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Charge Lattices and Consistency of 6D Supergravity

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ABSTRACT: We extend the known consistency conditions on the low-energy theory of six-dimensional $\mathcal{N} = 1$ supergravity. We review some facts about the theory of two-form gauge fields and conclude that the charge lattice Γ for such a theory has to be self-dual. The Green-Schwarz anomaly cancellation conditions in the supergravity theory determine a sublattice of Γ . The condition that this sublattice can be extended to a self-dual lattice Γ leads to a strong constraint on theories that otherwise appear to be self-consistent.

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1. Introduction

The space of chiral supergravity theories in six dimensions is strongly constrained by anomaly cancellation and other consistency conditions. It was shown in [1] that $\mathcal{N} = (2, 0)$ supersymmetry in six dimensions uniquely determines the massless spectrum of the theory. A systematic analysis of $\mathcal{N} = (1, 0)$ 6D supergravity theories and their string realizations was initiated in [2]-[7]. The space of low-energy 6D supergravity theories that are compatible with known anomaly and kinetic term sign conditions is quite constrained, but nevertheless contains a rich range of theories with different gauge groups and matter content. Many of these theories can be realized in string theory, through F-theory [8, 9, 10] and other constructions. There are, however, also many theories that satisfy these constraints with no known string realization.

If we find an apparently consistent supergravity theory that cannot be obtained in string theory we face three logical possibilities:

1. This low-energy theory is inconsistent because of a consideration that we are not yet aware of.
2. There is a string construction leading to this supergravity theory that is based on techniques we are not yet aware of.
3. There are consistent low-energy theories that cannot be obtained by any string construction. Of course, it is difficult to prove that this is the case because to prove this we would need to know all possible string constructions and understand all macroscopic constraints.

A productive research direction involves looking for such examples and trying to see which of the three possibilities above is the right one. This line of investigation can focus our thinking and point us either to new string constructions or to new macroscopic consistency conditions.

In [4, 5], an explicit correspondence between the discrete data describing the field content of low-energy supergravity theories and topological information about F-theory constructions was used to identify some classes of apparently consistent models without string realizations. One condition that is satisfied by all known 6D $\mathcal{N} = 1$ string theory vacua is that the lattice of allowed dyonic string charges in the low-energy theory must be a unimodular (self-dual) lattice. This raises the question of whether such self-duality is a necessary condition for consistency of the quantum theory.

An extremely interesting line of research starting with [11] and references therein has explored the consistency conditions on gauge theories based on higher form gauge fields. In particular, [12]-[17] have presented a general and beautiful description of such theories. This work leads to the conclusion that the natural framework for describing chiral fields is in a general class of self-dual cohomology theories. However, the discussion in these papers is fairly abstract, and more general than needed for our purposes here. Instead of attempting to frame the discussion here in the more general context of that work, we will take a lowbrow approach based on standard methods in field theory, which leads us to the conclusions we need. We should emphasize, however, that many of the observations made here are simple corollaries of the more general theory presented in these papers.

In this note we show by elementary arguments that any consistent 6D $\mathcal{N} = 1$ theory must have a lattice Γ of allowed dyonic string charges that is unimodular; *i.e.* $\Gamma = \Gamma^*$. This result is independent of string theory; the dyonic strings are simply the possible states charged under the two-form fields in the theory. We describe the self-duality of the charge lattice in Section 2. The consequences of this condition for constraining the space of 6D supergravity theories are discussed in Section 3. Examples of theories that are ruled out from this consideration are given in Section 4. Section 5 contains concluding remarks.

2. Consistency conditions for six-dimensional two-form gauge fields

Our goal in this section is to find some necessary conditions on the set of charges that can couple to six-dimensional two-form gauge fields. As stated in the introduction, this is a lowbrow version of the discussion in [11]-[17]. We will start by studying two warmup problems in two and four dimensions and then we will turn to the six-dimensional theory. We show that a 6D field theory with self-dual and anti-self-dual two-form fields cannot be consistent after dimensional reduction to 2D or 4D unless the charges lie in a self-dual lattice.

2.1 Scalar fields in two dimensions

The theory of free compact scalar fields in two dimensions is well known. Following [18, 19] we study p left moving scalars ϕ^{+a} and q right moving scalars ϕ^{-r} with standard two-point

functions

$$\langle \phi^{+a}(0)\phi^{+b}(z) \rangle = -\delta^{ab} \log z \quad ; \quad \langle \phi^{-r}(0)\phi^{-s}(\bar{z}) \rangle = -\delta^{rs} \log \bar{z} . \quad (2.1)$$

Typical operators in the theory are of the form

$$\mathcal{O}_{v_a, \tilde{v}_r} = \exp (i v_a \phi^{+a} + i \tilde{v}_r \phi^{-r}) . \quad (2.2)$$

Using (2.1) their operator product expansion is

$$\mathcal{O}_{v_a, \tilde{v}_r} \mathcal{O}_{u_a, \tilde{u}_r} \sim z^{v_a u_a} \bar{z}^{\tilde{v}_r \tilde{u}_r} \mathcal{O}_{v_a+u_a, \tilde{v}_r+\tilde{u}_r} + \dots \quad (2.3)$$

Therefore, completeness of the operator product expansion shows that (v_a, \tilde{v}_r) are vectors in a $p+q$ dimensional lattice Γ . Furthermore, single valuedness of (2.3) means that

$$v_a u_a - \tilde{v}_r \tilde{u}_r \in \mathbb{Z} . \quad (2.4)$$

This means that the lattice Γ has signature (p, q) and locality means that it must be an integral lattice. Note that the bilinear form on the lattice was determined by our choice of two point functions for the scalars (2.1).

As long as we limit ourselves to correlation functions in \mathbb{R}^2 , equation (2.4) is the only consistency condition. Studying the theory on more complicated surfaces leads to further constraints. In particular, imposing modular invariance on the partition function when the theory is on \mathbb{T}^2 leads to two additional conditions:

1. Invariance under the S transformation in the modular group forces the lattice Γ to be self-dual.
2. Invariance under the T transformation in the modular group forces the lattice Γ to be not only integral but also even. Without this requirement, the torus partition function is invariant under T^2 but not under T .

Should we impose these two requirements? In the context of string theory, the proper spacetime interpretation of the theory forces us to sum over the spin structures on the worldsheet. Hence, the torus partition function must be fully modular invariant and therefore Γ should be even and self-dual. However, if we are interested in a two-dimensional field theory without insisting on another interpretation of the theory in the target space, there is no need to be so restrictive.

Let us consider the situation without imposing T invariance. The operators (2.2) with even $v_a^2 - \tilde{v}_r^2$ have integer spin and they are bosons, but those with odd $v_a^2 - \tilde{v}_r^2$ have half integral spin and they are fermions. The presence of operators with half integer spin forces us to specify the spin structure on the torus. If we impose invariance under T the operators (2.2) include only bosons and we do not need to specify the spin structure on the torus. But if the theory is not T invariant, additional data, in particular the spin structure, can be included in the definition of the theory. With such data there is no need to impose T invariance for the theory to be consistent.

The situation with S invariance is different. With a choice of one-cycles we view the torus as $\mathbb{S}^1 \times \mathbb{S}^1$ and interpret the first \mathbb{S}^1 as space and the second \mathbb{S}^1 as Euclidean time. The standard state/operator correspondence of the operators (2.2) leads us to choose the spin structure in which the half-integer spin operators are anti-periodic in space. The trace over the Hilbert space corresponds to making these operators anti-periodic in Euclidean time. In this case the S transformation, which exchanges these two circles, must be a symmetry of the theory.

We conclude that the second requirement above of T invariance is optional, but we must always impose S invariance. Therefore, Γ must be integral and self-dual but not necessarily even.

An example of a consistent theory that is S invariant but not T invariant is the theory of a free left moving Weyl fermion. In its bosonized form it fits the discussion above about the free scalars with $p = 1$, $q = 0$. The corresponding lattice is integral and self-dual but it is not even.

2.2 Maxwell fields in four dimensions

As our second warmup example we remind the reader of the theory of free Maxwell fields in four dimensions A^a . The analog of (2.2) are loop operators including Wilson loops and 't Hooft loops. The former are of the form

$$W_{v_a} = \exp \left(i v_a \oint_{\Sigma_1} A^a \right) , \quad (2.5)$$

where Σ_1 is a closed loop. This operator describes the effect of a probe particle with “electric charges” v_a . Again, the vector v_a belongs to a lattice L . The 't Hooft operators describe the effect of probe magnetic particles with magnetic charges u^a . The Dirac quantization condition can be understood in terms of the relative locality of Wilson and 't Hooft operators. It implies that

$$v_a u^a \in \mathbb{Z} , \quad (2.6)$$

which is similar to (2.4) and states that u^a belongs to the dual lattice L^* .

More generally, states in the 4D theory can have dyonic charges (v_a, u^b) . The Dirac quantization condition states that for any two dyonic charges (v, u) and (v', u') the anti-symmetric charge product

$$v \cdot u' - u \cdot v' \in \mathbb{Z} \quad (2.7)$$

is integral. This describes a skew-symmetric inner product on the charge space. If the gauge group is $U(1)^n$, the skew-symmetric inner product can be written in the canonical form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (2.8)$$

where I is the $n \times n$ identity matrix. Note that the matrix (2.8) has unit determinant. The restriction of Dirac quantization alone does not uniquely determine the form of (2.8). Other integral skew-symmetric matrices might be considered for describing charges of candidate theories. Such models are incompatible, however, with the condition that Wilson loops

and 't Hooft loops are the proper observables for the theory. From representation theory of the relativistic symmetry group, in 4D these are the only available classes of operators. We discuss below how the situation is different in 6D.

2.3 Two-form fields in six dimensions

We now come to the theory of two-form gauge fields in six dimensions. This case is almost identical to the two-dimensional example of chiral bosons we started with. More generally, a similar situation arises in a theory of self-dual and anti-self-dual fields in any dimension of the form $4n + 2$. The conclusion in all these cases is that the charge lattice Γ must be integral and self-dual (but not necessarily even). Our analysis relies only on the structure of the free field theory of two-form fields, and is unchanged by interactions or coupling to gravity. We discuss the situation in the context of gravity in more detail in Section 5.

As in two dimensions, we consider p fields B^{+a} whose field strength $H^{+a} = dB^{+a}$ is self-dual and q fields B^{-r} whose field strength $H^{-r} = dB^{-r}$ is anti-self-dual. The analog of the local operators (2.2) of the two-dimensional theory and the line operators (2.5) of the four-dimensional theory are the surface operators

$$W_{v_a, \tilde{v}_r} = \exp \left(i \oint_{\Sigma_2} (v_a B^{+a} + \tilde{v}_r B^{-r}) \right). \quad (2.9)$$

These operators describe the effect of a probe string with worldsheet Σ_2 that carries charges (v_a, \tilde{v}_r) . Because of the (anti)self-dual properties of the fields, there is no need to add independent 't Hooft operators.

Dirac quantization, or equivalently locality of the operators (2.9), leads to [20]

$$v_a u_a - \tilde{v}_r \tilde{u}_r \in \mathbb{Z}. \quad (2.10)$$

This condition is very similar to (2.4). It states that the signature (p, q) lattice of charges Γ has to be an integral lattice.

As in the two-dimensional case above, additional restrictions on the lattice Γ cannot be seen by studying the theory on \mathbb{R}^6 . However, imposing consistency of the theory once it is compactified on various spaces shows that

$$\Gamma = \Gamma^* \quad (2.11)$$

i.e. Γ should be self-dual, as we now demonstrate.

Following the approach taken in [21], we can relate the 6D theory to a 2D theory by compactifying the 6D theory on $\mathbb{C}\mathbb{P}^2$. The resulting two-dimensional theory is precisely the two-dimensional example we studied at the beginning of this section. Each self-dual/anti-self-dual two-form field reduces to a left/right-moving scalar field in 2D. The operators (2.9) reduce to operators (2.2), with a corresponding reduction of the charges (v_a, \tilde{v}_r) to elements of the 2D lattice Γ . The self-duality of the charge lattice of the two-dimensional theory leads to the conclusion (2.11) for the self-duality of the lattice of the six-dimensional theory.

This simple but complete argument shows that a 6D field theory with self-dual/anti-self-dual two-form fields must have a charge lattice satisfying (2.11). The condition that

the charge lattice of any six-dimensional theory is self-dual is, however, central to our analysis. Therefore it makes sense to view this conclusion from different perspectives. We now describe several other compactifications which also lead to the condition (2.11).

As one alternative approach, we could attempt to imitate the argument in two dimensions, which was based on the partition function on $\mathbb{S}^1 \times \mathbb{S}^1$, by considering the partition function of the 6D theory on $\mathbb{S}^3 \times \mathbb{S}^3$. Then, invariance under the exchange of the two \mathbb{S}^3 factors could also lead to the self-duality condition (2.11). This analysis must differ in some aspects from the 2D story, since there is no analogue of the Hamiltonian picture. We have not completed the details of such an approach, but this could give another interesting perspective on this problem.

Next, we consider the theory compactified on various tori. (A related compactification of the theory of a single self-dual field on \mathbb{T}^6 was considered in [22].) As with chiral scalar fields in 2D, there is a question of boundary conditions when compactifying a theory of 6D chiral and anti-chiral two-form fields on a torus. Indeed, much of the formalism developed in [11]-[17] is designed to address issues of this kind. These works show that there is a \mathbb{Z}_2 structure needed for B fields in six dimensions analogous to the spin structure needed for fermions in 2D. In the following subsection we discuss an odd charge lattice that requires this generalization, analogous to the presence of operators of half-integral spin in 2D. Such odd lattices still satisfy (2.11). There does not seem to be any way to use nontrivial boundary conditions to accommodate non-self-dual lattices, though it would be nice to have a rigorous proof of this assertion.

Perhaps the most intuitive route to deriving (2.11) follows from considering the theory on $\mathbb{R}^4 \times \mathbb{T}^2$. Each field B^{+a} or B^{-r} in six dimensions leads to a single Maxwell field in four dimensions, and we can use these Maxwell fields to describe the operators. One set of operators is obtained by taking the worldsheet Σ_2 in (2.9) to be spanned by one of the cycles in the compact \mathbb{T}^2 and a loop Σ_1 in \mathbb{R}^4 . This leads to a Wilson loop (2.5) in four dimensions. If instead, we let Σ_2 wrap another cycle in \mathbb{T}^2 , we find an 't Hooft loop in four dimensions. Therefore, the lattice Γ of the six-dimensional theory includes both the electric and the magnetic charges of the four dimensional theory. Since in four dimensions the lattice of electric charges must be dual to the lattice of magnetic charges (2.6), we see again that the original six-dimensional lattice must be self-dual. We give a more explicit description of how the self-dual lattice reduces to the electromagnetic charge lattice in 4D in terms of fields in the following subsection.

We can also consider compactification of the 6D theory on \mathbb{T}^4 . In this case we get several copies of the 6D charge lattice in 2D from windings of each H^\pm on different two-cycles in \mathbb{T}^4 . The resulting 2D charge lattice is a tensor product of the 6D charge lattice with the unimodular lattice given by $H_2(\mathbb{T}^4, \mathbb{Z})$ with the associated intersection form. The condition that the 2D charge lattice is unimodular again shows self-duality of the 6D charge lattice. Note that a similar argument for the 4D theories considered above compactified on \mathbb{T}^2 gives an alternate proof that all 4D charge lattices have an inner product of the form (2.8).

The arguments presented here show from several points of view that any field theory with self-dual and/or anti-self-dual two-form fields in six dimensions must have a self-dual

dyonic string charge lattice. In particular, the dimensional reductions on \mathbb{CP}^2 and tori give simple yet complete demonstrations of this result.

2.4 Explicit example: (1, 1) signature two-form fields

It may be helpful to consider a specific example in more detail. Consider the case of one B^+ and one B^- in six dimensions, *i.e.* $(p, q) = (1, 1)$. As discussed above, each of the two fields B^\pm reduces on \mathbb{T}^2 to a single $U(1)$ gauge field in four dimensions. We denote the compactified dimensions by 4, 5, and note that the (anti)-self-duality condition states that the 6D field strengths $H^\pm = dB^\pm$ satisfy

$$H_{4\mu\nu}^\pm = \pm \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} H_{5\rho\sigma}^\pm. \quad (2.12)$$

Thus, if we denote the field strength of the corresponding $U(1)$ in 4D by

$$F_{\mu\nu} = H_{4\mu\nu}, \quad (2.13)$$

then strings wrapped around the 4-direction of the torus (which carry string charge under H_{40i} , with i in the radial spatial direction) carry electric charge under the 4D $U(1)$, while strings wrapped around the 5-direction carry magnetic charge under the same $U(1)$. Since strings wrapped in either direction carry charges in the same 6D charge lattice Γ , the resulting theory in 4D has a single $U(1)$ field coming from the dimensional reduction of each of the two 6D B fields. The bilinear form on allowed electric and magnetic charges for the resulting $U(1)^2$ gauge group is then

$$\Gamma^{4D} = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}. \quad (2.14)$$

As we noted above, the determinant of the skew-symmetric charge product in 4D is always 1. It follows that $\det \Gamma = \pm 1$, so $\Gamma = \Gamma^*$. By performing a linear transformation on magnetic charges to put them in the dual basis to the electric charges, it is always possible to transform (2.14) into the form (2.8).

There are two distinct choices for the integral lattice Γ with signature (1, 1). We can have the odd lattice

$$\Gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.15)$$

Or we can have the even lattice

$$\Gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.16)$$

Note that in the even case the two fields B^\pm can be combined into a single B field with a Lagrangian description analogous to that of a single 4D massless vector. In this case, the analogue of Wilson and 't Hooft surface operators for the B field are adequate for describing the theory. In the odd case, however, we must use the surface operators for B^\pm in eq. (2.9). Related to that is the fact that upon compactification the odd theory needs additional data like spin structure.

The 6D case should be contrasted with the situation in 4D. In six dimensions, as in two dimensions, the dyonic charge lattice has a symmetric inner product, and both (2.15) and (2.16) are possible charge lattices. In four dimensions, the bilinear form on the electric-magnetic charge space is antisymmetric, and there is only one possible inner product structure on electric and magnetic charges.

Each of the two lattices Γ_1, Γ_0 is realized in certain classes of string compactifications. For Γ_1 , after dimensional reduction to 4D the bilinear form on charges is

$$\Gamma_1^{4D} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2.17)$$

The analogous form arising from Γ_0 is

$$\Gamma_0^{4D} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.18)$$

In each case, the matrix describing the form can be put into the canonical 4D form of eq. (2.8) by a change of basis for the magnetic charges. For eq. (2.17) this can be done by changing the sign on the fourth charge, while for eq. (2.18) this is done by switching the third and fourth charges.

3. Application to 6D supergravity

In six dimensions, antisymmetric two-index tensor fields appear in two distinct $\mathcal{N} = 1$ supersymmetry multiplets. The gravity multiplet contains a two-form field B^+ with a self-dual field strength. Theories may also have any number T of tensor multiplets each containing a two-form B^- with an anti-self-dual field strength. The set of allowed charges carried by dyonic string excitations of the theory must live in a lattice Γ of signature $(1, T)$. There is no guarantee that all allowed dyonic charges on this lattice are realized in the spectrum of the quantum theory, though strong arguments suggest that in a consistent theory of gravity there must be quantum excitations realizing all allowed charges in the charge lattice (see e.g. [23, 24]).

In a general $\mathcal{N} = 1$ 6D supergravity theory, in addition to the gravity multiplet and T tensor multiplets there are V vector multiplets describing a general gauge group G with simple non-Abelian factors $G_1 \times \cdots \times G_k$, and H hypermultiplets with scalars transforming in a representation of G . The presence or absence of Abelian factors does not affect the discussion here, which depends only on the non-Abelian part of the gauge group.

We now review the Green-Schwarz anomaly cancellation conditions for 6D supergravity theories [25, 26, 27, 28], following [5]. The anomaly cancellation condition can be written in terms of the 8-form anomaly polynomial as

$$I_8(R, F) = \frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta. \quad (3.1)$$

Here

$$X_4^\alpha = \frac{1}{2}a^\alpha \text{tr}R^2 + \sum_i b_i^\alpha \left(\frac{2}{\lambda_i} \text{tr}F_i^2 \right) \quad (3.2)$$

with a^α , b_i^α transforming as vectors in the space $\mathbb{R}^{1,T}$ with symmetric inner product $\Omega_{\alpha\beta}$; “tr” denotes the trace in the fundamental representation, and λ_i are normalization constants depending on the type of each simple group factor ($\lambda = 1$ for $SU(N)$ factors). Cancellation of the individual terms in (3.1) gives

$$H - V = 273 - 29T \quad (3.3)$$

$$0 = B_{\text{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i B_{\mathbf{R}}^i \quad (3.4)$$

$$a \cdot a = 9 - T \quad (3.5)$$

$$a \cdot b_i = \frac{1}{6}\lambda_i \left(A_{\text{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i A_{\mathbf{R}}^i \right) \quad (3.6)$$

$$b_i \cdot b_i = -\frac{1}{3}\lambda_i^2 \left(C_{\text{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i C_{\mathbf{R}}^i \right) \quad (3.7)$$

$$b_i \cdot b_j = \lambda_i \lambda_j \sum_{\mathbf{RS}} x_{\mathbf{RS}}^{ij} A_{\mathbf{R}}^i A_{\mathbf{S}}^j \quad (3.8)$$

where $A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}$ are group theory coefficients defined through

$$\text{tr}_{\mathbf{R}}F^2 = A_{\mathbf{R}}\text{tr}F^2 \quad (3.9)$$

$$\text{tr}_{\mathbf{R}}F^4 = B_{\mathbf{R}}\text{tr}F^4 + C_{\mathbf{R}}(\text{tr}F^2)^2, \quad (3.10)$$

and where $x_{\mathbf{R}}^i$ and $x_{\mathbf{RS}}^{ij}$ denote the number of matter fields that transform in the irreducible representation \mathbf{R} of gauge group factor G_i , and (\mathbf{R}, \mathbf{S}) of $G_i \otimes G_j$ respectively. Note that for groups such as $SU(2)$ and $SU(3)$, which lack a fourth order invariant, $B_{\mathbf{R}} = 0$ and there is no condition (3.4).

It is shown in [5] using elementary group theory that the inner products on the LHS of conditions (3.5-3.8) are all integral as a consequence of global and local anomaly cancellation. Thus, independent of any UV realization of the theory, any apparently consistent low-energy 6D supergravity theory contains an integral lattice Λ formed from vectors $a, b_i \in \mathbb{R}^{1,T}$. We refer to this lattice as the *anomaly lattice* of the theory. The vector a is associated with a coupling $B \cdot a \text{tr}R^2$ of the B fields to space-time curvature, while the vectors b_i are associated with couplings $B \cdot b_i \text{tr}F_i^2$ of the B fields to the field strengths F_i of the various factors in the gauge group.

The 6D theory has gauge dyonic string excitations with charge b_i associated with instantons in the factors of the gauge group [29]. The charges b_i therefore are contained within the dyonic string charge lattice of the 6D theory. While we do not have a complete argument that the charge a also corresponds to a dyonic string*, there are several heuristic arguments that suggest that this is the case. Locally, a gravitational instanton will carry

*We thank Ashoke Sen for useful discussions on this point.

charge a and should live in the charge lattice. The fact that a has an integer inner product with itself and all vectors b_i indicates that it satisfies Dirac quantization with respect to the other charges, which also strongly suggests that it should also be an element of the charge lattice.

F-theory [8, 9, 10] gives a very general approach to constructing 6D $\mathcal{N} = 1$ supergravity theories by describing nonperturbative IIB string compactifications on a complex surface \mathcal{B} containing 7-branes. Geometrically, monodromies around the 7-branes describe a Calabi-Yau threefold that is elliptically fibered over the base \mathcal{B} . For any F-theory construction, the charge lattice $\Gamma = H_2(\mathcal{B}, \mathbb{Z})$ is spanned by dyonic strings in six dimensions arising when 3-branes are wrapped on the 2-cycles of \mathcal{B} . Self-duality of Γ is then an automatic consequence of Poincaré duality. The lattice Λ spanned by the vectors a, b_i is embedded in a simple fashion in the lattice Γ for any F-theory model [5]. In this embedding, a corresponds to the canonical class of the space \mathcal{B} and is always associated with a state in the charge lattice.

The condition that the lattice of charges Γ for the two-form fields of any 6D $\mathcal{N} = 1$ theory must be self-dual is a strong constraint on 6D supergravity theories. This constraint rules out a large class of models that satisfy the other known consistency constraints of anomaly cancellation and proper-sign gauge kinetic terms. In particular, this constraint implies that the anomaly lattice Λ should admit an embedding into a self-dual lattice Γ . In [4, 5] several theories were identified that have anomaly lattices Λ violating this condition. The result of the previous section shows that these theories are inconsistent. Thus, these theories are not consistent quantum supergravity theories for any type of UV completion. In the following section we give some examples of such theories.

4. Examples of inconsistent theories

We give several examples of theories that satisfy anomaly cancellation and gauge kinetic term sign constraints, but that violate the self-dual charge lattice condition. In the first example we assume that the charge a is contained in the charge lattice. The second example is very similar but contains a charge lattice that is disallowed even when a is not considered. The third example is a model with two tensor multiplets, and the final example contains a more exotic matter representation. The examples given have $T = 1$ and $T = 2$. For $T = 0$ the only integral lattice is \mathbb{Z} , which is automatically self-dual, so no models in this class are ruled out by the self-duality constraint. For $T > 1$ it seems likely that a large fraction of anomaly-free models violate the self-duality constraint, though it is more difficult to prove in cases where the number of simple factors in the gauge group is less than T .

Example 1:

One example of a problematic model, first identified in [4] and discussed further in [5], is a 6D supergravity theory with $T = 1$, gauge group

$$G = SU(4) \tag{4.1}$$

and matter content

$$1 \times (\mathbf{15}) + 40 \times (\mathbf{4}) + 10 \times (\mathbf{6}) + 24 \times (\mathbf{1}). \tag{4.2}$$

It is easy to check that $H - V = 244$ as required by anomaly cancellation. The anomaly lattice for this theory consists of vectors a, b spanning the lattice

$$\Lambda = \begin{pmatrix} a \cdot a & -a \cdot b \\ -a \cdot b & b \cdot b \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ 10 & 10 \end{pmatrix}. \quad (4.3)$$

This lattice cannot be embedded in any unimodular lattice of signature $(1, 1)$. The easiest way to see this is to note that if Λ is embedded in a unimodular lattice of the same dimensionality as itself, then the volume of the unit cell spanned by $-a, b$ in Λ must be an integer. But the determinant of Λ is the volume squared, and therefore must be a perfect square (up to a sign). In this case $\det \Lambda = -20$, which is not a perfect square, so Λ does not admit an embedding into any two-dimensional unimodular lattice.

We can also directly prove this assertion by examining all cases as follows. The only signature $(1, 1)$ unimodular lattices are $\Gamma_{0,1}$ from (2.16, 2.15). In the case of Γ_0 , the only choices for $a \cdot a = 8$ are (up to overall sign) $(-2, -2)$ and (up to sign and exchange of coordinates) $(-1, -4)$. If $a = (-2, -2)$ then $b = (x, y)$ with $2xy = 10, 2(x + y) = 10$, which have no integral solutions for x, y . Similarly, if $a = (-1, -4)$ then $2xy = 10, x + 4y = 10$, which again has no integer solutions. For Γ_1 , the only possibility (up to signs) for a is $a = (-3, 1)$, which gives $x^2 - y^2 = 10, 3x + y = 10$, which once again has no integer solutions. Thus, Λ cannot be realized by any set of vectors in a unimodular lattice Γ , so this 6D theory cannot be consistent when reduced on a torus to 4D, and therefore is inconsistent in 6D.

Example 2:

We now consider a closely related example, where the charge a is not needed to rule out the lattice. Basically, the idea is to construct a theory like the previous example, but with a second gauge group factor having a dyonic string instanton with charge proportional to a .

In this example the gauge group is

$$G = SU(3) \times SU(3). \quad (4.4)$$

The matter content is

$$15 \times (\mathbf{3}, \mathbf{1}) + 45 \times (\mathbf{1}, \mathbf{3}) + 5 \times (\mathbf{3}, \mathbf{3}) + 1 \times (\mathbf{1}, \mathbf{8}) + 27 \times (\mathbf{1}, \mathbf{1}). \quad (4.5)$$

It can be verified that both local and global anomalies cancel for this model, and the anomaly lattice is spanned by vectors $-a, b_1, b_2$ with inner products

$$\Lambda = \begin{pmatrix} 8 & 4 & 10 \\ 4 & 2 & 5 \\ 10 & 5 & 10 \end{pmatrix}. \quad (4.6)$$

Note that this lattice is degenerate, with $b_1 = -a/2$ and hence we can focus on the two-dimensional lattice spanned by the dyonic string charges b_1, b_2 ,

$$\hat{\Lambda} = \begin{pmatrix} 2 & 5 \\ 5 & 10 \end{pmatrix}. \quad (4.7)$$

The arguments used in the previous example show that this lattice cannot be embedded in any unimodular lattice, so the theory is inconsistent.

Example 3:

For another example, take the theory with $T = 2$ with gauge group

$$G = SU(N) \times SU(N). \quad (4.8)$$

and charged matter content

$$2N \times (\mathbf{N}, \mathbf{1}) + 2N \times (\mathbf{1}, \mathbf{N}). \quad (4.9)$$

The anomaly lattice for this model is spanned by vectors $-a, b_1, b_2$ with inner products

$$\Lambda = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (4.10)$$

Once again, this does not admit a unimodular embedding since the determinant is 28, which is not a perfect square. This is another example in which the inconsistency of the model depends upon the appearance of a in the charge lattice. (Note for comparison that the theory with one $SU(N)$ factor and $2N$ fundamental representations with $T = 1$ has an anomaly lattice $\text{diag}(8, -2)$ with determinant -16, which does admit an embedding into a unimodular lattice.)

Example 4:

For another example, take the theory with $T = 1$ with gauge group

$$G = SU(3) \times SU(4). \quad (4.11)$$

and matter content

$$2 \times (\mathbf{3}, \mathbf{4}) + 22 \times (\mathbf{3}, \mathbf{1}) + 13 \times (\mathbf{1}, \mathbf{4}) + 1 \times (\mathbf{1}, \mathbf{20}' \left[\begin{array}{|c|} \hline \square \\ \hline \end{array} \right]) + 105 \times (\mathbf{1}, \mathbf{1}). \quad (4.12)$$

The anomaly lattice for this model is spanned by vectors $-a, b_1, b_2$ with inner products

$$\Lambda = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 2 & 2 \\ 4 & 2 & 6 \end{pmatrix}. \quad (4.13)$$

As in example 2, this lattice is degenerate, with $b_1 = -a/2$. Considering only the lattice spanned by dyonic string charges b_1, b_2 , a similar argument to the other examples shows that the lattice cannot be embedded in any unimodular lattice, so the theory is not consistent.

5. Concluding remarks

Following on the work of [11]-[17] we have given simple physical arguments showing that the charge lattice of two-form fields in six dimensions should be integral and self-dual (but not necessarily even). The criteria we have used, based on consistency of the theory upon

compactification to lower dimensions, can be interpreted as physical requirements for the theory. Despite the substantial progress of [11]-[17], we feel that a simpler mathematical treatment connecting the general formalism to the physical properties of these theories is still desirable. But even without such a deeper understanding, we can use these consistency conditions to derive powerful constraints on supergravity theories and therefore also on possible string constructions. The results of this paper show that a certain class of 6D supergravity theories are inconsistent. This narrows the gap between the set of 6D theories that are not provably inconsistent and the set of theories with known string realizations.

The arguments given in Section 2 for self-duality of the 6D charge lattice were based on principles of field theory. It is important to stress that while coupling to gravity can lead to additional consistency conditions, the conditions derived from field theory are still necessary in the presence of gravity and other fields. In particular, when we consider compactification of a theory with two-forms and gravity on \mathbb{CP}^2 we can take the size of the compact space to be much larger than the Planck length, so that all gravitational effects are negligible.

When compactifying on tori there is also the issue that new charges appear coupling to the gauge fields arising from the 6D metric. This extends the charge lattice of the lower dimensional theory. Kaluza-Klein momenta produce electric charges for these additional gauge fields. We can choose, however, to perform the reduction on a square torus (or d -dimensional generalization thereof) in the absence of background fields, in which case the resulting extension of the charge lattice is only by a direct sum with a trivial $2d \times 2d$ matrix of the form (2.8). This does not change the determinant, so the unimodularity constraint still holds when gravity is added.

KK monopoles can also appear which naively will carry a dyonic charge associated with the vector a appearing in the coupling $a \cdot BR^2$. This does not change the self-dual structure of the charge lattice, but does give another reason to suspect that a is in the charge lattice. The role of the dyonic charge a in the theory is, however, somewhat subtle. Although this charge was shown in [5] to satisfy Dirac quantization with respect to gauge dyonic strings for any theory satisfying anomaly cancellation, we do not have a rigorous proof that this dyonic charge must appear in the charge lattice. It would be nice to have a more complete understanding of the role of the dyonic string charge a . One promising approach to understanding this and related questions is through a systematic understanding of the world-sheet theory on the dyonic strings themselves.

We have used the condition that the dyon lattice must be self-dual to rule out a number of 6D supergravity theories which otherwise obey the known consistency conditions. In some cases, the gauge instantons in the 6D theory provide a set of dyonic string charges that is sufficient to rule out the theory. In other cases, the full anomaly lattice cannot be embedded in a self-dual lattice only because of the vector a associated with gravitational physics. In the latter set of cases, a better understanding of a is needed to rigorously rule out these theories.

One might also wonder whether the modifications of the sum over topological sectors of [30] and [24] can affect our conclusions. These papers coupled a discrete gauge theory to the topological charges, which has the effect of excluding certain topological sectors. We

can contemplate coupling the various four-form string currents in (3.2) to such a discrete gauge theory. This amounts to enlarging the gauge symmetry of the two-forms B^\pm by adding a discrete gauge symmetry for such forms. One way to think about such discrete gauge symmetries [31, 24] is to embed them in continuous symmetries for additional B fields and then Higgs them down to the discrete symmetry. For example, if the discrete gauge symmetry is \mathbb{Z}_p , we add to the theory a two-form field B , which includes both a self-dual and an anti-self-dual $H = dB$, as well as its Higgs field, which is a one-form A . The presence of the new B fields enlarges the charge lattice. But since the added fields are “vector like”, they cannot affect our conclusion about the original system and the charge lattice of the massless B^\pm should still be self-dual.

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References

- [1] N. Seiberg, “Observations On The Moduli Space Of Superconformal Field Theories,” Nucl. Phys. B **303**, 286 (1988).
- [2] V. Kumar and W. Taylor, “String Universality in Six Dimensions,” [arXiv:0906.0987](#) [[hep-th](#)].
- [3] V. Kumar and W. Taylor, “A bound on 6D $\mathcal{N} = 1$ supergravities,” JHEP **0912**, 050 (2009) [arXiv:0910.1586](#) [[hep-th](#)].
- [4] D. R. Morrison, V. Kumar and W. Taylor, “Mapping 6D $\mathcal{N} = 1$ supergravity to F-theory,” [arXiv:0911.3393](#) [[hep-th](#)].
- [5] D. R. Morrison, V. Kumar and W. Taylor, “Global aspects of the space of $\mathcal{N} = 1$ supergravities,” [arXiv:1008.1062](#) [[hep-th](#)].
- [6] W. Taylor, “Anomaly constraints and string/F-theory geometry in 6D quantum gravity,” to appear in proceedings of “*Perspectives in Mathematics and Physics*,” conference in honor of I. M. Singer, [arXiv:1009.1246](#) [[hep-th](#)].
- [7] V. Kumar, D. Park and W. Taylor, “6D supergravity without tensor multiplets,” [arXiv:1011.0726](#) [[hep-th](#)].
- [8] C. Vafa, “Evidence for F-Theory,” Nucl. Phys. B **469**, 403 (1996) [arXiv:hep-th/9602022](#).
- [9] D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau Threefolds – I,” Nucl. Phys. B **473**, 74 (1996) [arXiv:hep-th/9602114](#).
- [10] D. R. Morrison and C. Vafa, “Compactifications of F-Theory on Calabi–Yau Threefolds – II,” Nucl. Phys. B **476**, 437 (1996) [arXiv:hep-th/9603161](#).
- [11] E. Witten, “Five-brane effective action in M theory,” J. Geom. Phys. **22**, 103-133 (1997). [hep-th/9610234](#).

- [12] M. J. Hopkins, I. M. Singer, “Quadratic functions in geometry, topology, and M theory,” *J. Diff. Geom.* **70**, 329-452 (2005). [math/0211216](#) [[math-at](#)].
- [13] D. E. Diaconescu, G. W. Moore and E. Witten, “E(8) gauge theory, and a derivation of K-theory from M-theory,” *Adv. Theor. Math. Phys.* **6**, 1031 (2003) [arXiv:hep-th/0005090](#)
- [14] G. W. Moore and E. Witten, “Self-duality, Ramond-Ramond fields, and K-theory,” *JHEP* **0005**, 032 (2000) [arXiv:hep-th/9912279](#)
- [15] D. S. Freed, G. W. Moore, G. Segal, “The Uncertainty of Fluxes,” *Commun. Math. Phys.* **271**, 247-274 (2007). [hep-th/0605198](#).
- [16] D. S. Freed, G. W. Moore, G. Segal, “Heisenberg Groups and Noncommutative Fluxes,” *Annals Phys.* **322**, 236-285 (2007). [[hep-th/0605200](#)].
- [17] G. Moore, “Overview of the theory of Self-dual fields” and “The RR charge of an orientifold”, which are available through <http://www.physics.rutgers.edu/~gmoore/>.
- [18] K. S. Narain, “New Heterotic String Theories In Uncompactified Dimensions < 10 ,” *Phys. Lett. B* **169**, 41 (1986).
- [19] K. S. Narain, M. H. Sarmadi and E. Witten, “A Note on Toroidal Compactification of Heterotic String Theory,” *Nucl. Phys. B* **279**, 369 (1987).
- [20] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, “p-brane dyons and electric-magnetic duality,” *Nucl. Phys. B* **520**, 179 (1998) [arXiv:hep-th/9712189](#).
- [21] E. Witten, “Duality relations among topological effects in string theory,” *JHEP* **0005**, 031 (2000) [arXiv:hep-th/9912086](#)
- [22] L. Dolan and C. R. Nappi, “A modular invariant partition function for the fivebrane,” *Nucl. Phys. B* **530**, 683 (1998) [arXiv:hep-th/9806016](#).
- [23] J. Polchinski, “Monopoles, duality, and string theory,” *Int. J. Mod. Phys. A* **19S1**, 145 (2004) [arXiv:hep-th/0304042](#).
- [24] T. Banks and N. Seiberg, “Symmetries and Strings in Field Theory and Gravity,” [arXiv:1011.5120](#) [[hep-th](#)].
- [25] M. B. Green, J. H. Schwarz and P. C. West, “Anomaly Free Chiral Theories In Six-Dimensions,” *Nucl. Phys. B* **254**, 327 (1985).
- [26] A. Sagnotti, “A Note on the Green-Schwarz mechanism in open string theories,” *Phys. Lett. B* **294**, 196 (1992) [arXiv:hep-th/9210127](#).
- [27] V. Sadov, “Generalized Green-Schwarz mechanism in F theory,” *Phys. Lett. B* **388**, 45 (1996) [arXiv:hep-th/9606008](#).
- [28] G. Honecker, “Massive U(1)s and heterotic five-branes on K3,” *Nucl. Phys. B* **748**, 126 (2006) [arXiv:hep-th/0602101](#).
- [29] M. J. Duff, H. Lu and C. N. Pope, “Heterotic phase transitions and singularities of the gauge dyonic string,” *Phys. Lett. B* **378**, 101 (1996) [arXiv:hep-th/9603037](#).
- [30] N. Seiberg, “Modifying the Sum Over Topological Sectors and Constraints on Supergravity,” *JHEP* **1007**, 070 (2010). [arXiv:1005.0002](#) [[hep-th](#)].
- [31] J. M. Maldacena, G. W. Moore, N. Seiberg, “D-brane charges in five-brane backgrounds,” *JHEP* **0110**, 005 (2001). [[hep-th/0108152](#)].