Equitably Coloured Balanced Incomplete Block Designs

by

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Abstract

In this thesis we determine necessary and sufficient conditions for the existence of an equitably ℓ-colourable balanced incomplete block design for any positive integer $\ell \geqslant 2$. In particular, we present a method for constructing non-trivial equitably ℓ-colourable BIBDs and prove that these designs are the only non-trivial equitably ℓ-colourable BIBDs that exist. We also observe that every equitable ℓ-colouring of a BIBD yields both an equalised ℓ-colouring and a proper 2-colouring of the same BIBD. We also discuss generalisations of these concepts including open questions for further research. The main results presented in this thesis also appear in [7].

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Chapter 1

Introduction

1.1 Definitions and History

A *balanced incomplete block design*, or BIBD, with parameters v, k, and λ, is a pair $(V, \mathcal{B})$ such that V is a set of v distinct elements called points and $\mathcal{B}$ is a collection of k-subsets of V called blocks such that each pair of points of V occurs in exactly λ blocks of $\mathcal{B}$. Every (v, k, λ)-BIBD consequently has two additional parameters; the *replication number* $r = \frac{\lambda(v-1)}{k-1}$ is the number of occurances of any given point among the blocks of $\mathcal{B}$, and the number of blocks $b = |\mathcal{B}| = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$. For this reason, (v, k, λ)-BIBDs are sometimes identified as (v, b, r, k, λ)-BIBDs.

A *colouring* of a (v, k, λ)-BIBD, $\mathcal{D} = (V, \mathcal{B})$, is a function $f : V \to C$ where $C = \{c_1, c_2, ..., c_\ell\}$ is a finite set of distinct elements called *colours*. Rosa and Colbourn surveyed many results on colourings of block designs in [9] including weak colourings, strong colourings and many results on colourings of $(v, 3, 1)$-BIBDs (commonly referred to as Steiner triple systems). Most results on the colourings of block designs are concerned with colourings in which each block has at least one pair of points of different colours. Such colourings are known as *weak colourings*. The *chromatic index* of a block design is the fewest number of colours required to colour the points in this way. A survey on weak colourings of Steiner triple system can be found in [3]. For some results on the chromatic index of balanced incomplete block designs see [6].

A block $B \in \mathcal{B}$ is *equitably ℓ-coloured* if B has n_i points coloured with colour c_i, $i = 1, ..., \ell$, and $|n_i - n_j| \leqslant 1$ for any distinct $i, j \in \{1, ..., \ell\}$. A design is *equitably ℓ-colourable* if the points can be coloured with ℓ colours such that every block is equitably ℓ-coloured. For example, consider the design $\mathcal{D} = (V, \mathcal{B})$ where $V = \{a, b, c, d, e, f\}$ and $\mathcal{B}$ consists of the blocks

$$\{a, b, c, d, e\},\ \{a, b, c, d, f\},\ \{a, b, c, e, f\}$$
$$\{a, b, d, e, f\},\ \{a, c, d, e, f\},\ \{b, c, d, e, f\}.$$

Now if we let $C_1 = \{a, b\}$, $C_2 = \{c, d\}$, and $C_3 = \{e, f\}$, then the colouring $\phi : V \to \{c_1, c_2, c_3\}$ defined by $\phi(x) = c_i$ for $x \in C_i$ is an equitable 3-colouring of $\mathcal{D}$ and hence $\mathcal{D}$ is an equitably 3-colourable $(6, 5, 4)$-BIBD.

Observe that an equitable colouring with at least two colours would also be a weak colouring with a specific restriction on the number of times a colour can appear on a block. As a part of the proof of Theorem 4.2, we will show that an equitably coloured block must adhere to a specific colour pattern. Colourings of Steiner triple systems with prescribed colour patterns are discussed in [8].

Let C_i denote the colour class of colour c_i, that is, C_i is the set of points that have been coloured with colour c_i. We can equivalently define a colouring $(C_1, C_2, ..., C_\ell)$ on a design $\mathcal{D} = (V, \mathcal{B})$ to be equitable if and only if $\left\lfloor \frac{k}{\ell} \right\rfloor \leqslant |B \cap C_i| \leqslant \left\lceil \frac{k}{\ell} \right\rceil$ for each $i \in \{1, 2, ..., \ell\}$ and $B \in \mathcal{B}$. Similarly, we may define an *equalised* colouring of a (v, k, λ)-BIBD to be a colouring of the points in which $\left\lfloor \frac{v}{\ell} \right\rfloor \leqslant |C_i| \leqslant \left\lceil \frac{v}{\ell} \right\rceil$. In Chapter 4, we establish that every equitable colouring of a BIBD is also an equalised colouring.

Note that when $k = 3$, the notion of a (v, k, λ)-BIBD and a 3-cycle decomposition of λK_v coincide. Adams, Bryant, Lefevre, and Waterhouse have investigated equitably ℓ-colourable m-cycle decompositions of the complete graph and the complete graph with the edges of a 1-factor removed. In particular, they have settled the spectrum problem for m-cycle systems with $m = 4, 5, 6$ when $\ell = 2$ and $\ell = 3$ in [2] and [1] respectively. It was their work on equitably ℓ-colourable m-cycle systems that inspired our investigation of equitably ℓ-colourable BIBDs. In Chapter 2, we generalise a statement made in [1] which only concerned Steiner triple systems to BIBDs with an arbitrary block size and index λ. In particular, we prove that if $\mathcal{D}$ is a (v, k, λ)-BIBD and $\ell \geqslant k$, then $\mathcal{D}$ can be equitably ℓ-coloured if and only if $\ell = v$.

1.2 Outline

The associated spectrum problem for equitably ℓ-colourable BIBDs is the problem of determining necessary and sufficient conditions on v such that an equitably ℓ-colourable (v, k, λ)-BIBD can and will exist for fixed ℓ, k, and λ. In Chapter 2, we prove an important preliminary result that will assist in narrowing down the necessary conditions for the existence of an equitably ℓ-colourable BIBD. In Chapter 3, we present a sufficient condition for the existence of a non-trivial equitably ℓ-colourable BIBD and in Chapter 4, we prove that this condition is also necessary. Finally in Chapter 5, we discuss the potential future of this research including generalisations and related open problems.

Chapter 2

A Generalisation From Triple Systems

Recall that $(v, 3, 1)$-BIBDs are commonly referred to as Steiner triple systems of order v or simply STS(v). It is well known that a STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$. In [1], Adams, Bryant, Lefevre, and Waterhouse mention that the only equitably 2-colourable Steiner triple system is the trivial system of order 3. They also state that if $\ell \geqslant 3$, then there exists an equitably ℓ-colourable STS(v) if and only if $\ell = v$ and $v \equiv 1, 3 \pmod 6$.

We prove an analogous statement for arbitrary (v, k, λ)-BIBDs when attempting to equitably colour designs with at least k colours. The generalised statement is the following theorem.

Theorem 2.1 *If $\mathcal{D}$ is a (v, k, λ)-BIBD and $\ell \geqslant k$, then $\mathcal{D}$ can be equitably ℓ-coloured if and only if $\ell = v$.*

Proof. It suffices to show that no two points can receive the same colour given that $\mathcal{D}$ is equitably ℓ-coloured and $\ell \geqslant k$.

Suppose that two points, say x and y, do receive the same colour, say c_i. Now let B be any block containing both x and y. Observe that there are $k-2$ other points in B and at least $k-1$ other colours that must occur among the points of B. So some colour, say c_j, must be absent from the block B. Now since $n_i \geqslant 2$ and $n_j = 0$ we have $|n_i - n_j| > 1$, which is a contradiction to B being a block of an equitably coloured design.

So it must be the case that every distinct point is coloured with a distinct colour, *i.e.*, $\ell = v$. $\square$

The significance of Theorem 2.1 is that it provides an upper bound on the number of colours that we may consider when attempting to equitably ℓ-colour a (v, k, λ)-BIBD. For instance, when seeking a non-trivial example of an equitably ℓ-colourable (v, k, λ)-BIBD (as we do in Chapter 3), only values of ℓ strictly less than k should be considered, for otherwise, the only examples occur when each point receives a distinct colour.

Chapter 3

A Non-trivial Construction

In this chapter, we illustrate a class of non-trivial examples of equitably ℓ-colourable BIBDs. In particular, we prove that for any positive integer $\ell \geqslant 2$, there exists an equitably ℓ-colourable $(k+1, k, \lambda)$-BIBD where $k \equiv \ell - 1 \pmod{\ell}$. To do this, we begin with a proposition.

Proposition 3.1 *If there exists a (v, k, λ)-BIBD where $v = k+1$, then $\lambda = u(k-1)$ where u is a positive integer.*

Proof. Consider the replication number r of such a design. In any design, $r = \frac{\lambda(v-1)}{k-1}$, so if $v = k+1$, then $r = \frac{\lambda k}{k-1}$. Since r must be an integer, we must have $\lambda k \equiv 0 \pmod{k-1}$. Now since $k-1$ and k are consecutive integers, they are relatively prime. So $\lambda k \equiv 0 \pmod{k-1}$ if and only if $\lambda \equiv 0 \pmod{k-1}$. That is, $\lambda = u(k-1)$ for some positive integer u. $\square$

In fact, the condition that $\lambda = u(k-1)$ is also sufficient for the existence of a $(k+1, k, \lambda)$-BIBD. Indeed, let $V = \{1, 2, ..., k+1\}$ be a set of points and for each $i \in V$, let $B_i = V \setminus \{i\}$. Then for each i, $|B_i| = |V| - 1 = k$ and for each pair of points $\{i, j\}$, we have that $i \notin B_i$ and $j \notin B_j$ but both i and j are elements of the remaining $k-1$ sets of the form B_x, where $x \in V \setminus \{i,j\}$. So if we let $\mathcal{B} = \cup_{i \in V} B_i$, then $(V, \mathcal{B})$ is a $(k+1, k, k-1)$-BIBD. Now simply take u copies of $\mathcal{B}$ to form a $(k+1, k, u(k-1))$-BIBD.

It remains to check which $(k+1, k, u(k-1))$-BIBDs are equitably ℓ-colourable when $k \equiv \ell - 1 \pmod{\ell}$. If we use the construction just described and let

$$C_i = \left\{ i, i+\ell, i+2\ell, ..., i + \left(\frac{k+1}{\ell} - 1 \right) \ell \right\} \subset V,$$

then $(C_1, C_2, ..., C_\ell)$ is an equitable ℓ-colouring. Indeed, if B_i is a block, then $i \in C_i$, so B_i has $(k+1)/\ell - 1$ points of C_i and $(k+1)/\ell$ points of C_j for each $j \neq i$. So B_i is an equitably ℓ-coloured block and therefore the design itself is equitably ℓ-coloured.

One application of the existence of non-trivial equitably ℓ-coloured designs is to find proper colourings of mixed hypergraphs. Colourings of mixed hypergraphs have been described extensively in [10]. A *mixed hypergraph* is a triple $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ where V is a set of vertices, and both $\mathcal{C}$ and $\mathcal{D}$ are collections of subsets of V called hyperedges. A *proper ℓ-colouring* of a mixed hypergraph $\mathcal{H}$ is a colouring of the vertices of V with ℓ distinct colours such that every element of $\mathcal{C}$ has at least two vertices of a common colour and every element of $\mathcal{D}$ has at least two vertices of a different colour. A *bihypergraph* is a mixed hypergraph in which $\mathcal{C} = \mathcal{D}$. Furthermore, if every element of $\mathcal{C} \cup \mathcal{D}$ has cardinality k, then the hypergraph is said to be *k-uniform.*

It is not hard to see that every (v, k, λ)-BIBD, $(V, \mathcal{B})$, is also a k-uniform mixed hypergraph where the point set of the design is the vertex set of the hypergraph and $\mathcal{B} = \mathcal{C} = \mathcal{D}$. Note that the above example of a non-trivial equitably ℓ-colourable BIBD is also an example of a properly ℓ-colourable bihypergraph. Indeed, the equitable colouring ensures that every hyperedge has at least two vertices of a common colour and at least two vertices of a different colour. For example, the equitably 2-colourable

$(4, 3, 2)$-BIBD obtained from this construction would be an example of a 2-colourable bihypergraph on four vertices with four hyperedges, each of size three.

Observe also that these non-trivial equitably ℓ-colourable designs are all 2-chromatic in the sense of weak colourings, that is, the fewest number of colours required in a proper colouring of these designs is two colours. Indeed, by Theorem 2.1, we know that $\ell < k$ and so it is true that $\left\lfloor \frac{k}{\ell} \right\rfloor \geqslant 1$. Hence it must be true that every colour is represented at least once on every block in the design. Now if we recolour the points of colours $c_3, ..., c_\ell$ so that they have colour c_2, then the design becomes properly 2-coloured as every block then has at least one point coloured with colour c_1 and at least one point coloured with colour c_2.

In fact, the construction detailed in this chapter is the only source of non-trivial examples of equitably ℓ-colourable BIBDs. The proof of this statement is the main result in this thesis and is presented in the following chapter. As a corollary of this fact, it is also true that every equitably ℓ-colourable (v, k, λ)-BIBD with $\ell < k$ must be 2-chromatic as the trivial (k, k, λ)-BIBDs are 2-chromatic and the only non-trivial equitably ℓ-coloured BIBDs are also 2-chromatic.

Chapter 4

Main Result

In this chapter we prove that the only non-trivial equitably ℓ-colourable BIBDs are those described in Chapter 3. Here, we assume that $\ell < k$, as Theorem 2.1 handled the case of $\ell \geqslant k$. In order to prove our main result, we will split the problem into two cases, whether or not $k \equiv 0 \pmod{\ell}$.

4.1 A Straightforward Case

When $k \equiv 0 \pmod{\ell}$, it is relatively straightforward to show that the only equitably ℓ-colourable BIBDs are the trivial BIBDs.

Theorem 4.1 *If $\mathcal{D}$ is an equitably ℓ-coloured (v, k, λ)-BIBD and $k \equiv 0 \pmod{\ell}$, then the size of each colour class is equal to v/ℓ. Moreover, such a design exists if and only if $v = k$.*

Proof. Recall that C_i is the colour class of colour c_i, $i = 1, 2, ..., \ell$. Note that each block of $\mathcal{D}$ must have exactly k/ℓ points of each colour class. A pair of points where both points are elements of the same colour class will be referred to as a *pure pair*. Now in each block there are $\binom{k/\ell}{2}$ pure pairs of C_i, for each i. Likewise, each of the

$\binom{|C_i|}{2}$ pure pairs must occur λ times each in the design, for each i. So by counting the total number of pure pairs of C_i in two ways, we see that

$$\lambda\binom{|C_i|}{2} = \binom{k/\ell}{2}b,$$

where b is the total number of blocks in $\mathcal{D}$. Since the right hand side is independent of i, we have established that $|C_i| = |C_j|$ for each $i, j \in \{1, 2, ..., \ell\}$. Now since $|C_1| + |C_2| + \cdots + |C_\ell| = v$, it must be that $|C_i| = v/\ell$ for each $i \in \{1, 2, ..., \ell\}$.

For the second claim, it suffices to prove the forward direction, as any (k, k, λ)-BIBD is trivially equitably ℓ-colourable for any ℓ. A pair of points where the two points are elements of different colour classes will be referred to as a *mixed pair*. We proceed by counting the total number of mixed pairs in two ways. On one hand, the total number of mixed pairs is $\lambda \sum_{i,j} |C_i||C_j|$. On the other hand, the total number of mixed pairs is $\left(\frac{k}{\ell}\right)^2 \binom{\ell}{2}b$ as there are $\left(\frac{k}{\ell}\right)\left(\frac{k}{\ell}\right)\binom{\ell}{2}$ mixed pairs in each block and $b = \frac{\lambda v(v-1)}{k(k-1)}$ blocks in $\mathcal{D}$. Since $|C_i| = v/\ell$ for each i, it follows that

$$\begin{array}{rrcl} & \lambda\binom{\ell}{2}\left(\frac{v}{\ell}\right)^2 & = & \left(\frac{k}{\ell}\right)^2\binom{\ell}{2}\frac{\lambda v(v-1)}{k(k-1)} \\ \iff & \frac{v}{v-1} & = & \frac{k}{k-1} \\ \iff & v & = & k. \end{array}$$

□

So when $k \equiv 0 \pmod{\ell}$, the only equitably ℓ-colourable (v, k, λ)-BIBDs are the trivial (k, k, λ)-BIBDs. We already know (by the construction in Chapter 3) that non-trivial examples do exist. It remains to prove that these are in fact the only non-trivial examples of equitably ℓ-colourable BIBDs.

4.2 A Less Than Straightforward Case

Recall that if $\mathcal{D}$ is an equitably ℓ-coloured (v, k, λ)-BIBD with the colouring $(C_1, C_2, ..., C_\ell)$ then $\lfloor \frac{k}{\ell} \rfloor \leqslant |B \cap C_i| \leqslant \lceil \frac{k}{\ell} \rceil$ for each block B and each colour class C_i. Now if $k \not\equiv 0 \pmod{\ell}$ then $k = q\ell + m$ for unique integers q and m such that $0 < m < \ell$. Note that $q = \lfloor \frac{k}{\ell} \rfloor$ and $q+1 = \lceil \frac{k}{\ell} \rceil$ and so, for each block B, we have that $|B \cap C_i| \in \{q, q+1\}$ for each $i = 1, 2, ..., \ell$. Furthermore, each block must have $\ell - m$ colours that are represented on q points each and m colours that are represented on $q+1$ points each. This can easily be observed by solving the system of equations:

$$\begin{aligned} xq + y(q+1) &= k \\ x + y &= \ell, \end{aligned}$$

where x and y denote the number of colours that are represented on q and $q+1$ points each respectively.

For example if $\ell = 3$ and $m = 2$, each block must have $\ell - m = 1$ colour represented on q points and $m = 2$ colours represented on $q+1$ points each.

Theorem 4.2 *If $\mathcal{D}$ is an equitably ℓ-coloured (v, k, λ)-BIBD and $k \not\equiv 0 \pmod{\ell}$ then either $v = k$ or the size of each colour class is equal to v/ℓ.*

Proof. Let α_i (resp. β_i) denote the number of blocks of $\mathcal{D}$ in which points of C_i form a subset of exactly size q (resp. $q+1$). Observe that $\alpha_i + \beta_i = b$, the number of blocks in the design.

By counting the number of points of colour i in the entire design, regardless of repetition, we find that

$$r|C_i| = \alpha_i q + \beta_i(q+1),$$

where r is the replication number of $\mathcal{D}$. Likewise, by counting the total number of pure pairs of points of colour i in the entire design, regardless of repetition, we find that

$$\lambda\binom{|C_i|}{2} = \alpha_i\binom{q}{2} + \beta_i\binom{q+1}{2}.$$

By substituting $\beta_i = b - \alpha_i$, these two equations simplify into the following system

$$\begin{aligned} r|C_i| &= b(q+1) - \alpha_i, \\ \lambda\binom{|C_i|}{2} &= b\binom{q+1}{2} - \alpha_i\left[\binom{q+1}{2} - \binom{q}{2}\right]. \end{aligned}$$

Simplifying this system in terms of $|C_i|$ gives

$$\lambda|C_i|^2 - (\lambda + 2qr)|C_i| + bq(q+1) = 0.$$

Solving for $|C_i|$ yields

$$|C_i| = \frac{1}{2} + \frac{qr}{\lambda} \pm \frac{1}{2\lambda}\sqrt{(\lambda+2qr)^2 - 4\lambda bq(q+1)}.$$

By recalling the formulae for the replication number r and the number of blocks b, we may eliminate the parameter λ from the formulation of $|C_i|$ and represent the discriminant as a quadratic in v as follows:

$$|C_i| = \frac{1}{2} + \frac{q(v-1)}{k-1} \pm \sqrt{D},$$

where

$$D = v^2\frac{q(q+1-k)}{k(k-1)^2} - v\frac{q(q+1-k)(k+1)}{k(k-1)^2} + \frac{q(q+1-k)}{(k-1)^2} + \frac{1}{4}.$$

The coefficient of v^2 in D is always negative as $m \geqslant 1$ and $\ell > 1$ imply that $k = q\ell + m > q + 1$ and so $0 > q + 1 - k$. Also the vertex of D lies at $v = (k+1)/2$, which is less than k for $k \geqslant 2$. But in any design, we must have $v \geqslant k$, so the maximum value of D is actually attained when $v = k$. When $v = k$, $D = 1/4$ and

$$|C_i| = \frac{1}{2} + q \pm \frac{1}{2}.$$

Any value of v greater than k will result in $|C_i|$ having only one admissible value, since the maximum value of D is $1/4$. Indeed, there is only one integer in the interval

$$\left[\frac{1}{2} + \frac{q(v-1)}{k-1} - \sqrt{D}, \frac{1}{2} + \frac{q(v-1)}{k-1} + \sqrt{D}\right] \subset \mathbb{R}$$

when $D < 1/4$.

We conclude from this argument that $v \neq k$ if and only if $|C_i|$ is independent of i. Now since $|C_1| + |C_2| + \cdots + |C_\ell| = v$, then either $v = k$ or $|C_i| = v/\ell$ for each i. $\square$

At this point it is clear that an equitable colouring of a BIBD is always an equalised colouring as well. Indeed, each type of equitably colourable BIBD satisfies $\left\lfloor \frac{v}{\ell} \right\rfloor \leqslant |C_i| \leqslant \left\lceil \frac{v}{\ell} \right\rceil$.

In order to continue towards proving the theorem that the only non-trivial equitably ℓ-colourable BIBDs are those described in Chapter 3, we will require some further necessary conditions on the number of points v.

Lemma 4.1 *If $\mathcal{D}$ is an equitably ℓ-coloured (v,k,λ)-BIBD with $v \neq k$ and $k \not\equiv 0 \pmod{\ell}$ then*

$$v = \frac{(\ell-1)k^2 - m(\ell-m)}{(\ell-1)k - m(\ell-m)}.$$

Proof. By Theorem 4.2, if $v \neq k$ and $k \equiv m \pmod{\ell}$, where $0 < m < \ell$, then $|C_i| = v/\ell$ for each $i = 1, 2, ..., \ell$. We proceed by counting the total number of pure pairs in $\mathcal{D}$ in two ways. On one hand, the total number of pure pairs is

$$\sum_i \lambda \binom{|C_i|}{2}$$

and on the other hand, the total number of pure pairs is

$$b\left[x\binom{q}{2}+y\binom{q+1}{2}\right],$$

where x and y denote the number of colours that are represented on q and $q+1$ points each respectively (as was previously discussed at the beginning of Section 4.2). Now recall that $x=\ell-m$, $y=m$, and $q=(k-m)/\ell$ and so, by counting pure pairs, we get

$$\lambda\ell\binom{v/\ell}{2}=b\left[\frac{\ell-m}{2}\cdot\frac{k-m}{\ell}\cdot\frac{k-m-\ell}{\ell}+\frac{m}{2}\cdot\frac{k-m+\ell}{\ell}\cdot\frac{k-m}{\ell}\right].$$

Simplifying and rearranging gives

$$v^2((\ell-1)k-m(\ell-m))-v((\ell-1)k^2-m(\ell-m))=0.$$

So either $v=0$ (which cannot happen) or

$$v=\frac{(\ell-1)k^2-m(\ell-m)}{(\ell-1)k-m(\ell-m)}.$$

□

Lemma 4.2 *If $\mathcal{D}$ is an equitably ℓ-coloured (v,k,λ)-BIBD with $v\neq k$ and $k\not\equiv 0 \pmod{\ell}$ then either $k\equiv 1 \pmod{\ell}$ or $k\equiv \ell-1 \pmod{\ell}$.*

Proof. By Lemma 4.1, we know that

$$v = \frac{(\ell-1)k^2 - m(\ell-m)}{(\ell-1)k - m(\ell-m)},$$

where $k \equiv m \pmod{\ell}$ and $0 < m < \ell$. So it suffices to prove that $m = 1$ or $m = \ell - 1$. Now let $a = \ell - 1$ and $b = m(\ell - m)$, so $v = \frac{ak^2-b}{ak-b}$. Note that $k = b/a$ is a root of $ak - b = 0$ and since $ak^2 - b = v(ak - b)$, $k = b/a$ is also a root of $ak^2 - b = 0$. This means that $a(b/a)^2 - b = 0$, or equivalently, that $\frac{b}{a}(b - a) = 0$. So either $b = 0$ (which cannot happen) or $b = a$.

Now $b = a$ implies that

$$\begin{aligned}
m(\ell-m) &= \ell - 1 \\
\Rightarrow \quad m^2 - \ell m + \ell - 1 &= 0 \\
\Rightarrow \quad m &= \tfrac{\ell \pm \sqrt{\ell^2 - 4(\ell-1)}}{2} \\
&= \tfrac{\ell \pm \sqrt{(\ell-2)^2}}{2} \\
&= 1 \text{ or } \ell - 1.
\end{aligned}$$

□

At this point, we are able to combine our results in order to establish the necessary condition for non-trivial equitably ℓ-colourable BIBDs.

Theorem 4.3 *The trivial BIBDs are always equitably ℓ-colourable for any ℓ and the only non-trivial equitably ℓ-colourable (v, k, λ)-BIBDs (with $k > \ell$) are those described in Chapter 3.*

Proof. By Theorem 4.1, we know that any non-trivial equitably ℓ-colourable (v, k, λ)-BIBD (with $k > \ell$) must satisfy $k \not\equiv 0 \pmod{\ell}$. Also, if $k \not\equiv 0 \pmod{\ell}$ and $v \neq k$, then by Theorem 4.2, $v \equiv 0 \pmod{\ell}$. By Lemma 4.2, either $k \equiv 1 \pmod{\ell}$ or

$k \equiv \ell - 1 \pmod{\ell}$. Also, by Lemma 4.1, we know that

$$v = \frac{(\ell-1)k^2 - m(\ell-m)}{(\ell-1)k - m(\ell-m)},$$

which in either case ($m = 1$ or $m = \ell - 1$), we must have $v = k + 1$.

If $k \equiv 1 \pmod{\ell}$ and $v = k + 1$, we cannot satisfy $v \equiv 0 \pmod{\ell}$ unless $\ell = 2$, so for any $\ell > 2$ there cannot exist an equitably ℓ-colourable BIBD with $k \equiv 1 \pmod{\ell}$ (note that for $\ell = 2$, $1 \equiv \ell - 1 \pmod{\ell}$).

If $k \equiv \ell-1 \pmod{\ell}$ and $v = k+1$, then we may construct an equitably ℓ-colourable BIBD using the construction detailed in Chapter 3. That is, the only non-trivial equitably ℓ-colourable BIBDs are $(k + 1, k, \lambda)$-BIBDs where $\lambda \equiv 0 \pmod{k-1}$ and $k + 1 \equiv 0 \pmod{\ell}$. □

Chapter 5

Final Remarks and Open Problems

The main focus of this thesis has been to solve the spectrum problem of equitably ℓ-colourable BIBDs. Having achieved this, it is reasonable to turn our attention to any possible generalisations and relaxations of the definition of an equitably ℓ-colourable BIBD. One such relaxation would be the idea of a partially equitably ℓ-colourable BIBD. A block B of a BIBD is *partially* equitably ℓ-coloured if B has n_i points coloured with colour c_i, $i = 1, ..., \ell$, and $|n_i - n_j| \leqslant 1$ for any distinct $i, j \in \{1, ..., \ell\}$ such that $n_i \neq 0$ and $n_j \neq 0$. A design is *partially* equitably ℓ-colourable if in a weak colouring of the design, the points can be coloured with ℓ colours such that every block is partially equitably ℓ-coloured. This differs from our definition of an equitably ℓ-colourable BIBD in that we are allowing some colours to be absent from a given block. In other words, the equitable property of the colouring only applies to the colours that actually appear on each block whereas in the standard definition, we require each colour to appear at least once on each block. Using this terminology, we may refer to the standard definition of an equitably ℓ-colourable BIBD as a *fully* equitably ℓ-colourable BIBD, to emphasise the difference between that of a partially equitably ℓ-colourable design.

One interesting observation is that an analogous version of Theorem 2.1 does not immediately hold for partially equitably ℓ-colourable BIBDs, as one of the main observations in the proof of Theorem 2.1 is that each colour must occur at least once on each block in a fully equitably ℓ-colourable BIBD. This leads to a few interesting open questions. What is the spectrum of partially equitably ℓ-colourable BIBDs? Is there an upper bound on the number of colours such a design could use in terms of the block size k?

Another interesting relaxation is that of a d-equitably ℓ-colourable BIBD. A block B of a BIBD is *d-equitably* ℓ-coloured if B has n_i points coloured with colour c_i, $i = 1, ..., \ell$, and $|n_i - n_j| \leqslant d$ for any distinct $i, j \in \{1, ..., \ell\}$ and d an integer at least 1. A design is *d-equitably* ℓ-colourable if the points can be coloured with ℓ colours such that every block is d-equitably ℓ-coloured. Again, we may observe that our standard definition of an equitably ℓ-colourable BIBD is actually a d-equitably ℓ-colourable BIBD with $d = 1$. This leads to the question; what is the spectrum of d-equitably ℓ-colourable BIBDs?

Similarly, we may adapt the notion of semi-equitable colourings that has recently arisen in the context of graph colourings in [4] and [5]. A design is *semi-equitably* ℓ-colourable if the design can be coloured with ℓ colours such that each block is permitted to have one nonconformist colour while the remaining $\ell - 1$ colours must each occur equitably within that block. That is, one colour per block can occur any number of times, while the other colours must occur within one of each other within that block. Again we may ask, what is the spectrum of semi-equitably ℓ-colourable BIBDs?

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