr}F_\kappa^2 + \sum_{ij} 2b_{ij}F_iF_j)^2\,. \tag{2.73}$$

The gravitational anomaly constraint becomes

$$H - V = 273\,. \tag{2.74}$$

The factorization equations coming from $U(1)$'s can be written out by using the quadratic form

$$P(x_i) = \sum_{ij} b_{ij}x_ix_j\,, \tag{2.75}$$

as the polynomial identities

$$0 = \sum_I (\mathcal{M}_I^\kappa E_\kappa^I) f_I(x_i) \qquad \text{for all } \kappa \tag{2.76}$$

$$P(x_i) = \frac{1}{18}\sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.77}$$

$$P(x_i) = \frac{\lambda_\kappa}{b_\kappa}\sum_I (\mathcal{M}_I^\kappa A_\kappa^I) f_I(x_i)^2 \qquad \text{for all } \kappa \tag{2.78}$$

$$P(x_i)^2 = \frac{1}{3}\sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.79}$$

The basis chosen for the $U(1)$ factors is as usual defined up to $GL(V_A, \mathbb{R})$ through (2.34). Since $T = 0 < 9$, the charge vectors $\{\vec{q}_i\}$ are linearly independent for solutions of

the factorization equations that give a non-degenerate kinetic term for some value of the dilaton.

As in the $T = 1$ case it is useful to look at a few examples in which the anomaly equations come into play. We first consider a theory with gauge group $SU(6) \times U(1)$ with 1 adjoint, 9 two-index anti-symmetric, 18 fundamental and 31 singlet representations of $SU(6)$. These solve the non-abelian anomaly factorization equations. The factorized non-abelian anomaly polynomial is

$$-\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + 6\mathrm{tr}F_{SU(6)}^2)^2 \,. \tag{2.80}$$

Denoting the charge of hypermultiplets in the adjoint/antisymmetric/fundamental/singlet representation as $d/a_x(x = 1, \cdots, 9)/f_y(y = 1, \cdots, 9)/s_z(z = 1, \cdots, 31)$ the anomaly equations become

$$0 = \sum_x 5a_x + \sum_y f_y \tag{2.81}$$

$$b = \frac{1}{18}(35d^2 + \sum_x 15a_x^2 + \sum_y 6f_y^2 + \sum_z s_z^2) \tag{2.82}$$

$$3b = (12d^2 + \sum_x 4a_x^2 + \sum_y f_y^2) \tag{2.83}$$

$$b^2 = \frac{1}{3}(35d^4 + \sum_x 15a_x^4 + \sum_y 6f_y^4 + \sum_z s_z^4) \tag{2.84}$$

For charges and b satisfying these equations, the anomaly polynomial of the theory factorizes into

$$-\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + 6\mathrm{tr}F_{SU(6)}^2 + 2bF_{U(1)}^2)^2 \,. \tag{2.85}$$

Finding an apparently consistent supergravity theory with this gauge group amounts to identifying values for b and the charges d, a_x, f_y, s_z so that (2.81) through (2.84) are satisfied. If we assume that the $U(1)$ is compact and the charges are integers then this is a system of Diophantine equations over the integers. In general, classifying solutions to such a system of equations can be a highly nontrivial problem in number theory.

A particularly interesting class of examples are pure abelian theories. In this case the only non-trivial abelian anomaly equations are equations (2.22) and (2.25) ((2.81) and (2.84) for $T = 0$). For a theory with a given number of abelian vector multiplets, there is a lower bound on the number of charged multiplets it must have. When the number of charged hypermultiplets saturate this bound, the charges that the hypermultiplets carry is severely restricted. Such theories have a particularly simple structure and are interesting to study further.

As an example, consider the case of a purely abelian theory when $T = 0$ and $V_A = 1$. We denote the charges of the X charged hypermultiplets in the theory by $q_1, \cdots, q_X \neq 0$.

Then we must solve

$$\begin{aligned} 18b &= \sum_I q_I^2, \\ 3b^2 &= \sum_I q_I^4. \end{aligned} \tag{2.86}$$

Using the inequality

$$(\sum_I q_I^2)^2 \leq X(\sum_I q_I^4), \tag{2.87}$$

we see that

$$X \geq 108. \tag{2.88}$$

When X is equal to 108, *i.e.,* when the number of charged hypermultiplets saturates the lower bound, the only solutions to the equations (2.86) are

$$q_I = \pm Q \quad \text{for all } I, \quad b = 6Q^2. \tag{2.89}$$

Similarly for any pure abelian $T = 0$ theory, using (2.76) and (2.79) we can show that the following relation between V_A and the number of charged hypermultiplets X holds

$$\frac{324\, V_A}{V_A + 2} \leq X \leq V_A + 273. \tag{2.90}$$

The proof is given in appendix B. Hence, as above, when $V_A = 1$ there must be at least 108 charged hypermultiplets; likewise, when $V_A = 2$ there must be at least 162 charged hypermultiplets. As seen in the $V_A = 1$ case, in the marginal cases when X exactly saturates this bound, the solutions to the charge equations are particularly simple. From (2.90) it follows that the maximum possible number of $U(1)$ factors that can be included in any $T = 0$ theory with no nonabelian gauge group is $V_A \leq 17$.[4]

A family of marginal/nearly marginal $T = 0$ theories with gauge group $U(1)^k, k \leq 7$ can be obtained by Higgsing an $SU(8)$ theory with one adjoint hypermultiplet and nine antisymmetric hypermultiplets. The number of charged hypermultiplets X for the various pure abelian theories one obtains by Higgsing the adjoint of this theory in different ways is summarized in table 3. An F-theory construction of this $SU(8)$ model, which has $b = 3$, through an explicit Weierstrass model is described in [34]. In principle the adjoint in this construction can be Higgsed to give an F-theory description of the full family of marginal/near marginal $U(1)^k$ models, though we have not worked out the details of this Higgsing.

Explicit F-theory compactifications are known for the first four theories on this table. Six-dimensional $\mathcal{N} = (1, 0)$ theories with $T = 0$ can be obtained by F-theory compactifications on Calabi-Yau threefolds that are elliptic fibrations of $\mathbb{P}^2$ [21]. Such a Calabi-Yau

[4] (2.90) alone implies that $V_A \leq 17$ or $V_A \geq 32$. An additional constraint following from equations (2.76) and (2.79) is needed to obtain the desired bound. We derive this constraint and show that indeed $V_A \leq 17$ in appendix B.

Gauge Group	·	$U(1)$	$U(1)^2$	$U(1)^3$	$U(1)^4$	$U(1)^5$	$U(1)^6$	$U(1)^7$
X	0	108	162	198	225	243	252	252
$324\,V_A/(V_A+2)$	0	108	162	194.4...	216	231.4...	243	252
X'	273	166	113	78	52	35	27	28

Table 3: The number of charged hypermultiplets X for pure abelian theories obtained by Higgsing the adjoint of the $SU(8)$ theory with one adjoint and nine antisymmetrics. We have also tabulated the number of uncharged hypermultiplets in the theory, $X' = (273 + V_A - X)$.

threefold that is non-singular can be expressed as a degree 18 hypersurface in the projective space $\mathbb{P}[1,1,1,6,9]$ [51], which is denoted by

$$X_{18}[1,1,1,6,9]^{2,272}. \tag{2.91}$$

The subscript denotes the degree of the hypersurface, the number in the brackets parametrize the projective space, and the two superscripts denote the $h_{1,1}$ and $h_{2,1}$ values of the manifold. For $T = 0$ vacua, the total rank of the gauge group is given by $(h_{1,1} - 2)$ and the number of uncharged hypermultiplet is given by $(h_{2,1} + 1)$ [21]. It is easy to check that the data of this manifold reproduces the first theory in table 3.

There is a general process by which one can replace the fiber-type of an elliptically fibered manifold to generate a different manifold [51]. From the point of view of stringy geometry, one can understand this as a conifold transition between topologically distinct manifolds [52]. Three manifolds can be generated from $X_{18}[1,1,1,6,9]^{2,272}$ by successive conifold transitions. They are given by

$$X_{12}[1,1,1,3,6]^{3,165}, \qquad X_9[1,1,1,3,3]^{4,112}, \qquad X_{6,6}[1,1,1,3,3,3]^{5,77}. \tag{2.92}$$

At a generic point in the complex moduli space, theories obtained by compactifying on these manifolds do not have nonabelian gauge symmetry. Comparing the numbers with table 3, we find that the massless spectrum of the six-dimensional theories obtained by F-theory compactifications on the three manifolds of (2.92) coincides with the massless spectrum of the second, third and fourth theories of table 3 with gauge groups $U(1)^k, k = 1, 2, 3$. We do not know how to continue this process to construct an explicit geometry realizing a theory with gauge group $U(1)^4$.

2.7 Linear Multiplets and Generalized Green-Schwarz Anomaly Cancellation

In this section we discuss linear multiplets and their role in the generalized Green-Schwarz anomaly cancellation mechanism. We first discuss how two different types of hypermultiplets can be distinguished when we consider their representation under $SU(2)_R$. Then we show how each multiplet couples to vector multiplets. In particular, we show how a linear multiplet can couple to an abelian vector multiplet and form a long multiplet. Next we depict the role that linear multiplets play in the generalized Green-Schwarz anomaly cancellation mechanism. Lastly we show that we may ignore long multiplets formed in this way and the generalized Green-Schwarz mechanism when we are discussing the massless

spectrum of the theory. Most of the information on linear multiplets given in this section can be found in [53].

There are two different kinds of hypermultiplets in supersymmetric 6D theories with 8 supercharges. The scalar components of the hypermultiplet can transform either as a complex **2** or a real $\mathbf{3}+\mathbf{1}$ under the $SU(2)_R$ symmetry of the theory. We refer to the first type of hypermultiplet simply as hypermultiplets, and the second kind of hypermultiplet as linear multiplets. As far as their contribution to the gravitational anomaly are concerned, the two kinds of hypermultiplets behave identically. The fermions of the linear multiplet are not charged under any gauge group, so the contribution of the linear multiplet to the anomaly is equivalent to that of a neutral hypermultiplet, as shown shortly.

As stated above, under the $SU(2)_R$ symmetry, the scalar components of the hypermultiplet transform as a complex **2**. The spinors, on the other hand are neutral, *i.e.,* singlets (**1**). Meanwhile, the scalar components of the linear multiplet transform as a real $\mathbf{3}+\mathbf{1}$. The spinors transform as **2**'s.

To see how these multiplets couple to other fields, it is useful to reduce to four dimensions on a two-torus and write out the Lagrangian in terms of $\mathcal{N}=1$ superfields. Both multiplets, when dimensionally reduced, are $\mathcal{N}=2$ fields that consist of two chiral superfields. The hypermultiplets can transform in a non-trivial representation of the gauge group and consist of two chiral superfields Q and $\tilde{Q}$. In this case, the representation of Q must be the conjugate of that of $\tilde{Q}$. It is well known that this multiplet couples to the $\mathcal{N}=2$ vector multiplet that consists of a vector multiplet V and a chiral multiplet Φ in the adjoint representation as

$$\int d^4x d^4\theta \, (Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V}\tilde{Q}) + \int d^4x d^2\theta \, \tilde{Q}^T \Phi Q + (h.c.)\,. \tag{2.93}$$

Meanwhile, the linear multiplets couple to other fields in quite a different manner [53]. They cannot couple to gauge fields in the standard way, as the Lagrangian would not be $SU(2)_R$ invariant in this case. They can couple to $U(1)$ gauge fields, however. The linear multiplet consists of two chiral fields C and B and couples to $U(1)$ gauge fields as

$$\int d^4x d^4\theta \, (\frac{1}{2}(iC - iC^\dagger - V)^2 + B^\dagger B) - \frac{1}{\sqrt{2}} \int d^4x d^2\theta \, B\Phi + (h.c.)\,. \tag{2.94}$$

Writing the scalar of C as $(\pi_3 + i\phi)$ and the scalar of B as $(\pi_1 + i\pi_2)$, the kinetic terms for the scalars become

$$-\int d^4x ((\partial_\mu \phi - \frac{1}{2}A_\mu)^2 + (\partial_\mu \pi_i)^2)\,. \tag{2.95}$$

The ϕ can be gauged away using the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda, \qquad \phi \to \phi + \frac{1}{2}\Lambda\,, \tag{2.96}$$

and the $U(1)$ gauge field obtains mass 1/2. The $U(1)$ gauge field has recived a mass by the Stückelberg mechanism.

By integrating out the F-terms of the linear multiplet, we see that the scalar in Φ recieves the same mass(1/2). Meanwhile, the fermions do not couple to the gauge field,

and hence only contribute to gravitational anomalies. They only couple to the fermions in V and Φ through Dirac mass terms, *i.e.,* fermions of C and B pair up with fermions of V and Φ into two Dirac fermions of mass $1/2$.

Combining the auxiliary fields of V and Φ, we get three real auxiliary fields (the 'D fields' for the $\mathcal{N} = 2$ vector multiplet) that are in the **3** of $SU(2)_R$. These couple to the scalars transforming as the **3**:

$$-\int d^4x(\pi_i D^i) \tag{2.97}$$

Expanding around a vacuum with $\pi_i = 0$, the $U(1)$ vector multiplet and linear multiplet together form a long $\mathcal{N} = 2$ multiplet with 5 scalars, 2 Dirac fermions, and a vector field, all of mass $1/2$ in units of the mass parameter. Note that this long massive spin-1 multiplet is not chiral, as the fermions are Dirac.

When we have linear multiplets, they may be used to cancel anomalies. As discussed in section 2.3, it is possible to cancel anomalies of the form

$$F_i \wedge X_6\,, \tag{2.98}$$

where X_6 is a six form, by adding the term

$$-\phi \wedge X_6\,. \tag{2.99}$$

ϕ is a Stückelberg 0-form inside a linear multiplet.

In order for the generalized anomaly cancellation to work, we must have a linear multiplet at our disposal. If we do not have such a linear multiplet, we cannot get rid of the term and hence the theory would be anomalous. In case we have such a multiplet, through the Stückelberg mechanism, we expect the linear multiplet to be eaten to form a long massive spin-1 multiplet. Schematically, we may write

$$L_i = V_i + H_i\,, \tag{2.100}$$

where L_i denotes the long multiplet, V_i denotes the $U(1)$ vector multiplet, and H_i the linear multiplet.

So we see that all the vector bosons of $U(1)$ gauge symmetries whose anomalies are cancelled in this fashion must be massive and must form a long multiplet. These long multiplets are non-chiral and hence do not contribute to gravitational anomalies. Furthermore none of the fields inside this multiplet are charged under other gauge groups. Therefore, we see that these multiplets contribute neither to gravitational anomalies nor to unbroken gauge/mixed anomalies.

By this logic we can further state that all long multiplets obtained by $U(1)$ gauge bosons coupling to linear multiplets do not contribute to the anomaly polynomial. Therefore, we may ignore all the long multiplets—or vector/linear multiplet pairs that couple—when we are discussing gravitational anomalies and gauge/mixed anomalies concerning unbroken gauge symmetry, *i.e.,* gauge symmetry of the massless spectrum.

Long multiplets and hence the generalized Green-Schwarz mechanism may thus be ignored when we are discussing the massless spectrum of the theory. In other words, when

we are constructing low-energy effective theories, writing down anomalous $U(1)$'s and then lifting them is a redundant procedure. We may safely restrict our attention to the massless spectrum whose anomalies are all cancelled by two-forms; the factorization condition (2.6) should hold for these theories.

3. Bounds on $T < 9$ Theories With $U(1)$'s

We now address the first (I) of the four questions raised in the introduction. That is, we prove that the number of different gauge/matter structures— specified by the gauge group and the non-abelian representation of the matter—is finite for theories with $T < 9$, when we ignore the charge of the matter under the $U(1)$'s.

The strategy we pursue is the following. First in section 3.1, we prove that in a non-anomalous theory, the number of $U(1)$'s is bounded by a number determined by the non-abelian gauge/matter content. We prove that the relations

$$V_A \leq (T+2)\sqrt{2N} + 2(T+2) \tag{3.1}$$

$$V_A \leq (T+2)(T+\frac{7}{2}) + (T+2)\sqrt{2V_{NA} + (T^2 - 51T + \frac{2225}{4})} \tag{3.2}$$

hold for non-anomalous theories with $T < 9$, where V_A is the rank of the abelian gauge group, V_{NA} is the number of nonabelian vector multiplets, and N is the number of hypermultiplet representations. These bounds imply that the number of $U(1)$'s one could add to a non-abelian theory is finite. We note that these bounds are in no sense optimal; they could be improved by a more careful analysis. These inequalities, however, will be sufficient for the purpose of proving that there is a finite bound on theories with $T < 9$.

In section 3.2 we define the concept of 'curable theories' as non-abelian theories with $H - V > 273 - 29T$ that can be made non-anomalous by adding $U(1)$ vector fields and without changing the non-abelian gauge/matter structure. Curable theories are defined so that all non-anomalous theories with abelian gauge symmetry can be obtained by adding $U(1)$'s either to non-anomalous theories, or to curable theories. We then show that the number of curable theories is finite for $T < 9$, which combined with our other results implies that the number of gauge/matter structures possible for non-anomalous theories with $T < 9$ is finite.

In section 3.3 we construct an infinite class of non-anomalous theories with an unbounded number of $U(1)$'s and $T \geq 9$.

3.1 Bound on Number of $U(1)$ Factors

In this section we prove equations (3.1) and (3.2) for non-anomalous theories with $T < 9$. Given a gauge group

$$\mathcal{G} = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa \times \prod_{i=1}^{V_A} U(1)_i \,, \tag{3.3}$$

we show that the bound on V_A can be given as a function of the number of nonabelian vector multiplets

$$V_{NA} = \sum_\kappa d_{\mathrm{Adj}_\kappa} \tag{3.4}$$

and N, the number of nonabelian matter representations.

This can be done by making use of equation (2.30), which is equivalent to (2.25) :

$$\begin{aligned} P(x_i)\cdot P(x_i) &= \frac{1}{3}\sum_I \mathcal{M}_I f_I(x_i)^4 \\ \Leftrightarrow b_{ij}\cdot b_{kl} + b_{ik}\cdot b_{jl} + b_{il}\cdot b_{jk} &= \sum_I \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} \end{aligned} \tag{3.5}$$

We should be looking for integral solutions of this equation for $b_{ij}, q_{I,i}$, but for now we simply determine the conditions for the equations to have real solutions. These conditions impose a bound on V_A, which also is a bound for integral solutions. These equations have a $GL(n,\mathbb{R})$ invariance summarized by (2.57) where the matrices M now can be taken to be real.

We first state the following useful

Fact : For $(T+1)$ symmetric $n\times n$ matrices $S_1,\cdots,S_{T+1}$, there exists a matrix $M\in GL(n,\mathbb{R})$ such that for $\tau\equiv\lceil n/(T+2)\rceil$ the matrices $S'_\alpha = M^t S_\alpha M$ satisfy

$$(S'_\alpha)_{kl} = 0 \qquad \text{for distinct } k,l\le\tau \tag{3.6}$$

for all $\alpha = 1,\cdots,(T+1)$.

Proof : First pick an arbitrary n-dimensional vector e_1. Then generate the set of $(T+2)$ vectors

$$V_1 = \{e_1, S_1 e_1, \cdots, S_{(T+1)} e_1\}\,. \tag{3.7}$$

When $1 < n/(T+2)$ there always exists a non-zero vector that is orthogonal to these $(T+2)$ vectors. Pick one and call it e_2. Then generate the set of $(T+2)$ vectors

$$V_2 = \{e_2, S_1 e_2, \cdots, S_{(T+1)} e_2\}\,. \tag{3.8}$$

When $2 < n/(T+2)$ there always exists a non-zero vector that is orthogonal to the set $V_1\cup V_2$ of vectors. Pick one and call it e_3. By iterating this process we can obtain τ non-zero mutually orthogonal vectors,

$$e_1,\cdots,e_\tau \tag{3.9}$$

such that

$$e_i^t S_\alpha e_j = 0 \quad \text{for } i\ne j \tag{3.10}$$

for all α. We can then choose vectors $e_{\tau+1},\cdots,e_n$ that together with $e_1,\cdots e_\tau$ form a basis of $\mathbb{R}^n$. Define

$$M = (e_1\cdots e_n)\,, \tag{3.11}$$

where e_i are column vectors. It is clear that $\det M\ne 0$ and that for $S'_\alpha = M^t S_\alpha M$

$$(S'_\alpha)_{kl} = 0 \qquad \text{for distinct } k,l\le\tau \tag{3.12}$$

for all α. □

Due to this fact, there exists a matrix $M \in GL(V_A, \mathbb{R})$ such that for all α

$$M_{ik} M_{jl} b^{\alpha}_{kl} = 0 \quad \text{for distinct } i, j \leq \tau \tag{3.13}$$

for any solution of (3.5). We have defined

$$\tau = \left\lceil \frac{V_A}{T+2} \right\rceil . \tag{3.14}$$

This means that the existence of a solution of (3.5) implies the existence of a solution of the same equations with

$$\vec{b}_{kl} = 0 \quad \text{for distinct } k, l \leq \tau \, . \tag{3.15}$$

Therefore, we may from now on assume that this condition is true.

For ordered pairs (i, j) with $i < j \leq \tau$, we define the vectors

$$\vec{Q}_{ij} \equiv (\sqrt{\mathcal{M}_1} q_{1,i} q_{1,j}, \sqrt{\mathcal{M}_2} q_{2,i} q_{2,j}, \cdots, \sqrt{\mathcal{M}_N} q_{N,i} q_{N,j}) \, . \tag{3.16}$$

Then we have

$$\vec{Q}_{ij} \cdot \vec{Q}_{kl} = \sum_I \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} = b_{ij} \cdot b_{kl} + b_{ik} \cdot b_{jl} + b_{il} \cdot b_{jk} = 0 \tag{3.17}$$

for ordered pairs $(i, j) \neq (k, l)$. Also from equation (2.25), we have

$$\vec{Q}_{ij} \cdot \vec{Q}_{ij} = \sum_I \mathcal{M}_I q^2_{I,i} q^2_{I,j} = b_{ii} \cdot b_{jj} + 2 b_{ij} \cdot b_{ij} = b_{ii} \cdot b_{jj} > 0. \tag{3.18}$$

The last inequality holds due to the fact that $\vec{b}_{ii}$ are timelike vectors since

$$|\vec{b}_{ii}|^2 = \frac{1}{3} \sum_I \mathcal{M}_I q^4_{I,i} > 0 \, , \tag{3.19}$$

as $\vec{q}_i$ cannot be a zero vector. The inner-product of two timelike $SO(1, T)$ vectors cannot be zero, and $\sum_I \mathcal{M}_I q^2_{I,i} q^2_{I,j}$ cannot be negative and hence the inequality in (3.18).

Therefore, $\vec{Q}_{ij}$ are non-zero mutually orthogonal vectors for $i < j \leq \tau$. Thus, we have $\tau(\tau - 1)/2$ non-zero orthogonal vectors in an N-dimensional space. Hence

$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq \frac{\tau(\tau-1)}{2} \leq N \leq H \leq V_{NA} + V_A + 273 - 29T. \tag{3.20}$$

Using the two inequalities

$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq N \tag{3.21}$$

$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq V_{NA} + V_A + 273 - 29T, \tag{3.22}$$

we obtain the bounds

$$V_A \leq (T+2)\sqrt{2N} + 2(T+2) \tag{3.23}$$

$$V_A \leq (T+2)(T + \frac{7}{2}) + (T+2)\sqrt{2V_{NA} + (T^2 - 51T + \frac{2225}{4})} \, , \tag{3.24}$$

as promised. We have used the fact that $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$ for non-negative a, b to simplify the first inequality. The second inequality simply follows from solving (3.22) for V_A when the inequality is saturated. This result implies that given a non-anomalous non-abelian theory, the number of $U(1)$'s one could add to the theory keeping it non-anomalous is bounded.

The equations we have used also apply to pure abelian theories. This is because we have not used any constraint coming from the non-abelian structure of the theory; we have only used the equation (3.5). Hence we can obtain a bound on the number of $U(1)$'s when the theory is purely abelian:

$$V_A \leq (T+2)(T+\frac{7}{2}) + (T+2)\sqrt{T^2-51T+\frac{2225}{4}} \tag{3.25}$$

Note that this bound is substantially weaker than the tighter bound $V_A \leq 17$ for $T=0$. (For $T=0$ this bound states that $V_A \leq 54$, while we show that $V_A \leq 17$ in appendix B.)

3.2 Curability and Finiteness of Curable Theories

We define 'curable' theories to be non-abelian theories that violate the gravitational anomaly bound $H-V > 273-29T$, but whose anomaly polynomial can nonetheless be made factorizable by adding $U(1)$ vector multiplets and some singlet hypermultiplets in such a way that the gravitational bound is satisfied. There should also exist values for the scalars in the tensor multiplets that make the kinetic terms of all gauge fields positive in the resulting non-anomalous theory. We also assume that these theories do not have any hypermultiplets that are singlets under the non-abelian gauge group.

From this definition it is clear that all non-anomalous theories with abelian gauge symmetry can be obtained by the following steps.

1. Begin with a theory without abelian gauge group factors that is either non-anomalous or curable.

2. Add abelian vector multiplets and (possibly) hypermultiplets in the trivial representation of the non-abelian gauge group.

3. Assign $U(1)$ charges to the matter.

We note that it is clear that the number of $U(1)$'s one could add to a given curable theory is finite, since it is bounded by (3.23) and (3.24). From this it is evident that the crucial remaining step in obtaining bounds on theories with abelian gauge symmetry is showing that the number of curable theories is bounded.

As an example of a curable theory, consider the $T=0$ theory with gauge group and matter content

$$SU(9): \qquad 26\times\square + 1\times\begin{smallmatrix}\square\\\square\end{smallmatrix} + 1\times\begin{smallmatrix}\square\\\square\end{smallmatrix}, \qquad (H-V=274)\,. \tag{3.26}$$

Although this theory violates the gravitational anomaly bound, it satisfies the other gauge/mixed anomaly equations with $b=2$. (Note that, as explained in section 2.6, when $T=0$ the

anomaly coefficients a, b are numbers.) The theory (3.35) can be cured by adding a single $U(1)$ vector multiplet and assigning charges to the matter in the following way

$$\begin{aligned} SU(9)\times U(1): \quad & 6\times(\square,+1)+6\times(\square,-1)+14\times(\square,\cdot)+ \\ & 1\times(\begin{smallmatrix}\square\\\square\end{smallmatrix},\cdot)+1\times(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\cdot), \qquad (H-V=273)\,. \end{aligned} \tag{3.27}$$

The anomaly polynomial of the final theory factorizes to

$$I_8=-\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2+2\mathrm{tr}F_{SU(9)}^2+6F_{U(1)}^2)^2\,. \tag{3.28}$$

A systematic classification of $T=0$ supergravity models with $SU(N)$ gauge groups and without anomalies or other inconsistencies was given in [42], and F-theory constructions of such models were analyzed in[34]. The methods and results of those papers, in which abelian gauge group factors were not treated, can be expanded to include curable models such as this $SU(9)$ theory. In particular, this presents a particularly simple example of a model with a 3-index antisymmetric representation of $SU(9)$ for which an F-theory realization might be constructed.

Since abelian vector multiplets and singlet hypermultiplets do not appear in a curable theory and do not contribute to the nonabelian gauge/mixed anomalies, it is clear that the anomaly polynomial of a curable theory takes the form

$$\begin{aligned} I_8=&-\frac{(H-V-273+29T)}{5760}\left(\mathrm{tr}R^4+\frac{5}{4}(\mathrm{tr}R^2)^2\right) \\ &-\frac{1}{32}\Omega_{\alpha\beta}\left(\frac{1}{2}a^\alpha\mathrm{tr}R^2+\sum_\kappa(\frac{2b_\kappa^\alpha}{\lambda_\kappa})\mathrm{tr}F_\kappa^2\right)\left(\frac{1}{2}a^\beta\mathrm{tr}R^2+\sum_\kappa(\frac{2b_\kappa^\beta}{\lambda_\kappa})\mathrm{tr}F_\kappa^2\right)\,, \end{aligned} \tag{3.29}$$

where $H-V$ is larger than $273-29T$. One might think that any theory of this type is naively curable, since we could apparently add an arbitrary number of $U(1)$ vector multiplets under which no matter field is charged, so that $H-V'=273-29T$. The kinetic term for these vector fields, however, would be degenerate—in fact zero—if we do so. In fact, in many cases the bounds on the number of $U(1)$ factors that can be added to a theory make it impossible to cure nonabelian theories with anomalies of the form (3.29).

In [38, 40] it was proven that the number of distinct nonabelian gauge groups and matter representations possible for theories with $T<9$ and no $U(1)$ factors is finite. The bound $H_{\mathrm{charged}}-V\leq 273-29T$ from the gravitational anomaly condition played a key role in this proof, limiting the number of charged hypermultiplets that could appear in a theory with any given nonabelian gauge group. To prove that the number of curable theories is also finite for $T<9$ we need an analogous constraint on the number of hypermultiplets for theories with $U(1)$ factors. We now find such a bound, using the bounds (3.23) and (3.24) on the number of $U(1)$ factors that can be added to a curable theory.

Suppose a theory is curable by adding V_A $U(1)$ vector multiplets and H' hypermulti-

plets. Then using (3.23) we obtain

$$\begin{aligned} 273 - 29T \geq (H-V)_{\text{cured theory}} &= H - V + H' - V_A \\ &\geq H - V + H' - \sqrt{2}(T+2)\sqrt{H'+N} - 2(T+2) \\ &= (H-V) + (\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N - (\frac{1}{2}T^2 + 4T + 6)\,, \end{aligned} \tag{3.30}$$

where H, V and N denote the numbers of hypermultiplets, vector multiplets and hypermultiplet representations in the initial non-abelian theory. Since $H' \geq 0$, when $N \leq (T+2)^2/2$ we have

$$(\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N \geq -\frac{(T+2)^2}{2}\,, \tag{3.31}$$

while when $N \geq (T+2)^2/2$ we have

$$(\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N \geq (\sqrt{N} - \frac{(T+2)}{\sqrt{2}})^2 - N = -\sqrt{2}(T+2)\sqrt{N} + \frac{(T+2)^2}{2}\,. \tag{3.32}$$

Thus, any curable theory satisfies one of the following two constraints:

$$H - V \leq 273 - 29T + (T^2 + 6T + 8) \tag{3.33}$$

$$H - V - \sqrt{2}(T+2)\sqrt{N} \leq 273 - 29T + (2T+4) \tag{3.34}$$

Curable theories therefore must satisfy the non-abelian factorization equations (2.15)–(2.19) and one of these modified gravitational anomaly constraints.

This result suggests that the proof in [38, 40] can be modified to show that the number of curable theories are in fact finite. There it was shown that the H of theories that obey the non-abelian factorization equations—and can have a positive kinetic term—grew faster than the V of the theory when V became large. This in turn implied that V must be bounded for theories that satisfy the the non-abelian factorization equations and respect the $H - V$ bound. We have shown that curable theories must obey the same non-abelian factorization equations with the $H - V$ constraint modified. Fortunately, this constant is only modified by a term subleading in $N < H$. This suggests that the boundedness of curable theories can be shown along the same lines as the proof of boundedness of non-abelian theories. This is indeed the case, though the added term proportional to $\sqrt{N}$ complicate some parts of the analysis. The details of the full proof of this statement are presented in appendix A.

We note that the equations (3.33) and (3.34) enable us to identify many uncurable theories with ease. For example, it can be shown that the $T = 0$ theory with gauge group and matter content

$$SU(7): \qquad 27 \times \square + 1 \times \begin{array}{|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}\,, \qquad (H - V = 351) \tag{3.35}$$

is uncurable, since

$$H - V > 273 - 29T + (T^2 + 6T + 8) = 281 \tag{3.36}$$

$$H - V > 273 - 29T + (2T+4) + \sqrt{2}(T+2)\sqrt{N} = 277 + 2\sqrt{56} = 291.9... \tag{3.37}$$

To summarize, we have defined 'curable theories' to be supergravity theories that satisfy the following conditions:

1. The gauge group is non-abelian.
2. The theory has no singlet hypermultiplets.
3. $H - V > 273 - 29T$
4. The theory can be made non-anomalous by adding $U(1)$ vector fields that are independent degrees of freedom, as well as possibly adding singlet hypermultiplets.
5. In the resulting non-anomalous theory, there exists a choice for the scalars in the tensor multiplets that makes the kinetic terms of all gauge fields positive.

We have proven the following facts:

1. The number of non-anomalous non-abelian theories is finite [38].
2. The number of $U(1)$'s one can add to non-anomalous theories is finite.
3. The number of curable theories is finite.
4. The number of $U(1)$'s one can add to curable theories is finite.

As pointed out in the beginning of this section, any non-anomalous theory with $U(1)$'s can be constructed by adding abelian vector multiplets and neutral hypermultiplets to a non-anomalous or curable theory with no abelian gauge symmetry. Hence it follows that there is only a finite number of distinct gauge/matter structures a 6D $(1,0)$ theory could have even when we allow abelian components to the gauge group. In particular, this implies that the total rank of the gauge group is bounded, even when we admit abelian factors in the gauge group.

3.3 $T \geq 9$

In this section, we show that for $T \geq 9$ a bound cannot be imposed on the number of $U(1)$'s as we have done in the case $T < 9$. We first show that there are certain classes of theories to which one could add an arbitrary number of $U(1)$'s, and discuss why this is not possible when $T < 9$. We end with an example of an infinite class of non-anomalous theories with an unbounded number of $U(1)$'s.

Suppose we have a theory $\mathcal{T}_0$ with gauge group $\mathcal{G}_0$ that satisfies all the anomaly equations and has an $SO(1,T)$ unit vector j_0 that satisfies $j_0 \cdot b_\kappa > 0$ for all gauge groups κ. Denote the number of vector and hypermultiplets of this theory as V_0 and H_0.

Suppose an $SO(1,T)$ vector b that satisfies the following conditions exists:

1. b is light-like, *i.e.,* $b^2 = 0$.
2. $a \cdot b = 0$.
3. $b_\kappa \cdot b = 0$ for all κ.

4. $b \cdot j_0 > 0$.

Recall that in the case $T < 9$ it is impossible for a vector b to satisfy conditions 1, 2 and 4 at the same time. In that case a is a time-like vector and if 1 and 2 are satisfied, b must be a zero vector. This is what prevented us from having a $U(1)$ with nothing charged under it.

The situation is quite different when $T \geq 9$; in this case a vector b satisfying the four conditions above is not ruled out in general. Once such a b is available one could construct theory $\mathcal{T}_k$ from $\mathcal{T}_0$ with the following properties.

1. The gauge group is $\mathcal{G}_k = \mathcal{G}_0 \times U(1)^k$.
2. The matter content is that of $\mathcal{T}_0$ with k neutral hypermultiplets added.
3. Nothing is charged under the $U(1)$'s, *i.e.,* $q_{I,j} = 0$ for all I, j.
4. The non-abelian anomaly coefficients are given by b_κ.
5. The abelian anomaly coefficients are given by $b_{ij} = \delta_{ij} b$.
6. The tensor multiplet scalar vacuum expectation value is given by j_0.

By adding the k neutral hypermultiplets, the gravitational anomaly condition,

$$H_k - V_k = (H_0 + k) - (V_0 + k) = H_0 - V_0 = 273 - 29T \tag{3.38}$$

is satisfied. The non-abelian anomaly factorization conditions are all satisfied by definition. We find that all the $U(1)$ anomaly equations (2.22)-(2.25) are also satisfied as both sides of the equation turn out to be 0. Also,

$$j_0 \cdot b_{ij} = (j_0 \cdot b)\delta_{ij} \tag{3.39}$$

is a positive definite matrix by the assumption that $b \cdot j_0 > 0$. Therefore, this theory satisfies all the anomaly equations and has a sensible kinetic term. Since this is true for any k we find that we could add an infinite number of $U(1)$'s to $\mathcal{T}_0$.

We conclude this section by presenting an explicit example of an infinite class of theories with an unbounded number of $U(1)$ factors in the gauge group. The simplest case, when there are no non-abelian factors, turns out to serve our purpose. A $U(1)^k$ theory with $273 - 29T + k$ neutral hypermultiplets and a, b_{ij} given by

$$a = (-3, 1 \times T, 0, \cdots, 0), \quad b_{ij} = b\delta_{ij} \quad \text{for } b = (3, (-1) \times 9, 0, \cdots, 0) \tag{3.40}$$

satisfies all the factorization equations. $x \times n$ denotes that n consecutive components have the same value x. Defining

$$j = (1, 0, 0, 0, 0, \cdots, 0)\,, \tag{3.41}$$

we find that the matrix for the kinetic term of the vector multiplets

$$j \cdot b_{ij} = 3\,\delta_{ij} \tag{3.42}$$

is positive definite. k is bounded below by $29T - 273$ but has no upper-bound.

4. Infinite Classes of Non-anomalous Theories with $U(1)$'s

In this section, we investigate the second and third questions (II and III) posed in the introduction, beginning with II: Given the gauge/matter content of the theory—by which we mean that we fix the gauge group and the representations of the hypermultiplets with respect to the non-abelian part of the gauge group—are there an infinite number of solutions to the $U(1)$ charge equations? We denote these $U(1)$ charge equations 'hypercharge' equations.

As pointed out in the introduction, there are infinite families of solutions that may be 'trivially generated' in the following sense. There certainly exist solutions of the anomaly equations with gauge group $\mathcal{G} = \mathcal{G}_0 \times U(1)^2$. In such a case, denoting the charge vectors with respect to the two $U(1)$'s $\vec{q}_1$ and $\vec{q}_2$, any linear combination $\vec{Q} = r\vec{q}_1 + s\vec{q}_2$ solves the anomaly equation for $\mathcal{G}' = \mathcal{G}_0 \times U(1)$ with the same matter structure. On top of the anomaly cancellation conditions, we may demand that additional consistency conditions be obeyed [54, 55, 56, 57]. Three such conditions are applicable to six-dimensional supergravity theories with compact $U(1)$ abelian factors:

1. **Charge Integrality Constraint :** All charges of particles should be integral with respect to the minimal charge of the $U(1)$'s.

2. **Minimal Charge Constraint :** The greatest common divisor of the charges of all particles under each $U(1)$ should coincide with the minimal charge–or inverse of the periodicity–of the $U(1)$.

3. **Unimodularity Constraint :** The string charge lattice spanned by the anomaly coefficients should be embeddable in a unimodular lattice.

The first and second constraints do not stop us from generating an infinite familiy because if the initial theory with $\mathcal{G} = \mathcal{G}_0 \times U(1)^2$ satisfied the charge integrality constraint and the minimal charge constraint, the new theory would also satisfy this constraint when r, s are taken to be mutually prime integers. In many cases the unimodularity constraint does not help either, as we see shortly.

Let us depict the situation with the simplest example. For $\mathcal{G} = U(1)^2$, $T = 1$ the following charges on the 246 hypermultiplets of the theory solve the anomaly equations. Assume that there are 48 hypermultiplets with charge $(0, 1)$, 48 hypermultiplets with charge $(1, 0)$, 48 hypermultiplets with charge $(1, 1)$ and 102 neutral hypermultiplets. Written in terms of charge vectors

$$\vec{q}_1 = (1 \times 96, 0 \times 48, 0 \times 102), \qquad \vec{q}_2 = (0 \times 48, 1 \times 96, 0 \times 102)\,, \tag{4.1}$$

where $q \times n$ denotes that n consecutive components have the same value q. The only

non-trivial anomaly equations concerned are

$$\frac{1}{6}(48x_1^2+48(x_1+x_2)^2+48x_2^2)=(\alpha_{11}+\tilde{\alpha}_{11})x_1^2+2(\alpha_{12}+\tilde{\alpha}_{12})x_1x_2+(\alpha_{22}+\tilde{\alpha}_{22})x_2^2 \tag{4.2}$$

$$\frac{2}{3}(48x_1^4+48(x_1+x_2)^4+48x_2^4)=(\alpha_{11}x_1^2+2\alpha_{12}x_1x_2+\alpha_{22}x_2^2)(\tilde{\alpha}_{11}x_1^2+2\tilde{\alpha}_{12}x_1x_2+\tilde{\alpha}_{22}x_2^2) \tag{4.3}$$

Both equations are satisfied by the choice

$$\alpha_{11}=\alpha_{22}=2\alpha_{12}=\tilde{\alpha}_{11}=\tilde{\alpha}_{22}=2\tilde{\alpha}_{12}=8\,. \tag{4.4}$$

Therefore

$$\vec{Q}=(r\times 48,(r+s)\times 48,s\times 48,0\times 102) \tag{4.5}$$

satisfy the equations

$$\frac{1}{6}(48r^2+48(r+s)^2+48s^2)=16r^2+16rs+16s^2 \tag{4.6}$$

$$\frac{2}{3}(48r^4+48(r+s)^4+48s^4)=(8r^2+8rs+8s^2)^2\,. \tag{4.7}$$

It is easy to see that this choice of charges solves the anomaly equation for $\mathcal{G}=U(1)$ with $\alpha=\tilde{\alpha}=(8r^2+8rs+8s^2)$. Therefore, we obtain an infinite class of solutions to the anomaly equations for $\mathcal{G}=U(1)$.

It is clear that imposing the charge integrality constraint and the minimal charge constraint does not stop us from generating this infinite family as we may take r and s to be mutually prime integers. Now we show that the unimodularity constraint is also satisfied when r and s are integers.

It is useful to notice that when $T=1$, a sufficient condition for the unimodularity constraint is that all the anomaly coefficients α and $\tilde{\alpha}$ defined in section 2.5 are even integers. This is because if all α and $\tilde{\alpha}$ are even integers, all string charge vectors

$$a=\begin{pmatrix}-2\\-2\end{pmatrix},\quad b=\frac{1}{2}\begin{pmatrix}\alpha\\\tilde{\alpha}\end{pmatrix} \tag{4.8}$$

are embeddable in the unimodular lattice spanned by

$$\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.9}$$

with inner product structure Ω as defined in (2.41). When r,s are integers, α and $\tilde{\alpha}$ of the $U(1)$ are both equal to $(8r^2+8rs+8s^2)$, which is an even integer. Therefore, the unimodularity constraint does not rule out this infinite class of theories.

The natural follow-up question to ask is whether there is some gauge/matter structure that permits an infinite number of distinct solutions to the hypercharge equations that cannot be lifted to a theory with more $U(1)$'s. It turns out that there are infinite classes

of solutions to anomaly equations of a theory with gauge group $\mathcal{G}_0 \times U(1)$ that cannot be lifted to $\mathcal{G}_0 \times U(1)^2$. The example we examine is the theory with gauge group $SU(13) \times U(1)$ that we presented at the end of section 2.5. There we found a solution to the non-abelian factorization condition with 4 antisymmetrics, 6 fundamentals and 23 singlets in the $SU(13)$. The non-abelian part of the factorized polynomial is

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)})\,. \tag{4.10}$$

Denoting the charge of hypermultiplets in the antisymmetric/fundamental/singlet representation as $a_x(x = 1, \cdots, 4)/f_y(y = 1, \cdots, 6)/s_z(z = 1, \cdots, 23)$ the anomaly equations become

$$9\sum_x a_x + \sum_y f_y = 0 \tag{4.11}$$

$$78\sum_x a_x^4 + 13\sum_y f_y^4 + \sum_z s_z^4 = \frac{3}{2}\alpha\tilde{\alpha} \tag{4.12}$$

$$78\sum_x a_x^2 + 13\sum_y f_y^2 + \sum_z s_z^2 = 6\alpha + 6\tilde{\alpha} \tag{4.13}$$

$$44\sum_x a_x^2 + 4\sum_y f_y^2 = 2\alpha + 2\tilde{\alpha} \tag{4.14}$$

There is an ansatz that solves this equation given by

$$(a_x) = (-3a - \frac{2}{3}f, a, a, a) \tag{4.15}$$

$$(f_y) = (f, f, f, f, f, f) \tag{4.16}$$

$$(s_z) = ((6a + f) \times 18, 0, 0, 0, 0, 0) \tag{4.17}$$

where in the last line we mean that 18 of the s_z take the value $(6a+f)$ while five take 0. This ansatz satisfies the first equation and renders the third and fourth equations equivalent. Then the second and third equation can be solved with respect to $\alpha, \tilde{\alpha}$ to yield

$$\frac{\alpha}{f^2}, \frac{\tilde{\alpha}}{f^2} = \frac{2}{9}\left[(49 + 198t + 594t^2) \pm \sqrt{39}(1 + 6t)\sqrt{23 - 24t - 36t^2}\right] \tag{4.18}$$

where we have defined $t = a/f$. It is easy to see that $\alpha, \tilde{\alpha}$ are real as long as

$$\frac{-2 - 3\sqrt{3}}{6} \leq t \leq \frac{-2 + 3\sqrt{3}}{6}\,. \tag{4.19}$$

Both α, $\tilde{\alpha}$ are positive when t is in this range. Hence we see that there are an infinite number of integral hypercharge solutions to the equations (4.11)-(4.14) that give allowed values of $\alpha, \tilde{\alpha}$.

It is clear that this theory cannot be lifted to a theory with gauge group $SU(13)\times U(1)^2$. Although the ansatz for the given solution seems to imply that this theory can be lifted, for example by choosing the charges for one $U(1)$ to be proportional to a and the charges for the other $U(1)$ to be proportional to f, the fact that a/f must lie in a certain range

implies that there must be an obstruction to doing this. The obstruction is that if one tries to lift the theory to a theory with gauge group $SU(13) \times U(1)^2$, the matrices α_{ij} and $\tilde{\alpha}_{ij}$ of this theory cannot be made into positive definite real matrices as is required for the $U(1)$ gauge fields to have positive-definite kinetic terms.

The next question to ask is whether there is an infinite subclass of these theories that satisfy all three quantum consistency conditions introduced at the beginning of this section. Generating a subclass of theories that satisfy the integrality constraint and the minimum charge constraint is not difficult. For example, by taking f and a to be mutually prime integers and f to be a multiple of 3, one can generate an infinite class of solutions that satisfy these two constraints. These conditions, however, do not lead to the unimodularity constraint.

In order to construct a subclass of theories that satisfy all three constraints, let us examine whether there exists an infinite number of rational values of t that make the right hand side of (4.18) rational. This problem boils down to the question of whether the equation

$$23 - 24t - 36t^2 = 39q^2 \tag{4.20}$$

admits an infinite number of solutions with rational t and q. We find that there indeed are an infinite number of rational solutions to this equation using methods outlined in chapter 7 of [58]. When

$$\frac{a}{f} = t = \frac{13k^2 - 234k - 51}{24(13k^2 + 3)} \tag{4.21}$$

for k rational we find that

$$\frac{\alpha}{f^2} = \frac{13}{144(3 + 13k^2)^2}\left[6687 + 54756k + 94458k^2 - 124956k^3 + 39455k^4\right] \tag{4.22}$$

$$\frac{\tilde{\alpha}}{f^2} = \frac{13}{144(3 + 13k^2)^2}\left[2475 + 37908k + 170274k^2 - 29484k^3 + 9035k^4\right] . \tag{4.23}$$

Hence we find that the number of non-anomalous theories with $SU(13) \times U(1)$ with this particular type of matter content is infinite.

To be clear, we now spell out the explicit subclass of theories that satisfy all three quantum consistency conditions. Setting $k = r/s$ for integers r and s in the above equations, we find that when

$$a = 13r^2 - 234rs - 51s^2 \tag{4.24}$$

$$f = 24(13r^2 + 3s^2)\,, \tag{4.25}$$

α and $\tilde{\alpha}$ take on the values

$$\alpha = 52\left[6687s^4 + 54756s^3r + 94458s^2r^2 - 124956sr^3 + 39455r^4\right] \tag{4.26}$$

$$\tilde{\alpha} = 52\left[2475s^4 + 37908s^3r + 170274s^2r^2 - 29484sr^3 + 9035r^4\right] , \tag{4.27}$$

which are even integers. As discussed early on in this section, this implies that the string charge lattice can be embedded in a unimodular lattice. It is clear that this ansatz assigns integer charges to all the fields and hence the charge integrality constraint is also satisfied.

If a and $(-3a - 2f/3)$ are mutually prime, the minimal charge constraint is also satisfied. There are an infinite number of integer pairs (r, s) that render a and $(-3a-2f/3)$ mutually prime. In fact, we can show that when

$$r = 84n + 43 \tag{4.28}$$

$$s = 182n + 92 \tag{4.29}$$

for integer n, a and $(-3a - 2f/3)$ are mutually prime. This fact is proven in appendix C.

We have found a particular gauge/matter structure with one $U(1)$ that has an infinite number of distinct solutions to the hypercharge equations for $T = 1$. Furthermore the theory cannot be lifted to a theory with two $U(1)$'s for these hypercharge assignments.

The situation is rather subtle for the case of $T = 0$. The equations (2.76)-(2.79) make it clear that any infinite class of solutions to the anomaly equation with charge vectors of the form

$$\vec{Q} = r\vec{q}_1 + s\vec{q}_2 \tag{4.30}$$

for one $U(1)$ can be lifted to $U(1)^2$. As in the $T = 1$ case there are a plethora of examples of gauge/matter structure that admit an infinite family of hypercharge solutions in this way. If, however, we want to identify an infinite class of theories that satisfy anomaly equations for a single $U(1)$ factor that cannot be extended to $U(1)^2$, we cannot have a simple linear ansatz as in the $T = 1$ case. Examining some specific examples of $T = 0$ theories gives interesting number theory problems that in some cases seem to have infinite $U(1)$ families that cannot be extended to $U(1)^2$ models, but we do not go into the details of these constructions here.

5. Conclusions

We have considered 6D supergravity theories with $(1, 0)$ supersymmetry with abelian as well as nonabelian gauge group factors. The following statements have been proven for such theories when the number of tensor multiplets T satisfies $T < 9$:

1. The number of abelian vector multiplets is bounded above by (3.23) and (3.24). The upper bound is determined by the nonabelian gauge/matter content.

2. The number of possible gauge groups and nonabelian matter content is finite, though there are families with infinite numbers of possible distinct $U(1)$ charges.

From (2), it immediately follows that

3. There is a global bound on the rank of the gauge group of any non-anomalous 6D $\mathcal{N} = (1, 0)$ theory with $T < 9$.

The conclusions we have reached for theories with abelian factors are in some ways closely parallel to the analogous results bounding the space of 6D supergravity theories with only nonabelian factors [38, 39]. Adding abelian factors does not change the basic result that the set of possible gauge groups and nonabelian matter representations is finite

for $T < 9$. The biggest difference for theories with abelian gauge group factors is that we cannot bound the number of distinct possibilities for $U(1)$ charges. Even this result, however, fits naturally into the pattern of theories with nonabelian gauge groups for small T. In [42], we analyzed the set of allowed 6D theories with $SU(N)$ gauge groups and no tensor fields ($T = 0$). For large N the bounds on the set of allowed representations of $SU(N)$ under which matter fields transform are quite stringent. As N decreases, however, more and more exotic matter representations are allowed by the anomaly conditions and other known low-energy constraints. For $SU(3)$ there are over 10,000 different matter combinations possible, and for $SU(2)$ there are many millions of combinations possible, including, for example, matter in the 113-index symmetric tensor representation (**114**). The infinite range of possible $U(1)$ charges seems like a natural divergent limit to this range of theories.

These results naturally lead to the question of whether it is possible to place stronger bounds on the set of consistent 6D theories than those understood from anomaly cancellation and other known constraints, either from string theory or from other macroscopic considerations. For theories without abelian factors, this question has led to an improved understanding of the space of 6D theories. F-theory [21, 59] gives a method for constructing a very general class of 6D theories. By relating discrete structure of the low-energy theory to topological structure of F-theory constructions, additional constraints placed by F-theory on the low-energy 6D theory have been identified for theories with nonabelian gauge groups [39]. One such condition is that the dyonic string charge lattice of the low-energy theory, which contains the anomaly lattice spanned by the $SO(1,T)$ vectors b_i, must be self-dual. This condition was shown to be a requirement for quantum consistency of any 6D supergravity theory in [57]. A second condition that is imposed by F-theory on supersymmetric theories is the Kodaira constraint that the total elliptically fibered space be Calabi-Yau. For theories with $T = 0$ and $SU(N)$ gauge group, for example, this constraint implies that $-12a = 36 \geq Nb$, where b is the anomaly coefficient for the $SU(N)$ gauge group. This condition places an additional constraint on the set of possible $SU(N)$ theories beyond the conditions imposed by anomaly cancellation. While all $T = 0$ $SU(N)$ models with $N > 8$ that satisfy anomaly cancellation also automatically satisfy this Kodaira constraint, this constraint places increasingly strong additional restrictions on $SU(N)$ theories for small N. In particular, this condition reduces the millions of possible models with exotic matter representations for theories with $SU(2)$ gauge group to less than 200 models, with the largest representation appearing being the **6** of $SU(2)$ [42]. More generally, the Kodaira constraint rules out all known infinite families of 6D theories with nonabelian gauge groups, even for $T \geq 9$, consistent with the known fact that there are a finite number of different possible gauge groups and matter content for F-theory constructions [39]. It is an open question whether (a) the Kodaira constraint can be realized from the point of view of 6D supergravity as a general consistency condition on any quantum theory, (b) this constraint depends crucially on the UV-completion and represents a constraint intrinsic to string theory, or (c) there are other more exotic string constructions beyond F-theory that can realize theories violating the Kodaira constraint.

Since F-theory can also only allow a finite number of possible $U(1)$ charges for any

class of theories, it seems that a constraint analogous to the Kodaira constraint must hold for theories with abelian gauge group factors. Such a constraint should place bounds on the $U(1)$ anomaly coefficients b_{ij}, leading to constraints on abelian charges through equation (2.22). The origin of such a constraint in F-theory is less clear for abelian factors than for nonabelian factors, however, since abelian factors arise in a global and less transparent fashion in F-theory than nonabelian factors. Some progress in formulating abelian factors geometrically in F-theory that may be relevant to this problem will appear in [60].

Another question whose resolution may help shed light on the issues addressed by this paper is the determination of the precise limit on the number of $U(1)$ factors that may arise in a 6D supergravity theory with fixed nonabelian gauge group and matter structure. The simplest case of this question is when $T = 0$ and there is no nonabelian gauge group. We know that F-theory models exist with up to $k = 7$ $U(1)$ factors, though the explicit geometry is only known up to $k = 3$, and the upper bound found here of $k < 17$ is probably not optimal. It would be interesting to find methods for decreasing the upper bound and/or constructing explicit F-theory models with larger k both in this simplest case and more generally. Recently, an impressive list of toric Calabi-Yau threefolds that are elliptic fibrations over $\mathbb{P}^2$ have been put together in [50]. Obtaining an upper bound on k for F-theory compactifications on these manifolds seems to be a goal attainable in the near future.

Finally, understanding $U(1)$ factors in supergravity and string constructions presents a similar challenge in four dimensions, though with additional subtleties. It seems likely that further progress on understanding $U(1)$ factors in six dimensions will also shed light on the 4D problem.

Acknowledgements

We would like to thank Allan Adams, Mboyo Esole, Iñaki García-Etxebarria, Thomas Grimm, James Halverson, Vijay Kumar, Joe Marsano, John McGreevy, and David Morrison for helpful discussions. DP would especially like to thank Koushik Balasubramanian for kindly listening to and commenting on his ramblings on the subject of this paper. We would like to thank the Ohio State University physics department and the organizers of the 2010 String Vacuum Project Fall Meeting for hospitality during the course of writing this paper. DP acknowledges support as a String Vacuum Project Graduate Fellow, funded through NSF grant PHY/0917807. This research was supported by the DOE under contract #DE-FC02-94ER40818.

A. Proof of Bound on Curable Theories

We prove that the number of curable theories as defined in section 3.2 is finite for $T < 9$. The crucial fact we use is that for curable theories one of the two following conditions must hold:

$$H - V \leq 273 - 29T + (T^2 + 6T + 8) \tag{A.1}$$

$$H - V - \sqrt{2}(T+2)\sqrt{N} \leq 273 - 29T + (2T + 4) \tag{A.2}$$

where N is the number of hypermultiplet representations of the theory. It is clear from [38, 40] that there could not be an infinite family of theories for which the first condition holds as it requires $H - V$ to be bounded. Therefore, it is sufficient to show that there does not exist an infinite family of curable theories for which the second condition (A.2) holds. It proves convenient to define

$$c(T) \equiv \frac{T+2}{\sqrt{2}} \,. \tag{A.3}$$

Before presenting the proof of the desired result, we will point out that the proof is very similar to that given for non-abelian theories in [38, 40]. Proofs of the existence of bounds on non-anomalous theories are carried out by two steps in these references. First, the authors identify infinite classes of theories that satisfy all the anomaly equations other than the gravitational anomaly bound, and that have positive kinetic terms for the gauge fields. Then they show that it is impossible for all the theories in that infinite class to satisfy the gravitational anomaly bound. This is proven by showing that as the total rank of the gauge group increases, the increase of H is much faster than V. This proves that constructing an infinite class of theories that satisfy all the anomaly cancellation conditions and that have positive kinetic terms for the gauge fields does not exist.

We also take the same approach in proving our bounds. In our case, however, we must prove that the increase of $H - c(T)\sqrt{N}$ is much faster than V for the infinite classes of theories one could construct. Most of our effort will be put in to showing that $\sqrt{N}$ does not increase so fast as to affect the growth of H.

There are infinite classes of theories that this is easy to show. For example, for the class of theories whose H and V exhibit a scaling behavior with respect to the rank of the total gauge group when it becomes large, the arguments presented in [38, 40] can be virtually repeated. This is because equation (A.2) implies that

$$(\sqrt{H} - c(T))^2 - V \leq 277 - 27T + c(T)^2 \,. \tag{A.4}$$

This is because as we have assumed there exist no singlets in curable theories, and hence

$$N \leq \frac{H}{2} < H \tag{A.5}$$

holds. Therefore, the scaling behavior of $(\sqrt{H} - c(T))^2 - V$ and $H - V$ with respect to the rank is equivalent and the boundedness argument for these classes of theories are essentially the same. In particular, for an infinite class of theories whose simple group factors have bounded rank, the proof of boundedness given in [38, 40] can be used with very little adjustments. This is presented in section A.1.

It is, however, worth pointing out that for some infinite class of theories, the situation is rather subtle. When there exist simple group factors with unbounded rank in the infinite class of theories, the bound (A.4) becomes too delicate to use. In that case the stronger bound (A.2) turns out to be more useful in proving the existence of bounds of curable theories. We will carry this out in section A.2.

We now turn to presenting the complete proof of the bound on curable theories. We proceed by *reductio ad absurdum*. Let us assume there is an infinite family of curable non-abelian theories with gauge group $\prod_\kappa \mathcal{G}_\kappa$. Due to the bound on $H - V - 2c\sqrt{N} > (\sqrt{H} - c)^2 - V - c^2$, we see that theories in an infinite family of curable theories should be unbounded in the dimension of the gauge group. If not, H and V are both bounded, and hence only a finite number of theories can be constructed. There are two ways an unbounded family can occur. These are given as the following:

1. The dimension of each $\mathcal{G}_\kappa$, or equivalently, d_{Adj_κ} is bounded, but the number of simple factors is unbounded in this family.

2. The dimension of a single simple group factor $\mathcal{G}_\kappa$ is unbounded.

We show that both kinds of families cannot exist in the following subsections.

An important fact we use throughout the proof is the fact that,

$$\begin{aligned} H &\geq \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{for which there exists a jointly charged hypermultiplet.} \end{pmatrix} \\ &= (\text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \text{ with } b_\iota \cdot b_\kappa \neq 0.) \end{aligned} \tag{A.6}$$

This can be shown in the following way.

Suppose a hypermultiplet representation I is charged under $\lambda \geq 2$ gauge groups. Then

$$\mathcal{M}_I \geq 2^\lambda \geq \lambda(\lambda - 1) = \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{that } I \text{ is charged jointly under.} \end{pmatrix} \tag{A.7}$$

and therefore

$$\begin{aligned} H = \sum_I \mathcal{M}_I &\geq \sum_I \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{that } I \text{ is charged jointly under.} \end{pmatrix} \\ &\geq \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{for which there exist a jointly charged hypermultiplet.} \end{pmatrix} \end{aligned} \tag{A.8}$$

This means that any ordered pair of gauge groups that has matter jointly charged under it contributes at least 1 to H. This proves the inequality in the first line of (A.6).

Since A_κ^R for any representation R of any simple Lie group $\mathcal{G}_\kappa$ is positive and since,

$$b_\iota \cdot b_\kappa = \sum \lambda_\kappa \lambda_\iota \sum_I \mathcal{M}_I^{\iota\kappa} A_\iota^I A_\kappa^I \geq 0 \tag{A.9}$$

the necessary sufficient condition for two gauge groups $\mathcal{G}_\iota, \mathcal{G}_\kappa$ to have jointly charged matter is $b_\iota \cdot b_\kappa \neq 0$. Therefore

$$H \geq \sum_{\iota \neq \kappa,\ b_\iota \cdot b_\kappa \neq 0} 1 = (\text{Number of pairs of } \iota \neq \kappa \text{ with } b_\iota \cdot b_\kappa \neq 0.) \tag{A.10}$$

This proves the equality in the second line of (A.6).

A.1 Case 1 : Bounded Simple Group Factors

Let us assume that there exists an infinite number of curable theories with bounded simple group factors but with unbounded total dimension.

Let's denote the gauge group of this infinite family of theories $\{\mathcal{T}_\Phi\}$ as $\mathcal{G}_\Phi = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa$ with $d_{\mathrm{Adj}_\kappa} < D$. Notice that we are denoting the number of gauge group factors, ν. So as $\Phi \to \infty$, $\nu \to \infty$. It is useful to classify the gauge group factors into three types according to their b^2 value:

1. **Type Z :** $b_\kappa^2 = 0$
2. **Type N :** $b_\kappa^2 < 0$
3. **Type P :** $b_\kappa^2 > 0$

Since the dimension of each factor is bounded,

$$(\sqrt{H} - c)^2 \leq H - 2c\sqrt{N} + c^2 \leq 273 - 29T + \nu D + c^2 \equiv B \sim \mathcal{O}(\nu)\,. \tag{A.11}$$

Therefore, the dimension of any representation is bounded also by

$$B' \equiv (\sqrt{B} + c)^2 \sim \mathcal{O}(\nu)\,. \tag{A.12}$$

Let's denote the number of N, Z, P type factors as ν_N, ν_Z, ν_P. Then

$$\nu = \nu_N + \nu_Z + \nu_P\,. \tag{A.13}$$

It is shown in [40] that b_ι, b_κ of any two P type factors $\mathcal{G}_\iota, \mathcal{G}_\kappa$ satisfy $b_\iota \cdot b_\kappa > 0$. Also it is shown that there exists $\nu_N^2/T - \nu_N$ distinct ordered pairs of type N gauge group factors that have matter jointly charged under them. Therefore, using (A.6) we can show that

$$\nu_P(\nu_P - 1) + (\frac{\nu_N^2}{T} - \nu_N) \leq H \leq B' \sim \mathcal{O}(\nu)\,. \tag{A.14}$$

Thus when ν is large,

$$\nu_P, \nu_N \leq \mathcal{O}(\sqrt{\nu}) \ll \nu \tag{A.15}$$

and therefore the majority of gauge group factors are of type Z:

$$\nu_Z \sim \mathcal{O}(\nu) \tag{A.16}$$

From the fact that two lightlike vectors cannot have zero inner product unless they are parallel, it is clear that in order for two Z type gauge groups to have no jointly charged matter their b vectors must be parallel. When we denote the size of the largest collection of parallel type Z vectors as μ there are at least $\nu_Z(\nu_Z - \mu)$ ordered pairs of type Z gauge groups with $b_\iota \cdot b_\kappa \neq 0$. This means that

$$\nu_Z(\nu_Z - \mu) = (\nu_Z - \mu)(\nu - \nu_P - \nu_N) \leq B'\,, \tag{A.17}$$

Group	Matter content	$H' - V'$	$2c\sqrt{N'}$	$a \cdot b$	b^2
$SU(\mathcal{N})$	$2\mathcal{N}$ $\square$	$\mathcal{N}^2 + 1$	$\leq 2c\sqrt{2\mathcal{N}}$	0	-2
	$(\mathcal{N}+8)$ $\square + 1$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{1}{2}\mathcal{N}^2 + \frac{15}{2}\mathcal{N} + 1$	$\leq 2c\sqrt{\mathcal{N}+9}$	1	-1
	$(\mathcal{N}-8)$ $\square + 1$ $\square\square$	$\frac{1}{2}\mathcal{N}^2 - \frac{15}{2}\mathcal{N} + 1$	$\leq 2c\sqrt{\mathcal{N}-7}$	-1	-1
	16 $\square + 2$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15\mathcal{N} + 1$	$\leq 2c\sqrt{18}$	2	0
$SO(\mathcal{N})$	$(\mathcal{N}-8)$ $\square$	$\frac{1}{2}\mathcal{N}^2 - \frac{15}{2}\mathcal{N}$	$\leq 2c\sqrt{\mathcal{N}-8}$	-1	-1
$Sp(\mathcal{N}/2)$	$(\mathcal{N}+8)$ $\square$	$\frac{1}{2}\mathcal{N}^2 + \frac{15}{2}\mathcal{N}$	$\leq 2c\sqrt{\mathcal{N}+8}$	1	-1
	16 $\square + 1$ $\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15\mathcal{N}$	$\leq 2c\sqrt{17}$	2	0

Table 4: Allowed charged matter for an infinite family of models with gauge group $H(\mathcal{N})$. The last column gives the values of $\alpha, \tilde{\alpha}$ in the factorized anomaly polynomial. H', V' and N' are as defined in the text.

so from (A.16) we see that $\nu_Z - \mu$ is of order at most $\mathcal{O}(1)$, *i.e.,* $\nu_Z - \mu$ is bounded as a function of D as we take $\nu \to \infty$. Therfore

$$\mu \sim \nu_Z \sim \mathcal{O}(\nu)\,. \tag{A.18}$$

Meanwhile, it was shown in [38] that all Z factors satisfy $H - V > 0$ on their own. Since the μ Z factors have no jointly charged matter among themselves,

$$H - V > \mu - D \times [(\nu_Z - \mu) + \nu_p + \nu_N] \sim \mathcal{O}(\nu)\,, \tag{A.19}$$

i.e., the right hand side of the inequality is unbounded as a function of ν. Then it is clear that,

$$H - V - 2c\sqrt{N} > H - 2c\sqrt{H} - V = (\sqrt{H} - c)^2 - V - c^2 \tag{A.20}$$

is also unbounded as a function of ν. Therefore, $H - V - 2c\sqrt{N}$ cannot be bounded when each simple group factor of the infinite family has bounded rank. This rules out case 1.

A.2 Case 2 : Unbounded Simple Group Factors

Let us assume that there exists an infinite number of curable theories with a simple group factor that is unbounded. This is possible if the gauge group contains a classical group $H(\mathcal{N})$ (which is either SU, SO or Sp) with unbounded rank. In this case, there would be an infinite subfamily whose gauge group is given by $H(\mathcal{N}) \times \mathcal{G}_{\mathcal{N}}$ with fixed classical group type H, and $\mathcal{G}_{\mathcal{N}} = \prod_{\kappa=1}^{\nu(\mathcal{N})} \mathcal{G}_\kappa$ an arbitrary product of simple gauge groups with $\mathcal{N}$ unbounded. It is shown in [38] that when $\mathcal{N}$ is large, the $\mathcal{H}(\mathcal{N})$ block must be among those given in table 4.

Let us enumerate the hypermultiplet representations charged under $H(\mathcal{N})$ with indices, $I' = 1, \cdots, N'$ and the ones uncharged(and hence charged only under other gauge group factors) as $I'' = N' + 1, \cdots, (N' + N'')$. Note that $(N' + N'') = N$. We call the former

hypermultiplets I' hypermultiplets and the latter I'' hypermultiplets. We also define

$$H' \equiv \sum_{I'} \mathcal{M}_{I'} \qquad V' \equiv d_{\mathrm{Adj}_{H(\mathcal{N})}} \tag{A.21}$$

$$H'' \equiv \sum_{I''} \mathcal{M}_{I''} \qquad V'' \equiv \sum_{\kappa} d_{\mathrm{Adj}_{\mathcal{G}_\kappa}} \tag{A.22}$$

Then

$$H - V = (H' - V') + (H'' - V'') \tag{A.23}$$

$$H - V - 2c\sqrt{N} \geq (H' - V' - 2c\sqrt{N'}) + (H'' - V'' - 2c\sqrt{2N''})\,, \tag{A.24}$$

where we have used the fact that for positive x and y, $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$.

From the $H - V - 2c\sqrt{N}$ constraint, we see that additional gauge groups have to be added to make the theory curable since for large $\mathcal{N}$, $(H' - V' - 2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N}^2)$ or $\sim \mathcal{O}(\mathcal{N})$. We shortly see that this is not possible for arbitrary large $\mathcal{N}$. We explicitly work out the proof for the cases when $(H' - V' - 2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N}^2)$, but the proof generalizes to the other case straightforwardly.

Let us see whether we can find an infinite class of $\mathcal{G}_\mathcal{N}$ such that $(H'' - V'' - 2c\sqrt{N''}) \sim -\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. There are again two kinds of behavior of $\mathcal{G}_\mathcal{N}$ under $\mathcal{N} \to \infty$. It can consist of simple gauge factors of bounded rank, or it can have a simple gauge factor whose rank is unbounded. We consider these cases separately.

A.2.1 Case 2-1 : Rank of Simple Gauge Group Factors of $\mathcal{G}_\mathcal{N}$ Bounded

Assume that the dimensions of of the simple gauge group factors are bounded by D. Denoting $\nu(\mathcal{N})$ as the number of gauge group factors, as previously mentioned, we see that

$$H'' - V'' - 2c\sqrt{N} > (\sqrt{H''} - c)^2 - c^2 - D\nu(\mathcal{N}) \tag{A.25}$$

must behave as $-\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. Therefore,

$$\nu(\mathcal{N}) \geq \mathcal{O}(\mathcal{N}^2)\,. \tag{A.26}$$

Hence we must have an infinite family of theories where the number of simple gauge group factors of $\mathcal{G}_\mathcal{N}$ increase at least as $\mathcal{O}(\mathcal{N}^2)$. Also it is clear that

$$H'' \leq \mathcal{O}(\nu) \tag{A.27}$$

for large $\mathcal{N}$ and therefore,

$$H = H' + H'' \leq \mathcal{O}(\mathcal{N}^2) + \mathcal{O}(\nu) \leq \mathcal{O}(\nu)\,. \tag{A.28}$$

Meanwhile, we know from table 4 that the size of the representation of I' hypermultiplets with respect to the κ gauge groups can be at most of order $\mathcal{O}(\mathcal{N})$ in our case.[5] Therefore, the maximum number of gauge groups an I' hypermultiplet could be charged

[5]When $(H' - V' - 2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N})$, the number of I' hypermultiplets can be at most of order $\mathcal{O}(1)$.

under is given by $\mathcal{O}(\log \mathcal{N}) \leq \mathcal{O}(\log \nu)$. Since there are at most $\mathcal{O}(\mathcal{N} \log \mathcal{N}) \leq \mathcal{O}(\sqrt{\nu} \log \nu)$ such factors, there exists an order $\mathcal{O}(\nu) \geq \mathcal{O}(\mathcal{N}^2)$ number of gauge groups among the ν gauge groups $\mathcal{G}_\kappa$ under which only the I'' hypermultiplets are charged. We denote theses gauge groups as $\{\mathcal{G}_{\kappa''}\}$.

Let us denote the size of the set ν'' and the total number of vector multiplets in $\{\mathcal{G}_{\kappa''}\}$ as V'''. Note that $\nu'' \sim \mathcal{O}(\nu)$. Then

$$H'' - V'' - 2c\sqrt{N''} \geq H'' - V''' - 2c\sqrt{N''} - D\mathcal{O}(\sqrt{\nu} \log \nu)\,. \tag{A.29}$$

Defining the number of P, N, and Z type factors in $\{\mathcal{G}_{\kappa''}\}$ as ν''_P, ν''_N and ν''_Z as before repeating the steps of case 1 we can show that

$$\nu''_P, \nu''_N \leq \mathcal{O}(\sqrt{\nu''}) \ll \nu''\,, \tag{A.30}$$

and therefore that

$$\nu''_Z \sim \mathcal{O}(\nu'')\,. \tag{A.31}$$

Also, denoting the size of the largest collection of parallel type Z vectors as μ'' we may show that

$$\mu'' \sim \mathcal{O}(\nu'') \tag{A.32}$$

as in case 1. We may finally show as in case 1 that

$$\begin{aligned} H'' - V''' &> \mu'' - D \times [(\nu_Z - \mu'') + \nu''_P + \nu''_N] \\ &\sim \mathcal{O}(\nu'') \sim \mathcal{O}(\nu)\,, \end{aligned} \tag{A.33}$$

and hence that

$$H'' - V''' - 2c\sqrt{N''} > (\sqrt{H''} - c)^2 - V''' - c^2 \geq \mathcal{O}(\nu)\,. \tag{A.34}$$

Putting this result together with (A.29) we find that

$$H'' - V'' - 2c\sqrt{N''} > \mathcal{O}(\nu) - D\mathcal{O}(\sqrt{\nu} \log \nu) \sim \mathcal{O}(\nu)\,. \tag{A.35}$$

Hence $(H'' - V'' - 2c\sqrt{N''})$ cannot behave as $-\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. We have come a long way to show that there exists a simple gauge group factor in $\mathcal{G}_\mathcal{N}$ that is unbounded in rank.

A.2.2 Case 2-2 : A Simple Gauge Group Factor of Unbounded Rank in $\mathcal{G}_\mathcal{N}$

In this case, there must be an infinite family of theories with

$$\hat{H}(\mathcal{N}) \times H(\mathcal{P}) \times \mathcal{G}_{\mathcal{N},\mathcal{P}} \tag{A.36}$$

with unbounded $\mathcal{N}$ and $\mathcal{P}$ where $\hat{H}$ and H are given classical groups. It is clear that both gauge groups have to come from table 4.

Unless $H' - V' - 2c\sqrt{N'}$ for $\hat{H}(\mathcal{N}) \times H(\mathcal{P})$ is bounded, by the same arguments as case 2-1 we can show that $\mathcal{G}_{\mathcal{N},\mathcal{P}}$ contains a gauge group factor of unbounded rank. By the same investigation as in [40] we find that all combinations that have bounded $H' - V' - 2c\sqrt{N'}$ cannot have positive definite kinetic terms.

Hence we are led to the conclusion that there must be an infinite family of theories with

$$\tilde{H}(\mathcal{N}) \times \hat{H}(\mathcal{P}) \times H(\mathcal{Q}) \times \mathcal{G}_{\mathcal{N},\mathcal{P},\mathcal{Q}} \,. \tag{A.37}$$

$\mathcal{N}$, $\mathcal{P}$ and $\mathcal{Q}$ are unbounded and $\tilde{H}$, $\hat{H}$ and H are given classical groups. All three unbounded gauge groups must be from table 4.

It also is the case that $H' - V' - 2c\sqrt{N'}$ for $\tilde{H}(\mathcal{N}) \times \hat{H}(\mathcal{P}) \times H(\mathcal{Q})$ must be bounded in order for $\mathcal{G}_{\mathcal{N},\mathcal{P},\mathcal{Q}}$ to have no gauge group factor of unbounded rank. It turns out to be impossible to find such a family with bounded $H' - V' - 2c\sqrt{N'}$ for which there exists a j vector that gives a positive kinetic term.

Our proof is concluded by the fact that one cannot construct an infinite family of theories that consists of four blocks from table 4 for which all the ranks of the individual gauge group factors go to infinity. □

B. A Bound on the Number of Vector Multiplets for Pure Abelian Theories with $T = 0$

In the case that $T = 0$ and the gauge group is purely abelian, we can obtain a lower bound on the number of charged hypermultiplets as a function of the the total rank of the gauge group. Similar bounds may be obtained for other values of $T < 9$ though they are less stringent.

We label the $U(1)$ gauge groups by $i = 1, \cdots, V_A$. The gravitational anomaly condition imposes that the number of hypermultiplets is equal to $V_A + 273$. We denote the number of charged hypermultiplets to be $X \leq (V_A + 273)$, and label them by $I = 1, \cdots, X$.

For $T = 0$, the vectors

$$\vec{q}_i \equiv (q_{1,i}, q_{2,i}, \cdots, q_{X,i}) \,, \tag{B.1}$$

whose components are the charges of the X charged hypermultiplets under $U(1)_i$, must satisfy

$$108 \sum_I f_I(x_i)^4 = (\sum_I f_I(x_i)^2)^2 \,. \tag{B.2}$$

This follows from (2.77) and (2.79) where, as before, we have defined

$$f_I(x_i) = q_{I,i} x_i \,. \tag{B.3}$$

In order for the kinetic term matrix proportional to

$$b_{ij} = \sum_I q_{I,i} q_{I,j} \tag{B.4}$$

to be positive definite, $\vec{q}_i$ must be linearly independent. This was explained at the end of section 2.4. Therefore, using the $GL(V_A)$ invariance of the equation, we can redefine $\vec{q}_i$ so these vectors become orthogonal. It is convenient to normalize them to have norm $\sqrt{108}$, *i.e.,*

$$\sum_I q_{I,i} q_{I,j} = \sqrt{108} \delta_{ij} \,. \tag{B.5}$$

(Note that the $q's$ are not necessarily integers in this basis.)

Plugging this into (B.2) and expanding, we find that $q_{I,i}$ must also satisfy

$$\sum_I q_{I,i}^4 = 1 \tag{B.6}$$

$$\sum_I q_{I,i}^2 q_{I,j}^2 = \frac{1}{3} \qquad \text{for } i,j \text{ distinct} \tag{B.7}$$

$$\sum_I q_{I,i}^3 q_{I,j} = 0 \qquad \text{for } i,j \text{ distinct} \tag{B.8}$$

$$\sum_I q_{I,i}^2 q_{I,j} q_{I,k} = 0 \qquad \text{for } i,j,k \text{ distinct} \tag{B.9}$$

$$\sum_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} = 0 \qquad \text{for } i,j,k,l \text{ distinct}\,. \tag{B.10}$$

Defining the vectors

$$\vec{Q}_i \equiv (q_{1,i}^2, q_{2,i}^2, \cdots, q_{X,i}^2) \tag{B.11}$$

$$\vec{A} \equiv \frac{1}{\sqrt{X}}(1,1,\cdots,1)\,, \tag{B.12}$$

and

$$\vec{Q}_{ij} \equiv (q_{1,i}q_{1,j}, q_{2,i}q_{2,j}, \cdots, q_{X,i}q_{X,j}) \tag{B.13}$$

for $i<j$, the equations obtained from (B.2) and (B.5) can be re-written as,

$$\vec{A}\cdot\vec{Q}_i = \sqrt{\frac{108}{X}} \tag{B.14}$$

$$\vec{Q}_i^2 = 1 \tag{B.15}$$

$$\vec{Q}_i \cdot \vec{Q}_j = \vec{Q}_{ij}^2 = \frac{1}{3} \qquad \text{for } i,j \text{ distinct} \tag{B.16}$$

$$\vec{A}\cdot\vec{Q}_{ij} = \vec{q}_i \cdot \vec{q}_j = 0 \qquad \text{for } i,j \text{ distinct} \tag{B.17}$$

$$\vec{Q}_i \cdot \vec{Q}_{ij} = 0 \qquad \text{for } i,j \text{ distinct} \tag{B.18}$$

$$\vec{Q}_i \cdot \vec{Q}_{jk} = \vec{Q}_{ij}\cdot\vec{Q}_{ik} = 0 \qquad \text{for } i,j,k \text{ distinct} \tag{B.19}$$

$$\vec{Q}_{ij}\cdot\vec{Q}_{kl} = 0 \qquad \text{for } i,j,k,l \text{ distinct} \tag{B.20}$$

It is easy to see that $\vec{Q}_i$ and $\vec{Q}_{ij}$ are all non-zero, since all $\vec{q}_i \neq \vec{0}$. Also $\vec{Q}_i$ and $\vec{Q}_{ij}$ are X-dimensional vectors by definition.

Using the given inner products we can show that

$$|\vec{Q}_1 + \cdots + \vec{Q}_{V_A}|^2 = V_A + \frac{1}{3}V_A(V_A-1) = \frac{1}{3}V_A(V_A+2)\,. \tag{B.21}$$

Since $\vec{A}$ is a unit vector by definition,

$$V_A\sqrt{\frac{108}{X}} = \vec{A}\cdot(\sum_i \vec{Q}_i) \leq |\sum_i \vec{Q}_i| = \sqrt{\frac{1}{3}V_A(V_A+2)}\,. \tag{B.22}$$

Hence

$$\frac{324V_A}{V_A+2} \leq X \leq V_A + 273\,, \tag{B.23}$$

as promised. This equation implies that

$$V_A \leq 17 \quad \text{or} \quad V_A \geq 32\,. \tag{B.24}$$

An additional constraint is needed to obtain an upper bound on V_A.

The additional constraint can be obtained by utilizing the full set of vectors $\vec{Q}_{ij}$ and $\vec{Q}_i$ we have defined. Note that by (B.19) and (B.20), $\vec{Q}_{ij}$ are mutually orthogonal. They are also orthogonal to $\vec{Q}_i$ and $\vec{A}$ as can be seen in (B.17), (B.18) and (B.19).

Also, all $\vec{Q}_i$ must be linearly independent. This is because if we assume

$$k_1\vec{Q}_1 + \cdots + k_{V_A}\vec{Q}_{V_A} = 0\,, \tag{B.25}$$

then for non-zero k_i

$$k_1^2 + \cdots k_{V_A}^2 + \frac{2}{3}(k_1k_2 + \cdots + k_{V_A-1}k_{V_A}) = 0\,. \tag{B.26}$$

But the l.h.s. can be rewritten as

$$\frac{2}{3}(k_1^2 + \cdots k_{V_A}^2) + \frac{1}{3}(k_1 + \cdots k_{V_A})^2 = 0 \tag{B.27}$$

and hence the equality cannot hold for non-zero k_i.

Therefore, we find that $\vec{Q}_i$ together with $\vec{Q}_{ij}$ form a set of linearly independent vectors. This means that we must have $V_A(V_A+1)/2$ linearly independent vectors in $X \leq V_A + 273$ dimensional space. Hence,

$$\frac{V_A(V_A+1)}{2} \leq X \leq V_A + 273. \tag{B.28}$$

From this we obtain the bound $V_A \leq 24$.

Put together with the bound (B.24) we obtain

$$V_A \leq 17\,, \tag{B.29}$$

as desired. □

C. Proof of Minimal Charge Condition for $SU(13) \times U(1)$ Models

In this section we prove that when

$$r = 84n + 43 = 2 \times 3 \times 7 \times (2n+1) + 1 \tag{C.1}$$

$$s = 182n + 92 = 7 \times 13 \times (2n+1) + 1 \tag{C.2}$$

for integer n, then the integers a and $(-3a - 2f/3)$ for

$$a = 13r^2 - 234rs - 51s^2 \tag{C.3}$$

$$f = 24(13r^2 + 3s^2) \tag{C.4}$$

are mutually prime. Let us define

$$g \equiv gcd(a, -3a - \frac{2}{3}f) = gcd(a, \frac{2}{3}f)\,. \tag{C.5}$$

Our goal is to show that $g = 1$.

We first acknowledge that r and s are mutually prime. This is because

$$gcd(r, s)|(13r - 6s) \tag{C.6}$$

and

$$13r - 6s = 7\,. \tag{C.7}$$

It is clear, however, that 7 and r are mutually prime. Therefore, $gcd(r, s)$ must be 1, and hence r and s must be mutually prime. Meanwhile, a is odd since r is even and s is odd. Therefore g must also be odd, i.e. $2 \nmid g$. Also, g is not divisible by 3. We can show $3 \nmid g$ by noting that $g \mid (13r^2 + 3s^2)$ and that

$$13r^2 + 3s^2 \equiv 1 \pmod 3\,. \tag{C.8}$$

Let us show that $g = 1$. By definition

$$\begin{aligned} g &= gcd(a, \frac{2f}{3}) = gcd(13r^2 - 234rs - 51s^2, 16(13r^2 + 3s^2)) \\ &= gcd(13r^2 - 234rs - 51s^2, 13r^2 + 3s^2)\,, \end{aligned} \tag{C.9}$$

where we have used the fact that $2 \nmid g$. Using standard properties of the greatest common divisor, we further find that

$$\begin{aligned} g &= gcd(13r^2 - 234rs - 51s^2, 13r^2 + 3s^2) \\ &= gcd(-234rs - 54s^2, 13r^2 + 3s^2) \\ &= gcd(-234r - 54s, 13r^2 + 3s^2) = gcd(-18(13r + 3s), 13r^2 + 3s^2) \\ &= gcd(13r + 3s, 13r^2 + 3s^2) \end{aligned} \tag{C.10}$$

In the penultimate line we have used the fact that

$$gcd(s, 13r^2 + 3s^2) = gcd(s, 13r^2) = 1 \tag{C.11}$$

since $s \equiv 1 \pmod{13}$ and s and r are mutually prime. In the last line we have used $2 \nmid g$ and $3 \nmid g$.

Therefore, g must be a divisor of

$$-(13r + 3s)(13 - 3s) + 13(13r^2 + 3s^2) = 48s^2\,. \tag{C.12}$$

We have seen in (C.11) that s is mutually prime with $13r^2 + 3s^2$. Therefore, g is mutually prime with s and hence

$$g|2^3 \times 3\,. \tag{C.13}$$

We, however, know that $2 \nmid g$ and $3 \nmid g$. This proves that $g = 1$, as desired. □

References

[1] L. Alvarez-Gaume and E. Witten, "Gravitational Anomalies," Nucl. Phys. B **234**, 269 (1984).

[2] M. B. Green, J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," Phys. Lett. **B149**, 117-122 (1984).

[3] D. J. Gross, J. A. Harvey, E. J. Martinec, R. Rohm, "The Heterotic String," Phys. Rev. Lett. **54**, 502-505 (1985).

D. J. Gross, J. A. Harvey, E. J. Martinec, R. Rohm, "Heterotic String Theory. 1. The Free Heterotic String," Nucl. Phys. **B256**, 253 (1985).

D. J. Gross, J. A. Harvey, E. J. Martinec, R. Rohm, "Heterotic String Theory. 2. The Interacting Heterotic String," Nucl. Phys. **B267**, 75 (1986).

[4] S. L. Adler, "Axial vector vertex in spinor electrodynamics," Phys. Rev. **177**, 2426 (1969).

J. S. Bell and R. Jackiw, "A PCAC puzzle: pi0 $\to$ gamma gamma in the sigma model," Nuovo Cim. A **60**, 47 (1969).

[5] M. B. Green, J. H. Schwarz, E. Witten, "Superstring Theory. Vol. 1: Introduction," Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

M. B. Green, J. H. Schwarz, E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics).

[6] A. Adams, O. DeWolfe and W. Taylor, "String universality in ten dimensions," Phys. Rev. Lett. **105**, 071601 (2010) [arXiv:1006.1352 [hep-th]].

[7] N. Seiberg, "Observations on the Moduli Space of Superconformal Field Theories," Nucl. Phys. **B303**, 286 (1988).

[8] H. Nishino and E. Sezgin, "Matter And Gauge Couplings Of N=2 Supergravity In Six-Dimensions," Phys. Lett. B **144**, 187 (1984).

H. Nishino and E. Sezgin, "The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings," Nucl. Phys. B **278**, 353 (1986).

[9] M. B. Green, J. H. Schwarz, P. C. West, "Anomaly Free Chiral Theories in Six-Dimensions," Nucl. Phys. **B254**, 327-348 (1985).

[10] A. Strominger, "Superstrings with Torsion," Nucl. Phys. **B274**, 253 (1986).

[11] L. J. Romans, "Selfduality For Interacting Fields: Covariant Field Equations For Six-dimensional Chiral Supergravities," Nucl. Phys. **B276**, 71 (1986).

[12] M. A. Walton, "The Heterotic String On The Simplest Calabi-yau Manifold And Its Orbifold Limits," Phys. Rev. **D37**, 377 (1988).

[13] M. Bianchi and A. Sagnotti, "On the systematics of open string theories," Phys. Lett. B **247** (1990) 517.

M. Bianchi and A. Sagnotti, "Twist symmetry and open string Wilson lines," Nucl. Phys. B **361** (1991) 519.

[14] A. Sagnotti, "A Note on the Green-Schwarz mechanism in open string theories," Phys. Lett. B **294**, 196 (1992) [arXiv:hep-th/9210127].

[15] J. Erler, "Anomaly Cancellation In Six-Dimensions," J. Math. Phys. **35**, 1819 (1994) [arXiv:hep-th/9304104].

[16] G. Aldazabal, A. Font, L. E. Ibanez, F. Quevedo, "Chains of N=2, D = 4 heterotic type II duals," Nucl. Phys. **B461**, 85-100 (1996). [hep-th/9510093].

[17] E. Witten, "Small instantons in string theory," Nucl. Phys. **B460**, 541-559 (1996). [hep-th/9511030].

[18] M. J. Duff, R. Minasian, E. Witten, "Evidence for heterotic / heterotic duality," Nucl. Phys. **B465**, 413-438 (1996). [hep-th/9601036].

[19] E. G. Gimon, J. Polchinski, "Consistency conditions for orientifolds and d manifolds," Phys. Rev. **D54**, 1667-1676 (1996). [hep-th/9601038].

[20] A. Dabholkar and J. Park, "An Orientifold of Type-IIB Theory on $K3$," Nucl. Phys. B **472**, 207 (1996) [arXiv:hep-th/9602030].

[21] D. R. Morrison, C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 1," Nucl. Phys. **B473**, 74-92 (1996), [hep-th/9602114].

D. R. Morrison, C. Vafa, "Compactifications of F theory on Calabi-Yau threefolds. 2.," Nucl. Phys. **B476**, 437-469 (1996). [hep-th/9603161].

[22] N. Seiberg, E. Witten, "Comments on string dynamics in six-dimensions," Nucl. Phys. **B471**, 121-134 (1996). [hep-th/9603003].

[23] E. G. Gimon and C. V. Johnson, "K3 Orientifolds," Nucl. Phys. B **477**, 715 (1996) [arXiv:hep-th/9604129].

[24] P. S. Aspinwall, M. Gross, "The SO(32) heterotic string on a K3 surface," Phys. Lett. **B387**, 735-742 (1996). [hep-th/9605131].

[25] M. Berkooz, R. G. Leigh, J. Polchinski, J. H. Schwarz, N. Seiberg and E. Witten, "Anomalies, Dualities, and Topology of D=6 N=1 Superstring Vacua," Nucl. Phys. B **475**, 115 (1996) [arXiv:hep-th/9605184].

[26] V. Sadov, "Generalized Green-Schwarz mechanism in F theory," Phys. Lett. **B388**, 45-50 (1996). [hep-th/9606008].

[27] G. Aldazabal, A. Font, L. E. Ibanez, A. M. Uranga, "New branches of string compactifications and their F theory duals," Nucl. Phys. **B492**, 119-151 (1997) [hep-th/9607121].

[28] R. Friedman, J. Morgan, E. Witten, "Vector bundles and F theory," Commun. Math. Phys. **187**, 679-743 (1997). [hep-th/9701162].

[29] P. Berglund, A. Klemm, P. Mayr, S. Theisen, "On type IIB vacua with varying coupling constant," Nucl. Phys. **B558**, 178-204 (1999) [hep-th/9805189].

[30] F. Riccioni, "Abelian vector multiplets in six-dimensional supergravity," Phys. Lett. B **474**, 79 (2000) [arXiv:hep-th/9910246].

F. Riccioni, "All couplings of minimal six-dimensional supergravity," Nucl. Phys. B **605**, 245 (2001) [arXiv:hep-th/0101074].

[31] P. S. Aspinwall, "An analysis of fluxes by duality," [hep-th/0504036].

[32] S. D. Avramis and A. Kehagias, "A systematic search for anomaly-free supergravities in six dimensions," JHEP **0510**, 052 (2005) [arXiv:hep-th/0508172].

[33] G. Honecker, "Massive U(1)s and heterotic five-branes on K3," Nucl. Phys. **B748**, 126-148 (2006) [hep-th/0602101].

G. Honecker, M. Trapletti, "Merging Heterotic Orbifolds and K3 Compactifications with Line Bundles," JHEP **0701**, 051 (2007) [hep-th/0612030].

G. Honecker, "Orbifolds versus smooth heterotic compactifications," [arXiv:0709.2037 [hep-th]].

[34] D. R. Morrison, W. Taylor, "Matter and singularities," [arXiv:1106.3563 [hep-th]].

[35] L. E. Ibanez, A. M. Uranga, "D = 6, N=1 string vacua and duality," [hep-th/9707075].

[36] W. Taylor, "TASI Lectures on Supergravity and String Vacua in Various Dimensions," [arXiv:1104.2051 [hep-th]].

[37] V. Kumar and W. Taylor, "String Universality in Six Dimensions," arXiv:0906.0987 [hep-th].

[38] V. Kumar and W. Taylor, "A bound on 6D N=1 supergravities," JHEP **0912**, 050 (2009) [arXiv:0910.1586 [hep-th]].

[39] V. Kumar, D. R. Morrison and W. Taylor, "Mapping 6D N = 1 supergravities to F-theory," arXiv:0911.3393 [hep-th].

[40] V. Kumar, D. R. Morrison and W. Taylor, "Global aspects of the space of 6D N = 1 supergravities," arXiv:1008.1062 [hep-th].

[41] W. Taylor, "Anomaly constraints and string/F-theory geometry in 6D quantum gravity," arXiv:1009.1246 [hep-th].

[42] V. Kumar, D. Park, W. Taylor, "6D supergravity without tensor multiplets," [arXiv:1011.0726 [hep-th]].

[43] R. Blumenhagen, G. Honecker, T. Weigand, "Loop-corrected compactifications of the heterotic string with line bundles," JHEP **0506**, 020 (2005). [hep-th/0504232].

R. Blumenhagen, G. Honecker, T. Weigand, "Supersymmetric (non-)Abelian bundles in the Type I and SO(32) heterotic string," JHEP **0508**, 009 (2005). [hep-th/0507041].

R. Blumenhagen, G. Honecker, T. Weigand, "Non-Abelian brane worlds: The Heterotic string story," JHEP **0510**, 086 (2005). [hep-th/0510049].

T. Weigand, "Heterotic vacua from general (non-) Abelian bundles," Fortsch. Phys. **54**, 505-513 (2006). [hep-th/0512191].

[44] M. Buican, D. Malyshev, D. R. Morrison, H. Verlinde, M. Wijnholt, "D-branes at Singularities, Compactification, and Hypercharge," JHEP **0701**, 107 (2007). [hep-th/0610007].

[45] T. W. Grimm, T. Weigand, "On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs," Phys. Rev. **D82**, 086009 (2010). [arXiv:1006.0226 [hep-th]].

[46] H. Hayashi, T. Kawano, Y. Tsuchiya, T. Watari, "More on Dimension-4 Proton Decay Problem in F-theory – Spectral Surface, Discriminant Locus and Monodromy," Nucl. Phys. **B840**, 304-348 (2010). [arXiv:1004.3870 [hep-th]].

[47] J. Marsano, N. Saulina, S. Schafer-Nameki, "A Note on G-Fluxes for F-theory Model Building," JHEP **1011**, 088 (2010), [arXiv:1006.0483 [hep-th]].

J. Marsano, "Hypercharge Flux, Exotics, and Anomaly Cancellation in F-theory GUTs," Phys. Rev. Lett. **106**, 081601 (2011). [arXiv:1011.2212 [hep-th]].

M. J. Dolan, J. Marsano, N. Saulina, S. Schafer-Nameki, "F-theory GUTs with U(1) Symmetries: Generalities and Survey," [arXiv:1102.0290 [hep-th]].

[48] E. Witten, "Five-brane effective action in M theory," J. Geom. Phys. **22**, 103-133 (1997). [hep-th/9610234].

D. Belov, G. W. Moore, "Holographic Action for the Self-Dual Field," [hep-th/0605038].

[49] P. S. Howe, G. Sierra and P. K. Townsend, "Supersymmetry In Six-Dimensions," Nucl. Phys. B **221**, 331 (1983).

[50] V. Braun, "Toric Elliptic Fibrations and F-Theory Compactifications," [arXiv:1110.4883 [hep-th]].

[51] A. Klemm, B. Lian, S. S. Roan, S. -T. Yau, "Calabi-Yau fourfolds for M theory and F theory compactifications," Nucl. Phys. **B518**, 515-574 (1998). [hep-th/9701023].

[52] A. Strominger, "Open p-branes," Phys. Lett. **B383**, 44-47 (1996). [hep-th/9512059].

[53] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," arXiv:hep-th/9603167.

[54] J. Polchinski, "Monopoles, duality, and string theory," Int. J. Mod. Phys. **A19S1**, 145-156 (2004). [hep-th/0304042].

[55] T. Banks, N. Seiberg, "Symmetries and Strings in Field Theory and Gravity," Phys. Rev. **D83**, 084019 (2011). [arXiv:1011.5120 [hep-th]].

[56] S. Hellerman, E. Sharpe, "Sums over topological sectors and quantization of Fayet-Iliopoulos parameters," [arXiv:1012.5999 [hep-th]].

[57] N. Seiberg, W. Taylor, "Charge Lattices and Consistency of 6D Supergravity," [arXiv:1103.0019 [hep-th]].

[58] L. J. Mordell, "Diophantine Equations," London, UK: Academic Press (1969) 312 p.

[59] C. Vafa, "Evidence for F-Theory," Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022].

[60] D. S. Park, "Anomaly Equations and Intersection Theory," *to appear*.