Nonequilibrium quantum fluctuations of a dispersive medium: Spontaneous emission, photon statistics, entropy generation, and stochastic motion

Mohammad F. Maghrebi,[1,2,*] Robert L. Jaffe,[1] and Mehran Kardar[2]

[1]*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
[2]*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*
(Received 7 January 2014; published 16 July 2014)

We study the implications of quantum fluctuations of a dispersive medium, under steady rotation, either in or out of thermal equilibrium with its environment. A rotating object exhibits a quantum instability by dissipating its mechanical motion via spontaneous emission of photons, as well as internal heat generation. Universal relations are derived for the radiated energy and angular momentum as trace formulas involving the object's scattering matrix. We also compute the quantum noise by deriving the full statistics of the radiated photons out of thermal and/or dynamic equilibrium. The (entanglement) entropy generation is quantified and the total entropy is shown to be always increasing. Furthermore, we derive a Fokker-Planck equation governing the stochastic angular motion resulting from the fluctuating backreaction frictional torque. As a result, we find a quantum limit on the uncertainty of the object's angular velocity in steady rotation. Finally, we show in some detail that a rotating object drags nearby objects, making them spin parallel to its axis of rotation. A scalar toy model is introduced to simplify the technicalities and ease the conceptual complexities and then a detailed discussion of quantum electrodynamics is presented.

I. INTRODUCTION AND SUMMARY

Fluctuation-induced phenomena have been widely explored in equilibrium where a global temperature exists and the medium (consisting of one or more objects) is static. Out of equilibrium, quantum and thermal fluctuations can give rise to a rich set of phenomena. A special case of interest is stationary nonequilibrium where there is a temperature gradient, or a medium in steady motion. Energy radiation, friction, and dissipation are among the most common themes in this realm.

Here we explore both thermal and and dynamic nonequilibrium with an emphasis on the latter. Specifically, when neutral objects are set in motion, they interact with quantum fluctuations in the background environment in a time-dependent fashion that may excite photons from the vacuum and lead to quantum radiation. The creation of photons by moving mirrors in one dimension was first discussed by Moore [1]. Accelerated neutral boundaries radiate energy and thus experience a backreaction force, or quantum friction [1–12]. Recent experiments mimicking such dynamical Casimir effects rely on quantum interference devices for rapidly changing boundary conditions of a cavity [13].

While a substantial amount of literature is devoted to the dynamical Casimir effect in the context of ideal mirrors with perfect boundary conditions [14–19], dielectric and dispersive materials have also been studied in several cases [20,21]. In general, the latter is more complicated since a quantum system is usually described by a Hamiltonian, which is lacking for a lossy system. A path-integral formulation is also not trivial since the physical system is out of equilibrium, necessitating the more complicated formalism developed by Schwinger and Keldysh [22,23]; several applications of this formalism to quantum friction are investigated in Refs. [24,25]. Interestingly, dispersive objects experience quantum friction even when they move at a constant relative velocity: Two parallel plates moving laterally with respect to each other experience a (noncontact) frictional force [26,27]. Noncontact friction is usually treated within the framework of the Rytov formalism, which is grounded in application of the fluctuation-dissipation theorem to electrodynamics [28]. Recently, it was shown that a quantum analog of Cherenkov effect appears when two planar objects are in relative motion beyond a threshold velocity set by the speed of light inside the medium [29]; this effect was originally discovered by Ginzburg and Frank [30]. A recent work [31] also explores quantum friction beyond Cherenkov velocity. There are discrepancies between the reported results, which need to be further investigated.

While a constant translational motion requires at least two bodies (otherwise it is trivial due to Lorentz symmetry), a single spinning object can experience friction [32], a phenomenon closely connected to superradiance first introduced by Zel'dovich [33]. He argued that a rotating object amplifies certain incident waves and speculated that this would lead to spontaneous emission when quantum mechanics is considered. In the context of general relativity, the Penrose process provides a mechanism similar to superradiance to extract energy from a rotating black hole [34], which also leads to quantum spontaneous emission [35]. This radiation, however, is different in nature from Hawking radiation, which is due to the existence of event horizons [36]. One can also find similar effects for a superfluid where a rotating object experiences friction even at zero temperature [37].[1]

In this article we expand on a previous work [39] dealing with quantum fluctuations of a rotating object. We treat

---

*Present address: Joint Quantum Institute, NIST and University of Maryland, College Park, Maryland 20742, USA.

[1]For another proposal related to Casimir-like forces in a slowly moving superfluid, see Ref. [38].

vacuum fluctuations in the presence of a dispersive object under rotation exactly, except for the assumption of small enough velocities to avoid complications of relativity, thus going beyond the approximate treatments of previous works in Refs. [32,40]. By incorporating the Green's function techniques into the Rytov formalism [28], we show that a rotating object spontaneously emits photons.

Since we aim to present in detail computations leading to the results briefly described in Ref. [39], this paper is necessarily heavy in technical content. Given that we also derive a number of results (pertaining to counting statistics, entropy, and stochastic motion), it is important that the mathematical formalism does not obscure their conceptual simplicity. As such, in the remainder of the Introduction we summarize the important results, in the order in which they appear in the main text.

We consider a solid of revolution rotating around its axis of symmetry at a rate Ω. For the sake of definiteness, we assume that the object is at a finite temperature T immersed in a zero-temperature vacuum. Generalization to a finite environment temperature is straightforward. The rotating object is characterized by its scattering amplitude S, which, according to the symmetries of the problem, is diagonal in frequency ω and the angular momentum along the rotation axis (in units of $\hbar$) m. We use α as a shorthand for all quantum numbers including ω and m and denote $\mathcal{I}_\alpha$ as the operator corresponding to the current, or the number of radiated photons, in a partial wave α.

(i) A dispersive object under rotation is unstable due to quantum fluctuations. We show that the object spontaneously emits photons at the rate

$$\frac{dN_\alpha}{d\omega dt} = \mathcal{N}_\alpha = \langle \mathcal{I}_\alpha \rangle = n(\omega - \Omega m, T)(1 - |S_\alpha|^2), \quad (1)$$

where $n(\omega, T) = 1/(e^{\hbar\omega/kT} - 1)$ is the Bose-Einstein distribution function at temperature T. Note the shift in frequency due to rotation. This effect persists at zero temperature,

$$\lim_{T \to 0} \mathcal{N}_\alpha = \Theta(\Omega m - \omega)(|S_\alpha|^2 - 1), \quad (2)$$

pointing to its quantum nature. The scattering matrix is in fact superunitary $|S_\alpha| > 1$ when $\omega < \Omega m$, hence superradiance.

(ii) The rate of energy and angular momentum radiation, and heat generation are obtained by integrating the current multiplied by the corresponding quantum number and can be expressed as trace formulas. The energy radiation, for example, can be written as

$$\mathcal{P} = \int \frac{d\omega}{2\pi} \hbar\omega \, \mathrm{Tr}[n(\omega - \Omega\hat{l}_z, T)(1 - \mathbb{S}\mathbb{S}^\dagger)]. \quad (3)$$

The angular momentum radiation and heat generation can be computed by replacing $\hbar\omega \to \hbar m$ and $\hbar\omega \to \hbar(\Omega m - \omega)$, respectively. The loss of angular momentum manifests itself in a quantum friction torque that opposes the rotation of the object. Initially at zero temperature, the object loses energy and angular momentum and heats up at the same time. The energy conservation is respected as the mechanical energy due to rotation is converted into radiation and heat; see Sec. II B 2 for a detailed discussion.

(iii) We go beyond the averaged value of the radiation and compute the fluctuations of the (fluctuation-induced) radiation,

i.e., higher moments of the current-current correlators. We find the cumulants of factorial moments as

$$\kappa_p = \left\langle \hat{\mathcal{I}}_\alpha^p \right\rangle_c = (p - 1)! \mathcal{N}_\alpha^p, \quad (4)$$

with the subscript c indicating the connected component of the p-point function. Remarkably, the average current $\mathcal{N}_\alpha$ also determines all higher moments of fluctuations.

(iv) Photon statistics can be derived from the knowledge of higher-moment fluctuations. The probability that n photons are radiated in a mode α is given by

$$P_\alpha(n) = \frac{\mathcal{N}_\alpha^n}{(\mathcal{N}_\alpha + 1)^{n+1}}. \quad (5)$$

(v) As the result of radiation, entropy is increased in the environment. The entropy generation can be obtained from photon statistics as

$$\mathcal{S} \equiv \frac{dS}{dt} = k_B \sum_\alpha \!\!\!\!\int [(\mathcal{N}_\alpha + 1) \ln(\mathcal{N}_\alpha + 1) - \mathcal{N}_\alpha \ln \mathcal{N}_\alpha]. \quad (6)$$

The symbol $\sum\!\!\!\!\int$ indicates an integral over frequency as well as a sum over other quantum numbers. This entropy can be interpreted as the entanglement entropy between the object and the environment consisting of radiated photons; see Sec. II D.

(vi) A freely rotating object slows down as the result of the quantum friction torque M and also undergoes a stochastic motion due to the fluctuational variance of the torque $\mathrm{Var}M$,

$$M = \hbar \sum_\alpha \!\!\!\!\int m\mathcal{N}_\alpha,$$

$$\mathrm{Var}M = \hbar^2 \sum_\alpha \!\!\!\!\int m^2 \mathcal{N}_\alpha(\mathcal{N}_\alpha + 1). \quad (7)$$

The equation of rotation is then a Langevin equation (I being the moment of inertia)

$$I\dot{\Omega}(t) = -M(\Omega(t)) + \eta(t; \Omega(t)). \quad (8)$$

The noise $\eta(t; \Omega(t))$ has zero mean, is independent at different times, and correlated for equal times via

$$\langle \eta(t; \Omega(t)) \rangle = 0,$$

$$\langle \eta(t; \Omega(t)) \eta(t'; \Omega(t')) \rangle = \mathrm{Var}M(\Omega(t))\delta(t - t'). \quad (9)$$

Equivalently, a Fokker-Planck equation describes the probability distribution as a function of angular velocity; for a detailed discussion, see Sec. I E. Even at zero temperature, we find a quantum limit on how sharply the angular momentum can be defined for a single object in steady rotation,

$$I\Delta\Omega = \sqrt{I \frac{\mathrm{Var}M(\Omega_0)}{\partial M/\partial \Omega_0}} \propto \sqrt{\hbar I \Omega_0}. \quad (10)$$

Thus the uncertainty in the angular momentum is proportional to the geometrical mean of $\hbar$ and the object's angular momentum (and not $\hbar$ itself).

(vii) A rotating object makes nearby bodies orbit around its center and also spin parallel to its rotations axis. For the details, see Sec. II F.

Our starting point is the Rytov formalism [28], which relates fluctuations of the electromagnetic field to fluctuating sources within the material bodies, and in turn to the material's

TABLE I. Comparison of scalar and electromagnetic formulations for a static medium: the Lagrangian in free space, the current density in vacuum, the Lagrangian terms due to the linear response of the medium and fluctuating sources, and the fluctuation-dissipation relation for scalar and electromagnetic fields, respectively. Here $a(\omega) = \hbar \coth(\hbar\omega/2k_B T)$.

Formulation	Scalar	EM
Free Lagrangian	$\frac{\omega^2}{c^2}|\Phi_\omega|^2 - |\nabla\Phi_\omega|^2$	$|\mathbf{E}_\omega|^2 - \frac{c^2}{\omega^2}|\nabla\times\mathbf{E}_\omega|^2$
Current density	$\mathrm{Im}\,\Phi_\omega^*\nabla\Phi_\omega$	$\frac{c^2}{\omega^2}\mathrm{Im}\,\mathbf{E}_\omega^*\times(\nabla\times\mathbf{E}_\omega)$
Medium's Lagrangian	$[\epsilon(\omega)-1]\frac{\omega^2}{c^2}|\Phi_\omega|^2$	$[\epsilon(\omega)-1]|\mathbf{E}_\omega|^2$
Random source Lagrangian	$-i\frac{\omega}{c}\Phi_\omega^*\varrho_\omega$	$\mathbf{E}_\omega^* \cdot \mathbf{K}_\omega$
Fluctuation-dissipation	$\langle\varrho_\omega(\mathbf{x})\varrho_\omega^*(\mathbf{y})\rangle = a(\omega)\mathrm{Im}\,\epsilon(\omega)\delta(\mathbf{x}-\mathbf{y})$	$\langle\mathbf{K}_\omega(\mathbf{x})\otimes\mathbf{K}_\omega^*(\mathbf{y})\rangle = a(\omega)\mathrm{Im}\,\epsilon(\omega)\delta(\mathbf{x}-\mathbf{y})\mathbb{I}$

dispersive properties, via the fluctuation-dissipation theorem. While our main focus is moving objects, we find it useful to first consider thermal nonequilibrium where the object and the environment are at different temperatures. For the sake of simplicity and clarity, we start with a toy model based on a scalar field in Sec. II and postpone the full discussion of electrodynamics to Sec. III. Readers interested only in the electromagnetic case may wish to skip the pedagogical Sec. II. While qualitative results are similar in the two cases, electromagnetism is distinguished by its vector nature, which gives rise to two different polarizations. For the reader's convenience, the analogy between scalar and electromagnetic fields and static vs rotating media is summarized in Tables I and II.

II. A TOY MODEL: A DIELECTRIC OBJECT INTERACTING WITH A SCALAR FIELD

We consider a scalar field that interacts with an object characterized by a response, or dielectric, function ϵ. The response function is in principle a function of both frequency and position and fully characterizes the object's dispersive properties. The field equation for this model in the frequency domain reads

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\epsilon(\omega,\mathbf{x})\right)\Phi(\omega,\mathbf{x}) = 0, \qquad (11)$$

with ϵ being 1 in the vacuum and a frequency-dependent function inside the object.

In order to describe quantum[2] (and thermal) fluctuations, one can consider the field as a stochastic entity whose fluctuations are governed by a random source. From this perspective, quantum fluctuations are cast into a Langevin-like

---

[2]We shall refer to the quanta of the scalar field as photons.

equation (similar to the random force in the theory Brownian motion). For the electromagnetic field, the Rytov formalism provides such a stochastic formulation [28]. We introduce a similar approach for the scalar field theory, the central subject of this section. The field equation coupled to a (random) source ϱ is given by

$$-\left(\triangle + \frac{\omega^2}{c^2}\epsilon(\omega,\mathbf{x})\right)\Phi(\omega,\mathbf{x}) = -\frac{i\omega}{c}\varrho_\omega(\mathbf{x}), \qquad (12)$$

where the source satisfies a δ-function correlation function in space

$$\langle\varrho_\omega(\mathbf{x})\varrho_\omega^*(\mathbf{y})\rangle = a(\omega)\mathrm{Im}\,\epsilon(\omega,\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \qquad (13)$$

with

$$a(\omega) = 2\hbar\left(n(\omega,T) + \frac{1}{2}\right) = \hbar\coth\left(\frac{\hbar\omega}{2k_B T}\right). \qquad (14)$$

Note that source fluctuations are related to the imaginary part of the response function in harmony with the fluctuation-dissipation theorem (FDT). At a finite temperature T, the Bose-Einstein distribution function $n(\omega,T) = [\exp(\hbar\omega/k_B T)-1]^{-1}$ captures thermal fluctuations; the additional $1/2$ is due to quantum zero-point fluctuations.

The field is related to the source via the Green's function G, defined as

$$-\left(\triangle + \frac{\omega^2}{c^2}\epsilon(\omega,\mathbf{x})\right)G(\omega,\mathbf{x},\mathbf{z}) = \delta(\mathbf{x}-\mathbf{z}). \qquad (15)$$

TABLE II. Comparison of scalar and electromagnetic formulations for a rotating medium: The last three rows of Table I (pertaining to the interior of the medium) are modified under steady rotation. Note that $\omega' = \omega - \Omega m$ is the shifted frequency in the comoving frame rotating at the rate Ω.

Formulation	Scalar	EM
Medium's Lagrangian	$[\epsilon(\omega')-1]\frac{\omega'^2}{c^2}|\Phi_\omega|^2$	$[\epsilon(\omega')-1]|\mathbf{E}_\omega|^2$
Random source Lagrangian	$-i\frac{\omega'}{c}\Phi_\omega^*\varrho_\omega$	$[\mathbf{E}_\omega^* + \frac{i}{\omega}\mathbf{v}\times(\nabla\times\mathbf{E}_\omega^*)] \cdot \mathbf{K}_\omega$
Fluctuation-dissipation	$\langle\varrho_\omega(\mathbf{x})\varrho_\omega^*(\mathbf{y})\rangle = a(\omega')\mathrm{Im}\,\epsilon(\omega')\delta(\mathbf{x}-\mathbf{y})$	$\langle\mathbf{K}_\omega(\mathbf{x})\otimes\mathbf{K}_\omega^*(\mathbf{y})\rangle = a(\omega')\mathrm{Im}\,\epsilon(\omega')\delta(\mathbf{x}-\mathbf{y})\mathbb{I}$

In equilibrium (uniform temperature with static objects), the field correlation function is obtained as

$$
\langle \Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle
$$
$$
= \frac{\omega^2}{c^2}\int\int_{\text{all space}} d\mathbf{z}\,d\mathbf{w}\,G(\omega,\mathbf{x},\mathbf{z})G^*(\omega,\mathbf{y},\mathbf{w})\langle\varrho_\omega(\mathbf{z})\varrho_\omega^*(\mathbf{w})\rangle
$$
$$
= \frac{\omega^2}{c^2}a(\omega)\int_{\text{all space}} d\mathbf{z}\,G(\omega,\mathbf{x},\mathbf{z})\text{Im}\,\epsilon(\omega,\mathbf{z})G^*(\omega,\mathbf{y},\mathbf{z})
$$
$$
= a(\omega)\text{Im}\,G(\omega,\mathbf{x},\mathbf{y}). \tag{16}
$$

Note that the second line in Eq. (16) follows from $\frac{\omega^2}{c^2}\text{Im}\,\epsilon = -\text{Im}\,G^{-1}$ according to Eq. (15). This equation manifests the FDT by relating field fluctuations to the imaginary part of the Green's function. However, Eq. (16) requires the system to be in equilibrium while Eq. (13) is formulated locally and makes no assumption about global properties of the system such as overall equilibrium. Therefore, we shall employ Eq. (13) to study nonequilibrium systems.

In the following sections we explore the interplay between geometry, motion, and temperature. While our main interest is the consequences of fluctuations in the context of moving objects, we make a detour to study quantum and thermal fluctuations for a static object. Out of thermal equilibrium, the object is at a temperature different from that of the environment. The techniques we develop in the following section are useful when we consider moving objects in or out of thermal equilibrium. For simplicity, we consider a disk in two-dimensional space; generalization to realistic objects is discussed in the context of electromagnetism.

In the following, we make the convention that $c = 1$ unless stated otherwise.

A. Field fluctuations for static objects

According to the Rytov formalism, field fluctuations are induced by random sources that fluctuate according to the object's local properties (encoded by the imaginary part of the response function) and temperature (through the Bose-Einstein factor). It is then natural to divide the space into the object and the environment (vacuum) and to compute the source fluctuations in each region separately.

1. Vacuum fluctuations

In this section we consider field fluctuations due to random sources only in the vacuum. The scalar field is coupled to fluctuating sources outside the object as

$$
-[\triangle + \omega^2\epsilon(\omega,\mathbf{x})]\Phi(\omega,\mathbf{x}) = \begin{cases} 0, & |\mathbf{x}| < R \\ -i\omega\varrho_\omega(\mathbf{x}), & |\mathbf{x}| > R, \end{cases} \tag{17}
$$

with R being the radius of the disk. Source fluctuations, according to the Rytov formalism, are determined by

$$
\langle\varrho_\omega(\mathbf{x})\varrho_\omega^*(\mathbf{y})\rangle = a_{\text{out}}(\omega)\text{Im}\,\epsilon_D(\omega)\delta(\mathbf{x}-\mathbf{y}), \tag{18}
$$

where the points $\mathbf{x}$ and $\mathbf{y}$ are outside the object, a_{out} corresponds to the temperature of the environment, and ϵ_D represents the response functions in the vacuum. It might seem that this function is 1 and $\text{Im}\,\epsilon_D = 0$, hence there are no source fluctuations outside the object. However, even in empty space,

we need sources to give rise to zero-point fluctuations. Indeed, as one has to integrate over infinite volume, the limit of $\text{Im}\,\epsilon_D \to 0$ should be taken with care. The corresponding field correlation function outside the object is given by

$$
\langle\Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}
$$
$$
= \omega^2 a_{\text{out}}(\omega)\text{Im}\,\epsilon_D(\omega)\int_{|\mathbf{z}|>R} d\mathbf{z}\,G(\omega,\mathbf{x},\mathbf{z})G^*(\omega,\mathbf{y},\mathbf{z}). \tag{19}
$$

Note that the Green's functions are evaluated outside the object. Let (r,ϕ) and (ξ,ψ) be the polar coordinates of $\mathbf{x}$ and $\mathbf{y}$, respectively. The (retarded) Green's function can be cast as a sum over partial waves in the cylindrical basis as

$$
G(\omega,\mathbf{x},\mathbf{y}) = \sum_{m=-\infty}^{\infty} \frac{i}{8}\big[H_m^{(2)}(\omega r) + S_m(\omega)H_m^{(1)}(\omega r)\big]
$$
$$
\times e^{im\phi}H_m^{(1)}(\omega\xi)e^{-im\psi}, \quad R < r < \xi, \tag{20}
$$

where $H_m^{(1,2)}$ are the Hankel functions of the first and second kind and $S_m(\omega)$ is the scattering matrix. Furthermore, we have assumed that the point $\mathbf{y}$ is located at a larger radius from the origin without loss of generality. In empty space $S = 1$ and we recover the free Green's function as

$$
G(\omega,\mathbf{x},\mathbf{y}) = \sum_{m=-\infty}^{\infty} \frac{i}{4}J_m(\omega r)e^{im\phi}H_m^{(1)}(\omega\xi)e^{-im\psi}, \quad r < \xi.
$$

To compute the integral in Eq. (19), one should integrate over $R < |\mathbf{z}| < \infty$; however, we take the limit that $\text{Im}\,\epsilon_D \to 0$ and only a singular contribution, due to the integral over $|\mathbf{z}| \to \infty$, survives. We can then safely choose the domain of integration as $|\mathbf{z}| > r,\xi$. We stress that in the intermediate steps, the argument of the Hankel function should be modified to $\sqrt{\epsilon_D}\,\omega r$ with the limit $\epsilon_D \to 1$ taken in the end. A little algebra yields

$$
\langle\Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}
$$
$$
= \frac{1}{16}a_{\text{out}}(\omega)\sum_{m=-\infty}^{\infty}\big[H_m^{(2)}(\omega r) + S_m(\omega)H_m^{(1)}(\omega r)\big]e^{im\phi}
$$
$$
\times \overline{\big[H_m^{(2)}(\omega\xi) + S_m(\omega)H_m^{(1)}(\omega\xi)\big]e^{im\psi}}. \tag{21}
$$

(The bar indicates complex conjugation.) The correlation function is then a bilinear sum over incoming plus scattered waves. In fact, in the absence of the object, this equation reduces to a bilinear sum over Bessel functions

$$
\langle\Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{empty space}}
$$
$$
= \frac{\hbar}{2}\left(n(\omega,T) + \frac{1}{2}\right)\sum_{m=-\infty}^{\infty} J_m(\omega r)J_m(\omega\xi)e^{im(\phi-\psi)}
$$
$$
= \frac{\hbar}{2}\left(n(\omega,T) + \frac{1}{2}\right)J_0(\omega|\mathbf{x}-\mathbf{y}|)
$$
$$
= \frac{\hbar}{2}\left(n(\omega,T) + \frac{1}{2}\right)\int_0^{2\pi}\frac{d\alpha}{2\pi}\,e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}, \tag{22}
$$

where $\mathbf{k}$ is the wave vector with $|\mathbf{k}| = \omega$ and $\angle\mathbf{k} = \alpha$. Being a complete basis, the Bessel functions can be recast into another basis such as planar waves in Eq. (22). In other words, quantum fluctuations in (empty) space can be written as a uniformly weighted sum over a complete set of functions. In the presence

of the object, vacuum fluctuations are organized into a sum over incoming plus scattered waves as in Eq. (21).

2. Inside fluctuations

Next we turn to study the source fluctuations inside the object:

$$- [\triangle + \omega^2 \epsilon(\omega, \mathbf{x})]\Phi(\omega, \mathbf{x}) = \begin{cases} -i\omega \varrho_\omega(\mathbf{x}), & |\mathbf{x}| < R \\ 0, & |\mathbf{x}| > R, \end{cases} \quad (23)$$

with

$$\langle \varrho_\omega(\mathbf{x})\varrho_\omega^*(\mathbf{y})\rangle = a_{\text{in}}(\omega)\text{Im}\,\epsilon(\omega, \mathbf{x})\delta(\mathbf{x} - \mathbf{y}), \quad (24)$$

where the sources' arguments are inside the object and $a_{\text{in}}(\omega)$ is defined with respect to the object's temperature. Similar to the previous section, the field correlation function for $\mathbf{x}$ and $\mathbf{y}$ outside the object can be computed via Green's functions

$$\langle \Phi(\omega, \mathbf{x})\Phi^*(\omega, \mathbf{y})\rangle_{\text{in fluc}}$$
$$= \omega^2 a_{\text{in}}(\omega) \int_{|\mathbf{z}| < R} d\mathbf{z}\, G(\omega, \mathbf{x}, \mathbf{z})\text{Im}\,\epsilon(\omega, \mathbf{z})G^*(\omega, \mathbf{y}, \mathbf{z}), \quad (25)$$

where ϵ is a possibly position-dependent response function. The Green's function in this equation involves a point inside and another outside the object. As the two points (inside and outside the object) cannot coincide, the Green's function satisfies a homogeneous equation (11) inside and a free (Helmholtz) equation outside. Hence, we can expand the Green's function as

$$G(\omega, \mathbf{x}, \mathbf{x}') = \sum_{m=-\infty}^{\infty} \frac{i}{8} f_{\omega,m}(r)e^{im\phi}\left[A H_m^{(1)}(\omega\xi) + B H_m^{(2)}(\omega\xi)\right]$$
$$\times e^{-im\psi}, \quad r < R < \xi, \quad (26)$$

where the prefactor is chosen for future convenience. Here $f_{\omega,m}(\omega)$ is the regular (at the origin) solution to[3]

$$- [\triangle + \omega^2\epsilon(\omega, r)]f_{\omega,m}(r)e^{im\phi} = 0,$$
$$- (\triangle + \omega^2)H_m^{(1,2)}(\omega r)e^{im\phi} = 0. \quad (27)$$

The coefficients A and B and the normalization of the function f are determined by matching the Green's functions approaching a point on the boundary from inside and outside the object

$$G(\omega, \mathbf{x}, \mathbf{y})|_{|\mathbf{x}|\to R^-} = G(\omega, \mathbf{x}, \mathbf{y})|_{|\mathbf{x}|\to R^+}. \quad (28)$$

Comparing Eqs. (20) and (26), we find

$$G(\omega, \mathbf{x}, \mathbf{y}) = \sum_{m=-\infty}^{\infty} \frac{i}{8} f_{\omega,m}(r)e^{im\phi} H_m^{(1)}(\omega\xi)e^{-im\psi},$$
$$r < R < \xi, \quad (29)$$

---

[3]For simplicity, we have assumed that the dielectric function is rotationally symmetric. This assumption is not essential for a static object, but is essential for rotating objects.

where the function f is constrained by continuity equations as

$$f_{\omega,m}(R) = H_m^{(2)}(\omega R) + S_m(\omega)H_m^{(1)}(\omega R),$$
$$\left[\frac{\partial}{\partial r}f_{\omega,m}(r) = \frac{\partial}{\partial r}\left[H_m^{(2)}(\omega r) + S_m(\omega)H_m^{(1)}(\omega r)\right]\right]_{r=R}. \quad (30)$$

In short, the differential equations (27) plus the boundary conditions in Eqs. (30) and the regularity of f at the origin determine both the function f and the elements of the S matrix. We then expand the Green's function in Eq. (25) in terms of partial waves from Eq. (29). Keeping in mind that $\rho \equiv |\mathbf{z}| < r, \xi$, we find

$$\langle \Phi(\omega, \mathbf{x})\Phi^*(\omega, \mathbf{y})\rangle_{\text{in fluc}}$$
$$= \frac{1}{64}\omega^2 a_{\text{in}}(\omega) \sum_{m=-\infty}^{\infty} H_m^{(1)}(\omega r)e^{-im\phi}\overline{H_m^{(1)}(\omega\xi)e^{-im\psi}}$$
$$\times 2\pi \int_0^R d\rho\, \rho f_{\omega,m}(\rho)\text{Im}\,\epsilon(\omega, \rho)\overline{f_{\omega,m}(\rho)}. \quad (31)$$

By virtue of the field equation, the integral in the last line of this equation can be converted to an expression on the boundary of the object: The conjugate of the function f satisfies the conjugated wave equation with $\epsilon \to \epsilon^*$. By subtracting the conjugated from the original equation, one can see that the integrand is equal to a total derivative. The integral then becomes

$$\frac{1}{-2i\omega^2}W(f_{\omega,m}(R), \overline{f_{\omega,m}(R)}), \quad (32)$$

with W being the Wronskian with respect to the radius. The continuity relations of Eq. (30) can be exploited to compute the Wronskian

$$W(f_{\omega,m}(R), \overline{f_{\omega,m}(R)}) = -\frac{4i}{\pi R}[1 - |S_m(\omega)|^2], \quad (33)$$

where we used the identity $W(H_m^{(1)}(x), H_m^{(2)}(x)) = -4i/\pi x$. Rather remarkably, this equation shows that all the relevant details of the inside solutions f can be encoded in the scattering matrix, i.e., fluctuations inside the object affect the correlation function only through the scattering matrix S. Combining the previous steps, we arrive at the (outside) correlation function due to the inside source fluctuations,

$$\langle \Phi(\omega, \mathbf{x})\Phi^*(\omega, \mathbf{y})\rangle_{\text{in fluc}} = \frac{1}{16}a_{\text{in}}(\omega) \sum_{m=-\infty}^{\infty} [1 - |S_m(\omega)|^2]$$
$$\times H_m^{(1)}(\omega r)e^{im\phi}\overline{H_m^{(1)}(\omega\xi)e^{im\psi}}. \quad (34)$$

The correlation function is a bilinear sum over outgoing (first kind of Hankel) functions; this is reasonable as the sources in the object must produce outgoing waves in the vacuum. The coefficient is, however, more interesting: It depends on the scattering matrix through $1 - |S|^2$ and vanishes for a nonlossy object, i.e., when the scattering matrix is unitary $|S| = 1$. We shall revisit this point later when we study radiation out of thermal or dynamic equilibrium.

3. Thermal radiation

In this section we employ the results from the previous sections to compute the radiation out of thermal equilibrium when the object is at rest, though at a temperature T different from that of the environment T_0. However, we first show that the equilibrium behavior is consistent with the FDT. At $T = T_0$, the distribution functions $a_{\text{in}}(\omega) = a_{\text{out}}(\omega) \equiv a(\omega)$ are equal. A sum over Eqs. (21) and (34) yields (for **x** and **y** outside the body)

$$\langle \Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle$$
$$= \langle \Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{out fluc}} + \langle \Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{in fluc}}$$
$$= a(\omega)\text{Im} \sum_{m=-\infty}^{\infty} \frac{i}{8}\big[H_m^{(2)}(\omega r) + S_m(\omega)H_m^{(1)}(\omega r)\big]$$
$$\times e^{im\phi} H_m^{(1)}(\omega\xi)e^{-im\psi}$$
$$= a(\omega)\text{Im}\, G(\omega,\mathbf{x},\mathbf{y}), \tag{35}$$

in agreement with the FDT.

Out of thermal equilibrium, the Poynting vector quantifies the radiation flux from the object into the environment. In our model for the scalar field, the radial component of the Poynting vector is given by

$$\langle \partial_t \Phi(t,\mathbf{x})\partial_r \Phi(t,\mathbf{x})\rangle = \frac{1}{\pi}\int_0^\infty d\omega\, \omega\, \text{Im} \langle \Phi(\omega,\mathbf{x})\partial_r \Phi^*(\omega,\mathbf{x})\rangle. \tag{36}$$

The total radiation rate is obtained by integrating over a closed surface enclosing the object. We compute the contribution due to inside and outside source fluctuations separately by inserting the corresponding correlation functions in Eq. (36). The radiated energy per unit time is then

$$\mathcal{P}_{\text{in fluc/out fluc}} = \pm\frac{1}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty d\omega\, \omega a_{\text{in/out}}(\omega)[1-|S_m(\omega)|^2], \tag{37}$$

with the upper (lower) sign corresponding to inside (outside) fluctuations, where we have used the expression for the Wronskian of Hankel functions. Note that the signs indicate that the flux due to the inside sources is outgoing while the vacuum fluctuations induce an incoming flux. In the absence of loss, i.e., when $|S| = 1$, there is no flux in either direction since the object lacks an exchange mechanism with the environment. In equilibrium, detailed balance prevails and there is no net radiation. One can also see that the reality of the correlation function in Eq. (35) guarantees that the corresponding Poynting vector in Eq. (36) vanishes. Out of thermal equilibrium, the total radiation to the environment is given by

$$\mathcal{P} = \sum_{m=-\infty}^{\infty}\int_0^\infty \frac{d\omega}{2\pi}\hbar\omega[n(\omega,T) - n(\omega,T_0)][1 - |S_m(\omega)|^2]. \tag{38}$$

We have expressed the radiation in terms of the Bose-Einstein distribution number $n(\omega,T)$. Clearly the net flux is in a direction opposite to the temperature gradient. The relation between the thermal emission and the absorptivity $1 - |S|^2$,

characterized by the deviation of the scattering matrix from unitarity, is Kirchhoff's law [41,42]. In the blackbody limit, the object perfectly absorbs an incoming wave and does not reflect back, leading to the vanishing of the scattering matrix S. This is possible only if the dielectric function slightly deviates from 1 (otherwise it leads to a finite scattering amplitude) with Im $\epsilon \ll 1$. While an infinite medium can be a perfect absorber at all frequencies and wave numbers, a compact object can act as a blackbody only in certain frequency regimes. At high temperatures the thermal radiation is dominated by large frequencies so we can assume Im $\epsilon\, \omega R \gg 1$. Within these limits, it can be shown that the scattering matrix is almost unitary for $|m| > \omega R$ while it is approximately zero when $|m| < \omega R$. Therefore, the sum over m at a fixed ω gives a factor of $2\omega R$ proportional to the circumference of the disk in harmony with the blackbody radiation and Stefan-Boltzmann law [43].

In the following sections, we apply the techniques that we have developed here to rotating objects.

B. Field fluctuations for moving objects

We first devise a Lagrangian from which Eq. (11) follows for a static object and then, with the guidance of Lorentz invariance, generalize it to a moving object. Schematically, the Lagrangian can be written as[4]

$$\mathcal{L} = \tfrac{1}{2}\epsilon(\partial_t\Phi)^2 - \tfrac{1}{2}(\nabla\Phi)^2$$
$$= \tfrac{1}{2}[(\partial_t\Phi)^2 - (\nabla\Phi)^2] + \tfrac{1}{2}(\epsilon - 1)(\partial_t\Phi)^2. \tag{39}$$

The second line breaks the Lagrangian into two parts: The first term is merely the free Lagrangian (in empty space) while the second term contributes only within the material, hence defining the interaction of the field with the object. In generalizing to moving objects, the free Lagrangian remains invariant. The interaction, however, should be defined with respect to the rest frame of the object. The latter is cast into a covariant form so that it reduces to the familiar expression in the rest frame

$$\mathcal{L} = \tfrac{1}{2}(\partial_t\Phi)^2 - \tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}(\epsilon' - 1)(U^\mu\partial_\mu\Phi)^2, \tag{40}$$

with U being the four-velocity [or three-velocity in (2+1)-dimensional space-time] of the object. Note that Φ is scalar, i.e., $\Phi'(t',\mathbf{x}') = \Phi(t,\mathbf{x})$ with the (unprimed) primed coordinates defined in the (laboratory) comoving frame. Also the dielectric function $\epsilon' = \epsilon(\omega',\mathbf{x}')$ is naturally defined in the comoving frame and should be transformed to the coordinates in the laboratory frame. Equation (40) introduces a minimal coupling between the object's motion and the scalar field in the background. For an object in uniform motion, this Lagrangian is obtained by an obvious Lorentz transformation. One might think that this equation should be further elaborated for an accelerating object. However, if the acceleration rate is small compared to the object's internal frequencies (plasma frequency, for example) the motion can be implemented by a local Lorentz transformation, hence Eq. (40). The field

---

[4]The response function may be nonlocal in time; the Lagrangian merely serves as a guide to obtain the field equation.

equation is deduced from the Lagrangian as

$$\left[\triangle - \partial_t^2 - (\epsilon' - 1)(U^\mu \partial_\mu)^2 \right] \Phi(t,\mathbf{x}) = 0.$$

This is the homogenous field equation in the presence of a moving object. We should also incorporate the coupling to random sources for applications of the Rytov formalism since the source is naturally defined in the comoving frame; a similar argument suggests a minimal coupling by adding $\Delta \mathcal{L} = -\varrho \, U^\mu \partial_\mu \Phi$ to the Lagrangian. The governing equation for the scalar field is then

$$-\left[\triangle - \partial_t^2 - (\epsilon' - 1)(U^\mu \partial_\mu)^2 \right] \Phi(t,\mathbf{x}) = U^\mu \partial_\mu \varrho(t,\mathbf{x}), \quad (41)$$

which reduces to Eq. (12) for an object at rest. Here we have defined $\varrho'(t',\mathbf{x}') \equiv \varrho(t,\mathbf{x})$. Source fluctuations are distributed according to Eq. (13) but with respect to the comoving frame

$$\langle \varrho'_{\omega'}(\mathbf{x}) \varrho'^*_{\omega'}(\mathbf{y}) \rangle = a(\omega') \mathrm{Im}\, \epsilon(\omega',\mathbf{x}') \delta(\mathbf{x}' - \mathbf{y}'), \quad (42)$$

with primed quantities defined in the moving frame. The two sets of coordinates are related via

$$t' = t, \quad r' = r, \quad \phi' = \phi - \Omega t. \quad (43)$$

We shall limit ourselves only to objects moving at velocities small compared to the speed of light, in which case $U \approx (1,\mathbf{v})$ with $\mathbf{v}$ being the local velocity. Rotating at an angular frequency Ω, $\mathbf{v} = \Omega \times \mathbf{x}$, Eq. (41) becomes

$$-\left[\triangle - \partial_t^2 - (\epsilon' - 1)(\partial_t + \Omega \partial_\phi)^2 \right] \Phi(t,\mathbf{x})$$
$$= (\partial_t + \Omega \partial_\phi) \varrho(t,\mathbf{x}). \quad (44)$$

Let us expand the random source $\varrho(t,\mathbf{x})$ in the laboratory frame as

$$\varrho(t,\mathbf{x}) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \varrho_\omega(\mathbf{x}) = \sum_m \int \frac{d\omega}{2\pi} e^{-i\omega t + im\phi} \varrho_{\omega,m}(r).$$
$$(45)$$

Similarly, we define $\varrho'_{\omega',m'}$ in the comoving frame with ω' and m' being conjugate to the time and angular variables in the same frame. The coordinate transformations in Eq. (43) along with the definition $\varrho'(t',\mathbf{x}') \equiv \varrho(t,\mathbf{x})$ yield $\varrho_{\omega,m}(r) = \varrho'_{\omega - \Omega m, m}(r)$. Therefore, fluctuations in the comoving frame (42) translate to

$$\langle \varrho_{\omega,m}(r) \varrho^*_{\omega,m}(\xi) \rangle = a(\omega - \Omega m) \mathrm{Im}\, \epsilon(\omega - \Omega m, r) \frac{r^{-1}\delta(r - \xi)}{2\pi}$$
$$(46)$$

in the laboratory frame. This equation is indeed similar to source fluctuations in a static object with ω being replaced by $\omega - \Omega m$. In other words, zero-point fluctuations in the object are centered at a frequency shifted from that of the vacuum.

Having formulated field equations and their corresponding source fluctuations, we compute correlation functions in the next section.

1. Field correlations

Similar to Sec. II A, we compute the field correlation functions separately for source fluctuations outside and inside the object. The treatment of the vacuum (outside) fluctuation is entirely identical to the case of a static object described by Eq. (21), while the scattering matrix is generally different when rotating.

For inside source fluctuations, the argument should be modified slightly. Let us define the (new) functions f as solutions to the wave equation inside the object

$$\left[\triangle - \partial_t^2 - (\epsilon' - 1)(\partial_t + \Omega \partial_\phi)^2 \right] e^{-i\omega t} e^{im\phi} f_{\omega,m}(r) = 0. \quad (47)$$

The Green's function for one point inside and the other outside the object takes a form similar to the static case

$$G(\omega,\mathbf{x},\mathbf{y}) = \sum_{m=-\infty}^{\infty} \frac{i}{8} f_{\omega,m}(r) e^{im\phi} H_m^{(1)}(\omega\xi) e^{-im\psi},$$
$$r < R < \xi, \quad (48)$$

with the function f satisfying continuity relations similar to Eq. (30) with $S_m(\omega)$ replaced by $S_{-m}(\omega)$.[5] The field correlation function is then related to source fluctuations as

$$\langle \Phi(\omega,\mathbf{x}) \Phi^*(\omega,\mathbf{y}) \rangle_{\text{in fluc}}$$
$$= \frac{1}{64} \sum_{m=-\infty}^{\infty} (\omega - \Omega m)^2 a_{\text{in}}(\omega - \Omega m) H_m^{(1)}(\omega r)$$
$$\times e^{im\phi} \overline{H_m^{(1)}(\omega\xi) e^{im\psi}} 2\pi \int_0^R d\rho \, \rho \, f_{\omega,m}(\rho)$$
$$\times \mathrm{Im}\, \epsilon(\omega - \Omega m, \rho) \overline{f_{\omega,m}(\rho)}, \quad (49)$$

where we have used Eq. (46). As before, we can exploit the wave equation to convert the integral in the preceding equation to a boundary term. The correlation function can be then cast in terms of the scattering matrix as

$$\langle \Phi(\omega,\mathbf{x}) \Phi^*(\omega,\mathbf{y}) \rangle_{\text{in fluc}}$$
$$= \frac{1}{16} \sum_{m=-\infty}^{\infty} a_{\text{in}}(\omega - \Omega m)[1 - |S_m(\omega)|^2]$$
$$\times H_m^{(1)}(\omega r) e^{im\phi} \overline{H_m^{(1)}(\omega\xi) e^{im\psi}}. \quad (50)$$

This equation is similar to the expression for a static object (34), with the important difference that the distribution a is a function of a shifted frequency defined from the point of view of the rotating frame.

2. Radiation, spontaneous emission, and superradiance

In a Gaussian theory, two-point correlation functions define the complete structure of fluctuations and can be used to compute force, torque, or radiation. Specifically, the energy radiation per unit time is obtained by the integral of $\langle \partial_t \Phi \partial_r \Phi \rangle$ over a surface enclosing the object. For a rotating object, the correlation functions derived in the previous section yield

$$\mathcal{P} = \sum_{m=-\infty}^{\infty} \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega [n_{\text{in}}(\omega - \Omega m) - n_{\text{out}}(\omega)]$$
$$\times [1 - |S_m(\omega)|^2]. \quad (51)$$

----

[5]With time-reversal invariance, the Green's function $G(\omega,\mathbf{x},\mathbf{y})$ is symmetric in its spatial arguments $G(\omega,\mathbf{x},\mathbf{y}) = G(\omega,\mathbf{y},\mathbf{x})$. For a rotating object, time reversal is no longer a symmetry; however, time reversal followed by reversing the angular velocity forms a symmetry that yields $G(\omega,r,\phi,\xi,\psi) = G(\omega,\xi,-\psi,r,-\phi)$. The negative sign carries through to the sign of the angular momentum m.

Similarly the torque, or the rate of angular momentum radiation, is given by integrating $\langle \partial_t \Phi \partial_\phi \Phi \rangle$ over the surface. We find an expression similar to Eq. (51) by replacing $\hbar\omega$ by $\hbar m$,

$$M = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{d\omega}{2\pi} \hbar m [n_{\text{in}}(\omega - \Omega m) - n_{\text{out}}(\omega)]$$
$$\times [1 - |S_m(\omega)|^2]. \qquad (52)$$

The function n_{in} is singular at $\omega = \Omega m$; however, at this frequency $\text{Im}\,\epsilon(\omega - \Omega m) = \text{Im}\,\epsilon(0) = 0$, which results in no loss. Therefore, $1 - |S|^2$ is zero at $\omega = \Omega m$ removing the singularity and rendering the above expressions well defined. We stress that, at zero temperature, Eqs. (51) and (52) should be understood only to the leading order in $\Omega R/c$ as computing higher orders in this quantity requires a more careful treatment of the field equations in higher orders of velocity. At $T = T_0 = 0$, the sum over partial waves is restricted to positive m, where the leading contribution comes from $m = 1$, while higher values of m give the leading radiation at multipolarity m. At a finite temperature, the contribution due to higher partial values can be important and even dominant, in which case they should be included. In the rest of this paper, summation over all partial waves should be understood in similar terms. Nevertheless, the more general input-output formalism precisely gives Eqs. (51) and (52) without any approximations regarding the velocity of the rotating object [44], hence their validity goes beyond the analysis provided here.

Let us consider the limit of zero temperature so that thermal radiation can be neglected. In this limit $n(\omega) = -\Theta(-\omega)$, that is, the distribution function vanishes for positive frequency but becomes 1 for negative frequencies. This distribution defines a vacuum state in which all positive-energy states are empty and, figuratively, all negative-energy states are occupied. Now the distribution function pertaining to inside fluctuations is defined with respect to a frequency shifted by a multiple of rotation frequency and thus can find negative values even when ω is positive. The difference of the Bose-Einstein distributions contributes in a frequency window of $[0, \Omega m]$. Therefore, even at zero temperature, a rotating object emits photons and loses energy; the number of photons emitted at frequency $\omega(>0)$ and partial wave m is given by

$$\mathcal{N}_m(\omega) \equiv \frac{dN_m(\omega)}{d\omega dt} = \Theta(\Omega m - \omega)[|S_m(\omega)|^2 - 1]. \quad (53)$$

The corresponding radiated energy or angular momentum is obtained by integrating over photon number multiplied by $\hbar\omega$ or $\hbar m$, respectively. It follows from Eq. (53) that a (physically acceptable) positive outflux of photons requires a superunitary scattering matrix $|S_m(\omega)| > 1$. Indeed, Zel'dovich argued that classical waves should amplify upon scattering from a rotating object exactly for frequencies in a range $0 < \omega < \Omega m$, a phenomenon that is called superradiance [33]. While spontaneous emission by a rotating object is a purely quantum effect, superradiance can be understood entirely within classical mechanics: A system is lossy if the imaginary part of its response function is positive (negative) for positive (negative) frequencies. For a rotating object, $\text{Im}\,\epsilon(\omega')$ has the same sign as $\omega' = \omega - \Omega m$, the frequency defined in the comoving frame; however, for (positive) ω smaller than Ωm, the argument of the dielectric function is negative and thus the object amplifies the

corresponding incident waves, hence superradiance. In fact, incoming waves in the superradiating regime extract energy from a rotating object and slow it down.

Superradiance and spontaneous emission are intimately related. When the object is at rest, it absorbs energy by getting excited to a higher level and deexcites by emitting a photon. For a rotating body, this picture breaks down, that is, the object can emit a photon while being excited to a higher level: The energy of the emitted photon is $\hbar\omega > 0$ in the laboratory frame; however, a rotating observer sees the same particle at a shifted frequency $\omega' = \omega - \Omega m$. In the superradiant regime where $\omega < \Omega m$, the frequency is negative in the comoving frame, hence the object has gained (positive) energy. This gain should be interpreted as heat generated inside the body. The energy conservation still holds because the energy of the emitted photon as well as heat is extracted from the rotational energy of the object. This observation is also at the heart of the superradiance phenomenon when incoming waves are enhanced upon scattering from a rotating object. The above argument shows that spontaneous emission conserves the energy and thus is (energetically) possible. In fact, as the object spontaneously emits photons (and heats up), it also slows down unless kept in steady motion by an external agent. In the context of general relativity, the Penrose process provides a similar mechanism to extract energy from a rotating black hole [34], which also leads to spontaneous emission [35].

We define E and E' as the energy of the object in the laboratory frame and the rotating frame, respectively. The two are related by $E' = E - \Omega L$, where L is the angular momentum of the object [45]. Hence, the heat generated per unit time $\mathcal{Q} \equiv dE'/dt$ is given by

$$\mathcal{Q} \equiv \frac{dE'}{dt} = \frac{dE}{dt} - \Omega\frac{dL}{dt} = \Omega M - \mathcal{P}. \qquad (54)$$

In order to maintain a steady rotation, one should exert a constant torque M. The work done is equal to the radiated energy plus heat $\Omega M = \mathcal{P} + \mathcal{Q}$. Note that the object loses energy to the environment $dE/dt = -\mathcal{P} < 0$ as well as angular momentum $dL/dt = -M < 0$. The rate of the energy gain in the object's rest frame can be obtained from Eqs. (51) and (52) as

$$\mathcal{Q} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \frac{d\omega}{2\pi} \hbar(\Omega m - \omega)\mathcal{N}_m(\omega). \qquad (55)$$

At zero temperature, the photon number production (53) has nonzero support only for $0 < \omega < \Omega m$ and thus the heat generation is manifestly positive. In brief, the object heats up while it loses energy (E decreases) if not connected to an infinite thermal bath. This suggests that the heat capacity from the point of view of the laboratory frame is negative; however, thermodynamic quantities are well defined in the comoving frame where the energy E' increases, hence the heat capacity is indeed positive.

We have argued that spontaneous emission is energetically possible, consistent with the energy conservation. This process also generates heat inside the object and photons in the environment, hence entropy is increasing. Notice that the line of argument can be reversed: A phenomenon that satisfies requirements of energy conservation and is

thermodynamically favored due to entropy production should occur. This observation completes the link between superradiance and spontaneous emission; see also Refs. [33,46]. In Sec. II D we study the statistics of radiated photons in some detail. In particular, we compute the entropy generation due to the creation of photons.

3. Radiation: Rotating disk

In this section we study quantum radiation by a rotating disk of radius R described by a spatially uniform but frequency-dependent dielectric function $\epsilon(\omega)$. Of course, the results provided here have little overlap with the real world. Readers who are interested in the specifics of a real dielectric material coupled to electromagnetism may skip directly to Sec. III D. We find solutions to the field equation inside and outside the object and match them on the boundary to compute the scattering amplitude. When linear velocities are small, Eq. (44) for the field equation [with the source term on the right-hand side (RHS) set to zero] yields

$$\left[\triangle - \partial_t^2 - (\epsilon' - 1)(\partial_t + \Omega \partial_\phi)^2 \right] \Phi(t, \mathbf{x}) = 0. \quad (56)$$

A solution characterized by frequency ω and the angular momentum m, i.e., of the form $\Phi = f(r) e^{-i\omega t} e^{im\phi}$, casts this equation into

$$\left[\frac{1}{r} \partial_r r \partial_r - \frac{m^2}{r^2} + \tilde{\omega}_m^2 \right] f(r) = 0. \quad (57)$$

Here we have defined a new m-dependent (possibly complex) frequency $\tilde{\omega}_m$ as

$$\tilde{\omega}_m^2 = (\epsilon' - 1)(\omega - \Omega m)^2 + \omega^2, \quad (58)$$

which is a constant for a fixed ω and m and position independent $\epsilon' = \epsilon(\omega') = \epsilon(\omega - \Omega m)$. Therefore, the equation that governs the field dynamics inside the object is a Helmholtz equation whose regular solutions are Bessel J functions, with the frequency replaced by $\tilde{\omega}_m$. Note that both the order and the argument of the Bessel functions depend on m, the latter through $\tilde{\omega}_m$. We define a scattering ansatz as

$$\Phi(\omega, \mathbf{x}) = \begin{cases} V_m(\omega) J_m(\tilde{\omega}_m r) e^{im\phi}, & r < R \\ H_m^{(2)}(\omega r) e^{im\phi} + S_m(\omega) H_m^{(1)}(\omega r) e^{im\phi}, & r > R, \end{cases} \quad (59)$$

with the outside solutions being a linear combination of incoming and (with the scattering matrix as the amplitude) outgoing waves. The scattering matrix can be easily obtained by matching boundary conditions

$$S_m(\omega) = -\frac{\partial_R J_m(\tilde{\omega}_m R) H_m^{(2)}(\omega R) - J_m(\tilde{\omega}_m R) \partial_R H_m^{(2)}(\omega R)}{\partial_R J_m(\tilde{\omega}_m R) H_m^{(1)}(\omega R) - J_m(\tilde{\omega}_m R) \partial_R H_m^{(1)}(\omega R)}. \quad (60)$$

When ϵ is real, i.e., for a lossless material, the denominator is merely the complex conjugate of the numerator and the scattering is unitary. Conversely, if ϵ has an imaginary part the scattering matrix is nonunitary. For a lossy object at rest, $\text{Im}\, \tilde{\omega}_m = |\omega| \text{Im} \sqrt{\epsilon} > 0$ (for positive frequency) and $|S|^2 < 1$. For a spinning object, $\text{Im}\, \tilde{\omega}_m \propto \text{Im}\, \epsilon' \propto \text{sgn}(\omega - \Omega m)$, hence the scattering matrix is subunitary for $\omega > \Omega m$ but superunitary $|S|^2 > 1$ in the superradiating range $\omega < \Omega m$.

One can now compute the radiation from the S matrix. Assuming that the object's linear velocity is small, the radiation is strongest at frequencies comparable to Ω, thus the first partial wave $m = 1$ suffices and the Bessel J functions can be expanded. The scattering matrix deviates from unitarity (by restoring units of c) as

$$|S_1(\omega)|^2 - 1 \approx -\frac{\pi}{8} \frac{\omega^2 (\omega - \Omega)^2 R^4}{c^4} \text{Im}\, \epsilon(\omega - \Omega). \quad (61)$$

This expression is manifestly negative for $\omega > \Omega$ but positive when $\omega < \Omega$ for any causal ϵ. One can then compute various quantities of interest such as torque, heat generation, and radiation. In particular, energy radiation per unit time is given by Eq. (51) as

$$\mathcal{P} \approx \frac{\hbar R^4}{16 c^4} \int_0^\Omega d\omega\, \omega^3 (\omega - \Omega)^2 |\text{Im}\, \epsilon(\omega - \Omega)|. \quad (62)$$

For a specific dielectric function, the radiation can be computed explicitly.

C. Higher dimensions, nonscalar field theories, and trace formulas

The above results can be readily generalized to higher dimensions. For a cylinder extended along the third dimension, quantum radiation is given by

$$\mathcal{P} = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega \sum_{m=-\infty}^\infty \int_{-\omega}^\omega \frac{L dk_z}{2\pi} [n_{\text{in}}(\omega - \Omega m) - n_{\text{out}}(\omega)]$$
$$\times [1 - |S_{mk_z}(\omega)|^2], \quad (63)$$

where L is the length of the cylinder and k_z is the wave vector along the z direction. Note that $|k_z|$ is bounded by ω (we have set $c = 1$), corresponding to propagating waves as opposed to evanescent waves that affect short distances from the cylinder but do not contribute to the radiation at infinity.

If the rotating object is not translationally symmetric in the z direction (while rotationally symmetric), the scattering matrix is no longer diagonal in k_z, leading to a more complicated analog of Eq. (63). Nevertheless, the S matrix can always be diagonalized in some basis. Indeed, one can write a general trace formula for the quantum radiation that is independent of a particular basis,

$$\mathcal{P} = \int_0^\infty \frac{d\omega}{2\pi} \hbar\omega \, \text{Tr}\{[n_{\text{in}}(\omega - \Omega \hat{l}_z) - n_{\text{out}}(\omega)](1 - \mathbb{S}\mathbb{S}^\dagger)\}, \quad (64)$$

where we trace over all the propagating modes. In this equation, $\hat{l}_z = \frac{1}{i} \frac{\partial}{\partial \phi}$ is the angular momentum operator (in units of $\hbar$) projecting out the rotational index m. The scattering matrix $\mathbb{S}$ is written in a general basis-free notation. Equation (64) is not specific to scalar fields or translationally symmetric objects but also holds for arbitrary shapes (though rotationally symmetric) and electromagnetism; the latter requires tracing over polarizations too. We present a general derivation of Eq. (64) in Sec. III in the context of electrodynamics.

D. Photon statistics and entropy generation

Heretofore we have studied in some detail an object out of thermal or dynamic equilibrium with the environment, where it is shown that the object emits photons. In this section we turn to a different aspect of this problem, namely, the statistics of radiated photons.

We first note that the field correlation function receives contributions from photons as well as zero-point and (at finite temperature) thermal fluctuations and can be broken up as

$$\langle \Phi \Phi \rangle = \langle \Phi \Phi \rangle_{\text{nonrad}} + \langle \Phi \Phi \rangle_{\text{rad}}. \tag{65}$$

The first term on the RHS is the nonradiative term

$$\langle \Phi(\omega,\mathbf{x}) \Phi^*(\omega,\mathbf{y}) \rangle_{\text{nonrad}} = \hbar \coth\left(\frac{\hbar\omega}{2k_B T}\right) \text{Im}\, G(\omega,\mathbf{x},\mathbf{y}). \tag{66}$$

This expression is purely real and thus does not contribute to the radiation. Equation (66) is similar to the fluctuation-dissipation relation in equilibrium; cf. Eq. (16). However, out of equilibrium, the total correlation function receives another contribution that cannot be written in the above form. For a disk rotating at a rate Ω, possibly at a finite temperature T, the radiation term can be deduced from the total correlation function (see Sec. II B 1) and using the above definition we find

$$\langle \Phi(\omega,\mathbf{x}) \Phi^*(\omega,\mathbf{y}) \rangle_{\text{rad}} = \frac{\hbar}{8} \sum_{m=-\infty}^{\infty} n(\omega - \Omega m, T)[1 - |S_m(\omega)|^2]$$
$$\times H_m^{(1)}(\omega r) e^{im\phi} \overline{H_m^{(1)}(\omega\xi) e^{im\psi}}. \tag{67}$$

This term is entirely composed of outgoing fields as expected. In the remainder of this section, we focus on the ensemble of radiated photons.

Radiation can be quantified by the photon current, or the number of photons radiated per unit time. Different frequencies and partial waves are statistically independent, thus we consider the current of a single mode of frequency ω and angular momentum m,

$$\mathcal{I}_{\omega,m} = \frac{2\pi r}{i} [\Phi_m^*(\omega,\mathbf{x}) \partial_r \Phi_m(\omega,\mathbf{x}) - \text{c.c.}], \tag{68}$$

where the field is expanded over partial waves as $\Phi(\omega,\mathbf{x}) = \sum_m \Phi_m(\omega,\mathbf{x})$. When averaged over the radiation field, this expression reproduces Eq. (53) for a rotating object at $T = 0$ or, more generally, at a finite T,

$$\mathcal{N}_m(\omega) = \langle \mathcal{I}_{\omega,m} \rangle = n(\omega - \Omega m, T)[1 - |S_m(\omega)|^2]. \tag{69}$$

We are interested in higher statistical moments for which we have to compute the corresponding correlation functions of currents. Since fluctuations are Gaussian distributed, current correlation functions can be reduced to a product of two-point functions of fields according to Wick's theorem.

We compute the fluctuations of the current at the radiation zone far away from the object [keeping in mind that the radiation field in Eq. (67) is strictly outgoing], in which limit the radial derivative acting on Φ gives a factor of $i\omega/c$. Therefore, far from the object, the current defined in Eq. (68)

can be cast as

$$\mathcal{I}_{\omega,m} = \lim_{r\to\infty} \frac{4\pi\omega r}{c} \Phi_m^*(\omega,\mathbf{x}) \Phi_m(\omega,\mathbf{x}); \tag{70}$$

this expression is useful in evaluating n-point correlation functions.

We can also define the probability distribution function $P(n)$, with n being the number of photons per mode emitted in a time duration t. We drop the subscript indices as the statistics can be computed independently for each mode. The probability distribution is related to current correlators by the Glauber-Kelley-Kleiner formula [47,48]

$$P(n) = \frac{1}{n!} \langle I^n e^{-I} \rangle_{\text{rad}}. \tag{71}$$

We introduce a generating function $F(\eta)$,

$$e^{F(\eta)} = \langle e^{\eta I} \rangle, \tag{72}$$

which allows us to compute the probability distribution from the generating function as

$$P(n) = \lim_{\eta\to-1} \frac{1}{n!} \frac{d^n}{d\eta^n} e^{F(\eta)}. \tag{73}$$

Taylor expanding F in η generates the cumulants of factorial moments as [49]

$$F(\eta) = \sum_{p=1}^{\infty} \frac{\kappa_p \eta^p}{p!}. \tag{74}$$

For a single object discussed above, the current I in Eq. (70) is a bilinear term in the field Φ and its conjugate. Diagrammatically, we can represent I as a vertex with an incoming and an outgoing line corresponding to Φ^* and Φ, respectively. From Eq. (72) it is then clear that the cumulants κ_p are given by

$$\kappa_p = \langle I^p \rangle_c, \tag{75}$$

with the subscript c indicating that the connected component of the p-point function should be computed. A little thought shows that the connected correlation function in the preceding equation yields

$$\kappa_p = (p-1)! \mathcal{N}^p, \tag{76}$$

with $\mathcal{N} = \langle I \rangle$ being the average current per mode. The generating function is then

$$F(\eta) = -\ln(1 - \eta\mathcal{N}) \quad \text{or} \quad e^{F(\eta)} = \frac{1}{1 - \eta\mathcal{N}}. \tag{77}$$

These equations indicate that the counting distribution $P(n)$ is solely determined from the mean value of the radiation. This strong version of Kirchhoff's law is due to Bekenstein and Schiffer [50]; see also Ref. [49]. Here F can also be interpreted as a one-loop effective action in a background defined by ηI. Adopting this point of view, Eqs. (75) and (77) follow immediately. The probability distribution is easily deduced from Eqs. (73) and (77) as

$$P(n) = \frac{\mathcal{N}^n}{(\mathcal{N}+1)^{n+1}}. \tag{78}$$

This equation completely determines photon number statistics [50]. In particular, it yields the average and the variance of the

number of radiated photons per mode per unit time as

$$\langle \mathcal{I} \rangle = \mathcal{N}, \quad \text{Var}\, \mathcal{I} \equiv \langle (\mathcal{I} - \langle \mathcal{I} \rangle)^2 \rangle = \mathcal{N}(\mathcal{N} + 1). \quad (79)$$

Having the full statistics, we can compute the entropy of radiated photons as

$$\frac{S}{k_B} = -\sum_{n=0}^{\infty} P(n) \ln P(n)$$
$$= (\mathcal{N} + 1) \ln(\mathcal{N} + 1) - \mathcal{N} \ln \mathcal{N}. \quad (80)$$

In fact, this equation describes the entropy of a bosonic system out of equilibrium [51]. If the occupation number $\mathcal{N}$ obeys the Bose-Einstein distribution, Eq. (80) indeed produces the entropy of a gas of thermal bosons.

Quantum or thermal radiation from a single object consists of photons across the whole spectrum. Therefore, we should sum over all frequencies and quantum numbers

$$\sum_{\omega, m} \to t \int \frac{d\omega}{2\pi} \sum_m,$$

where t is the time interval under consideration. The entropy from Eq. (80) is then linearly increasing over time, giving rise to a constant rate of entropy generation (restoring ω and m) as

$$\mathcal{S} \equiv \frac{dS}{dt} = k_B \sum_m \int_0^{\infty} \frac{d\omega}{2\pi} \{ [\mathcal{N}_m(\omega) + 1] \ln[\mathcal{N}_m(\omega) + 1]$$
$$- \mathcal{N}_m(\omega) \ln \mathcal{N}_m(\omega) \}. \quad (81)$$

In the blackbody limit (for a perfectly absorbing object at rest), we recover the entropy associated with Planckian radiation. For a finite-size object (comparable with thermal wavelength), the spectrum approaches that of the graybody radiation where one should include the dependence on absorptivity $r \equiv 1 - |S|^2$. Equation (81) then depends on temperature, the object's length scale, and material properties in a complicated way. Additionally, the object loses energy and thus contributes negatively to entropy generation as $\mathcal{S}_{\text{object}} = -\mathcal{P}/T$, with $\mathcal{P}$ being the (mean) power. The total entropy increase per mode is then

$$\frac{\mathcal{S}_{\text{total}}}{k_B} = \left(\frac{r}{e^x - 1} + 1 \right) \ln \left(\frac{r}{e^x - 1} + 1 \right)$$
$$- \frac{r}{e^x - 1} \ln \frac{r}{e^x - 1} - \frac{xr}{e^x - 1}, \quad (82)$$

where $x = \hbar\omega/k_B T$ and r is the absorptivity of the corresponding mode. It can be shown that this expression is positive for all $0 \leqslant r \leqslant 1$ as expected.

Equation (81) can be understood as the entanglement entropy between the object and the environment consisting of radiated photons. In Ref. [52] Klich and Levitov suggested that the entanglement entropy can be obtained from the full quantum statistics, or the quantum noise. Specifically, the entanglement entropy generation at a quantum point contact (allowing electrons to transport between two leads) in the presence of a dc voltage V was found to be $\frac{dS}{dt} = -\frac{eV}{h}[D \ln D + (1 - D) \ln(1 - D)]$, with D being the transmission [53]. This expression is completely determined by the fluctuation of the electric current, thus providing a link between quantum noise and entanglement entropy [52]. Equation (81) indeed gives

the bosonic analog of the results in Ref. [52], where the two leads should be thought of as the object and the environment. With the above picture in mind, Eq. (82) offers an alternative interpretation: While the thermodynamic entropy of the object [the last term in Eq. (82)] decreases as the object loses energy, the sum of the entanglement entropy and the thermodynamic entropy always increases, indicating that the former indeed should be interpreted as entropy.

We are mainly interested in a rotating object at zero temperature with the radiation given by Eq. (53). Defining $\sigma \equiv |S|^2$, the entropy generation due to radiation from a rotating object is given by [with $\mathcal{N}_m(\omega) = \sigma_m(\omega) - 1$]

$$\mathcal{S} = k_B \sum_{m=1}^{\infty} \int_0^{\Omega m} \frac{d\omega}{2\pi} \{ \sigma_m(\omega) \ln \sigma_m(\omega) - [\sigma_m(\omega) - 1] $$
$$\times \ln[\sigma_m(\omega) - 1] \}. \quad (83)$$

Similar to thermal radiation, there is another contribution to entropy due to the object itself. In this case, however, the latter is also increasing in time since the object heats up. Hence, as we have argued in Sec. IIB2, a rotating object tends to emit radiation for purely thermodynamic reasons.

Before concluding this section, we note that Eq. (83) can also be written as a trace formula similar to the expression (64) for the energy radiation, which should be valid in higher dimensions and other field theories including electrodynamics.

E. Diffusion equation for rotation

We now examine the angular fluctuations of a spinning object as the result of the backreaction force due to the radiation. Specifically, we find the probability distribution as a function of the angular velocity for a macroscopic object spinning freely or under a constant torque. Our discussion here applies to both zero temperature, which is dominated by zero-point quantum fluctuations, and finite temperature.

We first consider an object freely rotating at an angular frequency Ω_0. The radiation by the object carries away angular momentum parallel to the axis of rotation, resulting in a decrease in angular velocity. We shall assume that the time duration under consideration t is much longer that $1/\Omega_0$ such that the radiated photons have definite frequencies. For simplicity, we first take this time sufficiently small such that the angular velocity does not change significantly. The frictional torque is obtained from the radiation current by $\hbar m \mathcal{I}$ summed (integrated) over all quantum numbers, so the average change in angular velocity is

$$I(\Omega_0 - \overline{\Omega(t)}) = \hbar t \sum_m \int_0^{\infty} \frac{d\omega}{2\pi} m \mathcal{N}_m(\omega) \equiv \hbar t \bar{M}(\Omega_0), \quad (84)$$

where I is the moment of inertia around the rotation axis and $\bar{M} = M/\hbar$, the torque in units of $\hbar$, can be read off from Eq. (52) with $n_{\text{out}} = 0$. Note that the dependence of the torque on the angular velocity Ω_0 is made explicit. The variance of the angular momentum can be obtained from the corresponding variance of the current as

$$\text{Var}\, I\Omega(t) = \hbar^2 t \sum_m \int_0^{\infty} \frac{d\omega}{2\pi} m^2 \mathcal{N}_m(\omega)[\mathcal{N}_m(\omega) + 1]$$
$$\equiv \hbar^2 t \bar{M}_2(\Omega_0), \quad (85)$$

where $\bar{M}_2$ is defined for future reference. Exploiting the methods of the previous section, higher moments can be readily computed. For long times, however, the central limit theorem guarantees that the statistics is entirely determined by the mean and the variance of the distribution provided that the radiated photons are statistically independent. An extension of this theorem due to Lyapunov gives the statistical distribution even for long times when the initial angular velocity has changed significantly. The Lyapunov central limit theorem requires the random variables to be statistically independent but not necessarily identically distributed. With this assumption, the average of the random variables converges to a normal distribution with a mean value given by the sum of each variable's mean and a variance as the sum of all the variances [54]. Equations (84) and (85) then take the forms

$$I(\Omega_0 - \overline{\Omega(t)}) = \hbar \int^t dt' \bar{M}(\Omega(t')),$$

$$\mathrm{Var}\, I\,\Omega(t) = \hbar^2 \int^t dt' \bar{M}_2(\Omega(t')), \qquad (86)$$

which describe the deterministic decrease in the angular velocity as well as its uncertainty. Notice that the integrand in the above equations depends on the instantaneous value of the angular velocity. We stress that the above discussion is based on the adiabaticity of motion, namely, the rate at which the angular velocity $\Omega(t)$ changes is taken to be much smaller that $\Omega(t)$ itself.

The rotating object undergoes a stochastic motion due to the inherent quantum (and, at finite temperature, also thermal) fluctuations. Equivalently, the equation of motion can be cast into a Langevin equation subject to noise as

$$I\dot{\Omega}(t) = -\hbar \bar{M}(\Omega(t)) + \eta(t; \Omega(t)). \qquad (87)$$

The noise $\eta(t; \Omega(t))$ has zero mean

$$\langle \eta(t; \Omega(t)) \rangle = 0, \qquad (88)$$

is independent at different times, and its covariance is

$$\langle \eta(t; \Omega(t))\eta(t'; \Omega(t')) \rangle = \hbar^2 \bar{M}_2(\Omega(t))\delta(t - t'). \qquad (89)$$

The δ-function correlation in time implies that the radiated photons are not correlated over long times ($\gg 1/\Omega$). One can easily check that Eq. (86) follows directly from the Langevin equation (87). This equation is reminiscent of the Brownian motion for a particle due to its thermal motion where the angular velocity plays the role of the displacement. The Brownian motion is the prototype of the fluctuation-dissipation condition where the response function is related to the fluctuations in equilibrium. Equation (87) is rather distinct due to the fact that noise is evaluated out of equilibrium as the object rotates, hence the explicit dependence of the noise on $\Omega(t)$.[6] Nevertheless, we can deduce the distribution in angular velocity and its evolution just as one can find the probability distribution for a particle's position in a thermal bath. The Fokker-Planck equation offers a systematic derivation of the distribution function [55], which

we denote by $\mathscr{P}(\Omega, t)$, making explicit the dependence on the angular velocity as a function of time. The master equation governing the probability distribution is

$$\frac{\partial \mathscr{P}}{\partial t} + \frac{\partial}{\partial \Omega}\left[\frac{\hbar}{I}\bar{M}(\Omega)\mathscr{P} + \frac{\hbar^2}{I^2}\frac{\partial}{\partial \Omega}[\bar{M}_2(\Omega)\mathscr{P}]\right] = 0. \qquad (90)$$

Notice that this equation reproduces the average and the variance in Eq. (86) provided that the probability distribution is sharply peaked around the instantaneous average angular velocity. In other words, for an object starting to spin with a definite angular frequency, i.e., a δ function as $\delta(\Omega - \Omega_0)$, the time evolution of the probability distribution is, at long times, governed by a Gaussian function with the average and the variance given above.

In the presence of an external torque $M_0 = \hbar\bar{M}(\Omega_0)$ that tries to keep the object at a constant angular velocity Ω_0, the Fokker-Planck equation is modified as

$$\frac{\partial \mathscr{P}}{\partial t} + \frac{\partial}{\partial \Omega}\left[\frac{\hbar}{I}[\bar{M}(\Omega) - \bar{M}(\Omega_0)]\mathscr{P} + \frac{\hbar^2}{I^2}\frac{\partial}{\partial \Omega}[\bar{M}_2(\Omega)\mathscr{P}]\right]$$
$$= 0. \qquad (91)$$

In the steady state where the probability distribution is constant in time, we find

$$\frac{\hbar}{I}[\bar{M}(\Omega) - \bar{M}(\Omega_0)]\mathscr{P} + \frac{\hbar^2}{I^2}\frac{\partial}{\partial \Omega}[\bar{M}_2(\Omega)\mathscr{P}] = 0. \qquad (92)$$

This equation can be solved exactly to obtain

$$\mathscr{P}(\Omega) = \frac{C}{\bar{M}_2(\Omega)}\exp\left[-\frac{I}{\hbar}\int_0^\Omega d\Omega'\,\frac{\bar{M}(\Omega') - \bar{M}(\Omega_0)}{\bar{M}_2(\Omega')}\right], \qquad (93)$$

where the normalization constant C is determined by the condition $\int_0^\infty d\Omega\,\mathscr{P}(\Omega) = 1$. For an object with a large moment of inertia, the distribution is sharply peaked near Ω_0. In this case, the distribution function becomes a Gaussian in Ω as

$$\mathscr{P}(\Omega) \approx \sqrt{2\pi}\Delta\exp\left[-\frac{(\Omega - \Omega_0)^2}{2(\Delta\Omega)^2}\right], \qquad (94)$$

with

$$I\Delta\Omega = \sqrt{\hbar I\,\frac{\bar{M}_2(\Omega_0)}{\partial \bar{M}/\partial \Omega_0}}. \qquad (95)$$

This equation sets a quantum limit on how close to an eigenstate the angular velocity of a spinning object (driven by a constant torque) can be. Being in a regime that the angular velocity is small compared to other frequency scales, one can assume that the scattering matrix only slightly deviates from unity such that $\mathscr{N}_m(\omega) \ll 1$. At zero temperature, the leading contribution in $\Omega R/c$ is given by $m = 1$. One can see from Eqs. (84) and (85) that $\bar{M}_2 \approx \bar{M}$; hence, Eq. (95) becomes

$$I\Delta\Omega = \sqrt{\hbar I\,\frac{1}{\partial \ln \bar{M}/\partial \Omega_0}} \propto \sqrt{\hbar I\Omega_0}. \qquad (96)$$

This relation follows from the fact that usually $\bar{M}(\Omega_0)$ is a power law in Ω_0; we will discuss this in more detail for the electromagnetic case in Sec. III D. This means that the uncertainty in the angular momentum of a single object is not proportional to $\hbar$, but to the geometrical mean of $\hbar$ and the

---

[6] In the fluctuation-dissipation theorem, the noise is usually taken to be independent of the position and velocity of the particle within the linear-response regime.

total angular momentum, which is much greater than $\hbar$. Note that any infinitesimal dissipation gives rise to the uncertainty in Eq. (96) independent of the details and strength of loss.

F. Test object: Torque and tangential force

In this section, we consider a second, or a test, object in the vicinity of the rotating body and study the interaction between the two. Such an interaction goes beyond the Casimir-Polder force [56] between two polarizable objects due to the presence of the radiation field near the test object. As we argue below, the latter is the dominant contribution to the force when the two objects are far apart. Let the objects be two disks of radii R and a separated by a distance d. We shall assume that $d \gg R, a$ and that the test object is at rest. Our starting point is Eq. (65), where the correlation function is broken into nonradiative and radiative parts: The former is related to the imaginary part of the Green's function via Eq. (66), while the latter is given by

Eq. (67). We shall assume that the two objects are separated far enough that a single-reflection computation of the radiation field off of the second object suffices.

It is useful to expand the radiation field around this object in order to compute its scattering. Hence, we introduce translation matrices relating wave functions around two different origins, through

$$H_m^{(1)}(\omega r_1)e^{im\phi_1} = \sum_{n=-\infty}^{\infty} H_{n-m}^{(1)}(\omega d)J_n(\omega r_2)e^{in\phi_2}, \quad (97)$$

with (r_2,ϕ_2) being the coordinates with respect to the center of the second object. Upon scattering off the test object, the amplitude of the outgoing waves is given by the object's S matrix designated as $\mathfrak{S}$,

$$\begin{aligned}J_m(\omega r)e^{im\phi} &= \tfrac{1}{2}\big[H_m^{(2)}(x) + H_m^{(1)}(x)\big]e^{im\phi} \\ &\to \tfrac{1}{2}\big[H_m^{(2)}(\omega r) + \mathfrak{S}_m(\omega)H_m^{(1)}(\omega r)\big]e^{im\phi}. \quad (98)\end{aligned}$$

We can then write the scattering off of the second object as

$$\langle \Phi(\omega,\mathbf{x})\Phi^*(\omega,\mathbf{y})\rangle_{\text{scat}} = \frac{\hbar}{32}\sum_{m=1}^{\infty} n(\Omega m - \omega, T)[1 - |S_m(\omega)|^2]\left(\sum_{n=-\infty}^{\infty} H_{n-m}^{(1)}(\omega d)\big[H_n^{(2)}(\omega r) + \mathfrak{S}_n(\omega)H_n^{(1)}(\omega r)\big]e^{in\phi}\right)$$

$$\times \left(\sum_{p=-\infty}^{\infty} H_{p-m}^{(1)}(\omega d)\big[H_p^{(2)}(\omega\xi) + \mathfrak{S}_p(\omega)H_p^{(1)}(\omega\xi)\big]e^{ip\psi}\right). \quad (99)$$

Next we compute the torque exerted on the test object by the radiation field of the rotating body. Note that we have neglected the nonradiative term in Eq. (65) because it is given by the imaginary part of the Green's function, which can be reduced to a potential energy. The two objects being symmetric, the energy function is indifferent to a rotation of the disk and thus makes no contribution to the torque. The radiation field, on the other hand, exerts a torque that is the integral of $\langle \partial_\phi \Phi \partial_r \Phi \rangle$ over a closed contour around the test object. Note that $\partial_\phi \to in$ and ∂_r combine into the Wronskian of Bessel H functions of the first and second kind. A little algebra yields, in the first-reflection approximation,

$$M_{2\leftarrow 1} = \frac{\hbar}{8\pi}\sum_{m>0,n} n\int_0^\infty d\omega\, n(\omega - \Omega m, T)[1 - |S_m(\omega)|^2]$$

$$\times \big|H_{n-m}^{(1)}(\omega d)\big|^2 [1 - |\mathfrak{S}_n(\omega)|^2]. \quad (100)$$

The subscript indicates that the torque is exerted due to the radiation field of the first on the second object. As explained above, for a slowly rotating object at zero temperature, we may restrict to $m = 1$. Further, $n = 1$ is dominant at large separation. We then find the torque at $T = 0$ as

$$M_{2\leftarrow 1} = \frac{\hbar}{8\pi}\int_0^\Omega d\omega[|S_1(\omega)|^2 - 1]|H_0^{(1)}(\omega d)|^2[1 - |\mathfrak{S}_1(\omega)|^2]. \quad (101)$$

At close separations, one should include higher-order reflections. In the opposite extreme of large separations $\Omega d/c \gg 1$,

the torque falls off with distance as

$$M_{2\leftarrow 1} \sim \frac{\hbar c}{4\pi^2 d}\int_0^\Omega d\omega \frac{1}{\omega}[|S_1(\omega)|^2 - 1][1 - |\mathfrak{S}_1(\omega)|^2], \quad (102)$$

where we have made the factors of c explicit. Note that a nonvanishing torque requires the test object to be lossy, i.e., $|\mathfrak{S}_1(\omega)| < 1$.

One can also compute the force exerted on the test object. Let the two objects be separated along the x axis. Geometrically, they are symmetric with respect to the axis connecting them, nevertheless, a tangential force arises in the perpendicular direction along the y axis due to the radiation field. This force can be computed from the expectation value of the stress tensor

$$T_{ij} = \partial_i \Phi \partial_j \Phi + \tfrac{1}{2}\delta_{ij}((\partial_t \Phi)^2 - (\nabla \Phi)^2). \quad (103)$$

To compute the force parallel to the y axis, one should integrate the expectation value of the stress tensor over a closed contour around the test object:

$$\begin{aligned}F_y &= r\int_0^{2\pi} d\phi\, \langle T_{ij}\rangle \hat{r}_i \hat{y}_j \\ &= r\int_0^{2\pi} d\phi\bigg\langle \frac{1}{2}\sin\phi\bigg((\partial_t\Phi)^2 + (\partial_r\Phi)^2 - \frac{1}{r^2}(\partial_\phi\Phi)^2\bigg) \\ &\quad + \cos\phi\frac{1}{r}\partial_r\Phi\partial_\phi\Phi\bigg\rangle. \quad (104)\end{aligned}$$

Again, nonradiative terms do not contribute on the basis of symmetry. We can compute the tangential force explicitly; however, the algebra is rather long and the result is not very illuminating for our toy model of scalar fields. We postpone the discussion of the force to Sec. III C in the context of electromagnetism.

III. ELECTRODYNAMICS

In this section we generalize the methods and techniques developed in application to a scalar field to electromagnetism. The vector character of the latter complicates mathematical expressions, but the underlying concepts are identical to Sec. II, with the techniques straightforwardly extended to electrodynamics. We start from electromagnetic fluctuations and the corresponding correlation functions in the context of static objects and generalize them to spinning objects. Throughout this section we consider objects of arbitrary shape (rotationally symmetric in the case of spinning bodies) in a general basis of partial waves in three dimensions. We derive general trace formulas for the quantum and thermal radiation from a single object. We shall also explicitly keep the dependence on c.

A. Static objects

Quantum fluctuations of the electromagnetic field can be formulated in a number of ways. In a lossy medium such as a dielectric object, there are subtle complications requiring a careful treatment [57–60]. A convenient starting point for our purposes is the Rytov formalism [28], which relates quantum fluctuations of the fields to those of the sources and currents. For a dielectric object (with no magnetic response $\mu = 1$), Maxwell equations in the presence of sources are

$$\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B},$$

$$\nabla \times \mathbf{B} = -i\epsilon(\omega)\frac{\omega}{c}\mathbf{E} - i\frac{\omega}{c}\mathbf{K}, \quad (105)$$

or equivalently

$$\left(\nabla \times \nabla \times - \frac{\omega^2}{c^2}\epsilon(\omega)\mathbb{I}\right)\mathbf{E} = \frac{\omega^2}{c^2}\mathbf{K}. \quad (106)$$

Then, according to the Rytov formalism, source fluctuations are related to the imaginary part of the local dielectric function by

$$\langle \mathbf{K}(\omega,\mathbf{x}) \otimes \mathbf{K}^*(\omega,\mathbf{y})\rangle = a(\omega)\mathrm{Im}\,\epsilon(\omega,\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\mathbb{I}, \quad (107)$$

where the distributions a and n are defined as before. Current fluctuations are independent at different points (hence the δ function in space) and also independent for different vector components, hence the 3×3 unit matrix $\mathbb{I}$. The corresponding fluctuations of the electromagnetic (EM) field can be described in terms of the sources from Eq. (106) via the EM Green's function $\mathbf{E} = \frac{\omega^2}{c^2}\int\mathbb{G}\mathbf{K}$. We are mainly interested in the EM-field fluctuations outside the object from which we can compute the quantum radiation. As we have discussed in the previous section, field fluctuations receive contributions both from the fluctuating sources within the object and from fluctuations (zero point and at finite temperature, thermal)

in the vacuum outside the object. In the following, we first consider source fluctuations outside the object.

The dyadic EM Green's function is defined by

$$\left(\nabla \times \nabla \times - \frac{\omega^2}{c^2}\epsilon(\omega)\mathbb{I}\right)\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = \mathbb{I}\delta(\mathbf{x}-\mathbf{z}). \quad (108)$$

In an appropriate coordinate system (ξ_1,ξ_2,ξ_3), the outgoing wave Green's function (in empty space) can be broken up along the coordinate ξ_1 as [61,62]

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = i \begin{cases} \sum_\alpha \mathbf{E}^{\mathrm{out}}_{\bar\alpha}(\omega,\mathbf{x}) \otimes \mathbf{E}^{\mathrm{reg}}_\alpha(\omega,\mathbf{z}), & \xi_1(\mathbf{x}) > \xi_1(\mathbf{z}) \\ \sum_\alpha \mathbf{E}^{\mathrm{reg}}_\alpha(\omega,\mathbf{x}) \otimes \mathbf{E}^{\mathrm{out}}_{\bar\alpha}(\omega,\mathbf{z}), & \xi_1(\mathbf{x}) < \xi_1(\mathbf{z}), \end{cases}$$
$$(109)$$

where $\mathbf{E}^{\mathrm{out(in)}}$ is the outgoing (incoming) electric field normalized as

$$\frac{i}{2}\oint d\mathbf{\Sigma} \cdot \left\{\left[\nabla\times\mathbf{E}^{\mathrm{out/in}}_\alpha(\omega,\mathbf{z})\right] \times \mathbf{E}^{\mathrm{out/in}*}_\beta(\omega,\mathbf{z})\right.$$
$$\left. + \mathbf{E}^{\mathrm{out/in}}_\alpha(\omega,\mathbf{z})\times\nabla\times\mathbf{E}^{\mathrm{out/in}*}_\beta(\omega,\mathbf{z})\right\} = \pm\delta_{\alpha\beta}, \quad (110)$$

$$\oint d\mathbf{\Sigma} \cdot \left\{\left[\nabla\times\mathbf{E}^{\mathrm{out/in}}_\alpha(\omega,\mathbf{z})\right]\times\mathbf{E}^{\mathrm{in/out}*}_\beta(\omega,\mathbf{z})\right.$$
$$\left. + \mathbf{E}^{\mathrm{out/in}}_\alpha(\omega,\mathbf{z})\times\nabla\times\mathbf{E}^{\mathrm{in/out}*}_\beta(\omega,\mathbf{z})\right\} = 0, \quad (111)$$

which can be derived by a vector form of the Green's theorem; see the Appendix. In other words, the vector field is normalized such that the corresponding current is unity up to a sign. Also $\mathbf{E}^{\mathrm{reg}}$ defines a solution to the EM field regular everywhere in space. The index α runs over partial waves and $\bar\alpha$ indicates the partial wave that is related to α by time reversal. In the presence of an external object, the free Green's function for both points outside the object should be modified to incorporate the scattering from the object [with $\xi_1(\mathbf{x}) < \xi_1(\mathbf{z})$]

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = \frac{i}{2}\sum_\alpha \left[\mathbf{E}^{\mathrm{in}}_\alpha(\omega,\mathbf{x}) + S_\alpha(\omega)\mathbf{E}^{\mathrm{out}}_\alpha(\omega,\mathbf{x})\right] \otimes \mathbf{E}^{\mathrm{out}}_{\bar\alpha}(\omega,\mathbf{z}),$$
$$(112)$$

where $S_\alpha(\omega)$ is the scattering matrix as a function of the frequency ω and partial wave α. Note that we have assumed that the scattering matrix is diagonal in partial waves.[7] In general, one should sum over all β such that $S_{\beta\alpha} \neq 0$ with the rest of the derivation closely following the remainder of this section. The incoming wave $\mathbf{E}^{\mathrm{in}}$ is normalized to ensure that the corresponding energy flux is $-\omega$. When the object is not present, $S_\alpha(\omega) = 1$ and Eq. (112) reduces to the free Green's function with $\mathbf{E}^{\mathrm{reg}}_\alpha = (\mathbf{E}^{\mathrm{in}}_\alpha + \mathbf{E}^{\mathrm{out}}_\alpha)/2$. We stress that the Green's function in Eq. (112) is defined with both points outside the object.

---

[7] We choose an appropriate coordinate system where Maxwell equations are separable and take ξ_1 to be constant on the object's surface.

The EM-field correlation function due to the outside source fluctuations is then given by

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}$$

$$= \frac{\omega^4}{c^4} \int d\mathbf{z} \langle \mathbb{G}(\omega,\mathbf{x},\mathbf{z})\mathbf{K}(\omega,\mathbf{z}) \otimes \mathbb{G}^*(\omega,\mathbf{y},\mathbf{z})\mathbf{K}^*(\omega,\mathbf{z})\rangle$$

$$= a_{\text{out}}(\omega) \frac{\omega^4}{c^4} \text{Im}\, \epsilon_D \int_{\text{out}} d\mathbf{z}\, \mathbb{G}(\omega,\mathbf{x},\mathbf{z}) \cdot \mathbb{G}^*(\omega,\mathbf{y},\mathbf{z}), \quad (113)$$

where the dot product indicates the contraction of the second subindex of the two dyadic functions. The last line in this equation is obtained according to Eq. (107), where a_{out} corresponds to the distribution function at the environment temperature and ϵ_D is the dielectric function of the vacuum dust. The latter can be set to one only in the end as explained in the previous section: The integral over infinite space brings down a factor of $1/\text{Im}\,\epsilon_D$ [42], while the integration over any finite region vanishes as we take the limit $\text{Im}\,\epsilon_D \to 0$. Therefore, we can choose the domain of integration over $\mathbf{z}$ such that $\xi_1(\mathbf{z}) > \xi_1(\mathbf{x}), \xi_1(\mathbf{y})$. This allows us to use the partial-wave expansion of the Green's function in Eq. (112) to find

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}$$

$$= \frac{\omega^4}{4c^4} a_{\text{out}}(\omega) \sum_{\alpha,\beta} \left[\mathbf{E}_\alpha^{\text{in}}(\omega,\mathbf{x}) + S_\alpha(\omega)\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{x}) \right]$$

$$\otimes \left[\mathbf{E}_\beta^{\text{in}*}(\omega,\mathbf{y}) + S_\beta^*\mathbf{E}_\beta^{\text{out}*}(\omega,\mathbf{y}) \right]$$

$$\times \text{Im}\, \epsilon_D \int_{\text{out}} d\mathbf{z}\, \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z}). \quad (114)$$

Here and in subsequent parts, we frequently compute volume integrals similar to the last line of this equation, which can be cast as

$$\text{Im}\, \epsilon_D \int_{\text{out}} d\mathbf{z}\, \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z})$$

$$= \frac{1}{2i} \int_{\text{out}} d\mathbf{z} \big[\epsilon_D \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z})$$

$$- \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \epsilon_D^* \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z}) \big]$$

$$= \frac{c^2}{2i\omega^2} \int_{\text{out}} d\mathbf{z} \big\{ \big[\nabla\times\nabla\times\mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \big] \cdot \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z})$$

$$- \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \nabla\times\nabla\times\mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z}) \big\},$$

where in the last line we have used the homogenous version of Eq. (106) with the RHS set to zero. The volume integration can be then recast as a surface integral with two boundaries, one at the infinity and another at a finite distance from the object. The infinitesimal imaginary part of the dielectric function guarantees that outgoing functions are exponentially decaying at large distances and thus the surface integral at infinity does not contribute. We then obtain

$$\text{Im}\, \epsilon_D \int_{\text{out}} d\mathbf{z}\, \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \cdot \mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z})$$

$$= \frac{ic^2}{2\omega^2} \oint d\mathbf{\Sigma} \cdot \big\{ \big[\nabla\times\mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z}) \big] \times\mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z})$$

$$+ \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z})\times\nabla\times\mathbf{E}_{\bar{\beta}}^{\text{out}*}(\omega,\mathbf{z}) \big\}. \quad (115)$$

Therefore, from Eq. (110), the correlation function of the EM fields takes the form

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}$$

$$= a_{\text{out}}(\omega) \frac{\omega^2}{4c^2} \sum_\alpha \left[\mathbf{E}_\alpha^{\text{in}}(\omega,\mathbf{x}) + S_\alpha(\omega)\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{x}) \right]$$

$$\otimes \left[\mathbf{E}_\alpha^{\text{in}*}(\omega,\mathbf{y}) + S_\alpha^*(\omega)\mathbf{E}_\alpha^{\text{out}*}(\omega,\mathbf{y}) \right]. \quad (116)$$

The radiation due to the outside fluctuations can be computed by integrating over the Poynting vector $\mathbf{S} = c\,\mathbf{E}\times\mathbf{B}$ of the corresponding correlation function

$$\mathcal{P}_{\text{out fluc}} = \int_0^\infty \frac{d\omega}{2\pi} \oint d\mathbf{\Sigma} \cdot \frac{ic^2}{\omega} \langle (\nabla \times \mathbf{E})$$

$$\times\mathbf{E}^* + \mathbf{E}\times\nabla\times\mathbf{E}^*\rangle_{\text{out fluc}}$$

$$= \int_0^\infty \frac{d\omega}{2\pi} \sum_\alpha [-1 + |S_\alpha(\omega)|^2] a_{\text{out}}(\omega) \frac{i\omega}{4} \oint d\mathbf{\Sigma}$$

$$\cdot \big[(\nabla\times\mathbf{E}_\alpha^{\text{out}})\times\mathbf{E}_\alpha^{\text{out}*} + \mathbf{E}_\alpha^{\text{out}}\times\nabla\times\mathbf{E}_\alpha^{\text{out}*} \big]$$

$$= \frac{1}{4\pi} \int_0^\infty d\omega\, \omega a_{\text{out}}(\omega) \sum_\alpha [-1 + |S_\alpha(\omega)|^2], \quad (117)$$

where we used Eqs. (110) and (111).

The field correlation function induced by the inside fluctuations can be computed similarly. In this case, however, we need the Green's function with one point inside the object. Following an argument similar to the scalar case, we note that as the two points do not coincide, the Green's function satisfies a homogeneous equation inside with respect to the smaller coordinate while it satisfies the free EM equation outside the object in the larger coordinate. Hence, we can expand the Green's function as [with $\xi_1(\mathbf{z}) < \xi_1(\mathbf{x})$]

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = \frac{i}{2} \sum_\alpha \left[A\mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{x}) + B\mathbf{E}_{\bar{\alpha}}^{\text{in}}(\omega,\mathbf{x}) \right] \otimes \mathbf{F}_\alpha(\omega,\mathbf{z}),$$

$$(118)$$

where the prefactor $i/2$ is chosen for convenience, A and B are constants to be determined, and $\mathbf{F}_\alpha$ is defined as the regular (at the origin) solution to the EM equation inside the object

$$\left(\nabla\times\nabla\times - \frac{\omega^2}{c^2} \epsilon(\omega,\mathbf{x})\mathbb{I} \right) \mathbf{F}_\alpha(\omega,\mathbf{x}) = 0. \quad (119)$$

We can determine the coefficients A and B and the normalization of $\mathbf{F}$ by matching the Green's functions approaching a point on the boundary from inside and outside the object

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{y})|_{\mathbf{y}\to\Sigma^-} = \mathbb{G}(\omega,\mathbf{x},\mathbf{y})|_{\mathbf{y}\to\Sigma^+}, \quad \xi_1(\mathbf{x}) > \xi_1(\mathbf{y}), \quad (120)$$

where Σ represents the boundary. Comparing the two Green's functions given by Eqs. (112) and (118), we find ($A = 1$, $B = 0$)

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z}) = \frac{i}{2} \sum_\alpha \mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{x}) \otimes \mathbf{F}_\alpha(\omega,\mathbf{z}), \quad (121)$$

where $\mathbf{F}_\alpha$ and the S-matrix element S_α are determined by the continuity of the Green's function, which requires parallel components of electric and magnetic (the latter because $\mu = 1$)

fields to match at the boundary

$$\mathbf{F}_\alpha(\omega,\mathbf{z})_\| = \left[\mathbf{E}_\alpha^{\text{in}}(\omega,\mathbf{z}) + S_\alpha(\omega)\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{z})\right]_\|,$$

$$\left[\boldsymbol{\nabla}\times\mathbf{F}_\alpha(\omega,\mathbf{z})\right]_\| = \left[\boldsymbol{\nabla}\times\mathbf{E}_\alpha^{\text{in}}(\omega,\mathbf{z}) + S_\alpha(\omega)\boldsymbol{\nabla}\times\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{z})\right]_\|.$$

(122)

The correlation function due to the inside fluctuations is then given by

$$\langle\mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{in fluc}}$$

$$= a_{\text{in}}(\omega)\frac{\omega^4}{4c^4}\sum_{\alpha,\beta}\mathbf{E}_{\bar\alpha}^{\text{out}}(\omega,\mathbf{x})$$

$$\otimes \mathbf{E}_{\bar\beta}^{\text{out}*}(\omega,\mathbf{y})\int_{\text{in}}d\mathbf{z}\,\mathbf{F}_\alpha(\mathbf{z})\cdot\text{Im}\,\epsilon(\omega,\mathbf{z})\mathbf{F}_\beta^*(\mathbf{z}),\quad(123)$$

where a is the distribution function defined at the object's temperature. Again exploiting the wave equation for $\mathbf{F}$, the volume integral can be cast as a surface term

$$\int_{\text{in}}d\mathbf{z}\,\mathbf{F}_\alpha(\mathbf{z})\cdot\text{Im}\,\epsilon(\omega,\mathbf{z})\mathbf{F}_\beta^*(\omega,\mathbf{z})$$

$$= \frac{c^2}{2i\omega^2}\oint d\boldsymbol{\Sigma}\cdot\{[\boldsymbol{\nabla}\times\mathbf{F}_\alpha(\omega,\mathbf{z})]\times\mathbf{F}_\beta^*(\omega,\mathbf{z})$$

$$+ \mathbf{F}_\alpha(\omega,\mathbf{z})\times\boldsymbol{\nabla}\times\mathbf{F}_\beta^*(\omega,\mathbf{z})\}.\quad(124)$$

The continuity equations can be used to evaluate the surface integral

$$\int_{\text{in}}d\mathbf{z}\,\mathbf{F}_\alpha(\omega,\mathbf{z})\cdot\text{Im}\,\epsilon(\omega,\mathbf{z})\mathbf{F}_\beta^*(\omega,\mathbf{z}) = \frac{c^2}{\omega^2}\delta_{\alpha\beta}(1-|S_\alpha(\omega)|^2).$$

(125)

The field correlation function then becomes[8]

$$\langle\mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{in fluc}}$$

$$= a_{\text{in}}(\omega)\frac{\omega^2}{4c^2}\sum_\alpha[1-|S_\alpha(\omega)|^2]\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{x})\otimes\mathbf{E}_\alpha^{\text{out}*}(\omega,\mathbf{y}).$$

(126)

The radiation power due to the inside fluctuations can be computed from the corresponding correlation function as

$$\mathcal{P}_{\text{in fluc}} = \frac{1}{4\pi}\int_0^\infty d\omega\,\omega a_{\text{in}}(\omega)\sum_\alpha[1-|S_\alpha(\omega)|^2].\quad(127)$$

The total radiation per unit time is given by

$$\mathcal{P} = \frac{1}{4\pi}\int_0^\infty d\omega\,\omega(a_{\text{in}}(\omega)-a_{\text{out}}(\omega))\sum_\alpha[1-|S_\alpha(\omega)|^2]$$

$$= \int_0^\infty\frac{d\omega}{2\pi}\hbar\omega(n(\omega,T)-n(\omega,T_0))\sum_\alpha[1-|S_\alpha(\omega)|^2],$$

(128)

where in the last line the radiation is expressed in terms of the Bose-Einstein distribution function. In brief, we have derived

---

[8]We have changed $\alpha \to \bar\alpha$; note that $|S_{\bar\alpha}| = |S_\alpha|$ due to time-reversal symmetry.

Kirchhoff's law in the context of electrodynamics [41,42] and the partial waves also include electromagnetic polarizations. Notice that Eq. (128) is independent of the coordinate system and the shape of the object. In a general basis that the scattering matrix is not diagonal, the sum over α is replaced by a double sum over incoming and outgoing modes as

$$\sum_{\alpha,\beta}[\delta_{\beta\alpha} - |S_{\beta\alpha}(\omega)|^2] = \text{Tr}[\mathbb{I} - \mathbb{S}^\dagger(\omega)\mathbb{S}(\omega)],\quad(129)$$

which is cast as a manifestly invariant (trace) formula in the last line.

B. Moving objects

For bodies in uniform motion, the equations in the previous section are applied in the rest frame of the object and then transformed to describe the EM-field fluctuations in the appropriate laboratory frame. With all contributions of the field correlation functions in a single frame, one can then compute various physical quantities of interest, such as forces, or energy transfer from one object to another or to the vacuum. For nonuniform motion, we assume that the same equations apply locally to the instantaneous rest frame of the body [63]. This assumption should be valid as long as the rate of acceleration is less than typical internal frequencies characterizing the object, which are normally quite large. The EM wave equation for a moving medium can be inferred from a Lagrangian. A dielectric object is described by

$$\mathcal{L} = \tfrac{1}{2}\epsilon'\mathbf{E}'^2 - \tfrac{1}{2}\mathbf{B}'^2,\quad(130)$$

where $\mathbf{E}'$ and $\mathbf{B}'$ are the EM fields in the comoving frame related to the EM fields in the laboratory frame as

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B},\quad \mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E},\quad(131)$$

to lowest order in velocity. Note that $\epsilon' = \epsilon(\omega',\mathbf{x}')$ is the dielectric function defined in the moving frame similarly defined in Sec. II. The Lagrangian can be cast as

$$\mathcal{L} = \mathcal{L}_0 + \tfrac{1}{2}(\epsilon'-1)\mathbf{E}'^2,$$

where $\mathcal{L}_0 = \tfrac{1}{2}\mathbf{E}'^2 - \tfrac{1}{2}\mathbf{B}'^2$ is the free Lagrangian. Notice that $\mathcal{L}_0$ is invariant under the transformation in Eq. (131) to first order in v/c, i.e., $\mathcal{L}_0 = \tfrac{1}{2}\mathbf{E}^2 - \tfrac{1}{2}\mathbf{B}^2 + O(v^2/c^2)$, while the second term is related to the EM field in the laboratory frame by Eq. (131). The modified Maxwell equations are then obtained from the Lagrangian as

$$\left[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times - \frac{\omega^2}{c^2}\mathbb{I} - \frac{\omega^2}{c^2}\tilde{\mathbb{D}}(\epsilon'-1)\mathbb{D}\right]\mathbf{E} = 0,\quad(132)$$

where

$$\mathbb{D} = \mathbb{I} + \frac{1}{i\omega}\mathbf{v}\times\boldsymbol{\nabla}\times,\quad \tilde{\mathbb{D}} = \mathbb{I} + \frac{1}{i\omega}\boldsymbol{\nabla}\times\mathbf{v}\times.$$

(133)

The coupling with the (fluctuating) currents can be formulated by adding to the Lagrangian

$$\Delta\mathcal{L} = \mathbf{K}'\cdot\mathbf{E}',\quad(134)$$

where $\mathbf{K}'$ is defined in the moving frame. This equation follows from the assumption that a local current density is coupled

to the electric field in the instantaneous rest frame of the corresponding point in the moving object; see the discussion in Sec. II B. The inhomogeneous EM equation in the presence of random currents follows from the Lagrangian as

$$\left[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\mathbb{I}-\frac{\omega^2}{c^2}\tilde{\mathbb{D}}(\epsilon'-1)\mathbb{D}\right]\mathbf{E}=\frac{\omega^2}{c^2}\tilde{\mathbb{D}}\mathbf{K}'. \quad (135)$$

Again we should compute field correlation functions due to the outside and inside current fluctuations separately. The former can be easily deduced from Eq. (116) simply by inserting the scattering matrix for a rotating object

$$\langle\mathbf{E}(\omega,\mathbf{x})\otimes\mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{out fluc}}$$

$$=a_{\text{out}}(\omega)\frac{\omega^2}{4c^2}\sum_{\alpha_m}\left[\mathbf{E}_{\alpha_m}^{\text{in}}(\omega,\mathbf{x})+S_{\alpha_m}(\omega)\mathbf{E}_{\alpha_m}^{\text{out}}(\omega,\mathbf{x})\right]$$

$$\otimes\left[\mathbf{E}_{\alpha_m}^{\text{in}*}(\omega,\mathbf{y})+S_{\alpha_m}^*(\omega)\mathbf{E}_{\alpha_m}^{\text{out}*}(\omega,\mathbf{y})\right], \quad (136)$$

where the partial-wave index α_m includes m, the eigenvalue of the angular momentum along the z direction (in units of $\hbar$).

The inside fluctuations, on the other hand, are defined with respect to the rest frame of the object

$$\langle\mathbf{K}'(\omega',\mathbf{x}')\otimes\mathbf{K}'^*(\omega',\mathbf{y}')\rangle=a_{\text{in}}(\omega')\text{Im}\,\epsilon(\omega',\mathbf{x}')\delta(\mathbf{x}'-\mathbf{y}')\mathbb{I}. \quad (137)$$

Consider $\mathbf{K}'_{\omega'm'}(t',\mathbf{x}')$, a fluctuation of the current characterized by the angular momentum m' and frequency ω' in the rotating frame; we do not make the dependence of the current $\mathbf{K}$ on other quantum numbers explicit as its fluctuations depend only on ω and m as it will become clear shortly. For the sake of notational convenience, we define $\mathbf{K}(t,\mathbf{x})\equiv\mathbf{K}'(t',\mathbf{x}')$, which captures current fluctuations in the laboratory-frame coordinates. Note that the two sets of reference frame are related by Eq. (43) and supplemented by $z=z'$ along the symmetry axis of the object. One can then see that the partial wave m is invariant with respect to the reference frame while the frequency is shifted as

$$\omega'=\omega-\Omega m.$$

This modifies the spectral density of source fluctuations simply by replacing the frequency in ϵ and a by $\omega-\Omega m$. Therefore, the inside source fluctuations from the point of view of the laboratory-frame observer are given by

$$\langle\mathbf{K}_m(\omega,\mathbf{x})\otimes\mathbf{K}_m^*(\omega,\mathbf{y})\rangle$$

$$=a_T(\omega-\Omega m)\text{Im}\,\epsilon(\omega-\Omega m,r,z)\frac{\delta(r_\mathbf{x}-r_\mathbf{y})\delta(z_\mathbf{x}-z_\mathbf{y})}{2\pi r}\mathbb{I}. \quad (138)$$

Henceforth, we shall use the same notation $\mathbb{G}$ for the Green's function in the presence of a moving object corresponding to Eq. (132). The EM-field correlation function is given by

$$\langle\mathbf{E}(\omega,\mathbf{x})\otimes\mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{in fluc}}$$

$$=\frac{\omega^4}{c^4}\int_{\text{in}}d\mathbf{z}\,\langle\mathbb{G}(\omega,\mathbf{x},\mathbf{z})\tilde{\mathbb{D}}\mathbf{K}(\omega,\mathbf{z})\cdot\mathbb{G}^*(\omega,\mathbf{y},\mathbf{z})\tilde{\mathbb{D}}^*\mathbf{K}^*(\omega,\mathbf{z})\rangle. \quad (139)$$

We can expand the Green's function similarly to the previous section as

$$\mathbb{G}(\omega,\mathbf{x},\mathbf{z})=\frac{i}{2}\sum_{\alpha_m}\mathbf{E}_{\bar{\alpha}_m}^{\text{out}}(\omega,\mathbf{x})\otimes\mathbf{F}_{\alpha_m}(\omega,\mathbf{z}), \quad (140)$$

where $\mathbf{F}$ is a solution to the modified EM equation inside the dielectric object

$$\left[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times-\frac{\omega^2}{c^2}\mathbb{I}-\frac{\omega^2}{c^2}\tilde{\mathbb{D}}(\epsilon'-1)\mathbb{D}\right]\mathbf{F}=0 \quad (141)$$

and satisfies boundary conditions similar to Eq. (122), albeit with the scattering matrices for a rotating object[9]

$$\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})_\parallel=\left[\mathbf{E}_{\alpha_m}^{\text{in}}(\omega,\mathbf{z})+S_{\bar{\alpha}_m}(\omega)\mathbf{E}_{\alpha_m}^{\text{out}}(\omega,\mathbf{z})\right]_\parallel,$$

$$\left[\boldsymbol{\nabla}\times\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\right]_\parallel=\left[\boldsymbol{\nabla}\times\mathbf{E}_{\alpha_m}^{\text{in}}(\omega,\mathbf{z})+S_{\bar{\alpha}_m}(\omega)\boldsymbol{\nabla}\times\mathbf{E}_{\alpha_m}^{\text{out}}(\omega,\mathbf{z})\right]_\parallel. \quad (142)$$

The correlation function of the EM fields is then given by

$$\langle\mathbf{E}(\omega,\mathbf{x})\otimes\mathbf{E}^*(\omega,\mathbf{y})\rangle_{\text{in fluc}}$$

$$=\frac{\omega^4}{4c^4}\sum_{\alpha_m,\beta_m}a_{\text{in}}(\omega-\Omega m)\mathbf{E}_{\bar{\alpha}_m}^{\text{out}}(\omega,\mathbf{x})\otimes\mathbf{E}_{\bar{\beta}_m}^{\text{out}*}(\omega,\mathbf{y})$$

$$\times\int_{\text{in}}d\mathbf{z}\,\mathbb{D}\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\cdot\text{Im}\,\epsilon(\omega-\Omega m,\mathbf{z})\mathbb{D}^*\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z}). \quad (143)$$

The volume integral can be computed similarly to that of the previous subsection. We write the second line of the preceding equation as

$$\frac{1}{2i}\int_{\text{in}}d\mathbf{z}[(\epsilon'-1)\mathbb{D}\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\cdot\mathbb{D}^*\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})$$

$$-\mathbb{D}\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\cdot(\epsilon'^*-1)\mathbb{D}^*\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})]$$

$$=\frac{1}{2i}\int_{\text{in}}d\mathbf{z}\{[\tilde{\mathbb{D}}(\epsilon'-1)\mathbb{D}\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})]\cdot\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})$$

$$-\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\cdot\tilde{\mathbb{D}}^*(\epsilon'^*-1)\mathbb{D}^*\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})\}$$

$$=\frac{c^2}{2i\omega^2}\int_{\text{in}}d\mathbf{z}\{[\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})]\cdot\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})$$

$$-\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})\}$$

$$=\frac{c^2}{2i\omega^2}\oint d\boldsymbol{\Sigma}\cdot\{[\boldsymbol{\nabla}\times\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})]\times\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})$$

$$+\mathbf{F}_{\alpha_m}(\omega,\mathbf{z})\times\boldsymbol{\nabla}\times\mathbf{F}_{\beta_m}^*(\omega,\mathbf{z})\}, \quad (144)$$

where in the step from the second to the third line, we have used Eq. (132). Using the continuity relations, the last line

---

[9]Note that the scattering matrix is given for $\bar{\alpha}$. This is because the Green's function in the presence of a moving object is no longer symmetric with respect to its spatial arguments but satisfies a rather different symmetry; see the discussion in Sec. IIB1. If Eq. (112) defines the Green's function with $\xi_1(\mathbf{x})<\xi_1(\mathbf{z})$, then $\mathbb{G}(\omega,\mathbf{z},\mathbf{x})=\frac{i}{2}\sum_\alpha\mathbf{E}_{\bar{\alpha}}^{\text{out}}(\omega,\mathbf{z})\otimes[\mathbf{E}_\alpha^{\text{in}}(\omega,\mathbf{x})+S_{\bar{\alpha}}(\omega)\mathbf{E}_\alpha^{\text{out}}(\omega,\mathbf{x})]$.

gives

$$\int_{\text{in}} d\mathbf{z} \, \mathbb{D}\mathbf{F}_{\alpha_m}(\omega,\mathbf{z}) \cdot \text{Im}\,\epsilon(\omega - \Omega m,\mathbf{z})\mathbb{D}^*\mathbf{F}^*_{\beta_m}(\omega,\mathbf{z})$$

$$= \frac{c^2}{\omega^2}\delta_{\alpha_m\beta_m}(1 - |S_{\bar{\alpha}_m}|^2), \tag{145}$$

which is the analog of Eq. (125) for moving objects. The EM-field correlation function corresponding to the inside fluctuations is then obtained as

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y}) \rangle_{\text{in fluc}}$$

$$= \frac{\omega^2}{4c^2}\sum_{\alpha_m} a_{\text{in}}(\omega - \Omega m)(1 - |S_{\alpha_m}|^2)\mathbf{E}^{\text{out}}_{\alpha_m}(\omega,\mathbf{x})$$

$$\otimes \mathbf{E}^{\text{out*}}_{\alpha_m}(\omega,\mathbf{y}). \tag{146}$$

The total radiation per unit time can be obtained by integrating over the Poynting vector as

$$\mathcal{P} = \frac{1}{4\pi}\int_0^\infty d\omega\,\omega \sum_{\alpha_m}[a_{\text{in}}(\omega - \Omega m) - a_{\text{out}}(\omega)]$$

$$\times [1 - |S_{\alpha_m}(\omega)|^2]$$

$$= \int_0^\infty \frac{d\omega}{2\pi}\hbar\omega \sum_{\alpha_m}[n(\omega - \Omega m,T) - n(\omega,T_0)]$$

$$\times [1 - |S_{\alpha_m}(\omega)|^2]. \tag{147}$$

At zero-temperature everywhere, $n(\omega - \Omega m,0) - n(\omega,0) = -\Theta(\Omega m - \omega)$ and the quantum radiation happens in the superradiating regime; see the discussion in Sec. IIB2. Again we note that our derivation leading to Eq. (147) is not specific to a coordinate system and shape as long as the object is a solid of revolution with the angular momentum m being a good quantum number. In a general basis where the scattering matrix is not diagonal (except in m), we have

$$\mathcal{P} = \int_0^\infty \frac{d\omega}{2\pi}\hbar\omega \sum_{\alpha_m,\beta_m}[n(\omega - \Omega m,T) - n(\omega,T_0)]$$

$$\times [\delta_{\beta_m\alpha_m} - |S_{\beta_m\alpha_m}(\omega)|^2]$$

$$= \int_0^\infty \frac{d\omega}{2\pi}\hbar\omega\,\text{Tr}\{[n(\omega - \Omega\hat{l}_z,T) - n(\omega,T_0)]$$

$$\times [\mathbb{I} - \mathbb{S}^\dagger(\omega)\mathbb{S}(\omega)]\}, \tag{148}$$

where $\hat{l}_z$ is the angular momentum operator. This equation casts the quantum (and thermal) radiation from a rotating object into a trace formula applicable to any shape with rotational symmetry. For detailed discussions on thermal radiation and the heat transfer for arbitrary objects, see Ref. [64].

C. A test object in the presence of a rotating body

In this section we study the interaction of the radiation field from a rotating body with a test object at rest and assume that both objects are dielectric spheres. The overall EM-field correlation function is given by the sum of Eqs. (136) and (146) as

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{in fluc}} + \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{out fluc}}. \tag{149}$$

In the following we consider the limit of zero temperature in both the object and the environment. The generalization to finite temperature is straightforward. Similar to Sec. II, the correlation function in Eq. (149) can be recast as

$$\langle \mathbf{E} \otimes \mathbf{E}^* \rangle = \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{nonrad}} + \langle \mathbf{E} \otimes \mathbf{E}^* \rangle_{\text{rad}}, \tag{150}$$

where we have broken up the correlation function into radiative (due to propagating photons) and nonradiative (due to zero-point fluctuations) parts. The latter is given by

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y}) \rangle_{\text{nonrad}} = \hbar\,\text{sgn}(\omega)\text{Im}\,\mathbb{G}(\omega,\mathbf{x},\mathbf{y}), \tag{151}$$

where $\mathbb{G}$ is the Green's function in the presence of a rotating object. This equation is reminiscent of the FDT in equilibrium; the term on the RHS is purely real and thus does not contribute to the radiation, but leads to a Casimir-like force between the rotating body and nearby objects. The radiative term in the correlation function can be obtained from Eqs. (149) and (150) as

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y}) \rangle_{\text{rad}} \approx \frac{\hbar\omega^2}{2c^2}\sum_{\alpha_m}\Theta(\Omega m - \omega)(|S_{\alpha_m}|^2 - 1)$$

$$\times \mathbf{E}^{\text{out}}_{\alpha_m}(\omega,\mathbf{x}) \otimes \mathbf{E}^{\text{out*}}_{\alpha_m}(\omega,\mathbf{y}) \tag{152}$$

and contributes to the Poynting vector in the superradiating regime $0 < \omega < \Omega m$.

To find the interaction with a test object, we only consider the radiative term in the correlation function for two reasons. First, radiation pressure exerts a force falling off more slowly with the separation distance compared to the nonradiative part. Second, nonradiative fluctuations give rise to a potential energy depending only on the separation distance akin to the Casimir energy. The test object being spherical, the corresponding tangential force or torque due to the corresponding term in Eq. (150) is identically zero.

The radiation from a (nonmagnetic) rotating sphere is dominated by the lowest (electric) partial wave ($l = 1$, $m = 1, P = E$) in which case Eq. (152) yields

$$\langle \mathbf{E}(\omega,\mathbf{x}) \otimes \mathbf{E}^*(\omega,\mathbf{y}) \rangle_{\text{rad}} = \frac{\hbar\omega^2}{2c^2}\Theta(\Omega - \omega)(|S_{11E}|^2 - 1)$$

$$\times \mathbf{E}^{\text{out}}_{11E}(\omega,\mathbf{x}) \otimes \mathbf{E}^{\text{out*}}_{11E}(\omega,\mathbf{y}). \tag{153}$$

The partial waves in Eq. (152) are defined in spherical basis as

$$\mathbf{E}^{\text{out}}_{lmM}(\omega,\mathbf{x}) = \frac{\sqrt{\omega/c}}{\sqrt{l(l+1)}}\nabla \times h_l^{(1)}\left(\frac{\omega r}{c}\right)Y_{lm}(\theta,\phi)\mathbf{x},$$

$$\mathbf{E}^{\text{out}}_{lmE}(\omega,\mathbf{x}) = -i\frac{\sqrt{c/\omega}}{\sqrt{l(l+1)}}\nabla \times \nabla \times h_l^{(1)}\left(\frac{\omega r}{c}\right)Y_{lm}(\theta,\phi)\mathbf{x}, \tag{154}$$

where Y_{lm} is usual spherical harmonic function and $h_l^{(1)}$ is the spherical Hankel function of the first kind. The normalization is chosen to ensure the conditions in Eqs. (110) and (111).

In order to find the scattering from the second object, we expand the EM field around its origin located at a separation d on the x axis. To the lowest order in frequency, we have

$$\mathbf{E}^{\text{out}}_{11E}(\omega,\mathbf{x}) = \mathcal{U}_{11E,11E}\mathbf{E}^{\text{reg}}_{11E}(\omega,\tilde{\mathbf{x}}) + \mathcal{U}_{10M,11E}\mathbf{E}^{\text{reg}}_{10M}(\omega,\tilde{\mathbf{x}}) + \cdots, \tag{155}$$

where $\tilde{\mathbf{x}}$ is defined with respect to the new origin. The regular functions are defined by replacing the spherical Hankel function $h_l^{(1)}$ in Eq. (154) by the spherical Bessel function j_l. The translation matrices are given by [62]

$$\mathcal{U}_{11E,11E} = h_0^{(1)}\left(\frac{\omega d}{c}\right), \quad \mathcal{U}_{10M,11E} = \frac{\sqrt{2}\omega d}{4c}h_0^{(1)}\left(\frac{\omega d}{c}\right). \tag{156}$$

Next we consider the scattering from the test object

$$\mathbf{E}_{lmP}^{\text{reg}}(\omega,\tilde{\mathbf{x}}) \to \frac{1}{2}\left[\mathbf{E}_{lmP}^{\text{in}}(\omega,\tilde{\mathbf{x}}) + \mathfrak{S}_{lmP}\mathbf{E}_{lmP}^{\text{out}}(\omega,\tilde{\mathbf{x}})\right], \tag{157}$$

where $\mathfrak{S}_{lmP}$ is the corresponding scattering matrix. We then find the EM-field correlation function upon one scattering from the test object as

$$\langle \mathbf{E}(\omega,\tilde{\mathbf{x}}) \otimes \mathbf{E}^*(\omega,\tilde{\mathbf{y}})\rangle_{\text{rad}}$$

$$= \frac{\hbar\omega^2}{8c^2}\Theta(\Omega-\omega)(|S_{11E}|^2 - 1)\{\mathcal{U}_{11E,11E}\left[\mathbf{E}_{11E}^{\text{in}}(\omega,\tilde{\mathbf{x}}) + \mathfrak{S}_{11E}\mathbf{E}_{11E}^{\text{out}}(\omega,\tilde{\mathbf{x}})\right] + \mathcal{U}_{10M,11E}\left[\mathbf{E}_{10M}^{\text{in}}(\omega,\tilde{\mathbf{x}}) + \mathfrak{S}_{10M}\mathbf{E}_{10M}^{\text{out}}(\omega,\tilde{\mathbf{x}})\right]\}$$

$$\otimes \overline{\{\mathcal{U}_{11E,11E}\left[\mathbf{E}_{11E}^{\text{in}}(\omega,\tilde{\mathbf{y}}) + \mathfrak{S}_{11E}\mathbf{E}_{11E}^{\text{out}}(\omega,\tilde{\mathbf{y}})\right] + \mathcal{U}_{10M,11E}\left[\mathbf{E}_{10M}^{\text{in}}(\omega,\tilde{\mathbf{y}}) + \mathfrak{S}_{10M}\mathbf{E}_{10M}^{\text{out}}(\omega,\tilde{\mathbf{y}})\right]\}}. \tag{158}$$

Having the correlation functions, we can compute physical quantities of interest. In computing the torque, the partial waves $(1,1,E)$ and $(1,0,M)$ decouple; however, the latter does not contribute since its angular momentum along the z axis is zero. We then find that the torque falls off as $1/d^2$ with the separation distance as

$$M \sim \frac{\hbar}{8\pi}\int_0^\Omega d\omega(|S_{11E}|^2 - 1)|\mathcal{U}_{11E,11E}|^2(1 - |\mathfrak{S}_{11E}|^2)$$

$$= \frac{\hbar c^2}{8\pi d^2}\int_0^\Omega d\omega\frac{1}{\omega^2}(|S_{11E}|^2 - 1)(1 - |\mathfrak{S}_{11E}|^2). \tag{159}$$

For small particles whose polarizability are α_1 and α_2 for rotating and static bodies, respectively, we find

$$M = \frac{8\hbar c^2}{9\pi d^2}\int_0^\Omega d\omega\,\omega^4|\text{Im}\,\alpha_1(\omega - \Omega)|\text{Im}\,\alpha_2(\omega). \tag{160}$$

Computing the force is more complicated since the two partial waves mix and one has to find their overlap via the Maxwell stress tensor

$$T_{ij}(\omega) = E_i(\omega)E_j^*(\omega) + B_i(\omega)B_j^*(\omega) - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\delta_{ij}. \tag{161}$$

The y component of the force, perpendicular the x axis connecting the two objects and the z axis along the rotation, is obtained as

$$F_y = \int \frac{d\omega}{2\pi}\int r^2 d\Omega_{\hat{r}}\langle T_{ij}\rangle\hat{r}_i\hat{y}_j, \tag{162}$$

with $\Omega_{\hat{r}}$ being the solid angle corresponding to the unit vector $\hat{r}$ from the origin of the test object. A lengthy, though straightforward, calculation leads to

$$F_y = \frac{\hbar}{4\pi c^2}\int_0^\Omega d\omega\,\omega^2(|S_{11E}|^2 - 1)$$

$$\times \mathcal{U}_{10M,11E}\mathcal{U}_{11E,11E}\mathcal{T}_{10M,11E}\text{Re}\left(-1 + \overline{\mathfrak{S}_{10M}}\mathfrak{S}_{11E}\right), \tag{163}$$

where $\mathcal{T}_{10M,11E}$ characterizes the stress tensor sandwiched between the two partial waves, whose dependence on frequency is given by

$$\mathcal{T}_{10M,11E} = -\frac{\pi c}{2\sqrt{2}\omega}. \tag{164}$$

For a nonmagnetic object, we can safely assume $\mathfrak{S}_{10M} \approx 1$ since its frequency dependence can be neglected compared to $\mathfrak{S}_{11E}$, hence

$$F_y = \frac{\hbar}{32\pi d}\int_0^\Omega d\omega(|S_{11E}|^2 - 1)(1 - \text{Re}\,\mathfrak{S}_{11E}). \tag{165}$$

Notice that the force falls off as the inverse separation distance while the usual Casimir force decays much faster. For small particles with polarizability α_1 and α_2, we find

$$F_y = \frac{\hbar}{9\pi d}\int_0^\Omega d\omega\,\omega^6|\text{Im}\,\alpha_1(\omega - \Omega)|\text{Im}\,\alpha_2(\omega). \tag{166}$$

D. Vacuum friction on a rotating object

For a rotating object, we have to solve a complicated equation (141), but in the lowest order in velocity we can neglect the explicit dependence on velocity and set $\mathbb{D} \approx \tilde{\mathbb{D}} \approx \mathbb{I}$, changing the argument of the dielectric function as $\epsilon(\omega) \to \epsilon(\omega - \Omega m)$. Finally, at zero temperature, only frequencies within the range $[0,\Omega]$ contribute. We examine the cases where the object is a sphere or a right circular cylinder.

1. Sphere

Electromagnetic scattering from a sphere is most conveniently described in a basis (l,m,P), where l corresponds to the total angular momentum, m is the angular momentum along the z axis, and P is the polarization. In the approximations made here, the lowest partial wave $l = 1$ gives the leading order, while larger l are suppressed by higher powers of the (linear) velocity divided by the speed of light. We assume a nonmagnetic object, thus the electric polarizability gives the leading contribution to scattering matrix as

$$S_{1mE}(\omega) = 1 + i\frac{4\omega^3}{3c^3}\alpha(\omega - \Omega m), \tag{167}$$

where $\alpha(\omega)$ is the polarizability of a spherical object at low frequencies (appropriate to the problem of a rotating object at a small angular velocity) depending solely on the dielectric function $\epsilon(\omega)$, which is assumed to be a homogenous but frequency-dependent function within the object. Note that, at zero temperature, only $m = 1$ (and not $m = 0, -1$) contributes to the radiation.

a. Radiation. The rate of energy radiation to the vacuum is obtained as

$$\mathcal{P} \approx \int_0^\Omega \frac{d\omega}{2\pi} \hbar\omega(|S_{11E}|^2 - 1)$$

$$\approx \frac{4\hbar}{3\pi c^3} \int_0^\Omega d\omega\, \omega^4 [-\mathrm{Im}\,\alpha(\omega - \Omega)], \qquad (168)$$

where we have kept only the leading term in powers of frequency. For a dielectric sphere of radius R, the polarizability is $\alpha(\omega) = R^3[\epsilon(\omega) - 1]/[\epsilon(\omega) + 2]$; the radiation is then given by [32]

$$\mathcal{P} \approx \frac{4\hbar R^3}{3\pi c^3} \int_0^\Omega d\omega\, \omega^4 \left| \mathrm{Im}\, \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 2} \right|. \qquad (169)$$

Similarly, the frictional torque is obtained as

$$M \approx \frac{4\hbar R^3}{3\pi c^3} \int_0^\Omega d\omega\, \omega^3 \left| \mathrm{Im}\, \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 2} \right|. \qquad (170)$$

For a metallic particle, $\epsilon = 1 + i4\pi\sigma/\omega$ and $\mathrm{Im}\,\alpha \approx 3\omega R^3/4\pi\sigma$. Hence, we find [32]

$$\mathcal{P} = \frac{\hbar R^3 \Omega^6}{30\pi^2 c^3 \sigma}, \quad M = \frac{\hbar R^3 \Omega^5}{20\pi^2 c^3 \sigma}. \qquad (171)$$

b. Entropy generation. Making the above approximation, the rate of entropy generation is given by

$$\mathcal{S} \approx -k_B \int_0^\Omega \frac{d\omega}{2\pi} (|S_{11E}|^2 - 1) \ln(|S_{11E}|^2 - 1)$$

$$= k_B \mathcal{N}_{11E} \left| \ln \frac{R^3 \Omega^4}{c^3 \sigma} \right|, \qquad (172)$$

with $\mathcal{N}_{11E} = \frac{R^3 \Omega^5}{30\pi^2 c^3 \sigma}$, where the second line is computed for a metallic particle and $\mathcal{N}_{11E}$ is the total number of photons with the quantum number $11E$ radiated per unit time.

c. Uncertainty in angular momentum. For small angular velocities, one can compute the uncertainty of angular momentum from Eq. (96). Notice that $M \propto \Omega^5$, thus

$$I\,\Delta\Omega \approx \sqrt{\frac{\hbar I \Omega_0}{5}}. \qquad (173)$$

2. Cylinder

For a cylinder, the scattering matrices are more complicated due to mixing between the two polarizations. A complete basis for cylindrical waves is (m, k_z, P), with k_z being the wave vector parallel to the z axis and polarizations labeled by P. In the limit of a thin or slowly rotating cylinder where $\Omega R/c, \epsilon\Omega R/c \ll 1$, the first partial wave $m = 1$ gives the leading contribution while k_z should be integrated over all propagating waves. The corresponding scattering matrices are

$$S_{1k_z MM}(\omega) = 1 + \frac{i\pi}{2} \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 1} \frac{\omega^2}{c^2} R^2,$$

$$S_{1k_z EE}(\omega) = 1 + \frac{i\pi}{2} \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 1} k_z^2 R^2, \qquad (174)$$

$$S_{1k_z EM}(\omega) = S_{1k_z ME}(\omega) = \frac{i\pi}{2} \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 1} \frac{\omega k_z}{c} R^2,$$

where the argument of the dielectric function is $\omega - \Omega$ corresponding to $m = 1$. Again we have assumed that the object is described by a spatially constant but frequency-dependent $\epsilon(\omega)$.

Radiation. The energy radiation per unit time is obtained as [39]

$$\mathcal{P} \approx \int_0^\Omega \frac{d\omega}{2\pi} \hbar\omega \int_{-\omega/c}^{\omega/c} \frac{L\,dk_z}{2\pi} \sum_{P,P' \in \{M,E\}} [|S_{1k_z PP'}(\omega)|^2 - \delta_{PP'}]$$

$$\approx \frac{2\hbar L R^2}{3\pi c^3} \int_0^\Omega d\omega\, \omega^4 \left| \mathrm{Im}\, \frac{\epsilon(\omega - \Omega) - 1}{\epsilon(\omega - \Omega) + 1} \right|, \qquad (175)$$

where we have neglected terms of the order of R^4. If the cylinder has a small conductivity described by the dielectric function $\epsilon = 1 + i4\pi\sigma/\omega$ with $\sigma \ll \Omega$, Eq. (175) yields

$$\mathcal{P} = \frac{8\hbar L R^2 \Omega^4 \sigma}{c^3} \ln \frac{\Omega}{\sigma}, \qquad (176)$$

in agreement with the results of Ref. [40].

In the opposite limit where $\Omega \ll \sigma$, we find

$$\mathcal{P} = \frac{\hbar L R^2 \Omega^6}{90\pi^2 c^3 \sigma}, \quad M = \frac{\hbar L R^2 \Omega^5}{60\pi^2 c^3 \sigma}. \qquad (177)$$

The radiation from a rotating cylinder indeed takes a form similar to that of a rotating sphere; however, while $\Omega R/c$ is constrained to be small in the nonrelativistic limit, $\Omega L/c$ is not.

The results for entropy generation and angular momentum uncertainty also bear a close resemblance to a rotating sphere. Specifically, with $M \sim \Omega^5$, we find the same relation in Eq. (173) for a rotating cylinder. One may speculate that this result holds for any rotating object with an arbitrary geometry as long as the angular velocity is small.

To get an estimate for the magnitude of radiation effects, we consider a rapidly spinning nanotube of radius R and length L and assume that $\Omega R/c$ is small. We then find that the rotation slows down by an order of magnitude over a time scale of $\tau \sim (I/\hbar)(c^3/LR^2\Omega^3)$. The moment of inertia of a nanotube can be as small as 10^{-33} in SI units [65] (compare with $\hbar \approx 10^{-34}$). So even at small velocities, τ can be of the order of a few hours.

APPENDIX: GREEN'S THEOREM

The vector Green's theorem reads

$$E_i(\mathbf{x}) = \oint d\boldsymbol{\Sigma} \cdot \{ [\boldsymbol{\nabla} \times \mathbf{G}_i(\mathbf{x}, \mathbf{z})] \times \mathbf{E}(\mathbf{z})$$

$$+ \mathbf{G}_i(\mathbf{x}, \mathbf{z}) \times [\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{z})] \}, \qquad (A1)$$

where E_i is the i component of the electric field $\mathbf{E}$ that satisfies the vector Helmholtz equation and $\mathbf{G}_i$ is a vector defined from the dyadic Green's function as $(\mathbf{G}_i)_j = \mathbb{G}_{ij}$. Also note that the point $\mathbf{x}$ is enclosed by the boundary of the integration.

We choose $\mathbf{E} = \mathbf{E}_\beta^{\mathrm{reg}}$, a partial wave indexed by β, and also employ the definition of the Green's function in Eq. (109) to find

$$\left[\mathbf{E}_\beta^{\mathrm{reg}}(\mathbf{x})\right]_i = i \sum_\alpha \left[\mathbf{E}_\alpha^{\mathrm{reg}}(\mathbf{x})\right]_i \oint d\mathbf{\Sigma} \cdot \left\{\left[\mathbf{\nabla}\times\mathbf{E}_{\bar\alpha}^{\mathrm{out}}(\mathbf{z})\right]\times\mathbf{E}_\beta^{\mathrm{reg}}(\mathbf{z})\right.$$
$$\left. + \mathbf{E}_{\bar\alpha}^{\mathrm{out}}(\mathbf{z})\times\left[\mathbf{\nabla}\times\mathbf{E}_\beta^{\mathrm{reg}}(\mathbf{z})\right]\right\}. \qquad \text{(A2)}$$

The vector fields $\mathbf{E}_\alpha^{\mathrm{reg}}$ constitute a complete set, hence

$$i \oint d\mathbf{\Sigma} \cdot \left\{\left[\mathbf{\nabla}\times\mathbf{E}_{\bar\alpha}^{\mathrm{out}}(\mathbf{z})\right]\times\mathbf{E}_\beta^{\mathrm{reg}}(\mathbf{z}) + \mathbf{E}_{\bar\alpha}^{\mathrm{out}}(\mathbf{z}) \times \left[\mathbf{\nabla}\times\mathbf{E}_\beta^{\mathrm{reg}}(\mathbf{z})\right]\right\}$$
$$= \delta_{\alpha\beta}. \qquad \text{(A3)}$$

Using the definition of the regular wave functions, we arrive at Eq. (110).

---

[1] G. T. Moore, J. Math. Phys. **11**, 2679 (1970).

[2] S. A. Fulling and P. C. W. Davies, Proc. R. Soc. London Ser. A **348**, 393 (1976).

[3] L. H. Ford and A. Vilenkin, Phys. Rev. D **25**, 2569 (1982).

[4] M.-T. Jaekel and S. Reynaud, Phys. Lett. A **167**, 227 (1992).

[5] G. Calucci, J. Phys. A: Math. Gen. **25**, 3873 (1992).

[6] P. A. Maia Neto and S. Reynaud, Phys. Rev. A **47**, 1639 (1993).

[7] C. K. Law, Phys. Rev. A **49**, 433 (1994).

[8] V. V. Dodonov, Phys. Lett. A **207**, 126 (1995).

[9] G. Barton and A. Calogeracos, Ann. Phys. (NY) **238**, 227 (1995).

[10] O. Méplan and C. Gignoux, Phys. Rev. Lett. **76**, 408 (1996).

[11] M. Kardar and R. Golestanian, Rev. Mod. Phys. **71**, 1233 (1999).

[12] M. G. Silveirinha and S. I. Maslovski, Phys. Rev. A **86**, 042118 (2012).

[13] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, Nature (London) **479**, 376 (2011); P. D. Nation, J. R. Johansson, M. P. Blencowe, and F. Nori, Rev. Mod. Phys. **84**, 1 (2012).

[14] A. Lambrecht, M.-T. Jaekel, and S. Reynaud, Phys. Rev. Lett. **77**, 615 (1996).

[15] P. A. Maia Neto and L. A. S. Machado, Phys. Rev. A **54**, 3420 (1996).

[16] M. Montazeri and M. F. Miri, Phys. Rev. A **77**, 053815 (2008).

[17] D. F. Mundarain and P. A. Maia Neto, Phys. Rev. A **57**, 1379 (1998).

[18] V. V. Dodonov and A. B. Klimov, Phys. Rev. A **53**, 2664 (1996).

[19] M. Crocce, D. A. R. Dalvit, and F. D. Mazzitelli, Phys. Rev. A **64**, 013808 (2001).

[20] G. Barton and C. Eberlein, Ann. Phys. (NY) **227**, 222 (1993).

[21] A. Calogeracos and G. Barton, Ann. Phys. (NY) **238**, 268 (1995).

[22] J. Schwinger, J. Math. Phys. **2**, 407 (1961).

[23] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964).

[24] C. D. Fosco, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. D **76**, 085007 (2007).

[25] V. E. Mkrtchian and C. Henkel, Ann. Phys. (Berlin) **526**, 87 (2014).

[26] J. B. Pendry, J. Phys.: Condens. Matter **9**, 10301 (1997).

[27] A. I. Volokitin and B. N. J. Persson, J. Phys.: Condens. Matter **11**, 345 (1999); Rev. Mod. Phys. **79**, 1291 (2007).

[28] S. M. Rytov, Y. A. Kravtsov, and V. I. Tatarskii, *Priniciples of Statistical Radiophysics 3: Elements of Random Fields* (Springer, Berlin, 1989).

[29] M. F. Maghrebi, R. Golestanian, and M. Kardar, Phys. Rev. A **88**, 042509 (2013).

[30] V. L. Ginzburg and I. M. Frank, J. Phys. (USSR) **9**, 353 (1945).

[31] M. G. Silveirinha, Phys. Rev. A **88**, 043846 (2013).

[32] A. Manjavacas and F. J. García de Abajo, Phys. Rev. Lett. **105**, 113601 (2010); Phys. Rev. A **82**, 063827 (2010).

[33] Y. B. Zel'dovich, JETP Lett. **14**, 180 (1971).

[34] R. Penrose, Nuovo Cimento Riv. Ser. **1**, 252 (1969).

[35] W. G. Unruh, Phys. Rev. D **10**, 3194 (1974).

[36] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[37] A. Calogeracos and G. E. Volovik, JETP Lett. **69**, 281 (1999).

[38] D. C. Roberts and Y. Pomeau, Phys. Rev. Lett. **95**, 145303 (2005).

[39] M. F. Maghrebi, R. L. Jaffe, and M. Kardar, Phys. Rev. Lett. **108**, 230403 (2012).

[40] Y. B. Zel'dovich, L. V. Rozhanskii, and A. A. Starobinskii, Radiophys. Quantum Electron. **29**, 761 (1986).

[41] C. W. J. Beenakker, Phys. Rev. Lett. **81**, 1829 (1998).

[42] M. Krüger, T. Emig, and M. Kardar, Phys. Rev. Lett. **106**, 210404 (2011).

[43] L. Boltzmann, Ann. Phys. (Leipzig) **258**, 291 (1884).

[44] M. F. Maghrebi, R. Golestanian, and M. Kardar, Phys. Rev. D **87**, 025016 (2013).

[45] L. D. Landau and E. M. Lifshitz, *Mechanics*, Vol. 1 (Pergamon, Oxford, 1969).

[46] J. D. Bekenstein and M. Schiffer, Phys. Rev. D **58**, 064014 (1998).

[47] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[48] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

[49] C. Beenakker, arXiv:quant-ph/9808066.

[50] J. D. Bekenstein and M. Schiffer, Phys. Rev. Lett. **72**, 2512 (1994).

[51] L. D. Landau and E. M. Lifshitz, *Statistical Physics Part 1*, Vol. 5 (Pergamon, Oxford, 1980).

[52] I. Klich and L. Levitov, Phys. Rev. Lett. **102**, 100502 (2009).

[53] C. W. J. Beenakker, in *Quantum Computers, Algorithms and Chaos, Proceedings of the International School of Physics "Enrico Fermi," Course CLXII, Varenna, 2005*, edited by G. Casati, D. Shepelyansky, P. Zoller, and G. Benenti (IOS, Amsterdam, 2006).

[54] P. Billingsley, *Probability and Measure* (Wiley, New York, 2008).

[55] M. Kardar, *Statistical Physics of Fields* (Cambridge University Press, Cambridge, 2007).

[56] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).

[57] R. Matloob, R. Loudon, S. M. Barnett, and J. Jeffers, Phys. Rev. A **52**, 4823 (1995).

[58] T. Gruner and D.-G. Welsch, Phys. Rev. A **54**, 1661 (1996).

[59] M. Artoni and R. Loudon, Phys. Rev. A **55**, 1347 (1997).

[60] A. Narayanaswamy and G. Chen, Phys. Rev. B **77**, 075125 (2008).

[61] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1998).

[62] S. J. Rahi, T. Emig, N. Graham, R. L. Jaffe, and M. Kardar, Phys. Rev. D **80**, 085021 (2009).

[63] J. van Bladel, Proc. IEEE **64**, 301 (1976).

[64] M. Krüger, G. Bimonte, T. Emig, and M. Kardar, Phys. Rev. B **86**, 115423 (2012).

[65] P. Král and H. R. Sadeghpour, Phys. Rev. B **65**, 161401 (2002).