Cyclic Packing Designs and Simple Cyclic Leaves Constructed from Skolem-Type Sequences

by

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Abstract

A Packing Design, or a $PD\,(v,k,\lambda)$ is a pair $(V,\mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a set of k-subsets *(blocks)* such that any 2-subset of V appears in at most λ blocks. $PD\,(v,k,\lambda)$ is cyclic if its automorphism group contains a v-cycle, and it is called a *cyclic packing design*. The edges in the multigraph λK_v not contained in the packing form the leaves of the $CPD(v,k,\lambda)$, denoted by leave (v,k,λ).

In 2012, Silvesan and Shalaby used Skolem-type sequences to provide a complete proof for the existence of cyclic $BIBD\,(v,3,\lambda)$ for all admissible orders v and λ.

In this thesis, we use Skolem-type sequences to find all cyclic packing designs with block size 3 for a cyclic $BIBD\,(v,3,\lambda)$ and find the spectrum of leaves graph of the cyclic packing designs, for all admissible orders v and λ with the optimal leaves, as well as determine the number of base blocks for every λ when $k=3$.

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Chapter 1

Introduction

Design theory has its roots in recreational mathematics. Many types of designs that are studied today were first considered in the context of mathematical puzzles or brain-teasers in the eighteenth and nineteenth centuries. Designs with the property of balance became known as balanced incomplete block designs ($BIBD$); some were listed in the statistical tables of Fisher and Yates in 1938. [17]

Designs have many applications, such as tournament scheduling, lotteries, mathematical biology, algorithm design and analysis, networking, group testing, and cryptography. Design theory makes use of tools from linear algebra, group theory, ring theory and field theory, number theory, and combinatorics. The basic concepts of design theory are quite simple, but the mathematics used to study designs is varied and rich.

Combinatorial design theory is thought to have started in 1779 when Euler posed the question of constructing two orthogonal latin squares of order 6. This was known as *Euler's 36 Officers Problem* [1]. Over the years, however, combinatorial researchers have discussed a wider range of designs. These have included: 1-factorizations, Room Squares, designs based on unordered pairs, as well as other designs.

Informally, one may define a combinatorial design to be a way of selecting subsets from a finite set such that some conditions are satisfied. As an example, suppose it is required to select 3-sets from the seven objects $\{a, b, c, d, e, f, g\}$, such that each object occurs in three of the 3-sets and every intersection of two 3-sets has precisely one member. The solution to such a problem is a combinatorial design. One possible example is $\{abc, ade, afg, bdf, beg, cdg, cef\}$, which is also called a Steiner triple system of order 7 and is denoted $STS(7)$.

Another subject systematically studied was triple systems and among the most important is the celebrated Kirkman [21] *schoolgirl problem* which fascinated mathematicians for many years, and is as follows: 'Fifteen young ladies of a school walk out three abreast for seven days in succession: it's required to arrange them daily, so that no two shall walk twice abreast.'

Without the requirement of arranging the triples in days, the configuration is a

Steiner triple system of order 15, and hence was known to Kirkman. The first person to publish a complete solution to the Kirkman schoolgirl problem was Cayley [12]. Triple systems are natural generalizations of graphs and much of their study has a graph theoretic flavour. Connections with geometry, algebra, group theory and finite fields provide other perspectives.

Cyclic triple systems were also studied by Heffter [19], who introduced his famous first and second difference problems in relation to the construction of cyclic Steiner triple systems of order $6n + 1$ and $6n + 3$. In 1897, Heffter [19] stated two difference problems. The solution to these problems is equivalent to the existence of cyclic Steiner triple systems. Heffter's first difference problem (denoted by $HDP_1(n)$) is as follows:

Can a set $\{1, \ldots, 3n\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leq i \leq n$ such that $a_i + b_i \equiv c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n + 1}$?.

If such a partition is possible then $\{\{0, a_i + n, b_i + n\} | 1 \leq i \leq n\}$ will be the base blocks of a cyclic Steiner triple system of order $6n + 1$, $CSTS(6n + 1)$.

Heffter's second difference problem (denoted by $HDP_2(n)$) is as follows:

Can a set $\{1, \ldots, 3n + 1\} \backslash \{2n + 1\}$ be partitioned into n ordered triples (a_i, b_i, c_i) with $1 \leq i \leq n$ such that $a_i + b_i \equiv c_i$ or $a_i + b_i + c_i \equiv 0 \pmod{6n + 3}$?.

If such a partition is possible then $\{\{0, a_i + n, b_i + n\} | 1 \leq i \leq n\}$ with the addition of the base block $\{0, 2n+1, 4n+2\}$ having a short orbit of length $3n+1$ will be the base blocks of a cyclic Steiner triple system of order $6n+3$, $CSTS(6n+3)$.

In 1939, Peltesohn [24] solved Heffter's two difference problems, showing that at least one solution exists for each case, and constructed cyclic Steiner triple systems of order v for $v \equiv 1, 3 \pmod 6), v \neq 9$.

Skolem [36] also had an interest in triple systems; he constructed $STS(v)$ for $v = 6n+1$. He introduced the idea of a Skolem sequence of order n, which is a sequence of integers that satisfies the following two properties: every integer i, $1 \leq i \leq n$, occurs exactly twice and the two occurrences of i are exactly i integers apart. The sequence $(4, 2, 3, 2, 4, 3, 1, 1)$ is equivalent to the partition of the numbers $1, \ldots, 8$ into the pairs $(7, 8), (2, 4), (3, 6), (1, 5)$. This sequence is now known as a Skolem sequence of order 4. In the literature, Skolem sequences are also known as pure or perfect Skolem sequences. Skolem found the necessary conditions for the existence of such a sequence to be $n \equiv 0, 1 \pmod 4)$. He credited his colleague Th. Bang [3] for finding this proof. Skolem [37] extended this idea to that of the hooked Skolem sequence which will be defined later, the existence of which, for all admissible n, along with that of Skolem sequences, would constitute a complete solution to Heffter's first difference problem, and would lead to the construction of cyclic $STS(6n+1)$. O'Keefe [23] proved that a hooked Skolem sequence of order n exists if and only if $n \equiv 2,\, 3\ (mod\ 4)$.

Rosa [29], in 1966, introduced other types of sequences called Rosa and hooked Rosa sequences, and proved that a Rosa sequence of order n exists if and only if $n \equiv 0,\, 3\ (\mod 4)$ and a hooked Rosa sequence of order n exists if and only if $n \equiv 1,\, 2\ (mod\ 4)$. These two types of sequences constitute a complete solution to Heffter's second difference problem, which leads to the construction of cyclic $STS(v)$ for $v = 6n+3$. Thus, the study of triple systems has grown into a major part of the study of combinatorial designs.

In 1966, Schönheim [30,31] gave the Schönheim upper bound and lower bound for the number of blocks and edges in optimal packing and covering designs. Hanani [18] and Stanton, Rogers, Quinn and Cowan [38] introduced several results related to covering and packing designs. In Section 2.3 we present the main theorems regarding covering and packing designs.

In 1981, Colbourn and Colbourn [16] solved the existence problem of cyclic $BIBD(v, 3, \lambda)$, in [16] the necessary and sufficient conditions of Theorem 1.0.1 were given using the Peltesohn's technique.

Theorem 1.0.1 *[M. Colbourn, C. Colbourn, [16]] Necessary and sufficient conditions for the existence of a cyclic $BIBD(v, 3, \lambda)$ are:*

1. $\lambda \equiv 1, 5, 7, 11 (\mathrm{mod}\ 12)$ *and* $v \equiv 1, 3 (\mathrm{mod}\ 6)$ *or*

2. $\lambda \equiv 2, 10 \pmod{12}$ *and* $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ *or*

3. $\lambda \equiv 3, 9 \pmod{12}$ *and* $v \equiv 1 \pmod{2}$ *or*

4. $\lambda \equiv 4, 8 \pmod{12}$ *and* $v \equiv 0, 1 \pmod{3}$ *or*

5. $\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0, 1, 3 \pmod{4}$ *or*

6. $\lambda \equiv 0 \pmod{12}$ *and* $v \geq 3$,

with only two exceptions: cyclic $BIBD(9, 3, 1)$ *and cyclic* $BIBD(9, 3, 2)$ *do not exist.*

In 1992, Colbourn, Hoffman and Rees [15] used Skolem type sequences to construct cyclic partial Steiner triple systems of order v. Rees and Shalaby [28], in 2000, constructed cyclic triple systems with $\lambda = 2$ using Skolem-type sequences.

In 2012, Silvesan and Shalaby [35] extended the techniques used in [28] and introduced several new constructions that use Skolem-type sequences to construct cyclic triple systems for all admissible $\lambda > 2$ when $k = 3$.

Silvesan and Shalaby [35] proved the sufficiency of Theorem **??** using Skolem-type sequences. They have arranged the necessary conditions in Table 1.1, with $-$ sign for the designs with no short orbits, and $+$ sign for the designs that have short orbits with length equal to 1/3 of the full orbit. An empty cell in the table means that such a design does not exist.

$v/\lambda \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11
0	-		+		+		-		+		+	
1	-	-	-	-	-	-	-	-	-	-	-	-
2	-											
3	-	+	+	-	+	+	-	+	+	-	+	+
4	-		-		-		-		-		-	
5	-			-			-			-		
6	-				+				+			
7	-	-	-	-	-	-	-	-	-	-	-	-
8	-						-					
9	-	+	+	-	+	+	-	+	+	-	+	+
10	-				-				-			
11	-			-			-			-		

Table 1.1: Necessary conditions for the existence of a cyclic $BIBD(v, 3, \lambda)$, [35]

In this thesis, Chapter II is an introduction to basic concepts and known results. In Chapter III, Skolem type sequences are used to construct cyclic packing designs with the optimal leaves, as well as determining the number of base blocks for every λ when $k = 3$. Chapter IV contains concluding remarks, our results and future work.

Chapter 2

Preliminaries

In this chapter we review the basic definitions and theorems regarding Balanced Incomplete Block Designs, difference sets of designs, cyclic block designs, packing and covering designs, Skolem type sequences and circulant graphs.

2.1 Balanced Incomplete Block Designs ($BIBDs$)

In 1835, Plücker [26], in a study of algebraic curves, observed that, given v elements, a family of subsets of size 3 in which every pair of elements occurs in exactly a of the subsets, will contains $\frac{1}{6}v(v-1)$ such subsets. Such a system is called a Balanced Incomplete Block Design, or $BIBD\,(v,3,\lambda)$. Later, Plücker conjectured (correctly) that $v \equiv 1, 3\,(mod\ 6)$ is the necessary condition for a $BIBD\,(v, 3, 1)$ to exist [27].

The first block designs to be studied in detail were Balanced Incomplete Block Designs ($BIBDs$) with block size 3. These are also called triple systems. Let us define these terms as follows:

Definition 2.1.1 *A* ***Balanced Incomplete Block Design*** $BIBD\,(v, k, \lambda)$ *is a collection of k-subsets (or blocks) from a set* V*, where* $|V| = v, k < v$*, where each pair from* V *occur in exactly* λ *of the blocks.*

Example 2.1.1 *A* $BIBD\,(7, 3, 1)$*,* $V = \{1, 2, 3, 4, 5, 6, 7\}$:

$B = \{\{1, 2, 4\}\,\{2, 3, 5\}\,\{3, 4, 6\}\,\{4, 5, 7\}\,\{5, 6, 1\}\,\{6, 7, 2\}\,\{7, 1, 3\}\}\,.$

Definition 2.1.2 *Two set systems* $(V, \mathcal{B})$ *and* (W, D) *are isomorphic if there is a bijection (isomorphism)* φ *from* V *to* W *so that the number of times* $\mathcal{B}$ *appears as a block in* $\mathcal{B}$ *is the same as the number of times* $\varphi(\mathcal{B}) = \{\varphi(\S) : \S \in \mathcal{B}\}$ *appears as a block in* D*.*

Definition 2.1.3 *An isomorphism from a set system to itself is an automorphism.*

Definition 2.1.4 *A* $BIBD\,(v, k, \lambda)$ *is* ***cyclic*** *if it admits an automorphism, if* $(V, \mathcal{B})$ *is a cyclic* $BIBD\,(v, k, \lambda)$*, one may assume* $V = \mathbb{Z}_v$*, and* $\alpha : i \to i+1 \;\; (mod\ v)$ *is its cyclic automorphism. Let* $B = \{b_1, b_2, \cdots, b_k\}$ *be a block of a cyclic* $BIBD\,(v, k, \lambda)$*.*

The block orbit containing B is defined by the set of distinct blocks $B + i = \{b_1 + i, \cdots, b_k + i\} \pmod{v}$ for $i \in \mathbb{Z}_v$. If a block orbit has v blocks, then the block orbit is said to be full; otherwise it's said to be short. An arbitrary block from a block orbit is called a base block. A base block is also referred to as a starter block or an initial block. The block orbit that contains the block $\left\{0, \frac{v}{2}, \frac{2v}{2}, \cdots, \frac{(k-1)v}{2}\right\}$ is called a regular short orbit.

Theorem 2.1.1 *[Anderson, [1]] For all $BIBD\,(v, k, \lambda)$ with b blocks, each element occurs in r blocks where*

1. $r\,(k-1) = \lambda\,(v-1)$,

2. $bk = vr$.

Theorem 2.1.2 *[Anderson, [1]] The necessary conditions for a $BIBD\,(v, k, \lambda)$ to exist are $\lambda\,(v-1) \equiv 0 \pmod{(k-1)}$ and $\lambda v\,(v-1) \equiv 0 \pmod{k\,(k-1)}$.*

2.2 Difference Sets

Definition 2.2.1 *Suppose $(G, +)$ is a finite group of order v in which the identity element is denoted 0. Unless explicitly stated, we will not require that G be an abelian group. (In many examples, however, we will take $G = (\mathbb{Z}_v, +)$, the integers modulo* *v.) Let k and λ be positive integers such that $2 \leq k < v$. A $(v, k, \lambda)-$difference set in $(G, +)$ is a subset $D \subseteq G$ that satisfies the following properties:*

1. *$|D| = k$,*

2. *the multiset $\{x - y : x, y \in D, x \neq y\}$ contains every element in $G - \{0\}$ exactly λ times.*

Note that $\lambda (v - 1) = k (k - 1)$ if a (v, k, λ)-difference set exists.

Example 2.2.1 *A $(21, 5, 1)$-difference set in $(\mathbb{Z}_{21}, +)$: $DS = \{0, 1, 6, 8, 18\}$. If we compute the differences (modulo 21) we get from pairs of distinct elements in DS, we obtain the following:*

$1 - 0 = 1, 0 - 1 = 20$

$6 - 0 = 6, 0 - 6 = 15$

$8 - 0 = 8, 0 - 8 = 13$

$18 - 0 = 18, 0 - 18 = 3$

$6 - 1 = 5, 1 - 6 = 16$

$8 - 1 = 7, 1 - 8 = 14$

$18 - 1 = 17, 1 - 18 = 4$

$8 - 6 = 2, 6 - 8 = 19$

$18 - 6 = 12, 6 - 18 = 9$

$18-8=10, 8-18=11$.

So we get every element of $(\mathbb{Z}_{21}-\{0\})$ *exactly once as a difference of two elements in* DS.

Difference sets can be used to construct symmetric $BIBDs$ as follows: let DS be a (v,k,λ)-difference set in a group $(G,+)$. For any $g \in G$, define $D+g = \{x+g : x \in D\}$. Any set $D+g$ is called a **translate** of DS. Then, define $Dev(D)$ to be the collection of all v translates of DS. $Dev(D)$ is called the **development** of DS.

Theorem 2.2.1 *[Stinson, [39]] Let* DS *be a* (v,k,λ)*-difference set in an abelian group* $(G,+)$*. Then* $(G, Dev(D))$ *is a symmetric* $(v,k,\lambda)-BIBD$.

Theorem 2.2.2 *[Stinson, [39]] Suppose* DS *is a* (v,k,λ)*-difference set in an abelian group* $(G,+)$*. Then* $Dev(D)$ *consists of* v *distinct blocks.*

Example 2.2.2 *The* $(21,5,1)-BIBD$ *developed from the difference set of Example 2.2.1 has* 21 *distinct blocks:* $\{0,1,6,8,18\}, \{1,2,7,9,19\}, \ldots, \{20,0,5,7,17\}$.

Definition 2.2.2 *Let* $D_1, D_2, \cdots, D_t$ *be sets of size* k *in an additive abelian group* G *of order* v *such that the differences arising from the* D_i *give each non-zero element of* G *exactly* λ *times. Then* $D_1, D_2, \cdots, D_t$ *are said to form a* (v,k,λ)**-difference system** *in* G.

Example 2.2.3 $\{1, 2, 5\}$ *and* $\{1, 3, 9\}$ *form a* $(13, 3, 1)$ *difference system in* $\mathbb{Z}_{13}$.

2.3 Cyclic Block Designs

Definition 2.3.1 *A* ***Steiner Triple System*** *of order* v*, denoted* $STS\,(v)$*, is a pair* $(V, \mathcal{B})$ *where* V *is a set of* v *elements, and* B *is a family of* 3*-subsets from* v *such that every pair in* V *is in exactly one of these subsets. This corresponds to a* $BIBD\,(v, 3, 1)$.

Theorem 2.3.1 *[Kirkman, [22]] Steiner triple systems of order* v *exist for* $v = 1, 3\,(mod\ 6)$.

These two cases of $v = 1, 3\,(mod\ 6)$ are split into two further sub-cases with respect to values $(mod\ 4)$.

Definition 2.3.2 *A* $BIBD$ *is called a* ***Symmetric*** $BIBD$ *(or Symmetric Design) when* $b = v$ *and therefore* $r = k$*, where* r *is the number of blocks in which an elements appears.*

Definition 2.3.3 *A* $BIBD(v, 3, \lambda)$ *is* ***cyclic*** *if its automorphism group contains a* v*-cycle.*

Remark 2.3.1 *If* $\lambda = 1$*, an* $S_\lambda(2, k, v)$ *is called a* ***Steiner 2-design*** *and is denoted by* $S(2, k, v)$*. An* $S(2, k, v)$ *with* $k = 3$ *is a* ***Steiner triple system*** *of order* v*,* $STS(v)$*.*

Theorem 2.3.2 *[Stinson, [39]] The existence of a* (v, k, λ) *difference system in* $\mathbb{Z}_v$ *implies the existence of a cyclic* $(v, k, \lambda) - BIBD$ *in* $\mathbb{Z}_v$*.*

Proof Let the set in the difference system be $D_i = \{d_{i1}, d_{i2}, ..., d_{ik}\}, 1 \leq i \leq t$. It will be shown that the blocks $D_i, D_{i+1}, ..., D_{i+(v-1)}$ form a cyclic $(v, k, \lambda) - BIBD$. It's easy to see that all such sets are of size k, with elements chosen from the v elements in $\mathbb{Z}_v$. Let $a, b \in \mathrm{Z}_v$ *with* $\mathrm{a} \neq b$. Since $a = d_{ij} + (a - d_{ij})$ for any $1 \leq i \leq t, 1 \leq j \leq k$, $a \in D_i + (a - d_{ij})$. Similarly, $b \in D_i + (b - d_{ij})$ for any $1 \leq i \leq t, 1 \leq j \leq k$. So a and b are both elements of a translates $D_i + c$ exactly if $c = a - d_{ij} = b - d_{ih}$ for some $1 \leq j, h \leq k$. But $a - d_{ij} = b - d_{ih}$ iff $a - b = d_{ij} - d_{ih}$.

Since the $D_i, 1 \leq i \leq t$, form a (v, k, λ)-difference system, this occurs exactly λ times. Thus, a and b must occur together in exactly λ blocks. Thus, the translate form a $(v, k, \lambda) - BIBD$. ■

2.4 Packing and Covering designs

It's natural to ask what designs we could obtain if the necessary conditions for the existence of a $BIBD$ are not satisfied. We now define packing and covering designs.

Definition 2.4.1 *A* **Packing Design***, or a* $PD\,(v,k,\lambda)$*, is a family of* k*-subsets, called blocks, of a* v*-set* S*, such that every* 2*-subset, called a pair, of* S *is contained in at most* λ *blocks. The edges in the multigraph* λK_v *not contained in the packing form the* ***leave*** *of the* $PD(v,k,\lambda)$*, denoted by leave* (v,k,λ)*.*

Example 2.4.1 *For a* $PD\,(6,3,1)$*, the blocks are as follows:*

$\{1,5,6\},\{2,4,6\},\{1,3,4\},\{2,5,3\}$*, and the leaves are* $\{1,2\},\{3,6\},\{4,5\}$*.*

Definition 2.4.2 *A* **Covering Design***, or a* $CD\,(v,k,\lambda)$*, is a family of* k*-subsets, called blocks, of a* v*-set* S*, such that every* 2*-subset, called a pair, of* S *is contained in at least* λ *blocks.*

The extra edges added to the multigraph λK_v *in the covering form the excess of the* $CD(v,k,\lambda)$*, denoted by* ***excess*** (v,k,λ)*.*

Example 2.4.2 *A* $CD\,(6,3,1)$*: the blocks are* $\{1,2,4\},\{2,3,5\},\{3,4,6\},\{5,6,2\},$ $\{1,3,6\},\{1,4,5\}$*, and the excess is* $\{1,4\},\{2,5\},\{3,6\}$*.*

Schönheim [30, 31] gave an upper bound and lower bound for the number of blocks and edges in optimal packing and covering designs.

Theorem 2.4.1 *[Schönheim, [30]] The upper bound of the number blocks in a $PD(v,k,\lambda)$ is $\gamma(v,k,\lambda) = \left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{(k-1)} \right\rfloor \right\rfloor$.*

Proof A vertex x in the complete multigraph λK_v has degree $\lambda(v-1)$, and each block containing x takes $(k-1)$ pairs, so x can be contained in no more than $\left\lfloor \frac{\lambda(v-1)}{(k-1)} \right\rfloor$ blocks. Since there are v vertices, then the total number of appearance of the vertices in a $PD(v,k,\lambda)$ is no more than $v\left\lfloor \frac{\lambda(v-1)}{(k-1)} \right\rfloor$.

Finally, each block has k vertices, so the number of blocks is no more than $\gamma(v,k,\lambda) = \left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{(k-1)} \right\rfloor \right\rfloor$. ■

Example 2.4.3 *Suppose $v = 10$, $\lambda = 1$ and $k = 3$. If we apply Theorem 2.4.1 we obtain: $\gamma(v,k,\lambda) = \left\lfloor \frac{10}{3} \left\lfloor \frac{(9)}{(2)} \right\rfloor \right\rfloor$ so, $\gamma(v,k,\lambda) = 13$.*

Theorem 2.4.2 *[Schönheim, [31]] The lower bound of the number of blocks in a $CD(v,k,\lambda)$ is $\delta(v,k,\lambda) = \left\lceil \frac{v}{k} \left\lceil \frac{\lambda(v-1)}{(k-1)} \right\rceil \right\rceil$.*

Proof A vertex x in the complete multigraph λK_v has degree $\lambda(v-1)$, and each block containing x takes $(k-1)$ pairs, so x can be contained in no less than $\left\lceil \frac{\lambda(v-1)}{(k-1)} \right\rceil$ blocks. Since there are v vertices, then the total number of appearance of the vertices in a $CD(v,k,\lambda)$ is no less than $v\left\lceil \frac{\lambda(v-1)}{(k-1)} \right\rceil$.

Finally, each block has k vertices, so the number of blocks is no less than $\delta(v,k,\lambda) = \left\lceil \frac{v}{k} \left\lceil \frac{\lambda(v-1)}{(k-1)} \right\rceil \right\rceil$. ■

In Chapter 3, we will give the definition and necessary conditions for cyclic packing designs.

2.5 Skolem Type Sequences

In 1957, Thoralf Skolem studied various types of combinatorial designs. The most notable of his results came in the form of what are now known as Skolem sequences. Skolem sequences were first studied for use in constructing cyclic Steiner triple systems. Later, these sequences were generalized in many ways and are applied in several areas such as: triple systems [14], factorizations of complete graphs [25], balanced ternary designs [4, 5], and design of statistical models, such as a balanced sampling plan excluding contiguous units [40]. Some other papers in which these sequences have been very useful are [2, 7, 8, 10, 11].

Some special Skolem-type sequences are described below. In this section, we use the definitions from the Handbook of Combinatorial Designs [34], although equivalent definitions can be found in the literature (see for example [10]).

Definition 2.5.1 *A **Skolem sequence** of order n is a sequence $S = (s_1, s_2, ..., s_{2n})$ of $2n$ integers satisfying the conditions:*

1. *For every $k \in 1, 2, ..., n$, there exist exactly two elements s_i, s_j such that $s_i =$*

$s_j = k$.

2. *If* $s_i = s_j = k$, *with* $i < j$, *then* $j - i = k$.

Example 2.5.1 $(4,1,1,5,4,7,8,3,5,6,3,2,7,2,8,6)$ *is a Skolem sequence of order* 8.

Theorem 2.5.1 *[Skolem, [36]] A Skolem sequence of order* n *exists if and only if* $n \equiv 0, 1\,(mod\ 4)$.

Definition 2.5.2 *A **hooked Skolem sequence** of order* n *is a sequence* $S = (s_1, s_2, ..., s_{2n+1})$ *of* $2n+1$ *integers satisfying the conditions:*

1. *For every* $k \in 1, 2, ..., n$, *there exist exactly two elements* s_i, s_j *such that* $s_i = s_j = k$.

2. *If* $s_i = s_j = k$, *with* $i < j$, *then* $j - i = k$.

3. *If* $s_{2n} = 0$.

Example 2.5.2 $(9,7,1,1,3,5,10,3,7,9,5,11,8,6,4,2,10,2,4,6,8,0,11)$ *is a hooked Skolem sequence of order* 11.

Theorem 2.5.2 *[Shalaby, [34]] A hooked Skolem sequence of order* n *exists if and only if* $n \equiv 2, 3\,(mod\ 4)$.

Definition 2.5.3 ***A Rosa sequence*** *of order n is a sequence*

$R = (r_1, ..., r_n, 0, r_{n+2}, ..., r_{2n+1})$ *of $2n+1$ integers satisfying the conditions:*

1. *For every $k \in 1, 2, ..., n$, there exist exactly two elements s_i, s_j such that $r_i = r_j = k$*

2. *If $r_i = r_j = k$, with $i < j$, then $j - i = k$.*

3. *If $r_{n+1} = 0$.*

Example 2.5.3 $(6, 4, 2, 7, 2, 4, 6, 0, 3, 5, 7, 3, 1, 1, 5)$ *is a Rosa sequence of order 7.*

Theorem 2.5.3 *[Shalaby, [34]] A Rosa sequence of order n exists if and only if $n \equiv 0, 3\,(mod\ 4)$.*

Definition 2.5.4 *A* ***hooked Rosa sequence*** *of order n is a sequence*

$HR = (r_1, ..., r_n, 0, r_{n+2}, ..., r_{2n}, 0, r_{2n+2})$ *of $2n+2$ integers satisfying the conditions:*

1. *For every $k \in 1, 2, ..., n$, there exist exactly two elements r_i, r_j such that $r_i = r_j = k$*

2. *If $s_i = s_j = k$, with $i < j$, then $j - i = k$.*

3. *If $s_{n+1} = s_{2n+1} = 0$.*

Example 2.5.4 $(9,7,5,3,10,8,3,5,7,9,0,4,6,8,10,4,1,1,6,2,0,2)$ *is a hooked Rosa sequence of order* 10.

Theorem 2.5.4 *[Shalaby, [34]] A hooked Rosa sequence of order* n *exists if and only if* $n \equiv 1, 2\,(mod\ 4)$.

Theorem 2.5.5 *[Skolem, [37]]The existence of a Skolem sequence of order* n *implies the existence of a cyclic* $STS\,(6n+1)$.

Proof Suppose there exists a Skolem sequence of order n. Consider the pairs (a_i, b_i), with $a_i < b_i$, $1 \leq i \leq n$. The sets of differences $(mod\ 6n+1)$ of elements in the blocks $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, are:

$A = \{\pm[(b_i + n) - (a_i + n)] : 1 \leq i \leq n\} = \{\pm(b_i - a_i) : 1 \leq i \leq n\}$

$B = \{\pm[(a_i + n) - 0] : 1 \leq i \leq n\} = \{\pm(a_i + n)\}$

$C = \{\pm[(b_i + n) - 0] : 1 \leq i \leq n\} = \{\pm(b_i + n)\}$.

Now, the set A consists of all differences of the form $\pm(b_i - a_i)$. But by the definition of a Skolem sequence, the differences $b_i - a_i$, for $1 \leq i \leq n$, are exactly $1, 2, ..., n$. So $A = \{\pm 1, \pm 2, ..., \pm n\}$.

Now, together a_i and b_i, $1 \leq i \leq n$, consist of all elements $1, 2, ..., 2n$. Therefore, $a_i + n$ and $b_i + n$, for $1 \leq i \leq n$, are exactly the elements $n+1, n+2, ..., 3n$. So $B \cup C = \{\pm(n+1), \pm(n+2), ..., \pm(3n)\}$.

Therefore, $A \cup B \cup C = \{\pm 1, \pm 2, ..., \pm 3n\} = \mathbb{Z}_{6n+1} - \{0\}$, that is, all the nonzero elements of $\mathbb{Z}_{6n+1}$. Further, each of these elements occurs exactly once.

Thus the sets $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, form a $(6n+1, 3, 1)$ difference system, and by Theorem 1.12, there is a $(6n+1, 3, 1) - BIBD$, that is, an $STS(6n+1)$ formed by the translates $\{x, x + a_i + n, x + bi + n\}$, where $x \in \mathbb{Z}_{6n+1}$. ■

The proof of Theorem 2.5.6 is similar to the proof of the previous theorem and its omitted.

Theorem 2.5.6 *[Skolem, [37]]The existence of a hooked Skolem sequence of order* n *implies the existence of a cyclic* $STS\,(6n+1)$.

Theorem 2.5.7 *[Rosa, [29]]The existence of a Rosa sequences of order* n *implies the existence of a cyclic* $STS\,(6n+3)$.

Proof Suppose there exists a Rosa sequence of order n. Consider the pairs (a_i, b_i), with $a_i < b_i$, $1 \leq i \leq n$. The sets of differences $(mod\ 6n+3)$ of elements in the blocks $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, are:

$A = \{\pm [(b_i + n) - (a_i + n)] : 1 \leq i \leq n\} = \{\pm (b_i - a_i) : 1 \leq i \leq n\}$

$B = \{\pm [(a_i + n) - 0] : 1 \leq i \leq n\} = \{\pm (a_i + n)\}$

$C = \{\pm [(b_i + n) - 0] : 1 \leq i \leq n\} = \{\pm (b_i + n)\}$.

By the definition of a Rosa sequence, A consists of exactly the numbers

$\pm\ 1, \pm\ 2, ..., \pm\ n$. Since a_i and b_i make up the numbers $1, 2, ..., n, n+2, n+3, ..., 2n$, together B and C consist of exactly the numbers $\pm(n+1)$, $\pm(n+2)$, $..., \pm(2n), \pm(2n+2), ..., \pm(3n)$. Thus, all elements of $\mathbb{Z}_{6n+3} - \{0\}$ occur as differences exactly once, except for $2n+1$ and its inverse $4n+2$. So, the translates of these blocks give all pairs of elements except for those which differ by $2n+1$. Adding the short orbit base block $\{0, 2n+1, 4n+2\}$ with its translates $\{j, (2n+1)+j, (4n+2)+j\}$, where $0 \leq j \leq 2n$, gives the remaining pairs. ■

The proof of Theorem 2.5.8 is similar to the proof of the previous theorem and its omitted.

Theorem 2.5.8 *[Rosa, [29]]The existence of a hooked Rosa sequence of order n implies the existence of a cyclic $STS(6n+3)$.*

Definition 2.5.5 *An m-**near-Skolem sequence** of order n and defect m is a sequence $m - S_n = (s_1, s_2, \ldots, s_{2n-2})$ of $2n-2$ non-negative integers such that the following conditions hold:*

1. *for each $k \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$,*

2. *if $s_i = s_j = k$, then $|i - j| = k$.*

Example 2.5.5 $(1, 1, 6, 3, 7, 5, 3, 2, 6, 2, 5, 7)$ *is a near Skolem sequence of order* 7 *and defect* 4.

Theorem 2.5.9 *[Shalaby, [34]]An m-near Skolem sequence of order* n *exists if and only if either:*

1. $n \equiv 0, 1\,(mod\ 4)$ *when* m *is odd, or*

2. $n \equiv 2, 3\,(mod\ 4)$ *when* m *is even.*

Definition 2.5.6 *A **hooked near-Skolem sequence** of order* n *and defect* m *is a sequence* $m - near\, hS_n = (s_1, s_2, \ldots, s_{2n-1})$ *of integers* $s_i \in \{1, 2, \ldots, m-1, m+1, \ldots, n\}$ *satisfying conditions* (1) *and* (2) *above, as well as* $s_{2n-2} = 0$.

Example 2.5.6 $(2, 5, 2, 4, 6, 7, 5, 4, 1, 1, 6, 0, 7)$ *is a hooked near Skolem sequence of order* 7 *and defect* 3.

Theorem 2.5.10 *[Shalaby, [34]] A hooked m-near Skolem sequence of order* n *exists if and only if either:*

1. $n \equiv 2, 3\,(mod\ 4)$ *when* m *is odd, or*

2. $n \equiv 0, 1\,(mod\ 4)$ *when* m *is even.*

2.6 Circulant Graphs

In this section, we present the definition of and some basic properties related to circulant graphs.

Definition 2.6.1 *A cycle that travels exactly once over each vertex in a graph is called* ***Hamiltonian****. A graph containing a Hamiltonian cycle is called a* ***Hamiltonian graph****.*

Assume $\{ab\}$ to be any edge of a graph G with $V(G) \in \mathbb{Z}_v$. We shall use $\pm|a-b|$ to denote the *difference* of the edge $\{a,b\}$ in G. The number of distinct differences in a graph G defined on $\mathbb{Z}_v$ is called the *weight* of G, denoted $W(G)$. Let $C = (c_0, c_1, ..., c_{m-1})$ be an $m-$cycle of G and let $C + i = (c_0 + i, c_1 + i, ..., c_{m-1} + i)$ (mod v), where $i \in \mathbb{Z}_v$. A **cycle orbit** O of C is a collection of distinct m-cycles in $\{C + i|\ i \in \mathbb{Z}_v\}$. The length of a cycle orbit is its cardinality, the minimum positive integer k such that $C + k = C$. A **base cycle** of a cycle orbit O is a cycle $C \in O$ that is chosen arbitrarily. A cycle k-orbit is a cycle orbit of length k. A cycle v-orbit of C on G is said to be **full** and otherwise **short**.

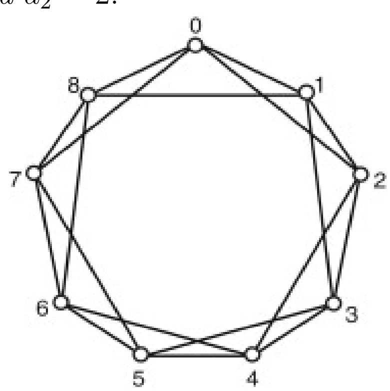
Given a subset S of $\mathbb{Z}_v - \{0\}$ with $S = -S$, the *circulant graph* $C(\mathbb{Z}_v, S)$ of order v is the Cayley graph Cay$[\mathbb{Z}_v, S]$, that is, the graph with vertex set $\mathbb{Z}_v$ and all possible edges of the form $\{a, a+w\}$, with $w \in S$. The set S is called the *connection set* and its size is the degree of $C_v(\mathbb{Z}_v, S)$.

In other words a circulant graph of order v with differences $\pm d_1, \pm d_2, \cdots, \pm d_t$, $C_v\langle d_1, d_2, \cdots, d_t\rangle$, is a graph with vertex set $\mathbb{Z}_v$ and edge set

$\{\{a, (a+d_i) \pmod{v}\} \mid a \in \mathbb{Z}_v, 1 \leq i \leq t\}$.

A graph G is circulant if and only if the automorphism group of G contains at least one permutation consisting of a minimal cycle of length $|V(G)|$. Clearly, a circulant graph of order v with a single difference d is a 2-factor when $d \neq \frac{v}{2}$ and is a 1-factor when $d = \frac{v}{2}$.

Example 2.6.1 *The following graph is a circulant graph on 9 vertices $C_9 < 1, 2 >$, with $d_1 = 1$ and $d_2 = 2$.*



Circulant graph $C_9 < 1, 2 >$

Lemma 2.6.1 is essential for the next chapter.

Lemma 2.6.1 *[H. Fu and S. Wu, [41]] Suppose* $S = \pm\{d\}$ *with* $d \in \mathbb{Z}_{\lfloor \frac{v}{2} \rfloor}$ *and let* $k = \frac{v}{gcd(v,d)}$. *Then the circulant graph* $C(\mathbb{Z}_v, S)$ *is the union of* $\frac{v}{k}$ *edge-disjoint k-cycles. If* $gcd(v,d) = 1$, *then* $C(\mathbb{Z}_v, S)$ *is exactly a Hamiltonian cycle in* k_v *and, if* $d = \frac{v}{m}$, *then* $C(\mathbb{Z}_v, S)$ *is the union of d edge-disjoint m-cycles.*

In this thesis, the symbols F, nH, and C denote, respectively, a 1-factor, n Hamiltonian cycles, and the union of $gcd(v,d)$ cycles of length $\frac{v}{gcd(v,d)}$.

Chapter 3

Cyclic Packing Designs with block size 3 from Skolem-type sequences

In this chapter, we use Skolem-type sequences to construct cyclic packing designs with optimal leaves, for all admissible λ.

3.1 Cyclic Packing Designs and optical orthogonal codes

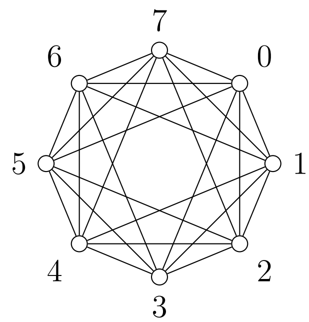
In this section, we give the definition and some fundamental properties of cyclic packing designs and optical orthogonal codes and we find the necessary conditions for cyclic packing designs.

Definition 3.1.1 *A **Packing Design**, or a $PD(v,k,\lambda)$, is a pair $(V,\mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a set of k-subsets* (blocks) *such that any 2-subset of V appears in at most λ blocks. A $PD(v,k,\lambda)$ is cyclic if its automorphism group contains a v-cycle; this is called a **cyclic packing design**,CPD. The edges in the multigraph λK_v are not contained in the packing form the leaves of the $CPD(v,k,\lambda)$, denoted by leave (v,k,λ).*

Definition 3.1.2 *The **leave** of the graph is a graph whose edges are the unordered pairs not appearing in a block of the system.*

For a cyclic (v,k,λ) packing design $(V,\mathcal{B})$ the point set V can be identified with $\mathbb{Z}_v$, the residue ring of integers modulo v. In this case, the packing design has an automorphism $w : i \to i+1 \pmod v$. For a cyclic packing design $CPD(v,k,\lambda)$, let $V = \mathbb{Z}_v$, $(\mathbb{Z}_v,\mathcal{B})$, let $B = \{b_1, b_2, \cdots, b_t\}$ be a block in $\mathcal{B}$. The block orbit containing B is defined to be the set of the following distinct blocks: $B + i = \{b_1 + i, \cdots, b_t + i\} \pmod v$, for $i \in \mathbb{Z}_v$. If a block orbit has v distinct blocks, then the block orbits is said to be **full**; otherwise it is said to be **short**. An arbitrary block from a block orbit is called a **base block**. A $CPD(v,k,\lambda)$ is uniquely determined by its base blocks. Given an arbitrary set of base blocks of a $CPD(v,k,\lambda)$, one can obtain the packing by applying the cycle to each base block.

Example 3.1.1 *For a* $CPD\,(8,3,1)$ *, the blocks are* $\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},$ $\{4,5,7\},\{5,6,0\},\{6,7,1\},\{7,0,2\}$ *. The leaves are* $\{\{0,4\},\{1,5\},\{2,6\},\{3,7\}\}$*.*



A very useful method of viewing a $CPD\,(v,k,\lambda)$ is using the definition of the difference set. A $CPD\,(v,k,\lambda)$ can be defined equivalently as a set of subsets of $\mathbb{Z}_v$, $\mathcal{B}$, each of cardinality from k, such that the list of the differences that emerge from $\mathcal{B}$, $\Delta\mathcal{B} =$

$\{a-b : a,b \in B, a \neq b, B \in \mathcal{B}\}$, cover each nonzero residue of integers modulo v at most the value of λ times. The pair $(\mathbb{Z}_v,\mathcal{B})$ is called a $CPD\,(v,k,\lambda)$ if the cardinality of $\Delta\mathcal{B}$ is exactly $\lambda k\,(k-1)\,t$ and $0 \notin \Delta\mathcal{B}$. A $CPD\,(v,k,1)$ is termed $g-$regular if the difference leave $(\mathbb{Z}_v - \Delta\mathcal{B})$ along with 0 forms an additive subgroup of $\mathbb{Z}_v$ having order g.

In [42], Yin discussed the number of base blocks in a $CPD\,(v,k,1)$ and its upper bounded by $\lfloor \frac{v-1}{k(k-1)} \rfloor$. Optical orthogonal codes can be constructed by different methods, however in the case of $\lambda = 1$, the problem of constructing optical orthogonal codes is equivalent to the problem of cyclic packing designs. Closely related to an optimal $CPD\,(v, k, 1)$ is an optimal $(v, k, 1)$ *optical orthogonal code,* for which we give the following definition:

Definition 3.1.3 *A* (v, k, λ) ***optical orthogonal code****,* $(v, k, \lambda)-OOC$ *in short, is a family of* $(0, 1)$*-sequences, called codewords, of length* v *and hamming weight* k *satisfying the following two properties:*

1. *The Auto-Correlation Property:* $\sum_{i=0}^{v-1} x_i x_{i+t} \leq \lambda$ *for any* $x \in C$ *and any integer* $t \neq 0\,(mod\ v)$*, where* $x = (x_0, x_1, \cdots, x_{v-1})$.

2. *The Cross-Correlation Property:* $\sum_{i=0}^{v-1} x_i y_{i+t} \leq \lambda$ *for any* $x, y \in C$ *with* $x \neq y$ *and any integer* t*, where* $x = (x_0, x_1, \cdots, x_{v-1})$ *and* $y = (y_0, y_1, \cdots, y_{v-1})$.

Here, all subscripts are reduced modulo v. The *Hamming weight* of a codeword is the number of nonzero entries in the word.

For example, $C = \{1100100000000, 1010000100000\}$ is a $(13, 3, 1)$ code with two codewords. In set theoretic notation, $C = \{\{0, 2, 7\}, \{0, 1, 4\}\} \pmod{13}$. A survey of cyclic designs and their applications to optimal optical orthogonal codes is given in [6].

Brickell and Wei, in [9], proved that the optimal $(v, 3, 1)$ optical orthogonal codes always exist except when $v \equiv \ 14, 20\,(mod\ 24)$, where an optimal $(v, 3, 1)$ optical orthogonal code OOC is equivalent to a $CPD\,(v, 3, 1)$.

Theorem 3.1.1 *[Yin, [43]] An optimal $(v, k, 1)-OOC$ is equivalent to an optimal $CPD\,(v, k, 1)$.*

According to Definition 3.1.2, given a $(v, k, \lambda)-$OOC, C, if we take every sequence in C and all its cyclic shifts as codewords, we obtain a constant-weight binary error correcting code of length v and weight k, which contains $|C|\,v$ codewords.

Theorem 3.1.2 *[Johnson, [20]](Johnson bound)*
$A\,(v, 2\,(k-\lambda)\,, \lambda) \leq \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \lfloor \frac{v-2}{k-2} \lfloor \cdots \lfloor \frac{v-\lambda}{k-\lambda} \rfloor \cdots \rfloor \rfloor \rfloor \rfloor.$

Thus, by the Johnson bound we obtain $|C| \leq \lfloor \frac{v}{k} \lfloor \frac{v-1}{k-1} \lfloor \frac{v-2}{k-2} \lfloor \cdots \lfloor \frac{v-\lambda}{k-\lambda} \rfloor \cdots \rfloor \rfloor \rfloor \rfloor$; if $\lambda = 1$, this reduces to $|C| \leq \lfloor \frac{v-1}{k(k-1)} \rfloor$. When $|C| = \lfloor \frac{v-1}{k(k-1)} \rfloor$ the code is an optimal optical orthogonal code. A similar bound is related to cyclic packing designs. We will use Johnson bound to derive a general upper bound of the number of base blocks in CPD, $\theta\,(v, k, \lambda)$.

Theorem 3.1.3 *[Chung, Salehi and Wei, [13]] The general upper bound for $\theta\,(v, k, \lambda)$ is $\theta\,(v, k, \lambda) \leq \frac{1}{v} A\,(v, 2\,(k-\lambda)\,, \lambda) \leq \lfloor \frac{1}{k} \lfloor \frac{v-1}{k-1} \lfloor \frac{v-2}{k-2} \lfloor \cdots \lfloor \frac{v-\lambda}{k-\lambda} \rfloor \cdots \rfloor \rfloor \rfloor \rfloor$.*

For $\lambda = 1$ this reduces to $\theta\,(v, k, 1) \leq \lfloor \frac{v-1}{k(k-1)} \rfloor$. Chung and Salehi in [13] state the following theorem:

Theorem 3.1.4 $\theta\,(v,3,1) = \lfloor \frac{v-1}{6} \rfloor$ *if* $v \neq 2 \pmod 6$.

We will use Theorem 3.1.3 for the number of base blocks in Section 3.2.

The number of blocks in a packing design $\gamma\,(v,k,\lambda)$ is bounded by $\left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{(k-1)} \right\rfloor \right\rfloor$; also, we can note that every base block in a (strictly) cyclic packing design generates a full orbit (containing v blocks), so the number of base blocks in a (strictly) cyclic packing $\rho\,(v,k,\lambda)$ is bounded by $\rho\,(v,k,\lambda) = \lfloor \frac{\gamma(v,k,\lambda)}{v} \rfloor \leq \lfloor \frac{\lambda(v-1)}{k(k-1)} \rfloor$, where $\rho\,(v,k,\lambda)$ is the number of base blocks in a cyclic packing design.

Therefore, we may easily conclude that the upper bound on the number of base blocks in a cyclic packing design for all admissible λ and $k = 3$ is given by the following corollary:

Corollary 3.1.5 *The upper bound of the number of base blocks in a cyclic packing design is* $\rho\,(v,3,\lambda) \leq \lfloor \frac{\lambda(v-1)}{6} \rfloor$.

For example, if $v = 26$, $k = 3$ and $\lambda = 2$, then the number of base blocks equals 8.

3.1.1 Necessary conditions for the existence of Cyclic Packing Designs

In this section, we will discuss the necessary conditions for the existence of the minimum leave of a cyclic packing design, for all admissible λ.

In 1992, Colbourn, Hoffman and Rees [15], used Skolem-type sequences to construct cyclic packing designs for packing Steiner triple systems of order v and their results are included in the following theorem:

Theorem 3.1.6 *[C. Colbourn, Hoffman and Rees, [15]] There is a cyclic packing* $S(2,3,v)$ *whose leave is* $l-$*regular whenever*

1. $v = 6n+1$ *and* $l \equiv 0 \pmod 6$;
2. $v = 6n+2$ *and* $l \equiv 1 \pmod 6$, *and if* $n \equiv 2,3 \pmod 4$, *then* $l \geq 7$;
3. $v = 6n+3$ *and* $l \equiv 0,2 \pmod 6$ *except when* $v = 9$ *and* $l = 0$;
4. $v = 6n+4$ *and* $l \equiv 3 \pmod 6$;
5. $v = 6n+5$ *and* $l \equiv 4 \pmod 6$;
6. $v = 6n+6$ *and* $l \equiv 5 \pmod 6$.

Proof [15] For case (3) with $l \equiv 0 \pmod 6$, form a cyclic $BIBD(v,3,1)\,(v \neq 9)$. Delete $\frac{l}{6}$ of the orbits of v blocks (leaving the orbit of $\frac{v}{3}$ blocks present).

In the remaining cases, take (a_i, b_i) from a Skolem sequence or hooked Skolem sequence of order $n - \frac{l}{6}$. Then form starter blocks of an $S(2,3,v)$ by taking $\{0, a_i+n, b_i+n\} \pmod v$, for $i = 1, 2, \cdots, n - \frac{l}{6}$. ■

We use Skolem type sequences to construct a cyclic packing design of order v and $\lambda = 2$ as seen in the following theorem:

Theorem 3.1.7 *For $\lambda = 2 \pmod{12}$, there is a cyclic packing design $CPD\,(v, 3, 2)$ whose leave is $l-$regular whenever*

1. $v = 3n + 2$ *and* $l \equiv 2 \pmod 6$, *for* $v \equiv 2, 5, 8, 11 \pmod{12}$,

2. $v = 3n + 3$ *and* $l \equiv 4 \pmod 6$, *for* $v \equiv 6 \pmod{12}$,

3. $v = 3n + 4$ *and* $l \equiv 6 \pmod 6$, *for* $v \equiv 10 \pmod{12}$.

Proof From a Skolem sequence or a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD\,(v, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$. Also, $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD\,(v, 3, 2)$.■

Remark 3.1.1 *If $l \geq m$, then the number of base blocks equals $\rho\,(v, 3, \lambda) - 1$, where* $m \equiv 6 \pmod 6$.

As seen in the previous results in Theorem 3.1.5 and Theorem 3.1.6, cyclic packing designs for higher λ can be constructed by combining cyclic packing designs for $\lambda = 1$ and $\lambda = 2$.

Theorem 3.1.8 *The necessary conditions for the existence of the minimum leave of a cyclic packing design $CPD\,(v, 3, \lambda)$ are:*

1. *If* $\lambda = 3 \pmod{12}$ *and* $v \equiv 0, 2, 4, 6, 8, 10 \pmod{12}$,

2. *If* $\lambda = 4 \pmod{12}$ *and* $v \equiv 2, 5, 8, 11 \pmod{12}$,

3. *If* $\lambda = 5 \pmod{12}$ *and* $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$,

4. *If* $\lambda = 6 \pmod{12}$ *and* $v \equiv 2, 6, 10 \pmod{12}$,

5. *If* $\lambda = 7 \pmod{12}$ *and* $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$,

6. *If* $\lambda = 8 \pmod{12}$ *and* $v \equiv 2, 5, 8, 11 \pmod{12}$,

7. *If* $\lambda = 9 \pmod{12}$ *and* $v \equiv 0, 2, 4, 6, 8, 10 \pmod{12}$,

8. *If* $\lambda = 10 \pmod{12}$ *and* $v \equiv 2, 5, 6, 8, 10, 11 \pmod{12}$,

9. *If* $\lambda = 11 \pmod{12}$ *and* $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$.

Proof 1. If we combine the base blocks of a $CPD\,(v, 3, 1)$ and a $CPD\,(v, 3, 2)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 3)$, the type of leave for $\lambda = 3$ in Table 3.1;

2. if we combine the base blocks of a $CPD\,(v, 3, 2)$ two times we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 4)$, the type of leave for $\lambda = 4$ in Table 3.1;

3. if we combine the base blocks of a $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 3)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 5)$, the type of leave for $\lambda = 5$ in Table 3.1;

4. if we combine the base blocks of a $CPD\,(v, 3, 2)$ three times we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 6)$, the type of leave for $\lambda = 6$ in Table 3.1;

5. if we combine the base blocks of a $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 5)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 7)$, the type of leave for $\lambda = 7$ in Table 3.1;

6. if we combine the base blocks of a $CPD\,(v, 3, 3)$ and a $CPD\,(v, 3, 5)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 8)$, the type of leave for $\lambda = 8$ in Table 3.1;

7. if we combine the base blocks of a $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 7)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 9)$, the type of leave for $\lambda = 9$ in Table 3.1;

8. if we combine the base blocks of a $CPD\,(v, 3, 5)$ two times we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 10)$, the type of leave for $\lambda = 10$ in Table 3.1;

9. if we combine the base blocks of a $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 9)$ we get the minimum number of base blocks and the required leave for a $CPD\,(v, 3, 11)$, the type of leave for $\lambda = 11$ in Table 3.1.■

$v(mod\ 24)/\lambda(\text{mod } 12)$	0	1	2	3	4	5	6	7	8	9	10	11
2	-	F	H	$H \cup F$	$2H$	$2H \cup F$	$3H$	$3H \cup F$	C	$C \cup F$	$C \cup H$	$H \cup C \cup F$
8	-	F	C	$C \cup F$	C	$C \cup F$	-	F	C	$C \cup F$	C	$C \cup F$
14	-	$F \cup 2C \cup H$	H	$C \cup F$	$2H$	$F \cup C \cup H$	$3H$	$C \cup 2H \cup F$	C	$3H \cup C \cup F$	$C \cup 3H$	$4H \cup C \cup F$
20	-	$F \cup 2C \cup H$	C	$H \cup F$	C	$F \cup C \cup H$	-	$F \cup C \cup H$	$2H \cup C$	$H \cup F$	C	$H \cup C \cup F$
0	-	$F \cup C \cup H$	+	$C \cup F$	+	F	-	$C \cup F$	+	$3F$	+	F
6	-	$F \cup C \cup H$	$H \cup C$	$C \cup F$	+	$H \cup C \cup F$	$H \cup C$	$C \cup F$	+	$3F$	$2F$	F
12	-	$F \cup C \cup H$	+	$C \cup F$	+	$H \cup C \cup F$	-	$C \cup H \cup F$	+	$H \cup F$	+	$2H \cup C \cup F$
18	-	$F \cup C \cup H$	$H \cup C$	$H \cup F$	+	F	$2H \cup 2F$	$C \cup H \cup F$	+	$C \cup F$	$2F$	$H \cup C \cup F$
4	-	$F \cup H$	-	$H \cup F$	-	$H \cup F$	-	$F \cup H$	-	$H \cup F$	-	$H \cup F$
10	-	$F \cup C$	$H \cup 2C$	$H \cup F$	-	$C \cup F$	$2H \cup 2F$	$F \cup 3H \cup C$	-	$2H \cup C \cup F$	$C \cup 2F$	$H \cup C \cup F$
16	-	$F \cup C$	-	$C \cup F$	-	$C \cup F$	-	$F \cup C$	-	$C \cup F$	-	$C \cup F$
22	-	$F \cup H$	$H \cup 2C$	$C \cup F$	-	$H \cup F$	$C \cup 2F$	$F \cup C$	-	$H \cup F$	$2H \cup 2F$	$C \cup F$
5	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
11	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
17	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
23	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
1	-	-	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-	-	-
13	-	-	-	-	-	-	-	-	-	-	-	-
19	-	-	-	-	-	-	-	-	-	-	-	-
3	-	+	+	-	+	+	-	+	+	-	+	+
9	-	+	+	-	+	+	-	+	+	-	+	+
15	-	+	+	-	+	+	-	+	+	-	+	+
21	-	+	+	-	+	+	-	+	+	-	+	+

Table 3.1: The necessary conditions for the existence of a cyclic $BIBD(v, 3, \lambda)$ and optimal leaves for all admissible λ of a cyclic packing design.

Example 3.1.2 *For $v = 26$, $\lambda = 3$ and $k = 3$ we will use a Skolem sequence of order 4 as in the construction of a $CPD\,(26, 3, 1)$ and a Skolem sequence of order 8 as in the construction of a $CPD\,(26, 3, 2)$ as follows:*

$S_4 = (1,1,3,4,2,3,2,4)$, *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for a $CPD(26,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$, *and* $\{0,4,12\}(\text{mod } 26)$. *Also we need a Skolem-type sequence of order* 8 *as follows:* $S_8 = (8,6,4,2,7,2,4,6,8,3,5,7,3,1,1,5)$ *yields the pairs:* $\{(14,15),(4,6),(10,13),(3,7),(11,16),(2,8),(5,12),(1,9)\}$. *These pairs yield in turn the triples:* $\{(1,22,23),(2,12,14),(3,18,21),(4,11,15),(5,19,24),(6,10,16),$

(7,13,20),(8,9,17). These triples yield the base blocks for a $CPD(26,3,2)$*:*

$\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$ *and* $\{0,8,17\}$ $(\text{mod } 26)$. *The leave is the union of two circulant graphs* $C_{26}\langle 1\rangle \cup C_{26}\langle 13\rangle$, *which represent the union of one Hamiltonian cycle and a 1-factor.*

Remark 3.1.2 *In the following cases the number of base blocks equals* $\rho(v,k,\lambda) - 1$*:*

1. $v \equiv 14, 20 \pmod{24}$ *and* $\lambda \equiv 1 \pmod{12}$;

2. $v \equiv 10, 22 \pmod{24}$ *and* $\lambda \equiv 2 \pmod{12}$;

3. $v \equiv 6, 12 \pmod{24}$ *and* $\lambda \equiv 5 \pmod{12}$;

4. $v \equiv 2, 6, 10, 14, 18, 22 \pmod{24}$ *and* $\lambda \equiv 6 \pmod{12}$;

5. $v \equiv 2, 20 \pmod{24}$ *and* $\lambda \equiv 7 \pmod{12}$;

6. $v \equiv 10, 22 \pmod{24}$ *and* $\lambda \equiv 10 \pmod{12}$;

7. $v \equiv 12, 18 \pmod{24}$ *and* $\lambda \equiv 11 \pmod{12}$.

In the rest of Chapter 3 we will discuss the constructions and type of leave in detail for every value of λ. Sometimes we will use a different constructions that is equivalent, but slightly different from one used from the necessary conditions to obtain the optimal leaves.

The general template to construct cyclic packing designs from the Skolem-type sequence it will be as follows: from the Skolem-type sequence we can obtain the pairs (a_i, b_i) where $1 \leq i \leq n$, then we can construct the triples $(i, a_i + n, b_i + n)$, these triples yields the base blocks for a cyclic packing design. Compute the number of base blocks by Corollary 3.1.5 and Remark 3.1.1, finally determine the type of leaves.

3.2 Cyclic Packing Designs for $k = 3$ and $\lambda = 1$, $CPD\,(v, 3, 1)$

In this section, we use Skolem-type sequences to construct cyclic packing designs for $k = 3$ and $\lambda = 1$.

3.2.1 Case 1: $v \equiv 2, 8 \pmod{12}$

We will apply Skolem-type sequences for $v \equiv 2, 8, 14 \pmod{24}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yields the base blocks for a cyclic packing design $CPD\,(6n + 2, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD\,(6n + 2, 3, 1)$. If $v \equiv 2, 8 \pmod{24}$, the number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$ and the leave is one circulant graph $C_v \left\langle \frac{v}{2} \right\rangle$, which represents a 1-factor.

Example 3.2.1 *$S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ yields the pairs $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$. These pairs yield in turn the triples $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$. These triples yield the base blocks for a $CPD\,(26, 3, 1)$: $\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\}$, and $\{0, 4, 12\} \pmod{26}$. The leave is one circulant graph $C_{26}\langle 13 \rangle$, which represents a 1-factor.*

We will apply Skolem-type sequences for $v \equiv 14 \pmod{24}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD\,(6n+8,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD\,(6n+8,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor - 1$. The leave is the union of four circulant graphs $C_v\left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 2 \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 3 \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 4 disjoint cycles of length $\frac{v}{2}$.

If $v \equiv 20 \pmod{24}$, then we will apply hooked Skolem-type sequences to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$ for $1 \leq i \leq n$ yield the base blocks for a cyclic packing design $CPD\,(6n+8,3,\lambda)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD\,(6n+8,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor - 1$. The leave is the union of four circulant graphs $C_v\left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 2 \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 4 \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle, and 4 disjoint cycles of length $\frac{v}{2}$.

Example 3.2.2 $hS_2 = (1,1,2,*,2)$ *yields the pairs* $\{(1,2),(3,5)\}$*. These pairs yield in turn the triples* $\{(1,3,4),(2,5,7)\}$*. These triples yield the base blocks for a* $CPD(20,3,1)$*:* $\{0,1,4\}$ *and* $\{0,2,7\}$ (mod 20)*. The leave is the union of four circulant graphs* $C_{20}\langle 10\rangle \cup C_{20}\langle 9\rangle \cup C_{20}\langle 8\rangle \cup C_{20}\langle 6\rangle$*, which represent the union of a 1-factor, one Hamiltonian cycle and 4 disjoint cycles of length* 10*.*

3.2.2 Case 2: $v \equiv 4, 10 \pmod{12}$

If $v \equiv 4, 10 \pmod{24}$, then we will apply Skolem-type sequences. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$ yield the base blocks for a cyclic packing design $CPD(6n+4,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or$\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(6n+4,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$. If $v \equiv 4 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 1 \right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

If $v \equiv 10 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2} - 1 \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.2.3 $S_5 = (5,2,4,2,3,5,4,3,1,1)$ *yields the pairs* $\{(9,10),(2,4),(5,8),(3,7),(1,6)\}$. *These pairs yield in turn the triples* $\{(1,14,15),(2,7,9),(3,10,13),(4,8,12),(5,6,11)\}$. *These* *triples yield the base blocks for a* $CPD\,(34,3,1)$*:*

$\{\{0,1,15\},\{0,2,9\},\{0,3,13\},\{0,4,12\}$ *and* $\{0,5,11\}\}\pmod{34}$.

The leave is the union of two circulant graphs $C_{34}\langle\, 17\rangle \cup C_{34}\langle\, 16\rangle$*, which represent the union of a 1-factor and 2 disjoint cycles of length* 17*.*

If $v \equiv 16, 22 \pmod{24}$, then we will apply hooked Skolem-type sequences to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n+4,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n+4,3,1)$.

The number of base blocks in this case is $\lfloor\frac{(v-1)(1)}{(3)(2)}\rfloor$. If $v \equiv 16 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}-2\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

If $v \equiv 22 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}-2\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

Example 3.2.4 $hS_6 = (1,1,2,5,2,4,6,3,5,4,3,*,6)$

yields the pairs $\{(1,2),(3,5),(8,11),(6,10),(4,9),(7,13)\}$*. These pairs yield in turn the triples* $\{(1,7,8),(2,9,11),(3,14,17),(4,12,16),(5,10,15),(6,13,19)\}$*. These triples yield the base blocks for a* $CPD(40,3,1)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,416\},\{0,5,15\}$ *and* $\{0,6,19\} \pmod{40}$. *The leave is the union of two circulant graphs* $C_{40}\langle 20\rangle \cup C_{40}\langle 18\rangle$, *which represent the union of a 1-factor and 2 disjoint cycles of length* 20.

3.2.3 Case 3: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11 \pmod{24}$, we will apply Skolem-type sequences. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n+5,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n+5,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \lfloor \frac{v}{2} \rfloor \right\rangle \cup \mathrm{C}_v \left\langle \lfloor \frac{v}{2} - 1 \rfloor \right\rangle$, which represent two Hamiltonian cycles.

Example 3.2.5 $S_4 = (1,1,3,4,2,3,2,4)$, *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. *These triples yield the base blocks for a* $CPD(29,3,1)$*:* $\{0,1,6\},\{0,2,11\},\{0,3,10\}$, *and* $\{0,4,12\} \pmod{29}$. *The leave is the union of two circulant graphs* $C_{29}\langle\ 13\rangle \cup C_{29}\langle\ 14\rangle$, *which represent two Hamiltonian cycles.*

If $v \equiv 17, 23 \pmod{24}$ then we will apply hooked Skolem-type sequences to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n+5, 3, 1)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n+5, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \lfloor \frac{v}{2} \rfloor \right\rangle \cup \mathrm{C}_v \left\langle \lfloor \frac{v}{2} - 2 \rfloor \right\rangle$, which represent two Hamiltonian cycles.

Example 3.2.6 $hS_3 = (3, 1, 1, 3, 2, *, 2)$, *yields the pairs* $\{(2,3), (5,7), (2,4)\}$. *These pairs yield in turn the triples:* $(1,5,6), (2,8,10), (3,4,7)$. *These triples yield the base blocks for a* $CPD(23, 3, 1)$*:* $\{0,1,6\}, \{0,2,10\}, \{0,3,7\} \pmod{23}$. *The leave is the union of two circulant graphs* $C_{23}\langle 9 \rangle \cup C_{23}\langle 11 \rangle$*, which represent two Hamiltonian cycles.*

3.2.4 Case 4: $v \equiv 0, 6 \pmod{12}$

For $v \equiv 6, 12 \pmod{24}$, we will apply Skolem-type sequences. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n+6, 3, 1)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n+6, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$. The leave is the union of three circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-2\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.2.7 *$S_4 = (1,1,3,4,2,3,2,4)$, yields the pairs $\{(1,2),(5,7),(3,6),(4,8)\}$. These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. These triples yield the base blocks for a $CPD(30,3,1)$: $\{0,1,6\},\{0,2,11\},\{0,3,10\}$, and $\{0,4,12\}$ (mod 30). The leave is the union of three circulant graphs $C_{30}\langle\ 13\rangle$ $\cup$ $C_{30}\langle\ 14\rangle$ $\cup$ $C_{30}\langle\ 15\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length* 15

For $v \equiv 0, 18 \pmod{24}$, we will apply hooked Skolem-type sequences to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n+6,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$ is another set of base blocks of a $CPD(6n+6,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(1)}{(3)(2)} \rfloor$. If $v \equiv 0 \pmod{24}$, the leave is the union of three circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-3\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and the union of three cycles of length $\frac{v}{3}$.

If $v \equiv 18 \pmod{24}$, the leave is the union of three circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-1\right\rangle$

$\cup$ $\mathrm{C}_v \left\langle \frac{v}{2} - 3 \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and the union of three cycles of length $\frac{v}{3}$.

Example 3.2.8 $hS_3 = (3,1,1,3,2,*,2)$, *yields the pairs* $\{(2,3),(5,7),(2,4)\}$. *These pairs yield in turn the triples:* $(1,5,6),(2,8,10),(3,4,7)$. *These triples yield the base blocks for a* $CPD(24,3,1)$*:* $\{0,1,6\},\{0,2,10\},\{0,3,7\}$.(mod 24). *The leave is the union of three circulant graphs* $C_{24}\langle\ 9\rangle \cup C_{24}\langle\ 11\rangle \cup C_{24}\langle\ 12\rangle$, *which represent the union of a 1-factor, one Hamiltonian cycle and and union of 3 cycles of length 8.*

Finally, we have an exceptional case if $v = 9$, because no cyclic $STS(9)$, thus, we can take one base block $\{0,1,3\}$, and the leave is one circulant graph $C_9\langle 4\rangle$, which represents one Hamiltonian cycle.

In the following table, we summarize the number of base blocks, leave, and the type of sequence used, for every v, $\lambda = 1$ and $k = 3$.

v (mod 12)	v (mod 24)	Number of blocks	Type of sequence	Leave
$v \equiv 2, 8$	v≡2,8	$\rho\,(v, 3, 1)$	S(6n+2)	F
$v \equiv 2, 8$	v≡14	$\rho\,(v, 3, 1) - 1$	S(6n+8)	$F \cup\ 2C \cup\ H$
$v \equiv 2, 8$	v≡20	$\rho\,(v, 3, 1) - 1$	hS(6n+8)	$F \cup\ 2C \cup\ H$
$v \equiv 4, 10$	v≡4	$\rho\,(v, 3, 1)$	S(6n+4)	$F \cup\ H$
$v \equiv 4, 10$	v≡10	$\rho\,(v, 3, 1)$	S(6n+4)	$F \cup\ C$
$v \equiv 4, 10$	v≡16	$\rho\,(v, 3, 1)$	hS(6n+4)	$F \cup\ C$
$v \equiv 4, 10$	v≡22	$\rho\,(v, 3, 1)$	hS(6n+4)	$F \cup\ H$
$v \equiv 5, 11$	v≡5,11	$\rho\,(v, 3, 1)$	S(6n+5)	$2H$
$v \equiv 5, 11$	v≡17,23	$\rho\,(v, 3, 1)$	hS(6n+5)	$2H$
$v \equiv 0, 6$	v≡6,12	$\rho\,(v, 3, 1)$	S(6n+6)	$F \cup\ C \cup\ H$
$v \equiv 0, 6$	v≡0	$\rho\,(v, 3, 1)$	hS(6n+6)	$F \cup\ H \cup\ C$
$v \equiv 0, 6$	v≡18	$\rho\,(v, 3, 1)$	hS(6n+6)	$F \cup\ H \cup\ C$

Table 3.2: Constructions of a cyclic packing design $CPD(v, 3, 1)$

3.2.5 Example of leave

In this example, we present the minimum leave of the cyclic packing design for $\lambda = 1$ and $v = 14$, where the leaves are a 1-factor, one Hamiltonian cycle and four disjoint cycles of length 7, $(H \cup 2C \cup F)$.

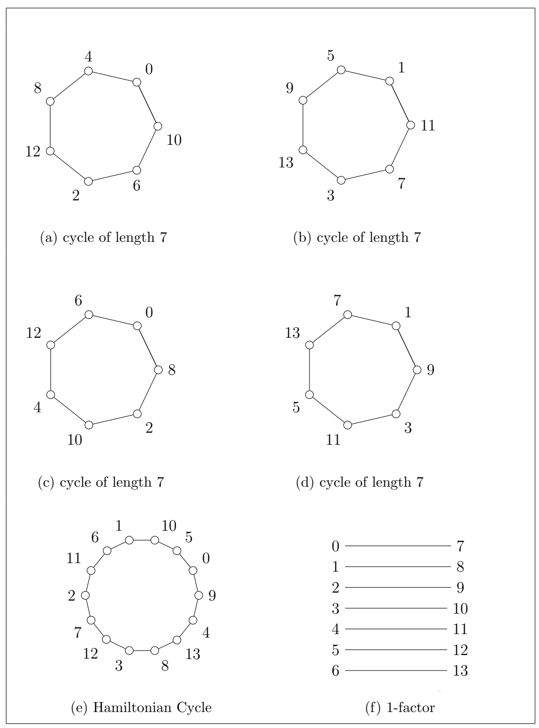


Figure 3.1: The minimum leave of $CPD\,(14, 3, 1)$, $(H \cup 2C \cup F)$

3.3 Cyclic Packing Designs for $k = 3$ and $\lambda = 2$, $CPD(v, 3, 2)$

3.3.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 14 \pmod{24}$, we will apply Skolem-type sequences to construct cyclic packing designs. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n + 2, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n + 2, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v \langle 1 \rangle$, which represents one Hamiltonian cycle.

Example 3.3.1 $S_{12} = (12, 10, 8, 6, 4, 2, 11, 2, 4, 6, 8, 10, 12, 5, 9, 7, 3, 11, 5, 3, 1, 1, 7, 9)$ *yields the pairs:*$\{(21, 22), (6, 8), (17, 20), (5, 9), (14, 19), (4, 10), (16, 23), (3, 11),$
$(15, 24), (2, 12), (7, 18), (1, 13).$

These pairs yield in turn the triples: $\{(1, 33, 34), (2, 18, 20), (3, 29, 32), (4, 17, 21), (5, 26, 31),$ $(6, 16, 22), (7, 28, 35), (8, 15, 23), (9, 27, 36), (10, 14, 24), (11, 19, 30), (12, 13, 25)\}$. *These triples yield the base blocks for a* $CPD(38, 3, 2)$:

$\{0,1,34\},\{0,2,20\},\{0,3,32\},\{0,4,21\},\{0,5,31\},\{0,6,22\},\{0,7,35\},\{0,8,23\},$ $\{0,9,36\},0,10,24\},\{0,11,30\},$ *and* $\{0,12,25\}$ (mod 38).

The leave is one circulant graph $C_{38}\langle 1\rangle$*, which represents one Hamiltonian cycle.*

For $v \equiv 8, 20 \pmod{24}$, we will apply hooked Skolem-type sequences to construct cyclic packing designs. From hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+2,3,2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n+2,3,2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v\langle 2\rangle$, which represents 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.3.2 $hS_6 = (1,1,2,5,2,4,6,3,5,4,3,*,6)$ *yields the pairs* $\{(1,2),(3,5),(8,11),(6,10),(4,9),(7,13)\}$*. These pairs yield in turn the triples* $\{(1,7,8),(2,9,11),(3,14,17),(4,12,16),(5,10,15),(6,13,19)\}$*. These triples yield the base blocks for a* $CPD(20,3,2)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\}$ (mod 20).

The leave is one circulant graph $C_{20}\langle 2\rangle$*, which represents 2 disjoint cycles of length* 10.

3.3.2 Case 2: $v \equiv 10 \pmod{12}$

For $v \equiv 10, 22 \pmod{24}$, we will apply hooked Skolem-type sequences to construct cyclic packing designs. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+4, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n+4, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor - 1$. The leave is the union of three circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \langle 4 \rangle$, which represent the union of one Hamiltonian cycle and 4 disjoint cycles of length $\frac{v}{2}$.

Example 3.3.3 $hS_{10} = (10, 8, 6, 4, 2, 9, 2, 4, 6, 8, 10, 5, 3, 7, 9, 3, 5, 1, 1, 0, 7)$ *yields the pairs*$\{(18, 19), (5, 7), (13, 16), (4, 8), (12, 17), (3, 9), (14, 21), (2, 10), (6, 15), (1, 11)\}$. *These pairs yield in turn the triples* $\{(1, 28, 29), (2, 15, 17), (3, 23, 26), (4, 14, 18), (5, 22, 27),$ $(6, 13, 19), (7, 24, 31), (8, 12, 20), (9, 16, 25)(10, 11, 21)\}$. *These triples yield the base blocks for a* $CPD(34, 3, 2)$*:*

$\{0, 1, 29\}, \{0, 2, 17\}, \{0, 3, 26\}, \{0, 4, 18\}, \{0, 5, 27\}, \{0, 6, 19\}, \{0, 7, 31\}, \{0, 8, 20\}, \{0, 9, 25\}$ and $\{0, 10, 21\} \pmod{34}$. The leave is the union of three circulant graphs $C_{34} \langle 1 \rangle$ $\cup$ $\mathrm{C}_{34} \langle 2 \rangle \cup \mathrm{C}_{34} \langle 4 \rangle$, which represent the union of one Hamiltonian cycle and 4 disjoint cycles of length 17.

3.3.3 Case 3: $v \equiv 5; 11 \pmod{12}$

For $v \equiv 5, 17 \pmod{24}$ we will apply Skolem-type sequences to construct cyclic packing designs. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+2, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$ is another set of base blocks of a $CPD(3n + 2, 3, 2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v \langle 1 \rangle$, which represents one Hamiltonian cycle.

Example 3.3.4 $S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:* $\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (5, 14)\}$. *These pairs yield in turn the triples:*

$\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$.

These triples yield the base blocks for a $CPD(29, 3, 2)$*:*

$\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}, \{0, 8, 27\}$, *and* $\{0, 9, 23\} \pmod{29}$.

The leave is one circulant graph $C_{29} \langle 1 \rangle$*, which represents one Hamiltonian cycle.*

For $v \equiv 11, 23 \pmod{24}$, we will apply hooked Skolem-type sequences to construct cyclic packing designs. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+2, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+2, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v \langle 2 \rangle$, which represents one Hamiltonian cycle.

Example 3.3.5 $hS_{11} = (8, 9, 2, 6, 2, 1, 1, 10, 8, 6, 9, 11, 4, 7, 5, 3, 4, 10, 3, 5, 7, *, 11)$ *yields the pairs:*$\{(6, 7), (3, 5), (16, 19), (13, 17), (15, 20), (4, 10), (14, 21), (1, 9), (2, 11),$ *(8,18),*$(12, 23)$.

These pairs yield in turn the triples $\{(1, 17, 18), (2, 14, 16), (3, 27, 30), (4, 24, 28), (5, 26, 31),$ $(6, 15, 21), (7, 25, 32), (8, 12, 20), (9, 13, 22), (10, 19, 29), (11, 23, 34)$.

These triples yield the base blocks for a $CPD(35, 3, 2)$*:*

$\{0, 1, 18\}, \{0, 2, 14\}, \{0, 3, 30\}, \{0, 4, 28\}, \{0, 5, 31\}, \{0, 6, 21\}, \{0, 7, 32\}, \{0, 8, 20\}, \{0, 9, 22\},$ $\{0, 10, 29\}$ *and* $\{0, 11, 34\} \pmod{35}$*. The leave is one circulant graph* $C_{35} \langle 2 \rangle$*, which represents one Hamiltonian cycle.*

3.3.4 Case 4: $v \equiv 6 \pmod{12}$

For $v \equiv 6, 18 \pmod{24}$, we will apply Skolem-type sequences. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+3, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n+3, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(2)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v\langle 1 \rangle \cup \mathrm{C}_v\langle 2 \rangle$, which represent the union of one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.3.6 $S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$*, yields the pairs:* $\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (5, 14)\}$*. These pairs yield in turn the triples:* $\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$*. These triples yield the base blocks for a* $CPD(30, 3, 2)$*:*

$\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}, \{0, 8, 27\}$*, and* $\{0, 9, 23\} \pmod{30}$*.*

The leave is the union of two circulant graphs $C_{30}\langle 1 \rangle \cup C_{30}\langle 2 \rangle$*, which represent the union of one Hamiltonian cycle and 2 disjoint cycles of length* 15*.*

Finally, we have an exceptional case if $v = 9$ because no cyclic $STS\,(9)$ exists, thus, we can take two base blocks $\{0,1,3\}$ and $\{0,2,5\}$, and the leave is the union of two circulant graphs $C_9\langle 1\rangle \cup \mathrm{C}_9\langle 4\rangle$ which represent the union of two Hamiltonian cycles.

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 2$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	$Number of blocks$	$Type of sequence$	$Leaves$
$v \equiv 2,8$	v≡2,14	$\rho\,(v,3,2)$	S(3n+2)	H
$v \equiv 2,8$	v≡8,20	$\rho\,(v,3,2)$	hS(3n+2)	C
$v \equiv 10$	v≡10,22	$\rho\,(v,3,2)-1$	hS(3n+4)	$H \cup\ 2C$
$v \equiv 5,11$	v≡5,17	$\rho\,(v,3,2)$	S(3n+2)	H
$v \equiv 5,11$	v≡11,23	$\rho\,(v,3,2)$	hS(3n+2)	H
$v \equiv 6$	v≡6,18	$\rho\,(v,3,2)$	S(3n+3)	$H \cup\ C$

Table 3.3: Constructions of a cyclic packing design $CPD(v,3,2)$

3.4 Cyclic Packing Designs for $k = 3$ and $\lambda = 3$, $CPD(v, 3, 3)$

3.4.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8 \pmod{24}$, we will use the union between $CPD(v, 3, 1)$ and $CPD(v, 3, 2)$ obtained in Section 3.2 and Section 3.3, respectively to construct a cyclic packing design.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. For $v \equiv 2 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of one Hamiltonian cycle and a 1-factor. For $v \equiv 8 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.1 *Let $v = 26$. We will use a Skolem sequence of order 4 as in the construction of a $CPD(26, 3, 1)$ and a Skolem sequence of order 8 as in the construction of a $CPD(26, 3, 2)$ as follows:*

$S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$. *These pairs yield in turn the triples* $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$.

These triples yield the base blocks for a $CPD(26, 3, 1)$:

$\{0,1,6\}, \{0,2,11\}, \{0,3,10\}$, *and* $\{0,4,12\}(\text{mod } 26)$. *Also we need a Skolem-type sequence of order* 8 *as follows:* $S_8 = (8,6,4,2,7,2,4,6,8,3,5,7,3,1,1,5)$ *yields the pairs:* $\{(14,15),(4,6),(10,13),(3,7),(11,16),(2,8),(5,12),(1,9)\}$. *These pairs yield in turn the triples:* $\{(1,22,23),(2,12,14),(3,18,21),(4,11,15),(5,19,24),(6,10,16),$

(7,13,20),(8,9,17). These triples yield the base blocks for a $CPD(26,3,2)$*:*

$\{0,1,23\}, \{0,2,14\}, \{0,3,21\}, \{0,4,15\}, \{0,5,24\}, \{0,6,16\}, \{0,7,20\}$

and $\{0,8,17\}$ (mod 26). *The leave is the union of two circulant graphs* $C_{26}\langle 1\rangle \cup C_{26}\langle 13\rangle$*, which represent the union of one Hamiltonian cycle and a 1-factor.*

For $v \equiv 14 \pmod{24}$, we will use the union between $CPD(v,3,1)$, $CPD(v,3,2)$ obtained in Section 3.2 and Section 3.3, respectively and one copy of the base block $\left\{0, \frac{v}{2}-3, \frac{v}{2}-2\right\}$ to construct a cyclic packing design.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.2 *Let* $v = 38$. *We will use a Skolem sequence of order* 5 *as in the construction of a* $CPD(38,3,1)$*, a Skolem sequence of order* 12 *as in the construction of a* $CPD(38,3,2)$ *, and one copy of the base block* $\{0,16,17\}$ *as follows:*

$S_{12} = (12,10,8,6,4,2,11,2,4,6,8,10,12,5,9,7,3,11,5,3,1,1,7,9)$ *yields the pairs:*

$\{(21,22),(6,8),(17,20),(5,9),(14,19),(4,10),(16,23),(3,11),(15,24),(2,12),(7,18),(1,13)\}$.

These pairs yield in turn the triples:

$\{(1,33,34),(2,18,20),(3,29,32),(4,17,21),(5,26,31),(6,16,22),(7,28,35),$

$(8,15,23),(9,27,36),(10,14,24),(11,19,30),(12,13,25)\}$.

These triples yield the base blocks for a $CP(38,3,2)$*:*

$\{0,1,34\},\{0,2,20\},\{0,3,32\},\{0,4,21\},\{0,5,31\},\{0,6,22\},\{0,7,35\},\{0,8,23\},$

$\{0,9,36\},0,10,24\},\{0,11,30\}$*, and* $\{0,12,25\}\pmod{38}$. *Also we need a Skolem-type sequence of order* 5 *as follows*

$S_5=(1,1,3,4,5,3,2,4,2,5)$ *yields the pairs* $\{(1,2),(7,9),(3,6),(4,8),(5,10)\}$.

These pairs yield in turn the triples: $\{(1,6,7),(2,12,14),(3,8,11),(4,9,13),(5,10,15)\}$*.These triples yield the base blocks for a* $CPD(38,3,1)$*:*

$\{0,1,7\},\{0,2,14\},\{0,3,11\},\{0,4,13\}$ *and* $\{0,5,15\}\pmod{38}$. *Finally; add one copy of the base block* $\{0,16,17\}$.

The leave is the union of two circulant graphs $C_{38}\langle 19\rangle \cup C_{38}\langle 18\rangle$*, which represent the union of a 1-factor and 2 disjoint cycles of length* 19.

For $v\equiv 20\pmod{24}$, we will use the union between $CPD\,(v,3,1)$ and $CPD\,(v,3,2)$ obtained in Section 3.2 and Section 3.3, respectively and one copy of the base block $\left\{0,\frac{v}{2}-4,\frac{v}{2}-2\right\}$ to construct a cyclic packing design.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of one Hamiltonian cycle and a 1-factor.

Example 3.4.3 *Let $v = 44$. We will use a hooked Skolem sequence of order 6 as in the construction of a $CPD\,(38, 3, 1)$ hooked Skolem sequence of order 14 as in the construction of a $CPD\,(38, 3, 2)$ and one copy of the base block $\{0, 18, 20\}$ as follows:*

$hS_{14} = (14, 12, 10, 8, 6, 4, 2, 13, 2, 4, 6, 8, 10, 12, 14, 9, 3, 11, 7, 3, 13, 5, 1, 1, 9, 7, 5, *, 11)$

yields the pairs $\{(23, 24), (7, 9), (17, 20), (6, 10), (22, 27), (5, 11), (19, 26), (4, 12), (16, 25),$

(3,13),(18,29),(2,14),(8,21),$(1, 15)$.

These pairs yield in turn the triples

$\{(1, 37, 38), (2, 21, 23), (3, 31, 34), (4, 20, 24), (5, 36, 41), (6, 19, 25), (7, 33, 40), (8, 18, 26),$

(9,30,39),(10,17,27),(11,32,43),(12,16,28),(13,22,35),(14,15,29).

These triples yield the base blocks for a $CPD(44, 3, 2)$:

$\{0, 1, 38\}, \{0, 2, 23\}, \{0, 3, 34\}, \{0, 4, 24\}, \{0, 5, 41\}, \{0, 6, 25\}, \{0, 7, 40\}, \{0, 8, 26\},$

$\{0, 9, 39\}, \{0, 10, 27\}, \{0, 11, 43\}, \{0, 12, 28\}, \{0, 13, 35\}$ *and* $\{0, 14, 29\} (\text{mod } 44)$.

Also we need a hooked Skolem-type sequence of order 6 as following:

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, *, 6)$ *yields the pairs* $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$.

These pairs yield in turn the triples $\{(1, 7, 8), (2, 9, 11), (3, 14, 17), (4, 12, 16), (5, 10, 15), (6, 13, 19)\}$.

These triples yield the base blocks for a $CPD(44, 3, 1)$:

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\} \pmod{44}$*, finally we add one copy of the base block* $\{0,18,20\}$*. The leave is the union of two circulant graphs* $C_{44}\langle 22\rangle \cup C_{44}\langle 21\rangle$*, which represent the union of one Hamiltonian cycle and a 1-factor.*

3.4.2 Case 2: $v \equiv 0, 6 \pmod{12}$

For $v \equiv 0 \pmod{24}$, we will apply Skolem-type sequences, hooked Skolem-types sequences, and one copy of the following base blocks: $\left\{0, \frac{v}{2}-3, \frac{v}{2}-1\right\}$, $\{0,5,6\}$ and $\{0,4,7\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+9,3,3)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+9,3,3)$.

From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n+6,3,3)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(6n+6,3,3)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}-4\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.4 $S_5 = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$

yields the pairs $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$.

These pairs yield in turn the triples: $\{(1, 6, 7), (2, 12, 14), (3, 8, 11), (4, 9, 13), (5, 10, 15)\}$.

These triples yield the base blocks for a $CPD(24, 3, 3)$*:*

$\{0, 1, 7\}, \{0, 2, 14\}, \{0, 3, 11\}, \{0, 4, 13\}$ *and* $\{0, 5, 15\} \pmod{24}$,

Also we need the following sequence:

$hS_3 = (1, 1, 2, 3, 2, *, 3)$ *yields the pairs* $\{(1, 2), (3, 5), (4, 7)\}$.

These pairs yield in turn the triples $\{(1, 4, 5), (2, 6, 8), (3, 7, 10)\}$.

These triples yield the base blocks for a $CPD(24, 3, 3)$*:*

$\{0, 1, 5\}, \{0, 2, 8\}$ *and* $\{0, 3, 10\} \pmod{24}$. *Finally we add the following blocks:* $\{0, 5, 6\}$,$\{0, 4, 7\}$ *and* $\{0, 9, 11\}$. *The leave is the union of two circulant graphs* $C_{24}\langle 8\rangle \cup C_{24}\langle 12\rangle$*, which represent the union of a 1-factor and 2 disjoint cycles of length* 12.

For $v \equiv 12 \pmod{24}$, we will apply Skolem-type sequences, hooked Skolem type sequences, and one copy of the base block:$\{0, \frac{v}{2} - 2, \frac{v}{2} - 1\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n + 6, 3, 3)$: $\{0, a_i + n, b_i + n\}$, $1 \leq$

$i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$ is another set of base blocks of a $CPD(6n+6,3,3)$.

From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,3)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+3,3,3)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$.

The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v \langle 3 \rangle$, which represent the union of a 1-factor and union of three cycles of length $\frac{v}{3}$.

Example 3.4.5

$S_5 = (1,1,3,4,5,3,2,4,2,5)$ *yields the pairs* $\{(1,2),(7,9),(3,6),(4,8),(5,10)\}$.

These pairs yield in turn the triples: $\{(1,6,7),(2,12,14),(3,8,11),(4,9,13),(5,10,15)\}$.

These triples yield the base blocks for a $CPD(36,3,3)$*:*

$\{0,1,7\},\{0,2,14\},\{0,3,11\},\{0,4,13\}$ *and* $\{0,5,15\}$ (mod 36).

Also we need the following sequence:

$hS_{11} = (8,9,2,6,2,1,1,10,8,6,9,11,4,7,5,3,4,10,3,5,7,*,11)$ *yields the pairs*

$\{(6,7),(3,5),(16,19),(13,17),(15,20),(4,10),(14,21),(1,9),(2,11),(8,18),(12,23)\}$.

These pairs yield in turn the triples

$\{(1,17,18),(2,14,16),(3,27,30),(4,24,28),(5,26,31),(6,15,21),(7,25,32),(8,12,20),$ $(9,13,22),(10,19,29),(11,23,34)\}$. *These triples yield the base blocks for a*

$CPD(36,3,3)$:

$\{0,1,18\},\{0,2,14\},\{0,3,30\},\{0,4,28\},\{0,5,31\},\{0,6,21\},\{0,7,32\},\{0,8,20\},\{0,9,22\}$, $\{0,10,29\}$ *and* $\{0,11,34\} \pmod{36}$. *Finally we add one copy of the base block:* $\{0,16,17\}$. *The leave is the union of two circulant graphs* $C_{36}\langle 3\rangle \cup C_{36}\langle 18\rangle$, *which represent the union of a 1-factor and union of 3 cycles of length 12.*

For $v \equiv 6 \pmod{24}$, we will apply the union between $CPD(v,3,1)$, $CPD(v,3,2)$ obtained in Section 3.2 and Section 3.3, respectively and one copy of the base block $\left\{0,\frac{v}{2}-2,\frac{v}{2}-1\right\}$ to construct a cyclic packing design.

The number of base blocks in this case is $\lfloor\frac{(v-1)(3)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\langle 2\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.6 *Let* $v = 30$. *We will use a Skolem sequence of order* 9 *as in the construction of a* $CPD(30,3,1)$ *and a Skolem sequence of order* 4 *as in the construction of a* $CPD(30,3,2)$ *and one copy of the base block* $\{0,13,14\}$ *as follows:* $S_9 = (3,7,5,3,9,1,1,5,7,8,6,4,2,9,2,4,6,8)$ *yields the pairs:*$\{(6,7),(13,15),(1,4),$ $(12,16),(3,8),(11,17),(2,9),(10,18),(5,14)\}$. *These pairs yield in turn the triples:* $\{(1,15,16),(2,22,24),(3,10,13),(4,21,25),(5,12,17),(6,20,26),(7,11,18),$ $(8,19,27),(9,14,23)\}$. *These triples yield the base blocks for a* $CPD(30,3,2)$*:*

$\{0,1,16\},\{0,2,24\},\{0,3,13\},\{0,4,25\},\{0,5,17\},\{0,6,26\},\{0,7,18\},$

$\{0,8,27\}$, *and* $\{0,9,23\}(\text{mod } 30)$.

Also we need the following sequence: $S_4 = (1,1,3,4,2,3,2,4)$ *which yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$.

These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for a $CPD(30,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$, *and* $\{0,4,12\}(\text{mod } 30)$. *Finally, add one copy of the base block* $\{0,13,14\}$. *The leave is the union of two circulant graphs* $C_{30}\langle 2\rangle \cup C_{30}\langle 15\rangle$, *which represent the union of a 1-factor and 2 disjoint cycles of length* 15.

For $v \equiv 18 \pmod{24}$, we will use the union between $CPD(v,3,1)$, $CPD(v,3,2)$ obtained in Section 3.2 and Section 3.3, respectively and one copy of the base block $\left\{0, \frac{v}{2}-3, \frac{v}{2}-1\right\}$ to construct a cyclic packing design.

The number of base blocks in this case is $\lfloor\frac{(v-1)(3)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

Example 3.4.7 *Let* $v = 42$. *We will use a Skolem sequence of order* 13 *as in the construction of a* $CPD(42,3,1)$ *and a hooked Skolem sequence of order* 6 *as in the construction of a* $CPD(42,3,2)$ *and one copy of the base block* $\{0,18,20\}$ *as follows:*

$S_{13} = (5,11,9,7,3,5,13,3,1,1,7,9,11,12,10,8,6,4,2,13,2,4,6,8,10,12)$ *yields the pairs:* $\{(9,10),(19,21),(5,8),(18,22),(1,6),(17,23),(4,11),(16,24),(3,12),(15,25),$ $(2,13),(14,26),(7,20)\}$. *These pairs yield in turn the triples:*

$\{(1,22,23),(2,32,34),(3,18,21),(4,31,35),(5,14,19),(6,30,36),(7,17,24),(8,29,37),$ $(9,16,25),(10,28,38),(11,15,26),(12,27,39),(13,20,33)\}$.*These triples yield the base blocks for a* $CPD(42,3,2)$*:*

$\{0,1,23\},\{0,2,34\},\{0,3,21\},\{0,4,35\},\{0,5,19\},\{0,6,36\},\{0,7,24\},$ $\{0,8,37\},\{0,9,25\},0,10,38\},\{0,11,26\}$,{*0,12,39*}, *and* $\{0,13,33\}(\text{mod } 42)$.

Also we need a Hooked Skolem type sequence of order 6 *as following:*

$hS_6 = (1,1,2,5,2,4,6,3,5,4,3,*,6)$ *yields the pairs*$\{(1,2),(3,5),(8,11),(6,10),(4,9),(7,13)\}$. *These pairs yield in turn the triples* $\{(1,7,8),(2,9,11),(3,14,17),(4,12,16),(5,10,15),$

(6,13,19). These triples yield the base blocks for $CPD(42,3,1)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\}(\text{mod } 42)$. *Finally we add one copy of the base block* $\{0,18,20\}$.

The leave is the union of two circulant graphs $C_{42}\langle 1\rangle \cup C_{42}\langle 21\rangle$, *which represent the union of a 1-factor and one Hamiltonian cycle.*

3.4.3 Case 3: $v \equiv 4, 10 \pmod{12}$

For $v \equiv 4 \pmod{24}$, we will apply Skolem-type sequences two times and one copy of the base block $\{0, 2, 3\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n + 4, 3, 2)$ and $CPD(6n + 4, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n + 4, 3, 2)$ and $CPD(6n + 4, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

Example 3.4.8 *Let $v = 28$. We will use a Skolem sequence of order 8; a Skolem sequence of order 4 and one copy of the base block $\{0, 2, 3\}$ as follows:*

$S_8 = (8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$, *yields the pairs:*

$\{(14, 15), (4, 6), (10, 13), (3, 7), (11, 16), (2, 8), (5, 12), (1, 9)\}$. *These pairs yield in turn the triples:*

$\{(1, 22, 23), (2, 12, 14), (3, 18, 21), (4, 11, 15), (5, 19, 24), (6, 10, 16), (7, 13, 20), (8, 9, 17)\}$.

These triples yield the base blocks for a $CPD(28, 3, 2)$:

$\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$, *and* $\{0,8,17\} \pmod{28}$. *Also we need a Skolem type sequence of order* 4 *as follows:* $S_4 = (1,1,3,4,2,3,2,4)$ *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. *These triples yield the base blocks for a* $CPD(28,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$, *and* $\{0,4,12\} \pmod{28}$. *Finally, we will add the base block* $\{0,2,3\}$*, the leave is the union of two circulant graphs* $C_{28}\langle 13\rangle \cup C_{28}\langle 14\rangle$*, which represent the union of a 1-factor and one Hamiltonian cycle.*

For $v \equiv 16 \pmod{24}$, we will apply Skolem-type sequences, hooked Skolem-type sequences, and one copy of the base block: $\{0,2,3\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+4,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$ is another set of base blocks of a $CPD(3n+4,3,2)$.
From a hooked Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n+4,3,1)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(6n+4,3,1)$.

The number of base blocks in this case is $\lfloor\frac{(v-1)(3)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}-2\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.9 *Let* $v = 40$*. We will use a Skolem sequence of order* 12*, a hooked Skolem sequence of order* 6 *and one copy of the base block* $\{0, 2, 3\}$ *as follows:*

$S_{12} = (12, 10, 8, 6, 4, 2, 11, 2, 4, 6, 8, 10, 12, 5, 9, 7, 3, 11, 5, 3, 1, 1, 7, 9)$, *yields the pairs:*$\{(21, 22), (6, 8), (17, 20), (5, 9), (14, 19), (4, 10), (16, 23), (3, 11), (15, 24), (2, 12), (7, 18), (1, 13)\}$.

These pairs yield in turn the triples:

$\{(1, 33, 34), (2, 18, 20), (3, 29, 32), (4, 17, 21), (5, 26, 31), (6, 16, 22), (7, 28, 35),$

$(8, 15, 23), (9, 27, 36), (10, 14, 24), (11, 19, 30), (12, 13, 25)\}$.

These triples yield the base blocks for a $CPD(40, 3, 2)$*:*

$\{0, 1, 34\}, \{0, 2, 20\}, \{0, 3, 32\}, \{0, 4, 21\}, \{0, 5, 31\}, \{0, 6, 22\}, \{0, 7, 35\}, \{0, 8, 23\},$

$\{0, 9, 36\}, 0, 10, 24\}, \{0, 11, 30\}$, *and* $\{0, 12, 25\} \pmod{40}$.

Also we need a hooked Skolem type sequence of order 6 *as follows:*

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, *, 6)$,

yields the pairs $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$.

These pairs yield in turn the triples

$\{(1, 7, 8), (2, 9, 11), (3, 14, 17), (4, 12, 16), (5, 10, 15), (6, 13, 19)\}$.

These triples yield the base blocks for a $CPD(40, 3, 1)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\} \pmod{40}$. *Finally we add the base block* $\{0,2,3\}$, *the leave is the union of two circulant graphs* $C_{40}\langle 18\rangle \cup C_{40}\langle 20\rangle$, *which represent the union of a 1-factor and 2 disjoint cycles of length* 20.

For $v \equiv 10 \pmod{24}$, we will apply near Skolem-type sequences, $CPD(v,3,2)$ obtained in Section 3.3 and one copy of the base block: $\{0,2,4\}$ to construct a cyclic packing design. From a near Skolem sequence of order $(n,2)$, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n-2,3,1)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n-2,3,1)$.

The number of base blocks in this case is $\lfloor\frac{(v-1)(3)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}\right\rangle \cup \mathrm{C}_v\langle 1\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

Example 3.4.10 *Let* $v = 34$. *We will use a near Skolem sequence of order* 6, *a hooked Skolem sequence of order* 10 *as in the construction of a* $CPD(34,3,2)$ *and one copy of the base block* $\{0,2,4\}$ *as follows:*

$nS_6 = (4,5,3,6,4,3,5,1,1,6)$

yields the pairs $\{(8,9),(3,6),(1,5),(2,7),(4,10)\}$.*These pairs yield in turn the triples* $\{(1,14,15),(3,9,12),(4,7,11),(5,8,13),(6,10,16)\}$.

These triples yield the base blocks for a $CPD(34,3,1)$*:*

$\{0,1,15\}, \{0,3,12\}, \{0,4,11\}, \{0,5,13\}$ *and* $\{0,6,16\}$ (mod 34).

Also we need a hooked Skolem type sequence of order 10 *as following:*

$hS_{10} = (10,8,6,4,2,9,2,4,6,8,10,5,3,7,9,3,5,1,1,*,7)$ *yields the pairs*

$\{(18,19),(5,7),(13,16),(4,8),(12,17),(3,9),(14,21),(2,10),(6,15),(1,11)\}$.

These pairs yield in turn the triples $\{(1,28,29),(2,15,17),(3,23,26),(4,14,18),$ *(5,22,27),(6,13,19),(7,24,31),(8,12,20),(9,16,25),(10,11,21). These triples yield the base blocks for a* $CPD(34,3,2)$*:*

$\{0,1,29\}, \{0,2,17\}, \{0,3,26\}, \{0,4,18\}, \{0,5,27\}, \{0,6,19\}, \{0,7,31\}, \{0,8,20\}, \{0,9,25\}$, *and* $\{0,10,21\}$(mod 34). *Finally we add one copy of the base block* $\{0,2,4\}$.

The leave is the union of two circulant graphs $C_{34}\langle 1\rangle \cup C_{34}\langle 17\rangle$*, which represent the union of a 1-factor and one Hamiltonian cycle.*

For $v \equiv 22 \pmod{24}$, we will apply near Skolem-type sequences, $CPD(v,3,2)$ obtained in Section 3.3, respectively and one copy of the following base blocks: $\left\{0, \frac{v}{2}-3, \frac{v}{2}-2\right\}$,$\left\{0, \frac{v}{2}-6, \frac{v}{2}-4\right\}$ and $\left\{0, \frac{v}{2}-5, \frac{v}{2}-1\right\}$ to construct a cyclic packing design. From a near Skolem sequence of order $(n,2)$, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n+10,3,1)$:

$\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n + 10, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(3)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v \left\langle 2 \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.4.11 *Let $v = 46$ we will use a near Skolem sequence of order 6, a hooked Skolem sequence of order 14 as in the construction of a $CPD(46, 3, 2)$ and one copy of the base blocks $\{0, 20, 21\}$,$\{0, 17, 19\}$ and $\{0, 18, 22\}$ as follows:*

$nS_6 = (4, 5, 3, 6, 4, 3, 5, 1, 1, 6)$, *yields the pairs* $\{(8, 9), (3, 6), (1, 5), (2, 7), (4, 10)\}$.

These pairs yield in turn the triples: $\{(1, 14, 15), (3, 9, 12), (4, 7, 11), (5, 8, 13), (6, 10, 16)\}$.

These triples yield the base blocks for a $CPD(46, 3, 1)$:

$\{0, 1, 15\}, \{0, 3, 12\}, \{0, 4, 11\}, \{0, 5, 13\}$ *and* $\{0, 6, 16\}(\bmod 46)$.

Also we need a hooked Skolem type sequence of order 14 as follows:

$hS_{14} = (14, 12, 10, 8, 6, 4, 2, 13, 2, 4, 6, 8, 10, 12, 14, 9, 3, 11, 7, 3, 13, 5, 1, 1, 9, 7, 5, *, 11)$,

yields the pairs $\{(23, 24), (7, 9), (17, 20), (6, 10), (22, 27), (5, 11), (19, 26), (4, 12), (16, 25),$

(3,13),(18,29),(2,14),(8,21),(1,15). These pairs yield in turn the triples

$\{(1, 37, 38), (2, 21, 23), (3, 31, 34), (4, 20, 24), (5, 36, 41), (6, 19, 25), (7, 33, 40),$

$(8, 18, 26), (9, 30, 39), (10, 17, 27), (11, 32, 43), (12, 16, 28), (13, 22, 35), (14, 15, 29)\}$.

These triples yield the base blocks for a $CPD(46, 3, 2)$:

$\{0,1,38\},\{0,2,23\},\{0,3,34\},\{0,4,24\},\{0,5,41\},\{0,6,25\},\{0,7,40\},\{0,8,26\},\{0,9,39\},$ $\{0,10,27\},\{0,11,43\},\{0,12,28\},\{0,13,35\}$ *and* $\{0,14,29\} \pmod{46}$*, finally we add one copy of the following base blocks* $\{0,20,21\}$*,* $\{0,17,19\}$ *and* $\{0,18,22\}$*.*

The leave is the union of two circulant graphs $C_{46}\langle 2\rangle \cup C_{46}\langle 23\rangle$*, which represent the union of a 1-factor and 2 disjoint cycles of length* 23*.*

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 3$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	$Number of blocks$	$Type of sequence$	$Leaves$
$v \equiv 2,8$	v≡2	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2)$	$H \cup F$
$v \equiv 2,8$	v≡8	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2)$	$C \cup F$
$v \equiv 2,8$	v≡14	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2) \cup \{0,\frac{v}{2}-3,\frac{v}{2}-2\}$	$C \cup F$
$v \equiv 2,8$	v≡20	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2) \cup \{0,\frac{v}{2}-4,\frac{v}{2}-2\}$	$H \cup F$
$v \equiv 0,6$	v≡0	$\rho(v,3,3)$	S(3n+9), hS(6n+6),$\{0,\frac{v}{2}-3,\frac{v}{2}-1\}$,$\{0,5,6\}$,$\{0,4,7\}$	$C \cup F$
$v \equiv 0,6$	v≡12	$\rho(v,3,3)$	S(6n+6), hS(3n+3),$\{0,\frac{v}{2}-2,\frac{v}{2}-1\}$	$C \cup F$
$v \equiv 0,6$	v≡6	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2) \cup \{0,\frac{v}{2}-2,\frac{v}{2}-1\}$	$C \cup F$
$v \equiv 0,6$	v≡18	$\rho(v,3,3)$	$CPD(v,3,1) \cup CPD(v,3,2) \cup \{0,\frac{v}{2}-3,\frac{v}{2}-1\}$	$H \cup F$
$v \equiv 4,10$	v≡4	$\rho(v,3,3)$	$S(3n+4), S(6n+4)$,$\{0,2,3\}$	$H \cup F$
$v \equiv 4,10$	v≡16	$\rho(v,3,3)$	$S(3n+4), hS(6n+4)$,$\{0,2,3\}$	$C \cup F$
$v \equiv 4,10$	v≡10	$\rho(v,3,3)$	$nS(6n-2), CPD(v,3,2)$, $\{0,2,4\}$	$H \cup F$
$v \equiv 4,10$	v≡22	$\rho(v,3,3)$	nS(6n-2), $CPD(v,3,2)$, $\{0,\frac{v}{2}-3,\frac{v}{2}-2\}$, $\{0,\frac{v}{2}-6,\frac{v}{2}-4\}$, $\{0,\frac{v}{2}-5,\frac{v}{2}-1\}$	$C \cup F$

Table 3.4: Constructions of a cyclic packing design $CPD(v,3,3)$

3.4.4 Example of leaves

In this example, we present the minimum leave of the cyclic packing design for $\lambda = 3$ and $v = 8$, where the leaves are a 1-factor and two disjoint cycles of length 4, $(C \cup F)$.

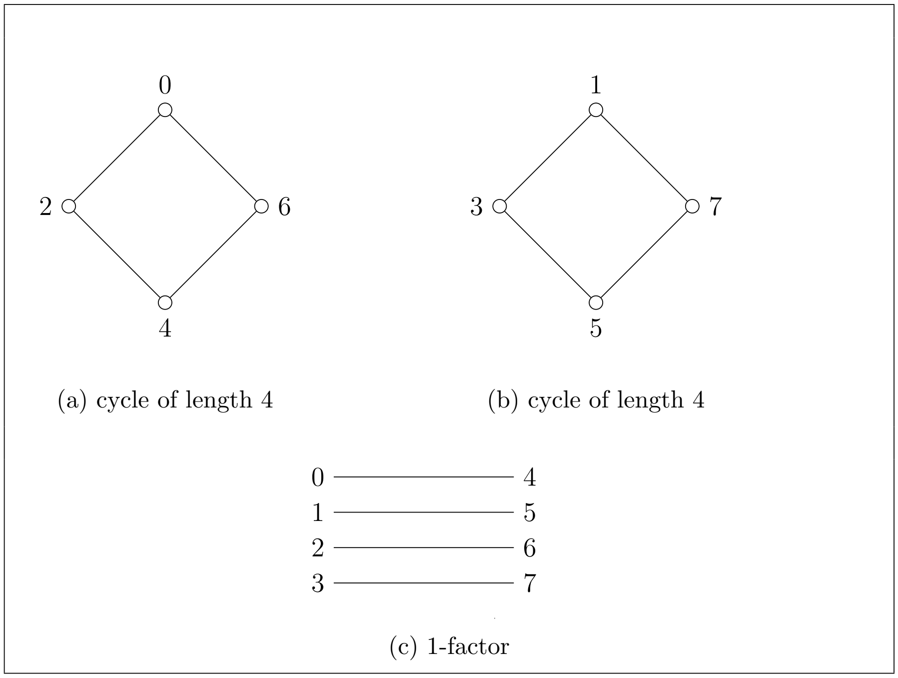


Figure 3.2: The minimum leave of $CPD\,(8,3,3)$, $(C \cup F)\,.$

3.5 Cyclic Packing Designs for $k = 3$ and $\lambda = 4$, $CPD\,(v, 3, 4)$

3.5.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8, 14, 20 \pmod{24}$, we will apply Skolem-type sequences or hooked Skolem types sequences to construct a cyclic packing design. From a Skolem sequence or a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. These triples yield the base blocks for a $CPD(3n + 2, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$. Also $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n + 2, 3, 2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(4)}{(3)(2)} \rfloor$.

If $v \equiv 2, 14 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle$, which represent the union of two Hamiltonian cycles.

If $v \equiv 8, 20 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle$, which represent the union of 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.5.1 *Let $v = 44$. We will apply a hooked Skolem sequence of order 14 as follows:*

$hS_{14} = (14, 12, 10, 8, 6, 4, 2, 13, 2, 4, 6, 8, 10, 12, 14, 9, 3, 11, 7, 3, 13, 5, 1, 1, 9, 7, 5, *, 11)$,

yields the pairs $\{(23, 24), (7, 9), (17, 20), (6, 10), (22, 27), (5, 11), (19, 26), (4, 12), (16, 25),$

(3,13),(18,29),(2,14),(8,21),(1,15). These pairs yield in turn the triples $\{(1,37,38),(2,21,23),(3,31,34),(4,20,24),(5,36,41),(6,19,25),(7,33,40),(8,18,26),$ *(9,30,39),(10,17,27),(11,32,43),(12,16,28),(13,22,35),(14,15,29). These triples yield the base blocks for two* $CPD(44,3,2)$*:*

1. $\{0,1,38\},\{0,2,23\},\{0,3,34\},\{0,4,24\},\{0,5,41\},\{0,6,25\},\{0,7,40\},\{0,8,26\},$ $\{0,9,39\},\{0,10,27\},\{0,11,43\},\{0,12,28\},\{0,13,35\}$*, and* $\{0,14,29\} \pmod{44}$*;*

2. $\{0,37,38\},\{0,21,23\},\{0,31,34\},\{0,20,24\},\{0,36,41\},\{0,19,25\},\{0,33,40\},$ $\{0,18,26\},\{0,30,39\},\{0,17,27\},\{0,32,43\},\{0,16,28\},\{0,22,35\}$*, and* $\{0,15,29\} \pmod{44}$.

The leave is the union of two circulant graphs $C_{44}\langle 2\rangle \cup C_{44}\langle 2\rangle$*, which represent the union of 2 disjoint cycles of length* 22*.*

3.5.2 Case 2: $v = 5; 11 \pmod{12}$

For $v \equiv 5, 11, 17, 23 \pmod{24}$, we will apply Skolem-type sequences or hooked Skolem type sequences to construct a cyclic packing design. From a Skolem sequence or a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. These triples yield the base blocks for a $CPD(3n+2,3,2)$: $\{0, a_i+n, b_i+n\}$,

$1 \leq i \leq n$. Also $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n+2, 3, 2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(4)}{(3)(2)} \rfloor$.

If $v \equiv 5, 17 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle$, which represent the union of two Hamiltonian cycles.

If $v \equiv 11, 23 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle$,which represent the union of two Hamiltonian cycles.

Example 3.5.2 *Let $v = 29$. We will use a Skolem sequence of order 9 as follows:* $S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:* $\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (15, 14)\}$.

These pairs yield in turn the triples: $\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17),$ $(6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$. *These triples yield the base blocks for two $CPD(29, 3, 2)$:*

1. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}, \{0, 8, 27\}$,*and* $\{0, 9, 23\} \pmod{29}$;

2. $\{0, 15, 16\}, \{0, 22, 24\}, \{0, 10, 13\}, \{0, 21, 25\}, \{0, 12, 17\}, \{0, 20, 26\}, \{0, 11, 18\}, \{0, 19, 27\}$,*and* $\{0, 14, 23\} \pmod{29}$.

The leave is the union of two circulant graphs $C_{29} \langle 1 \rangle \cup C_{29} \langle 1 \rangle$, which represent the union of two Hamiltonian cycles.

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 4$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	$Number of blocks$	$Type of sequence$	$Leaves$
$v \equiv 2, 8$	v≡2,14	$\rho\,(v, 3, 4)$	S(3n+2)	$2H$
$v \equiv 2, 8$	v≡8,20	$\rho\,(v, 3, 4)$	hS(3n+2)	C
$v \equiv 5, 11$	v≡5,17	$\rho\,(v, 3, 4)$	S(3n+2)	$2H$
$v \equiv 5, 11$	v≡11,23	$\rho\,(v, 3, 4)$	hS(3n+2)	$2H$

Table 3.5: Constructions of a cyclic packing design $CPD(v, 3, 4)$

3.6 Cyclic Packing Designs for $k = 3$ and $\lambda = 5$, $CPD\,(v, 3, 5)$

3.6.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8, 14, 20 \pmod{24}$, we will apply the union between $CPD\,(v, 3, 2)$ and $CPD\,(v, 3, 3)$ obtained in Section 3.3 and Section 3.4, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3)(2)} \rfloor$.

For $v \equiv 2 \pmod{24}$ the leave is the union of three circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 Hamiltonian cycles.

For $v \equiv 8 \pmod{24}$ the leave is the union of three circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 14 \pmod{24}$ the leave is the union of three circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 20 \pmod{24}$ the leave is the union of three circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.6.1 *If* $v = 26$ *then the number of base blocks equal* 20*, we will use a Skolem sequence of order* 4 *as in the construction of a* $CPD\,(26, 3, 2)$ *and a Skolem sequence of order* 8 *as in the construction of a* $CPD\,(26, 3, 3)$ *as follows:*

$S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$*. These pairs yield in turn the triples* $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$.

These triples yield the base blocks for a $CPD(26, 3, 1)$*:*

$\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\},$ *and* $\{0, 4, 12\} (\mathrm{mod}\ 26)$*. Also we need a Skolem-type sequence of order* 8 *as follows:* $S_8 = (8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$ *yields the pairs:* $\{(14, 15), (4, 6), (10, 13), (3, 7), (11, 16), (2, 8), (5, 12), (1, 9)\}$.

These pairs yield in turn the triples: $\{(1, 22, 23), (2, 12, 14), (3, 18, 21), (4, 11, 15), (5, 19, 24), (6, 10, 16), (7, 13, 20), (8, 9, 17)\}$.

These triples yield the base blocks for two $CPD(26, 3, 2)$*:*

1. $\{0, 1, 23\}, \{0, 2, 14\}, \{0, 3, 21\}, \{0, 4, 15\}, \{0, 5, 24\}, \{0, 6, 16\}, \{0, 7, 20\}$ *and* $\{0, 8, 17\} (\mathrm{mod}\ 26)$*,*

2. $\{0, 23, 24\}, \{0, 12, 14\}, \{0, 18, 21\}, \{0, 11, 15\}, \{0, 19, 24\}, \{0, 10, 16\}, \{0, 13, 20\}$ *and* $\{0, 9, 17\} (\mathrm{mod}\ 26)$.

The leave is the union of three circulant graphs $C_{26}\langle 1\rangle \cup C_{26}\langle 1\rangle \cup C_{26}\langle 13\rangle$*, which represent the union of a 1-factor and 2 Hamiltonian cycles.*

3.6.2 Case 2: $v \equiv 0; 6 \pmod{12}$

For $v \equiv 0 \pmod{24}$ we will use the sequences from $CPD\,(v,3,3)$ obtained in Section 3.4 also we apply Rosa types sequence, and one copy of the following base blocks: $\left\{0, \frac{v}{3}, \frac{v}{3}+1\right\}$ to construct a cyclic packing design. From Rosa sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(3n+3,3,2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3).(2)} \rfloor$.

The leave is one circulant graph $C_v \left\langle \frac{v}{2} \right\rangle$, which represents a 1-factor.

Example 3.6.2 *If $v = 24$ then the number of base blocks equal 19, we will use a Skolem sequence of order 5 and a hooked Skolem sequence of order 3 as in the construction of a $CPD\,(24,3,3)$ and we will use Rosa sequence as following: $R_7 = (5,1,1,3,7,5,3,*,6,4,2,7,2,4,6)$.*

yields the pairs $\{(2,3),(11,13),(4,7),(10,14),(1,6),(9,15),(5,12)\}$.

These pairs yield in turn the triples: $\{(1,9,10),(2,18,20),(3,11,14),(4,17,21),(5,8,13),(6,16,22),(7,12,19)\}$. These triples yield the base blocks for a $CPD(24,3,2)$: $\{0,1,10\},\{0,2,20\},\{0,3,14\},\{0,4,21\},\{0,5,13\},\{0,6,22\}$ and $\{0,7,19\} \pmod{24}$

$S_5 = (1,1,3,4,5,3,2,4,2,5)$ *yields the pairs* $\{(1,2),(7,9),(3,6),(4,8),(5,10)\}$.

These pairs yield in turn the triples: $\{(1,6,7),(2,12,14),(3,8,11),(4,9,13),(5,10,15)\}$. *These triples yield the base blocks for a* $CPD(24,3,1)$*:*

$\{0,1,7\},\{0,2,14\},\{0,3,11\},\{0,4,13\}$ *and* $\{0,5,15\}$ (mod 24),

$hS_3 = (1,1,2,3,2,*,3)$ *yields the pairs* $\{(1,2),(3,5),(4,7)\}$. *These pairs yield in turn the triples* $\{(1,4,5),(2,6,8),(3,7,10)\}$. *These triples yield the base blocks for a* $CPD(24,3,1)$*:*

$\{0,1,5\},\{0,2,8\}$ *and* $\{0,3,10\}$,(mod 24). *Finally, we add one copy of the base block* $\{0,8,9\}$. *The leave is one circulant graph* $C_{24}\langle 12\rangle$*, which represents a a 1-factor.*

For $v \equiv 12 \pmod{24}$, we will use the union between a $CPD\,(v,3,3)$ obtained in Section 3.4 and hooked skolem type sequences to construct a cyclic packing design. From hooked a Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+3,3,2)$. The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3).(2)} \rfloor - 1$. The leave is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 3\rangle \cup \mathrm{C}_v\langle 3\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and union of two of (three cycles of length $\frac{v}{3}$).

Example 3.6.3 *If* $v = 36$ *then the number of base blocks equal* 28*, we will use a Skolem sequence of order* 5 *and a hooked Skolem sequence of order* 11*.*

$S_5 = (1,1,3,4,5,3,2,4,2,5)$ *yields the pairs* $\{(1,2),(7,9),(3,6),(4,8),(5,10)\}$.

These pairs yield in turn the triples: $\{(1,6,7),(2,12,14),(3,8,11),(4,9,13),(5,10,15)\}$.

These triples yield the base blocks for a $CPD(36,3,1)$*:*

$\{0,1,7\},\{0,2,14\},\{0,3,11\},\{0,4,13\}$ *and* $\{0,5,15\} \pmod{36}$,

Also we need the following sequence:

$hS_{11} = (8,9,2,6,2,1,1,10,8,6,9,11,4,7,5,3,4,10,3,5,7,*,11)$

yields the pairs $\{(6,7),(3,5),(16,19),(13,17),(15,20),(4,10),(14,21),(1,9),(2,11),$ *(8,18),(12,23). These pairs yield in turn the triples*

$\{(1,17,18),(2,14,16),(3,27,30),(4,24,28),(5,26,31),(6,15,21),(7,25,32),$ *(8,12,20),(9,13,22),(10,19,29),(11,23,34)*$\}$.

These triples yield the base blocks for two $CPD(36,3,2)$*:*

1. $\{0,1,18\},\{0,2,14\},\{0,3,30\},\{0,4,28\},\{0,5,31\},\{0,6,21\},\{0,7,32\},\{0,8,20\},\{0,9,22\},$ $\{0,10,29\}$ *and* $\{0,11,34\} \pmod{36}$,

2. $\{0,17,18\},\{0,12,14\},\{0,27,30\},\{0,24,28\},\{0,26,31\},\{0,15,21\},\{0,27,32\},\{0,12,20\},$ $\{0,13,22\},\{0,19,29\}$,*and* $\{0,23,34\} \pmod{36}$.

Finally we add one copy of the base block: $\{0,16,17\}$. *The leave is the union of four* *circulant graphs* $C_{36}\langle 1\rangle \cup C_{36}\langle 3\rangle \cup C_{36}\langle 3\rangle \cup C_{36}\langle 18\rangle$*, which represent the union of a 1-factor, one Hamiltonian cycle and union of 6 cycles of length 12.*

For $v \equiv 6 \pmod{24}$ we will use the sequences from a $CPD(v,3,3)$ obtained in Section 3.4. Also we apply a Skolem-type sequence to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$ is another set of base blocks of a $CPD(3n+3,3,1)$.
The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3).(2)} \rfloor - 1$. The leaves is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, 2 disjoint cycles graph of length $\frac{v}{2}$ and one Hamiltonian cycles.

Example 3.6.4 *Let* $v = 30$. *We will use a Skolem sequence of order* 9 *and a Skolem sequence of order* 4 *as follows:*

$S_9 = (3,7,5,3,9,1,1,5,7,8,6,4,2,9,2,4,6,8)$ *yields the pairs:*$\{(6,7),(13,15),(1,4),(12,16),(3,8),(11,17),(2,9),(10,18),(5,14)\}$. *These pairs yield in turn the triples:* $\{(1,15,16),(2,22,24),(3,10,13),(4,21,25),(5,12,17),(6,20,26),(7,11,18),(8,19,27),(9,14,23)\}$. *These triples yield the base blocks for two* $CPD(30,3,2)$*s:*

1. $\{0,1,16\},\{0,2,24\},\{0,3,13\},\{0,4,25\},\{0,5,17\},\{0,6,26\},\{0,7,18\},$

$\{0,8,27\}$ *and* $\{0,9,23\}$ *(*mod *30),*

2. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},$ $\{0,19,27\}$ *and* $\{0,14,23\}$ *(*mod *30).*

   Also we need the following sequence: $S_4 = (1,1,3,4,2,3,2,4)$ *which yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$.

   These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

   These triples yield the base blocks for a $CPD(30,3,1)$*:*

   $\{0,1,6\},\{0,2,11\},\{0,3,10\}$, *and* $\{0,4,12\}(\text{mod } 30)$. *Finally, add one copy of the base block* $\{0,13,14\}$.

   The leave is the union of four circulant graphs $C_{30}\langle 1\rangle \cup 2C_{30}\langle 2\rangle \cup C_{30}\langle 15\rangle$*, which represent the union of a 1-factor, 2 disjoint cycles graph of length* 15 *and one Hamiltonian cycles.*

For $v \equiv 18 \pmod{24}$ we will use the sequences from a $CPD(v,3,3)$ obtained in Section 3.4 also we apply a Skolem types sequence and one block of the form $\{0, v-2, v-1\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,1)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+3,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v \left\langle \frac{v}{2} \right\rangle$, which represents a a 1-factor.

Example 3.6.5 *Let* $v = 42$. *We will use a Skolem sequence of order* 13*, a hooked Skolem sequence of order* 6 *and one copy of the base blocks* $\{0,1,2\}$*;* $\{0,18,20\}$ *as follows:*

$S_{13} = (5,11,9,7,3,5,13,3,1,1,7,9,11,12,10,8,6,4,2,13,2,4,6,8,10,12)$ *yields the pairs:*

$\{(9,10),(19,21),(5,8),(18,22),(1,6),(17,23),(4,11),(16,24),(3,12),(15,25),(2,13),$ $(14,26),(7,20)\}$. *These pairs yield in turn the triples:*

$\{(1,22,23),(2,32,34),(3,18,21),(4,31,35),(5,14,19),(6,30,36),(7,17,24),(8,29,37),$ $(9,16,25),(10,28,38),(11,15,26),(12,27,39),(13,20,33)\}$.

These triples yield the base blocks for two $CPD(42,3,2)$*:*

1. $\{0,1,23\},\{0,2,34\},\{0,3,21\},\{0,4,35\},\{0,5,19\},\{0,6,36\},\{0,7,24\},$ $\{0,8,37\},\{0,9,25\},\{0,10,38\},\{0,11,26\},\{0,12,39\}$ *and* $\{0,13,33\} \pmod{42}$,

2. $\{0,22,23\},\{0,32,34\},\{0,18,21\},\{0,31,35\},\{0,14,19\},\{0,30,36\},\{0,17,24\},$ $\{0,29,37\},\{0,16,25\},\{0,28,38\},\{0,15,26\},\{0,27,39\}$ *and* $\{0,20,33\} \pmod{42}$.

Also we need a hooked Skolem type sequence of order 6 *as follows:*

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, *, 6)$

yields the pairs $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$.

These pairs yield in turn the triples $\{(1, 7, 8), (2, 9, 11), (3, 14, 17), (4, 12, 16), (5, 10, 15)$ $, (6, 13, 19)$. *These triples yield the base blocks for a* $CPD(42, 3, 1)$*:*

$\{0, 1, 8\}, \{0, 2, 11\}, \{0, 3, 17\}, \{0, 4, 16\}, \{0, 5, 15\}$ *and* $\{0, 6, 19\} \pmod{42}$. *Finally we add one copy of the base blocks* $\{0, 1, 2\}$, $\{0, 18, 20\}$.

The leave is one Circulant Graph $C_{42}\langle 21\rangle$*, which represents a a 1-factor.*

3.6.3 Case 3: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11, 23 \pmod{24}$, we will use the union between a $CPD(v, 3, 4)$, $CPD(v, 3, 1)$ obtained in Section 3.5 and Section 3.2, respectively and we add one base block to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3).(2)} \rfloor$.

For $v \equiv 5 \pmod{24}$ we add $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor\right\}$ to construct a cyclic packing design. The leave in this case is one circulant graph $C_v\langle 1\rangle$, which represents one Hamiltonian cycle.

For $v \equiv 11 \pmod{24}$ we add $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor + 1\right\}$ to construct a cyclic packing design. The leave in this case is one circulant graph $C_v\langle 2\rangle$, which represents one Hamiltonian cycle.

For $v \equiv 23 \pmod{24}$ we add $\left\{0, \lfloor\frac{v}{2}\rfloor - 1, \lfloor\frac{v}{2}\rfloor\right\}$ to construct a cyclic packing design. The leave in this case is one circulant graph $C_v\langle 2\rangle$, which represents one Hamiltonian cycle.

Example 3.6.6 *If $v = 29$ then the number of base blocks equal 23, we will use a Skolem sequence of order 4 and a Skolem sequence of order 9 as in the construction of a $CPD(26, 3, 4)$ and one copy of the base block $\{0, 13, 14\}$ as follows:*

$S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:*

$\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (15, 14)\}$. *These pairs yield in turn the triples:*

$\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$. *These triples yield the base blocks for two $CPD(29, 3, 2)$:*

1. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}, \{0, 8, 27\}$ *and* $\{0, 9, 23\} \pmod{29}$,

2. $\{0, 15, 16\}, \{0, 22, 24\}, \{0, 10, 13\}, \{0, 21, 25\}, \{0, 12, 17\}, \{0, 20, 26\}, \{0, 11, 18\}, \{0, 19, 27\}$ *and* $\{0, 14, 23\} \pmod{29}$.

Also we need a Skolem type sequence of order 4 as follows: $S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$. *These pairs yield in turn the triples* $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$.

These triples yield the base blocks for a $CPD(29, 3, 1)$*:*

$\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\},$ *and* $\{0, 4, 12\}(\text{mod } 29)$.

Finally we add the block $\{0, 13, 14\}$ *so we have* 23 *blocks.*

The leave is one circulant graph of $C_{29}\langle 1\rangle$*, which represents one Hamiltonian cycle.*

For $v \equiv 17 \pmod{24}$, we will use the sequences from a $CPD\,(v, 3, 4)$ obtained in Section 3.5 also we apply the Skolem type sequence and we add the following base blocks: $\left\{0, \lfloor\frac{v}{2}\rfloor - 2, \lfloor\frac{v}{2}\rfloor - 1\right\}$, $\left\{0, \lfloor\frac{v}{2}\rfloor - 4, \lfloor\frac{v}{2}\rfloor - 3\right\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a $CPD(6n + 11, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n + 11, 3, 1)$.
The number of base blocks in this case is $\lfloor\frac{(v-1).(5)}{(3).(2)}\rfloor$. The leave in this case is one circulant graph $C_v\left\langle\lfloor\frac{v}{2}\rfloor\right\rangle$, which represents one Hamiltonian cycle.

3.6.4 Case 4: $v \equiv 4, 10 \pmod{12}$

For $v \equiv 10, 22 \pmod{24}$, we will use the union between a $CPD\,(v, 3, 2)$ and $CPD\,(v, 3, 3)$ obtained in Section 3.3 and Section 3.4, respectively and one block to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1).(5)}{(3).(2)}\rfloor$.

For $v \equiv 10 \pmod{24}$ we add $\{0, 1, 2\}$ to construct a cyclic packing design. The leave is the union of two circulant graphs $C_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles graph of length $\frac{v}{2}$.

For $v \equiv 22 \pmod{24}$ we add $\{0, 2, 4\}$ to construct a cyclic packing design. The leaves is the union of two circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle graph.

Example 3.6.7 *Let* $v = 34$. *We will use a near Skolem sequence of order* 6*; a hooked Skolem sequence of order* 10 *from a* $CPD(34, 3, 2)$ *and a* $CPD(34, 3, 3)$ *and one copy of the base block* $\{0, 1, 2\}$ *as follows:*

$nS_6 = (4, 5, 3, 6, 4, 3, 5, 1, 1, 6)$

yields the pairs $\{(8, 9), (3, 6), (1, 5), (2, 7), (4, 10)\}$. *These pairs yield in turn the triples* $\{(1, 14, 15), (3, 9, 12), (4, 7, 11), (5, 8, 13), (6, 10, 16)\}$.

These triples yield the base blocks for a $CPD(34, 3, 1)$*:*

$\{0, 1, 15\}, \{0, 3, 12\}, \{0, 4, 11\}, \{0, 5, 13\}$ *and* $\{0, 6, 16\} \pmod{34}$.

Also we need a hooked Skolem type sequence of order 10 *as follows:*

$hS_{10} = (10, 8, 6, 4, 2, 9, 2, 4, 6, 8, 10, 5, 3, 7, 9, 3, 5, 1, 1, *, 7)$*; yields the pairs*

$\{(18, 19), (5, 7), (13, 16), (4, 8), (12, 17), (3, 9), (14, 21), (2, 10), (6, 15), (1, 11)\}$.

These pairs yield in turn the triples $\{(1, 28, 29), (2, 15, 17), (3, 23, 26), (4, 14, 18),$ $(5, 22, 27), (6, 13, 19), (7, 24, 31), (8, 12, 20), (9, 16, 25), (10, 11, 21)$.

These triples yield the base blocks for two $CPD(34, 3, 2)$*:*

1. $\{0,1,29\}, \{0,2,17\}, \{0,3,26\}, \{0,4,18\}, \{0,5,27\},$

   $\{0,6,19\}, \{0,7,31\}, \{0,8,20\}, \{0,9,25\}$ *and* $\{0,10,21\} \pmod{34}$*,*

2. $\{0,28,29\}, \{0,15,17\}, \{0,23,26\}, \{0,14,18\}, \{0,22,27\},$

   $\{0,13,19\}, \{0,24,31\}, \{0,12,20\}, \{0,16,25\}$ *and* $\{0,11,21\} \pmod{34}$*.*

Finally we add one copy of the base block $\{0,1,2\}$*.*

The leave is the union of two circulant graphs $C_{34}\langle 4\rangle \cup C_{34}\langle 17\rangle$*, which represent the union of a 1-factor and 2 disjoint cycles graph of length* 17*.*

For $v \equiv 4, 16 \pmod{24}$, we will use the sequences from a $CPD(v, 3, 3)$ obtained in Section 3.4, also we apply the Skolem type sequence to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+1, 3, 2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+1, 3, 2)$.
The number of base blocks in this case is $\lfloor \frac{(v-1).(5)}{(3).(2)} \rfloor$.
For $v \equiv 4 \pmod{24}$ the leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle graph.
For $v \equiv 16 \pmod{24}$ the leave is the union of two circulant graphs $C_v\left\langle \frac{v}{2}-2\right\rangle$

$\cup$ $C_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles graph of length $\frac{v}{2}$.

Example 3.6.8 *Let* $v = 28$. *We will use a Skolem sequence of order* 9, *a Skolem sequence of order* 8 *and a Skolem sequence of order* 4 *as follows:*

$S_9 = (3,7,5,3,9,1,1,5,7,8,6,4,2,9,2,4,6,8)$ *yields the pairs:*$\{(6,7),(13,15),(1,4),$ $(12,16),(3,8),(11,17),(2,9),(10,18),(5,14)\}$. *These pairs yield in turn the triples:*

$\{(1,15,16),(2,22,24),(3,10,13),(4,21,25),(5,12,17),(6,20,26),(7,11,18),$ $(8,19,27),(9,14,23)\}$. *These triples yield the base blocks for a* $CPD(28,3,2)$*:*

$\{0,1,16\},\{0,2,24\},\{0,3,13\},\{0,4,25\},\{0,5,17\},\{0,6,26\},\{0,7,18\},$ $\{0,8,27\}$ *and* $\{0,9,23\}(\text{mod } 28)$,

$S_8 = (8,6,4,2,7,2,4,6,8,3,5,7,3,1,1,5)$ *yields the pairs:*

$\{(14,15),(4,6),(10,13),(3,7),(11,16),(2,8),(5,12),(1,9)\}$. *These pairs yield in turn the triples:*

$\{(1,22,23),(2,12,14),(3,18,21),(4,11,15),(5,19,24),(6,10,16),(7,13,20),(8,9,17)\}$.

These triples yield the base blocks for a $CPD(28,3,2)$*:*

$\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$ *and* $\{0,8,17\}$ $(\text{mod } 28)$. *Also we need a Skolem type sequence of order* 4 *as follows:*

$S_4 = (1,1,3,4,2,3,2,4)$ *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. *These triples yield the base blocks for a* $CPD(28,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}$ (mod 28)*, the leave is the union of two circulant graphs* $C_{28}\langle 13\rangle \cup C_{28}\langle 14\rangle$*, which represent the union of a 1-factor and one Hamiltonian cycle graph.*

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 5$, and $k = 3$.

v (mod 12)	v (mod 24)	Number of blocks	Type of sequence	Leaves
$v \equiv 2,8$	v≡2	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3)$	$2H \cup F$
$v \equiv 2,8$	v≡8	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3)$	$C \cup F$
$v \equiv 2,8$	v≡14	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3)$	$H \cup C \cup F$
$v \equiv 2,8$	v≡20	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3)$	$H \cup C \cup F$
$v \equiv 4,10$	v≡10	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3) \cup \{0,1,2\}$	$C \cup F$
$v \equiv 4,10$	v≡22	$\rho(v,3,5)$	$CPD(v,3,2) \cup CPD(v,3,3) \cup \{0,2,4\}$	$H \cup F$
$v \equiv 4,10$	v≡4	$\rho(v,3,5)$	$CPD(v,3,3) \cup S(3n+1)$	$H \cup F$
$v \equiv 4,10$	v≡16	$\rho(v,3,5)$	$CPD(v,3,3) \cup S(3n+1)$	$C \cup F$
$v \equiv 0,6$	v≡0	$\rho(v,3,5)$	$CPD(v,3,3) \cup R(3n+3) \cup \left\{0, \frac{v}{3}, \frac{v}{3}+1\right\}$	F
$v \equiv 0,6$	v≡12	$\rho(v,3,5)-1$	$CPD(v,3,3) \cup hS(3n+3)$	$H \cup C \cup F$
$v \equiv 0,6$	v≡6	$\rho(v,3,5)-1$	$CPD(v,3,3) \cup S(3n+3)$	$H \cup C \cup F$
$v \equiv 0,6$	v≡18	$\rho(v,3,5)$	$CPD(v,3,3) \cup hS(3n+3) \cup \{0, v-2, v-1\}$	F
$v \equiv 5,11$	v≡5	$\rho(v,3,5)$	$CPD(v,3,4) \cup CPD(v,3,1) \cup \left\{0, \lfloor\frac{v}{2}\rfloor -1, \lfloor\frac{v}{2}\rfloor\right\}$	H
$v \equiv 5,11$	v≡11	$\rho(v,3,5)$	$CPD(v,3,4) \cup CPD(v,3,1) \cup \left\{0, \lfloor\frac{v}{2}\rfloor -1, \lfloor\frac{v}{2}\rfloor +1\right\}$	H
$v \equiv 5,11$	v≡23	$\rho(v,3,5)$	$CPD(v,3,4) \cup CPD(v,3,1) \cup \left\{0, \lfloor\frac{v}{2}\rfloor -1, \lfloor\frac{v}{2}\rfloor\right\}$	H
$v \equiv 5,11$	v≡17	$\rho(v,3,5)$	$CPD(v,3,4) \cup S(6n+11) \cup \left\{0, \lfloor\frac{v}{2}\rfloor -2, \lfloor\frac{v}{2}\rfloor -1\right\} \cup \left\{0, \lfloor\frac{v}{2}\rfloor -4, \lfloor\frac{v}{2}\rfloor -3\right\}$	H

Table 3.6: Constructions of a cyclic packing design $CPD(v,3,5)$

3.6.5 Example of leaves

In this example, we present the leaves of the graph of the cyclic packing design for $\lambda = 5$ and $v = 12$, where the leaves are a 1-factor, one Hamiltonian cycle and two of each of 3 disjoint cycles of length 4, $(H \cup C \cup F)$.

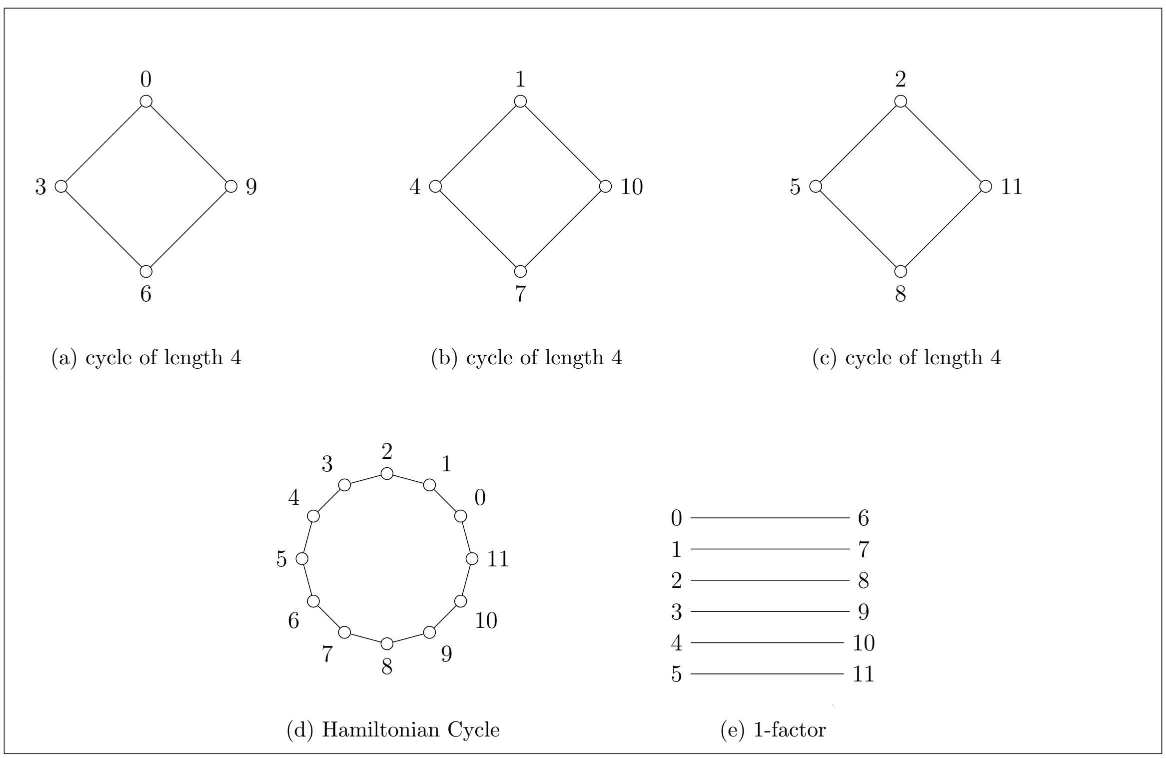


Figure 3.3: The minimum leave of $CPD\,(12, 3, 5)$, $(H \cup C \cup F)$.

3.7 Cyclic Packing Designs for $k = 3$ and $\lambda = 6$, $CPD\,(v, 3, 6)$

3.7.1 Case 1: $v \equiv 2 \pmod{12}$

For $v \equiv 2, 14 \pmod{24}$, we will use the union between a $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 4)$ obtained in Section 3.3 and Section 3.5, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1).(6)}{(3)(2)} \rfloor - 1$. The leave is the union of three circulant graphs $C_v\langle 1 \rangle \cup \mathrm{C}_v\langle 1 \rangle \cup \mathrm{C}_v\langle 1 \rangle$, which represent the union of three Hamiltonian cycles.

Example 3.7.1 *If $v = 26$ then the number of base blocks equal 24, we will take three copies of a Skolem sequence of order 8 as follows:*

$S_8 = (8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$ *yields the pairs:*

$\{(14, 15), (4, 6), (10, 13), (3, 7), (11, 16), (2, 8), (5, 12), (1, 9)\}$.

These pairs yield in turn the triples:

$\{(1, 22, 23), (2, 12, 14), (3, 18, 21), (4, 11, 15), (5, 19, 24), (6, 10, 16), (7, 13, 20), (8, 9, 17)\}$.

These triples yield the base blocks for three $CPD(26, 3, 2)$:

1. $\{0, 1, 23\}, \{0, 2, 14\}, \{0, 3, 21\}, \{0, 4, 15\}, \{0, 5, 24\}, \{0, 6, 16\}, \{0, 7, 20\}$ *and* $\{0, 8, 17\} \pmod{26}$,

2. $\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$ *and* $\{0,8,17\}(\text{mod } 26)$,

3. $\{0,23,24\},\{0,12,14\},\{0,18,21\},\{0,11,15\},\{0,19,24\},\{0,10,16\},\{0,13,20\}$ *and* $\{0,9,17\}(\text{mod } 26)$.

The leave is the union of three circulant graphs $C_{26}\langle 1\rangle \cup C_{26}\langle 1\rangle \cup C_{26}\langle 1\rangle$*, which represent the union of three Hamiltonian cycles.*

3.7.2 Case 2: $v \equiv 6 \pmod{12}$

For $v \equiv 6 \pmod{24}$, we will use the sequences as in the construction of a $CPD(v,3,5)$ obtained in Section 3.6, and apply a Skolem types sequence to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n,3,1)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(6)}{(3)(2)} \rfloor - 1$. The leave is the union of three circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\langle 2\rangle$, which represent the union of one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.7.2 *Let* $v = 30$*. We will use a Skolem sequence of order* 5*, a Skolem* *sequence of order* 9 *and a Skolem sequence of order* 4 *as follows:*

$S_5 = (5, 2, 4, 2, 3, 5, 4, 3, 1, 1)$ *yields the pairs* $\{(9, 10), (2, 4), (5, 8), (3, 7), (1, 6)\}$. *These pairs yield in turn the triples* $\{(1, 14, 15), (2, 7, 9), (3, 10, 13), (4, 8, 12), (5, 6, 11)\}$. *These triples yield the base blocks for a* $CPD(30, 3, 1)$*:*

$\{0, 1, 15\}, \{0, 2, 9\}, \{0, 3, 13\}, \{0, 4, 12\}$ *and* $\{0, 5, 11\}$ (mod 30).

$S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:*$\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (5, 14)\}$. *These pairs yield in turn the triples:* $\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$. *These triples yield the base blocks for two* $CPD(30, 3, 2)$*:*

1. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}, \{0, 8, 27\} and \{0, 9, 23\}$ (mod 30),

2. $\{0, 15, 16\}, \{0, 22, 24\}, \{0, 10, 13\}, \{0, 21, 25\}, \{0, 12, 17\}, \{0, 20, 26\}, \{0, 11, 18\}, \{0, 19, 27\} and \{0, 14, 23\}$ (mod 30).

Also we need the following sequence: $S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$.

These pairs yield in turn the triples $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$.

These triples yield the base blocks for a $CPD(30, 3, 1)$*:*

$\{0, 1, 6\}, \{0, 2, 11\}, \{0, 3, 10\}$ *and* $\{0, 4, 12\}$(mod 30).

Finally, we add one copy of the base block $\{0, 13, 14\}$. *The leave is the union of three circulant graphs* $C_{30}\langle 1\rangle \cup C_{30}\langle 2\rangle \cup C_{30}\langle 2\rangle$, *which represent the union of one Hamiltonian cycle and 2 disjoint cycles of length* 15.

For $v \equiv 18 \pmod{24}$, we will use the union between a $CPD(v, 3, 3)$ and a $CPD(v, 3, 3)$ obtained in Section 3.4 to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1).(6)}{(3).(2)} \rfloor - 1$.

The leave is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of two Hamiltonian cycles and 2-factors.

Example 3.7.3 *If* $v = 42$ *then the number of base blocks* 40 *we will use a Skolem sequence of order* 13, *a hooked Skolem sequence of order* 6 *and two copies of the block* $\{0, 18, 20\}$ *as follows:*

$S_{13} = (5, 11, 9, 7, 3, 5, 13, 3, 1, 1, 7, 9, 11, 12, 10, 8, 6, 4, 2, 13, 2, 4, 6, 8, 10, 12)$ *yields the pairs:* $\{(9, 10), (19, 21), (5, 8), (18, 22), (1, 6), (17, 23), (4, 11), (16, 24), (3, 12), (15, 25), (2, 13), (14, 26), (7, 20)\}$.

These pairs yield in turn the triples: $\{(1, 22, 23), (2, 32, 34), (3, 18, 21), (4, 31, 35), (5, 14, 19), (6, 30, 36), (7, 17, 24), (8, 29, 37), (9, 16, 25), (10, 28, 38), (11, 15, 26), (12, 27, 39), (13, 20, 33)\}$.

These triples yield the base blocks for two $CPD(42, 3, 2)$*:*

1. $\{0, 1, 23\}, \{0, 2, 34\}, \{0, 3, 21\}, \{0, 4, 35\}, \{0, 5, 19\}, \{0, 6, 36\}, \{0, 7, 24\}$,

$\{0,8,37\},\{0,9,25\},0,10,38\},\{0,11,26\},\{0,12,39\}$ *and* $\{0,13,33\} \pmod{42}$,

2. $\{0,22,23\},\{0,32,34\},\{0,18,21\},\{0,31,35\},\{0,14,19\},\{0,30,36\},\{0,17,24\}$, $\{0,29,37\},\{0,16,25\},0,28,38\},\{0,15,26\},\{0,27,39\}$ *and* $\{0,20,33\} \pmod{42}$.

Also we need a hooked Skolem type sequence of order 6 *as follows:*

$hS_6 = (1,1,2,5,2,4,6,3,5,4,3,*,6)$

yields the pairs $\{(1,2),(3,5),(8,11),(6,10),(4,9),(7,13)\}$.

These pairs yield in turn the triples $\{(1,7,8),(2,9,11),(3,14,17),(4,12,16),$ *(5,10,15),(6,13,19). These triples yield the base blocks for a* $CPD(42,3,1)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\}(\text{mod } 42)$.

Finally we add two copies of the block $\{0,18,20\}$*, the leave is the union of four circulant graph* $C_{42}\langle 1\rangle \cup C_{42}\langle 1\rangle \cup C_{42}\langle 21\rangle \cup C_{42}\langle 21\rangle$*, which represent the union of two Hamiltonian cycles and 2-factors.*

3.7.3 Case 3: $v \equiv 10 \pmod{12}$

For $v \equiv 10, 22 \pmod{24}$, we will use the union between a $CPD(v,3,3)$ and a $CPD(v,3,3)$ obtained in Section 3.4 to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1).(6)}{(3).(2)}\rfloor - 1$.

For $v \equiv 10 \pmod{24}$ the leave is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle$

$\cup\ C_v\left\langle\frac{v}{2}\right\rangle\cup C_v\left\langle\frac{v}{2}\right\rangle$, which represent the union of two Hamiltonian cycles and 2-factors. For $v\equiv 22 \pmod{24}$ the leave is the union of four circulant graphs $C_v\langle 2\rangle\cup C_v\langle 2\rangle$ $\cup\ C_v\left\langle\frac{v}{2}\right\rangle\cup C_v\left\langle\frac{v}{2}\right\rangle$, which represent the union of 2 disjoint cycles graph of length $\frac{v}{2}$ and 2-factors.

Example 3.7.4 *If* $v = 34$ *then the number of base blocks equal* 32 *we will use a near Skolem sequence of order* 6, *a hooked Skolem sequence of order* 10 *from* $CPD\,(34,3,3)$ *and one copy of the base block* $\{0,2,4\}$ *as follows:*

$nS_6=(4,5,3,6,4,3,5,1,1,6)$ *yields the pairs* $\{(8,9),(3,6),(1,5),(2,7),(4,10)\}$.

These pairs yield in turn the triples $\{(1,14,15),(3,9,12),(4,7,11),(5,8,13),(6,10,16)\}$.

These triples yield the base blocks for two $CPD(34,3,1)$*:*

1. $\{0,1,15\},\{0,3,12\},\{0,4,11\},\{0,5,13\}$ *and* $\{0,6,16\} \pmod{34}$,

2. $\{0,14,15\},\{0,9,12\},\{0,7,11\},\{0,8,13\}$ *and* $\{0,10,16\} \pmod{34}$.

Also we need a hooked Skolem type sequence of order 10 *as follows:*

$hS_{10}=(10,8,6,4,2,9,2,4,6,8,10,5,3,7,9,3,5,1,1,*,7)$

yields the pairs $\{(18,19),(5,7),(13,16),(4,8),(12,17),(3,9),(14,21),(2,10),(6,15),(1,11)\}$.

These pairs yield in turn the triples $\{(1,28,29),(2,15,17),(3,23,26),(4,14,18),(5,22,27),$ *(6,13,19),(7,24,31),(8,12,20),(9,16,25),(10,11,21). These triples yield the base blocks for two* $CPD(34,3,2)$*:*

1. $\{0,1,29\}, \{0,2,17\}, \{0,3,26\}, \{0,4,18\}, \{0,5,27\}, \{0,6,19\}, \{0,7,31\}$, $\{0,8,20\}, \{0,9,25\}$ *and* $\{0,10,21\} \pmod{34}$

2. $\{0,28,29\}, \{0,15,17\}, \{0,23,26\}, \{0,14,18\}, \{0,22,27\}, \{0,13,19\}, \{0,24,31\}$, $\{0,12,20\}, \{0,16,25\}$ *and* $\{0,11,21\} \pmod{34}$.

Finally we add two copies of the block $\{0,2,4\}$. *The leave is the union of four circulant graphs* $C_{34}\langle 1\rangle \cup C_{34}\langle 1\rangle \cup C_{34}\langle 17\rangle \cup C_{34}\langle 17\rangle$, *which represent the union of two Hamiltonian cycles and 2-factors.*

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 6$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	$Number of blocks$	$Type of sequence$	$Leaves$
$v \equiv 2$	v≡2,14	$\rho(v,3,6)-1$	$CPD(v,3,2) \cup CPD(v,3,4)$	$3H$
$v \equiv 10$	v≡10	$\rho(v,3,6)-1$	$CPD(v,3,3) \cup CPD(v,3,3)$	$2H \cup 2F$
$v \equiv 10$	v≡22	$\rho(v,3,6)-1$	$CPD(v,3,3) \cup CPD(v,3,3)$	$C \cup 2F$
$v \equiv 6$	v≡6	$\rho(v,3,6)-1$	$CPD(v,3,5) \cup S(6n)$	$H \cup C$
$v \equiv 6$	v≡18	$\rho(v,3,6)-1$	$CPD(v,3,3) \cup CPD(v,3,3)$	$2H \cup 2F$

Table 3.7: Constructions of a cyclic packing design $CPD(v,3,6)$

3.8 Cyclic Packing Designs for $k = 3$ and $\lambda = 7$, $CPD(v,3,7)$

3.8.1 Case 1: $v \equiv 2, 8$ (mod 12)

For $v \equiv 2, 20 \pmod{24}$, we will use the union between a $CPD(v,3,5)$ and a $CPD(v,3,2)$ obtained in Section 3.6 and Section 3.3, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor - 1$.

For $v \equiv 2 \pmod{24}$ the leave is the union of four circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and three Hamiltonian cycles.

For $v \equiv 20 \pmod{24}$ the leave is the union of four circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.1 *If $v = 26$ then the number of base blocks equal 28, we will use a Skolem sequence of order 4 as in the construction of a $CPD(26,3,2)$ and a Skolem sequence of order 8 as in the construction of a $CPD(26,3,3)$ as follows:*

$S_4 = (1,1,3,4,2,3,2,4)$ yields the pairs $\{(1,2),(5,7),(3,6),(4,8)\}$. These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for a $CPD(26,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}(\text{mod }26)$. *Also we need a Skolem type sequence of order* 8 *as follows:* $S_8 = (8,6,4,2,7,2,4,6,8,3,5,7,3,1,1,5)$ *yields the pairs:* $\{(14,15),(4,6),(10,13),(3,7),(11,16),(2,8),(5,12),(1,9)\}$.

These pairs yield in turn the triples: $\{(1,22,23),(2,12,14),(3,18,21),(4,11,15),(5,19,24),$ $(6,10,16),(7,13,20),(8,9,17)\}$. *These triples yield the base blocks for three* $CPD(26,3,2)$*:*

1. $\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$ *and* $\{0,8,17\}(\text{mod }26)$,

2. $\{0,1,23\},\{0,2,14\},\{0,3,21\},\{0,4,15\},\{0,5,24\},\{0,6,16\},\{0,7,20\}$ *and* $\{0,8,17\}(\text{mod }26)$,

3. $\{0,23,24\},\{0,12,14\},\{0,18,21\},\{0,11,15\},\{0,19,24\},\{0,10,16\},\{0,13,20\}$ *and* $\{0,9,17\}(\text{mod }26)$.

The leave is the union of four circulant graphs $C_{26}\langle 1\rangle \cup C_{26}\langle 1\rangle \cup C_{26}\langle 1\rangle \cup C_{26}\langle 13\rangle$, *which represent the union of a 1-factor and three Hamiltonian cycles.*

For $v \equiv 14(\text{mod }24)$, we will use a Skolem-type sequence, $3CPD\,(v,3,2)$ obtained in Section 3.3 and one copy of the base blocks: $\left\{0,\frac{v}{2}-3,\frac{v}{2}-2\right\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n + 8, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$ is another set of base blocks of a $CPD(6n + 8, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$. The leave is the union of four circulant graph $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, two Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.2 *If* $v = 38$ *then the number of base blocks equal* 43*, we will use a Skolem sequence of order* 12 *as in the construction of a* $CPD(38, 3, 2)$ *and a Skolem sequence of order* 5 *as follows:* $S_5 = (5, 2, 4, 2, 3, 5, 4, 3, 1, 1)$ *yields the pairs* $\{(9, 10), (2, 4), (5, 8), (3, 7), (1, 6)\}$. *These pairs yield in turn the triples* $\{(1, 14, 15), (2, 7, 9), (3, 10, 13), (4, 8, 12), (5, 6, 11)\}$.*These triples yield the base blocks for a* $CPD(38, 3, 1)$*:*$\{0, 1, 15\}, \{0, 2, 9\}, \{0, 3, 13\}, \{0, 4, 12\}$ *and* $\{0, 5, 11\} \pmod{38}$.

Also we need a Skolem type sequence of order 12 *as follows:*

$S_{12} = (12, 10, 8, 6, 4, 2, 11, 2, 4, 6, 8, 10, 12, 5, 9, 7, 3, 11, 5, 3, 1, 1, 7, 9)$ *yields the pairs:*

$\{(21, 22), (6, 8), (17, 20), (5, 9), (14, 19), (4, 10), (16, 23), (3, 11), (15, 24), (2, 12), (7, 18), (1, 13)\}$.

These pairs yield in turn the triples:

$\{(1,33,34),(2,18,20),(3,29,32),(4,17,21),(5,26,31),(6,16,22),(7,28,35),$

$(8,15,23),(9,27,36),(10,14,24),(11,19,30),(12,13,25)\}.$

These triples yield the base blocks for three $CPD(38,3,2)$*:*

1. $\{0,1,34\},\{0,2,20\},\{0,3,32\},\{0,4,21\},\{0,5,31\},\{0,6,22\},\{0,7,35\},\{0,8,23\},$ $\{0,9,36\},0,10,24\},\{0,11,30\}$ *and* $\{0,12,25\}(\text{mod } 38),$

2. $\{0,33,34\},\{0,18,20\},\{0,29,32\},\{0,17,21\},\{0,26,31\},\{0,16,22\},\{0,28,35\},\{0,15,23\},$ $\{0,27,36\},0,14,24\},\{0,19,30\}$ *and* $\{0,13,25\}(\text{mod } 38),$

3. $\{0,33,34\},\{0,18,20\},\{0,29,32\},\{0,17,21\},\{0,26,31\},\{0,16,22\},\{0,28,35\},\{0,15,23\},$ $\{0,27,36\},0,14,24\},\{0,19,30\}$ *and* $\{0,13,25\}(\text{mod } 38).$

Finally we add the following block: $\{0,16,17\}$*. The leave is the union of four circulant graph* $C_{38}\langle 1\rangle \cup C_{38}\langle 1\rangle \cup (C_{38}\langle 18\rangle) \cup C_{38}\langle 19\rangle$*, which represent the union of a 1-factor, two Hamiltonian cycle and 2 disjoint cycles of length 19.*

For $v \equiv 8 \pmod{24}$, we will use a Skolem-type sequence, $2CPD\,(v,3,3)$ obtained in Section 3.4 and one copy of the base blocks: $\left\{0,\frac{v}{2}-3,\frac{v}{2}-1\right\}$, $\left\{0,\frac{v}{2}-2,\frac{v}{2}\right\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i-a_i=i$, for $1\le i\le n$. The set of all triples (i,a_i+n,b_i+n), for $1\le i\le n$, yield the base blocks for a cyclic packing design $CPD(6n+8,3,1)$: $\{0,a_i+n,b_i+n\}$, $1\le i\le n$, or $\{0,i,b_i+n\}$, $1\le i\le n$, is another set of base blocks of a $CPD(6n+8,3,1)$.

The number of base blocks in this case is $\lfloor\frac{(v-1).(7)}{(3).(2)}\rfloor$. The leave is one circulant graph $C_v\left\langle\frac{v}{2}\right\rangle$, which represents a 1-factor.

3.8.2 Case 2: $v \equiv 0; 6 \pmod{12}$

For $v \equiv 0$ (mod 24), we will use the sequences from a $CPD\,(v,3,3)$ twice obtained in Section 3.4 and a Skolem type sequence to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n,3,1)$.

The number of base blocks in this case is $\lfloor\frac{(v-1)(7)}{(3)(2)}\rfloor$. The leave is the union of three circulant graphs $C_v\left\langle\frac{v}{2}, \frac{v}{2}-4, \frac{v}{2}-4\right\rangle$, which represent the union of a 1-factor and union of $gcd(v,d)$ cycles of length $\frac{v}{gcd(v,d)}$.

Example 3.8.3 *If $v = 24$ then the number of base blocks equal 26, we will use a Skolem sequence of order 5 and a hooked a Skolem sequence of order 3 as in the construction of a $CPD\,(24,3,3)$ and we will use a Skolem sequence of order 4 as follows: $S_4 = (1,1,3,4,2,3,2,4)$ yields the pairs $\{(1,2),(5,7),(3,6),(4,8)\}$. These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. These triples yield the base* *blocks for a* $CPD(24,3,1)$*:* $\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}(\text{mod } 24)$*.*

$S_5 = (1,1,3,4,5,3,2,4,2,5)$ *yields the pairs* $\{(1,2),(7,9),(3,6),(4,8),(5,10)\}$*.*

These pairs yield in turn the triples: $\{(1,6,7),(2,12,14),(3,8,11),(4,9,13),(5,10,15)\}$*. These triples yield the base blocks for two* $CPD(24,3,1)$*:*

1. $\{0,1,7\},\{0,2,14\},\{0,3,11\},\{0,4,13\}$ *and* $\{0,5,15\}$ $(\text{mod } 24)$*,*

2. $\{0,6,7\},\{0,12,14\},\{0,8,11\},\{0,9,13\}$ *and* $\{0,10,15\}$ $(\text{mod } 24)$*.*

$hS_3 = (1,1,2,3,2,*,3)$ *yields the pairs* $\{(1,2),(3,5),(4,7)\}$*. These pairs yield in turn the triples* $\{(1,4,5),(2,6,8),(3,7,10)\}$*. These triples yield the base blocks for two* $CPD(24,3,1)$*:*

1. $\{0,1,5\},\{0,2,8\}$ *and* $\{0,3,10\}$ $(\text{mod } 24)$*,*

2. $\{0,4,5\},\{0,6,8\}$ *and* $\{0,7,10\}$ $(\text{mod } 24)$*.*

Finally, we add the one copy of the base blocks $\{0,5,6\}$*,*$\{0,4,7\}$*,*$\{0,9,11\}$*. The leave is the union of three circulant graphs* $C_{24}\langle 8\rangle \cup C_{24}\langle 8\rangle \cup C_{24}\langle 12\rangle$*, which represent the union of a 1-factor and 8 disjoint cycles of length* 3*.*

For $v \equiv 12 \pmod{24}$, we will use the sequences from a $CPD\,(v,3,3)$ twice obtained in Section 3.4, a Skolem type sequence and we add one copy of the base blocks: $\left\{0, \frac{v}{2}-4, \frac{v}{2}-1\right\}$, $\left\{0, \frac{v}{2}-3, \frac{v}{2}\right\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n+12, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n+12, 3, 1)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$.

The leave is the union of three circulant graphs $C_v \left\langle \frac{v}{2} - 5 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 2 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and union of $gcd(v, d)$ cycles of length $\frac{v}{gcd(v,d)}$.

Example 3.8.4 *If* $v = 36$ *then the number of base blocks equal* 40*, we will use a Skolem sequence of order* 5*,* 4 *and a hooked Skolem sequence of order* 11.

$S_5 = (1, 1, 3, 4, 5, 3, 2, 4, 2, 5)$ *yields the pairs* $\{(1, 2), (7, 9), (3, 6), (4, 8), (5, 10)\}$.

These pairs yield in turn the triples: $\{(1, 6, 7), (2, 12, 14), (3, 8, 11), (4, 9, 13), (5, 10, 15)\}$.

These triples yield the base blocks for two $CPD(36, 3, 1)$*:*

1. $\{0, 1, 7\}, \{0, 2, 14\}, \{0, 3, 11\}, \{0, 4, 13\}$ *and* $\{0, 5, 15\} \pmod{36}$,

2. $\{0, 6, 7\}, \{0, 12, 14\}, \{0, 8, 11\}, \{0, 9, 13\}$ *and* $\{0, 10, 15\} \pmod{36}$.

Also we need the following sequences:

$S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$. *These pairs yield in turn the triples* $\{(1, 5, 6), (2, 9, 11), (3, 7, 10), (4, 8, 12)\}$.

These triples yield the base blocks for a $CPD(36,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}(\text{mod } 36)$.

$hS_{11} = (8,9,2,6,2,1,1,10,8,6,9,11,4,7,5,3,4,10,3,5,7,*,11)$ *yields the pairs*

$\{(6,7),(3,5),(16,19),(13,17),(15,20),(4,10),(14,21),(1,9),(2,11),(8,18),(12,23)\}$.

These pairs yield in turn the triples

$\{(1,17,18),(2,14,16),(3,27,30),(4,24,28),(5,26,31),(6,15,21),(7,25,32),(8,12,20),$ $(9,13,22),(10,19,29),(11,23,34)$. *These triples yield the base blocks for two* $CPD(36,3,2)$*:*

1. $\{0,1,18\},\{0,2,14\},\{0,3,30\},\{0,4,28\},\{0,5,31\},\{0,6,21\},\{0,7,32\},\{0,8,20\},$ $\{0,9,22\},\{0,10,29\}$ *and* $\{0,11,34\}\,(\text{mod } 36)$,

2. $\{0,17,18\},\{0,12,14\},\{0,27,30\},\{0,24,28\},\{0,26,31\},\{0,15,21\},\{0,27,32\},\{0,12,20\},$ $\{0,13,22\},\{0,19,29\}$ *and* $\{0,23,34\}\,(\text{mod } 36)$.

Finally we add one copy of the base blocks:$\{0,16,17\},\{0,16,17\},\{0,14,17\},\{0,15,18\}$

The leave is the union of three circulant graphs $C_{36}\langle 13\rangle \cup C_{36}\langle 16\rangle \cup C_{36}\langle 18\rangle$*, which represent the union of a 1-factor, one Hamiltonian cycle and 4 disjoint cycles of length* 9.

For $v \equiv 6 \pmod{24}$, we will use the sequences as in the construction of a $CPD(v,3,2)$, $CPD(v,3,5)$ obtained in Section 3.3 and Section 3.6, respectively and we add one copy of the base block $\{0, v-2, v-1\}$ to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$.

The leave is the union of three circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.5 *Let* $v = 30$. *We will use a Skolem sequence of order* 9, *a Skolem sequence of order* 4 *as follows:*

$S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:*$\{(6, 7), (13, 15), (1, 4),$ $(12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (5, 14)\}$. *These pairs yield in turn the triples:* $\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18),$ $(8, 19, 27), (9, 14, 23)\}$. *These triples yield the base blocks for three* $CPD(30, 3, 2)$*:*

1. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\},$ $\{0, 8, 27\}$ *and* $\{0, 9, 23\} \pmod{30}$,

2. $\{0, 15, 16\}, \{0, 22, 24\}, \{0, 10, 13\}, \{0, 21, 25\}, \{0, 12, 17\}, \{0, 20, 26\}, \{0, 11, 18\},$ $\{0, 19, 27\}$ *and* $\{0, 14, 23\} \pmod{30}$,

3. $\{0, 15, 16\}, \{0, 22, 24\}, \{0, 10, 13\}, \{0, 21, 25\}, \{0, 12, 17\}, \{0, 20, 26\}, \{0, 11, 18\},$ $\{0, 19, 27\}$ *and* $\{0, 14, 23\} \pmod{30}$.

Also we need the following sequence: $S_4 = (1, 1, 3, 4, 2, 3, 2, 4)$ *yields the pairs* $\{(1, 2), (5, 7), (3, 6), (4, 8)\}$.

These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for a $CPD(30,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}(\text{mod } 30)$. *Finally we add one copy of the base blocks* $\{0,1,2\}$,$\{0,13,14\}$. *The leave is the union of three circulant graphs* $C_{30}\langle 2\rangle$ $\cup$ $C_{30}\langle 2\rangle \cup C_{30}\langle 15\rangle$, *which represent the union of a 1-factor and 2 disjoint cycles of length* 15.

For $v \equiv 18 \pmod{24}$, we will use the sequences as in the construction of a $CPD(v,3,5) \cup$ a $CPD(v,3,2)$ obtained in Section 3.6 and Section 3.3, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1).(7)}{(3).(2)}\rfloor$. The leave is the union of three circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.6 *Let* $v = 42$. *We will use a Skolem sequence of order* 13, *a hooked Skolem sequence of order* 6 *and one copy of the base blocks* $\{0,1,2\}$, $\{0,18,20\}$ *as follows:*

$S_{13} = (5,11,9,7,3,5,13,3,1,1,7,9,11,12,10,8,6,4,2,13,2,4,6,8,10,12)$ *yields the pairs:*

$\{(9,10),(19,21),(5,8),(18,22),(1,6),(17,23),(4,11),(16,24),$

$(3,12),(15,25),(2,13),(14,26),(7,20)\}$. *These pairs yield in turn the triples:*

$\{(1,22,23),(2,32,34),(3,18,21),(4,31,35),(5,14,19),(6,30,36),(7,17,24),$

$(8,29,37),(9,16,25),(10,28,38),(11,15,26),(12,27,39),(13,20,33)\}$.

These triples yield the base blocks for three $CPD(42,3,2)$*:*

1. $\{0,1,23\},\{0,2,34\},\{0,3,21\},\{0,4,35\},\{0,5,19\},\{0,6,36\},\{0,7,24\},$ $\{0,8,37\},\{0,9,25\},0,10,38\},\{0,11,26\}$,{*0,12,39*} *and* $\{0,13,33\}(\text{mod } 42)$,

2. $\{0,22,23\},\{0,32,34\},\{0,18,21\},\{0,31,35\},\{0,14,19\},\{0,30,36\},\{0,17,24\},$ $\{0,29,37\},\{0,16,25\},0,28,38\},\{0,15,26\}$,{*0,27,39*} *and* $\{0,20,33\}(\text{mod } 42)$,

3. $\{0,22,23\},\{0,32,34\},\{0,18,21\},\{0,31,35\},\{0,14,19\},\{0,30,36\},\{0,17,24\},$ $\{0,29,37\},\{0,16,25\},0,28,38\},\{0,15,26\}$,{*0,27,39*} *and* $\{0,20,33\}(\text{mod } 42)$.

Also we need a hooked Skolem type sequence of order 6 *as follows:*

$hS_6 = (1,1,2,5,2,4,6,3,5,4,3,*,6)$

yields the pairs $\{(1,2),(3,5),(8,11),(6,10),(4,9),(7,13)\}$. *These pairs yield in turn the triples* $\{(1,7,8),(2,9,11),(3,14,17),(4,12,16),(5,10,15),(6,13,19)\}$.

These triples yield the base blocks for a $CPD(42,3,1)$*:*

$\{0,1,8\},\{0,2,11\},\{0,3,17\},\{0,4,16\},\{0,5,15\}$ *and* $\{0,6,19\}(\text{mod } 42)$. *Finally, we add one copy of the base blocks* $\{0,1,2\}$, $\{0,18,20\}$.

The leave is the union of three circulant graphs $C_{42}\langle 1\rangle \cup C_{42}\langle 2\rangle \cup C_{42}\langle 21\rangle$, *which* *represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length 21.*

3.8.3 Case 3: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11, 17, 23 \pmod{24}$, we will use the union between a $CPD\,(v, 3, 5)$ and a $CPD\,(v, 3, 2)$ obtained in Section 3.6 and Section 3.3, respectively. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$.

For $v \equiv 5 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle$, which represent the union of two Hamiltonian cycles.

For $v \equiv 11, 23 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 2 \rangle$ $\cup\ 2(C_v \langle 2 \rangle)$, which represent the union of two Hamiltonian cycles.

For $v \equiv 17 \pmod{24}$ the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \lfloor \frac{v}{2} \rfloor \right\rangle$, which represent the union of two Hamiltonian cycles.

Example 3.8.7 *If* $v = 29$ *then the number of base blocks equal* 32*, we will use a Skolem sequence of order* 4 *and a Skolem sequence of order* 9 *as in the construction of a* $CPD\,(26, 3, 2)$ *and one copy of the base block* $\{0, 13, 14\}$ *as follows:*

$S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:*

$\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (15, 14)\}$.

These pairs yield in turn the triples:

$\{(1,15,16),(2,22,24),(3,10,13),(4,21,25),(5,12,17),(6,20,26),(7,11,18),$ $(8,19,27),(9,14,23)\}$. *These triples yield the base blocks for three* $CPD(29,3,2)$*:*

1. $\{0,1,16\},\{0,2,24\},\{0,3,13\},\{0,4,25\},\{0,5,17\},\{0,6,26\},\{0,7,18\},\{0,8,27\}$ *and* $\{0,9,23\} (\text{mod } 29)$,

2. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},\{0,19,27\}$ *and* $\{0,14,23\} (\text{mod } 29)$,

3. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},\{0,19,27\}$ *and* $\{0,14,23\} (\text{mod } 29)$.

Also we need a Skolem type sequence of order 4 *as follows:*

$S_4 = (1,1,3,4,2,3,2,4)$ *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for a $CPD(29,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\}$ $(\text{mod } 29)$. *Finally we add the block* $\{0,13,14\}$, *so we have* 32 *blocks and the leave is the union of two circulant graphs* $C_{29}\langle 1\rangle \cup C_{29}\langle 1\rangle$, *which represent the union of two Hamiltonian cycles.*

3.8.4 Case 4: $v \equiv 4, 10 \pmod{12}$

For $v \equiv 4 \pmod{24}$, we will apply Skolem type sequences of order n three times and Skolem type sequences of order m to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+1, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n + 1, 3, 2)$.

From a Skolem sequence of order m, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq m$. The set of all triples $(i, a_i + m, b_i + m)$, for $1 \leq i \leq m$, yield the base blocks for a cyclic packing design $CPD(6m+4, 3, 1)$: $\{0, a_i + m, b_i + m\}$, $1 \leq i \leq m$, or $\{0, i, b_i + m\}$, $1 \leq i \leq m$, is another set of base blocks of a $CPD(6m + 4, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

For $v \equiv 16 \pmod{24}$, we will apply Skolem type sequences of order n three times and hooked Skolem type sequences of order m to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+1, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or

$\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+1, 3, 1)$.

From a hooked Skolem sequence of order m, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + m, b_i + m)$, for $1 \le i \le m$, yield the base blocks for a cyclic packing design $CPD(6m+4, 3, 1)$: $\{0, a_i + m, b_i + m\}$, $1 \le i \le m$, or $\{0, i, b_i + m\}$, $1 \le i \le m$, is another set of base blocks of a $CPD(6m+4, 3, 1)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} - 2 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.8 *Let* $v = 28$. *We will use a Skolem sequence of order* 9 *and a Skolem sequence of order* 4 *as follows:*

$S_9 = (3, 7, 5, 3, 9, 1, 1, 5, 7, 8, 6, 4, 2, 9, 2, 4, 6, 8)$ *yields the pairs:*$\{(6, 7), (13, 15), (1, 4), (12, 16), (3, 8), (11, 17), (2, 9), (10, 18), (5, 14)\}$. *These pairs yield in turn the triples:* $\{(1, 15, 16), (2, 22, 24), (3, 10, 13), (4, 21, 25), (5, 12, 17), (6, 20, 26), (7, 11, 18), (8, 19, 27), (9, 14, 23)\}$. *These triples yield the base blocks for three* $CPD(28, 3, 2)$*:*

1. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}$, $\{0, 8, 27\}$ *and* $\{0, 9, 23\}$ (mod 28),

2. $\{0, 1, 16\}, \{0, 2, 24\}, \{0, 3, 13\}, \{0, 4, 25\}, \{0, 5, 17\}, \{0, 6, 26\}, \{0, 7, 18\}$, $\{0, 8, 27\}$ *and* $\{0, 9, 23\}$ (mod 28),

3. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},$ $\{0,19,27\}$ *and* $\{0,14,23\} \pmod{28}$.

Also we need a Skolem-type sequence of order 4 *as follows:* $S_4 = (1,1,3,4,2,3,2,4)$ *yields the pairs* $\{(1,2),(5,7),(3,6),(4,8)\}$. *These pairs yield in turn the triples* $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$. *These triples yield the base blocks for a* $CPD(28,3,1)$*:*

$\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\} \pmod{28}$. *The leave is the union of two circulant graphs* $C_{28}\langle 13\rangle \cup C_{28}\langle 14\rangle$*, which represent the union of a 1-factor and one Hamiltonian cycle.*

For $v \equiv 10 \pmod{24}$, we will apply hooked Skolem type sequences of order m three times, near Skolem type sequences of order n and two copies of the base block: $\{0,2,4\}$ to construct cyclic packing designs. From a hooked Skolem sequence of order m, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+m, b_i+m), for $1 \leq i \leq m$, yield the base blocks for a cyclic packing design $CPD(3m+4,3,2)$: $\{0, a_i+m, b_i+m\}$, $1 \leq i \leq m$, or $\{0, i, b_i+m\}$, $1 \leq i \leq m$, is another set of base blocks of a $CPD(3m+4,3,2)$.

From a near Skolem sequence of order n and defect 2, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for

$1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n-2,3,1)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n-2,3,1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$. The leave is the union of five circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, three Hamiltonian cycles and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.8.9 *Let $v=34$. We will use a near Skolem sequence of order 6 as in the construction of a $CPD(34,3,2)$, a hooked Skolem sequence of order 10 as in the construction of a $CPD(34,3,3)$ and one copy of the base block $\{0,1,2\}$ as follows: $nS_6 = (4,5,3,6,4,3,5,1,1,6)$ yields the pairs $\{(8,9),(3,6),(1,5),(2,7),(4,10)\}$. These pairs yield in turn the triples $\{(1,14,15),(3,9,12),(4,7,11),(5,8,13),(6,10,16)\}$.*

These triples yield the base blocks for a $CPD(34,3,1)$:

$\{0,1,15\},\{0,3,12\},\{0,4,11\},\{0,5,13\}$ *and* $\{0,6,16\}$ (mod 34).

Also we need hooked a Skolem type sequence of order 10 as follows:

$hS_{10} = (10,8,6,4,2,9,2,4,6,8,10,5,3,7,9,3,5,1,1,*,7)$*; yields the pairs*

$\{(18,19),(5,7),(13,16),(4,8),(12,17),(3,9),(14,21),(2,10),(6,15),(1,11)\}$.

These pairs yield in turn the triples $\{(1,28,29),(2,15,17),(3,23,26),(4,14,18),(5,22,27),(6,13,19),(7,24,31),(8,12,20),(9,16,25),(10,11,21)$.

These triples yield the base blocks for three $CPD(34,3,2)$:

1. $\{0,1,29\}, \{0,2,17\}, \{0,3,26\}, \{0,4,18\}, \{0,5,27\}, \{0,6,19\}, \{0,7,31\},$
   $\{0,8,20\}, \{0,9,25\}$ *and* $\{0,10,21\} \pmod{34}$,

2. $\{0,28,29\}, \{0,15,17\}, \{0,23,26\}, \{0,14,18\}, \{0,22,27\}, \{0,13,19\}, \{0,24,31\},$
   $\{0,12,20\}, \{0,16,25\}$ *and* $\{0,11,21\} \pmod{34}$,

3. $\{0,28,29\}, \{0,15,17\}, \{0,23,26\}, \{0,14,18\}, \{0,22,27\}, \{0,13,19\}, \{0,24,31\},$
   $\{0,12,20\}, \{0,16,25\}$ *and* $\{0,11,21\} \pmod{34}$.

Finally, add two copies of the block $\{0,2,4\}$. *The leave is the union of five circulant graphs* $C_{34}\langle 1\rangle \cup C_{34}\langle 1\rangle \cup C_{34}\langle 1\rangle \cup C_{34}\langle 4\rangle \cup C_{34}\langle 17\rangle$, *which represent the union of a 1-factor, three Hamiltonian cycles and 2 disjoint cycles of length* 17.

For $v \equiv 22 \pmod{24}$, we will use the sequences from a $CPD(v,3,5)$, $CPD(v,3,2)$ obtained in Section 3.5 and Section 3.3, respectively and one copy of the base block: $\{0, v-2, v-1\}$. The number of base blocks in this case is $\lfloor \frac{(v-1)(7)}{(3)(2)} \rfloor$.
The leave is the union of two circulant graphs $C_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 7$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	Number of blocks	Type of sequence	Leaves
$v \equiv 2, 8$	v≡2	$\rho(v,3,7) - 1$	$CPD(v,3,2) \cup CPD(v,3,5)$	$3H \cup F$
$v \equiv 2, 8$	v≡20	$\rho(v,3,7) - 1$	$CPD(v,3,2) \cup CPD(v,3,5)$	$C \cup H \cup F$
$v \equiv 2, 8$	v≡8	$\rho(v,3,7)$	$2CPD(v,3,3) \cup S(6n+8) \cup \left\{0, \frac{v}{2} - 3, \frac{v}{2} - 1\right\} \cup \left\{0, \frac{v}{2} - 2, \frac{v}{2}\right\}$	F
$v \equiv 2, 8$	v≡14	$\rho(v,3,7)$	$3CPD(v,3,2) \cup \left\{0, \frac{v}{2} - 6, \frac{v}{2} - 5\right\} \cup \left\{0, \frac{v}{2} - 4, \frac{v}{2} - 3\right\} \cup \left\{0, \frac{v}{2} - 2, \frac{v}{2}\right\}$	$C \cup\ 2H \cup\ F$
$v \equiv 4, 10$	v≡4	$\rho(v,3,7)$	$3S(3n+1) \cup S(6n+4)$	$H \cup F$
$v \equiv 4, 10$	v≡16	$\rho(v,3,7)$	$3S(3n+1) \cup hS(6n+4)$	$C \cup F$
$v \equiv 4, 10$	v≡10	$\rho(v,3,7)$	$3S(3n+4) \cup nS(6n-2) \cup \{0,2,4\} \cup \{0,2,4\}$	$3H \cup C \cup F$
$v \equiv 4, 10$	v≡22	$\rho(v,3,7)$	$CPD(v,3,2) \cup CPD(v,3,5) \cup \{0, v-2, v-1\}$	$C \cup F$
$v \equiv 0, 6$	v≡0	$\rho(v,3,7)$	$2CPD(v,3,3) \cup S(6n)$	$C \cup F$
$v \equiv 0, 6$	v≡12	$\rho(v,3,7)$	$2CPD(v,3,3) \cup S(6n+12)$	$C \cup H \cup F$
$v \equiv 0, 6$	v≡6	$\rho(v,3,7)$	$CPD(v,3,2) \cup CPD(v,3,5) \cup \{0, v-2, v-1\}$	$C \cup F$
$v \equiv 0, 6$	v≡18	$\rho(v,3,7)$	$CPD(v,3,2) \cup CPD(v,3,5)$	$C \cup H \cup F$
$v \equiv 5, 11$	v≡5	$\rho(v,3,7)$	$CPD(v,3,5) \cup CPD(v,3,2)$	$2H$
$v \equiv 5, 11$	v≡11,23	$\rho(v,3,7)$	$CPD(v,3,5) \cup CPD(v,3,2)$	$2H$
$v \equiv 5, 11$	v≡17	$\rho(v,3,7)$	$CPD(v,3,5) \cup CPD(v,3,2)$	$2H$

Table 3.8: Constructions of a cyclic packing design $CPD(v,3,7)$

3.9 Cyclic Packing Designs for $k = 3$ and $\lambda = 8$, $CPD\,(v, 3, 8)$

3.9.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2 \pmod{24}$, we will use three copies of $CPD\,(v, 3, 2)$ obtained in Section 3.3, near Skolem-type sequences, and one copy of the following base block: $\{0, v-2, v-1\}$, $\{0, v-4, v-3\}$, $\{0, v-6, v-5\}$ to construct a cyclic packing design. From a near Skolem sequence of order n and defect 2, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n + 5, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n + 5, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(8)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v\,\langle 2 \rangle$, which represent 2 disjoint cycles graph of length $\frac{v}{2}$.

For $v \equiv 8 \pmod{24}$, we will use three copies of a $CPD\,(v, 3, 2)$, one copy of a $CPD\,(v, 3, 1)$ obtained in Section 3.3 and Section 3.2, respectively, Skolem-type sequences and one copy of the base blocks: $\left\{0, \frac{v}{2} - 3, \frac{v}{2} - 1\right\}$, $\left\{0, \frac{v}{2} - 2, \frac{v}{2}\right\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n + 8, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n + 8, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(8)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v \langle 2 \rangle$, which represent 2 disjoint cycles graph of length $\frac{v}{2}$.

For $v \equiv 14$, (mod 24), we will apply hooked Skolem-type sequences of order n and $CPD\,(v, 3, 7)$ obtained in Section 3.8 to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n + 2, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n + 2, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(8)}{(3)(2)} \rfloor$. The leave is the union of four circulant graph $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle$, which represent the union two Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.9.1 *If* $v = 38$ *then the number of base blocks equal* 49*, we will use a Skolem sequence of order* 4 *and a Skolem sequence of order* 12 *as in the construction of a* $CPD\,(38, 3, 7)$ *as follows:* $S_5 = (5, 2, 4, 2, 3, 5, 4, 3, 1, 1)$

yields the pairs $\{(9,10),(2,4),(5,8),(3,7),(1,6)\}$. *These pairs yield in turn the triples* $\{(1,14,15),(2,7,9),(3,10,13),(4,8,12),(5,6,11)\}$. *These triples yield the base blocks for a* $CPD(38,3,1)$:$\{0,1,15\},\{0,2,9\},\{0,3,13\},\{0,4,12\}$ *and* $\{0,5,11\}\ (\text{mod } 38)$.

Also we need a Skolem type sequence of order 12 *as follows:*

$S_{12} = (12,10,8,6,4,2,11,2,4,6,8,10,12,5,9,7,3,11,5,3,1,1,7,9)$

yields the pairs: $\{(21,22),(6,8),(17,20),(5,9),(14,19),(4,10),(16,23),(3,11),$ *(15,24),(2,12),(7,18),(1,13)*$\}$.

These pairs yield in turn the triples: $\{(1,33,34),(2,18,20),(3,29,32),(4,17,21),(5,26,31),(6,16,22),(7,28,35),(8,15,23),(9,27,36),(10,14,24),(11,19,30),(12,13,25)\}$.

These triples yield the base blocks for three $CPD(38,3,2)$*:*

1. $\{0,1,34\},\{0,2,20\},\{0,3,32\},\{0,4,21\},\{0,5,31\},\{0,6,22\},\{0,7,35\},\{0,8,23\},$ $\{0,9,36\},\{0,10,24\},\{0,11,30\}$ *and* $\{0,12,25\}$ (mod 38),

2. $\{0,33,34\},\{0,18,20\},\{0,29,32\},\{0,17,21\},\{0,26,31\},\{0,16,22\},\{0,28,35\},\{0,15,23\},$ $\{0,27,36\},0,14,24\},\{0,19,30\}$ *and* $\{0,13,25\}$ (mod 38),

3. $\{0,33,34\},\{0,18,20\},\{0,29,32\},\{0,17,21\},\{0,26,31\},\{0,16,22\},\{0,28,35\},\{0,15,23\},$ $\{0,27,36\},0,14,24\},\{0,19,30\}$ *and* $\{0,13,25\}$ (mod 38).

Also we need a hooked Skolem type sequence of order 6 *as follows:*

$hS_6 = (1, 1, 2, 5, 2, 4, 6, 3, 5, 4, 3, *, 6)$

yields the pairs $\{(1, 2), (3, 5), (8, 11), (6, 10), (4, 9), (7, 13)\}$. *These pairs yield in turn the triples* $\{(1, 7, 8), (2, 9, 11), (3, 14, 17), (4, 12, 16), (5, 10, 15), (6, 13, 19)\}$.

These triples yield the base blocks for a $CPD(38, 3, 1)$*:*

$\{0, 1, 8\}, \{0, 2, 11\}, \{0, 3, 17\}, \{0, 4, 16\}, \{0, 5, 15\}$ *and* $\{0, 6, 19\} \pmod{38}$. *Finally we add the following base block:*$\{0, 16, 17\}$.

The leave is the union of four circulant graph $C_{38}\langle 1\rangle \cup C_{38}\langle 1\rangle \cup C_{38}\langle 18\rangle \cup C_{38}\langle 18\rangle$*, which represent the union of two Hamiltonian cycle and 2 disjoint cycles of length 19.*

For $v \equiv 20 \pmod{24}$, we will apply a hooked Skolem-type sequence of order n and we add one base block of the form $\left\{0, \frac{v}{2} - 1, \frac{v}{2} + 1\right\}$ and a $CPD(v, 3, 7)$ obtained in Section 3.8 to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n + 2, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n + 2, 3, 1)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(8)}{(3)(2)} \rfloor$. The leave is one circulant graphs $C_v\langle 2\rangle$, which represent 2 disjoint cycles graphs of length $\frac{v}{2}$.

3.9.2 Case 2: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11, 17, 23 \pmod{24}$, we will use the union of a $CPD\,(v, 3, 5)$, $CPD\,(v, 3, 2)$ and a $CPD\,(v, 3, 1)$ obtained in Section 3.6, Section 3.3 and Section 3.2, respectively to construct cyclic packing designs. The number of base blocks in this case is $\lfloor \frac{(v-1)(8)}{(3)(2)} \rfloor$.

For $v \equiv 5 \pmod{24}$, we add $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor \right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 1 \rangle$, which represents one Hamiltonian cycle graph.

For $v \equiv 11 \pmod{24}$, we add $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor + 1 \right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 2 \rangle$, which represents one Hamiltonian cycle graph.

For $v \equiv 17 \pmod{24}$, we add $\left\{0, \lfloor \frac{v}{2} \rfloor, \lfloor \frac{v}{2} \rfloor + 1 \right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \left\langle \lfloor \frac{v}{2} - 2 \rfloor \right\rangle$, which represents one Hamiltonian cycle graph.

For $v \equiv 23 \pmod{24}$, we add $\left\{0, \lfloor \frac{v}{2} \rfloor - 2, \lfloor \frac{v}{2} \rfloor \right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 2 \rangle$, which represents one Hamiltonian cycle graph.

Example 3.9.2 *If $v = 29$ then the number of base blocks equal 37, we will use a Skolem sequence of order 4, a Skolem sequence of order 9 and one copy of the base block $\{0, 13, 14\}$ as follows:*

$S_9 = (3,7,5,3,9,1,1,5,7,8,6,4,2,9,2,4,6,8)$, yields the pairs:

$\{(6,7),(13,15),(1,4),(12,16),(3,8),(11,17),(2,9),(10,18),(15,14)\}$.

These pairs yield in turn the triples:

$\{(1,15,16),(2,22,24),(3,10,13),(4,21,25),(5,12,17),(6,20,26),(7,11,18),(8,19,27),(9,14,23)\}$.

These triples yield the base blocks for three $CPD(29,3,2)$:

1. $\{0,1,16\},\{0,2,24\},\{0,3,13\},\{0,4,25\},\{0,5,17\},\{0,6,26\},\{0,7,18\},\{0,8,27\}$ *and* $\{0,9,23\} \pmod{29}$,
2. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},\{0,19,27\}$ *and* $\{0,14,23\} \pmod{29}$,
3. $\{0,15,16\},\{0,22,24\},\{0,10,13\},\{0,21,25\},\{0,12,17\},\{0,20,26\},\{0,11,18\},\{0,19,27\}$ *and* $\{0,14,23\} \pmod{29}$.

Also we need a Skolem type sequence of order 4 as follows: $S_4 = (1,1,3,4,2,3,2,4)$, yields the pairs $\{(1,2),(5,7),(3,6),(4,8)\}$.

These pairs yield in turn the triples $\{(1,5,6),(2,9,11),(3,7,10),(4,8,12)\}$.

These triples yield the base blocks for two $CPD(29,3,1)$:

1. $\{0,1,6\},\{0,2,11\},\{0,3,10\}$ *and* $\{0,4,12\} \pmod{29}$,
2. $\{0,5,6\},\{0,9,11\},\{0,7,10\}$ *and* $\{0,8,12\} \pmod{29}$.

Finally we add the block $\{0, 13, 14\}$ *so, we have* 37 *blocks and the leave is one circulant graph of* $C_{29}\langle 1\rangle$*, which represents one Hamiltonian cycle graph.*

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 8$, and $k = 3$.

v (mod 12)	v (mod 24)	Number of blocks	Type of sequence	Leaves
$v \equiv 2, 8$	v≡2	$\rho(v, 3, 8)$	$3CPD(v,3,2) \cup nS(3n+5) \cup \{0, v-2, v-1\} \cup \{0, v-4, v-3\} \cup \{0, v-6, v-5\}$	C
$v \equiv 2, 8$	v≡8	$\rho(v, 3, 8)$	$3CPD(v,3,2) \cup CPD(v,3,1) \cup S(6n+8) \cup \left\{0, \frac{v}{2}-3, \frac{v}{2}-1\right\} \cup \left\{0, \frac{v}{2}-2, \frac{v}{2}\right\}$	C
$v \equiv 2, 8$	v≡14	$\rho(v, 3, 8)$	$CPD(v,3,7) \cup hS(6n+2)$	C
$v \equiv 2, 8$	v≡20	$\rho(v, 3, 8)$	$CPD(v,3,7) \cup hS(6n+2)$	$2H \cup C$
$v \equiv 5, 11$	v≡5	$\rho(v, 3, 8)$	$CPD(v,3,5) \cup CPD(v,3,2) \cup CPD(v,3,1)$	H
$v \equiv 5, 11$	v≡11,23	$\rho(v, 3, 8)$	$CPD(v,3,5) \cup CPD(v,3,2) \cup CPD(v,3,1)$	H
$v \equiv 5, 11$	v≡17	$\rho(v, 3, 8)$	$CPD(v,3,5) \cup CPD(v,3,2) \cup CPD(v,3,1) \cup \left\{0, \lfloor\frac{v}{2}\rfloor, \lfloor\frac{v}{2}\rfloor+1\right\}$	H

Table 3.9: Constructions of a cyclic packing design $CPD(v, 3, 8)$

3.10 Cyclic Packing Designs for $k = 3$ and $\lambda = 9$, $CPD(v, 3, 9)$

3.10.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8 \pmod{24}$, we will use the sequences from a $CPD(v, 3, 8)$ and a $CPD(v, 3, 1)$ obtained in Section 3.9 and Section 3.2, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(9)}{(3)(2)} \rfloor$.
The leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$

Example 3.10.1 *Let $v = 26$. We have 33 blocks from a $CPD(v, 3, 8)$ and 4 blocks from a $CPD(v, 3, 1)$, then totally we have 37 blocks.*
The leave is the union of two circulant graphs $C_{26} \langle 2 \rangle \cup C_{26} \langle 13 \rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length 13.

For $v \equiv 14 \pmod{24}$, we will use the sequences from a $CPD(v, 3, 7)$ union a $CPD(v, 3, 2)$ obtained in Section 3.8 and Section 3.3, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(9)}{(3)(2)} \rfloor$.
The leave is the union of five circulant graph $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle$ $\cup\ \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, two Hamiltonian cycle and 2 dis-

joint cycles of length $\frac{v}{2}$.

For $v \equiv 20 \pmod{24}$, we will use the sequences from a $CPD\,(v,3,8)$ and a $CPD\,(v,3,1)$ obtained in Section 3.9 and Section 3.2, respectively and one copy of the base block: $\left\{0, \frac{v}{2}-4, \frac{v}{2}-2\right\}$ to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1)(9)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}\right\rangle$, which represent the union of a 1-factor and Hamiltonian cycle.

3.10.2 Case 2: $v \equiv 4, 10$ (mod 12)

For $v \equiv 4, 16 \pmod{24}$, we will use the sequences from $CPD\,(v,3,7)$ obtained in Section 3.8 and a Skolem-type sequence of order n to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+1,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+1,3,2)$.

The number of base blocks in this case is $\lfloor\frac{(v-1)(9)}{(3)(2)}\rfloor$.

For $v \equiv 4 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}-1\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle

For $v \equiv 16 \pmod{24}$, the leave is the union of two circulant graphs $C_v\left\langle\frac{v}{2}-2\right\rangle \cup \mathrm{C}_v\left\langle\frac{v}{2}\right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.10.2 *Let* $v = 28$. *We have 31 blocks from a* $CPD\,(28,3,7)$ *and from a Skolem sequence we have 9 blocks then totally we have 40 blocks.*

The leave is the union of two circulant graphs $C_{28}\langle 13\rangle \cup C_{28}\langle 14\rangle$, *which represent the union of a 1-factor and one Hamiltonian cycle.*

For $v \equiv 10 \pmod{24}$, we will use the sequences from a $CPD\,(v,3,7)$ and $CPD\,(v,3,2)$ obtained in Section 3.8 and Section 3.3, respectively and one copy of the base block: $\{0, v-2, v-1\}$. The number of base blocks in this case is $\lfloor \frac{(v-1)(9)}{(3)(2)} \rfloor$. The leave is the union of five circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, 2 Hamiltonian cycles and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.10.3 *Let* $v = 34$. *We have 38 blocks from a* $CPD\,(34,3,7)$, *10 blocks from a* $CPD\,(34,3,2)$ *and we add the block* $\{0,1,2\}$ *then totally we have 49 block.*

The leave is the union of five circulant graphs $C_{34}\langle 1\rangle \cup C_{34}\langle 1\rangle \cup C_{34}\langle 4\rangle \cup C_{34}\langle 4\rangle \cup C_{34}\langle 17\rangle$, *which represent the union of a 1-factor, 2 Hamiltonian cycles and 2 disjoint cycles of length* 17.

For $v \equiv 22 \pmod{24}$, we will use the sequences from a $CPD\,(v,3,6)$, $CPD\,(v,3,2)$, and a $CPD\,(v,3,1)$ obtained in Section 3.7, Section 3.3 and Section 3.2, respectively and one copy of the following base block: $\{0,2,4\}$ and $\left\{0, \frac{v}{2}-2, \frac{v}{2}\right\}$. The number of base blocks in this case is $\lfloor\frac{(v-1)(9)}{(3)(2)}\rfloor$. The leave is the union of two circulant graphs $C_v\langle 1\rangle \cup C_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

3.10.3 Case 3: $v \equiv 0, 6 \pmod{12}$

For $v \equiv 0 \pmod{24}$, we will take three copies of a $CPD(v, 3, 3)$ obtained in Section 3.4 and one copy of the base block $\left\{0, \frac{v}{3}, v - \frac{v}{3}\right\}$ to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1)(9)}{(3)(2)}\rfloor$. The leave is the union of three circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup C_v\left\langle \frac{v}{2}\right\rangle \cup C_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of 3-factors.

For $v \equiv 6 \pmod{24}$, we will take three copies of a $CPD(v, 3, 3)$ obtained in Section 3.4 and one copy of the base block $\{0, 2, v - 2\}$ to construct a cyclic packing design. The number of base blocks in this case is $\lfloor\frac{(v-1)(9)}{(3)(2)}\rfloor$.

The leave is the union of three circulant graphs $C_v\left\langle \frac{v}{2}\right\rangle \cup C_v\left\langle \frac{v}{2}\right\rangle \cup C_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of 3-factors.

Example 3.10.4 *Let $v = 30$. We have 42 blocks from $CPD(30, 3, 3)$ and we add the block $\{0, 10, 20\}$ then totally we have 43 block.*

The leave is the union of three circulant graphs $C_{30}\langle 15\rangle \cup C_{30}\langle 15\rangle \cup C_{30}\langle 15\rangle$, which represent the union of 3-factors.

For $v \equiv 12 \pmod{24}$, we will apply hooked a Skolem type sequence of order n, a $CPD(v, 3, 7)$ obtained in Section 3.8 and one copy of the base block of the form $\left\{0, \frac{v}{2}-5, \frac{v}{2}-2\right\}$ to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+3, 3, 2)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n+3, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(9)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

For $v \equiv 18 \pmod{24}$, we will apply a Skolem type sequence of order n, a $CPD\,(v, 3, 7)$ obtained in Section 3.8, and one copy of the base block of the form $\{0, v-2, v-1\}$ to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples (i, a_i+n, b_i+n), for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+3, 3, 2)$: $\{0, a_i+n, b_i+n\}$, $1 \leq i \leq n$, or $\{0, i, b_i+n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n+3, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(9)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.10.5 *Let $v = 42$. We have 47 block from a $CPD\,(42, 3, 7)$, 13 blocks* *from a Skolem sequence of order* 13 *and we add one block* $\{0,1,2\}$ *then totally we have 61 blocks. The leave is the union of two circulant graphs* $C_{42}\langle 2\rangle \cup C_{42}\langle 21\rangle$, *which represent the union of a 1-factor and 2 disjoint cycles of length* 21.

In this table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 9$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	Number of blocks	Type of sequence	Leaves
$v \equiv 2,8$	v≡2,8	$\rho(v,3,9)$	$CPD(v,3,8) \cup CPD(v,3,1)$	$C \cup F$
$v \equiv 2,8$	v≡14	$\rho(v,3,9)$	$CPD(v,3,7) \cup CPD(v,3,2)$	$3H \cup C \cup F$
$v \equiv 2,8$	v≡20	$\rho(v,3,9)$	$CPD(v,3,8) \cup CPD(v,3,1) \cup \{0,\frac{v}{2}-4,\frac{v}{2}-2\}$	$H \cup F$
$v \equiv 4,10$	v≡4	$\rho(v,3,9)$	$CPD(v,3,7) \cup S(3n+1)$	$H \cup F$
$v \equiv 4,10$	v≡16	$\rho(v,3,9)$	$CPD(v,3,7) \cup S(3n+1)$	$C \cup F$
$v \equiv 4,10$	v≡10	$\rho(v,3,9)$	$CPD(v,3,7) \cup S(3n+1) \cup \{0,v-2,v-1\}$	$2H \cup C \cup F$
$v \equiv 4,10$	v≡22	$\rho(v,3,9)$	$CPD(v,3,6) \cup CPD(v,3,2) \cup CPD(v,3,1) \cup \{0,2,4\} \cup \{0,\frac{v}{2}-2,\frac{v}{2}\}$	$H \cup F$
$v \equiv 0,6$	v≡0,6	$\rho(v,3,9)$	$3CPD(v,3,3) \cup \{0,\frac{v}{3},v-\frac{v}{3}\}$	$3F$
$v \equiv 0,6$	v≡12	$\rho(v,3,9)$	$CPD(v,3,7) \cup hS(3n+3) \cup \{0,\frac{v}{2}-5,\frac{v}{2}-2\}$	$H \cup F$
$v \equiv 0,6$	v≡18	$\rho(v,3,9)$	$CPD(v,3,7) \cup S(3n+3) \cup \{0,v-2,v-1\}$	$C \cup F$

Table 3.10: Constructions of a cyclic packing design $CPD(v,3,9)$

3.10.4 Example of leave

In this example, we present the minimum leave of the cyclic packing design for $\lambda = 9$ and $v = 28$, where the leaves are a 1-factor and one Hamiltonian cycle. $(H \cup F)$.

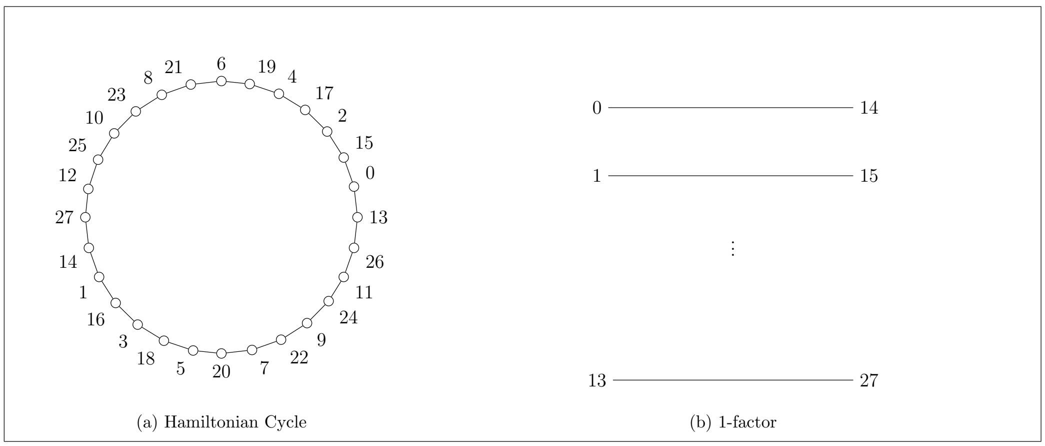


Figure 3.4: The minimum leave of $CPD\,(28, 3, 9)$, $(H \cup F)$

3.11 Cyclic Packing Designs for $k = 3$ and $\lambda = 10$, $CPD\,(v, 3, 10)$

3.11.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8, 14, 20 \pmod{24}$, we will use the sequences from a $CPD\,(v, 3, 8)$ and a $CPD\,(v, 3, 2)$ obtained in Section 3.9 and Section 3.3, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(10)}{(3)(2)} \rfloor$.

For $v \equiv 2 \pmod{24}$, the leave is the union of two circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 2 \rangle$, which represent the union of one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 8 \pmod{24}$, the leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle$, which represent the union of 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 14 \pmod{24}$, the leave is the union of five circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle$, which represent the union of three Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 20 \pmod{24}$, the leave is the union of two circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle$, which represent the union of 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.11.1 *Let $v = 38$. We have 49 blocks from a $CPD\,(v, 3, 8)$ and from a $CPD\,(v, 3, 2)$ we have 12 blocks then totally we have 61 block.*

The leave is the union of five circulant graphs $C_{38}\langle 1\rangle \cup C_{38}\langle 1\rangle \cup C_{38}\langle 1\rangle \cup C_{38}\langle 18\rangle$ $\cup\ C_{38}\langle 18\rangle$, *which represent the union of three Hamiltonian cycle and 2 disjoint cycles of length* 19.

3.11.2 Case 2: $v \equiv 10$ (mod 12)

For $v \equiv 10, 22 \pmod{24}$, we will use the sequences from a $CPD(v, 3, 5)$ twice obtained in Section 3.6 to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(10)}{(3)(2)} \rfloor - 1$.

For $v \equiv 10 \pmod{24}$, the leave is the union of four circulant graphs $C_v\langle 4\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of 2-factors and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 22 \pmod{24}$, the leaves is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of two Hamiltonian cycles and 2-factors.

Example 3.11.2 *Let* $v = 46$. *We will use a sequences as in the construction of a* $CPD(v, 3, 5)$ *twice. So, we have 74 block.*

The leave is the union of four circulant graphs $C_{46}\langle 1\rangle \cup C_{46}\langle 1\rangle \cup C_{46}\langle 23\rangle \cup C_{46}\langle 23\rangle$, *which represent the union of two Hamiltonian cycles and 2-factors.*

3.11.3 Case 3: $v \equiv 6 \pmod{12}$

For $v \equiv 6 \pmod{24}$, we will apply Skolem-type sequences and the sequences from a $CPD\,(v, 3, 9)$ obtained in Section 3.10 to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n, 3, 1)$: $\{0, a_i + n, b_i + n\}$, $1 \le i \le n$, or $\{0, i, b_i + n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(6n, 3, 1)$.
The number of base blocks in this case is $\lfloor \frac{(v-1)(10)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of 2-factors.

Example 3.11.3 *For $v = 30$ we have 43 blocks from a $CPD\,(v, 3, 9)$ and we have 5 blocks from a Skolem type sequence of order 5. So, we have 48 block.*
The leave is the union of two circulant graphs $C_{30}\,\langle 15 \rangle \cup C_{30}\,\langle 15 \rangle$, which represent the union of 2-factors.

For $v \equiv 18 \pmod{24}$, we will use the sequences from a $CPD\,(v, 3, 7)$, a $CPD\,(v, 3, 3)$ obtained in Section 3.8 and Section 3.4, respectively and one copy of the base block $\{0, v-2, v-1\}$ to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(10)}{(3)(2)} \rfloor$. The leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of 2-factors.

Example 3.11.4 *For $v = 42$ we have 47 blocks from a $CPD\,(v, 3, 7)$, we have 20 blocks from a $CPD\,(v, 3, 3)$ and $\{0, 1, 2\}$ then totally we have 61 block.*

The leave is the union of two circulant graphs $C_{42}\,\langle 21\rangle \cup C_{42}\,\langle 21\rangle$, which represent the union of 2-factors.

3.11.4 Case 4: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11, 17, 23 \pmod{24}$, we will use the sequences as in the construction of a $CPD\,(v, 3, 5)$ twice obtained in Section 3.6 to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(10)}{(3)(2)} \rfloor$.

For $v \equiv 5 \pmod{24}$, the leave is the union of two circulant graphs $C_v\,\langle 1\rangle \cup \mathrm{C}_v\,\langle 1\rangle$, which represent the union of two Hamiltonian cycles.

For $v \equiv 11, 23 \pmod{24}$, the leave is the union of two circulant graphs $C_v\,\langle 2\rangle \cup \mathrm{C}_v\,\langle 2\rangle$, which represent the union of two Hamiltonian cycles.

For $v \equiv 17 \pmod{24}$, the leave is the union of two circulant graphs $C_v \left\langle \lfloor \frac{v}{2} \rfloor \right\rangle \cup \mathrm{C}_v \left\langle \lfloor \frac{v}{2} \rfloor \right\rangle$, which represent the union of two Hamiltonian cycles.

Example 3.11.5 *Let $v = 41$. We will use a sequences as in the construction of a $CPD\,(v, 3, 5)$ twice. So, we have 66 blocks. The leave is the union of two circulant graphs $C_{41}\,\langle 20\rangle \cup C_{41}\,\langle 20\rangle$, which represent the union of two Hamiltonian cycles.*

In the following table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 10$, and $k = 3$.

v (mod 12)	v (mod 24)	$Number of blocks$	$Type of sequence$	$Leaves$
$v \equiv 2, 8$	v≡2	$\rho(v, 3, 10)$	$CPD(v, 3, 8) \cup CPD(v, 3, 2)$	$C \cup H$
$v \equiv 2, 8$	v≡8	$\rho(v, 3, 10)$	$CPD(v, 3, 8) \cup CPD(v, 3, 2)$	C
$v \equiv 2, 8$	v≡14	$\rho(v, 3, 10)$	$CPD(v, 3, 8) \cup CPD(v, 3, 2)$	$C \cup 3H$
$v \equiv 2, 8$	v≡20	$\rho(v, 3, 10)$	$CPD(v, 3, 8) \cup CPD(v, 3, 2)$	C
$v \equiv 10$	v≡10	$\rho(v, 3, 10) - 1$	$2CPD(v, 3, 5)$	$C \cup 2F$
$v \equiv 10$	v≡22	$\rho(v, 3, 10) - 1$	$2CPD(v, 3, 5)$	$2H \cup 2F$
$v \equiv 6$	v≡6	$\rho(v, 3, 10)$	$CPD(v, 3, 9) \cup S(6n)$	$2F$
$v \equiv 6$	v≡18	$\rho(v, 3, 10)$	$CPD(v, 3, 7) \cup CPD(v, 3, 3) \cup \{0, v-2, v-1\}$	$2F$
$v \equiv 5, 11$	v≡5,11,17,23	$\rho(v, 3, 10)$	$2CPD(v, 3, 5)$	$2H$

Table 3.11: Constructions of a cyclic packing design $CPD(v, 3, 10)$

3.11.5 Example of leave

In this example, we present the minimum leave of the cyclic packing design for $\lambda = 10$ and $v = 17$, where the leaves are two Hamiltonian cycles, $(2H)$.

Figure 3.5: The minimum leave of $CPD\,(17, 3, 10)$, $(2H)$

3.12 Cyclic Packing Designs for $k = 3$ and $\lambda = 11$, $CPD(v, 3, 11)$

3.12.1 Case 1: $v \equiv 2, 8 \pmod{12}$

For $v \equiv 2, 8, 14, 20 \pmod{24}$, we will use the sequences as in the construction of a $CPD(v, 3, 9)$ and the sequences as in the construction of a $CPD(v, 3, 2)$ obtained in Section 3.10 and Section 3.3, respectively to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor$.

For $v \equiv 2 \pmod{24}$, the leave is the union of three circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 8 \pmod{24}$, the leave is the union of three circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

For $v \equiv 14 \pmod{24}$, the leave is the union of six circulant graphs $C_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \langle 1 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, and 2 disjoint cycles of length $\frac{v}{2}$ and 4 Hamiltonian cycles .

For $v \equiv 20 \pmod{24}$, the leave is the union of three circulant graphs $C_v \langle 2 \rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} - 1 \right\rangle \cup \mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.12.1 *For $v = 32$, we have 46 block from $CPD\,(v, 3, 9)$ and from a $CPD\,(v, 3, 2)$ we have 10 blocks then totally we have 56 block. The leave is the union of three circulant graphs $C_{32}\langle 2\rangle \cup C_{32}\langle 2\rangle \cup C_{32}\langle 16\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length* 16.

3.12.2 Case 2: $v \equiv 4, 10 \pmod{12}$

For $v \equiv 4, 16 \pmod{24}$, we will apply a Skolem-type sequence and a $CPD\,(v, 3, 9)$ obtained in Section 3.10 to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n + 1, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n + 1, 3, 2)$.

The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor$.

For $v \equiv 4 \pmod{24}$, the leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} - 1 \right\rangle$ $\cup$ $\mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and one Hamiltonian cycle.

For $v \equiv 16 \pmod{24}$, the leave is the union of two circulant graphs $C_v \left\langle \frac{v}{2} - 2 \right\rangle$ $\cup$ $\mathrm{C}_v \left\langle \frac{v}{2} \right\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.12.2 *For $v = 40$, we have 58 block from a $CPD\,(v, 3, 9)$ and we have 13 blocks from a Skolem-type sequence of order 13, So we have 71 block.*
The leave is the union of two circulant graphs $C_{40}\langle 18\rangle \cup C_{40}\langle 20\rangle$, which represent the union of a 1-factor and 2 disjoint cycles of length 20.

For $v \equiv 10, 22 \pmod{24}$, we will apply a hooked Skolem-type sequence, a $CPD\,(v, 3, 9)$ obtained in Section 3.10 and we add one copy of the base block $\{0, v-2, v-1\}$ to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(3n+4, 3, 2)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(3n+4, 3, 2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor$.
For $v \equiv 10 \pmod{24}$, the leave is the union of five circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 6 disjoint cycles of length $\frac{v}{2}$.
For $v \equiv 22 \pmod{24}$, the leave is the union of two circulant graphs $C_v\langle 4\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represents a 1-factor union 2 disjoint cycles of length $\frac{v}{2}$.

Example 3.12.3 *For $v = 46$, we have 67 block from a $CPD\,(v, 3, 9)$ also we have 14* *block from a hooked Skolem-type sequence of order 14, and we add the block* $\{0,1,2\}$*,* *So we have 82 block. The leave is the union of two circulant graphs* $C_v\langle 4\rangle \cup C_{46}\langle 23\rangle$*, which represents a 1-factor union 2 disjoint cycles of length 23.*

3.12.3 Case 3: $v \equiv 0,6 \pmod{12}$

For $v \equiv 0,6 \pmod{24}$, we will apply a Skolem-type sequence and a $CPD(v,3,10)$ obtained in Section 3.11 to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(6n,3,1)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(6n,3,1)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor$. The leave is one circulant graph $C_v\left\langle \frac{v}{2}\right\rangle$, which represents a 1-factor.

Example 3.12.4 *For* $v = 30$*, we have 48 block from a* $CPD(v,3,10)$ *and we have 5 blocks from a Skolem-type sequence of order 5, So we have 53 block.*

The leave is one circulant graph $C_{30}\langle 15\rangle$*, which represents a a 1-factor.*

For $v \equiv 12 \pmod{24}$, we will apply a hooked Skolem-type sequence and a $CPD(v,3,9)$ obtained in Section 3.10 to construct a cyclic packing design. From a hooked Skolem sequence of order n, construct the pairs (a_i,b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+3,3,2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor - 1$. The leave is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 1\rangle \cup \mathrm{C}_v\langle 3\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, two Hamiltonian cycles and union of $gcd(v,d)$ cycles of length $\frac{v}{gcd(v,d)}$.

Example 3.12.5 *For $v = 36$, we have 52 block from a $CPD(v,3,9)$ and we have 11 block from a hooked Skolem-type sequence of order 11, So we have 63 block. The leave is the union of four circulant graphs $C_{36}\langle 1\rangle \cup C_{36}\langle 1\rangle \cup C_{36}\langle 3\rangle \cup C_{36}\langle 18\rangle$, which represent the union of a 1-factor, two Hamiltonian cycles and union of* 3 *cycles of length* 12*.*

For $v \equiv 18 \pmod{24}$, we will use a Skolem-type sequence and a $CPD(v,3,9)$ obtained in Section 3.10 to construct a cyclic packing design. From a Skolem sequence of order n, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \le i \le n$. The set of all triples (i, a_i+n, b_i+n), for $1 \le i \le n$, yield the base blocks for a cyclic packing design $CPD(3n+3,3,2)$: $\{0, a_i+n, b_i+n\}$, $1 \le i \le n$, or $\{0, i, b_i+n\}$, $1 \le i \le n$, is another set of base blocks of a $CPD(3n+3,3,2)$. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor - 1$. The leave is the union of four circulant graphs $C_v\langle 1\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\langle 2\rangle \cup \mathrm{C}_v\left\langle \frac{v}{2}\right\rangle$, which represent the union of a 1-factor, one Hamiltonian cycle and 2 disjoint cycles of length $\frac{v}{2}$.

3.12.4 Case 4: $v \equiv 5, 11 \pmod{12}$

For $v \equiv 5, 11, 23 \pmod{24}$, we will use the union between a $CPD\,(v, 3, 10)$ and a $CPD\,(v, 3, 1)$ obtained in Section 3.11 and Section 3.2, respectively and we add one base block to construct a cyclic packing design. The number of base blocks in this case is $\lfloor \frac{(v-1)(11)}{(3)(2)} \rfloor$.

For $v \equiv 5 \pmod{24}$, we add one block of the form $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor + 1\right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 1 \rangle$, which represents a one Hamiltonian cycle.

For $v \equiv 11 \pmod{24}$, we add one block of the form $\left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor\right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 2 \rangle$, which represents a one Hamiltonian cycle.

For $v \equiv 23 \pmod{24}$, we add one block of the form $\left\{0, \lfloor \frac{v}{2} \rfloor - 2, \lfloor \frac{v}{2} \rfloor\right\}$ to construct a cyclic packing design. The leave is one circulant graph $C_v \langle 2 \rangle$, which represents a one Hamiltonian cycle.

Example 3.12.6 *For $v = 47$, we have 76 block from a $CPD\,(v, 3, 10)$, we have 7 blocks from a $CPD\,(v, 3, 1)$ and we add the block $\{0, 21, 23\}$, So we have 84 block. The leave is one circulant graphs $C_{47} \langle 2 \rangle$, which represents a one Hamiltonian cycle.*

For $v \equiv 17 \pmod{24}$, we will apply a near Skolem-type sequence of order n and defect 2, also we take the sequences from a $CPD\,(v,3,8)$, a $CPD\,(v,3,2)$ obtained in Section 3.9 and Section 3.3, respectively and we add one base block of the form $\left\{0, \lfloor\frac{v}{2}\rfloor - 2, \lfloor\frac{v}{2}\rfloor\right\}$ to construct a cyclic packing design. From a near Skolem sequence of order n and defect 2, construct the pairs (a_i, b_i) such that $b_i - a_i = i$, for $1 \leq i \leq n$. The set of all triples $(i, a_i + n, b_i + n)$, for $1 \leq i \leq n$, yield the base blocks for a cyclic packing design $CPD(6n-1,3,1)$: $\{0, a_i + n, b_i + n\}$, $1 \leq i \leq n$, or $\{0, i, b_i + n\}$, $1 \leq i \leq n$, is another set of base blocks of a $CPD(6n-1,3,1)$. The number of base blocks in this case is $\lfloor\frac{(v-1)(11)}{(3)(2)}\rfloor$. The leave is one circulant graph $C_v\langle 1\rangle$, which represents a one Hamiltonian cycle.

Example 3.12.7 *For $v = 41$, we have 53 block from a $CPD\,(v,3,8)$, 13 block from a $CPD\,(v,3,2)$, also from a near Skolem-type sequence of order 7 and defect 2, we have 6 blocks and we add the block $\{0, 18, 20\}$, So we have 73 block. The leave is one circulant graph $C_{41}\langle 1\rangle$, which represents a one Hamiltonian cycle.*

In this table, we summarize the number of base blocks, leaves, and the type of sequence used, for every v, $\lambda = 11$, and $k = 3$.

$v \pmod{12}$	$v \pmod{24}$	Number of blocks	Type of sequence	Leaves
$v \equiv 2, 8$	v≡2	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup CPD(v, 3, 2)$	$H \cup C \cup F$
$v \equiv 2, 8$	v≡8	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup CPD(v, 3, 2)$	$C \cup F$
$v \equiv 2, 8$	v≡14	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup CPD(v, 3, 2)$	$4H \cup C \cup F$
$v \equiv 2, 8$	v≡20	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup CPD(v, 3, 2)$	$H \cup C \cup F$
$v \equiv 4, 10$	v≡4	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup S(3n + 1)$	$H \cup F$
$v \equiv 4, 10$	v≡16	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup S(3n + 1)$	$C \cup F$
$v \equiv 4, 10$	v≡10	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup hS(3n + 1) \cup \{0, v - 2, v - 1\}$	$H \cup C \cup F$
$v \equiv 4, 10$	v≡22	$\rho(v, 3, 11)$	$CPD(v, 3, 9) \cup hS(3n + 1) \cup \{0, v - 2, v - 1\}$	$C \cup F$
$v \equiv 0, 6$	v≡0,6	$\rho(v, 3, 11)$	$CPD(v, 3, 10) \cup S(6n)$	F
$v \equiv 0, 6$	v≡12	$\rho(v, 3, 11) - 1$	$CPD(v, 3, 10) \cup hS(3n + 3)$	$2H \cup C \cup F$
$v \equiv 0, 6$	v≡18	$\rho(v, 3, 11) - 1$	$CPD(v, 3, 10) \cup S(3n + 3)$	$H \cup C \cup F$
$v \equiv 5, 11$	v≡5	$\rho(v, 3, 11)$	$CPD(v, 3, 10) \cup CPD(v, 3, 1) \cup \left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor\right\}$	H
$v \equiv 5, 11$	v≡11	$\rho(v, 3, 11)$	$CPD(v, 3, 10) \cup CPD(v, 3, 1) \cup \left\{0, \lfloor \frac{v}{2} \rfloor - 1, \lfloor \frac{v}{2} \rfloor\right\}$	H
$v \equiv 5, 11$	v≡23	$\rho(v, 3, 11)$	$CPD(v, 3, 10) \cup CPD(v, 3, 1) \cup \left\{0, \lfloor \frac{v}{2} \rfloor - 2, \lfloor \frac{v}{2} \rfloor\right\}$	H
$v \equiv 5, 11$	v≡17	$\rho(v, 3, 11)$	$CPD(v, 3, 8) \cup CPD(v, 3, 2) \cup nS(6n - 1) \cup \left\{0, \lfloor \frac{v}{2} \rfloor - 2, \lfloor \frac{v}{2} \rfloor\right\}$	H

Table 3.12: Constructions of a cyclic packing design $CPD(v, 3, 11)$

Chapter 4

Conclusions and Future work

4.1 Conclusions

The constructions and the existence of a $BIBD(v, 3, \lambda)$ have been studied by many researchers. In 1981, Colbourn and Colbourn [16] solved the existence problem of cyclic $BIBD(v, 3, \lambda)$ In [16], the necessary and sufficient conditions of Theorem 1.0.1 were given using Peltesohn's technique. In 1992, Colbourn, Hoffman and Rees [15] used Skolem type sequences to construct cyclic partial Steiner triple systems of order v. Rees and Shalaby [28], in 2000, constructed cyclic triple systems with $\lambda = 2$ using Skolem-type sequences.

In 2012, Silvesan and Shalaby [35] extended the techniques used before and introduced several new constructions that use Skolem-type sequences to construct cyclic triple systems for all admissible $\lambda > 2$ when $k = 3$. Silvesan and Shalaby [35] proved the sufficiency of Theorem **??** using Skolem-type sequences. They have arranged the necessary conditions in Table 1.1, with $-$ sign for the designs with no short orbits, and $+$ sign for the designs that have short orbits with length equal to $1/3$ of the full orbit. An empty cell in the table means that such a design does not exist.

In this thesis, we used Skolem-type sequences to construct cyclic packing designs with block size 3 for a cyclic $BIBD\,(v, 3, \lambda)$, and found the spectra of the leave graphs of the cyclic packing designs for all admissible orders v and λ with the optimal leaves. As well, we determined the upper bound on the number of base blocks in a cyclic packing design by Corollary 3.1.4.

In the following table, we present the necessary conditions for the existence of a cyclic $BIBD(v, 3, \lambda)$ and the optimal leaves for all admissible λ of a cyclic packing design.

$v(mod\ 24)/\lambda(\text{mod } 12)$	0	1	2	3	4	5	6	7	8	9	10	11
2	-	F	H	$H \cup F$	$2H$	$2H \cup F$	$3H$	$3H \cup F$	C	$C \cup F$	$C \cup H$	$H \cup C \cup F$
8	-	F	C	$C \cup F$	C	$C \cup F$	-	F	C	$C \cup F$	C	$C \cup F$
14	-	$F \cup 2C \cup H$	H	$C \cup F$	$2H$	$F \cup C \cup H$	$3H$	$C \cup 2H \cup F$	C	$3H \cup C \cup F$	$C \cup 3H$	$4H \cup C \cup F$
20	-	$F \cup 2C \cup H$	C	$H \cup F$	C	$F \cup C \cup H$	-	$F \cup C \cup H$	$2H \cup C$	$H \cup F$	C	$H \cup C \cup F$
0	-	$F \cup C \cup H$	$+$	$C \cup F$	$+$	F	-	$C \cup F$	$+$	$3F$	$+$	F
6	-	$F \cup C \cup H$	$H \cup C$	$C \cup F$	$+$	$H \cup C \cup F$	$H \cup C$	$C \cup F$	$+$	$3F$	$2F$	F
12	-	$F \cup C \cup H$	$+$	$C \cup F$	$+$	$H \cup C \cup F$	-	$C \cup H \cup F$	$+$	$H \cup F$	$+$	$2H \cup C \cup F$
18	-	$F \cup C \cup H$	$H \cup C$	$H \cup F$	$+$	F	$2H \cup 2F$	$C \cup H \cup F$	$+$	$C \cup F$	$2F$	$H \cup C \cup F$
4	-	$F \cup H$	-	$H \cup F$	-	$H \cup F$	-	$F \cup H$	-	$H \cup F$	-	$H \cup F$
10	-	$F \cup C$	$H \cup 2C$	$H \cup F$	-	$C \cup F$	$2H \cup 2F$	$F \cup 3H \cup C$	-	$2H \cup C \cup F$	$C \cup 2F$	$H \cup C \cup F$
16	-	$F \cup C$	-	$C \cup F$	-	$C \cup F$	-	$F \cup C$	-	$C \cup F$	-	$C \cup F$
22	-	$F \cup H$	$H \cup 2C$	$C \cup F$	-	$H \cup F$	$C \cup 2F$	$F \cup C$	-	$H \cup F$	$2H \cup 2F$	$C \cup F$
5	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
11	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
17	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
23	-	$2H$	H	-	$2H$	H	-	$2H$	H	-	$2H$	H
1	-	-	-	-	-	-	-	-	-	-	-	-
7	-	-	-	-	-	-	-	-	-	-	-	-
13	-	-	-	-	-	-	-	-	-	-	-	-
19	-	-	-	-	-	-	-	-	-	-	-	-
3	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$
9	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$
15	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$
21	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$	-	$+$	$+$

Table 4.1: The necessary conditions for the existence of a cyclic $BIBD(v, 3, \lambda)$ and optimal leaves for all admissible λ of a cyclic packing design

Theorem 4.1.1 *The necessary conditions of Table 4.1 are sufficient for the existence of the leave of a cyclic packing design $CPD\,(v, 3, \lambda)$.*

Proof Case 1: $\lambda = 1 \pmod{12}$ and $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.2.1.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.2.2.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.2.3.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.2.4.

Case 2: $\lambda = 2 \pmod{12}$ and $v \equiv 2, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.3.1.

For $v \equiv 10 \pmod{12}$, see the constructions in Section 3.3.2.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.3.3.

For $v \equiv 6 \pmod{12}$, see the constructions in Section 3.3.4.

Case 3: $\lambda = 3 \pmod{12}$ and $v \equiv 0, 2, 4, 6, 8, 10 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.4.1.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.4.2.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.4.3.

Case 4: $\lambda = 4 \pmod{12}$ and $v \equiv 2, 5, 8, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.5.1.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.5.2.

Case 5: $\lambda = 5 \pmod{12}$ and $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.6.1.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.6.2.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.6.3.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.6.4.

Case 6: $\lambda = 6 \pmod{12}$ and $v \equiv 2, 6, 10 \pmod{12}$.

For $v \equiv 2 \pmod{12}$, see the constructions in Section 3.7.1.

For $v \equiv 6 \pmod{12}$, see the constructions in Section 3.7.2.

For $v \equiv 10 \pmod{12}$, see the constructions in Section 3.7.3.

Case 7: $\lambda = 7 \pmod{12}$ and $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.8.1.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.8.2.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.8.3.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.8.4.

Case 8: $\lambda = 8 \pmod{12}$ and $v \equiv 2, 5, 8, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.9.1.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.9.2.

Case 9: $\lambda = 9 \pmod{12}$ and $v \equiv 0, 2, 4, 6, 8, 10 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.10.1.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.10.2.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.10.3.

Case 10: $\lambda = 10 \pmod{12}$ and $v \equiv 2, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.11.1.

For $v \equiv 10 \pmod{12}$, see the constructions in Section 3.11.2.

For $v \equiv 6 \pmod{12}$, see the constructions in Section 3.11.3.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.11.4.

Case 11: $\lambda = 11 \pmod{12}$ and $v \equiv 0, 2, 4, 5, 6, 8, 10, 11 \pmod{12}$.

For $v \equiv 2, 8 \pmod{12}$, see the constructions in Section 3.12.1.

For $v \equiv 4, 10 \pmod{12}$, see the constructions in Section 3.12.2.

For $v \equiv 0, 6 \pmod{12}$, see the constructions in Section 3.12.3.

For $v \equiv 5, 11 \pmod{12}$, see the constructions in Section 3.12.4.■

4.2 Future Work

Definition 4.2.1 *A Covering Design, $CD(v, k, \lambda)$ is a pair $(V, \mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a set of k-subsets* (blocks) *such that any 2-subset of V appears in at least λ blocks. A $CD(v, k, \lambda)$ is cyclic if its automorphism group contains a v-cycle and is called a cyclic covering design.*

In Chapter 3, we used Skolem type sequences to construct cyclic packing designs with block size 3 for all admissible λ. As a part of our future work, the following points will be investigated:

1. Use Skolem-type sequences to construct the cyclic covering designs for $BIBD(v, 3, \lambda)$ with the optimal excess.

2. Use Skolem-type sequences to construct the cyclic spectrum of leaves and excess for $k = 4$ and all admissible λ.

3. Find more applications for cyclic packing and covering designs.

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