

# Multistage Air Traffic Flow Management under Capacity Uncertainty: A Robust and Adaptive Optimization Approach\*

Shubham Gupta<sup>†</sup> Dimitris Bertsimas<sup>‡</sup>

May 31, 2011

## Abstract

In this paper, we study the first application of robust and adaptive optimization in the Air Traffic Flow Management (ATFM) problem. The existing models for network-wide ATFM assume deterministic capacity estimates across airports and sectors without taking into account the uncertainty in capacities induced by weather. We introduce a weather-front based approach to model the uncertainty inherent in airspace capacity estimates resulting from the impact of a small number of weather fronts moving across the National Airspace (NAS). The key advantage of our uncertainty set construction is the low-dimensionality (uncertainty in only two parameters govern the overall uncertainty set for each airspace element). We formulate the consequent ATFM problem under capacity uncertainty within the robust and adaptive optimization framework and propose tractable solution methodologies. Our theoretical contributions are as follows: i) we propose a polyhedral description of the convex hull of the discrete uncertainty set; ii) we prove the equivalence of the robust problem to a modified instance of the deterministic problem; and iii) we solve optimally the LP relaxation of the adaptive problem using piece-wise affine policies where the number of pieces in an optimal policy are governed by the number of extreme points in the uncertainty set. A particularly attractive feature is that for most practically encountered instances, an affine policy suffices to solve the adaptive problem optimally. Finally, we report empirical results from the proposed models on real-world flight schedules augmented with simulated weather fronts that illuminate the merits of our proposal. The key insights from our computational results are: i) the robust problem inherits all the attractive properties of the deterministic problem (e.g., superior integrality properties and fast computational times); and ii) the price of robustness and adaptability is typically small.

*Key words: robust optimization; multi-stage adaptive optimization; air transportation*

---

\*Research funded by the NSF Grant EFRI-0735905.

<sup>†</sup>Operations Research Center, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, shubhamg@mit.edu

<sup>‡</sup>Boeing Professor of Operations Research, Sloan School of Management, Co-director of the Operations Research Center, Massachusetts Institute of Technology, E40-147, Cambridge, MA 02139-4307, dbertsim@mit.edu

# 1 Introduction

Weather accounts for the majority of the total air traffic delays caused due to terminal, en-route congestion and a myriad of other operational factors. To assess the impact of weather on the total aviation delays, we consider the OPSNET<sup>1</sup> delays data for the year 2009. As evidenced in the monthly delays plot in Figure 1, there is a significant spike in the delays for the summer months (May-July), when there is pronounced convective weather activity. Another statistic from the same source indicates that approximately 60% of total delays is attributable to weather across 12 months of 2009. These two observations highlight the importance of addressing weather induced capacity uncertainty for mitigating aviation delays.

Air Traffic Flow Management (ATFM) refers to the set of tools and processes that are used to alleviate congestion costs and ensure the goal of safe and expeditious aircraft movements. Some of the main ATFM tools currently being deployed include Ground Delay Programs (GDPs), Airspace Flow Programs (AFPs) and rerouting of aircrafts. The existing academic models for network ATFM assume a deterministic estimate on the available capacities at airports and sectors (*airspace elements* henceforth). Not accounting for capacity uncertainty may lead to suboptimal and possibly infeasible solutions. This state of affairs invites a new mathematical approach that incorporates the uncertainty inherent in the estimates of the airspace resources to come up with a robust schedule.

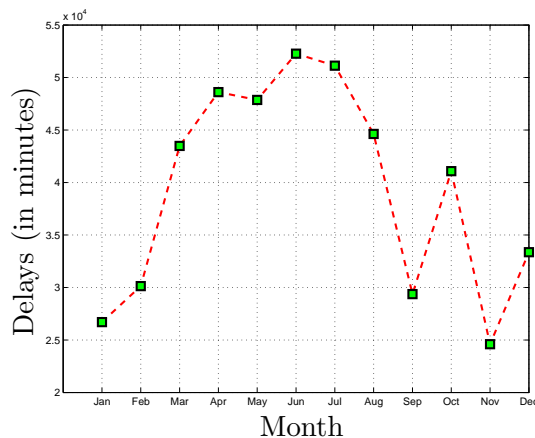


Figure 1: OPSNET monthly delays for 2009.

The research literature on ATFM is rich and spans across more than two decades. Starting with Odoni [18], who first conceptualized the problem of scheduling flights in real time in order to minimize congestion costs, a plethora of models have been proposed to handle different versions of the problem. The problem of assigning ground-delays in the context of a single-airport (*Single-Airport Ground-Holding Problem*, SAGHP henceforth) has been studied in

---

<sup>1</sup>The Operations Network (OPSNET) is the official source of National Airspace (NAS) air traffic operations and delay data.

Terrab and Odoni [21]; and in the multiple airport setting in Terrab and Paulose [22], Vranas et al. [23]. The problem of controlling release times and speed adjustments of aircraft while air-borne for a network of airports taking into account the capacitated airspace has been studied in Bertsimas and Stock-Patterson [11], Lindsay et al. [16]. For a detailed survey of the various contributions and a taxonomy of all the problems, see Bertsimas and Odoni [9] and Hoffman et al. [14]. Recently, the research community has attempted to overcome the critical hurdle of fairness in ATFM models which had been conspicuously unaddressed so far. Towards this goal, Bertsimas and Gupta [8] propose deterministic formulations for the network-wide ATFM problem that incorporate fairness and collaboration. Specifically, they propose discrete optimization models capable of controlling the number of reversals and amount of overtaking in resulting flight sequences.

In contrast, the literature dealing simultaneously with stochastic airspace capacities has been rather sparse. One of the first attempts at dealing with Stochastic SAGHP was by Richetta and Odoni [19], [20]. Subsequently Ball et al. [1] proposed another model for the same problem. Recently, Mukherjee and Hansen [17] study the SAGHP in a dynamic stochastic setting. Kotnyek and Richetta [15] present equitable models for the stochastic SAGHP and prove the equivalence of integrality and equity in the model presented in [20]. While there has been some work in a single-airport setting, we are not aware of any paper which takes into account uncertainty in a network setting. Our study *marks the first attempt at modeling the network ATFM problem in a stochastic setting*.

There are primarily two approaches in the literature to address decision-making under uncertainty, namely, i) *Stochastic Programming*; and ii) *Robust Optimization*. Dantzig [13] proposed the approach of stochastic programming which entails generating scenarios for uncertain data with appropriate probabilities. Unfortunately, this approach suffers significantly from the practical difficulty of not knowing the exact distribution of the data to generate relevant scenarios. Furthermore, it generally becomes intractable quickly as the number of scenarios increases, thereby posing substantial computational challenges. In the last two decades, an alternative approach by the name of robust optimization has been studied to overcome these challenges (see Bertsimas et al. [4] and the recent book by Ben Tal et al. [3] for extensive literature review and references). The key idea of robust optimization is to construct appropriate uncertainty sets for the uncertain parameters that captures the probabilistic properties of the problem. The goal subsequently is to construct a solution that is feasible for all outcomes of the uncertainty set and which optimizes the worst-case objective. The key advantage of robust optimization is that it presents a tractable framework to model optimization problems under uncertainty. Specifically, the robust counterpart of a linear optimization problem (LOP) is still a LOP [4].

Although, robust optimization has been successful in tractably solving many class of optimization problems with uncertainty, it may suffer from a pitfall of the possibility of highly conservative solutions. This is a consequence of the optimization over the worst-case realization of the uncertain parameters. This drawback is further aggravated in multi-stage

problems as robust optimization produces a single (static) solution. Consequently, there is an alternative paradigm for multi-period decision-making called *adaptive optimization* wherein decisions are adapted to capture the progressive information revealed over time. There are two classes of models within adaptive optimization: i) *policy-based full adaptability*; and ii) *finite adaptability*.

Within the fully-adaptable framework, the most extensively studied class of policies is *affine*<sup>2</sup>. The success of affine policies is due to its computational tractability and strong empirical evidence reported in a variety of application settings. Bertsimas et al. [5] study the performance of affine policies in two-stage adaptive problem. They show that affine policies are optimal for two-stage adaptive optimization problem for simplex uncertainty sets. These advances in robust and multi-stage adaptive optimization theory and technology facilitates our endeavor of addressing uncertainty in the typically large scale ATFM models.

## 1.1 Our Proposal

Our overall strategy to address capacity uncertainty in the network-wide ATFM problem under the robust and adaptive paradigm consists of two inter-related issues:

- **Issue I:** *Model of weather-induced uncertain capacity.* We propose a low-dimensional uncertainty model (viz., which can be described by few underlying parameters) that intuitively captures the dynamics of moving weather fronts. Normally, the most severe disruptions occur due to the presence of an ongoing weather front which traverses through some part of the NAS. We believe that the uncertainty governing the movement of this weather front is dictated by the uncertainty in the time of arrival, the duration of impact and the reduction in capacity commensurate with the intensity of the front. The overall uncertainty model is the result of the impact of a small number of weather fronts moving across the NAS. We refer to this model as a *weather-front based approach*.
- **Issue II:** *Tractable solution methodologies for the robust and adaptive ATFM problem.* Given the model for capacity uncertainty, we invoke the recent advances in the theory of robust and adaptive optimization as surveyed earlier to solve the uncertain ATFM problem. Specifically, we prove the equivalence of the robust problem to a modified instance of the deterministic problem and solve optimally the LP relaxation of the adaptive problem using affine policies. The latter task is achieved by counting the number of extreme points in our uncertainty set which govern the number of affine pieces needed in an optimal recourse.

**Contributions and Outline.** The key contributions of this paper are as follows: a) we model weather uncertainty in a way that is consistent with existing data, yet it

---

<sup>2</sup>Affine policies have been studied extensively in control theory (please refer to survey by Bemporad et al. [2] for details).

allows the development of a tractable robust and adaptive optimization approach; b) our approach maintains the strong integrality properties of the deterministic model by developing appropriate theoretical results; and c) we report computational results to validate that our approach is promising.

The paper is structured as follows: Section 2 introduces our modeling approach for uncertain capacity and formalizes the robust and adaptive ATFM problem. Section 3 describes our uncertainty model under a mathematical programming framework. Section 4 presents the tractable solution methodologies for the robust and adaptive problem. Section 5 reports computational results. Section 6 presents the conclusions and directions for further research.

**Preliminaries.** We fix some notation now that we use throughout the rest of the paper. We denote by  $\mathbf{e}$  the unit vector  $(1, \dots, 1)$  comprising of all ones. The dimension is implicit in the context (e.g.,  $\mathbf{e}'\mathbf{w}$  where  $\mathbf{w} \in \mathbb{R}^3 \implies \mathbf{e} \in \mathbb{R}^3$ ). Furthermore, *vertical vector concatenation* is denoted by the comma  $(,)$  operator, e.g.,  $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$  and  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ , then  $(\mathbf{u}, \mathbf{v}) \triangleq (u_1, \dots, u_n, v_1, \dots, v_n)$ .

We refer to vectors specific to time-period  $t$  in two ways: i) including the index in parenthesis, e.g.,  $\mathbf{w}(t)$ , or ii) as a subscript, e.g.,  $\mathbf{w}_t$ . We define, for any time-varying vector quantity  $\{\mathbf{b}_t \in \mathbb{R}^n\}_{t=1, \dots, T}$ , the following *stacked vector*  $\mathbf{b}_{[t]} \triangleq (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_{t-1}) \in \mathbb{R}^{n \times (t-1)}$ , which represents measurements available at the beginning of period  $t$ . We use  $\oplus$  to denote *polyhedron concatenation*. More precisely, for two polyhedron  $P_1$  and  $P_2$ ,  $P_1 \oplus P_2 = \{(x, y) \mid x \in P_1, y \in P_2\}$ .

## 2 ATFM Problem under Capacity Uncertainty

In this section, we first describe the deterministic ATFM problem. Subsequently, we introduce our model of capacity uncertainty. Finally, we end by formalizing the robust and adaptive models we intend to solve.

### 2.1 Deterministic ATFM Problem: Notation and Model

Here, we reproduce an augmented version of the well-studied Bertsimas Stock-Patterson model [11] for the deterministic ATFM problem proposed recently by Bertsimas and Gupta [8]. As mentioned in the Introduction, this latter model incorporates the critical aspect of fairness amongst airlines by controlling the number of reversals in the resulting flight sequences.

**Notation.** The model's formulation requires definition of the following notation:

- $\mathcal{K}$  : set of airports,
- $\mathcal{F}$  : set of flights,
- $\mathcal{T}$  : set of time periods,
- $\mathcal{S}$  : set of sectors,
- $\mathcal{S}^f \subseteq \mathcal{S}$  : set of sectors that can be flown by flight  $f$ ,
- $\mathcal{C}$  : set of pairs of flights that are continued,
- $\mathcal{R}$  : set of pairs of flights that are reversible (definition below),
- $P_i^f$  : sector  $i$ 's preceding sector in flight  $f$ 's path,
- $L_i^f$  : sector  $i$ 's subsequent sector in flight  $f$ 's path,
- $D_k(t)$  : departure capacity of airport  $k$  at time  $t$ ,
- $A_k(t)$  : arrival capacity of airport  $k$  at time  $t$ ,
- $S_j(t)$  : capacity of sector  $j$  at time  $t$ ,
- $d_f$  : scheduled departure time of flight  $f$ ,
- $a_f$  : scheduled arrival time of flight  $f$ ,
- $s_f$  : turnaround time of an airplane after flight  $f$ ,
- $\text{orig}_f$  : airport of departure of flight  $f$ ,
- $\text{dest}_f$  : airport of arrival of flight  $f$ ,
- $l_{fj}$  : minimum number of time units that flight  $f$  must spend in sector  $j$ ,
- $M$  : maximum permissible delay for a flight,
- $\delta$  : tradeoff parameter between ground-hold and air-borne delay costs,
- $T_j^f = [\underline{T}_j^f, \overline{T}_j^f]$  : set of feasible time periods for flight  $f$  to arrive in sector  $j$ ,
- $T_{f,f'}^{\text{reversal}}$  : set of time-periods common for flights  $f$  and  $f'$  where a reversal could occur.

**Definition 2.1.** A pair of flights  $(f, f')$  belongs to  $\mathcal{R}$  if  $\text{dest}_f = \text{dest}_{f'}$  and  $a_f \leq a_{f'} \leq a_f + M$ .

For each pair of flights  $(f, f') \in \mathcal{R}$ , we count a *reversal*, if in the resulting solution, flight  $f'$  arrives before flight  $f$  (i.e.,  $\exists t$  such that  $w_{\text{dest}_{f'}, t}^{f'} > w_{\text{dest}_f, t}^f$ ). Figure 2 depicts a reversible pair of flights  $(f, f') \in \mathcal{R}$ .

**The Decision Variables.** The decision variables are:

$$w_{j,t}^f = \begin{cases} 1, & \text{if flight } f \text{ arrives at sector } j \text{ by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

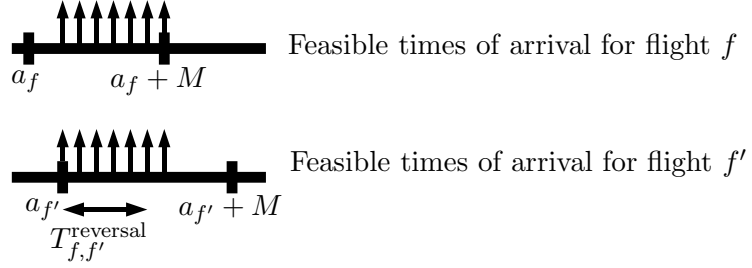


Figure 2: A reversible pair of flights  $(f, f') \in \mathcal{R}$ .

This definition of the decision variables, using “ $by$ ” instead of “ $at$ ”, is critical to the understanding of the formulation. The variables are defined only for the set of sectors an aircraft may fly through on its route to the destination airports. In addition, variables are used for the departure and the arrival airports, in order to determine the optimal times for departure and for arrival.

In addition, for each element  $(f, f') \in \mathcal{R}$ , we introduce the following new variable:

$$s_{f,f'} = \begin{cases} 1, & \text{if there is a reversal,} \\ 0, & \text{otherwise.} \end{cases}$$

**The TFMP model.** The cost function for each flight consists of a linear combination of ground-hold delay (GD) and air-borne delay (AD) with a tradeoff parameter  $\delta > 1$  as air-borne delay is more costly. Consequently, the complete description of the TFMP model is as follows:

$$IZ_{\text{TFMP}} = \min_{\mathbf{w}, \mathbf{s}} \sum_{f \in \mathcal{F}} \left( \delta \cdot \left( \sum_{t \in T_{\text{dest}_f}^f} t \cdot (w_{\text{dest}_f, t}^f - w_{\text{dest}_f, t-1}^f) - a_f \right) + (1 - \delta) \right. \\ \left. \left( \sum_{t \in T_{\text{orig}_f}^f} t \cdot (w_{\text{orig}_f, t}^f - w_{\text{orig}_f, t-1}^f) - d_f \right) \right) + \lambda \cdot \left( \sum_{(f, f') \in \mathcal{R}} s_{f,f'} \right)$$

subject to:

$$\sum_{f \in \mathcal{F}: \text{orig}_f = k} (w_{k,t}^f - w_{k,t-1}^f) \leq D_k(t), \quad \forall k \in \mathcal{K}, t \in \mathcal{T}, \quad (1a)$$

$$\sum_{f \in \mathcal{F}: \text{dest}_f = k} (w_{k,t}^f - w_{k,t-1}^f) \leq A_k(t), \quad \forall k \in \mathcal{K}, t \in \mathcal{T}, \quad (1b)$$

$$\sum_{f \in \mathcal{F}: j \in \mathcal{S}_f, j' = L_j^f} (w_{j,t}^f - w_{j',t}^f) \leq S_j(t), \quad \forall j \in \mathcal{S}, t \in \mathcal{T}, \quad (1c)$$

$$w_{j,t}^f - w_{j',t-l_{fj'}}^f \leq 0, \quad \forall f \in \mathcal{F}, t \in T_j^f, j \in \mathcal{S}^f : j \neq \text{orig}_f, j' = P_j^f, \quad (1d)$$

$$w_{\text{orig}_f,t}^f - w_{\text{dest}_{f'},t-s_f}^{f'} \leq 0, \quad \forall (f, f') \in \mathcal{C}, \forall t \in T_k^f, \quad (1e)$$

$$w_{j,t-1}^f - w_{j,t}^f \leq 0, \quad \forall f \in \mathcal{F}, j \in \mathcal{S}^f, t \in T_j^f, \quad (1f)$$

$$w_{\text{dest}_{f'},t}^{f'} \leq w_{\text{dest}_f,t}^f + s_{f,f'}, \quad \forall (f, f') \in \mathcal{R}, t \in T_{f,f'}^{\text{reversal}}, \quad (1g)$$

$$w_{\text{dest}_f,t}^f \leq w_{\text{dest}_{f'},t}^{f'} + 1 - s_{f,f'}, \quad \forall (f, f') \in \mathcal{R}, t \in T_{f,f'}^{\text{reversal}}, \quad (1h)$$

$$s_{f,f'} \in \{0, 1\}, \quad \forall (f, f') \in \mathcal{R}, \quad (1i)$$

$$w_{j,t}^f \in \{0, 1\}, \quad \forall f \in \mathcal{F}, j \in \mathcal{S}^f, t \in T_j^f. \quad (1j)$$

The first three sets of constraints take into account the capacities of the various elements of the system. Constraints (1a) ensure that the number of flights which may take off from airport  $k$  at time  $t$ , will not exceed the departure capacity of airport  $k$  at time  $t$ . Likewise, Constraints (1b) ensure that the number of flights which may arrive at airport  $k$  at time  $t$ , will not exceed the arrival capacity of airport  $k$  at time  $t$ . Finally, Constraints (1c) ensure that the total number of flights which may feasibly be in Sector  $j$  at time  $t$  will not exceed the capacity of Sector  $j$  at time  $t$ .

The next three sets of constraints capture the various connectivities - namely sector, flight and time connectivity. Constraints (1d) stipulate that a flight cannot arrive at Sector  $j$  by time  $t$  if it has not arrived at the preceding sector by time  $t - l_{fj'}$ . In other words, a flight cannot enter the next sector on its path until it has spent at least  $l_{fj'}$  time units (the minimum possible) traveling through one of the preceding sectors on its current path. Constraints (1e) represent connectivity between flights. They handle the cases in which a flight is continued, i.e., the flight's aircraft is scheduled to perform a subsequent flight within some user-specified time interval. The first flight in such cases is denoted as  $f'$  and the subsequent flight as  $f$ , while  $s_f$  is the minimum amount of time needed to prepare flight  $f$  for departure, following the landing of flight  $f'$ . Constraints (1f) ensure connectivity in time. Thus, if a flight has arrived at element  $j$  by time  $\tilde{t}$ , then  $w_{j,t}^f$  has to have a value of 1 for all later time periods ( $t \geq \tilde{t}$ ).

If there is a reversal between flights  $f$  and  $f'$ , i.e.,  $s_{f,f'} = 1$ , then Constraint (1g) becomes redundant and Constraint (1h) stipulates that if flight  $f$  has arrived by time



$t$ , then flight  $f'$  has to arrive by that time, hence ensuring that flight  $f$  cannot arrive before flight  $f'$ . Similarly, if there is no reversal, i.e.,  $s_{f,f'} = 0$ , then Constraint (1h) becomes redundant and Constraint (1g) stipulates that if flight  $f'$  has arrived by time  $t$ , then flight  $f$  has to arrive by that time, hence ensuring that flight  $f'$  cannot arrive before flight  $f$ . Thus, we are able to model a reversal with the addition of only one variable ( $s_{f,f'}$ ).

For clearer exposition, we work with the following concise description of (TFMP) in the remainder of this paper (the variables  $\mathbf{w}$  below are assumed to include  $\mathbf{s}$ ):

$$\begin{aligned}
 IZ_{\text{TFMP}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\
 \text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}, \\
 & \mathbf{w} \in \{0, 1\}^n.
 \end{aligned} \tag{2}$$

Given the deterministic capacity estimate  $\mathbf{b}$ , our next task is to construct an appropriate set of possible capacity realizations. We will denote this set by  $\mathcal{U}$ . We will refer to the TFMP problem with the added complexity of weather-induced uncertain capacity as TFMPWU where WU is the abbreviation of weather-induced uncertainty.

## 2.2 Model of weather-front induced capacity uncertainty

In this section, we model capacity uncertainty by considering the impact of a small number of weather fronts moving across the airspace. An important advantage of this approach is that a few parameters can be used to capture the dynamics of a weather-front; thereby, leading to a low-dimensional description of the uncertainty set.

To motivate our approach, consider a day in which it is known with high fidelity that the storm would start during a certain time-period of the day (say, 4 to 6 pm on a particular day). But, within this two hour interval, it is difficult to predict the exact time of arrival of storm, and thus it seems prudent to consider uncertainty in the time of arrival. Furthermore, the resulting drop in capacity over the planning period is not known exactly. An overly pessimistic estimate might result in loss of capacity, whereas, an optimistic estimate might impact the subsequent connections if the actual reduction is worse. Thus, there is uncertainty associated with the intensity of weather disruption. Finally, the last time of the impact of the storm (or alternatively, the duration of the storm) is not known precisely and should be captured in the uncertainty set. Thus, we feel, that a reasonable set of parameters which correctly capture the dynamics of a weather front are the time of arrival, the duration and the capacity reduction.

**Exact description for single airspace element.** Following this discussion, we next describe the uncertainty set for a single airspace element. The key parameters of the weather-front affecting the said airspace element are:

1. Actual time of arrival:  $T_a \in \{\underline{T}_a, \dots, \bar{T}_a\}$ .
2. Duration:  $d \in \{\underline{d}, \dots, \bar{d}\}$ .
3. Reduction in capacity:  $\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}$ .

Subsequently, we use  $T_b$  to denote  $T_a + d$  (time of revival of the weather-front), which implies that given the bounds on  $T_a$  and  $d$ , we have  $T_b \in \{\underline{T}_a + \underline{d}, \dots, \bar{T}_a + \bar{d}\}$ .

For a particular realization of the three parameters,  $T_a$ ,  $d$  and  $\alpha$ , the capacity vector  $\mathbf{b}$  can be written as follows:

$$\mathbf{b} = (C, \dots, C, \alpha C, \dots, \alpha C, C, \dots, C),$$

that is (see also Figure 3),

$$b_k = \begin{cases} \alpha C, & k \in \{T_a, \dots, T_b\}, \\ C, & k \in \mathcal{T} \setminus \{T_a, \dots, T_b\}. \end{cases}$$

*Assumption 1.* We assume that the value of  $\alpha$  is such that  $\alpha C$  is an integer because of two reasons: i) it is not practical to have fractional capacity and ii) the integrality proofs presented later become simplified as a result of this assumption.



Figure 3: Depiction of the capacity profile under a *weather-front* based uncertainty set for a single affected airspace element. The plot is for a particular realization of the parameters  $(T_a, d, \alpha)$ .

**Example 2.1.** We now give a concrete example of the uncertainty set we are trying to model. Suppose, we have a time-horizon of 5 periods, i.e,  $\mathcal{T} = 5$ . Moreover, let  $T_a \in \{3, 4\}$ ,  $d \in \{1, 2\}$ ,  $\alpha \in \{0.6, 0.8\}$  and  $C = 30$ . Then,  $\mathcal{U} = \{(30, 30, 18, 30, 30), (30, 30, 18, 18, 30), (30, 30, 30, 18, 30), (30, 30, 30, 18, 18), (30, 30, 24, 30, 30), (30, 30, 24, 24, 30), (30, 30, 30, 24, 30), (30, 30, 30, 24, 24)\}$ .

To concretely motivate the appropriateness of our proposal for modeling the capacity uncertainty, we present practical evidence of the applicability of fitting step functions to actual capacity profiles. Figure 4 plots the capacity profiles for two sectors from data obtained from Lincoln Labs. It should be evident from the plot that fitting these profiles by step functions is an appropriate engineering approximation.

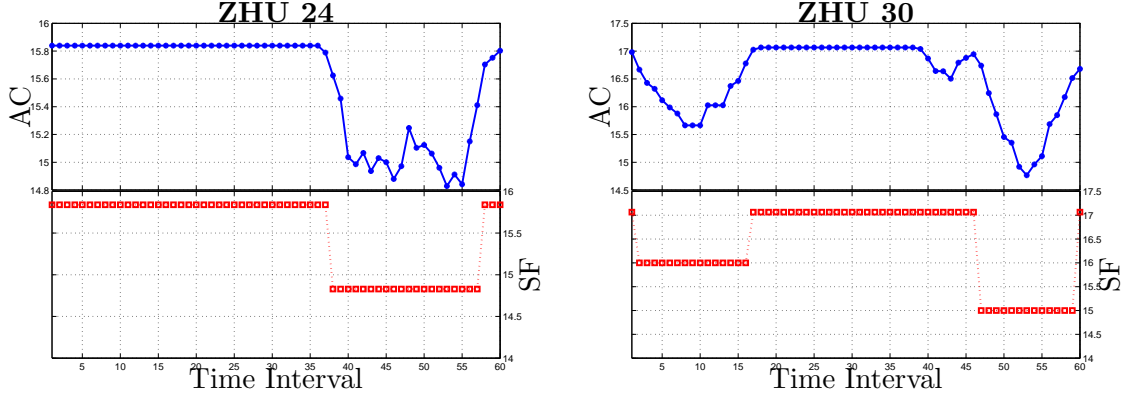


Figure 4: Illustration of the applicability of step functions to model capacity profiles. **AC** denotes actual sector capacity and **SF** denotes the step function capacity. ZHU 24 and ZHU 30 are sector titles.

**Extending to an airspace setting.** We now extend the uncertainty set proposed for a single airspace element to multiple airspace elements. We define a weather front as an entity which is independent of all weather disturbances from other fronts. Suppose we have  $k$  weather fronts (denoted by  $\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_k$ ) over the course of a day. On a typical day,  $k$  would take a small value (say  $k = 3, 4$ ). We model the traversal of a weather front through the day by dividing it into a number of *phases*, where a phase transition materializes when the spatial or temporal composition of the front changes substantially. Each weather front  $\mathcal{W}_i$  is further decomposed into phases:  $(\mathcal{W}_i^1, \dots, \mathcal{W}_i^{p_i})$ , where  $p_i$  is the number of phases of the  $i^{\text{th}}$  weather front.  $\mathcal{W}_i^j$  captures the snapshot of the weather front during phase  $j$  as follows:

- $\mathcal{W}_i^j(\mathcal{S})$  : set of sectors impacted by the  $i^{\text{th}}$  weather front during phase  $j$
- $\mathcal{W}_i^j(\mathcal{K})$  : set of airports impacted by the  $i^{\text{th}}$  weather front during phase  $j$

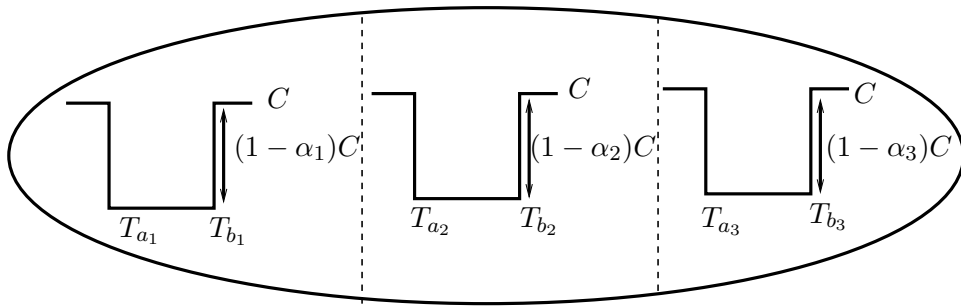


Figure 5: Traversal of a *weather-front* across the NAS. It has three phases over the course of its existence.

**Model of weather front propagation.** A key distinction between the modeling

approach for a single airspace element and multiple airspace elements is the augmented set of parameters governing the uncertainty set for the latter case. Specifically, we introduce another parameter  $L$  which captures the lag between the front's time of arrival across consecutive phases. During phase I, the parameters  $(T_{a_1}, d_1, \alpha_1)$  govern the capacity realizations for  $\mathcal{W}_i^1$ . Subsequently, the lag (denoted by  $L_1$ ) dictates the set of parameters  $(T_{a_2}, d_2, \alpha_2)$  of the weather front in phase II, and so on so forth. This way we model the strong correlation of  $T_{a_2}$  with  $T_{a_1}$ . More generally,

$$T_{a_k} = T_{a_{k-1}} + L_{k-1} = T_{a_1} + \sum_{j=1}^{k-1} L_j$$

We model the uncertainty inherent in  $L_j$  in an analogous way as  $T_a$ , i.e.,  $L_j \in \{\underline{L}_j, \dots, \overline{L}_j\}$ . Subsequently, the range of possible values for  $T_{a_k}$  is given by  $\{\underline{T}_{a_{k-1}} + \underline{L}_{k-1}, \dots, \overline{T}_{a_{k-1}} + \overline{L}_{k-1}\}$ . Note that, even though we use the additional parameter  $L_j$  for front propagation, the eventual description is described completely by  $(T_a, d, \alpha)$ .

### 2.3 Models

We now formalize the setting in which we intend to solve the robust and adaptive ATFM problem. Using the discretization  $\mathcal{T} = \{1, \dots, T\}$  introduced in the formulation of TFMP (the deterministic ATFM problem), the right-hand side capacity vector in the formulation TFMP can be decomposed into time indexed sub-vectors, i.e., it can be written as:

$$\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_T),$$

where  $\mathbf{b}_t$  has the following description:

$$\mathbf{b}_t = (\mathbf{b}_t^A, \mathbf{b}_t^D, \mathbf{b}_t^S).$$

$\mathbf{b}_t^A, \mathbf{b}_t^D, \mathbf{b}_t^S$  correspond to the components of  $\mathbf{b}_t$  pertaining to the arrival capacity at airports, departure capacity at airports and sector capacities respectively.

$$\mathbf{b}_t^A = (A_1(t), \dots, A_{|\mathcal{K}|}(t)); \quad \mathbf{b}_t^D = (D_1(t), \dots, D_{|\mathcal{K}|}(t)); \quad \mathbf{b}_t^S = (S_1(t), \dots, S_{|S|}(t));$$

**Multi-stage models.** We first present the multi-stage adaptive and robust optimiza-

tion models (over  $T$  periods):

$$\begin{aligned}
IZ_{\text{Adapt}} = \min_{\mathbf{w}_1, \mathbf{w}_i(\mathbf{b}_{[i]})} & \left[ \mathbf{c}'_1 \mathbf{w}_1 + \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_2 \mathbf{w}_2(\mathbf{b}_{[2]}) + \cdots + \right. \right. \\
& \left. \left. \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_{T-1} \mathbf{w}_{T-1}(\mathbf{b}_{[T-1]}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_T \mathbf{w}_T(\mathbf{b}_{[T]}) \right] \right] \right] \\
\text{s.t. } & \mathbf{A}_1 \mathbf{w}_1 + \sum_{i=2}^T \mathbf{A}_i \mathbf{w}_i(\mathbf{b}_{[i]}) \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\
& \mathbf{w}_i(\mathbf{b}_{[i]}) \in \{0, 1\}^{n_i}.
\end{aligned} \tag{\Pi_{\text{Adapt}}^T}$$

In the problem  $\Pi_{\text{Adapt}}^T$ , the uncertainty only affects the right-hand side capacity vector  $\mathbf{b}$ . The constraint matrices  $\mathbf{A}_i$  and the cost vectors  $\mathbf{c}_i$  are assumed certain. The only restriction on the uncertainty set  $\mathcal{U}$  is that  $\mathcal{U} \subseteq \mathbb{R}_+^n$  or  $\mathcal{U} \subseteq \mathbb{Z}_+^n$  (this corresponds to the requirement of non-negative capacities). The equivalent robust version of the multi-stage adaptive problem is as follows:

$$\begin{aligned}
IZ_{\text{Rob}} = \min_{\mathbf{w}} & \sum_{i=1}^T \mathbf{c}_i \mathbf{w}_i \\
\text{s.t. } & \sum_{i=1}^T \mathbf{A}_i \mathbf{w}_i \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\
& \mathbf{w}_i \in \{0, 1\}^{n_i}.
\end{aligned} \tag{\Pi_{\text{Rob}}}$$

Let  $Z_{\text{Adapt}}$  and  $Z_{\text{Rob}}$  denote the optimal objective function for the LP Relaxation.

In the remainder of the paper, we denote by TFMPWUAdapt, the TFMPWU problem incorporating adaptability and TFMPWURob, the TFMPWU problem incorporating robustness.

### 3 Characterization of Weather-front Induced Uncertainty Set

We now proceed towards incorporating the discrete uncertainty set introduced in Section 2 within a mathematical programming framework to solve TFMPWURob and TFMPWUAdapt.

#### 3.1 The need for $\text{conv}(\mathcal{U})$

A LOP under a discrete uncertainty set does not have a similar fate. In fact, the robust problem under discrete uncertainty set is  $\mathcal{NP}$ -Hard, in general. Consider the

following general robust problem (RobDU denotes robust optimization problem under discrete uncertainty set):

$$\begin{aligned} Z_{\text{RobDU}} = \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}_0. \end{aligned} \quad (3)$$

**Lemma 1.** *RobDU is  $\mathcal{NP}$ -Hard.*

Not surprisingly, we need an alternative tractable line of attack for the case of discrete uncertainty sets. A standard approach to overcome this hurdle is to work with its convex hull to enable a polyhedral description. Hence, if the convex hull of the discrete uncertainty set can be written as a polyhedral set, then the robust counterpart is efficiently solvable. We formalize below the equivalence of solving over a discrete set and over the convex hull of this set<sup>3</sup>.

**Lemma 2.** *Let  $\mathcal{U}_0$  denote a discrete set. Then, the robust problem  $Z_{\text{RobDU}}$  is equivalent to:*

$$\begin{aligned} Z_{\text{RobConvDU}} = \min_{\mathbf{x}} \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{Ax} \leq \mathbf{b}, \quad \forall \mathbf{b} \in \text{conv}(\mathcal{U}_0). \end{aligned} \quad (4)$$

where  $\text{conv}(\mathcal{U}_0)$  denotes the convex hull of  $\mathcal{U}_0$ .

Consequently, we subject ourselves to the challenge of generating a polyhedral description for  $\text{conv}(\mathcal{U})$  (under the discrete uncertainty set introduced in Section 2.2). Our first key contribution (documented in Section 3.2) culminates with the success in this challenge.

### 3.2 Polyhedral description of $\text{conv}(\mathcal{U})$

We use the following auxiliary variables:

$$y_t = \begin{cases} 1, & \text{if capacity drops by time } t, \\ 0, & \text{otherwise.} \end{cases} \quad z_t = \begin{cases} 1, & \text{if capacity revives by time } t, \\ 0, & \text{otherwise.} \end{cases}$$

The variables  $y_t$  and  $z_t$  are defined for  $t \in \{\underline{T}_a, \dots, \overline{T}_b\}$ . But, given that the capacity drop cannot occur after  $\overline{T}_a$  and capacity revival cannot occur before  $\underline{T}_b$ , we can set a few variables as parameters at the outset, namely,

$$\left\{ y_t = 1, t \in \{\overline{T}_a, \dots, \overline{T}_b\}; \quad z_t = 0, t \in \{\underline{T}_a, \dots, \underline{T}_b - 1\}; \quad z_t = 1, t \in \{\overline{T}_b\} \right\}$$

---

<sup>3</sup>the proofs of the two lemmas are available in the full version and can be requested from the author

We give a mathematical description of the uncertainty set for the capacity profile as depicted in Figure 3. We start by providing a description for a particular realization of  $\alpha$ :

$$\begin{aligned}\mathcal{U}_\alpha = \{ & \mathbf{b} \in \mathbb{Z}_+^m \mid b_t = C(1 - y_t) + \alpha C y_t + (1 - \alpha) C z_t, \forall t \in \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & b_t = C, \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t; \\ & y_{\overline{T}_a} = 1; z_{\underline{T}_b-1} = 0; z_{\overline{T}_b} = 1; y_t, z_t \in \{0, 1\}\}\end{aligned}$$

Let  $|\mathcal{U}_\alpha| = K$ , where  $K = (\overline{T}_a - \underline{T}_a + 1) \times (\overline{d} - \underline{d} + 1)$ . The  $K$  elements of  $\mathcal{U}_\alpha$  are indexed by  $(i, j)$  where  $i \in \{\underline{T}_a, \dots, \overline{T}_a\}$ ,  $j \in \{i + \underline{d}, \dots, i + \overline{d}\}$ .  $\mathbf{b}^{i,j}$  can be written concretely as follows:

$$b_k^{i,j} = \begin{cases} \alpha C, & k \in \{i, \dots, j\}, \\ C, & k \in \mathcal{T} \setminus \{i, \dots, j\}. \end{cases} \quad (5)$$

**A polyhedral description of  $\text{conv}(\mathcal{U}_\alpha)$ .** Our goal now is to come up with a polyhedral description of  $\text{conv}(\mathcal{U}_\alpha)$  (denoted by  $\mathcal{Q}_\alpha$ ). We claim that the following polyhedron is precisely  $\mathcal{Q}_\alpha$ :

$$\begin{aligned}\mathcal{P}_\alpha = \{ & \mathbf{b} \in \mathbb{R}_+^m \mid b_t = C(1 - y_t) + \alpha C y_t + (1 - \alpha) C z_t, \forall t \in \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & b_t = C, \forall t \in \mathcal{T} \setminus \{\underline{T}_a, \dots, \overline{T}_b\}; \\ & y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t; \\ & y_{\overline{T}_a} = 1; z_{\underline{T}_b-1} = 0; z_{\overline{T}_b} = 1; 0 \leq y_t, z_t \leq 1\}\end{aligned}$$

*Remark 1.* In the description of  $\mathcal{U}_\alpha$  and  $\mathcal{P}_\alpha$ , it is useful to note that if  $\overline{T}_a < \underline{T}_b$ , then, the constraints  $z_t \leq y_t$  are redundant and hence, can be removed from the polyhedral description.

**Proposition 1.**

$$\mathcal{Q}_\alpha = \mathcal{P}_\alpha \quad (6)$$

*Proof.* We use a technique to prove integrality of a polyhedron based on randomization known as *Randomized Rounding* (please refer to Chapter 3 of Bertsimas and Weisman-tel [12] for details). Let  $\mathbf{y}^*, \mathbf{z}^*$  be an optimal solution of the following problem:

$$\begin{aligned}Z_{\text{LP}} = \min_{\mathbf{y}, \mathbf{z}} & \quad \mathbf{c}'\mathbf{y} + \mathbf{d}'\mathbf{z} \\ \text{s.t.} & \quad (\mathbf{y}, \mathbf{z}) \in \mathcal{X}. \end{aligned} \quad (7)$$

where  $\mathcal{X} = \{\mathbf{y}, \mathbf{z} \mid y_t \leq y_{t+1}, z_t \leq z_{t+1}, z_t \leq y_t, t \in \{\underline{T}_a, \dots, \overline{T}_b\}; 0 \leq y_t, z_t \leq 1\}$ .

*Outline of the argument:* From  $\mathbf{y}^*, \mathbf{z}^*$ , we create a new random integer solution  $(\mathbf{y}, \mathbf{z})$ , that is feasible in  $\mathcal{X}$ . Subsequently, we show that  $\mathbb{E}[\mathbf{c}'\mathbf{y} + \mathbf{d}'\mathbf{z}] = Z_{\text{LP}} = \mathbf{c}'\mathbf{y}^* + \mathbf{d}'\mathbf{z}^*$ . This shows that  $Z_{\text{IP}} = Z_{\text{LP}}$ , and, since this is true for an arbitrary cost vector  $\mathbf{c}$  and  $\mathbf{d}$ , the polyhedron  $\mathcal{X}$  is integral.

The randomization we use is as follows: we sort the values  $\mathbf{y}^*, \mathbf{z}^*$  from smallest to largest in the interval  $[0, 1]$ . We generate a random variable  $U$  distributed uniformly in  $[0, 1]$ . The rounding is done as follows:

$$y_t = \begin{cases} 1, & y_t^* \geq U, \\ 0, & y_t^* < U. \end{cases} \quad z_t = \begin{cases} 1, & z_t^* \geq U, \\ 0, & z_t^* < U. \end{cases}$$

The solution produced is clearly feasible because  $y_t \leq y_{t+1}$  and  $z_t \leq z_{t+1}$  is trivially satisfied. Moreover,  $z_t \leq y_t$  because  $z_t^* \leq y_t^*$ , and after rounding  $z_t$  can never become 1 unless  $y_t$  becomes 1 (see Figure 6 for an easy visualization).

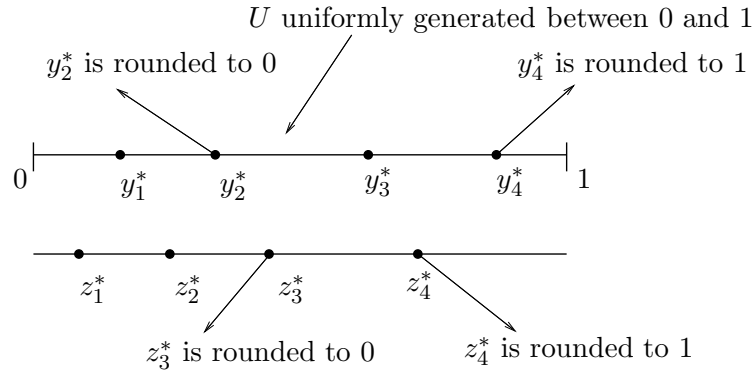


Figure 6: Illustration of the geometry of the *randomized rounding* algorithm for proving the integrality of polyhedron  $\mathcal{X}$ .  $\mathbf{y}^*, \mathbf{z}^*$  satisfy  $y_t^* \leq y_{t+1}^*$ ,  $z_t^* \leq z_{t+1}^*$  and  $z_t^* \leq y_t^*$ .

Let  $Z_{\text{H}}$  be the value of the solution produced. The expected value of the solution is:

$$\begin{aligned} \mathbb{E}[Z_{\text{H}}] &= \sum_{t \in \{\underline{T}_a, \dots, \bar{T}_b\}} c_t \mathbb{P}(y_t = 1) + d_t \mathbb{P}(z_t = 1) \\ &= \sum_{t \in \{\underline{T}_a, \dots, \bar{T}_b\}} c_t \mathbb{P}(y_t^* \leq U) + d_t \mathbb{P}(z_t^* \leq U) \\ &= \sum_{t \in \{\underline{T}_a, \dots, \bar{T}_b\}} c_t y_t^* + d_t z_t^* \\ &= Z_{\text{LP}} \end{aligned}$$

and thus  $Z_{\text{IP}} = Z_{\text{LP}}$ . Since,  $\mathbf{c}$  and  $\mathbf{d}$  are arbitrary, the polyhedron  $\mathcal{X}$  is integral. This implies that  $\mathcal{P}_\alpha$  is integral (using Assumption 1).



□

The overall uncertainty set for the capacity profile as depicted in Figure 3 is as follows:

$$\mathcal{U} = \left( \bigcup_{\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}} \mathcal{U}_\alpha \right)$$

**A polyhedral description of  $\text{conv}(\mathcal{U})$ .** Let  $\mathcal{Q} = \text{conv}(\mathcal{U})$  denote the convex hull of  $\mathcal{U}$ . We show that  $\mathcal{P}_\alpha$  is precisely  $\text{conv}(\mathcal{U})$ .

**Proposition 2.**

$$\mathcal{Q} = \mathcal{Q}_\alpha \tag{8}$$

*Proof.* Note the following relation,

$$\bigcup_{\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}} \mathcal{Q}_\alpha = \bigcup_{\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}} \text{conv}(\mathcal{U}_\alpha) \subseteq \text{conv}\left(\bigcup_{\alpha \in \{\underline{\alpha}, \dots, \bar{\alpha}\}} \mathcal{U}_\alpha\right) = \text{conv}(\mathcal{U}) = \mathcal{Q}$$

In particular, for  $\alpha = \underline{\alpha}$ ,

$$\mathcal{Q}_\alpha \subseteq \mathcal{Q} \tag{9}$$

Next, we show that  $\mathcal{U} \subseteq \mathcal{Q}_\alpha$  by using the equivalence of  $\mathcal{Q}_\alpha$  and  $\mathcal{P}_\alpha$  from Proposition 1. For a particular value of  $\alpha$ , let  $\mathbf{b}^{i,j} \in \mathcal{U}_\alpha$ . Consider the following<sup>4</sup>:

$$y_k = \begin{cases} 0, & \text{if } k \in \{\underline{T}_a, \dots, \underline{T}_a + i - 1\}, \\ \frac{1-\alpha}{1-\underline{\alpha}}, & \text{if } k \in \{\underline{T}_a + i, \dots, \min(\bar{T}_a - 1, \underline{T}_a + j - 1)\}, \\ 1, & \text{otherwise.} \end{cases}$$

$$z_k = \begin{cases} 0, & \text{if } k \in \{\underline{T}_a, \dots, \min(\bar{T}_a - 1, \underline{T}_a + j - 1)\}, \\ \frac{\alpha-\underline{\alpha}}{1-\underline{\alpha}}, & \text{if } k \in \{\min(\bar{T}_a, \underline{T}_a + j), \dots, \underline{T}_a + j - 1\}, \\ 1, & \text{otherwise.} \end{cases}$$

Moreover, we expand  $b_k$  using:

$$b_k = C(1 - y_k) + \underline{\alpha}C y_k + (1 - \underline{\alpha})C z_k$$

---

<sup>4</sup>The reason for the complicated indexing is that  $y_{\bar{T}_a}$  is set to 1 in the polyhedral description  $\mathcal{P}_\alpha$ . Therefore, the definitions of  $y_k$  and  $z_k$  need to account for whether the capacity drop occurs before or after  $\bar{T}_a$ . Hence, the need for  $\min(\bar{T}_a - 1, \underline{T}_a + j - 1)$ .

for all possible combinations of  $y_k$  and  $z_k$  over the various indices as shown below:

$$b_k = \begin{cases} C + 0 + 0 = C, & \text{if } k \in \{\underline{T}_a, \dots, \underline{T}_a + i - 1\}, \\ \left(\frac{\alpha - \underline{\alpha}}{1 - \underline{\alpha}}\right)C + \underline{\alpha}\left(\frac{1 - \alpha}{1 - \underline{\alpha}}\right)C + 0 = \alpha C, & \text{if } k \in \{\underline{T}_a + i, \dots, \min(\overline{T}_a - 1, \underline{T}_a + j - 1)\}, \\ 0 + \underline{\alpha}C + (1 - \underline{\alpha})\left(\frac{\alpha - \underline{\alpha}}{1 - \underline{\alpha}}\right)C = \alpha C, & \text{if } k \in \{\min(\overline{T}_a, \underline{T}_a + j), \dots, \underline{T}_a + j - 1\}, \\ 0 + \underline{\alpha}C + (1 - \underline{\alpha})C = C, & \text{if } k \in \{\underline{T}_a + j, \dots, \overline{T}_b\}. \end{cases} \quad (10)$$

But, (10) has exactly the form of (5), the element  $\mathbf{b}^{i,j} \in \mathcal{U}_\alpha$ . Hence,  $\mathbf{b}^{i,j} \in \mathcal{P}_\alpha$ . This implies  $\mathcal{U}_\alpha \subseteq \mathcal{P}_\alpha$ . Since  $\alpha$  is arbitrary, therefore,

$$\mathcal{U} \subseteq \underbrace{\mathcal{P}_\alpha = \mathcal{Q}_\alpha}_{\text{Using Proposition 1}}$$

Note that  $\mathcal{Q}_\alpha$  is a convex set (a polyhedron). This implies that all convex combinations of the set  $\mathcal{U}$  belongs to  $\mathcal{Q}_\alpha$ . But,  $\text{conv}(\mathcal{U})$  is precisely  $\mathcal{Q}$ . Hence,

$$\mathcal{Q} \subseteq \mathcal{Q}_\alpha \quad (11)$$

The proposition subsequently follows from (9) and (11).  $\square$

We now have the equipment to state the main theorem of this section:

**Theorem 1.**

$$\mathcal{Q} = \mathcal{P}_\alpha \quad (12)$$

*Proof.* The proof follows trivially from Proposition 1 and Proposition 2.  $\square$

*Remark 2.* An important takeaway of the polyhedral description  $\mathcal{P}_\alpha$  is that the explicit description of the uncertainty governing the parameter  $\alpha$  is not required. All that is needed as input is the worst cast realization of  $\alpha$  (viz.  $\underline{\alpha}$ ). This further simplifies the input requirements and implies that only two parameters  $T_a$  and  $d$  govern the overall uncertainty set.

Let  $\mathcal{P}_{\underline{\alpha}_{ij}}^e$  denote the uncertain realizations for airspace element  $e$ . The overall uncertainty set spanning all airspace elements is given by:

$$\mathcal{P}^{\text{overall}} = \bigoplus_{\substack{i=1, \dots, k, j=1, \dots, p_i, \\ e \in \mathcal{W}_i^j(\mathcal{S}) \cup \mathcal{W}_i^j(\mathcal{K})}} \mathcal{P}_{\underline{\alpha}_{ij}}^e$$

## 4 Solution Methodologies

In this section, we propose solution approaches for the robust and adaptive versions of the TFMP problem under the uncertainty sets constructed in Section 3.

## 4.1 Robust TFMP

We explicitly characterize an equivalent form of the robust TFMP problem as a specific instance of the deterministic TFMP problem for an arbitrary uncertainty set.

**Definition 4.1.**  $\mathbf{b}_{\min} = (\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m)$ , where  $\underline{b}_i = \min \{b_i \mid \mathbf{b} \in \mathcal{U}\}$ .

**Theorem 2.** *For an arbitrary uncertainty set  $\mathcal{U}$  for the right hand side capacity, TFMPWURob is equivalent to solving the following modified TFMP instance:*

$$\begin{aligned} IZ_{\text{TFMPWURob}} = \min_{\mathbf{w}} \quad & \mathbf{c}'\mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{w} \leq \mathbf{b}_{\min}, \\ & \mathbf{w} \in \{0, 1\}^n. \end{aligned} \tag{13}$$

*Proof.* We show the equivalence of the set of feasible solutions of TFMPWURob (denoted by  $\mathcal{W}$ ) with the instance of TFMP where the right-hand side is  $\mathbf{b}_{\min}$  (denoted by  $\overline{\mathcal{W}}$ ). Specifically,  $\mathcal{W} = \{\mathbf{w} \in \{0, 1\}^n \mid \mathbf{A}\mathbf{w} \leq \mathbf{b}, \mathbf{b} \in \mathcal{U}\}$  and  $\overline{\mathcal{W}} = \{\mathbf{w} \in \{0, 1\}^n \mid \mathbf{A}\mathbf{w} \leq \mathbf{b}_{\min}\}$ .

Suppose  $\mathbf{w} \in \mathcal{W}$ , then for the  $i^{\text{th}}$  constraint, we have:

$$\begin{aligned} \mathbf{a}'_i \mathbf{w} &\leq b_i, \forall \mathbf{b} \in \mathcal{U}, \\ \Rightarrow \mathbf{a}'_i \mathbf{w} &\leq \min_{\mathbf{b} \in \mathcal{U}} b_i = \underline{b}_i. \end{aligned} \tag{14}$$

Since  $i$  is arbitrary, (14) further implies the following:

$$\begin{aligned} \mathbf{a}'_i \mathbf{w} &\leq \min_{\mathbf{b} \in \mathcal{U}} b_i, \quad i = 1, \dots, m \\ \Rightarrow \mathbf{A}\mathbf{w} &\leq \mathbf{b}_{\min}, \\ \Rightarrow \mathbf{w} &\in \overline{\mathcal{W}}. \end{aligned} \tag{15}$$

Suppose  $\mathbf{w} \in \overline{\mathcal{W}}$ , then by definition of  $\mathbf{b}_{\min}$ :

$$\begin{aligned} \mathbf{b}_{\min} &\leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\ \Rightarrow \mathbf{A}\mathbf{w} &\leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}, \\ \Rightarrow \mathbf{w} &\in \mathcal{W}. \end{aligned} \tag{16}$$

From (15) and (16), we have  $\mathcal{W} = \overline{\mathcal{W}}$ . The proposition subsequently follows, as the set of feasible solutions of the two problems and the objective function being minimized is exactly the same.  $\square$

Since TFMP is efficiently solvable in practice and has superior integrality properties, Proposition 2 suggests that TFMPWURob is no harder to solve than TFMP.

## 4.2 Adaptive TFMP

In this section, we explicitly propose ways to solve the LP Relaxation of TFMP-WUAdapt optimally. Bertsimas and Goyal [6] study the two-stage adaptive optimization problem  $\Pi_{\text{Adapt}}^2$  and show the optimality of affine policy for a simplex<sup>5</sup> uncertainty set. Although, the results reported in Bertsimas and Goyal [6] are for a two-stage adaptive model, they are true for multi-stage adaptive problems too. Since, the TFMP problem is inherently a multi-period problem, it is appropriate that we extend the result for multiple periods as done below (please refer to Appendix A for the proof):

**Theorem 3.** *Consider the problem  $\Pi_{\text{Adapt}}^T(\mathcal{U})$  such that  $\mathcal{U}$  is a simplex. Then, there is an optimal multi-stage solution  $\hat{\mathbf{w}}_i(\mathbf{b})$  such that  $\hat{\mathbf{w}}_i(\mathbf{b})$  are affine functions of  $\mathbf{b}$ , i.e., for all  $\mathbf{b} \in \mathcal{U}$ ,*

$$\hat{\mathbf{w}}_i(\mathbf{b}) = \mathbf{P}_i \mathbf{b} + \mathbf{q}_i, \quad (17)$$

where  $\mathbf{P}_i \in \mathbb{R}^{n_i \times m}$ ,  $\mathbf{q}_i \in \mathbb{R}^{n_i}$ .

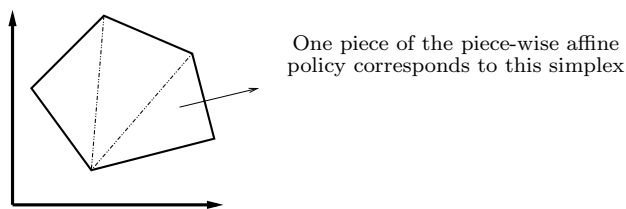


Figure 7: An example illustrating the partition of a polyhedral uncertainty set into multiple simplices. Shown is a polyhedral set in  $\mathbb{R}^2$  broken into three simplices.

Our next step is to quantify the number of extreme points in our uncertainty set for the TFMP problem. Table 1 enumerates the number of extreme points in the different constructions of the polyhedral description of the uncertainty set  $\mathcal{U}$ . Algorithm 1 summarizes a procedure to solve the LP Relaxation of TFMPWUAdapt after considering the value of  $\mathcal{E}$ .

---

**Algorithm 1** Solving the LP Relaxation of TFMPWUAdapt

---

- If  $\mathcal{E} \leq m + 1$ , then an affine policy for the recourse variables is optimal.
  - If  $\mathcal{E} > m + 1$ , then divide the uncertainty set into multiple simplices each having  $m + 1$  extreme points. Subsequently, a piecewise affine policy is optimal for this case.
- 

**Number of Extreme Points Relative to the Number of Constraints.** Table 1 implies that  $\mathcal{E}$  increases exponentially with an increase in the number of airspace

---

<sup>5</sup>A simplex is a set which is generated by  $m + 1$  extreme points, where  $m$  is the number of constraints.

Description (Denoted By)	Uncertainty Set	# of Extreme Points
1 WF, Single airspace element, one phase ( $EP_i$ )	$\mathcal{P}_{\underline{\alpha}_i} = \left\{ \mathbf{b} \in \mathbb{R}_+^m \mid b_t = C(1-y_t) + \underline{\alpha}_i C y_t \right.$ $\left. + (1-\underline{\alpha}_i) C z_t, \forall t \in \{\underline{T}_{a_i}, \dots, \overline{T}_{b_i}\}; \right.$ $b_t = C, \forall t \in \mathcal{T} \setminus \{\underline{T}_{a_i}, \dots, \overline{T}_{b_i}\};$ $\left. y_t \leq y_{t+1}; z_t \leq z_{t+1}; z_t \leq y_t; 0 \leq y_t, z_t \leq 1 \right\}$	$(\overline{T}_{a_i} - \underline{T}_{a_i} + 1) \times (\overline{d}_i - \underline{d}_i + 1)$
1 WF, Airspace, one phase ( $\mathcal{E}_i^j$ )	$\mathcal{P}_{\underline{\alpha}_i}^{\mathcal{AS}, \mathcal{OP}} = \bigoplus_{k \in \mathcal{A}} \mathcal{P}_{\underline{\alpha}_i}^k$	$EP_i^{ \mathcal{W}_i^j(\mathcal{S})  +  \mathcal{W}_i^j(\mathcal{K}) }$
1 WF, Airspace, all phases ( $\mathcal{E}_i$ )	$\mathcal{P}_{\underline{\alpha}_i}^{\mathcal{AS}, \mathcal{AP}} = \bigoplus_{j=1}^{p_i} \mathcal{P}_{\underline{\alpha}_i}^j$	$\prod_{j=1}^{p_i} \mathcal{E}_i^j$
$k$ WF, Airspace, all phases ( $\mathcal{E}$ )	$\mathcal{P}_{\underline{\alpha}}^{\mathcal{AS}, \mathcal{AP}} = \bigoplus_{i=1}^k \mathcal{P}_{\underline{\alpha}_i}$	$\prod_{i=1}^k \mathcal{E}_i$

Table 1: Number of extreme points for the weather-front based polytope.  $\mathcal{AS}$  denotes airspace,  $\mathcal{OP}$  denotes one-phase and  $\mathcal{AP}$  denotes all-phases.

elements affected by weather and an increase in the number of time-periods. In contrast,  $m \propto (2|\mathcal{K}| + |\mathcal{S}|)|\mathcal{T}|$  which implies that it increases linearly in the number of airspace elements and the total number of time-periods. Consequently, in the asymptotic regime,  $\mathcal{E}$  will dominate  $m$ . But, the case which is practically relevant pertains to the number of extreme points when there is an upper bound on  $|\mathcal{W}_i^j(\mathcal{K})|, |\mathcal{W}_i^j(\mathcal{S})|$  and the uncertainty set of  $T_a$  and  $d$ . Let,

$$\tau = \max_i \left\{ \overline{T}_{a_i} - \underline{T}_{a_i} + 1, \overline{d}_i - \underline{d}_i + 1 \right\} \quad \Delta = \max_{i,j} \left\{ |\mathcal{W}_i^j(\mathcal{S})|, |\mathcal{W}_i^j(\mathcal{K})| \right\}$$

$\mathcal{E}$  can now be upper bounded by  $\tau^{4k*\Delta*P}$  where  $P$  is the maximum number of phases across all fronts. Table 2 lists an upper bound on  $\mathcal{E}$  for various combinations of  $\tau$  and  $\Delta$ . The main observation is that if any of the two parameters  $\tau$  or  $\Delta$  is less than 3, then  $\mathcal{E}$  is in the thousands and is thus less than the number of constraints.

$\tau$	$\Delta$	Upper Bound on $\mathcal{E}$
$P = 1$		
1	1	1
2	2	256
2	3	4096
3	2	6561
3	3	531441

Table 2: Upper bound on the number of extreme points in the weather-front based uncertainty set ( $k = 1$ ).

### 4.3 Characterizing the Price of Robustness and Adaptability

In this section, we study the relation between the optimal objective function values of the deterministic, robust and adaptive versions of the TFMP problem.

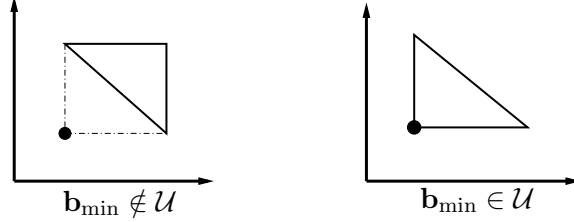


Figure 8: Example uncertainty sets with and without  $\mathbf{b}_{\min}$  (black filled circle denotes  $\mathbf{b}_{\min}$ ).

**Definition 4.2.** The price of robustness and adaptability is:

- $\mathcal{POR}(\text{TFMP}) = IZ_{\text{TFMPRob}} - IZ_{\text{TFMP}}$
- $\mathcal{POA}(\text{TFMP}) = IZ_{\text{TFMPAdapt}} - IZ_{\text{TFMP}}$

To quantify the relative performance of the robust and adaptability problems, we define the *adaptability gap*, denoted by  $\mathcal{AG}(\text{TFMP})$ , to be the ratio of the robust cost to the adaptive cost, i.e.:

**Definition 4.3.**  $\mathcal{AG}(\text{TFMP}) = \frac{IZ_{\text{TFMPRob}}}{IZ_{\text{TFMPAdapt}}}$

We now prove a result on the dependence of the relative optimal objective values of the robust and adaptive problems on a specific property of the uncertainty set.

**Proposition 3.** Consider the multi-stage adaptive optimization problem  $\Pi_{\text{Adapt}}^T$  and its robust counterpart  $\Pi_{\text{Rob}}$ . If  $\mathbf{b}_{\min} \in \mathcal{U}$ , then,

$$Z_{\text{Adapt}} = Z_{\text{Rob}}.$$

*Proof.* Let  $\mathbf{w}_1^*, \mathbf{w}_i^*, i \geq 2$  be an optimal solution of  $\Pi_{\text{Rob}}$  and  $\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_i(\mathbf{b}), \forall \mathbf{b} \in \mathcal{U}$  be an optimal solution of  $\Pi_{\text{Adapt}}^T$ . Since  $\mathbf{w}_1^*, \mathbf{w}_i^*, i \geq 2$  is optimal, it is also feasible for  $\Pi_{\text{Rob}}$ . Therefore,

$$\mathbf{A}_1 \mathbf{w}_1^* + \sum_{i=2}^T \mathbf{A}_i \mathbf{w}_i^* \leq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}$$

This implies that the solution  $\mathbf{w}_1 = \mathbf{w}_1^*, \mathbf{w}_i(\mathbf{b}) = \mathbf{w}_i^*, \forall \mathbf{b} \in \mathcal{U}$  is feasible for  $\Pi_{\text{Adapt}}^T$ .

$$\begin{aligned} Z_{\text{Adapt}} &\leq \mathbf{c}'_1 \mathbf{w}_1^* + \sum_{i=2}^T \mathbf{c}'_i \mathbf{w}_i^* \\ &= Z_{\text{Rob}} \end{aligned} \tag{18}$$

Since  $\mathbf{b}_{\min} \in \mathcal{U}$ ,

$$Z_{\text{Adapt}} = \mathbf{c}'_1 \hat{\mathbf{w}}_1 + \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_2 \hat{\mathbf{w}}_2(\mathbf{b}_2) + \cdots + \right. \quad (19)$$

$$\left. \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_{T-1} \hat{\mathbf{w}}_{T-1}(\mathbf{b}_{T-1}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_T \hat{\mathbf{w}}_T(\mathbf{b}_T) \right] \right]$$

$$\geq \mathbf{c}'_1 \hat{\mathbf{w}}_1 + \sum_{i=2}^T \mathbf{c}'_i \hat{\mathbf{w}}_i(\mathbf{b}_{\min}) \quad (20)$$

Now,  $(\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_i(\mathbf{b}_{\min}))$  is a feasible solution of  $\Pi_{\text{Rob}}$ ,

$$Z_{\text{Rob}} \leq \mathbf{c}'_1 \hat{\mathbf{w}}_1 + \sum_{i=2}^T \mathbf{c}'_i \hat{\mathbf{w}}_i(\mathbf{b}_{\min})$$

$$\leq Z_{\text{Adapt}} \quad (21)$$

From (18) and (21),

$$Z_{\text{Rob}} = Z_{\text{Adapt}} \quad (22)$$

□

Proposition 3 leads to the following corollary:

**Corollary 1.** *For the weather-front based uncertainty set with the underlying parameters  $(T_a, d, \alpha)$ , the following holds true:*

1. *if  $\mathbf{b}_{\min} \notin \mathcal{U}$ , then  $Z_{\text{TFMP}} \leq Z_{\text{TFMPWUAdapt}} \leq Z_{\text{TFMPWURob}}$ , and  $\mathcal{AG}(\text{TFMP}) \geq 1$ .*
2. *if  $\mathbf{b}_{\min} \in \mathcal{U}$ , then  $Z_{\text{TFMP}} \leq Z_{\text{TFMPWUAdapt}} = Z_{\text{TFMPWURob}}$ , and  $\mathcal{AG}(\text{TFMP}) = 1$ .*

## 5 Computational Results

In this section, we report computational results from the solution approaches introduced in Section 4.

### 5.1 Experimental Setup

In our setup, the airspace is subdivided into sectors of equal dimensions (10 by 10) that form a grid, thereby, having a total of 100 sectors. 55 major airports of the US are then mapped to one of these 100 sectors based on its geographical coordinates. The entire airspace is subsequently divided into three regions, namely, north-east,

south-west and central. Each of these regions is then subject to simulated weather-fronts. The time of arrival, duration and capacity reduction of each weather-front is generated randomly from appropriate intervals. This enables the construction of uncertainty sets with varying degree of extreme points to study the implications on the complexity of solving the robust/adaptive problems and the corresponding price. Finally, the capacity inputs used for all the instances are at the “*infeasibility border*”, i.e., values which when perturbed slightly on the conservative side lead to infeasibility of the overall problem.

To solve the TFMPWURob and TFMPWUAdapt models (which use various uncertainty sets), we use a new tool for robust optimization - Robust Optimization Made Easy (ROME). We use the same tool to compute optimal solutions for the deterministic problem. Table 3 reports the performance of TFMPWURob whereas Table 4 reports the performance of TFMPWUAdapt. The running times reported in the table correspond to both the solver time and the ROME input parsing time (which accounts for majority of the total time). The following conclusions can be drawn from these computational results:

- The integrality properties of the robust equivalent closely mimics the deterministic version which is known to have strong integrality properties (as evidenced by the fact that in all cases the % non-integral solutions are less than 1%). Furthermore, the running times for TFMPWURob compares favorably with the deterministic counterpart. These two observations validate the theoretical expectation listed in Section 4.1. Finally, the performance of TFMPWURob is not affected by the number of extreme points in the uncertainty set.
- The results for TFMPWUAdapt indicate that in terms of running times, TFMPWUAdapt is quite expensive when compared to TFMPWURob (this is expected as the transformed LP from the adaptive instance is of a much larger size). Furthermore, the complexity of the adaptive instance typically increases with the number of extreme points.

**Characteristics of Robust Solutions.** To study the nature of robust schedules, we define two quantities which qualitatively capture the factors impacting the price of robustness and the deviation in the robust schedule (relative to the deterministic counterpart). The first is the percentage capacity reduction in the capacity vector used to solve the robust problem (viz.  $\mathbf{b}_{\min}$ ) relative to the deterministic capacity estimate  $\mathbf{b}_{\det}$  (mathematically, this is  $100 * \frac{\mathbf{e}'(\mathbf{b}_{\det} - \mathbf{b}_{\min})}{\mathbf{e}'\mathbf{b}_{\det}}$  and will be denoted by  $C_{\text{Red}}$  henceforth). This quantity is expected to govern the price of robustness in that higher the reduction, higher is the objective cost for the robust problem ( $Z_{\text{Rob}} \rightarrow \infty$  as  $\mathbf{b}_{\min} \rightarrow 0$ ). The second quantity (somewhat correlated with the first one) is the difference in the deterministic and robust flight schedules. It is defined as  $\mathbf{e}'(\mathbf{a}^{\text{Det}} - \mathbf{a}^{\text{Rob}})$



<b>Region</b> (# of Flights)	$Z_{\text{Det}}$	<b>Sol. Time</b> (sec.)	$Z_{\text{Rob}}$	<b>Sol. Time</b> (sec.)	<b>% Nonint</b>	<b>Uncertainty Set</b> (# of EPs)
<i>North-East</i> (500-1000)	208	378	229	407	0	81
	227	515	230	643	0	256
	145	636	161.5	638	0.15	729
	935	4839	971.5	4721	0	20736000
<i>Central</i> (500-1000)	650	1996	677	1984	0	81
	647	1770	675	1779	0	729
	935	4685	1013	4607	0	4299816
	240	401	248	402	0.10	1679616
<i>South-West</i> (500-1000)	642	1937	647	1977	0	400
	137	578	144	566	0.10	256
	208	358	208	387	0	7962624
	935	4540	1044	4713	0	65536000

Table 3: Computational Experience with TFMPWURob.

<b>Region</b> (# of Flights)	$Z_{\text{Rob}}$	<b>Sol. Time</b> (sec.)	$Z_{\text{Adapt}}$	<b>Sol. Time</b> (sec.)	<b>% Nonint</b>	<b>Uncertainty Set</b> (# of EPs)
<i>North-East</i> (100-150)	92	84	92	265	0	81
	29.5	73	29.5	491	0	128
	54	55	54	369	0	256
	113	127	113	326	0	4096
<i>Central</i> (100-150)	27	43	27	813	0	1428
	22	34	22	287	0	225
	86	98	86	257	0	729
	35	63	35	382	0	1728
<i>South-West</i> (100-150)	75	86	75	250	0	64
	42	87	42	431	0	1296
	123	133	123	519	0	144
	99	106	99	294	0	1728

Table 4: Computational Experience with TFMPWUAdapt.

where the vector  $\mathbf{a}$  consists of the time of arrival of all flights at its destination airport (Det corresponds to deterministic and Rob to the robust problem). Figure 9 plots the price of robustness and schedule deviation as a function of  $C_{\text{Red}}$ . The key insight from the plot is that when  $\mathbf{b}_{\text{min}}$  is “close” to the deterministic capacity, the price of robustness is quite small and the deterministic schedule can be adjusted to make it feasible for uncertain scenarios.

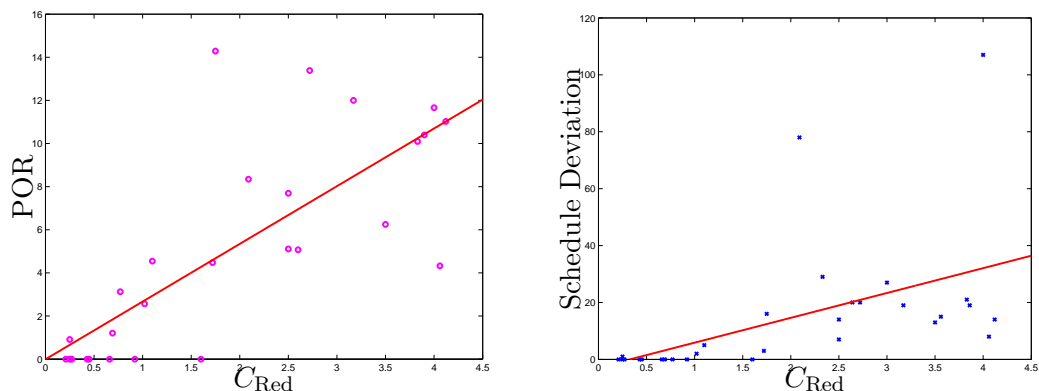


Figure 9: Characteristics of Robust Solutions. **Left:** POR as a function of capacity reduction. **Right:** Deviation in the robust schedule. The red line corresponds to the best linear fit.

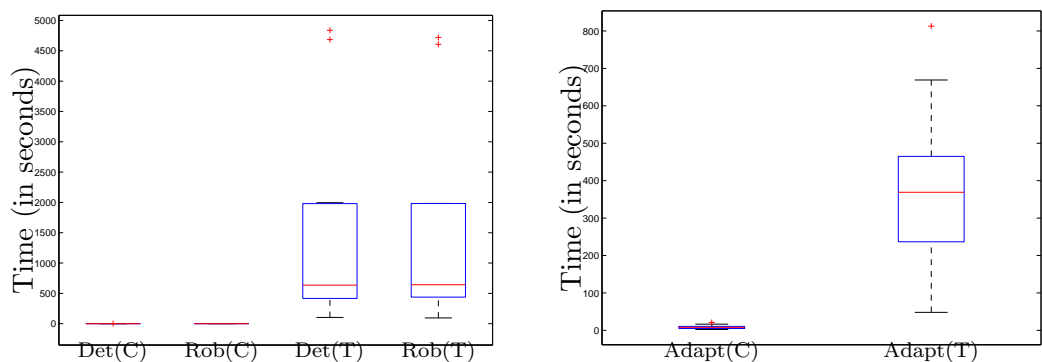


Figure 10: **Left:** Running times for deterministic and robust problems. **Right:** Running times for adaptive problem. C denotes CPLEX solver time and T denotes Total time (including parsing time).

## 6 Conclusions

This study attempts to present a framework for solving network air traffic flow management problems in a stochastic setting under the robust and adaptive optimization

paradigm. We introduce a new weather-front based approach to construct uncertainty set of possible capacity realizations. The key benefit of this approach is the low-dimensionality of the resulting discrete sets (as uncertainty in only two parameters govern the uncertainty set for each airspace element). We formulate the consequent robust and adaptive problems and propose tractable solution methodologies. We prove the equivalence of the robust problem to a new deterministic instance. For the adaptive problem, an important takeaway is that the number of extreme points is typically less than the number of constraints for most practical instances, thereby implying that an affine policy is optimal. Computational experience with the robust and adaptive models validate the practical promise of our proposal.

## Appendix

### A Power of Affine Policies in Multistage Adaptive Optimization

#### Proof of Theorem 3:

*Proof.* Let  $\mathbf{w}_i^*(\mathbf{b})$  be an optimal solution of  $\Pi_{\text{Adapt}}^T(\mathcal{U})$ . We will construct an alternative solution  $\hat{\mathbf{w}}_i(\mathbf{b})$  such that  $\hat{\mathbf{w}}_i(\mathbf{b})$  are affine functions of  $\mathbf{b}$  for  $i \geq 2$  and the worst case cost of this solution is equal to  $Z_{\text{Adapt}}$ . Let

$$\mathbf{Q} = \left[ (\mathbf{b}^1 - \mathbf{b}^{m+1}) \dots (\mathbf{b}^m - \mathbf{b}^{m+1}) \right] \quad (23)$$

Since  $\mathbf{b}^1, \dots, \mathbf{b}^{m+1}$  are affinely independent,  $(\mathbf{b}^1 - \mathbf{b}^{m+1}), \dots, (\mathbf{b}^m - \mathbf{b}^{m+1})$  are linearly independent and  $\mathbf{Q}$  is an invertible full-rank matrix. For any  $\mathbf{b} \in \mathcal{U}$ ,  $\exists \gamma$ ,  $0 \leq \gamma \leq 1$ ,  $\mathbf{e}'\gamma = 1$ ,

$$\mathbf{b} = \sum_{i=1}^{m+1} \gamma_i \mathbf{b}^i, \quad (24)$$

$$\mathbf{b} = \sum_{i=1}^m \gamma_i (\mathbf{b}^i - \mathbf{b}^{m+1}) + \mathbf{b}^{m+1}, \quad (25)$$

$$\mathbf{b} = \mathbf{Q} \cdot \gamma + \mathbf{b}^{m+1}, \quad \gamma = (\gamma_1, \dots, \gamma_m)^T. \quad (26)$$

Since  $\mathbf{Q}$  is invertible, we have,

$$\mathbf{Q}^{-1}(\mathbf{b} - \mathbf{b}^{m+1}) = \gamma \quad (27)$$

Let

$$\mathbf{W}_i = \left[ (\mathbf{w}_i^*(\mathbf{b}^1) - \mathbf{w}_i^*(\mathbf{b}^{m+1})) \dots (\mathbf{w}_i^*(\mathbf{b}^m) - \mathbf{w}_i^*(\mathbf{b}^{m+1})) \right]$$

For all  $\mathbf{b} \in \mathcal{U}$ , where  $\mathbf{b} = \sum_{j=1}^{m+1} \gamma_j \mathbf{b}^j$  for  $0 \leq \gamma \leq 1$ , consider the following solution:

$$\begin{aligned}\hat{\mathbf{w}}_i(\mathbf{b}) &= \mathbf{W}_i \mathbf{Q}^{-1}(\mathbf{b} - \mathbf{b}^{m+1}) + \mathbf{w}_i^*(\mathbf{b}^{m+1}) \\ &= \mathbf{W}_i \boldsymbol{\gamma} + \mathbf{w}_i^*(\mathbf{b}^{m+1}) \\ &= \sum_{j=1}^{m+1} \gamma_j \mathbf{w}_i^*(\mathbf{b}^j)\end{aligned}$$

The worst case cost of the solution  $\hat{\mathbf{w}}_i(\mathbf{b})$  can be bounded as follows:

$$\begin{aligned}\max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_i \hat{\mathbf{w}}_i(\mathbf{b}) &= \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_i \sum_{j=1}^{m+1} \gamma_j \mathbf{w}_i^*(\mathbf{b}^j) \\ &= \max_{\mathbf{b} \in \mathcal{U}} \sum_{j=1}^{m+1} \gamma_j \mathbf{c}'_i \mathbf{w}_i^*(\mathbf{b}^j) \\ &\leq \max_{j=1, \dots, m+1} \mathbf{c}'_i \mathbf{w}_i^*(\mathbf{b}^j)\end{aligned}\tag{28}$$

Consider the worst-case objective cost for the last two stages:

$$\max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_{T-1} \hat{\mathbf{w}}_{T-1}(\mathbf{b}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_T \hat{\mathbf{w}}_T(\mathbf{b}) \right] \leq \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_{T-1} \hat{\mathbf{w}}_{T-1}(\mathbf{b}) + \max_{j=1, \dots, m+1} \mathbf{c}'_T \mathbf{w}_T^*(\mathbf{b}^j) \right]\tag{29}$$

$$\leq \max_{j=1, \dots, m+1} \mathbf{c}'_{T-1} \mathbf{w}_{T-1}^*(\mathbf{b}^j) + \max_{j=1, \dots, m+1} \mathbf{c}'_T \mathbf{w}_T^*(\mathbf{b}^j)\tag{30}$$

This indicates that the worst-case objective cost becomes separable. Using the same argument one step at a time in the backward direction (i.e., for  $i = T - 2, \dots, 1$ ), we have,

$$\begin{aligned}\mathbf{c}'_1 \hat{\mathbf{w}}_1 + \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_2 \hat{\mathbf{w}}_2(\mathbf{b}) + \dots + \right. \\ \left. \max_{\mathbf{b} \in \mathcal{U}} \left[ \mathbf{c}'_{T-1} \hat{\mathbf{w}}_{T-1}(\mathbf{b}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{c}'_T \hat{\mathbf{w}}_T(\mathbf{b}) \right] \right] \leq \mathbf{c}'_1 \mathbf{w}^* + \sum_{i=2}^T \left[ \max_{j=1, \dots, m+1} \mathbf{c}'_i \mathbf{w}_i^*(\mathbf{b}^j) \right] \\ \leq Z_{\text{Adapt}}^T\end{aligned}$$

Therefore, the worst case cost of the solution  $\hat{\mathbf{w}}_i(\mathbf{b})$  is equal to the optimal cost of  $\Pi_{\text{Adapt}}^T(\mathcal{U})$ , which implies that the affine policy for each stage (after first) is optimal.  $\square$

## References

- [1] Ball M., Hoffman R., Odoni A.R., Rifkin R. A Stochastic Integer Program With Dual Network Structure and its Application to the Ground Holding Problem. *Operations Research*. 51:161-171, 2003.

- [2] Bemporad A., Borrelli F. and Morari M. Min-max control of constrained uncertain discrete-time linear systems. *IEEE Transactions on Automatic Control*. 48(9):1600-1606, 2003.
- [3] Ben-Tal A., Ghaoui El L. and Nemirovski A. *Robust Optimization*. Princeton Univ. Press, 2009.
- [4] Bertsimas D., Brown. D.B. and Caramanis C. Theory and Applications of Robust Optimization. *SIAM Review*. (To Appear). 2010.
- [5] Bertsimas D., Goyal V. On the Power and Limitations of Affine Policies in Two-Stage Adaptive Optimization. *Math Programming*. 2009.
- [6] Bertsimas D., Goyal V. On the Power of Robust Solutions in Two-Stage Stochastic and Adaptive Optimization Problems. *Mathematics of Operations Research*. 35(2):284-305, 2010.
- [7] Bertsimas D., Goyal V. and Sun, Xu. Power of Finite Adaptability in Multi-Stage Problems. *Mathematics of Operations Research*. (To Appear), 2010.
- [8] Bertsimas D., Gupta S. A Proposal for Network Air Traffic Management Problem Incorporating Fairness and Airline Collaboration. *Operations Research*. (To Appear). 2010.
- [9] Bertsimas D., Odoni A.R. A critical survey of optimization models for tactical and strategic aspects of air traffic flow management. Technical report, NASA. 1997.
- [10] Bertsimas D., Sim M. The Price of Robustness. *Operations Research*. 52(2):35-53, 2004.
- [11] Bertsimas D., Stock S. The Air Traffic Management Problem with Enroute Capacities. *Operations Research*. 46:406-422, 1998.
- [12] Bertsimas D., Weismantel R. (2005). *Optimization Over Integers*. Dynamic Ideas.
- [13] Dantzig, G.B. Linear Programming under uncertainty. *Management Science*. 197-206, 1955.
- [14] Hoffman R., Mukherjee A. and Vossen T. Air Traffic Flow Management. Working Paper. 2007.
- [15] Kotnyek B., Richetta O. Equitable Models for the Stochastic Ground-Holding Problem Under Collaborative Decision Making. *Transportation Science*. 40:133-146, 2006.
- [16] Lindsay K., Boyd E., Burlingame R. Traffic flow management modeling with the time assignment model. *Air Traffic Control Quarterly*. 1(3):255-276, 1993.

- [17] Mukherjee A., Hansen M.,. A Dynamic Stochastic Model for the Single Airport Ground Holding Problem. *Transportation Science*. 41:444-456, 2007.
- [18] Odoni A.R. The Flow Management Problem in Air Traffic Control, in *Flow Control of Congested Networks*, A.R. Odoni, L. Bianco and G. Szego eds., 269-288, Springer-Verlag, Berlin. 1987.
- [19] Richetta O. and Odoni A.R. Solving Optimally the Static Ground-Holding Policy Problem in Air Traffic Control. *Transportation Science*. 27:228-238, 1993.
- [20] Richetta O. and Odoni A.R. Dynamic Solution to the Ground-Holding Policy Problem in Air Traffic Control. *Trans. Res.* 28a:167-185. 1994.
- [21] Terrab M. and Odoni A.R. Strategic Flow Control on an Air Traffic Network. *Operations Research*. 41:138-152, 1993.
- [22] Terrab M. and Paulose S. Dynamic Strategic and Tactical Air Traffic Flow Control. RPI Technical Report. 1993.
- [23] Vranas P.B., Bertsimas D., Odoni A.R. The Multi-Airport Ground Holding Problem in Air Traffic Control. *Operations Research*. 42:249-26, 1994.