



Conic relaxations for transmission system planning

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Abstract—We apply lift-and-project type relaxations for polynomial optimization problems to AC transmission system planning, and obtain new second-order cone and semidefinite models. The models are compared to linear and nonlinear approaches on a six-bus test system.

Index Terms—Transmission system planning, lift-and-project, second-order cone programming, semidefinite programming, AC power flow.

I. INTRODUCTION

Static transmission system planning entails choosing among a set of candidate lines whose addition to a network minimizes construction costs while meeting physical capacity requirements [1], [2]. The basic problem is nonlinear, and moreover nonconvex due to both the mathematical description of AC power flow and products between line and voltage variables.

Until recently, optimization-based approaches have almost exclusively been preceded by application of the DC power flow approximation [3], [4], because the full AC version was considered intractable. At present, there are few methods that do not first employ this simplification; of note is [5], in which the full AC formulation is solved using a heuristic based on nonlinear programming. An alternate recent approach is [6], where we used a lift-and-project procedure to derive efficient linear programming relaxations for both the simplified and full variants of the problem.

Second-order cone [7] and semidefinite [8] programming are generalizations of linear programming, which are more expressive but retain its polynomial time efficiency. The recent result of [9] has made AC power flow far more tractable via a semidefinite programming relaxation of optimal power flow, often capturing the true optimal solution exactly. In this paper we employ their power flow formulation inside a transmission system planning framework, and, using a lift-and-project procedure [6], [10], [11], construct a high-fidelity mixed-integer semidefinite programming relaxation. We then use the result of [12] to obtain a mixed-integer second-order cone relaxation directly from the semidefinite model. Each is demonstrated on the six-bus test system of [1], [5].

Although mixed-integer second-order cone and semidefinite programming algorithms are relatively immature today, their similarities with linear programming suggest that comparable sophistication is attainable. Indeed, mixed-integer second-order cone programming is an active area of research [13]–[15], and we can anticipate that robust and efficient tools will become available in coming years. At the same time, the speed

of semidefinite and second-order cone programming relative to nonlinear programming makes our relaxations directly applicable within generic branch-and-bound frameworks. Our paper does not address this approach.

II. AC POWER FLOW

In this section we provide the nonlinear AC transmission system planning model and review the semidefinite power flow relaxation of [9]. We work in rectangular coordinates, for which the voltage at bus i is represented by the complex number $w_i + jx_i$.

Let Ω_0 and Ω be the sets of existing and candidate lines. Unless otherwise specified, single subscripts are to be taken over all buses and double subscripts over members of Ω . Let c_{ij} be a line cost vector, p_i , \bar{p}_i , q_i , and \bar{q}_i be real and reactive bus power limits, v_i and \bar{v}_i be voltage limits, η_{ij}^0 be the number of lines present in the existing network, and $\bar{\eta}_{ij}$ be the number of new lines which may be constructed. Let $g_{ij} + jb_{ij}$ and $p_{ij} + jq_{ij}$ respectively be complex admittance and power flow, and let \bar{s}_{ij} be capacity per line increment. Finally, let g_{ij} be conductance and define $b_{ij}^s = b_{ij} + b_{ij}^{sh}$, where b^{sh} is line shunt susceptance. The AC transmission planning model in rectangular coordinates is given by

$$\text{NLAC} \quad \min_{\eta, p, q, w, x} \sum_{i \sim j} c_{ij} \eta_{ij} \quad (1)$$

$$\text{s.t.} \quad p_{ij} = (\eta_{ij}^0 + \eta_{ij}) (b_{ij}(w_i x_j - w_j x_i) - g_{ij}(x_i x_j + w_i w_j) + g_{ij}(w_i^2 + x_i^2)) \quad (2)$$

$$q_{ij} = (\eta_{ij}^0 + \eta_{ij}) (g_{ij}(w_i x_j - w_j x_i) + b_{ij}(x_i x_j + w_i w_j) - b_{ij}^s(w_i^2 + x_i^2)) \quad (3)$$

$$p_i \leq \sum_j p_{ij} \leq \bar{p}_i \quad (4)$$

$$q_i \leq \sum_j q_{ij} \leq \bar{q}_i \quad (5)$$

$$v_i^2 \leq w_i^2 + x_i^2 \leq \bar{v}_i^2 \quad (6)$$

$$\sqrt{p_{ij}^2 + q_{ij}^2} \leq (\eta_{ij}^0 + \eta_{ij}) \bar{s}_{ij} \\ (i, j) \in \Omega_0 \cup \Omega \quad (7)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (8)$$

The basic approach of [9] is to define a column vector of the real and imaginary bus voltages $X = [w^T x^T]^T$, and then substitute the symmetric matrix W for XX^T subject to the constraint $\text{rank}(W) = 1$ and $W \succeq 0$, where \succeq denotes

matrix semidefiniteness. Dropping the rank constraint yields the relaxation.

A dual formulation of the power flow equations is used in [9] to eliminate unnecessary variables; for the purpose of integration into a transmission planning framework, it is more convenient here to remain with the primal variables. The size of the problem can be reduced by sparsifying W and in place of (10) using the relaxed condition

$$\begin{bmatrix} W_{i,i} & W_{i,j} & W_{i,i+n} & W_{i,j+n} \\ W_{j,i} & W_{j,j} & W_{j,i+n} & W_{j,j+n} \\ W_{i+n,i} & W_{i+n,j} & W_{i+n,i+n} & W_{i+n,j+n} \\ W_{j+n,i} & W_{j+n,j} & W_{j+n,i+n} & W_{j+n,j+n} \end{bmatrix} \succeq 0$$

for each line (i, j) , where n is the number of buses. We denote the above submatrix ${}^{ij}W$.

Incorporating the substitutions into NLAC we have

$$\text{NLSDAC} \quad \min_{\eta, p, q, W} \sum_{i \sim j} c_{ij} \eta_{ij} \quad (9)$$

$$\text{s.t.} \quad W \succeq 0 \quad (10)$$

$$\begin{aligned} p_{ij} &= (\eta_{ij}^0 + \eta_{ij}) (b_{ij}(W_{i,j+n} - W_{j,i+n}) \\ &\quad - g_{ij}(W_{i+n,j+n} + W_{i,j}) \\ &\quad + g_{ij}(W_{i,i} + W_{i+n,i+n})) \end{aligned} \quad (11)$$

$$\begin{aligned} q_{ij} &= (\eta_{ij}^0 + \eta_{ij}) (g_{ij}(W_{i,j+n} - W_{j,i+n}) \\ &\quad + b_{ij}(W_{i+n,j+n} + W_{i,j}) \\ &\quad - b_{ij}^s(W_{i,i} + W_{i+n,i+n})) \end{aligned} \quad (12)$$

$$\underline{v}_i^2 \leq W_{i,i} + W_{i+n,i+n} \leq \bar{v}_i^2 \quad (13)$$

$$\begin{aligned} p_{ij}^2 + q_{ij}^2 &\leq (\eta_{ij}^0 + \eta_{ij})^2 \bar{s}_{ij}^2 \\ (i, j) &\in \Omega_0 \cup \Omega \end{aligned} \quad (14)$$

Constraints (4), (5), (8)

III. RELAXATIONS

Except for the semidefinite constraint and line capacity, NLSDAC only has bilinear nonlinearities, which are amenable to same linear relaxations as in [6]. An additional benefit of using second-order cone and semidefinite formulations is that the line capacity constraint (14) may be enforced exactly.

We proceed by substituting the 4×4 matrix Δ^{ij} for each instance of the product $\eta_{ij} {}^{ij}W$. In addition to being symmetric, Δ^{ij} is implicitly constrained so that $\Delta_{11}^{ij} = \Delta_{22}^{ji}$, $\Delta_{33}^{ij} = \Delta_{44}^{ji}$, $\Delta_{12}^{ij} = \Delta_{12}^{ji}$, $\Delta_{34}^{ij} = \Delta_{34}^{ji}$, $\Delta_{13}^{ij} = \Delta_{24}^{ji}$, and $\Delta_{14}^{ij} = \Delta_{23}^{ji}$. New constraints are formed by taking products of existing constraints, for example (17) below is obtained from multiplying (10) and (8), and then substituting Δ^{ij} . We

then have the following mixed-integer semidefinite program:

$$\text{SDAC} \quad \min_{\eta, \mu, \nu, \phi, \psi, W, \Delta} \sum_{i \sim j} c_{ij} \eta_{ij} \quad (15)$$

$$\text{s.t.} \quad W \succeq 0 \quad (16)$$

$$0 \leq \Delta^{ij} \leq \bar{\eta}_{ij} {}^{ij}W \quad (17)$$

$$\begin{aligned} \mu_{ij} &= b_{ij}(W_{i,j+n} - W_{j,i+n}) \\ &\quad - g_{ij}(W_{i,j} + W_{i+n,j+n}) \\ &\quad + g_{ij}(W_{i,i} + W_{i+n,i+n}) \end{aligned} \quad (18)$$

$$\begin{aligned} \nu_{ij} &= g_{ij}(W_{i,j+n} - W_{j,i+n}) \\ &\quad + b_{ij}(W_{i,j} + W_{i+n,j+n}) \\ &\quad - b_{ij}^s(W_{i,i} + W_{i+n,i+n}) \end{aligned} \quad (19)$$

$$\begin{aligned} \phi_{ij} &= b_{ij}(\Delta_{14}^{ij} - \Delta_{23}^{ij}) \\ &\quad - g_{ij}(\Delta_{12}^{ij} + \Delta_{34}^{ij}) \\ &\quad + g_{ij}(\Delta_{11}^{ij} + \Delta_{33}^{ij}) \end{aligned} \quad (20)$$

$$\begin{aligned} \psi_{ij} &= g_{ij}(\Delta_{14}^{ij} - \Delta_{23}^{ij}) \\ &\quad + b_{ij}(\Delta_{12}^{ij} + \Delta_{34}^{ij}) \\ &\quad - b_{ij}^s(\Delta_{11}^{ij} + \Delta_{33}^{ij}) \end{aligned} \quad (21)$$

$$\underline{p}_i \leq \sum_j \eta_{ij}^0 \mu_{ij} + \phi_{ij} \leq \bar{p}_i \quad (22)$$

$$\underline{q}_i \leq \sum_j \eta_{ij}^0 \nu_{ij} + \psi_{ij} \leq \bar{q}_i \quad (23)$$

$$\underline{v}_i^2 \leq W_{i,i} + W_{i+n,i+n} \leq \bar{v}_i^2 \quad (24)$$

$$\underline{v}_i^2 \eta_{ij} \leq \Delta_{11}^{ij} + \Delta_{33}^{ij} \leq \bar{v}_i^2 \eta_{ij} \quad (25)$$

$$\underline{v}_j^2 \eta_{ij} \leq \Delta_{22}^{ij} + \Delta_{44}^{ij} \leq \bar{v}_j^2 \eta_{ij} \quad (26)$$

$$\begin{aligned} \underline{v}_i^2 (\bar{\eta}_{ij} - \eta_{ij}) &\leq \bar{\eta}_{ij} (W_{i,i} + W_{i+n,i+n}) \\ &\quad - \Delta_{11}^{ij} - \Delta_{33}^{ij} \leq \bar{v}_i^2 (\bar{\eta}_{ij} - \eta_{ij}) \end{aligned} \quad (27)$$

$$\begin{aligned} \underline{v}_j^2 (\bar{\eta}_{ij} - \eta_{ij}) &\leq \bar{\eta}_{ij} (W_{j,j} + W_{j+n,j+n}) \\ &\quad - \Delta_{22}^{ij} - \Delta_{44}^{ij} \leq \bar{v}_j^2 (\bar{\eta}_{ij} - \eta_{ij}) \end{aligned} \quad (28)$$

$$\mu_{ij}^2 + \nu_{ij}^2 \leq \bar{s}_{ij}^2 \quad (i, j) \in \Omega_0 \quad (29)$$

$$\phi_{ij}^2 + \psi_{ij}^2 \leq \eta_{ij}^2 \bar{s}_{ij}^2 \quad (30)$$

$$\begin{aligned} (\bar{\eta}_{ij} \mu_{ij} - \phi_{ij})^2 &+ (\bar{\eta}_{ij} \nu_{ij} - \psi_{ij})^2 \leq \\ &(\bar{\eta}_{ij} - \eta_{ij})^2 \bar{s}_{ij}^2 \quad (i, j) \in \Omega_0 \end{aligned} \quad (31)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (32)$$

The new variables ϕ and ψ represent flows in the new, variable network, while the flows in the existing network are denoted μ and ν , and obey AC power flow. We remark that if the semidefinite constraints are omitted and (29) through (31) are replaced by piecewise linear approximations, we recover the linear model LAC of [6].

Larger, more accurate models may be constructed by taking higher-order products of constraints and performing similar substitutions [10], [11]; however, the number of new constraints and variables increases rapidly, and at some point a compromise must be struck. In this regard, SDAC is highly appealing: it is reasonably sized as far as lift-and-project relaxations go, and utilizes to the full extent today's broadest

polynomial-time conic optimization framework, semidefinite programming.

Using SDAC as a starting point, we can straightforwardly construct a second-order cone relaxation (SOCAC) via the fact that positivity of all two-by-two principal minors is a necessary condition for positive semidefiniteness [12]. The relaxation is thus obtained by replacing (16) and (17) with the second-order cone constraints

$$\begin{aligned} W_{i,i} &\geq 0 \\ W_{i+n,i+n} &\geq 0 \\ \Delta_{kk}^{ij} &\geq 0, \quad k = 1, \dots, 4 \end{aligned}$$

$$\begin{aligned} W_{i,j}^2 &\leq W_{i,i}W_{j,j} \\ W_{i,j+n}^2 &\leq W_{i,i}W_{j+n,j+n} \\ W_{i+n,j}^2 &\leq W_{i+n,i+n}W_{j,j} \\ W_{i+n,j+n}^2 &\leq W_{i+n,i+n}W_{j+n,j+n} \end{aligned}$$

$$\begin{aligned} \Delta_{12}^{ij^2} &\leq \Delta_{11}^{ij}\Delta_{22}^{ij} \\ \Delta_{14}^{ij^2} &\leq \Delta_{11}^{ij}\Delta_{44}^{ij} \\ \Delta_{23}^{ij^2} &\leq \Delta_{22}^{ij}\Delta_{33}^{ij} \\ \Delta_{34}^{ij^2} &\leq \Delta_{33}^{ij}\Delta_{44}^{ij} \end{aligned}$$

$$\begin{aligned} \left(\bar{\eta}_{ij}W_{i,j} - \Delta_{12}^{ij} \right)^2 &\leq \\ \left(\bar{\eta}_{ij}W_{i,i} - \Delta_{11}^{ij} \right) \left(\bar{\eta}_{ij}W_{j,j} - \Delta_{22}^{ij} \right) \end{aligned}$$

$$\begin{aligned} \left(\bar{\eta}_{ij}W_{i,j+n} - \Delta_{14}^{ij} \right)^2 &\leq \\ \left(\bar{\eta}_{ij}W_{i,i} - \Delta_{11}^{ij} \right) \left(\bar{\eta}_{ij}W_{j+n,j+n} - \Delta_{44}^{ij} \right) \end{aligned}$$

$$\begin{aligned} \left(\bar{\eta}_{ij}W_{i+n,j} - \Delta_{32}^{ij} \right)^2 &\leq \\ \left(\bar{\eta}_{ij}W_{j,j} - \Delta_{22}^{ij} \right) \left(\bar{\eta}_{ij}W_{i+n,i+n} - \Delta_{33}^{ij} \right) \end{aligned}$$

$$\begin{aligned} \left(\bar{\eta}_{ij}W_{i+n,j+n} - \Delta_{34}^{ij} \right)^2 &\leq \\ \left(\bar{\eta}_{ij}W_{i+n,i+n} - \Delta_{33}^{ij} \right) \left(\bar{\eta}_{ij}W_{j+n,j+n} - \Delta_{44}^{ij} \right) \end{aligned}$$

for each line (i, j) . Note that the line capacity constraints (29)-(31) are convex quadratic, and therefore already within the class of second-order cone constraints. As with analogous linear models, it is straightforward to formulate more accurate but less efficient binary versions of both SDAC and SOCAC [6].

IV. COMPUTATIONAL RESULTS

In this section we test SDAC and SOCAC on the six-bus example of [5], which is an adaptation of Garver's six-bus system [1]. We specifically consider the case in which there are preexisting lines. Also included is the linear model LAC of [6]. To facilitate comparison, the following piecewise linear relaxation for line limits, as required for LAC, has been used:

$$|p_{ij}| + |q_{ij}| \leq \sqrt{2}S_{ij}, \quad |p_{ij}| \leq S_{ij}, \quad |q_{ij}| \leq S_{ij}.$$

Mixed integer linear and second-order cone programs were solved using the modeling language AMPL [16] and solver CPLEX [17]. Mixed integer semidefinite programs were solved by enumerating over additions to lines $\{1-5, 2-3, 2-6, 3-5, 4-6\}$, which is the set of all lines appearing in one or more optimal solution from LAC. CVX [18] and SeDuMi [19] were then used to identify the feasible ones; in this fashion, it is partially determined how to prune and reinforce solutions from LAC.

Table I shows the objective value and solution reported for the nonlinear approach in [5] (NL) and obtained by LAC, SOCAC, and SDAC. LAC and SOCAC each found solutions with the same objective; SOCAC, however, found three, whereas LAC found six, indicating that SOCAC eliminated more low-quality solutions than did LAC. SDAC found three solutions with a higher objective than the other models. The bottom portion of Table I shows a single solution obtained by each model.

TABLE I
CONIC MODELS ON THE SIX-BUS TEST SYSTEM OF [5]

Model	NL [5]	LAC	SOCAC	SDAC
Number of solutions	-	6	3	3
Obj.	160	80	80	100
Line additions				
2 - 6	2	2	2	1
3 - 5	2	1	1	2
4 - 6	2	0	0	1

We restate that at present, only mixed-integer linear programming can be considered a mature technology, but that mixed-integer second-order cone programming is an active field of research [13]–[15], and we expect that mixed-integer semidefinite programming will follow. As such, the methods presented here are not currently scalable approaches, although we expect them to be in the near future.

V. CONCLUSION AND FUTURE WORK

We have derived mixed-integer semidefinite and second-order cone relaxations of AC transmission system planning using a lift-and-project procedure. On a small example, the new models are shown to be more accurate than a similarly constructed linear model. The models do yet not enjoy the maturity of linear algorithms, but advancements in mixed-integer conic optimization are to be expected, in which case these models will soon represent practical approaches to high-fidelity transmission system planning without the complications of nonconvexity. At the same time, second-order cone and semidefinite programming are very efficient compared with nonlinear algorithms, and hence the models presented here should also be attractive for use with general integer programming techniques such as branch-and-bound.

ACKNOWLEDGMENT

Work is supported by the Office of Naval Research Grant N00014-02-1-0623, monitored by Dr. T. Ericson.

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