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q-Integral inequalities associated with some fractional q-integral operators

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Abstract

In recent years fractional q-integral inequalities have been investigated by many authors. Therefore, the fractional q-integral inequalities have become one of the most powerful and far-reaching tools for the development of many branches of pure and applied mathematics. Here, we aim to establish some new fractional q-integral inequality by using fractional q-integral operators. Relevant connections of the results presented here with earlier ones are also pointed out.

MSC: Primary 26D10; 26A33; secondary 26D15

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1 Introduction and preliminaries

In recent years the study of fractional q-integral inequalities involving functions of independent variables has been an important research subject in mathematical analysis because the inequality technique is also one of the very useful tools in the study of special functions and theory of approximations. During the last two decades or so, several interesting and useful extensions of many of the fractional integral inequalities have been considered by several authors (see, for example, [1–12]; see also the very recent work [13]). The above-mentioned works have largely motivated our present study.

For our purpose, we begin by recalling the well-known celebrated functional considered by Chebyshev [14] and defined by

$$T(f,g) := \frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

where $f(x)$ and $g(x)$ are two integrable functions on $[a,b]$. If $f(x)$ and $g(x)$ are synchronous on $[a,b]$, *i.e.*,

$$\bigl(f(x)-f(y)\bigr)\bigl(g(x)-g(y)\bigr) \geq 0 \tag{1.2}$$

for any $x, y \in [a,b]$, then $T(f,g) \geq 0$.

The functional (1.1) has attracted many researchers' attention due to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities (see, *e.g.*, [15–22]; for a very recent work, see also [23]).

In 1935, Grüss [24] proved the inequality

$$\left|T(f,g)\right| \leq \frac{(M-m)(N-n)}{4}, \tag{1.3}$$

where $f(x)$ and $g(x)$ are two bounded functions, *i.e.*,

$$m \leq f(x) \leq M, \qquad n \leq g(x) \leq N \tag{1.4}$$

for any $m, M, n, N \in \mathbb{R}$ and $x, y \in [a, b]$.

Pólya and Szegö [25] obtained the following inequality defined as

$$\frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{(\int_a^b f(x)\,dx \int_a^b g(x)\,dx)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2, \tag{1.5}$$

provided f, g satisfy (1.4) and $m, n > 0$.

Similarly, Dragomir and Diamond proved that (see [26], p.28, Eq. 2.1)

$$\left|T(f,g)\right| \leq \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f(x)\,dx \int_a^b g(x)\,dx, \tag{1.6}$$

where $f(x)$ and $g(x)$ are two positive integrable functions so that

$$0 < m \leq f(x) \leq M < \infty, \qquad 0 < n \leq g(x) \leq N < \infty \tag{1.7}$$

for *a.e.* $x \in [a, b]$.

Recently, Anber and Dahmani [2], by using the Riemann-Liouville fractional integral, presented some interesting integral inequalities of Pólya and Szegö type. Here, motivated essentially by the above work, we aim at establishing certain (presumably) new Pólya-Szegö type q-inequalities associated with fractional q-integral operators.

For our purpose, we need the following definitions and some properties.

Definition 1 A real-valued function $f(t)$ $(t > 0)$ is said to be in the space C_μ^n $(n, \mu \in \mathbb{R})$ if there exists a real number $p > \mu$ such that $f^{(n)}(t) = t^p \phi(t)$, where $\phi(t) \in C(0, \infty)$.

Here, for the case $n = 1$, we use a simpler notation $C_\mu^1 = C_\mu$.

Definition 2 Let $\Re(\alpha) > 0$, β and η be real or complex numbers. Then a q-analogue of Saigo's fractional integral $I_q^{\alpha,\beta,\eta}$ is given for $|\frac{\tau}{t}| < 1$ by (see [27], p.172, Eq. (2.1))

$$\begin{aligned} I_q^{\alpha,\beta,\eta}\{f(t)\} :=& \frac{t^{-\beta-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t;q)_{\alpha-1} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q^\alpha;q)_m (q;q)_m} \\ & \cdot q^{(\eta-\beta)m} (-1)^m q^{-\binom{m}{2}} (\tau/t-1)_q^m f(\tau)\,d_q\tau. \end{aligned} \tag{1.8}$$

The integral operator$I_q^{\alpha,\beta,\eta}$ includes both the q-analogues of the Riemann-Liouville and Erdélyi-Kober fractional integral operators given by the following relationships:

$$\begin{aligned} I_q^{\alpha}\{f(t)\} &: \left(= I_q^{\alpha,-\alpha,0}\{f(t)\}\right) \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} f(\tau)\,d_q\tau \quad (\alpha > 0; 0 < q < 1), \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} I_q^{\eta,\alpha}\{f(t)\} &: \left(= I_q^{\alpha,0,\eta}\{f(t)\}\right) \\ &= \frac{t^{-\eta-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} \tau^{\eta} f(\tau)\,d_q\tau \quad (\alpha > 0; 0 < q < 1), \end{aligned} \tag{1.10}$$

where $(a;q)_\alpha$ is the *q-shifted factorial*.

The *q-shifted factorial* $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n = 0), \\ \prod_{k=0}^{n-1}(1 - aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1.11}$$

where $a, q \in \mathbb{C}$, and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$.

The *q-shifted factorial* for negative subscript is defined by

$$(a;q)_{-n} := \frac{1}{(1 - aq^{-1})(1 - aq^{-2}) \cdots (1 - aq^{-n})} \quad (n \in \mathbb{N}_0). \tag{1.12}$$

We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k) \quad \left(a, q \in \mathbb{C}; |q| < 1\right). \tag{1.13}$$

It follows from (1.11), (1.12) and (1.13) that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \quad (n \in \mathbb{Z}), \tag{1.14}$$

which can be extended to $n = \alpha \in \mathbb{C}$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad \left(\alpha \in \mathbb{C}; |q| < 1\right), \tag{1.15}$$

where the principal value of q^α is taken.

For $f(t) = t^\mu$ in (1.8), we get the known formula [28]

$$I_q^{\alpha,\beta,\eta}\left\{t^\mu\right\} := \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)} x^{\mu-\beta}. \tag{1.16}$$

Lemma 1 (Choi and Agarwal [28]) *Let* $0 < q < 1$ *and* $f : [0,\infty) \to \mathbb{R}$ *be a continuous function with* $f(t) \geq 0$ *for all* $t \in [0,\infty)$. *Then we have the following inequalities*:

(i) *The Saigo fractional q-integral operator of the function* $f(t)$ *in* (1.8)

$$I_q^{\alpha,\beta,\eta}\{f(t)\} \geq 0 \tag{1.17}$$

for all $\alpha > 0$ *and* $\beta, \eta \in \mathbb{R}$ *with* $\alpha + \beta > 0$ *and* $\eta < 0$;

(ii) *The q-analogue of Riemann-Liouville fractional integral operator of the function* $f(t)$ *of order* α *in* (1.9)

$$I_q^{\alpha}\{f(t)\} \geq 0 \tag{1.18}$$

for all $\alpha > 0$;

(iii) *The q-analogue of Erdélyi-Kober fractional integral operator of the function* $f(t)$ *in* (1.10)

$$I_q^{\eta,\alpha}\{f(t)\} \geq 0 \tag{1.19}$$

for all $\alpha > 0$ *and* $\eta \in \mathbb{R}$.

2 Certain fractional *q*-integral inequalities

In this section, we establish certain Pólya-Szegö type integral inequalities for the synchronous functions involving the hypergeometric fractional integral operator (1.8), some of which are presumably (new) ones. For our purpose, we begin with providing the following lemma involving a q-analogue of Saigo's fractional integral operator.

Lemma 2 *Let* $0 < q < 1$, u *and* v *be two continuous and positive integrable functions on* $[0,\infty)$ *with*

$$0 < m_1 \leq u(\tau) \leq M_1 < \infty, \qquad 0 < n_1 \leq v(\tau) \leq N_1 < \infty \quad \bigl(\tau \in [0,t], t > 0\bigr). \tag{2.1}$$

Then the following inequality holds true:

$$\frac{(I_q^{\alpha,\beta,\eta}\{u^2(t)\})(I_q^{\alpha,\beta,\eta}\{v^2(t)\})}{(I_q^{\alpha,\beta,\eta}\{u(t)\}\{v(t)\})^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1N_1}{m_1n_1}} + \sqrt{\frac{m_1n_1}{M_1N_1}}\right)^2 \tag{2.2}$$

for all $\alpha > 0$, *and* $\beta, \eta \in \mathbb{R}$ *with* $\alpha + \beta > 0$, *and* $\eta < 0$.

Proof From (2.1), for $\tau \in [0,t]$, $t > 0$, we have

$$\frac{u(\tau)}{v(\tau)} \leq \frac{m_1}{N_1}, \tag{2.3}$$

which yields

$$\bigl(N_1u(\tau) - m_1v(\tau)\bigr) \leq 0. \tag{2.4}$$

Analogously, we have

$$\frac{n_1}{M_1} \leq \frac{u(\tau)}{v(\tau)}, \tag{2.5}$$

from which one has

$$\big(n_1u(\tau) - M_1v(\tau)\big) \le 0. \tag{2.6}$$

Multiplying (2.4) and (2.6), we obtain

$$(M_1N_1 + m_1n_1)u(\tau)v(\tau) \ge M_1m_1v^2(\tau) + N_1n_1u^2(\tau). \tag{2.7}$$

Now, multiplying both sides of (2.7) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\tau/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m$$

and taking q-integration of the resulting inequality with respect to τ from 0 to t with the aid of Definition 2, we get

$$(M_1N_1 + m_1n_1)I_q^{\alpha,\beta,\eta}\big\{u(t)v(t)\big\} \ge M_1m_1I_q^{\alpha,\beta,\eta}\big\{v^2(t)\big\} + N_1n_1I_q^{\alpha,\beta,\eta}\big\{u^2(t)\big\}. \tag{2.8}$$

Applying the AM-GM inequality, *i.e.*, $a + b \ge 2\sqrt{ab}$, $a, b \in \mathbb{R}^+$, we have

$$(M_1N_1 + m_1n_1)I_q^{\alpha,\beta,\eta}\big\{u(t)v(t)\big\} \ge 2\sqrt{M_1m_1I_q^{\alpha,\beta,\eta}\big\{v^2(t)\big\}N_1n_1I_q^{\alpha,\beta,\eta}\big\{u^2(t)\big\}}. \tag{2.9}$$

This implies that after little simplification

$$\frac{I_q^{\alpha,\beta,\eta}\{u^2(t)\}I_q^{\alpha,\beta,\eta}\{v^2(t)\}}{\{I_q^{\alpha,\beta,\eta}\{u(t)v(t)\}\}^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1N_1}{m_1n_1}} + \sqrt{\frac{m_1n_1}{M_1N_1}}\right)^2. \tag{2.10}$$

This completes the proof of Lemma 2. □

Theorem 1 *Let* $0 < q < 1$, f *and* g *be two positive integrable functions on* $[0,\infty)$ *and* m, M, n, N *be positive real numbers with inequality* (2.1) *holds. Then the following inequality holds true*:

$$\begin{aligned}&\left|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\big\{f(t)g(t)\big\}\right| \\ &\quad \le \frac{(M-m)(N-n)}{4\sqrt{mMnN}}I_q^{\alpha,\beta,\eta}\big\{f(t)\big\}I_q^{\alpha,\beta,\eta}\big\{g(t)\big\}\end{aligned} \tag{2.11}$$

for all $\alpha > 0$, *and* $\beta, \eta \in \mathbb{R}$ *with* $\alpha + \beta > 0$, *and* $\eta < 0$.

Proof Let f and g be two positive integrable functions on $[0,\infty)$. Then, for all $\tau, \rho \in (0,t)$ with $t > 0$, we have

$$A(\tau,\rho) = \big(f(\tau) - f(\rho)\big)\big(g(\tau) - g(\rho)\big), \tag{2.12}$$

or, equivalently,

$$A(\tau,\rho) = f(\tau)g(\tau) + f(\rho)g(\rho) - f(\tau)g(\rho) - f(\rho)g(\tau). \tag{2.13}$$

Now, multiplying both sides of (2.13) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\tau/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m$$

and taking q-integration of the resulting inequality with respect to τ from 0 to t with the aid of Definition 2, we get

$$\begin{aligned}&\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^t(q\tau/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\\&\quad\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m A(\tau,\rho)\,d_q\tau\\&\quad=I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}+\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}f(\rho)g(\rho)\\&\quad\quad-g(\rho)I_q^{\alpha,\beta,\eta}\{f(t)\}-f(\rho)I_q^{\alpha,\beta,\eta}\{g(t)\}.\end{aligned}\tag{2.14}$$

Again, multiplying both sides of (2.14) by

$$\frac{t^{-\beta-1}}{\Gamma_q(\alpha)}(q\rho/t;q)_{\alpha-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\rho/t-1)_q^m$$

and taking q-integration of the resulting inequality with respect to ρ from 0 to t and using (1.8), we get

$$\begin{aligned}&\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t(q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\right.\\&\quad\left.\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\right\}^2(\tau/t-1)_q^m(\rho/t-1)_q^m A(\tau,\rho)\,d_q\tau\,d_q\rho\\&\quad=2\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}-2I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}.\end{aligned}\tag{2.15}$$

By using the Cauchy-Schwarz inequality for double integrals, we have

$$\begin{aligned}&\left|\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t(q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\right.\right.\\&\quad\left.\left.\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\right\}^2(\tau/t-1)_q^m(\rho/t-1)_q^m A(\tau,\rho)\,d_q\tau\,d_q\rho\right|\\&\quad\le\left[\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t(q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\right.\right.\\&\quad\left.\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\right\}^2(\tau/t-1)_q^m(\rho/t-1)_q^m f^2(\tau)\,d_q\tau\,d_q\rho\\&\quad\quad+\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t(q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\left\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\right.\end{aligned}$$

$$
\begin{aligned}
&\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\Bigg\}^2 (\tau/t-1)_q^m(\rho/t-1)_q^m f^2(\tau)\,d_q\tau\,d_q\rho \\
&-2\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t (q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\Bigg\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^\alpha;q)_m(q;q)_m} \\
&\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\Bigg\}^2 (\tau/t-1)_q^m(\rho/t-1)_q^m g^2(\tau)\,d_q\tau\,d_q\rho \\
&+\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t (q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\Bigg\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^\alpha;q)_m(q;q)_m} \\
&\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\Bigg\}^2 (\tau/t-1)_q^m(\rho/t-1)_q^m g^2(\tau)\,d_q\tau\,d_q\rho \\
&-2\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t (q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\Bigg\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^\alpha;q)_m(q;q)_m} \\
&\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\Bigg\}^2 (\tau/t-1)_q^m(\rho/t-1)_q^m g(\tau)g(\rho)\,d_q\tau\,d_q\rho\Bigg]^{\frac{1}{2}}.
\end{aligned}
\tag{2.16}
$$

Applying Definition 2, we get

$$
\begin{aligned}
&\Bigg|\frac{t^{-2(\beta+1)}}{\Gamma_q^2(\alpha)}\int_0^t\int_0^t (q\tau/t;q)_{\alpha-1}(q\rho/t;q)_{\alpha-1}\Bigg\{\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^\alpha;q)_m(q;q)_m} \\
&\quad\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}\Bigg\}^2 (\tau/t-1)_q^m(\rho/t-1)_q^m A(\tau,\rho)\,d_q\tau\,d_q\rho\Bigg| \\
&\le 2\left[\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\}-\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2\right]^{\frac{1}{2}} \\
&\quad\cdot\left[\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\{g^2(t)\}-\left(I_q^{\alpha,\beta,\eta}\{g(t)\}\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
\tag{2.17}
$$

By applying Lemma 2, we get

$$
\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\} \\
&\le\frac{1}{4}\left(\sqrt{\frac{M}{m}}+\sqrt{\frac{m}{M}}\right)^2\left(\{I_q^{\alpha,\beta,\eta}f(t)\}\right)^2 \\
&=\frac{(M+m)^2}{4mM}\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2.
\end{aligned}
\tag{2.18}
$$

After little simplification, we get

$$
\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\alpha,\beta,\eta}\{f^2(t)\}-\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2 \\
&\le\left(\frac{(M+m)^2}{4mM}-1\right)\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2
\end{aligned}
\tag{2.19}
$$

or

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} t^{-\beta} I_q^{\alpha,\beta,\eta}\{f^2(t)\} - \left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2 \\ \leq \frac{(M-m)^2}{4mM}\left(I_q^{\alpha,\beta,\eta}\{f(t)\}\right)^2. \tag{2.20}$$

Similarly, we get

$$\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} t^{-\beta} I_q^{\alpha,\beta,\eta}\{g^2(t)\} - \left(I_q^{\alpha,\beta,\eta}\{g(t)\}\right)^2 \\ \leq \frac{(N-n)^2}{4nN}\left(I_q^{\alpha,\beta,\eta}\{g(t)\}\right)^2. \tag{2.21}$$

Finally, by adding (2.14), (2.17), (2.20) and (2.21), side by side, we arrive at the desired result (2.11). □

In the sequel, we can present another inequality involving the q-fractional integral operator given in (1.8), asserted by the following lemma.

Lemma 3 *Let* $0 < q < 1$, u *and* v *be two continuous and positive integrable functions on* $[0,\infty)$ *with* (2.1) *holds. Then the following inequality holds true*:

$$\frac{(I_q^{\alpha,\beta,\eta}\{u^2(t)\})(I_q^{\gamma,\delta,\zeta}\{v^2(t)\})}{(I_q^{\alpha,\beta,\eta}\{u(t)\}\{v(t)\})(I_q^{\gamma,\delta,\zeta}\{u(t)\}\{v(t)\})} \leq \frac{1}{4}\left(\sqrt{\frac{M_1N_1}{m_1n_1}} + \sqrt{\frac{m_1n_1}{M_1N_1}}\right)^2 \tag{2.22}$$

for all $\alpha,\gamma > 0$, *and* $\beta,\eta,\delta,\zeta \in \mathbb{R}$ *with* $\alpha+\beta > 0$, $\gamma+\delta > 0$, *and* $\eta,\zeta < 0$.

Proof To prove Lemma 2, we start from the condition

$$\frac{m_1}{N_1} \leq \frac{u(\tau)}{v(\tau)} \quad \left(\tau \in [0,t], t > 0\right), \tag{2.23}$$

we get

$$\frac{m_1}{N_1}\left(v^2(\tau)\right) \leq u(\tau)v(\tau) \quad \left(\tau \in [0,t], t > 0\right). \tag{2.24}$$

Now, multiplying both sides of (2.24) by

$$\frac{t^{-\delta-1}}{(\Gamma_q(\gamma))}(q\rho/t;q)_{\gamma-1}\sum_{n=0}^{\infty}\frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^\gamma;q)_n(q;q)_n} \\ \cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n \quad \left(\rho \in (0,t); t > 0\right),$$

and integrating with respect to ρ from 0 to t, we get

$$\frac{m_1}{N_1} I_t^{\gamma,\delta,\zeta,\nu}\left\{v^2(t)\right\} \leq I_t^{\gamma,\delta,\zeta,\nu}\left\{u(t)v(t)\right\}. \tag{2.25}$$

Multiplying (2.24) and (2.25), we get the desired result (2.22). This completes the proof of Lemma 2. □

Theorem 2 *Let $0 < q < 1$, f and g be two positive integrable functions on $[0,\infty)$ and there exist positive real numbers m, n, M, N with inequality* (2.1) *holds. Then we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}t^{-\delta}I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}+\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}\\&\quad-I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\}-I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\}\Bigg|\\&\quad\le\frac{(M-m)(N-n)}{2\sqrt{mMnN}}I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\}\end{aligned}\tag{2.26}$$

for all $\alpha,\gamma>0$, and $\beta,\eta,\delta,\zeta\in\mathbb{R}$ with $\alpha+\beta>0$, $\gamma+\delta>0$, and $\eta,\zeta<0$.

Proof Multiplying both sides of (2.14) by

$$\begin{aligned}&\frac{t^{-\delta-1}}{(\Gamma_q(\gamma))}(q\rho/t;q)_{\gamma-1}\sum_{n=0}^{\infty}\frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n}\\&\quad\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n\quad\big(\rho\in(0,t);t>0\big),\end{aligned}$$

and integrating with respect to ρ from 0 to t, we get

$$\begin{aligned}&\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)}\int_0^t\int_0^t(q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1}\\&\quad\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m\\&\quad\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n A(\tau,\rho)\,d_q\tau\,d_q\rho\\&\quad=\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}t^{-\delta}I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}+\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}\\&\quad-I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\}-I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\}.\end{aligned}\tag{2.27}$$

By using the Cauchy-Schwarz inequality for double integrals, we have

$$\begin{aligned}&\Bigg|\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)}\int_0^t\int_0^t(q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1}\\&\quad\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m\\&\quad\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n A(\tau,\rho)\,d_q\tau\,d_q\rho\Bigg|\\&\quad\le\Bigg[\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)}\int_0^t\int_0^t(q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1}\\&\quad\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m}\frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n}\cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m\\&\quad\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n f^2(\rho)\,d_q\tau\,d_q\rho\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \\
&\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n f^2(\tau)\, d_q\tau\, d_q\rho \\
&- 2\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \\
&\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\left.\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n f(\tau)f(\rho)\, d_q\tau\, d_q\rho \right]^{\frac{1}{2}} \\
&\cdot \left[\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \right. \\
&\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n g^2(\rho)\, d_q\tau\, d_q\rho \\
&+ \frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \\
&\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n g^2(\tau)\, d_q\tau\, d_q\rho \\
&- 2\frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \\
&\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\left.\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n g(\tau)g(\rho)\, d_q\tau\, d_q\rho \right]^{\frac{1}{2}}. \qquad (2.28)
\end{aligned}
$$

Applying Definition 2, we get

$$
\begin{aligned}
&\left| \frac{t^{-\delta-1}t^{-\beta-1}}{\Gamma_q(\gamma)\Gamma_q(\alpha)} \int_0^t \int_0^t (q\rho/t;q)_{\gamma-1}(q\tau/t;q)_{\alpha-1} \right. \\
&\quad\cdot \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q^{\alpha};q)_m(q;q)_m} \frac{(q^{\gamma+\delta};q)_n(q^{-\zeta};q)_n}{(q^{\gamma};q)_n(q;q)_n} \cdot q^{(\eta-\beta)m}(-1)^m q^{-\binom{m}{2}}(\tau/t-1)_q^m \\
&\quad\left.\cdot q^{(\zeta-\delta)n}(-1)^n q^{-\binom{n}{2}}(\rho/t-1)_q^n A(\tau,\rho)\, d_q\tau\, d_q\rho \right| \\
&\quad\le 2\left[\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} t^{-\delta} I_q^{\alpha,\beta,\eta}\{f^2(t)\} + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} t^{-\beta} I_q^{\gamma,\delta,\zeta}\{f^2(t)\} \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2I_q^{\alpha,\beta,\eta}\{f(t)\}I_q^{\gamma,\delta,\zeta}\{f(t)\}\Bigg]^{\frac{1}{2}} \\
&\quad \cdot \Bigg[\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}t^{-\delta}I_q^{\alpha,\beta,\eta}\{g^2(t)\} + \frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_t^{\gamma,\delta,\zeta}\{g^2(t)\} \\
&\quad - 2I_q^{\alpha,\beta,\eta}\{g(t)\}I_q^{\gamma,\delta,\zeta}\{g(t)\}\Bigg]^{\frac{1}{2}}. \qquad (2.29)
\end{aligned}
$$

Applying Definition 2, we get

$$
\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\gamma,\delta,\zeta}\{f^2(t)\} - I_q^{\gamma,\delta,\zeta}\{f(t)\}I_q^{\alpha,\beta,\eta}\{f(t)\} \\
&\quad \le \frac{(M-m)^2}{4mM}\big(I_q^{\gamma,\delta,\zeta}\{f(t)\}\big)\big(I_q^{\alpha,\beta,\eta}\{f(t)\}\big), \qquad (2.30)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}t^{-\delta}I_q^{\alpha,\beta,\eta}\{f^2(t)\} - I_q^{\gamma,\delta,\zeta}\{f(t)\}I_q^{\alpha,\beta,\eta}\{f(t)\} \\
&\quad \le \frac{(M-m)^2}{4mM}\big(I_q^{\gamma,\delta,\zeta}\{f(t)\}\big)\big(I_q^{\alpha,\beta,\eta}\{f(t)\}\big). \qquad (2.31)
\end{aligned}
$$

Similarly, for the function $g(t)$, we get

$$
\begin{aligned}
&\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}t^{-\beta}I_q^{\gamma,\delta,\zeta}\{g^2(t)\} - I_q^{\gamma,\delta,\zeta}\{g(t)\}I_q^{\alpha,\beta,\eta}\{g(t)\} \\
&\quad \le \frac{(M-m)^2}{4mM}\big(I_q^{\gamma,\delta,\zeta}\{g(t)\}\big)\big(I_q^{\alpha,\beta,\eta}\{g(t)\}\big), \qquad (2.32)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}t^{-\delta}I_q^{\alpha,\beta,\eta}\{g^2(t)\} - I_q^{\gamma,\delta,\zeta}\{g(t)\}I_q^{\alpha,\beta,\eta}\{g(t)\} \\
&\quad \le \frac{(M-m)^2}{4mM}\big(I_q^{\gamma,\delta,\zeta}\{g(t)\}\big)\big(I_q^{\alpha,\beta,\eta}\{g(t)\}\big). \qquad (2.33)
\end{aligned}
$$

Finally, in view of (2.27) to (2.33), we arrive at the desired result (2.26). This completes the proof of Theorem 2. □

Remark 1 It may be noted that the inequality in (2.26) when $\zeta = \eta$ reduces immediately to that in (2.11).

3 Special cases and concluding remarks

By virtue of the unified nature of Saigo's fractional q-integral operator (1.8), a large number of new and known integral inequalities involving q-analogues of the Riemann-Liouville and Erdélyi-Kober fractional integral operators are seen to follow as special cases of our main result. Indeed, by suitably specializing the values of parameters α, β, η (and γ, δ, ζ in addition of Theorem 2), inequalities (2.11) and (2.26) in Theorems 1 and 2, respectively, would yield further Grüss type integral inequalities involving the above-mentioned integral operators.

If we put $\beta = 0$ (and $\delta = 0$ in addition in Theorem 2), using (1.10), inequalities (2.11) and (2.26) gives the following results involving q-analogues of the Erdélyi-Kober fractional integral operators, which are believed to be new.

Corollary 1 *Let $0 < q < 1$, f and g be two positive integrable functions on $[0,\infty)$ and m, M, n, N be positive real numbers with inequality* (2.1) *holds. Then the following inequality holds true*:

$$\left|\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I_q^{\eta,\alpha}\{f(t)g(t)\}\right| \le \frac{(M-m)(N-n)}{4\sqrt{mMnN}} I_q^{\eta,\alpha}\{f(t)\} I_q^{\eta,\alpha}\{g(t)\} \tag{3.1}$$

for all $\alpha > 0$, and $\eta \in \mathbb{R}$ with $\eta < 0$.

Corollary 2 *Let $0 < q < 1$, f and g be two positive integrable functions on $[0,\infty)$ and there exist positive real numbers m, n, M, N with inequality* (2.1) *holds. Then we have*

$$\begin{aligned} &\left|\frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)} I_q^{\eta,\alpha}\{f(t)g(t)\} + \frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)} I_q^{\zeta,\gamma}\{f(t)g(t)\}\right. \\ &\quad \left. - I_q^{\eta,\alpha}\{f(t)\} I_q^{\zeta,\gamma}\{g(t)\} - I_{0,t}^{\eta,\alpha}\{g(t)\} I_q^{\zeta,\gamma}\{f(t)\}\right| \\ &\quad \le \frac{(M-m)(N-n)}{2\sqrt{mMnN}} I_{0,t}^{\eta,\alpha}\{f(t)\} I_q^{\zeta,\gamma}\{g(t)\} \end{aligned} \tag{3.2}$$

for all $\alpha, \gamma > 0$, and $\eta, \zeta \in \mathbb{R}$ with $\eta, \zeta < 0$.

Similarly, if we set $\eta = 0$ and replace β by $-\alpha$ in Theorem 1 (and $\zeta = 0$ and replace δ by $-\gamma$ in addition in Theorem 2), using (1.9), inequalities (2.11) and (2.26) gives the following results involving q-analogues of the Riemann-Liouville and Erdélyi-Kober fractional integral operators, which are also believed to be new.

Corollary 3 *Let $0 < q < 1$, f and g be two positive integrable functions on $[0,\infty)$ and m, M, n, N be positive real numbers with inequality* (2.1) *holds. Then the following inequality holds true*:

$$\left|\frac{1}{\Gamma(1+\alpha)} t^{\alpha} I_q^{\alpha}\{f(t)g(t)\}\right| \le \frac{(M-m)(N-n)}{4\sqrt{mMnN}} I_q^{\alpha}\{f(t)\} I_q^{\alpha}\{g(t)\} \tag{3.3}$$

for all $\alpha > 0$.

Corollary 4 *Let $0 < q < 1$, f and g be two positive integrable functions on $[0,\infty)$ and there exist positive real numbers m, n, M, N with inequality* (2.1) *holds. Then we have*

$$\begin{aligned} &\left|\frac{1}{\Gamma(1+\gamma)} t^{\gamma} I_q^{\alpha}\{f(t)g(t)\} + \frac{1}{\Gamma(1+\alpha)} t^{\alpha} I_q^{\gamma}\{f(t)g(t)\}\right. \\ &\quad \left. - I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)\} - I_q^{\alpha}\{g(t)\} I_q^{\gamma}\{f(t)\}\right| \\ &\quad \le \frac{(M-m)(N-n)}{2\sqrt{mMnN}} I_q^{\alpha}\{f(t)\} I_q^{\gamma}\{g(t)\} \end{aligned} \tag{3.4}$$

for all $\alpha, \gamma > 0$.

We conclude this paper by emphasizing, again, that our main result here, being of a very general nature, can be specialized to yield numerous interesting fractional integral inequalities including q-analogues of some known results (see, for example [13]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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References

1. Anastassiou, G: q-Fractional inequalities. CUBO **13**, 61-71 (2011)
2. Anastassiou, GA: Fractional Differentiation Inequalities. Springer, Dordrecht (2009)
3. Cerone, P, Dragomir, SS: A refinement of the Grüss inequality and applications. Tamkang J. Math. **38**, 37-49 (2007)
4. Choi, J, Agarwal, P: Some new Saigo type fractional integral inequalities and their q-analogues. Abstr. Appl. Anal. **2014**, Article ID 579260 (2014)
5. Choi, J, Agarwal, P: Certain fractional integral inequalities involving hypergeometric operators. East Asian Math. J. **30**, 283-291 (2014)
6. Choi, J, Agarwal, P: Certain new pathway type fractional integral inequalities. Honam Math. J. **36**, 437-447 (2014)
7. Choi, J, Agarwal, P: Certain integral transform and fractional integral formulas for the generalized Gauss hypergeometric functions. Abstr. Appl. Anal. **2014**, Article ID 735946 (2014)
8. Mazouzi, S, Qi, F: On an open problem regarding an integral inequality. J. Inequal. Pure Appl. Math. **4**(2), 31 (2003)
9. Mitrinović, DS, Pečarić, JE, Fink, AM: Classical and New Inequalities in Analysis. Mathematics and Its Applications (East European Series), vol. 61. Kluwer Academic, Dordrecht (1993)
10. Pachpatte, BG: On multidimensional Grüss type integral inequalities. J. Inequal. Pure Appl. Math. **3**(2), 27 (2002)
11. Podlubny, I: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
12. Zhu, C, Yang, W, Zhao, Q: Some new fractional q-integral Grüss-type inequalities and other inequalities. J. Inequal. Appl. **2012**, Article ID 299 (2012)
13. Anber, A, Dahmani, Z: New integral results using Pólya-Szegö inequality. Acta Comment. Univ. Tartu Math. **17**(2), 171-178 (2013)
14. Chebyshev, PL: Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. Proc. Math. Soc. Charkov **2**, 93-98 (1882)
15. Anastassiou, GA: Advances on Fractional Inequalities. Springer Briefs in Mathematics. Springer, New York (2011)
16. Belarbi, S, Dahmani, Z: On some new fractional integral inequalities. J. Inequal. Pure Appl. Math. **10**(3), 86 (2009) (electronic)
17. Dahmani, Z, Mechouar, O, Brahami, S: Certain inequalities related to the Chebyshev's functional involving a type Riemann-Liouville operator. Bull. Math. Anal. Appl. **3**(4), 38-44 (2011)
18. Dragomir, SS: Some integral inequalities of Grüss type. Indian J. Pure Appl. Math. **31**(4), 397-415 (2000)
19. Kalla, SL, Rao, A: On Grüss type inequality for hypergeometric fractional integrals. Matematiche **66**(1), 57-64 (2011)
20. Lakshmikantham, V, Vatsala, AS: Theory of fractional differential inequalities and applications. Commun. Appl. Anal. **11**, 395-402 (2007)
21. Öğünmez, H, Özkan, UM: Fractional quantum integral inequalities. J. Inequal. Appl. **2011**, Article ID 787939 (2011)
22. Sulaiman, WT: Some new fractional integral inequalities. J. Math. Anal. **2**(2), 23-28 (2011)
23. Wang, G, Agarwal, P, Chand, M: Certain Grüss type inequalities involving the generalized fractional integral operator. J. Inequal. Appl. **2014**, Article ID 147 (2014)
24. Grüss, G: Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$. Math. Z. **39**, 215-226 (1935)
25. Pólya, G, Szegö, G: Aufgaben und Lehrsatze aus der Analysis. Band 1. Die Grundlehren der mathmatischen Wissenschaften, vol. 19. Springer, Berlin (1925)
26. Dragomir, SS, Diamond, NT: Integral inequalities of Grüss type via Pólya-Szegö and Shisha-Mond results. East Asian Math. J. **19**(1), 27-39 (2003)
27. Garg, M, Chanchlani, L: q-Analogues of Saigo's fractional calculus operators. Bull. Math. Anal. Appl. **3**(4), 169-179 (2011)
28. Choi, J, Agarwal, P: Some new Saigo type fractional integral inequalities and their q-analogues. Abstr. Appl. Anal. **2014**, Article ID 579260 (2014)