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On the Approximability of Adjustable Robust Convex Optimization under Uncertainty

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Abstract In this paper, we consider adjustable robust versions of convex optimization problems with uncertain constraints and objectives and show that under fairly general assumptions, a static robust solution provides a good approximation for these adjustable robust problems. An adjustable robust optimization problem is usually intractable since it requires to compute a solution for all possible realizations of uncertain parameters, while an optimal static solution can be computed efficiently in most cases if the corresponding deterministic problem is tractable. The performance of the optimal static robust solution is related to a fundamental geometric property, namely, the *symmetry* of the uncertainty set. Our work allows for the constraint and objective function coefficients to be uncertain and for the constraints and objective functions to be convex, thereby providing significant extensions of the results in [8] and [9] where only linear objective and linear constraints were considered. The models in this paper encompass a wide variety of problems in revenue management, resource allocation under uncertainty, scheduling problems with uncertain processing times, semidefinite optimization among many others. To the best of our knowledge, these are the first approximation bounds for adjustable robust convex optimization problems in such generality.

Keywords Robust Optimization · Static Policies · Adjustable Robust Policies

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1 Introduction

In most real world problems, problem parameters are uncertain at the optimization or decision-making phase. Solutions obtained via deterministic optimization might be sensitive to even small perturbations in the problem parameters that might render them highly infeasible or suboptimal. Stochastic optimization was introduced by Dantzig [12] and Beale [1], and since then has been extensively studied in the literature to address the uncertainty in problem parameters. A stochastic optimization approach assumes a probability distribution over the uncertain parameters and seeks to optimize the expected value of the objective function. We refer the reader to several textbooks including Infanger [18], Kall and Wallace [19], Prekopa [21], Shapiro [22], Shapiro et al. [23] and the references therein for a comprehensive review of stochastic optimization.

While the stochastic optimization approach has its merits, it is by and large computationally intractable even when the constraint and objective functions are linear. Shapiro and Nemirovski [24] give hardness results for two-stage and multi-stage stochastic optimization problems where they show that the multi-stage stochastic optimization problem is computationally intractable even if approximate solutions are desired. Dyer and Stougie [13] show that a multi-stage stochastic optimization problem where the distribution of uncertain parameters in any stage also depends on the decisions in past stages is PSPACE-hard. Even for the stochastic problems that can be solved efficiently, it is difficult to estimate the probability distributions of the uncertain parameters from historical data to formulate the problem.

More recently, the robust optimization approach has been considered to address optimization under uncertainty and has been studied extensively (see Ben-Tal and Nemirovski [4], [5] and [6], El Ghaoui and Lebret [14], Bertsimas and Sim [10, 11], Goldfarb and Iyengar [16]). In a robust optimization approach, the uncertain parameters are assumed to belong to some uncertainty set and the goal is to construct a single (static) solution that is feasible for all possible realizations of the uncertain parameters from the set and optimizes the worst-case objective. We point the reader to the survey by Bertsimas et al. [7] and the book by Ben-Tal et al. [2] and the references therein for an extensive review of the literature in robust optimization. This approach is significantly more tractable as compared to a stochastic optimization approach and the robust problem is equivalent to the corresponding deterministic problem in computational complexity for a large class of problems and uncertainty sets (Bertsimas et al. [7]). However, the robust optimization approach may have the following drawbacks. Since it optimizes over the worst-case realization of the uncertain parameters, it may produce conservative solutions that may not perform well in the expected value sense. Moreover, the robust approach computes a single (static) solution even for a multi-stage problem with several stages of decision-making as opposed a fully-adjustable solution where decisions in each stage depend on the actual realizations of the parameters in past stages. This may further add to the conservativeness.

Another approach is to consider solutions that are fully-adjustable in each stage and depend on the realizations of the parameters in past stages and optimize over the worst case. Such solution approaches have been considered in the literature and referred to as adjustable robust policies (see Ben-Tal et al. [3] and the book by Ben-Tal et al. [2] for a detailed discussion of these policies). Unfortunately, the adjustable robust problem is computationally intractable in general. Feige et al. [15] show that it is hard to approximate even a two-stage set covering linear program within a factor of $\Omega(\log n)$ for a general uncertainty set. Bertsimas and Goyal [8] show that a *static solution* (a single solution that is feasible for all possible realizations of the uncertain parameters) is a 2-approximation for the adjustable robust as well as a stochastic optimization problem with linear constraints and objective with right hand side uncertainty if the uncertainty set belongs to the non-negative orthant and is perfectly *symmetric* such as a hypercube or an ellipsoid. Bertsimas et al. [9] give two significant generalizations of the result in [8] where they show that the performance of the static solution for the adjustable robust and the stochastic problem depends on a fundamental geometric property of the uncertainty namely symmetry. The authors consider a generalized notion of symmetry introduced in Minkowski [20], where the symmetry of a convex set is a number between 0 and 1. The symmetry of a set being equal to one implies that it is perfectly symmetric (such as an ellipsoid). Furthermore, they also generalize the static robust solution policy to a finitely adjustable solution policy for the multi-stage stochastic and adjustable robust optimization problems and give a similar bound on its performance that is related to the symmetry of the uncertainty sets.

The models in [8] and [9] consider only covering constraints of the form $\boldsymbol{a}^T\boldsymbol{x} \geq b$, $\boldsymbol{a} \in \mathbb{R}^n$ and $b \geq 0$ and the uncertainty appears only in the right hand side of the constraints. While these are fairly general models, they do not handle packing constraints and uncertainty in constraint coefficients. They do not capture several important applications such as revenue management and resource allocation problems under uncertain resource requirements, where we have packing constraints and uncertainty in constraint coefficients. In a typical revenue management problem, we need to allocate of scarce resources to a demand with uncertain resource requirements such that the total revenue is maximized. As an example, consider a multi-period problem where there is a single resource with capacity $h \in \mathbb{R}_+$, and in each period $k = 1, \ldots, T$, a demand arrives and requires an uncertain amount, b_k of that resource and we obtain revenue d_k per unit of demand satisfied. The goal in each period is to decide on the fraction of demand that should be satisfied such that the worst case revenue over all future resource requirements is maximized.

Let x_k be the fraction of demand in period k that is satisfied for $k = 1, \ldots, K$. Note that in period k, we observe the resource requirements for demands in all periods up to period k but do not know the resource requirements for future demands. The multi-period adjustable robust revenue maximization problem can now be formulated as follows.

$$\begin{aligned}
&\max d_1 x_1(b_1) + \min_{b_2 \in \mathcal{U}_2} \max_{x_2(b_1,b_2)} \big(d_2 \cdot x_2(b_1,b_2) + \ldots \\
&\qquad\qquad\qquad\qquad + \min_{b_K \in \mathcal{U}_K} \max_{x_K(b_1,\ldots,b_K)} d_K \cdot x_K(b_1,\ldots,b_K)) \ldots\big) \\
&\quad \sum_{k=1}^{K} b_k \cdot x_k(b_1,\ldots,b_k) \leq h, \ \forall b_k \in \mathcal{U}_k, k = 2,\ldots,K \\
&\quad 0 \leq x_k(b_1,\ldots,b_k) \leq 1, \ \forall k = 1,\ldots,K.
\end{aligned} \tag{1.1}$$

In order to address such problems, we need to consider linear optimization problems under constraint and objective coefficient uncertainty. Moreover, the framework in [8] and [9] covers linear optimization problems, but not general convex optimization problems. In this paper, we study adjustable robust versions of fairly general convex optimization problems under uncertain convex constraints and objective. In particular, these include linear packing problems under constraint and objective uncertainty. While computing an optimal adjustable robust solution is intractable, we show that an optimal static robust solution provides a good approximation for the adjustable robust problem.

1.1 Models and Notation

We study the following adjustable robust version of a two-stage convex optimization problem $\Pi_{AR}^{\mathsf{cons}}(\mathcal{U})$ under uncertain constraints.

$$\begin{aligned} z_{AR}^{\mathsf{cons}}(\mathcal{U}) = \max_{\boldsymbol{x}\in X} f_1(\boldsymbol{x}) + \min_{\boldsymbol{u}\in\mathcal{U}} \max_{\boldsymbol{y}(\boldsymbol{u})\in Y} f_2\big(\boldsymbol{y}(\boldsymbol{u})\big) \\ g_i(\boldsymbol{x},\boldsymbol{y}(\boldsymbol{u}),\boldsymbol{u}) \le c_i,\ \forall \boldsymbol{u}\in\mathcal{U},\ \forall i\in I, \end{aligned} \tag{1.2}$$

where $\mathcal{U}\subset\mathbb{R}_+^p$ is a compact convex set, I is the set of indices for the constraints, $c_i\in\mathbb{R}_+$ for all $i\in I$ and functions $f_1:\mathbb{R}_+^{n_1}\to\mathbb{R}_+$, $f_2:\mathbb{R}_+^{n_2}\to\mathbb{R}_+$ and $g_i:\mathbb{R}_+^{n_1+n_2+p}\to\mathbb{R}_+, i\in I$ satisfy the following conditions:

(A1) The function f_1 is concave and satisfies $f_1(\mathbf{0})=0$.

(A2) f_2 is concave and satisfies $f_2(\mathbf{0})=0$.

(A3) For all $i\in I$, g_i is convex in $\boldsymbol{x}$ and $\boldsymbol{y}$, concave and increasing in $\boldsymbol{u}$ and satisfies,

$$\begin{aligned} g(\mathbf{0},\mathbf{0},\boldsymbol{u}) = 0,\ \forall \boldsymbol{u}\in\mathcal{U} \\ \frac{\partial g_i}{\partial u_j} \ge 0,\ \forall j=1,\ldots,p,\ \forall \boldsymbol{x}\in X, \boldsymbol{y}\in Y \end{aligned}$$

Moreover, given $\hat{\boldsymbol{x}}\in X$ and $\hat{\boldsymbol{y}}\in Y$, the problem $\max_{\boldsymbol{u}\in\mathcal{U}} g_i(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\boldsymbol{u})$ is solvable in polynomial time.

(A4) Sets X and Y are convex and $\mathbf{0}\in X$, $\mathbf{0}\in Y$.

Assumptions **(A1)-(A4)** are fairly general and model two-stage adjustable robust versions of a large class of deterministic convex optimization problems. In Section 1.2 we illustrate that they hold for several important application areas. The assumption that the objective functions f_1 and f_2 are concave and the constraint functions $g_i(\cdot), i\in I$ are convex are necessary for the above problem to be a convex optimization problem. The assumption that $f_1(\mathbf{0})=f_2(\mathbf{0})=0$ is without loss of generality as we can always translate f_1 and f_2 to satisfy this condition. In addition, we assume that for all $i\in I$, g_i is an increasing concave function of $\boldsymbol{u}$ and $g(\mathbf{0},\mathbf{0},\boldsymbol{u})=0$ for all $\boldsymbol{u}\in\mathcal{U}$. We require the concavity assumption for the tractability of the static-robust problem.

Note that $\Pi_{AR}^{\mathsf{cons}}(\mathcal{U})$ is a two-stage problem where $\boldsymbol{x}$ denotes the first-stage decision. The uncertain parameters $\boldsymbol{u}\in\mathcal{U}$ materialize after the first-stage decisions have been made and we can then select the second-stage or recourse decision $\boldsymbol{y}(\boldsymbol{u})$ that depends on $\boldsymbol{u}$. The goal is to find $\boldsymbol{x}\in X$ such that for all $\boldsymbol{u}\in\mathcal{U}$ there is a feasible recourse decision $\boldsymbol{y}(\boldsymbol{u})$ and the total objective: $f_1(\boldsymbol{x})+f_2(\boldsymbol{y}(\boldsymbol{u}))$ is maximized over the worst-case choice of $\boldsymbol{u}\in\mathcal{U}$. We note that in $\Pi_{AR}^{\mathsf{cons}}(\mathcal{U})$, we only model uncertainty in the constraints (for all $i\in I$, the constraint function g_i is a function of $\boldsymbol{u}$ in addition to the decision variables $\boldsymbol{x},\boldsymbol{y}(\boldsymbol{u})$). There is no uncertainty in the objective function. We also consider the problem where uncertainty appears in both constraints as well as objective functions. This model is described in (4.1).

Since computing an optimal fully-adjustable solution for (1.2) is intractable in general even when f_1, f_2 and $g_i, i\in I$ are linear (see Dyer and Stougie [13]), we consider the corresponding static-robust problem $\Pi_{Rob}(\mathcal{U})$:

$$
\begin{aligned}
z^{\mathsf{cons}}_{Rob}(\mathcal{U}) = \max_{\boldsymbol{x}\in X, \boldsymbol{y}\in Y} \; & f_1(\boldsymbol{x}) + f_2(\boldsymbol{y}) \\
& g_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) \le c_i, \; \forall \boldsymbol{u}\in\mathcal{U}, \; \forall i \in I,
\end{aligned}
\tag{1.3}
$$

We show that it is tractable to compute an optimal static robust solution to (1.3) and we compare the performance of an optimal static robust solution for the adjustable robust problem (1.2). Note that under assumptions $(\mathbf{A1}) - (\mathbf{A4})$, both (1.2) and (1.3) are feasible as $\boldsymbol{x} = \mathbf{0}, \boldsymbol{y}(\boldsymbol{u}) = \mathbf{0}$ for all $\boldsymbol{u} \in \mathcal{U}$ is a feasible solution for (1.2) and $\boldsymbol{x} = \mathbf{0}, \boldsymbol{y} = \mathbf{0}$ is a feasible solution for (1.3). Therefore,

$$
z^{\mathsf{cons}}_{AR}(\mathcal{U}) \ge z^{\mathsf{cons}}_{Rob}(\mathcal{U}) \ge 0.
$$

1.2 Applications

In this section, we illustrate the wide modeling flexibility of (4.1) by placing a variety of problem domains in this framework.

1.2.1 Two-Stage Linear Optimization under Constraint Uncertainty

We first consider the classical two-stage adjustable robust linear maximization problem with uncertainty in the constraint coefficients.

$$
\begin{aligned}
z^{\mathsf{Lin}}_{AR}(\mathcal{U}) = \; \max \; \boldsymbol{c}^T\boldsymbol{x} + \min_{\boldsymbol{B}\in\mathcal{U}} \; & \max_{\boldsymbol{y}(\boldsymbol{B})\in Y} \; \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{B}) \\
& \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}) \le \boldsymbol{h} \;\; \forall \boldsymbol{B}\in\mathcal{U} \\
& \boldsymbol{x} \ge \mathbf{0} \\
& \boldsymbol{y}(\boldsymbol{B}) \ge \mathbf{0},
\end{aligned}
\tag{1.4}
$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n_1}_+$ is the first-stage constraint matrix, $\boldsymbol{B}$ is the uncertain second-stage constraint matrix, $\boldsymbol{c} \in \mathbb{R}^{n_1}_+$, $\boldsymbol{d} \in \mathbb{R}^{n_2}_+$ and $\boldsymbol{h} \in \mathbb{R}^m_+$. In the framework (1.2), $f_1(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x}$ that satisfies **(A1)**, $f_2(\boldsymbol{y}(\boldsymbol{B})) = \boldsymbol{d}^T\boldsymbol{y}$ that satisfies **(A2)**, and

$$
g_i(\boldsymbol{x},\boldsymbol{y},\boldsymbol{B}) = \sum_{j=1}^{n_1} A_{ij}x_j + \sum_{j=1}^{n_2} B_{ij}y_j, i = 1,\ldots,m,
$$

that satisfies **(A3)** as it is convex (linear) in $\boldsymbol{x}$ and $\boldsymbol{y}$ and concave (linear) and increasing in $\boldsymbol{B}$ since $\boldsymbol{x} \ge 0, \boldsymbol{y} \ge 0$ and $\boldsymbol{B} \in \mathbb{R}^{m\times n_2}_+$. Before discussing next several applications of the framework of (1.4) we note that we can not model a linear covering constraint in the framework (1.2), i.e., a constraint of the form $\boldsymbol{a}^T\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{y} \ge 1$. To model this constraint in (1.2), we require

$$
g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) = 1 - \boldsymbol{a}^T\boldsymbol{x} - \boldsymbol{u}^T\boldsymbol{y},
$$

and the constraint can be formulated as $g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u}) \le 0$. However, in this case g does not satisfy **(A3)** as $g(\mathbf{0},\mathbf{0},\boldsymbol{u}) \ne 0$. Even with this limitation, we are able to model many interesting and important applications that can not be formulated using models in [8] and [9].

Revenue Management. We can model many revenue management problems in the framework (1.4) as discussed briefly earlier. Here, we formulate a two-stage version of the revenue management problem discussed in (1.1) in the framework (1.4) and we consider the case where each demand possibly requires multiple resources in uncertain amounts instead of a single resource. Suppose there are m resources with finite capacity $\boldsymbol{h} \in \mathbb{R}^m_+$ and two demand types: D_1 ($|D_1| = n_1$) and D_2 ($|D_2| = n_2$). Demands D_1 have a known resource requirement given by a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n_1}$ (A_{ij} denotes the amount of resource i required per unit of demand j) and $c_j \in \mathbb{R}_+$ is the revenue per unit of satisfied demand $j \in D_1$. The demands D_2 have uncertain resource requirements given by a matrix $\boldsymbol{B} \in \mathcal{U}$ and d_j is the revenue per unit of satisfied demand $j \in D_2$. The goal is to decide the optimal fraction of demands D_1 that must be satisfied such that the total worst case revenue is maximized.

To formulate this revenue management problem in the two-stage framework of (1.4), let $x_j,\ j = 1, \ldots, n_1$ denote the fraction of demand $j \in D_1$ satisfied and the second-stage variables $y_j(\boldsymbol{B}),\ j = 1, \ldots, n_2$ denote the fraction of demand $j \in D_2$ satisfied. The right-hand side of the constraints, $\boldsymbol{h} \in \mathbb{R}^m_+$, is the capacity vector of different resources. Also,

$$X = [0, 1]^n,\ Y = [0, 1]^n.$$

With these parameters, the problem (1.4) formulates the above two-stage revenue management problem. This formulation can be extended to a multi-stage problem and therefore, we can model a multi-stage resource allocation problem where there are multiple demand types and all demands of type j arrive in stage j after irrevocable allocation decisions have been made for demand types $1, \ldots, j-1$.

Production Planning. We can formulate production planning problems under uncertain production requirements as a special case of (1.4). In a production planning problem, we need to produce n different products from m raw materials. Product j generates a revenue of d_j per unit for all $j = 1, \ldots, n$ and h_i denotes the amount of raw material i available for all $i = 1, \ldots, m$. If matrix $\boldsymbol{B} \in \mathbb{R}^{m \times n}$ denotes the uncertain raw material requirements for all products, the robust production planning problem can be formulated in the framework (1.4) with $\boldsymbol{A} = 0$.

Network Design. We can also model a network design problem in the framework (1.4) where the goal is to decide on edge capacities for satisfying an uncertain demand in the framework of (1.4). In particular, consider the following problem: given an undirected graph $G = (V, E)$, let c_e be the cost of one unit of capacity and u_e be the maximum possible capacity of edge $e \in E$. For any $i, j \in V$, r_{ij} is the uncertain demand from i to j that realizes in the second-stage and let d_{ij} be the profit obtained from one unit of flow from i to j. The goal is to decide on the capacity of the edges in the first stage and the flow in the second-stage so that the total profit is maximized. This network design problem arises in many important applications including capacity planning problems under uncertain demand.

Let x_e denote the fraction of total edge capacity u_e that is not bought (recall we are modeling in a maximization framework). Therefore, for all edges $e \in E$, $(1 - x_e)$ is the fraction of edge capacity bought. Let $\mathcal{P}_{ij}$ denote the set of paths from i to j for $i, j \in V$. For any $i, j \in V$ and $P \in \mathcal{P}_{ij}$, let $f_P(\boldsymbol{r})$ denote the flow on path P, when the demand vector is $\boldsymbol{r}$. We can formulate the network design problem as follows.

$$\begin{aligned}
&\max\ \boldsymbol{c}^T\boldsymbol{x} + \min_{\boldsymbol{r}\in\mathcal{U}} \sum_{i,j\in V} d_{ij}\left(\sum_{P\in\mathcal{P}_{ij}} f_P(\boldsymbol{r})\right)\\
&\quad u_e x_e + \sum_{i,j\in V}\ \sum_{P\in\mathcal{P}_{ij}:e\in P} r_{ij} f_P(\boldsymbol{r}) \le u_e,\ \forall e\in E\\
&\quad 0 \le f_P \le 1,\ \forall P\in\mathcal{P}_{ij}, \forall i,j\in V,
\end{aligned}$$

which falls within the framework (1.4).

1.2.2 Quadratically Constrained Problems

In this section, we formulate quadratically constrained problems under uncertainty in the constraints in the framework of (1.2). Consider the following two-stage maximization problem, Π_{AR}^{Quad},

$$\begin{aligned}
z_{AR}^{\mathsf{Quad}} = \max_{\boldsymbol{x}}\ & \boldsymbol{c}^T\boldsymbol{x} + \min_{\boldsymbol{Q}\in\mathcal{U}}\ \max_{\boldsymbol{y}(\boldsymbol{Q})}\ \boldsymbol{d}^T\boldsymbol{y}(\boldsymbol{Q})\\
& \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{y}(\boldsymbol{Q})^T\boldsymbol{Q}\boldsymbol{y}(\boldsymbol{Q}) \le t,\ \forall\ \boldsymbol{Q}\in\mathcal{U}\\
& \boldsymbol{x}\in\mathbb{R}_+^{n_1}\\
& \boldsymbol{y}(\boldsymbol{Q})\in\mathbb{R}_+^{n_2},
\end{aligned} \tag{1.5}$$

where $\mathcal{U}\subset S_+^{n_2}\cap\mathbb{R}_+^{n_2\times n_2}$ is the set of uncertain values $\boldsymbol{Q}\in S_+^{n_2}\cap\mathbb{R}_+^{n_2\times n_2}$ and $\boldsymbol{P}\in S_+^{n_1}$ (S_+^n denotes the set of symmetric positive semidefinite matrices in dimension $n\times n$). In the framework of (1.2), $\boldsymbol{u}=\boldsymbol{Q}$ and we have the following functions:

$$\begin{aligned}
f_1(\boldsymbol{x}) &= \boldsymbol{c}^T\boldsymbol{x},\\
f_2(\boldsymbol{y}) &= \boldsymbol{d}^T\boldsymbol{y},\\
g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Q}) &= \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{y}^T\boldsymbol{Q}\boldsymbol{y}.
\end{aligned}$$

The objective functions f_1, f_2 are concave (linear) and satisfy **(A1)** and **(A2)**. The constraint function, g is convex in $\boldsymbol{x}$ and $\boldsymbol{y}$ since $\boldsymbol{P}$ and $\boldsymbol{Q}$ are positive semidefinite, and concave (linear) in $\boldsymbol{Q}$. Also, $g(\boldsymbol{0},\boldsymbol{0},\boldsymbol{Q})=0$ for all $\boldsymbol{Q}\in\mathcal{U}$ and it is increasing in $\boldsymbol{Q}$ as

$$\frac{\partial g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Q})}{\partial Q_{ij}} = y_i y_j \ge 0,\ \forall i,j=1,\ldots,n_2, \forall \boldsymbol{x},\boldsymbol{y}\ge\boldsymbol{0}.$$

Therefore, g satisfies Assumption **(A3)** and problem (1.5) can be modeled in the framework of (1.2). An example of (1.5) includes a portfolio optimization problem where the covariance matrix is uncertain and the goal is to find a portfolio such that the return is maximized and the worst-case variance is at most a given threshold t. Since this is a one-period problem and there is no recourse decision unlike (1.5), we can formulate it as a static-robust problem in the framework of (1.3) where $\boldsymbol{x}=\boldsymbol{0}$, $\boldsymbol{y}$ denotes the portfolio vector, and $\boldsymbol{u}=\boldsymbol{Q}$ where $\boldsymbol{Q}$ is the uncertain covariance matrix. As above, the functions f_2 and g are defined as

$$f_2(\boldsymbol{y}) = \boldsymbol{d}^T\boldsymbol{y},\ g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Q}) = \boldsymbol{y}^T\boldsymbol{Q}\boldsymbol{y}.$$

1.2.3 Semidefinite Programming

In this section, we formulate semidefinite optimization problems under uncertain constraints in the framework of (1.2). Consider the following two-stage maximization problem, Π_{AR}^{SDP}:

$$\begin{aligned} z_{AR}^{\mathsf{SDP}} = \max_{\boldsymbol{X}} \boldsymbol{C} \cdot \boldsymbol{X} + \min_{\boldsymbol{Q} \in \mathcal{U}} \max_{\boldsymbol{Y}(\boldsymbol{Q})} \ & \boldsymbol{D} \cdot \boldsymbol{Y}(\boldsymbol{Q}) \\ & \boldsymbol{P} \cdot \boldsymbol{X} + \boldsymbol{Q} \cdot \boldsymbol{Y}(\boldsymbol{Q}) \leq t, \ \forall \, \boldsymbol{Q} \in \mathcal{U} \\ & \boldsymbol{X} \in S_+^{n_1} \cap \mathbb{R}_+^{n_1 \times n_1} \\ & \boldsymbol{Y}(\boldsymbol{Q}) \in S_+^{n_2} \cap \mathbb{R}_+^{n_2 \times n_2}, \ \forall \boldsymbol{Q} \in \mathcal{U}, \end{aligned} \tag{1.6}$$

where $\mathcal{U} \subset \mathbb{R}_+^{n_2 \times n_2}$ and $\boldsymbol{P} \in \mathbb{R}_+^{n_1 \times n_1}$. Note that $\boldsymbol{C} \cdot \boldsymbol{X}$ denotes the trace of matrices $\boldsymbol{C}$ and $\boldsymbol{X}$. In the general framework of (1.2), we have the following functions:

$$\begin{aligned} f_1(\boldsymbol{X}) &= \boldsymbol{C} \cdot \boldsymbol{X}, \\ f_2(\boldsymbol{Y}) &= \boldsymbol{D} \cdot \boldsymbol{Y}, \\ g(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Q}) &= \boldsymbol{P} \cdot \boldsymbol{X} + \boldsymbol{Q} \cdot \boldsymbol{Y}. \end{aligned}$$

The functions f_1, f_2 are linear functions and satisfy **(A1)** and **(A2)**. The constraint function g is linear in $\boldsymbol{X}$, $\boldsymbol{Y}$ and also in $\boldsymbol{u} = \boldsymbol{Q}$. Also, $g(\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{Q}) = 0$ for all $\boldsymbol{Q} \in \mathcal{U}$ and g is increasing in $\boldsymbol{u} = \boldsymbol{Q}$ as

$$\frac{\partial g}{\partial Q_{ij}} = Y_{ij} \geq 0, \ \forall i, j = 1, \ldots, n_2, \forall \boldsymbol{Y} \in S_+^{n_2} \cap \mathbb{R}_+^{n_2 \times n_2}.$$

Therefore, g satisfies Assumption **(A3)**.

2 Contributions

In this paper, we show that the two-stage adjustable robust convex optimization problem under uncertainty as defined in (1.2) can be well approximated by an optimal solution to the corresponding robust version defined in (1.3) under Assumptions **(A1)-(A4)**. Furthermore, we show that an optimal static-robust solution can be computed in polynomial time. We relate the performance of the static robust solution for the adjustable robust problems to fundamental geometric properties of the uncertainty set, namely, *symmetry* and the *translation factor*. Let us introduce these geometric properties in order to discuss the results further.

Symmetry of a convex set. Given a nonempty compact convex set $\mathcal{U} \subset \mathbb{R}^m$ and a point $\boldsymbol{u} \in \mathcal{U}$, we define the symmetry of $\boldsymbol{u}$ with respect to $\mathcal{U}$ as follows:

$$\mathbf{sym}(\boldsymbol{u}, \mathcal{U}) := \max\{\alpha \geq 0 : \boldsymbol{u} + \alpha(\boldsymbol{u} - \boldsymbol{u'}) \in \mathcal{U}, \ \forall \boldsymbol{u'} \in \mathcal{U}\}. \tag{2.1}$$

Then, the symmetry of $\mathcal{U}$ is defined as follows.

$$\mathbf{sym}(\mathcal{U}) = \max_{\boldsymbol{u} \in \mathcal{U}} \mathbf{sym}(\boldsymbol{u}, \mathcal{U}), \tag{2.2}$$

and the maximizer of the above quantity is referred to as the *point of symmetry* of $\mathcal{U}$. This notion of symmetry was introduced in [20]. In general, $1/m \leq \mathbf{sym}(\mathcal{U}) \leq 1$ and if the set $\mathcal{U}$ is absolutely symmetric $\mathbf{sym}(\mathcal{U}) = 1$.

The translation factor $\rho(\boldsymbol{u},\mathcal{U})$. For a convex compact set $\mathcal{U} \subset \mathbb{R}^m_+$, following [9], we define a *translation factor* $\rho(\boldsymbol{u},\mathcal{U})$, the *translation factor* of $\boldsymbol{u} \in \mathcal{U}$ with respect to $\mathcal{U}$, as follows.

$$\rho(\boldsymbol{u},\mathcal{U}) = \min\{\alpha \in \mathbb{R}_+ \mid \mathcal{U} - (1-\alpha)\cdot \boldsymbol{u} \subset \mathbb{R}^m_+\}.$$

In other words, $\mathcal{U}' := \mathcal{U} - (1-\rho)\boldsymbol{u}$ is the maximum possible translation of $\mathcal{U}$ in the direction $-\boldsymbol{u}$ such that $\mathcal{U}' \subset \mathbb{R}^m_+$. Figure 1 gives a geometric picture. Note that for $\alpha = 1, \mathcal{U} - (1-\alpha)\cdot\boldsymbol{u} = \mathcal{U} \subset \mathbb{R}^m_+$. Therefore, $0 < \rho \leq 1$. And ρ approaches 0, when the set $\mathcal{U}$ moves away from the origin. If there exists $\boldsymbol{u} \in \mathcal{U}$ such that $\boldsymbol{u}$ is at the boundary of $\mathbb{R}^m_+$, then $\rho = 1$. We denote

$$\rho(\mathcal{U}) := \rho(\boldsymbol{u}_0,\mathcal{U}),$$

where $\boldsymbol{u}_0$ is the symmetry point of the set $\mathcal{U}$.

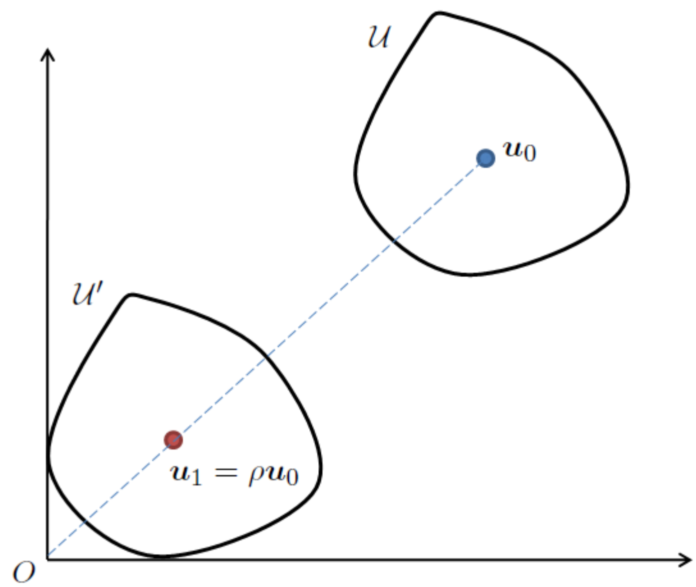

Fig. 1 Geometry of the translation factor.

For the two-stage adjustable robust convex optimization problem, $\Pi^{\mathsf{cons}}_{AR}(\mathcal{U})$ as defined in (1.2) where the constraints are uncertain, we show that

$$z^{\mathsf{cons}}_{Rob}(\mathcal{U}) \leq z^{\mathsf{cons}}_{AR}(\mathcal{U}) \leq \left(1 + \frac{\rho}{s}\right) \cdot z^{\mathsf{cons}}_{Rob}(\mathcal{U}),$$

where $z^{\mathsf{cons}}_{Rob}(\mathcal{U})$ is the objective value of an optimal static robust solution of (1.3) and $s = \mathbf{sym}(\mathcal{U})$ and $\rho = \rho(\mathcal{U})$ are the symmetry and the translation factors of the uncertainty set $\mathcal{U}$. In particular, if the uncertainty set $\mathcal{U}$ is symmetric (such as a hypercube, ellipsoid, or a norm-ball), the bound is 2, i.e., the objective value of the optimal adjustable robust solution is at most twice the objective value of an optimal static-robust solution.

We note that most commonly used uncertainty sets are absolutely symmetric. For instance, a parameter uncertainty is very commonly modeled as an interval leading to a hypercube uncertainty set. If we want to avoid possibly conservative corner points in the uncertainty sets, we consider uncertainty sets like ellipsoids and other norm-balls which are also absolutely symmetric. Therefore, an optimal static robust solution that can be computed efficiently provides a good approximation for the adjustable robust problem. This bounds are similar to the bounds presented in [8] and [9] for the performance of a static robust solution for adjustable robust linear minimization problems where the uncertainty appears in the right hand side of the constraints. However, the models studied in this paper are very general and address uncertain convex constraints and objectives as opposed to just linear problems. To the best of our knowledge, these are the first approximation bounds for adjustable robust convex optimization problems in such generality.

The rest of the paper is structured as follows. In Section 3, we show that for two-stage adjustable robust convex optimization problems under constraint uncertainty only, we show that the static solution provides a good approximation under fairly general assumptions. We further show in Section 3.1 that an optimal static solution can be arbitrarily worse than an optimal fully-adjustable solution for stochastic optimization problems under constraint uncertainty even for a linear model. In Section 4, we extend the results to incorporate uncertainty both in constraints and the objective. In Section 5, we extend further the results to multi-stage problems.

3 Two-stage Adjustable Robust Convex Optimization

In this section, we prove the following bound on the performance of static robust solutions with respect to an optimal fully-adjustable solution.

Theorem 1 *Consider the problem defined in* (1.2) *and the corresponding robust problem defined in* (1.3). *Let* $s = \mathbf{sym}(\mathcal{U})$ *and* $\rho = \rho(\mathcal{U})$. *Then, under Assumptions* **(A1)-(A4)**

$$z_{Rob}^{\mathsf{cons}}(\mathcal{U}) \le z_{AR}^{\mathsf{cons}}(\mathcal{U}) \le \left(1 + \frac{\rho}{s}\right) \cdot z_{Rob}^{\mathsf{cons}}(\mathcal{U}).$$

In preparation of the proof of Theorem 1 we present several lemmas.

Lemma 1 (Bertsimas et al. [9]) *Let* $\mathcal{U} \subset \mathbb{R}_+^p$ *be a compact convex set and let* $\boldsymbol{u}_0$ *be the point of symmetry of* $\mathcal{U}$. *Then,*

$$\boldsymbol{u} \le \left(1 + \frac{\rho(\mathcal{U})}{\mathbf{sym}(\mathcal{U})}\right) \boldsymbol{u}_0, \ \forall \boldsymbol{u} \in \mathcal{U}. \tag{3.1}$$

Lemma 2 *For any* $0 \le \alpha \le 1$, $\boldsymbol{x} \in X, \boldsymbol{y} \in Y$, $\boldsymbol{u} \in \mathcal{U}$,

(a) *Under Assumption* **(A1)**, $f_1(\alpha \boldsymbol{x}) \ge \alpha f_1(\boldsymbol{x})$.
(b) *Under Assumption* **(A2)**, $f_2(\alpha \boldsymbol{y}) \ge \alpha f_2(\boldsymbol{y})$.
(c) *Under Assumption* **(A3)**, *for all* $i \in I$, $g_i(\alpha \boldsymbol{x}, \alpha \boldsymbol{y}, \boldsymbol{u}) \le \alpha g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})$.

Proof **(a)** Recall that f_1 is concave and $f_1(\mathbf{0}) = 0$ from **(A1)**. Therefore, for any $0 \le \alpha \le 1$,

$$\begin{aligned} f_1(\alpha \boldsymbol{x}) &= f_1((1-\alpha)\mathbf{0} + \alpha \boldsymbol{x}) \\ &\ge (1-\alpha) f_1(\mathbf{0}) + \alpha f_1(\boldsymbol{x}) \\ &= \alpha f_1(\boldsymbol{x}), \end{aligned}$$

where the second last inequality follows from concavity of f_1 and the last equation follows as $f_1(\mathbf{0}) = 0$. A similar argument also proves part **(b)**, i.e.,

$$f_2(\alpha \boldsymbol{y}) \geq \alpha f_2(\boldsymbol{y}).$$

(c) For any $i \in I$, under Assumption **(A3)** g_i is jointly convex in $\boldsymbol{x}$ and $\boldsymbol{y}$ and $g_i(\mathbf{0}, \mathbf{0}, \boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in \mathcal{U}$. Therefore, for any $0 \leq \alpha \leq 1$,

$$\begin{aligned} g_i(\alpha\boldsymbol{x}, \alpha\boldsymbol{y}, \boldsymbol{u}) &= g_i\big((1-\alpha)(\mathbf{0}, \mathbf{0}, \boldsymbol{u}) + \alpha(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})\big) \\ &\leq (1-\alpha) g_i(\mathbf{0}, \mathbf{0}, \boldsymbol{u}) + \alpha g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \\ &= \alpha g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}). \end{aligned}$$

■

Lemma 3 *For any $\beta \geq 1$, $\boldsymbol{x} \in X, \boldsymbol{y} \in Y$, $\boldsymbol{u} \in \mathcal{U}$ and under Assumption* **(A3)**

$$g_i(\boldsymbol{x}, \boldsymbol{y}, \beta\boldsymbol{u}) \leq \beta g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}).$$

Proof Since g_i is concave and increasing in $\boldsymbol{u}$, we have

$$\begin{aligned} g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) &= g_i\left(\left(1 - \frac{1}{\beta}\right)\big(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0}\big) + \frac{1}{\beta}\big(\boldsymbol{x}, \boldsymbol{y}, \beta\boldsymbol{u}\big)\right) \\ &\geq \left(1 - \frac{1}{\beta}\right) g_i(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0}) + \frac{1}{\beta} g_i(\boldsymbol{x}, \boldsymbol{y}, \beta\boldsymbol{u}) \\ &\geq \frac{1}{\beta} g_i(\boldsymbol{x}, \boldsymbol{y}, \beta\boldsymbol{u}), \end{aligned}$$

where the second last inequality follows as g_i is concave in $\boldsymbol{u}$ and the last inequality follows as $g_i(\boldsymbol{x}, \boldsymbol{y}, \mathbf{0}) \geq 0$ for all $\boldsymbol{x} \in X, \boldsymbol{y} \in Y$. ■

Proof of Theorem 1 Let $\boldsymbol{u}_0$ be the point of symmetry of $\mathcal{U}$ (that is the maximizer in (2.2)) and $s = \mathbf{sym}(\boldsymbol{u}_0, \mathcal{U})$. Suppose $\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{u})$ for all $\boldsymbol{u} \in \mathcal{U}$ be an optimal fully adjustable solution for (4.1). Therefore,

$$z_{AR}^{\mathsf{cons}}(\mathcal{U}) = f_1(\boldsymbol{x}^*) + \min_{\boldsymbol{u} \in \mathcal{U}} f_2(\boldsymbol{y}^*(\boldsymbol{u})) \leq f_1(\boldsymbol{x}^*) + f_2(\boldsymbol{y}^*(\boldsymbol{u}_0)). \tag{3.2}$$

Let

$$\alpha = \frac{1}{(1 + \rho/s)},$$

and let $\hat{\boldsymbol{x}} = \alpha\boldsymbol{x}^*$ and $\hat{\boldsymbol{y}} = \alpha\boldsymbol{y}^*(\boldsymbol{u}_0)$. First, we show that $\hat{\boldsymbol{x}} \in X$ and $\hat{\boldsymbol{y}} \in Y$. Recall that X and Y are convex and $\mathbf{0} \in X, \mathbf{0} \in Y$. Also, $\boldsymbol{x}^* \in X, \boldsymbol{y}^*(\boldsymbol{u}_0) \in Y$. Therefore,

$$(1-\alpha)\mathbf{0} + \alpha\boldsymbol{x}^* = \hat{\boldsymbol{x}} \in X, \; (1-\alpha)\mathbf{0} + \alpha\boldsymbol{y}^*(\boldsymbol{u}_0) = \hat{\boldsymbol{y}} \in Y.$$

For any $i \in I$, $\boldsymbol{u} \in \mathcal{U}$ we have

$$\begin{aligned} g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}) &= g_i(\alpha\boldsymbol{x}^*, \alpha\boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}) \\ &\leq \alpha \cdot g_i(\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}) && (3.3) \\ &\leq \alpha \cdot g_i\left(\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{u}_0), \frac{1}{\alpha}\boldsymbol{u}_0\right) && (3.4) \\ &\leq \alpha \cdot \frac{1}{\alpha} \cdot g_i(\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}_0) && (3.5) \\ &\leq c_i, \end{aligned}$$

where (3.3) follows from Lemma 2(c). Inequality (3.4) follows as g_i is increasing in $\boldsymbol{u}$ and $\boldsymbol{u} \leq (1/\alpha)\boldsymbol{u}_0$ from (3.1). Inequality (3.5) follows from Lemma 3 and the last inequality follows as $\boldsymbol{x}^*, \boldsymbol{y}^*$ is a feasible fully-adjustable solution. Therefore, $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ is a feasible solution for (1.3).

Let $z_{Rob}^{\mathsf{cons}}(\mathcal{U})$ be as defined in (1.3). Now,

$$\begin{aligned} z_{Rob}^{\mathsf{cons}}(\mathcal{U}) &\geq f_1(\hat{\boldsymbol{x}}) + f_2(\hat{\boldsymbol{y}}) \\ &= f_1(\alpha \boldsymbol{x}^*) + f_2\left(\alpha \boldsymbol{y}^*(\boldsymbol{u}_0)\right) \\ &\geq \alpha f_1(\boldsymbol{x}^*) + \alpha f_2\left(\boldsymbol{y}^*(\boldsymbol{u}_0)\right) \qquad (3.6) \\ &= \alpha\big(f_1(\boldsymbol{x}^*) + f_2(\boldsymbol{y}^*(\boldsymbol{u}_0))\big) \\ &\geq \alpha \cdot z_{AR}^{\mathsf{cons}}(\mathcal{U}), \qquad (3.7) \end{aligned}$$

where (3.6) follows from Lemma 2(a) and 2(b) and (3.7) follows from (3.2). ■

We next show that we can efficiently compute an optimal static-robust solution to (1.3).

Theorem 2 *Under Assumptions* **(A1)-(A4)** *an optimal static robust solution to* (1.3) *can be computed in polynomial time.*

Proof We show that given any $\boldsymbol{x} \in X, \boldsymbol{y} \in Y$, we can solve the separation problem in polynomial time, i.e., verify that $\boldsymbol{x}, \boldsymbol{y}$ is feasible to (4.4) or find a violated constraint in polynomial time. From the equivalence between separation and optimization [17], we know that an optimal solution to (4.4) can be computed in polynomial time.

The separation problem is the following: given a solution $\hat{\boldsymbol{x}} \in X, \hat{\boldsymbol{y}} \in Y$, we need to decide whether there exist $i \in I$ and $\boldsymbol{u} \in \mathcal{U}$ such that $g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}) > c_i$. Let

$$z_i = \max_{\boldsymbol{u} \in \mathcal{U}} g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}).$$

Under Assumption **(A3)**, g_i is concave in $\boldsymbol{u}$ and $\mathcal{U}$ is convex, and the above maximization problem is a convex optimization problem that can be solved efficiently. Thus, we can compute z_i for all $i \in I$. If for any $i' \in I$, $z_{i'} > c_{i'}$, then there exists $\boldsymbol{u} \in \mathcal{U}$ such that

$$g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}) > c_i,$$

which implies that $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ is not feasible for (1.3). On the other hand, if $z_i \leq c_i$ for all $i \in I$, it implies that

$$g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}) \leq c_i, \ \forall \boldsymbol{u} \in \mathcal{U}, \ i \in I,$$

implying that $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ is feasible. Hence, the separation problem can be solved in polynomial time. ■

3.1 Large Gap Instance for Stochastic Optimization

While we show that the performance of a static robust solution for the adjustable robust maximization problem under uncertain convex constraints and objective is good under fairly general assumptions, the same is not true for the stochastic optimization problem under constraint uncertainty even for a linear problem. Consider the following instance.

$$\begin{aligned} \max \ & \mathbb{E}_\theta[x(\theta)] \\ & \theta x(\theta) \leq 1, \ \forall \theta \in [1/e^M, 1], \end{aligned} \qquad (3.8)$$

where θ is uniformly distributed between $1/e^M$ and 1 for some large $M \in \mathbb{R}$. It is easy to see that the optimal objective value of a fully-adjustable solution is $M(1-e^{-M})$ while the optimal static solution is $x = 1$ with objective value 1. Therefore, the static solution can be arbitrarily worse as compared to a fully-adjustable solution for the stochastic optimization problem under constraint coefficient uncertainty. Note that an adjustable robust solution can also be highly suboptimal for the stochastic optimization problem in general, however, the above example shows that we can not prove similar bounds on the performance of static solutions for the case of stochastic convex optimization problems under similar assumptions.

4 Two-stage Robust Convex Optimization: Constraint and Objective Uncertainty

In this section, we consider an extension to the model when both constraints and objective are uncertain. In particular, we study the following adjustable robust version of a two-stage convex optimization problem $\Pi_{AR}(\mathcal{U})$ under uncertain constraint and objective functions.

$$\begin{aligned} z_{AR}(\mathcal{U}) = \max_{\boldsymbol{x}\in X} f_1(\boldsymbol{x}) + \min_{\boldsymbol{u}\in\mathcal{U}} & \max_{\boldsymbol{y}(\boldsymbol{u})\in Y} f_2\big(\boldsymbol{y}(\boldsymbol{u}), \boldsymbol{u}\big) \\ & g_i(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{u}), \boldsymbol{u}) \le c_i,\ \forall \boldsymbol{u} \in \mathcal{U},\ \forall i \in I, \end{aligned} \tag{4.1}$$

where $\mathcal{U} \subset \mathbb{R}^p_+$ is a compact convex set, I is the set of indices for the constraints, $c_i \in \mathbb{R}_+$ for all $i \in I$. Similar to (1.2), the objective function $f_1 : \mathbb{R}^{n_1} \to \mathbb{R}_+$ satisfies **(A1)**, the constraint function $g_i : \mathbb{R}^{n_1+n_2+p} \to \mathbb{R}_+, i \in I$ satisfies **(A3)**, and sets X and Y satisfy **(A4)**. We also make the following assumption for f_2.

(A5) $f_2 : \mathbb{R}^{n_2+p} \to \mathbb{R}_+$ is concave in $\boldsymbol{y}$ and convex and decreasing in $\boldsymbol{u}$ and satisfies

$$\begin{aligned} & f_2(\boldsymbol{0}, \boldsymbol{u}) = 0, \forall \boldsymbol{u} \in \mathcal{U}, \\ & f_2(\boldsymbol{y}, \beta\boldsymbol{u}) \ge \frac{1}{\beta} \cdot f_2(\boldsymbol{y}, \boldsymbol{u}), \forall \beta \ge 1, \boldsymbol{y} \in Y, \boldsymbol{u} \in \mathcal{U}. \end{aligned} \tag{4.2}$$

Moreover, we assume that given $\hat{\boldsymbol{y}} \in Y$, the problem $\min_{\boldsymbol{u}\in\mathcal{U}} f_2(\hat{\boldsymbol{y}}, \boldsymbol{u})$ is solvable in polynomial time.

This assumption is necessary for the tractability of the separation problem for computing an optimal static robust solution. Note that the second-stage objective function, f_2 depends on both, the decision variable $\boldsymbol{y}(\boldsymbol{u})$, as well as the uncertain vector $\boldsymbol{u} \in \mathcal{U}$. Therefore, we can model uncertainty in the objective function in this framework while in (1.2), the objective function was not uncertain.

As an example of an application that can be formulated in the framework (4.1), consider the following maximization problem with a fractional objective under constraint and objective uncertainty. We first consider the classical two-stage adjustable robust linear maximization problem with uncertainty in the constraint coefficients and in the objective.

$$\begin{aligned} z_{AR}^{\mathsf{Frac}}(\mathcal{U}) = \max \boldsymbol{c}^T\boldsymbol{x} + \min_{(\boldsymbol{B},\boldsymbol{d})\in\mathcal{U}} & \max_{\boldsymbol{y}(\boldsymbol{B},\boldsymbol{d})\in Y} \sum_{j=1}^{n_2} \frac{1}{d_j} \cdot y(\boldsymbol{B}, \boldsymbol{d}) \\ & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}(\boldsymbol{B}, \boldsymbol{d}) \le \boldsymbol{h}\ \ \forall (\boldsymbol{B}, \boldsymbol{d}) \in \mathcal{U} \\ & \boldsymbol{x} \ge \boldsymbol{0} \\ & \boldsymbol{y}(\boldsymbol{B}, \boldsymbol{d}) \ge \boldsymbol{0}, \end{aligned} \tag{4.3}$$

where $\boldsymbol{A} \in \mathbb{R}_+^{m\times n_1}$ is the first-stage constraint matrix, $\boldsymbol{B}$ is the uncertain second-stage constraint matrix and $\boldsymbol{d}$ define the uncertain second-stage objective and $(\boldsymbol{B}, \boldsymbol{d}) \in \mathcal{U} \subset \mathbb{R}_+^{m\times n_2 \times n_2}$, $\boldsymbol{c} \in \mathbb{R}_+^{n_1}$ and $\boldsymbol{h} \in \mathbb{R}_+^{m}$. In the framework (4.1), $f_1(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x}$ that satisfies Assumption **(A1)**,

$$g_i(\boldsymbol{x}, \boldsymbol{y}, (\boldsymbol{B}, \boldsymbol{d})) = \sum_{j=1}^{n_1} A_{ij}x_j + \sum_{j=1}^{n_2} B_{ij}y_j, i = 1, \ldots, m,$$

that satisfies Assumption **(A3)** as it is convex (linear) in $\boldsymbol{x}$ and $\boldsymbol{y}$ and concave (linear) and increasing in $(\boldsymbol{B}, \boldsymbol{d})$ since $\boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0$ and $\boldsymbol{B} \in \mathbb{R}_+^{m\times n_2}$. Also,

$$f_2(\boldsymbol{y}, (\boldsymbol{B}, \boldsymbol{d})) = \sum_{j=1}^{n_2} \frac{1}{d_j} \cdot y_j.$$

Note that f_2 satisfies Assumption **(A5)**: f_2 is concave (linear) in $\boldsymbol{y}$ and convex in $\boldsymbol{d}$ (and therefore, $(\boldsymbol{B}, \boldsymbol{d})$). Also, f_2 is decreasing in $(\boldsymbol{B}, \boldsymbol{d})$ and for all $(\boldsymbol{B}, \boldsymbol{d}) \in \mathcal{U}$, $\beta \geq 1$, f_2 satisfies (4.2), i.e., $f_2(\boldsymbol{0}, (\boldsymbol{B}, \boldsymbol{d})) = 0$, and

$$\begin{aligned} f_2(\boldsymbol{y}, \beta(\boldsymbol{B}, \boldsymbol{d})) &= \sum_{j=1}^{n_2} \frac{1}{\beta d_j} y_j \\ &= \frac{1}{\beta} \sum_{j=1}^{n_2} \frac{1}{d_j} y_j \\ &= \frac{1}{\beta} f_2(\boldsymbol{y}, (\boldsymbol{B}, \boldsymbol{d})). \end{aligned}$$

Such a fractional objective can arise in problems where there is a functional relationship between constraint and objective uncertainty. For instance, consider a revenue management problem where the objective coefficients correspond to the price while constraint coefficients correspond to the resource requirements. The uncertainty in both these elements is driven by the uncertainty in demand: if a particular demand j is high, the resource requirement would be high which implies the constraint coefficients $B_{ij}, i = 1, \ldots, m$ are high, while the corresponding price, i.e., the objective coefficient of y_j is small. Therefore, we need a fractional objective function to model such a functional relationship.

Since it is intractable to compute an optimal fully-adjustable solution for (4.1) in general even when f_1, f_2 and $g_i, i \in I$ are linear (see Dyer and Stougie [13]), we consider the following corresponding static robust problem, $\Pi_{Rob}(\mathcal{U})$.

$$\begin{aligned} z_{Rob}^{\max}(\mathcal{U}) = \max_{\boldsymbol{x}\in X, \boldsymbol{y}\in Y} \; & f_1(\boldsymbol{x}) + \min_{\boldsymbol{u}\in\mathcal{U}} \; f_2(\boldsymbol{y}, \boldsymbol{u}) \\ & g_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}) \leq c_i, \; \forall \boldsymbol{u} \in \mathcal{U}, \; \forall i \in I, \end{aligned} \tag{4.4}$$

We prove a bound on the performance of static robust solutions with respect to an optimal fully-adjustable solution. In particular, we prove the following theorem.

Theorem 3 *Consider the problem defined in* (4.1) *and the corresponding robust problem defined in* (4.4). *Let* $s = \mathbf{sym}(\mathcal{U})$ *and* $\rho = \rho(\mathcal{U})$. *Then, under Assumptions* **(A1), (A3)-(A5)**

$$z_{Rob}(\mathcal{U}) \leq z_{AR}(\mathcal{U}) \leq \left(1 + \frac{\rho}{s}\right)^2 \cdot z_{Rob}(\mathcal{U}).$$

Note that the approximation bound is worse than the bound in Theorem 1 where we consider the model, $\Pi_{AR}^{\mathsf{cons}}(\mathcal{U})$ where only the constraints are uncertain. We first show that we can efficiently compute an optimal static-robust solution to (4.4).

Theorem 4 *Under Assumptions* **(A1), (A3)-(A5)***, an optimal static robust solution to* (4.4) *can be computed in polynomial time.*

Proof The separation problem is the following: given a solution $\hat{\boldsymbol{x}} \in X, \hat{\boldsymbol{y}} \in Y, \hat{\theta} \in \mathbb{R}$, we need to decide whether there exist $i \in I$ and $\boldsymbol{u} \in \mathcal{U}$ such that $g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}) > c_i$ or there exist $\boldsymbol{u} \in \mathcal{U}$ such that $f_2(\hat{\boldsymbol{y}}, \boldsymbol{u}) < \hat{\theta}$. The separation problem over the constraints g_i is same as in (1.3) and can be solved by computing

$$z_i = \max_{\boldsymbol{u} \in \mathcal{U}} g_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{u}),$$

which is a polynomial time solvable problem under Assumption **(A3)**.

The objective function f_2 is also uncertain and we need to verify whether $\hat{\theta} \geq f_2((\hat{\boldsymbol{y}}, \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{U}$. Let

$$\theta = \min_{\boldsymbol{u} \in \mathcal{U}} f_2(\hat{\boldsymbol{y}}, \boldsymbol{u}).$$

Since f_2 is convex in $\boldsymbol{u}$ and $\mathcal{U}$ is convex, the above maximization problem is a convex optimization problem and can be solved in polynomial time under Assumption **(A5)**. If $\theta < \hat{\theta}$, there exists $\boldsymbol{u} \in \mathcal{U}$ such that $f_2(\hat{\boldsymbol{y}}, \boldsymbol{u}) < \hat{\theta}$. ■

Let us first prove the following property for f_2.

Lemma 4 *For any* $\boldsymbol{y} \in Y$*,* $\boldsymbol{u} \in \mathcal{U}$ *and* $0 \leq \alpha \leq 1$*,*

$$f_2(\alpha \boldsymbol{y}, \boldsymbol{u}) \geq \alpha f_2(\boldsymbol{y}, \boldsymbol{u}).$$

Proof Recall that f_2 is concave in $\boldsymbol{y}$ and $f_2(\boldsymbol{0}, \boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in \mathcal{U}$. Therefore, for any $0 \leq \alpha \leq 1$,

$$\begin{aligned} f_2(\alpha \boldsymbol{y}, \boldsymbol{u}) &= f_2((1-\alpha)(\boldsymbol{0}, \boldsymbol{u}) + \alpha(\boldsymbol{y}, \boldsymbol{u})) \\ &\geq (1-\alpha) f_2(\boldsymbol{0}, \boldsymbol{u}) + \alpha f_2(\boldsymbol{y}, \boldsymbol{u}) \\ &= \alpha f_2(\boldsymbol{y}, \boldsymbol{u}), \end{aligned}$$

where the second last inequality follows from concavity of f_2 and the last equation follows as $f_2(\boldsymbol{0}, \boldsymbol{u}) = 0$. ■

Proof of Theorem 3 Let $\boldsymbol{u}_0$ be the point of symmetry of $\mathcal{U}$ and $s = \mathbf{sym}(\boldsymbol{u}_0, \mathcal{U})$. Suppose $\boldsymbol{x}^*, \boldsymbol{y}^*(\boldsymbol{u})$ for all $\boldsymbol{u} \in \mathcal{U}$ be an optimal fully adjustable solution for (4.1). Therefore,

$$z_{AR}(\mathcal{U}) = f_1(\boldsymbol{x}^*) + \min_{\boldsymbol{u} \in \mathcal{U}} f_2(\boldsymbol{y}^*(\boldsymbol{u}), \boldsymbol{u}) \leq f_1(\boldsymbol{x}^*) + f_2(\boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}_0). \tag{4.5}$$

Let $\alpha = \frac{1}{(1+\rho/s)}$, and let $\hat{\boldsymbol{x}} = \alpha \boldsymbol{x}^*$ and $\hat{\boldsymbol{y}} = \alpha \boldsymbol{y}^*(\boldsymbol{u}_0)$. Using an argument similar to the proof of Theorem 1, we can show that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ is a feasible solution for (4.4).

Therefore,

$$
\begin{aligned}
z_{Rob}(\mathcal{U}) &\geq f_1(\hat{\boldsymbol{x}}) + \min_{\boldsymbol{u}\in\mathcal{U}} f_2(\hat{\boldsymbol{y}}, \boldsymbol{u}) \\
&\geq f_1(\hat{\boldsymbol{x}}) + f_2\left(\hat{\boldsymbol{y}}, \frac{1}{\alpha}\boldsymbol{u}_0\right) && (4.6)\\
&= f_1(\alpha\boldsymbol{x}^*) + f_2\left(\alpha\boldsymbol{y}^*(\boldsymbol{u}_0), \frac{1}{\alpha}\boldsymbol{u}_0\right) \\
&\geq \alpha f_1(\boldsymbol{x}^*) + f_2\left(\alpha\boldsymbol{y}^*(\boldsymbol{u}_0), \frac{1}{\alpha}\boldsymbol{u}_0\right) && (4.7)\\
&\geq \alpha f_1(\boldsymbol{x}^*) + \alpha f_2\left(\boldsymbol{y}^*(\boldsymbol{u}_0), \frac{1}{\alpha}\boldsymbol{u}_0\right) && (4.8)\\
&\geq \alpha f_1(\boldsymbol{x}^*) + \alpha\cdot\alpha f_2(\boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}_0) && (4.9)\\
&\geq \alpha^2\big(f_1(\boldsymbol{x}^*) + f_2(\boldsymbol{y}^*(\boldsymbol{u}_0), \boldsymbol{u}_0)\big) && (4.10)\\
&\geq \alpha^2\cdot z_{AR}(\mathcal{U}), && (4.11)
\end{aligned}
$$

where (4.6) follows as f_2 is decreasing in $\boldsymbol{u}$ (Assumption **(A5)**), and for all $\boldsymbol{u}\in\mathcal{U}$, $\boldsymbol{u}\leq 1/\alpha\cdot\boldsymbol{u}_0$. Inequality (4.7) follows from Lemma 2(a) and from **(A1)**. Inequality (4.8) follows from Lemma 4. Inequality (4.9) follows from (4.2). Inequality (4.10) follows as $\alpha\leq 1$ and $f_1(\boldsymbol{x}^*)\geq 0$ and the last inequality follows from (4.5). ■

5 Multi-stage Adjustable Robust Convex Optimization

In this section, we consider the multi-stage extension to the two-stage adjustable robust convex optimization problems considered in previous sections. In a multi-stage problem, uncertainty is revealed in multiple stages and the decision in each stage depends only on the realization of the uncertain parameters in the past stages. Many applications including the dynamic knapsack problem and the applications in revenue management as discussed in (1.1) are examples of multi-stage problems.

Let us first introduce the multi-stage model. For each $k=1,\ldots,K$, $\boldsymbol{u}_k$ denotes the uncertain parameters in stage $k+1$ and $\boldsymbol{y}_k(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k)$ denote the decision in stage $k+1$. Note that the decision in stage $k+1$ depends on the uncertain parameter realization in stages 2 to $k+1$. The uncertain vector $\boldsymbol{u}_k\in\mathcal{U}_k$ for each $k=1,\ldots,K$. In general, the uncertainty set $\mathcal{U}_k$ in stage $k+1$ depends on the realization of the uncertain parameters in previous stages. Such a multi-stage uncertainty evolution can be represented as a directed *layered* network where each stage corresponds to a layer and the nodes in each layer represent the possible uncertainty sets in that stage. Such an uncertainty network is a generalization of the scenario tree model (see Shapiro et al. [23]) and is discussed in detail in [9]. For a general uncertainty evolution network, a static solution is not a good approximation of the fully-adjustable model. However, Bertsimas et al. [9] show that a *finitely adjustable* solution where instead of a single (static) solution, a small number of solutions is specified and for each possible realization of the uncertain parameters, at least one of the solution is a good approximation for the linear constrained model with right hand side and objective coefficient uncertainty. Moreover, the number of solutions in the finitely adjustable solution policy depend on the total number of paths from the root node in the uncertainty network to nodes corresponding to last stage.

Due to the space constraint and for the sake of notational convenience, we present only the case where the uncertainty network is a single path. The ideas extend to the case where there are multiple paths in the network in a straightforward manner using ideas from Bertsimas et al. [9] where we associate one solution for each path in the uncertainty network to construct a finitely adjustable solution. We consider the following adjustable robust maximization problem under constraint function uncertainty, Π_{AR}^{mult}.

$$
\begin{aligned}
z_{AR}^{\mathsf{mult}} = \max_{\boldsymbol{x}\in X}\ f_1(\boldsymbol{x}) + \min_{\boldsymbol{u}_1\in\mathcal{U}_1}\ \max_{\boldsymbol{y}_1(\boldsymbol{u}_1)} \Bigg(\min_{\boldsymbol{u}_2\in\mathcal{U}_2}\ \max_{\boldsymbol{y}_2(\boldsymbol{u}_1,\boldsymbol{u}_2)} \Big(\ldots \\
\min_{\boldsymbol{u}_K\in\mathcal{U}_K}\ \max_{\boldsymbol{y}_K(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K)} f_2(\boldsymbol{y}_1(\boldsymbol{u}_1),\ldots,\boldsymbol{y}_K(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K))\Big) \ldots \Bigg) \\
g_i(\boldsymbol{x},\boldsymbol{y}_1(\boldsymbol{u}_1),\ldots,\boldsymbol{y}_K(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K),\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K) \le c_i, \\
\forall \boldsymbol{u}_k\in\mathcal{U}_k,\ \forall i\in I, k\in[K]
\end{aligned}
\tag{5.1}
$$

where $\mathcal{U}_k \subset \mathbb{R}_+^{p_k}, k=1,\ldots,K$ is a compact convex set, I is the set of indices for the constraints, $c_i\in\mathbb{R}_+$ for all $i\in I$ and functions $f_1:\mathbb{R}^n\to\mathbb{R}_+$, $f_2:\mathbb{R}^{(n_1+\ldots+n_k)}\to\mathbb{R}_+$ and $g_i:\mathbb{R}^{n+(n_1+p_1)+\ldots+(n_k+p_k)}\to\mathbb{R}_+, i\in I$ satisfy Assumptions **(A1)-(A4)**. These assumptions can be translated to the assumptions in the multi-stage model by assuming that $\boldsymbol{u}=(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K)$ and $\boldsymbol{y}=(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K)$. Note that we consider that the uncertainty set $\mathcal{U}_k$ in stage $k+1$ does not depend on the realization of the uncertain parameters in the previous stages and thus, the uncertainty evolution network is a path. We can formulate the corresponding static robust optimization problem as follows.

$$
\begin{aligned}
z_{Rob}^{\mathsf{mult}} = \ \max\ & f_1(\boldsymbol{x}) + f_2(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K) \\
& g_i(\boldsymbol{x},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K,\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K) \le c_i,\ \forall \boldsymbol{u}_k\in\mathcal{U}_k, k=1,\ldots,K\ \forall i\in I \\
& \boldsymbol{x}\in X \\
& \boldsymbol{y}_k\in Y_k,\ k=1,\ldots,K.
\end{aligned}
\tag{5.2}
$$

The above multi-stage problem can be considered as a two-stage problem where the second-stage decision is $(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_K)\in Y_1\times\ldots\times Y_k$ and the uncertain vector is $(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_K)\in\mathcal{U}_1\times\ldots\times\mathcal{U}_K$. The multi-stage problem where the uncertainty evolution network is a path essentially reduces to a two-stage problem. From Theorem 2, we know that an optimal solution to (5.2) can be computed in polynomial time.

We show that a single (static) solution is a good approximation of the fully-adjustable problem Π_{AR}^{mult} when the uncertainty evolution network is a path. In particular, we prove the following theorem.

Theorem 5 *Consider the problem Π_{AR}^{mult} defined in* (5.1) *and the corresponding problem defined in* (5.2)*. For all $k=1,\ldots,K$, let $s_k=\mathbf{sym}(\mathcal{U}_k)$, $\rho_k=\rho(\mathcal{U}_k$ and*

$$s=\min_{k=1,\ldots,K} s_k,\ \ \rho=\max_{k=1,\ldots,K}\rho_k.$$

Then, under Assumptions **(A1)-(A4)**

$$z_{Rob}^{\mathsf{mult}} \le z_{AR}^{\mathsf{mult}} \le \left(1+\frac{\rho}{s}\right)\cdot z_{Rob}^{\mathsf{mult}}.$$

In the interest of space, we only include a sketch of the proof which proceed along similar lines as the proof of Theorem 1. We construct a feasible solution for (5.2) from an optimal fully-adjustable solution for (5.1). In particular, if $\boldsymbol{u}_k^0$ is the point of symmetry of $\mathcal{U}_k$, and $\boldsymbol{x}^*, \boldsymbol{y}_k^*(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ for all $\boldsymbol{u}_1 \in \mathcal{U}_1, \ldots, \boldsymbol{u}_k \in \mathcal{U}_k$, $k = 1, \ldots, K$ is an optimal fully-adjustable solution for Π_{AR}^{mult}. Then we show that the solution

$$\hat{\boldsymbol{x}} = \frac{1}{(1+\rho/s)}\boldsymbol{x}^*,\ \hat{\boldsymbol{y}}_k = \frac{1}{(1+\rho/s)}\boldsymbol{y}_k^*(\boldsymbol{u}_1^0, \ldots, \boldsymbol{u}_k^0),\ k = 1, \ldots, K,$$

is a feasible solution for (5.2) under Assumptions **(A3)** and **(A4)** satisfied by the constraint functions. Moreover, under Assumptions **(A1)** and **(A2)** for the objective functions, we show that this feasible solution has an objective value of $1/(1+\rho/s)$ fraction of z_{AR}^{mult}, thus, proving the desired performance bound for static solutions.

6 Conclusions

In this paper, we propose a tractable approximation for very general adjustable robust convex optimization problems where uncertainty appears both in the constraints as well as the objective. In fact, we show that a static robust solution is a good approximation for such problems under fairly general assumptions and the performance depends on the geometric properties of the uncertainty set, namely, the symmetry and the translation factor. The models considered in the paper are significant extensions of the results in [8] and [9] and allow us to formulate important applications such as revenue management, scheduling and portfolio optimization that were not addressed in the earlier work.

In any real-world problem, the choice of how to model uncertainty lies with the modeler. Since our results relate the performance bounds to the geometric properties of the uncertainty set, they provide useful guidance towards choosing a specific uncertainty set for a particular application such that the performance of static-robust is good. Moreover, since an optimal static solution can be computed efficiently in most cases, the results in this paper provide a strong justification for the practical applicability of the robust optimization approach in many important applications.

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