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Hamiltonian quantum simulation with bounded-strength controls

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Abstract

We propose dynamical control schemes for Hamiltonian simulation in many-body quantum systems that avoid instantaneous control operations and rely solely on realistic *bounded-strength control Hamiltonians*. Each simulation protocol consists of periodic repetitions of a basic control block, constructed as a modification of an 'Eulerian decoupling cycle,' that would otherwise implement a trivial (zero) target Hamiltonian. For an open quantum system coupled to an uncontrollable environment, our approach may be employed to engineer an effective evolution that simulates a target Hamiltonian on the system while suppressing unwanted decoherence to the leading order, thereby allowing for *dynamically corrected simulation*. We present illustrative applications to both closed- and open-system simulation settings, with emphasis on *simulation of non-local (two-body) Hamiltonians using only local (one-body) controls*. In particular, we provide simulation schemes applicable to Heisenberg-coupled spin chains exposed to general linear decoherence, and show how to simulate Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits, as potentially relevant to the dynamical generation of a topologically protected quantum memory. Additional implications for quantum information processing are discussed.

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1. Introduction

The ability to accurately engineer the Hamiltonian of complex quantum systems is both a fundamental control task and a prerequisite for quantum simulation, as originally envisioned by Feynman [1–4]. The basic idea underlying Hamiltonian simulation is to use an available quantum system, together with available (classical or quantum) control resources, to emulate the dynamical evolution that would have occurred under a different, desired Hamiltonian not directly accessible to implementation—in fact, this idea may be more generally applied to emulate a desired non-unitary (dissipative) evolution, see e.g. [5] for a recent survey. From a control-theory standpoint, the simplest setting is provided by *open-loop Hamiltonian engineering in the time domain* [6, 7], whereby coherent control over the system of interest is achieved solely based on suitably designed time-dependent modulation (most commonly sequences of control pulses), without access to ancillary quantum resources and/or measurement and feedback. While open-loop Hamiltonian engineering techniques have their origin and a long tradition in nuclear magnetic resonance (NMR) [8, 9], the underlying physical principles of 'coherent averaging' have recently found widespread use in the context of quantum information processing (QIP), leading in particular to dynamical symmetrization and dynamical decoupling (DD) schemes for control and decoherence suppression in open quantum systems [10–15].

As applications for both universally programmable ('digital') and purpose-built ('analog') quantum simulators continue to emerge across physics and chemistry [3, 4, 16–18], and implementations become closer to experimental reality [19–21], it is imperative to expand the repertoire of available quantum-simulation procedures, and scrutinize the validity of the underlying control assumptions. While existing approaches differ considerably in their details and an extended comparison is not our scope here, we are specifically interested in advancing open-loop (analog) Hamiltonian simulation schemes which, as mentioned, employ purely unitary control resources. With a few exceptions (notably, the use of so-called 'perturbation theory gadgets' [22]), such schemes have relied thus far on the ability to implement sequences of effectively *instantaneous*, 'bang-bang' (BB) control pulses [23–31]. Although this is a convenient and often reasonable first approximation, instantaneous pulses necessarily involve unbounded control amplitude and/or power, something which is out of reach for many control devices of interest and is fundamentally unphysical. In the context of DD, a general approach for achieving (to at least the leading order) the same dynamical symmetrization as in the BB limit was proposed in [32], based on the idea of continuously applying bounded-strength control Hamiltonians according to an Eulerian cycle, so-called *Eulerian DD* (EDD). From a Hamiltonian engineering perspective, EDD protocols translate directly into bounded-strength simulation schemes for *specific* effective Hamiltonians—most commonly, the trivial (zero) Hamiltonian in the case of 'non-selective averaging' for quantum memory (or 'time-suspension' in NMR terminology). More recently, EDD has also served as the starting point for constructing bounded-strength *gate simulation* schemes in the presence of decoherence, so-called *dynamically corrected gates* (DCGs) for universal quantum computation [33–36].

In this work, we show that the approach of Eulerian control can be further systematically exploited to construct *bounded-strength Hamiltonian simulation schemes* for a broad class of target evolutions on both closed and open (finite-dimensional) quantum systems. In addition to being device-independent, our approach requires rather limited control resources—basically, only *local* (single-qubit) Hamiltonians with bounded control strength, applied to a suitable subset of target qubits. As such, Eulerian simulation protocols may substantially expand the control toolbox for programming complex Hamiltonians into a broad variety of existing or near-term quantum simulators, subject to realistic control assumptions.

The content is organized as follows. We begin in section 2 by introducing the appropriate control-theoretic framework and by reviewing the basic principles of open-loop simulation via average Hamiltonian theory, along with its application to Hamiltonian simulation in the BB setting. Section 3 is devoted to constructing and analyzing simulation schemes that employ bounded-strength controls: while section 3.1 reviews the required background on EDD, section 3.2 introduces our new Eulerian simulation protocols for a general closed quantum system, and provides an explicit application to a simple two-qubit example. In section 3.3 we address the important problem of Hamiltonian simulation in the presence of slowly-correlated (non-Markovian) decoherence, by identifying conditions under which a desired Hamiltonian may be enacted on the target system while simultaneously decoupling the latter from its environment, and by contrasting Eulerian simulation protocols with DCGs. Section 4 presents a number of illustrative applications of Eulerian simulation in interacting multi-qubit networks. In particular, we provide explicit protocols to simulate a large family of two-body Hamiltonians in Heisenberg-coupled spin systems additionally exposed to arbitrary linear depolarization or dephasing, as well as to achieve Kitaev's honeycomb lattice Hamiltonian starting from Ising-coupled qubits. We conclude in section 5.

2. Principles of Hamiltonian simulation

2.1. Control-theoretic framework

We consider a quantum system $\mathcal{S}$, with associated Hilbert space $\mathcal{H}$, whose evolution is described by a time-independent Hamiltonian H. As mentioned, *Hamiltonian simulation* is the task of making $\mathcal{S}$ evolve under some other time-independent target Hamiltonian, say, $\tilde{H}$. Without loss of generality, both the input and the target Hamiltonians may be taken to be traceless. Two related scenarios are worth distinguishing for QIP purposes:

- Closed-system simulation, in which case $\mathcal{S}$ coincides with the quantum system of interest, S (also referred to as the 'target' henceforth), which undergoes purely *unitary* (coherent) dynamics;
- Open-system simulation, in which case $\mathcal{S}$ is a bipartite system on $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_B$, where B represents an uncontrollable environment (also referred to as 'bath' henceforth), and the reduced dynamics of the target system S is *non-unitary*.

In both cases, we shall assume the target system S to be a network of interacting qudits, hence $\mathcal{H}_S \simeq \left(\mathbb{C}^d\right)^{\otimes n}$, for finite d and n. In the general open-system scenario, the joint Hamiltonian on $\mathcal{H}$ may be expressed in the following form:

$$H = H_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + \sum_\alpha S_\alpha \otimes B_\alpha, \tag{1}$$

where the operators H_S (H_B) and S_α (B_α) act on $\mathcal{H}_S$ ($\mathcal{H}_B$) respectively, and all the bath operators are assumed to be norm-bounded, but otherwise unspecified (potentially unknown). The closed-system setting is recovered from equation (1) in the limit $S_\alpha \equiv 0$. Likewise, we may express the target Hamiltonian $\tilde{H}$ in a similar form, with two simulation tasks being of special relevance: $\tilde{S}_\alpha \equiv 0$, in which case the objective is to realize a desired system Hamiltonian $\tilde{H}_S$ while dynamically decoupling S from its bath B, thereby suppressing unwanted decoherence [12]; or, more generally, $H_S \mapsto \tilde{H}_S$ *and* $S_\alpha \mapsto \tilde{S}_\alpha$, where the simulated, dynamically symmetrized error generators $\tilde{S}_\alpha$ may for instance allow for decoherence-free subspaces or subsystems to exist [14, 37].

The free dynamics is modified by an open-loop controller acting on the target system according to

$$H \mapsto H(t) = H + H_c(t), \quad H_c(t) \equiv \sum_u h_u(t) = \sum_u f_u(t) X_u, \tag{2}$$

where the operators $\left\{X_u = X_u^\dagger\right\}$ and the (real) functions $\left\{f_u(t)\right\}$ represent the available control Hamiltonians and the corresponding, generally time-dependent, control inputs respectively. Let t and $\tilde{t}$ denote the actual and the simulated time, respectively, where in general we allow for $t \neq \tilde{t}$ in order to account for time-overhead in the simulation—for instance, an overall scale factor $t \equiv s\tilde{t}$, with $s > 0$, in the simplest case [26, 27]. Clearly, if the Hamiltonian $\left(\tilde{H} - H\right)$ is contained in the admissible control set, the corresponding simulation problem is trivial and the desired time-evolution,

$$\tilde{U}\left(\tilde{t}\right) = e^{-i\tilde{H}\tilde{t}} = e^{-iHt}, \quad t \geqslant 0,$$

can be exactly simulated continuously in time, with no overhead. However, this level of control need not be available in settings of interest, including open quantum systems where control actions are necessarily restricted to the target system S alone, hence formally $H_c(t) \equiv H_c(t) \otimes \mathbb{I}_B$ in equation (2). In line with the general idea of 'analog' quantum simulation [3], we shall assume in what follows a *restricted* set of control Hamiltonians (in a sense to be made more precise later) and focus on the task of *approximately* simulating the desired time evolution $\tilde{U}(t)$ at a single *final time* $t = \tilde{T}_f$, or more generally, *stroboscopically* at multiple times, that is, at instants $t = \tilde{t}_M$, where

$$\tilde{t}_M = M\tilde{T}, \quad M \in \mathbb{N},$$

and $\tilde{T}$ is a fixed minimum time interval. Choosing $\tilde{T}$ sufficiently small allows in principle any desired accuracy in the approximation to be met, with the limit $\tilde{T} \to 0$ formally recovering the continuous limit.

Specifically, let $U(t)$ and $U_c(t)$ denote the unitary propagators associated to the total and the control Hamiltonians in equation (2), respectively:

$$U(t) = \mathcal{T} \exp\left\{-i \int_0^t \left[H + H_c(\tau)\right] \mathrm{d}\tau\right\}, \tag{3}$$

$$U_c(t) = \mathcal{T}\exp\left\{-i\int_0^t H_c(\tau)\,\mathrm{d}\tau\right\}, \tag{4}$$

where we have set $\hbar = 1$ and $\mathcal{T}$ indicates time-ordering, as usual. Then, for a given pair $\left(H, \tilde{H}\right)$, we shall provide sufficient conditions for $\tilde{H}$ to be 'reachable' from H and, if so, devise a *cyclic* control procedure such that the resulting controlled propagator

$$U(t_M) \approx \tilde{U}\left(\tilde{t}_M\right), \qquad t_M = MT_c\,, \qquad M \in \mathbb{N}, \tag{5}$$

where T_c is the cycle time of the controller, that is, $U_c\left(t + T_c\right) = U_c(t)$. If, for a *fixed* input Hamiltonian H, *arbitrary* target Hamiltonians are reachable for given control resources, the simulation scheme is referred to as *universal*. In this case, complete controllability must be ensured by the tunable Hamiltonians X_u in conjunction with the always-on 'drift' H_S [7]. In contrast, we shall be especially interested in situations where control over S is more limited.

Similar to DD protocols, Hamiltonian simulation protocols are most easily constructed and analyzed by effecting a transformation to the 'toggling' frame associated to $U_c(t)$ in equation (4) [8, 12, 15]. That is, evolution in the toggling frame is generated by the time-dependent, control-modulated Hamiltonian

$$H'(t) = U_c^\dagger(t) H U_c(t), \tag{6}$$

with the toggling-frame propagator $U'(t)$ being related to the physical propagator in equation (3) by $U(t) = U_c(t)\,U'(t)$. Since $U_c(t)$ is cyclic and H is time-independent, $H'(t)$ acquires the periodicity of the controller, thus it follows that $U(t_M) = U'(t_M) = \left[U'\left(T_c\right)\right]^M$. The stroboscopic controlled dynamics of $\mathcal{S}$ is then determined by

$$U(t_M) = \left[U'\left(T_c\right)\right]^M\,. \tag{7}$$

Average Hamiltonian theory [8, 41] may be invoked to associate an effective *time-independent* Hamiltonian $\bar{H}$ to the evolution in the toggling-frame:

$$U\left(T_c\right) = U'\left(T_c\right) \equiv \exp\left(-i\bar{H}T_c\right), \tag{8}$$

where $\bar{H}$ is determined by the Magnus expansion [38], $\bar{H} = \bar{H}^{(0)} + \bar{H}^{(1)} + \bar{H}^{(2)} + \ldots$ Explicitly, the leading-order term, determining evolution over a cycle up to the first order in time, is given by

$$\bar{H}^{(0)} = \frac{1}{T_c}\int_0^{T_c} H'(\tau)\,\mathrm{d}\tau = \frac{1}{T_c}\int_0^{T_c} U_c^\dagger(\tau) H U_c(\tau)\,\mathrm{d}\tau\,, \tag{9}$$

with (absolute) convergence being ensured as long as $t\left\|H\right\| < \pi$ [40]. Subject to convergence condition, higher-order corrections for evolution over time t can also be upper-bounded by (see lemma 4 in [39])

$$\left\|\sum_{\ell=\kappa}^{\infty} t\bar{H}^{(\ell)}\right\| \leqslant c_\kappa\left[\left(t\left\|H\right\|\right)^{\kappa+1}\right], \qquad c_\kappa = O(1). \tag{10}$$

Ideally, one would like to achieve $\bar{H}T_c = \tilde{H}\tilde{T}$, so that equality would hold in equation (5) for all $M \in \mathbb{N}$. In what follow, we shall primarily focus on achieving *first-order simulation*

instead, by engineering the control propagator $U_c(t)$ in such a way that

$$\bar{H}T_c \approx \bar{H}^{(0)}T_c = \tilde{H}\tilde{T}, \tag{11}$$

whereby, using equation (10) with $\kappa = 1$,

$$U(t_M) = e^{-i\bar{H}MT_c} = e^{-i\bar{H}^{(0)}MT_c} + O\left[\left(MT_c\left\|H\right\|\right)^2\right] \approx \tilde{U}(\tilde{t}_M). \tag{12}$$

In general, the accuracy of the approximation in equation (11) improves as the 'fast control limit', $T_c \to 0$, is approached. Physically, this corresponds to requiring that the shortest control time scale (e.g., pulse separation) involved in the control sequence be sufficiently small relative to the shortest correlation time of the dynamics induced by H [41, 42]. While the problem of constructing arbitrary high-order Hamiltonian simulation schemes is of separate interest, *second-order simulation* can be readily achieved, in principle, by ensuring that $U_c(t)$ is time-symmetric, namely, $U_c(t) = U_c(T_c - t)$ for $t \in [0, T_c]$. Since all odd-order Magnus corrections vanish in this case [41], it follows (by using again equation (10), with $\kappa = 2$), that $\bar{H}T_c = \tilde{H}\tilde{T} + O\left[\left(\left\|H\right\|T_c\right)^3\right]$, as desired.

2.2. Hamiltonian simulation with BB controls

BB Hamiltonian simulation provides the simplest control setting for achieving the intended objective, given in equation (5). Two main assumptions are involved: (i) first, we must be able to express the target Hamiltonian $\tilde{H}$ as

$$\tilde{H} = \sum_{j=1}^{N} w_j U_j^\dagger H U_j, \qquad W \equiv \sum_j w_j > 0\,, \tag{13}$$

where $\{U_j\}$ are unitary operators on S and the $\{w_j\}$ non-negative real numbers (not all zero). (ii) Second, the available control resources include a discrete set of instantaneous pulses $\{P_j\}$ on S, whose application results in a piecewise-constant control propagator $U_c(t)$ over $[0, T_c]$, with corresponding toggling-frame propagators $\{U_j\}$, $U_j \equiv \prod_{k=1}^{j} P_k$, $U_1 = \mathbb{1}_S$ [10, 15]. Assumptions (i)–(ii) together allow for the time-average in equation (9) to be mapped to a convex (positive-weighted) sum. Equation (13) may be interpreted as a *sufficient* condition for the target Hamiltonian $\tilde{H}$ to be reachable from H given open-loop unitary control on S alone. Reachable Hamiltonians must thus be at least as 'disordered' as the input one in the sense of majorization [15, 26, 27].

Specifically, equation (13) leads naturally to the following BB simulation scheme. Given simulation weights $\{w_j\}$, define the following simulation intervals and timing pattern:

$$\tau_j \equiv w_j\tilde{T}, \qquad t_j \equiv \sum_{k=1}^{j}\tau_k, \qquad t_0 = 0, \qquad t_N \equiv T_c = W\tilde{T}. \tag{14}$$

A piecewise-constant control propagator for the basic simulation block to be repeated may then be constructed as follows:

$$U_c^{\mathrm{BB}}\left(t_{j-1}+\theta\right)=U_j\,,\quad \theta\in\left[0,\,\tau_j\right],\quad j=1,\,\ldots,\,N. \tag{15}$$

By using equation (9), it is immediate to verify that

$$\bar{H}^{(0)}=\frac{1}{T_c}\sum_{j=1}^{N}\tau_j U_j^{\dagger}HU_j=\frac{\tilde{T}}{T_c}\tilde{H}, \tag{16}$$

implementing the desired evolution, equations (11) and (12), with time overhead $s\equiv W$, provided that the convergence conditions for first-order simulation under H are obeyed. Since in practice, even in the absence of any control errors as we consider, technological limitations always constrain the cycle duration to a *finite* minimum value $T_c>0$ [42], such convergence conditions upper-bound the maximum simulated time $\tilde{t}_M$ up to which evolution under $\tilde{H}$ may be reliably simulated using the physical Hamiltonian H.

In analogy with BB DD schemes, realizing the prescription of equation (15) requires to discontinuously change the control propagator from U_j to $U_{j+1}=\left(U_{j+1}U_j^{\dagger}\right)U_j$, via an instantaneous BB pulse $U_{j+1}U_j^{\dagger}$ at the jth endpoint t_j. As a result, despite its conceptual simplicity, BB simulation is unrealistic whenever large control amplitudes are not an option, and the evolution induced by H *during* the application of a control pulse must be considered from the outset. This demands redesigning the basic control block in such a way that the actions of H and $H_c(t)$ are simultaneously accounted for.

3. Hamiltonian simulation with bounded controls

3.1. Eulerian simulation of the trivial Hamiltonian

The key to overcome the disadvantages of BB Hamiltonian simulation is to ensure that the control propagator varies continuously in time during each control cycle. We achieve this goal by invoking *Eulerian control design* [32]. We begin by revisiting how, for the special case of a target identity evolution (that is, $\tilde{H}\equiv 0$, also corresponding to a 'noop' gate, in terms of the end-time simulated propagator), EDD can be naturally interpreted as a bounded-strength simulation scheme.

In the Eulerian approach, the available control resources include a discrete set of unitary operations on S, say, $\left\{U_\gamma\right\}$, $\gamma=1,\,\ldots,\,L$, which are realized over a finite time interval Δ through application of bounded-strength control Hamiltonians $\left\{h_\gamma(t)\right\}$, $\gamma=1,\,\ldots,\,L$, with $\left|h_\gamma(t)\right|\leqslant h_{\max}<\infty$. That is,

$$U_\gamma\equiv u_\gamma(\Delta),\quad u_\gamma(\delta)=\mathcal{T}\exp\left\{-i\int_0^{\delta}h_\gamma(\tau)\,\mathrm{d}\tau\right\}. \tag{17}$$

Note that the choice of the control Hamiltonians $h_\gamma(t)$ is not unique, which allows for implementation flexibility. The unitaries $\left\{U_\gamma\right\}$ are identified with the image of a generating set of a finite group under a faithful, unitary, projective representation ρ [32]. That is, let $\mathcal{G}\equiv\{g\}$ be a finite group of order $\left|\mathcal{G}\right|$, such that each element may be written as an ordered product of elements in a generating set $\Gamma\equiv\{\gamma\}$ of order $\left|\Gamma\right|=L$, $g\mapsto\rho(g)\equiv U_g$ be the representation

map[4], and $G \equiv \{U_g\}$. The *Cayley graph* $C(\mathcal{G}, \Gamma)$ of $\mathcal{G}$ relative to Γ can be thought of as pictorially representing all elements of $\mathcal{G}$ as strings of generators in Γ. Each vertex represents a group element and a vertex g is connected to another vertex g' by a directed edge 'colored' (labeled) with generator γ if and only if $g' = \gamma g$. The number of edges in $C(\mathcal{G}, \Gamma)$ is thus equal to $N \equiv |\Gamma|\,|\mathcal{G}|$. Because a Cayley graph is regular, it always has an *Eulerian cycle* that visits each edge exactly once and starts (and ends) on the same vertex [43, 44]. Let us denote with $C \equiv (\gamma_1, \ldots, \gamma_N)$ the ordered list of generators defining an Eulerian cycle on $C(\mathcal{G}, \Gamma)$ which, without loss of generality, starts (and ends) at the identity element of $\mathcal{G}$.

Once a control Hamiltonian for implementing each generator as in equation (17) is chosen, an EDD protocol is constructed by assigning a cycle time as $T_c \equiv N\Delta$ and by applying the control Hamiltonians $h_\gamma(t)$ sequentially in time, following the order determined by the Eulerian cycle C. Thus,

$$U_c^{\mathrm{EDD}}(t_j) = U_{\gamma_j} U_c^{\mathrm{EDD}}(t_{j-1}), \quad j = 1, \ldots, N, \tag{18}$$

where U_{γ_j} is the image of the generator labeling the jth edge in C. As established in [32], the lowest-order average Hamiltonian associated to the above EDD cycle has the form $\bar{H}^{(0)} \equiv \Pi_{\mathcal{G}}[F_\Gamma(H)]$, where for any operator A acting on $\mathcal{H}_S$, the map

$$\Pi_{\mathcal{G}}(A) = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} U_g^\dagger A U_g \tag{19}$$

projects onto the centralizer of $\mathcal{G}$ (i.e., $\Pi_{\mathcal{G}}(A)$ commutes with all $U_g \in G$), and

$$F_\Gamma(H) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{1}{\Delta}\int_0^\Delta u_\gamma(\tau)^\dagger H u_\gamma(\tau)\,\mathrm{d}\tau \tag{20}$$

implements an average of H over both the control interval and the group generators. Accordingly, bounded-strength simulation of $\tilde{H} = 0$ is achieved provided that the following DD condition is obeyed:

$$\Pi_{\mathcal{G}}[F_\Gamma(H)] = 0. \tag{21}$$

By Schur's lemma, this is automatically ensured if the group representation acts irreducibly on $\mathcal{H}_S$. Formally, the BB limit may be recovered by letting $F_\Gamma(A) \equiv A$ for all A [32], reflecting the ability to directly implement all the group elements (with no overhead, as if $|\Gamma| = 1$) and corresponding to uniform simulation weights $w_j = 1/|\mathcal{G}|$.

3.2. Eulerian simulation protocols beyond noop: construction

We now show how the Eulerian cycle method can be extended to bounded-strength simulation of a non-trivial class of target Hamiltonians. We assume that $\tilde{H}$ may be expressed as a convex unitary mixture of the group representatives U_g,

[4] Recall that a projective representation need only be a homomorphism up to phase, i.e., it obeys $U_{gg'} \propto U_g U_{g'}$ for $g, g' \in \mathcal{G}$, with proportionality rather than equality.

	Δ	Θ_1	Δ	Θ_2	Δ	Θ_3	...	Δ	Θ_N
Hamiltonian, $H_c(t)$	$h_{\gamma_1}(\delta)$	0	$h_{\gamma_2}(\delta)$	0	$h_{\gamma_3}(\delta)$	0	...	$h_{\gamma_N}(\delta)$	0
Unitary, $U_c(t)$	$u_{\gamma_1}(\delta)$	U_{g_1}	$u_{\gamma_2}(\delta)U_{g_1}$	U_{g_2}	$u_{\gamma_3}(\delta)U_{g_2}$	U_{g_3}	...	$u_{\gamma_N}(\delta)U_{g_{N-1}}$	$\mathbb{1}$

Figure 1. Schematics of an Eulerian simulation protocol. The basic control block consists of N time intervals, each involving a 'ramping-up' sub-interval of fixed duration Δ, during which $H_c(t) \neq 0$, followed by a 'coasting' (free evolution) period of variable duration Θ_k, equation (24), during which no control is applied. During the jth ramping-up sub-interval we apply h_{γ_j}, i.e., the control Hamiltonian that realizes the generator γ_j, smoothly changing the control propagator from $U_{g_{j-1}}$ to U_{g_j}. In this way, the control protocol corresponding to equations (26)–(27) is implemented. By construction, a standard EDD cycle with $\tilde{H} = 0$ is recovered by letting $\Theta_k \to 0$ for all k, while in the limit $\Delta \to 0$ standard BB simulation of $\tilde{H}$ is implemented.

$$\tilde{H} = \sum_{g\in\mathcal{G}} w_g U_g^\dagger H U_g,\ w_g \geqslant 0,\quad W \equiv \sum_g w_g > 0\,. \tag{22}$$

We construct the desired protocol starting from an Eulerian cycle $C = \left(\gamma_1, \ldots, \gamma_N\right)$ on $C\left(\mathcal{G}, \Gamma\right)$. The idea is to append to each of the N control slots that define an EDD scheme a free-evolution (or 'coasting') period of suitable duration, in such a way that the net simulated Hamiltonian is modified from $\tilde{H} = 0$ to $\tilde{H} \neq 0$ as given in equation (22). A pictorial representation of the basic control block is given in figure 1. As in equation (17), let Δ denote the minimum time duration required to implement each generator, hence, to smoothly change the control propagator from U_g to $U_{g'}$ along the cycle. While such 'ramping up' control intervals have all the same length, each 'coasting' interval is designed to keep the control propagator constant at $U_{g'}$ for a duration determined by the corresponding weight $w_{g'}$. Since the control is switched off during coasting, continuity of the control Hamiltonian $H_c(t)$ may be ensured, if desired, by additionally requiring that

$$h_\gamma(0) = 0 = h_\gamma(\Delta), \quad \gamma = 1, \ldots, L. \tag{23}$$

An Eulerian simulation protocol may be specified as follows. As before, let the jth time interval be denoted as $\left[t_{j-1}, t_j\right]$, $j = 1, \ldots, N$, with $t_0 \equiv 0$ and $t_N \equiv T_c$. For each j, let $\tau_{g_j} \equiv w_{g_j}\tilde{T}$ as in the BB case. We assign the duration of the jth coasting period as

$$\Theta_j \equiv \frac{\tau_{g_j}}{|\Gamma|}, \tag{24}$$

resulting in the following timing pattern $\left\{t_j\right\}$ (compare to equation (14)):

$$t_j = \sum_{k=1}^{j}\left(\Delta + \Theta_k\right) = j\Delta + \frac{1}{|\Gamma|}\sum_{k=1}^{j}\tau_{g_k},\ t_N = T_c = N\Delta + W\tilde{T}\,. \tag{25}$$

As the expression for the cycle times makes it clear, the resulting protocol may be equivalently interpreted in two ways: starting from an EDD cycle, corresponding to $N\Delta$ and $\tilde{H} = 0$, we

introduce the coasting periods to allow for non-trivial simulated dynamics to emerge; or, starting from a BB simulation scheme for $\tilde{H}$, corresponding to $W\tilde{T}$, we introduce the ramping-up periods to allow for control Hamiltonians to be smoothly switched over Δ. Either way, bounded-strength protocols imply a time-overhead $N\Delta$ relative to the BB case, recovering the BB limit as $\Delta \to 0$ as expected. Explicitly, the control propagator for Eulerian simulation has the form:

$$U_c^{\mathrm{EUS}}\left(t_{j-1}+\delta\right) = u_{\gamma_j}(\delta)\, U_{g_{j-1}} \quad \text{for } \delta \in [0,\, \Delta]\,, \tag{26}$$

$$U_c^{\mathrm{EUS}}\left(t_{j-1}+\Delta+\theta\right) = U_{g_j} \quad \text{for } \theta \in \left[0,\, \Theta_j\right]. \tag{27}$$

The resulting first-order Hamiltonian $\bar{H}^{(0)}$ under Eulerian simulation is derived by evaluating the time-average in equation (9) with the control propagator given by equations (26)–(27). Direct calculation along the lines of [32] yields:

$$\begin{aligned}
\bar{H}^{(0)} &= \frac{1}{T_c}\sum_{j=1}^{N}\Bigg[\int_{\delta=0}^{\Delta} U_c\left(t_{j-1}+\delta\right)^{\dagger} H U_c\left(t_{j-1}+\delta\right)\mathrm{d}\delta \\
&\qquad + \int_{\theta=0}^{\Theta_j} U_c\left(t_{j-1}+\Delta+\theta\right)^{\dagger} H U_c\left(t_{j-1}+\Delta+\theta\right)\mathrm{d}\theta\Bigg] \\
&= \frac{1}{T_c}\sum_{j=1}^{N}\left[\int_{\delta=0}^{\Delta} U_{g_{j-1}}^{\dagger} u_{\gamma_j}(\delta)^{\dagger} H u_{\gamma_j}(\delta)\, U_{g_{j-1}}\mathrm{d}\delta + \int_{\theta=0}^{\Theta_j} U_{g_j}^{\dagger} H U_{g_j}\mathrm{d}\theta\right] \\
&= \frac{1}{T_c}\sum_{g\in\mathcal{G}}\left[U_g^{\dagger}\left(\sum_{\gamma\in\Gamma}\int_{\delta=0}^{\Delta} u_{\gamma}(\delta)^{\dagger} H u_{\gamma}(\delta)\,\mathrm{d}\delta\right) U_g + |\Gamma|\,\frac{\tau_g}{|\Gamma|}\, U_g^{\dagger} H U_g\right],
\end{aligned}$$

where the last equality follows from two basic properties of Eulerian cycles: firstly, the list $\{g_0, g_1, \ldots, g_{N-1}\}$ (and also $\{g_1, g_2, \ldots, g_N\}$) of the vertices that are being visited contains each element $g \in \mathcal{G}$ precisely $|\Gamma|$ times; secondly, in traversing the Cayley graph, each group element g is left exactly once by a γ-labeled edge for each generator $\gamma \in \Gamma$. Thus, by recalling the definitions given in equations (19) and (20), we finally obtain

$$\bar{H}^{(0)} = \frac{N\Delta}{T_c}\Pi_{\mathcal{G}}\left[F_{\Gamma}(H)\right] + \frac{\tilde{T}}{T_c}\sum_{g\in\mathcal{G}} w_g U_g^{\dagger} H U_g = \frac{\tilde{T}}{T_c}\tilde{H}\,, \tag{28}$$

which indeed achieves the intended first-order simulation goal, equations (11)–(12), as long as convergence holds and the DD condition of equation (21) is obeyed.

The simulation accuracy may be improved by symmetrizing $U_c^{\mathrm{EUS}}(t)$ in time. In analogy to symmetrized EDD protocols [10], this can be easily accomplished by running the protocol and then suitably running it again in reverse. Specifically, let the duration of the coasting interval be changed as $\Theta_j \mapsto \Theta_j/2$. Run the protocol as described above until time $t = N\Delta + \frac{1}{2}W\tilde{T}$. Then, from time $t = N\Delta + \frac{1}{2}W\tilde{T}$ until time $t = T_c = 2N\Delta + W\tilde{T}$, modify equations (26)–(27) as follows:

$$U_c^{\mathrm{EUS}}\left[T_c - \left(t_{j-1} + \Delta\right) + \delta\right] = u_{\gamma_j}\left(\Delta - \delta\right) U_{g_{j-1}} \quad \text{for } \delta \in \left[0, \Delta\right],$$
$$U_c^{\mathrm{EUS}}\left[T_c - \left(t_{j-1} + \Delta + \Theta_j\right) + \theta\right] = U_{g_j} \quad \text{for } \theta \in \left[0, \Theta_j\right],$$

for $j = N, \ldots, 1$. Provided that one is able to implement $u_{\gamma_j}\left(\Delta - \delta\right)$, we again obtain

$$\bar{H}^{(0)} = 2\frac{N\Delta}{T_c}\Pi_{\mathcal{G}}\left[F_\Gamma\left(H\right)\right] + \frac{\tilde{T}}{T_c}\sum_{g\in\mathcal{G}} w_g U_g^\dagger H U_g\,,$$

while satisfying $U_c\left(t\right) = U_c\left(T_c - t\right)$ for $t \in \left[0, T_c\right]$, hence ensuring that $\bar{H}^{(1)} = 0$.

Example. By way of concrete illustration, it is useful to consider an explicit first-order Eulerian simulation scheme, in the simplest instance of $n = 2$ qubits. Specifically, assume that the physical Hamiltonian is an isotropic Heisenberg coupling of the form

$$H = H_{\mathrm{iso}} = J\left(X\otimes X + Y\otimes Y + Z\otimes Z\right) \equiv J\left(X_1X_2 + Y_1Y_2 + Z_1Z_2\right),$$

where J has units of energy and the third equality defines an equivalent compact notation. We are interested in a class of target XYZ Hamiltonians of the form

$$\tilde{H} = H_{XYZ} = J_xX_1X_2 + J_yY_1Y_2 + J_zZ_1Z_2, \quad J_u \in \mathbb{R}. \tag{29}$$

For instance, $J_x = J_y = \pm J$, $J_z = 0$ corresponds to an isotropic XX model, whereas if $J_x = J_y$ with $J_z \neq 0$, an XXZ interaction is obtained, the special value $J_z = \mp 2J$ corresponding to the important case of a dipolar Hamiltonian. The construction of a simulation protocol starts from observing that Hamiltonians as in equation (29) are reachable from H, in the sense of equation (22), based on *single-qubit control only*.

Specifically, let $\mathcal{G} \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \equiv \mathbb{Z}_2^2$, and let the representation ρ map $(n, m) \in \mathcal{G}$ to $X^nZ^m \otimes \mathbb{I}$. That is, $\mathcal{G}$ is mapped to the following set of unitaries:

$$\left\{U_g\right\} = \left\{\mathbb{I}\otimes\mathbb{I}, X\otimes\mathbb{I}, Y\otimes\mathbb{I}, Z\otimes\mathbb{I}\right\} \equiv G_1 = \left\{\mathbb{I}, X_1, Y_1, Z_1\right\}. \tag{30}$$

Choosing the generators of $\mathcal{G}$ to be $\left(1, 0\right) \mapsto \gamma_{x,1} = X_1$ and $\left(0, 1\right) \mapsto \gamma_{z,1} = Z_1$, we assume that we have access to the control Hamiltonians

$$h_x\left(t\right) = f_x\left(t\right)X_1 \quad \text{and} \quad h_z\left(t\right) = f_z\left(t\right)Z_1\,,$$

where the control inputs $f_x\left(t\right)$ and $f_z\left(t\right)$ satisfy $f_u\left(0\right) = 0 = f_u\left(\Delta\right)$ and $\int_0^\Delta f_u\left(\tau\right)\mathrm{d}\tau = \pi/2$, for $u = x, z$. Recalling equation (17), this yields the control propagators

$$u_x\left(\delta\right) = \cos\left[\int_0^\delta f_x\left(\tau\right)\mathrm{d}\tau\right]\mathbb{I} - i\sin\left[\int_0^\delta f_x\left(\tau\right)\mathrm{d}\tau\right]X_1,$$
$$u_z\left(\delta\right) = \cos\left[\int_0^\delta f_z\left(\tau\right)\mathrm{d}\tau\right]\mathbb{I} - i\sin\left[\int_0^\delta f_z\left(\tau\right)\mathrm{d}\tau\right]Z_1,$$

with $u_x\left(\Delta\right) = X_1$ and $u_z\left(\Delta\right) = Z_1$ (up to phase), as desired.

Note that for any single-qubit Hamiltonians A and B, averaging over the unitary group in equation (30) results in the following projection super-operator:

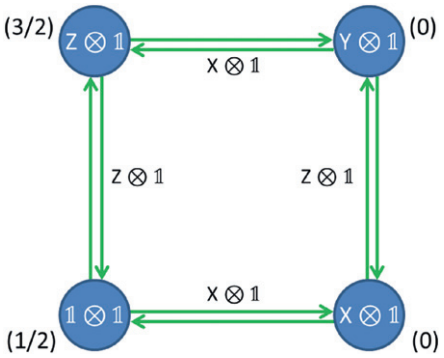

Figure 2. Cayley graph for the Eulerian simulation of the dipolar Hamiltonian in Heisenberg-coupled qubits. Vertices are labeled by group elements; edges are labeled by group generators. Numbers in parentheses next to vertices indicate the weights w_g of the corresponding group elements g in equation (29), which is proportional to the time $\tau_g = w_g \tilde{T}$ spent at vertex g during the coasting subinterval; see also figure 1.

$$\Pi_{\mathcal{G}}\left(A \otimes B\right) = \frac{1}{4}\sum_{U\in\{\mathbb{1},X,Y,Z\}} U^{\dagger}AU \otimes B = \frac{1}{2}\,\mathrm{tr}\left(A\right)\mathbb{1} \otimes B. \tag{31}$$

In general, the map F_Γ is trace-preserving and, in this case, it acts non-trivially only on the first qubit. Thus, F_Γ is trace-preserving on the first qubit. Since each term in H is traceless in the first qubit, the decoupling condition $\Pi_{\mathcal{G}}\left[F_\Gamma\left(H\right)\right] = 0$ follows directly from equation (31), even though the relevant representation ρ is, manifestly, reducible.

Having satisfied our main requirements, reachability of *XYZ* Hamiltonians as in equation (29) is equivalent to the existence of a solution to the following set of conditions:

$$\begin{aligned} J\left(w_{\mathbb{1}} + w_{X_1} - w_{Y_1} - w_{Z_1}\right) &= J_x, \\ J\left(w_{\mathbb{1}} - w_{X_1} + w_{Y_1} - w_{Z_1}\right) &= J_y, \\ J\left(w_{\mathbb{1}} - w_{X_1} - w_{Y_1} + w_{Z_1}\right) &= J_z, \end{aligned} \tag{32}$$

for non-negative weights w_g. While infinitely many choices exist in general, minimizing the total weight $W = \sum_g w_g$ keeps the simulation time overhead to a minimum. For instance, it is easy to verify that a dipolar Hamiltonian of the form

$$\tilde{H} = H_{\mathrm{dip}} = -J\left(X_1X_2 + Y_1Y_2 - 2Z_1Z_2\right)$$

may be simulated with minimum time overhead by choosing weights

$$w_{\mathbb{1}} = \frac{1}{2}, \quad w_{X_1} = 0 = w_{Y_1}, \quad w_{Z_1} = \frac{3}{2}.$$

The Cayley graph associated with the resulting Eulerian simulation protocol is depicted in figure 2, with the explicit timing structure of the control block as in figure 1 and $N = 2 \times 4 = 8$ control segments per block. It is worth observing that although the weights w_{X_1} and w_{Y_1} are zero in the particular case at hand, *all* group members of $\mathcal{G}$ are nonetheless required, and the

unitaries X_1 and Y_1 still show up in the simulation scheme (during the ramping-up sub-intervals, as evident from equation (26)). This is crucial to guarantee that the unwanted F_Γ term is projected out.

3.3. Eulerian simulation while decoupling from an environment

The ability to implement a desired Hamiltonian on the target system S, while switching off (at least to the leading order) the coupling to an uncontrollable environment B, is highly relevant to realistic QIP applications. That is, with reference to equation (1), the objective is now to *simultaneously* achieve $\tilde{H}_S \equiv H_{\text{target}}$, $\tilde{S}_\alpha \equiv 0$, by unitary control operations acting on S alone. Because the first-order Magnus term $\bar{H}^{(0)}$ is additive (recall equation (9)), it is appropriate to treat each summand of H individually, leading to a relevant average Hamiltonian of the form

$$\bar{H}^{(0)} = \bar{H}_S \otimes \mathbb{I}_B + \sum_\alpha \bar{S}_\alpha \otimes B_\alpha + \mathbb{I}_S \otimes H_B,$$

where for a generic operator on $\mathcal{H}_S$ we let

$$\bar{A} \equiv \frac{1}{T_c}\int_0^{T_c} U_c^\dagger(\tau) A U_c(\tau)\,\mathrm{d}\tau.$$

We can then apply the analysis of section 3.2 to the internal system Hamiltonian ($\bar{H}_S$) and each error generator ($\bar{S}_\alpha$) separately, to obtain in both cases a simulated operator of the form given in equation (28):

$$\bar{A} = \frac{N\Delta}{T_c}\Pi_{\mathcal{G}}\left[F_\Gamma(A)\right] + \frac{\tilde{T}}{T_c}\sum_{g\in\mathcal{G}} w_g U_g^\dagger A U_g.$$

Since the task is to decouple S from B while maintaining the non-trivial evolution due to $\tilde{H}_S = H_{\text{target}}$, the reachability condition of equation (22) must now ensure that

$$\tilde{H}_S = \sum_{g\in\mathcal{G}} w_g U_g^\dagger H_S U_g, \tag{33}$$

$$0 = \sum_{g\in\mathcal{G}} w_g U_g^\dagger S_\alpha U_g, \quad \forall\ \alpha. \tag{34}$$

Accordingly, it is necessary to extend the DD assumption of equation (21) to become

$$\Pi_{\mathcal{G}}\left[F_\Gamma(H_S)\right] = 0, \tag{35}$$

$$\Pi_{\mathcal{G}}\left[F_\Gamma(S_\alpha)\right] = 0, \quad \forall\ \alpha, \tag{36}$$

such that $\bar{A} = \left(\tilde{T}/T_c\right)\tilde{A}$ holds for each of the summands in H. Altogether we recover

$$\bar{H}^{(0)} = \frac{\tilde{T}}{T_c}\tilde{H}_S \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B.$$

It is interesting in this context to highlight some similarities and differences with DCGs [33], which also use Eulerian control as their starting point and are specifically designed to achieve a desired unitary evolution on the target system while simultaneously removing decoherence to the leading [33, 34, 36] or, in principle, arbitrarily high order [35]. By

construction, the open-system simulation procedure just described *does* provide a first-order DCG implementation for the target gate $Q \equiv \exp\left(-i\tilde{H}_S\tilde{T}_f\right)$, with $\tilde{T}_f = T_c = N\Delta + W\tilde{T}$: in particular, the requirement that equations (33)–(34) be obeyed together (for the *same* weights w_g) is effectively equivalent to evading the 'no-go theorem' for black-box DCG constructions established in [34], with the coasting intervals and the resulting 'augmented' Cayley graph playing a role similar in spirit to a (first-order) 'balance-pair' implementation. Despite these formal similarities, a key difference between the two approaches is that DCGs focus directly on synthesizing a desired unitary *final-time propagator* independently of the intervening dynamics, as opposed to approximating a desired Hamiltonian *generator* at intermediate times as well. As also discussed in [26, 27], gate synthesis is a weaker simulation notion in general, since inequivalent control protocols may lead to the same end-time propagator. Furthermore, while the internal system Hamiltonian, H_S, is a crucial input in a Hamiltonian simulation problem, it is effectively treated as an unwanted error contribution in analytical DCG constructions, in which case complete controllability over the target system S must be supplied by the controls alone. Although in more general (optimal-control inspired) DCG constructions [36], limited external control is assumed and H_S may become essential for universality to be maintained, emphasis remains, as noted above, on end-time synthesis of a target unitary propagator. Finally, a main intended application of DCGs is realizing low-error *single- and two-qubit gates* for use within fault-tolerant quantum computing architectures, as opposed to robust Hamiltonian engineering for many-body quantum simulators which is our focus here.

3.4. Eulerian simulation protocols: requirements

Before presenting explicit applications, we summarize and critically assess the various requirements that should be obeyed for Eulerian simulation to achieve the intended control objective of equation (5) in a closed or, respectively, open-system setting:

(i) *Time independence*. Both the internal Hamiltonian H and the target Hamiltonian $\tilde{H}$ are taken to be time-independent (and, without loss of generality, traceless).

(ii) *Reachability*. The target Hamiltonian $\tilde{H}$ must be reachable from H, that is, there must be a control group $\mathcal{G}$, with a faithful, unitary projective representation mapping $g \mapsto \rho(g) = U_g$, such that equation (22) holds. For dynamically-corrected Eulerian simulation in the presence of an environment, this requires, as noted, that for the *same* weights $\{w_g\}$, the desired system Hamiltonian is reachable from H_S while the trivial (zero) Hamiltonian is reachable from each error generator S_α separately, such that both equations (33)–(34) hold together.

(iii) *Bounded control*. For each generator γ of the chosen control group $\mathcal{G}$, we need access to *bounded* control Hamiltonians $h_\gamma(t)$, such that application of $h_\gamma(t)$ over a time interval of duration Δ realizes the group representative $U_\gamma = \rho(\gamma) = u_\gamma(\Delta)$, additionally subject (if desired) to the continuity condition of equation (23).

(iv) *Decoupling conditions*. Suitable DD conditions, equation (21) in a closed system or equations (35)–(36) in the open-system error-corrected case, must be fulfilled, in order for undesired contributions to the simulated Hamiltonians to be averaged out by symmetry to the leading order.

(v) *Time-efficiency*. If the choice of $\mathcal{G}$ is not unique for given $\left(H, \tilde{H}\right)$, the smallest group should be chosen, in order to keep the number of intervals per cycle, $N = \left|\mathcal{G}\right| \left|\Gamma\right|$, to a minimum. In particular, *efficient* Hamiltonian simulation requires that $\left|\mathcal{G}\right|$ (hence also $\left|\Gamma\right|$) scales (at most) *polynomially* with the number of subsystems n. If, for fixed $\mathcal{G}$, the choice of the simulation weights $\left\{w_g\right\}$ is not unique, then similar to the BB case the combination with the smallest total weight W should be chosen, in order to minimize the time overhead.

The key simplification that the time-independence assumption (i) introduces into the problem is that the periodicity of the control action is directly transferred to the toggling-frame Hamiltonian of equation (6), allowing one to simply focus on single-cycle evolution. Although this assumption is strictly not fundamental, general time-dependent Hamiltonians may need to be dealt with on a case-by-case basis (see also [45–47]). A situation of special practical relevance arises in this context for open systems exposed to *classical noise*, in which case $\mathcal{H}_B \simeq \mathbb{C}$ and the system-bath interaction in equation (1) is effectively replaced by a classical, time-dependent stochastic field. Similar to DD and DCG schemes, Eulerian simulation protocols remain applicable as long as the noise process is stationary and exhibiting correlations over sufficiently long time scales [10, 48].

The reachability assumption (ii) is a prerequisite for Eulerian Hamiltonian simulation schemes. Although BB Hamiltonian simulation need not be group-based, most BB schemes follow this design principle alike. Assumption (iii), restricting the admissible control resources to *physical* Hamiltonians with bounded amplitude (thus finite control durations, as opposed to instantaneous implementation of arbitrary group unitaries as in the BB case) is a basic assumption of the Eulerian control approach. As remarked, our premise is that the available Hamiltonian control is *limited*, restricted to only the target system if the latter is coupled to an environment, and typically *non-universal* on $\mathcal{H}_S$; in particular, we cannot directly express $\tilde{H} = H + H_c$ and apply $H_c = \tilde{H} - H$, or else the problem would be trivial. In addition to error-corrected Hamiltonian simulation in open quantum systems, scenarios of great practical interest may arise when the control Hamiltonians are subject to more restrictive *locality constraints* than the system and target Hamiltonians are (e.g., two-body simulation with only local controls, see also section 4.1).

The required decoupling conditions in assumption (iv) are automatically obeyed if the representation ρ acts irreducibly on $\mathcal{H}_S$. This follows from Schur's lemma, together with the fact that the map F_Γ defined in equation (20) is trace-preserving, and both H_S and S_α can be taken to be traceless. While convenient, irreducibility is not, however, a requirement, as already demonstrated by the two-qubit example of section 3.2. When the representation ρ is reducible, care must be taken in order to ensure that assumption (iv) is obeyed. It should be stressed that this is possible *independently* of the target Hamiltonian $\tilde{H}$. Therefore, if the choice $(\mathcal{G}, \rho)$ works for one Eulerian simulation scheme (whether ρ is irreducible or not), then it can be used for Eulerian simulation with any target $\tilde{H}$ that belongs to the reachable set from H, that is, that can satisfy equation (22).

We close this discussion by recalling that it is always possible for a finite-dimensional target system S to find a control group $\mathcal{G}$ for which both assumptions (ii) and (iv) are satisfied, by resorting to the concept of a *transformer* [15, 28]. A transformer is a pair $\left(\mathcal{G}, \rho\right)$, where $\mathcal{G}$ is a finite group and $\rho\colon \mathcal{G} \to \mathcal{U}\left(\mathcal{H}_S\right)$, $g \mapsto \rho\left(g\right) = U_g$ is a faithful, unitary, projective

representation such that, for *any* traceless Hermitian operators A and B on $\mathcal{H}_S$ with $A \neq 0$, one may express

$$B = \sum_{g\in\mathcal{G}} w_g U_g^\dagger A U_g, \qquad w_g \geqslant 0.$$

We illustrate this general idea in the simplest case of a single qubit, $\mathcal{H} = \mathcal{H}_S = \mathbb{C}^2$. Let X, Y, Z denote the Pauli matrices and R the unitary matrix defined by

$$R = \frac{i-1}{2}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \tag{37}$$

which corresponds to a rotation by an angle $4\pi/3$ about an axis $\hat{n} \equiv \left(1, 1, 1\right)\big/\sqrt{3}$. Direct calculation shows that $R^3 = I$ and that conjugation by R cyclically shifts the Pauli-matrices, i.e., $R^\dagger X R = Y$, $R^\dagger Y R = Z$, and $R^\dagger Z R = X$. Consider now the group $\mathcal{G}$ given by the presentation

$$\mathcal{G} = \left\langle x, y, z, r \,\middle|\, x^2 = y^2 = z^2 = r^3 = 1,\ xz = y,\ r^{-1}xr = y,\ r^{-1}yr = z,\ r^{-1}zr = x \right\rangle.$$

Using the defining relations of this group, its elements can always be written as $x^a z^b r^c$, where $a, b \in \{0, 1\}$ and $c \in \{0, 1, 2\}$. Clearly, the assignment ρ given by $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$, $r \mapsto R$ yields a faithful, unitary, irreducible projective representation. It is shown in [28] that the pair $(\mathcal{G}, \rho)$ defines a transformer, namely, any 2×2 traceless matrix B may be reached from any fixed 2×2 traceless, non-zero matrix A, for suitable non-negative weights w_g. The irreducibility property for *any* transformer pair can be established by contradiction[5].

Since general transformer groups tend to be large, purely transformer-based simulation schemes are inefficient. In practice, given the system Hamiltonian H_S, the challenge is to find a group $\mathcal{G}$ that grants a reasonably efficient scheme while satisfying assumptions (ii) and (iv), and subject to the ability to implement the required control operations. As we shall see, transformer-inspired ideas may still prove useful in devising simulation schemes in the presence, for instance, of additional symmetry conditions.

4. Illustrative applications

In this section, we analyze different paradigmatic Hamiltonian simulation tasks motivated by QIP applications. While a number of other interesting examples and generalizations may be envisioned (as also further discussed in the Conclusions), our goal here is to give a concrete sense of the usefulness and versatility of our Eulerian simulation approach in physically realistic control settings. In particular, we focus on achieving (first-order) *non-local Hamiltonian simulation using only bounded-strength local (single-qubit) control*, in both closed and open multi-qubit systems.

[5] If ρ were reducible, then there would exist a non-trivial invariant subspace $\mathcal{H}_{\text{inv}} \subset \mathcal{H}_S$ such that $\rho(\mathcal{G})\mathcal{H}_{\text{inv}} \subseteq \mathcal{H}_{\text{inv}}$. Hence, any Hamiltonian of the form $H = \sum_{ij} a_{ij} |v_i\rangle\langle v_j|$, with $\{|v_i\rangle\}$ being an orthonormal basis for $\mathcal{H}_{\text{inv}}$, could only be transformed to other Hamiltonians of this same form, preventing $(\mathcal{G}, \rho)$ from being a transformer.

4.1. Eulerian simulation in closed Heisenberg-coupled qubit networks

We begin by noting that the analysis and simulation protocols described for two Heisenberg-coupled qubits in section 3.2 may be easily generalized to a chain of n qubits (or spins), subject to nearest-neighbor (NN) homogeneous Heisenberg couplings, that is, described by a physical Hamiltonian of the form

$$H = H_{\text{iso}}^{(\text{NN})} = \sum_{i=1}^{n-1} J \left(X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1} \right) \equiv \sum_{i=1}^{n-1} J \, \vec{\sigma}_i \cdot \vec{\sigma}_{i+1},$$

where for later reference we have introduced the standard compact notation $\vec{\sigma}_i \equiv \left(X_i, Y_i, Z_i \right)$ and we assume for concreteness that n is even. In this case, we need only change the unitary representation ρ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ to be defined by the two generators

$$\left(1, 0\right) \mapsto \gamma_{x,\text{odd}} = X \otimes \mathbb{I} \otimes X \otimes \mathbb{I} \otimes \cdots \otimes X \otimes \mathbb{I} \equiv X_1 X_3 \ldots X_{n-1}$$

and

$$\left(0, 1\right) \mapsto \gamma_{z,\text{odd}} = Z \otimes \mathbb{I} \otimes Z \otimes \mathbb{I} \otimes \cdots \otimes Z \otimes \mathbb{I} \equiv Z_1 Z_3 \ldots Z_{n-1},$$

resulting in the set of unitaries [47]

$$\left\{ U_g \right\} = \{\mathbb{I}, X_1 X_3 \ldots X_{n-1}, Y_1 Y_3 \ldots Y_{n-1}, Z_1 Z_3 \ldots Z_{n-1}\} \equiv G_{\text{odd}}.$$

Physically, the required generators $\gamma_{x,\text{odd}}$ and $\gamma_{z,\text{odd}}$ correspond to control Hamiltonians that are still just sums of one-local terms, and that act non-trivially on odd qubits only:

$$h_x(t) = f_x(t)\left(X_1 + X_3 + \ldots + X_{n-1}\right), \; h_z(t) = f_z(t)\left(Z_1 + Z_3 + \ldots + Z_{n-1}\right).$$

We expect that the design of Eulerian simulation schemes for more general scenarios where both the input and the target $\left(H, \tilde{H}\right)$ are *arbitrary* two-body Hamiltonians (including, for instance, long-range couplings) will greatly benefit from the existence of combinatorial approaches for constructing efficient DD groups [46, 49]. A more in-depth analysis of this topic is, however, beyond our current purpose.

4.2. Error-corrected Eulerian simulation in open Heisenberg-coupled qubit networks

Imagine now that the Heisenberg-coupled system S considered in the previous section is coupled to an environment B, and the task is to achieve the desired XYZ Hamiltonian simulation while also removing *arbitrary linear decoherence* to the leading order. The total input Hamiltonian has the form

$$H = H_{\text{iso}}^{(\text{NN})} \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + \sum_{i=1}^{n} \vec{\sigma}_i \otimes \vec{B}_i, \quad \vec{B}_i \equiv \left(B_{x,i}, B_{y,i}, B_{z,i} \right), \tag{38}$$

where H_B and $B_{u,i}$, for each i and $u = x, y, z$, are operators acting on $\mathcal{H}_B$, whose norm is sufficiently small to ensure convergence of the relevant Magnus series, similar to first-order DCG constructions [33, 34]. The target Hamiltonian then reads

$$\tilde{H} = H_{\text{XYZ}} \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B,$$

in terms of suitable coupling-strength parameters J_u as in equation (29). As before, we start by analyzing the case of $n = 2$ qubits in full detail. Our strategy to synthesize a dynamically

corrected simulation scheme involves two stages: (i) we will first decouple S from B, while leaving the system Hamiltonian $H_S = H_{\rm iso}$ unaffected; (ii) we will then apply the closed-system protocol of section 4.1 to convert $H_{\rm iso}$ into the target system Hamiltonian $\tilde{H}_S = H_{XYZ}$. Once a suitable group and weights are identified in this way, both stages are carried out simultaneously in application.

A suitable DD group able to suppress general linear decoherence is provided by $\mathcal{G}_{\rm DD} = \mathbb{Z}_2 \times \mathbb{Z}_2$, under the n-fold tensor power representation yielding (see also [34]):

$$\{U_h\} \equiv G_{\rm GL} = \left\{\mathbb{I}, X^{(\rm all)}, Y^{(\rm all)}, Z^{(\rm all)}\right\} = \{\mathbb{I}, X_1X_2, Y_1Y_2, Z_1Z_2\},$$

generated, for instance, by $\gamma_{x,\rm all} = X^{(\rm all)} = X_1X_2$ and $\gamma_{z,\rm all} = Z^{(\rm all)} = Z_1Z_2$. In addition to the order of $G_{\rm GL}$ being minimal, with $|G_{\rm GL}| = 4$ independently of n, step (i) above is automatically accomplished for the input Hamiltonian at hand, since

$$[H_{\rm iso}, U_h] = 0, \quad \forall\, U_h \in G_{\rm GL}. \tag{39}$$

Given a generic operator A on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$, we may define the superoperator $\Phi_{\rm DD}$ as

$$\Phi_{\rm DD}(A) = \frac{1}{4}\left(A + X^{(\rm all)}AX^{(\rm all)} + Y^{(\rm all)}AY^{(\rm all)} + Z^{(\rm all)}AZ^{(\rm all)}\right),$$

corresponding to weights $\{w_h\}$ given by $w_{\mathbb{I}} = w_{X_1X_2} = w_{Y_1Y_2} = w_{Z_1Z_2} = 1/4$.

In step (ii), we still rely on the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, but now under a different representation. We choose the representation yielding the set G_1 of equation (30), with the same single-qubit generators $\gamma_{x,1} = X_1$, $\gamma_{z,1} = Z_1$, and the corresponding weights $\{w_{g_1}\}$ determined by the solution of equation (32). Define the superoperator Φ_1 to act as

$$\Phi_1(A) = w_{\mathbb{I}}A + w_{X_1}X_1AX_1 + w_{Y_1}Y_1AY_1 + w_{Z_1}Z_1AZ_1.$$

Then the combined action of the two superoperators $\Phi_{\rm DD}$ and Φ_1 yields

$$\Phi_1\left[\Phi_{\rm DD}(A)\right] = \sum_{U_{g_1}\in G_1}\sum_{U_h\in G_{\rm GL}} w_{g_1}w_h U_{g_1}^\dagger U_h^\dagger A U_h U_{g_1} \equiv \sum_{g\in\mathcal{G}} w_g U_g^\dagger A U_g, \tag{40}$$

where $\mathcal{G} \equiv [\mathbb{Z}_2 \times \mathbb{Z}_2] \times [\mathbb{Z}_2 \times \mathbb{Z}_2] \simeq \mathbb{Z}_2^4$, with unitary representation elements corresponding to the *full* Pauli group on two qubits:

$$\{U_g\} = \{\mathbb{I}_i, X_i, Y_i, Z_i\}^{\otimes 2}.$$

The above representation is irreducible, with $\Pi_{\mathcal{G}}$ implementing the complete depolarizing channel on two qubits:

$$\Pi_{\mathcal{G}}(A) = \frac{1}{16}\sum_{g\in\mathcal{G}} U_g^\dagger A U_g = \frac{\mathrm{tr}(A)}{4}\mathbb{I}, \quad \forall\, A.$$

Thanks to the fact that all of the system terms in H are traceless and F_Γ is trace-preserving, this ensures that the DD conditions of equations (35)–(36) are satisfied. Since $|\mathcal{G}| = 16$ and $|\Gamma| = 4$, each simulation cycle will involve in general $N = 64$ time segments, with the number of non-zero weights (hence W and the simulation time-overhead) being determined by the details of the error model and/or the target Hamiltonian.

A practically important case, where simpler simulation schemes are possible, occurs if qubits couple to their environment along a fixed axis, effectively corresponding to a *purely dephasing* interaction—say, for concreteness, that $B_{y,i} = 0 = B_{z,i}$ for $i = 1, 2$ in equation (38). A smaller DD group suffices in this case [34], namely $\mathcal{G}_{\rm DD} = \mathbb{Z}_2$, represented again in terms of collective qubit rotations,

$$\{U_h\} \equiv G_{\rm D} = \left\{\mathbb{I}, Z^{(\rm all)}\right\} = \{\mathbb{I}, Z_1 Z_2\},$$

and generated by the single element $\gamma_{z,\rm all}$. Clearly, the commutation relationship in equation (39) is maintained, still allowing our two-step procedure to be followed. In this case, the combined group for simulation is $\mathcal{G} \equiv \mathbb{Z}_2 \times \left[\mathbb{Z}_2 \times \mathbb{Z}_2\right] \simeq \mathbb{Z}_2^3$, with $|\mathcal{G}| = 8$, $|\Gamma| = 3$, *reducibly* represented as follows on the two-qubit space:

$$\left\{U_g\right\} = \{\mathbb{I}, X_1, Y_1, Z_1, Z_2, Z_1 Z_2, X_1 Z_2, Y_1 Z_2\}. \tag{41}$$

Suppose, for instance, that the task is to simulate a dipolar Hamiltonian $H_{\rm dip}$ as in section 4.1. By following the above general procedure, with weights $\{w_h\}$ given by $w_{\mathbb{I}} = w_{Z_1 Z_2} = 1/2$ for G_D alone, it is easy to see that equation (40) simplifies, leading to simulation weights $w_{\mathbb{I}} = 1/4$, $w_{Z_1} = 3/4 = w_{Z_2}$, $w_{Z_1 Z_2} = 1/4$, with the remaining four weights equal to 0. While this implies that the simulation can now be achieved with only $N = 8 \times 3 = 24$ segments per cycle and minimum weight $W = 2$, care is needed in ensuring that the DD conditions in (35)–(36), are *still* obeyed. This may be checked by inspection. In particular, the fact that $\Pi_{\mathcal{G}}\left[F_\Gamma\left(X_i\right)\right] = 0$ for $i = 1, 2$ follows by analyzing the structure of each toggling-frame 'error Hamiltonian', $u_{\gamma_j}^\dagger(t) X_i u_{\gamma_j}(t)$, for $\gamma_j \in \Gamma = \left\{X_1, X_2, Z_1 + Z_2\right\}$, and verifying that no term proportional to Z_2 is generated, that would be left uncorrected by averaging over the representation in equation (41). Likewise, the fact that $\Pi_{\mathcal{G}}\left[F_\Gamma\left(H_S\right)\right] = 0$ for $H_S = H_{\rm iso}$ may be verified by a similar calculation, or by using the trace argument in section 4.1 for the two group generators $\gamma_{x,1} = X_1$ and $\gamma_{z,1} = Z_1$, while also noting that for the third generator $\gamma_{z,\rm all} = Z_1 Z_2$, we have $F_{Z_1 Z_2}\left(H_{\rm iso}\right) = H_{\rm iso}$ and the latter is decoupled by the representation in equation (41), $\Pi_{\mathcal{G}}\left(H_{\rm iso}\right) = 0$. Thus, Eulerian Hamiltonian simulation in the presence of single-axis errors can be *efficiently* achieved.

Again, the schemes we have presented for $n = 2$ can be generalized to a chain consisting of n spins, which interact according to a NN Heisenberg interaction and are each linearly coupled to the environment, according to equation (38). In this case, exploiting the results of section 4.1, a useful group for simulation is provided by $\mathcal{G} \simeq \mathbb{Z}_2^4$, under the unitary representation

$$\left\{U_g\right\} \equiv G_{\rm GL} \times G_{\rm odd},$$

corresponding to generators $\gamma_{x,\rm all}$, $\gamma_{z,\rm all}$, $\gamma_{x,\rm odd}$, $\gamma_{z,\rm odd}$, all of which can be implemented using only one-local (single-qubit) Hamiltonians. As before, each simulation cycle will consist in the general case of arbitrary linear decoherence of $N = 16 \times 4 = 64$ time segments. Despite the reducibility of the above representation (with the full Pauli group on n qubits consisting of 4^n elements), the DD conditions given by equations (35)–(36) remain valid for reasons similar to those outlined for $n = 2$ under pure dephasing.

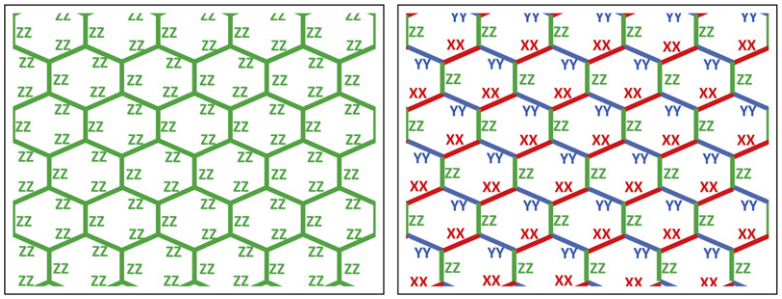

Figure 3. Input and target Hamiltonians on a 2D honeycomb lattice, where qubits are placed at each vertex. Left: the system Hamiltonian H describes a system where all adjacent vertices have ZZ Ising couplings. Right: the target Hamiltonian $\tilde{H}$ realizes Kitaev's honeycomb lattice model, with XX, YY, and ZZ couplings depending on the type of the edge.

4.3. Eulerian simulation of Kitaev's honeycomb lattice Hamiltonian

We return to Eulerian simulation in closed quantum systems, but tackle a more complicated Hamiltonian of paradigmatic relevance to topological quantum memories, namely, Kitaev's honeycomb lattice model [50]. Suppose that the target system consists of a network of qubits arranged on a honeycomb lattice and interacting via NN Ising couplings. The relevant Hamiltonian H is graphically displayed in figure 3 (left), where vertices represent qubits and edges represent two-qubit couplings of the form $Z_k Z_\ell$, with vertices k and ℓ being adjacent in the graph and Z_k indicating, as before, the Pauli Z operator acting non-trivially only on qubit k. The target Hamiltonian $\tilde{H}$ is shown in figure 3 (right), where some of the edges are now of the form $X_k X_\ell$ and $Y_k Y_\ell$. In accordance with the figure, we shall also call the XX-edges *forward-slashes*, the YY-edges *back-slashes*, and the ZZ-edges *verticals* henceforth.

The basic idea to accomplish this simulation is to exploit the matrix R given in equation (37), in conjunction with the symmetry of our problem: since all Hamiltonian terms are *precisely* two-local and of the homogeneous form $\sigma \otimes \sigma$, it will be possible to avoid using the full machinery of a transformer. Consider the group $\mathcal{G}$ generated by the three unitaries, ρ_X, τ_X, and R_{global}, where ρ_X, shown in figure 4 (left) with $\sigma = X$, has X's on every second forward-slash, τ_X, shown in figure 4 (center) with $\sigma = X$, has X's on every second back-slash, and R_{global}, shown in figure 4 (right), has R applied to every vertex. These unitaries can be generated by one-local (single-qubit) Hamiltonians. By repeatedly conjugating ρ_X and τ_X with R_{global}, we immediately see that we can also perform ρ_σ and τ_σ, shown in figure 4, for any $\sigma = X, Y, Z$. Note that up to phase, all such ρ and τ commute. Because conjugation by R maps Pauli matrices to Pauli matrices, for any Pauli σ we have $R\sigma = \left(R\sigma R^{-1}\right)R = \sigma' R$, where σ' is another Pauli matrix. Thus, up to phase, we can write any element of $\mathcal{G}$ in the canonical form

$$U_g = \rho\tau R_{\text{global}}^a, \quad a \in \{0, 1, 2\}, \tag{42}$$

where $\rho \in \{\mathbb{I}, \rho_X, \rho_Y, \rho_Z\}$, $\tau \in \{\mathbb{I}, \tau_X, \tau_Y, \tau_Z\}$, and R_{global}^a only appears on the right.

To construct an Eulerian simulation protocol we must be able to choose w_g so that $\tilde{H}$ is reachable from H, i.e., obeys equation (22), while ensuring that the DD condition of equation

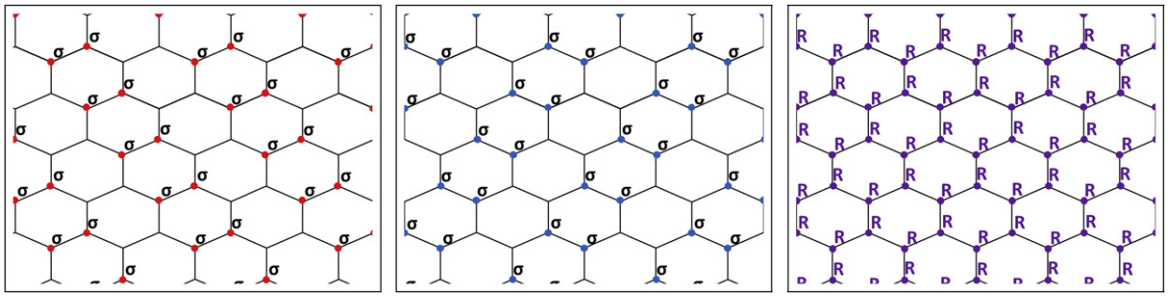

Figure 4. Pictorial representation of different control operations. Left: the unitary ρ_σ, with σ on the vertices of every second forward-slash and $\mathbb{I}$ on all other vertices, where σ is a fixed X, Y, or Z operator. When $\sigma = X$, this is the generator ρ_X. Center: the unitary τ_σ, with σ on the vertices of every second back-slash, where σ is a fixed X, Y, or Z operator. When $\sigma = X$ this is the generator τ_X. Right: the generator R_{global}, with R at every vertex.

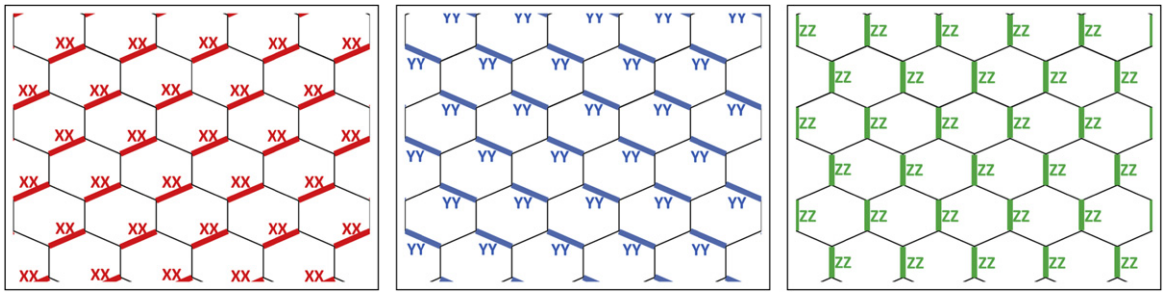

Figure 5. Pictorial representation of different simulation superoperators (see text). Left: action of the superoperator Φ_{XX}, leaving XX terms at forward-slashes only. Center: action of the superoperator Φ_{YY}, leaving YY terms at back-slashes only. Right: action of the superoperator Φ_{ZZ}, leaving ZZ terms at verticals only.

(21) is also fulfilled. We start from the fact that

$$\frac{1}{2}\mathbb{I}(Z\otimes Z)\mathbb{I} + \frac{1}{2}(X\otimes\sigma)(Z\otimes Z)(X\otimes\sigma) = \begin{cases} Z\otimes Z & \text{if } \sigma = \mathrm{X} \\ 0 & \text{if } \sigma = \mathbb{I}. \end{cases}$$

Observe that when $U_g = \rho_X$, all forward-slash edges connect vertices that are acted upon by either $\mathbb{I}\otimes\mathbb{I}$ or $X\otimes X$, while all other edges connect vertices that are operated by $X\otimes\mathbb{I}$. Consequently, $\frac{1}{2}\mathbb{I}^\dagger H\mathbb{I} + \frac{1}{2}\rho_X^\dagger H\rho_X$ removes all Hamiltonian terms except for those along the forward-slashes; upon conjugating by R_{global}, we may then convert these surviving ZZ terms to XX terms, as desired. To summarize,

$$\Phi_{XX}(H) \equiv \frac{1}{2}R_{\text{global}}^\dagger H R_{\text{global}} + \frac{1}{2}\left(\rho_X R_{\text{global}}\right)^\dagger H\left(\rho_X R_{\text{global}}\right)$$

gives the Hamiltonian shown in figure 5 (left). Similarly, the effect of $\frac{1}{2}\mathbb{I}^\dagger H\mathbb{I} + \frac{1}{2}\tau_X^\dagger H\tau_X$ is to leave precisely the back-slash edges, which can be converted from ZZ to YY by conjugation by R_{global}^2. Thus,

$$\Phi_{YY}(H) \equiv \frac{1}{2} R_{\text{global}}^{2\,\dagger} H R_{\text{global}}^{2} + \frac{1}{2}\left(\tau_X R_{\text{global}}^{2}\right)^{\dagger} H \left(\tau_X R_{\text{global}}^{2}\right)$$

gives the Hamiltonian shown in figure 5 (center). Lastly, it is not hard to see that the product $\rho_X \tau_X$ has X's on every second row of verticals; accordingly,

$$\Phi_{ZZ}(H) \equiv \frac{1}{2} \mathbb{I}^{\dagger} H \mathbb{I} + \frac{1}{2}\left(\rho_X \tau_X\right)^{\dagger} H \left(\rho_X \tau_X\right)$$

isolates precisely the verticals, giving the Hamiltonian shown in figure 5 (right). In this case, no R-conjugation is necessary since we wish to maintain ZZ edges along the verticals. Putting all these steps together, we conclude that

$$\begin{aligned} \tilde{H} \;=\; & \frac{1}{2} R_{\text{global}}^{\dagger} H R_{\text{global}} + \frac{1}{2}\left(\rho_X R_{\text{global}}\right)^{\dagger} H \left(\rho_X R_{\text{global}}\right) + \frac{1}{2} R_{\text{global}}^{2\,\dagger} H R_{\text{global}}^{2} \\ & + \frac{1}{2}\left(\tau_X R_{\text{global}}^{2}\right)^{\dagger} H \left(\tau_X R_{\text{global}}^{2}\right) + \frac{1}{2} \mathbb{I}^{\dagger} H \mathbb{I} + \frac{1}{2}\left(\rho_X \tau_X\right)^{\dagger} H \left(\rho_X \tau_X\right), \end{aligned}$$

thus providing the desired weights for the Eulerian protocol. Since there are $|\Gamma| = 3$ generators and, from equation (42), $|\mathcal{G}| = 4 \times 4 \times 3 = 48$ group elements, each control block consists of $N = 144$ time intervals.

Lastly, we must verify that equation (21) holds. Note that $F_\Gamma(H)$ acts via conjugating each vertex by unitaries (since the generating pulses are one-local), and since such an operation is trace-preserving at each vertex, this necessarily takes the precisely two-local terms in H to precisely two-local terms in $F_\Gamma(H)$. Since no one-local terms can arise, all terms are of the form $\sigma_u^{(k)} \otimes \sigma_v^{(\ell)}$, where k and ℓ are adjacent vertices and σ_u, $\sigma_v \in \{X, Y, Z\}$. Thus, we may write

$$F_\Gamma(H) = \sum_{k,\ell\,\text{adjacent}} \sum_{u,v} a_{u,v}^{(k,\ell)} \sigma_u^{(k)} \otimes \sigma_v^{(\ell)}.$$

Due to the canonical form of our group elements, equation (42), the action of $\Pi_{\mathcal{G}}$ reads

$$\Pi_{\mathcal{G}}\left[F_\Gamma(H)\right] = \frac{1}{|\mathcal{G}|} \sum_{a=0}^{2} \sum_{\tau,\rho} R^{a\dagger} \tau\rho\, F_\Gamma(H)\, \rho\tau R^{a},$$

where $\tau \in \{\mathbb{I}, \tau_X, \tau_Y, \tau_Z\}$ and $\rho \in \{\mathbb{I}, \rho_X, \rho_Y, \rho_Z\}$, respectively. Just as the map $\frac{1}{2}\mathbb{I}H\mathbb{I} + \frac{1}{2}\rho_X H \rho_X$ removes all non-forward-slash ZZ terms, the map $\sum_\rho \rho F_\Gamma(H) \rho$ depolarizes precisely one vertex of each pair of non-forward-slash vertices, and therefore suppresses all non-forward-slash terms. With only forward-slash terms remaining, $\sum_\tau \tau \left[\sum_\rho \rho F_\Gamma(H) \rho\right] \tau = 0$, since the τ-sum removes all non-back-slash terms. Thus, we conclude that $\Pi_{\mathcal{G}}\left[F_\Gamma(H)\right] = 0$, as desired.

5. Conclusion and outlook

We have shown that the Eulerian cycle technique successfully employed in both DD schemes and DCGs can be extended to also enable Hamiltonian quantum simulation with realistic *bounded-strength* controls. For given internal dynamics and control resources, we have characterized the family of reachable target Hamiltonians and provided constructive open-loop control protocols for stroboscopically implementing a desired evolution in the family with

accuracy (at least) up to the second order in the sense of average Hamiltonian theory. We have additionally shown how Hamiltonian simulation may be accomplished in an open quantum system while *simultaneously* suppressing unwanted decoherence, provided that appropriate time-scale requirements and decoupling conditions are fulfilled—paving the way to dynamically corrected quantum simulation. The usefulness and flexibility of our Eulerian simulation techniques have been explicitly illustrated through several QIP-motivated examples involving both unitary and open-system dynamics on interacting qubit networks. In all cases, access to purely *local* (single-qubit) control Hamiltonians is assumed, subject to finite-amplitude constraint and the ability to collectively apply such Hamiltonians to selected subsets of target qubits, for instance, qubits belonging to regular lattice patterns in one- or two-dimensional arrays. It is worth stressing that this level of control is in principle available in a variety of platforms for quantum simulation, with such 'spatially periodic' control operations being often amenable of simple implementation, e.g. in optical lattices via globally applied pulses [3, 5].

While it is our hope that our results may be of immediate relevance to ongoing efforts for developing and programming quantum simulators in the laboratory, several possible generalizations and further research questions are worth mentioning. As an additional simulation problem dual to the one we analyzed for Heisenberg-coupled spin chains, exploring schemes where a target Heisenberg Hamiltonian is generated out of Ising couplings only would be of interest, given the experimental availability of the latter in existing large-scale trapped-ion simulators [20]. Likewise, an interesting issue is to explore the extent to which the proposed Eulerian approach may find application in simulation schemes for more exotic Hamiltonians involving *higher-order interactions*, notably, as arising in the Kitaev toric code [16] and lattice gauge theories [51].

From an implementation perspective, our present results calls for further, dedicated analysis of the impact of *control errors*, as inevitably present in experiments and effectively limiting the maximum time over which the target dynamics may be reliably simulated. Since Eulerian control design is inherently robust (to the leading order) against *systematic* Hamiltonian errors along a full cycle [32, 34], a similar degree of robustness may be expected for the 'ramping-up' portion of a simulation block. While we also expect that the requisite timing precision in both 'coasting' and 'ramping-up' periods may be similar to the one demanded by DD protocols [52], a detailed analysis is needed to establish quantitative error bounds and venues for enhancing fault-tolerance in a given physical architecture. Partly related to that, an ambitious goal is to determine whether Hamiltonian simulation schemes able to *guarantee* a minimum fidelity over arbitrarily long simulation times may be devised, in the spirit of [52] for the particular case of the zero Hamiltonian.

Building on existing results for DD schemes [47], the use and possible advantages of *randomized* simulation schemes in terms of robustness and/or efficiency may be yet another venue of investigation, especially in connection with large control groups. Finally, it could be useful to explore whether bounded-strength simulation as proposed here may be made compatible with *open-loop filtering* techniques for modulating coupling strengths, such as recently proposed in [53], as well as in [54] in conjunction with non-unitary open-loop control via field gradients.

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