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Order generalised gradient and operator inequalities

Sever S Dragomir[1] and Eder Kikianty[2*]

*Correspondence:
ekikianty@uj.ac.za;
eder.kikianty@gmail.com
[2]Department of Pure and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park, 2006, South Africa
Full list of author information is available at the end of the article

Abstract
We introduce the notion of order generalised gradient, a generalisation of the notion of subgradient, in the context of operator-valued functions. We state some operator inequalities of Hermite-Hadamard and Jensen types. We discuss the connection between the notion of order generalised gradient and the Gâteaux derivative of operator-valued functions. We state a characterisation of operator convexity via an inequality concerning the order generalised gradient.
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1 Background

Convex functions play a crucial role in many fields of mathematics, most prominently in optimisation theory. There are two main important inequalities which characterise convex functions, namely Jensen's and Hermite-Hadamard's inequalities. In 1905 (1906), Jensen defined convex functions as follows: $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function if and only if

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} \quad \text{for any } a, b \in I. \tag{1}$$

Inequality (1) is referred to as Jensen's inequality. Hermite-Hadamard's inequality provides a refinement for Jensen's inequality, namely, for a convex function $f : I \subset \mathbb{R} \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \quad \text{for any } a, b \in I. \tag{2}$$

We refer the reader to Section 2 for further details regarding these inequalities.

Similarly to the case of real-valued functions, the operator convexity can be characterised by some operator inequalities. Hansen and Pedersen [1] characterise operator convexity via a non-commutative generalisation of Jensen's inequality. If f is a real continuous function on an interval I, and $\mathcal{A}(H)$ is the set of bounded self-adjoint operators on a Hilbert space H with spectra in I, then f is operator convex if and only if

$$f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i$$

for $x_1,\dots,x_n \in \mathcal{A}(H)$ and $a_1,\dots,a_n \in \mathcal{B}(H)$ with $\sum_{i=1}^n a_i^* a_i = \mathbf{1}$. We refer the reader to Section 2 for further details regarding this characterisation.

One of the useful differential properties of convex functions is the fact that their one-sided directional derivatives exist universally [2, p.213]. Just as the ordinary two-sided directional derivatives of a differentiable function can be described in terms of gradient vectors, the one-sided directional derivatives can be described in terms of 'subgradient' vectors [2, p.213]. A vector x^* is said to be a subgradient of a convex function $f : K \subset \mathbb{R}^n \to \mathbb{R}$ at point x if

$$f(x) - f(y) \geq x^* \cdot (x - y) \quad \text{for all } y \in K. \tag{3}$$

This condition is referred to as the *subgradient inequality* [2, p.214]. If (3) holds for every $x \in K$, then (3) characterises the convexity of f (cf. Eisenberg [3, Theorem 1]).

In this paper, we introduce the notion of *order generalised gradient* (*cf.* Section 3) for operator-valued functions, which is a generalisation of (3) (without the assumption of convexity) in the settings of bounded self-adjoint operators on a Hilbert space. Furthermore, we state some inequalities of Hermite-Hadamard and Jensen types for the order generalised gradient in Section 4. Finally, in Section 5, we state the connection between the order generalised gradient and Gâteaux derivative of operator-valued functions. We state a characterisation of convexity analogues to (3) in the context of operator-valued functions.

2 Inequalities for convex functions

This section serves as a point of reference for known results regarding some inequalities related to convex functions (both real-valued and operator-valued functions).

2.1 Jensen's inequality

Jensen's inequality for convex functions plays a crucial role in the theory of inequalities due to the fact that other inequalities, such as the arithmetic-geometric mean, Hölder, Minkowski and Ky Fan's inequalities, can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f be a convex function on C. If $\mathbf{p} = (p_1,\dots,p_n)$ is a probability sequence and $\mathbf{x} = (x_1,\dots,x_n) \in C^n$, then

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i). \tag{4}$$

This inequality is referred to as *Jensen's inequality*. Recently, Dragomir [4] obtained the following refinement of Jensen's inequality:

$$\begin{aligned} f\left(\sum_{j=1}^n p_j x_j\right) &\leq \min_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^n p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right] \\ &\leq \frac{1}{n}\left[\sum_{k=1}^n (1-p_k) f\left(\frac{\sum_{j=1}^n p_j x_j - p_k x_k}{1-p_k}\right) + \sum_{k=1}^n p_k f(x_k)\right] \end{aligned}$$

$$\begin{aligned}
&\leq \max_{k\in\{1,\ldots,n\}}\left[(1-p_k)f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right] \\
&\leq \sum_{j=1}^{n} p_j f(x_j), \qquad (5)
\end{aligned}$$

where f, x_k and p_k are as defined above. For other refinements of Jensen's inequality, we refer the reader to Pečarić and Dragomir [5] and Dragomir [6].

The above result provides a different approach to the one that Pečarić and Dragomir [5] obtained in 1989

$$\begin{aligned}
f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\leq \cdots \leq \sum_{i=1}^{n} p_i f(x_i) \qquad (6)
\end{aligned}$$

for $k \geq 1$, and $\mathbf{p}$, $\mathbf{x}$ are as defined above.

If $q_1,\ldots,q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following refinement obtained in 1994 by Dragomir [6] also holds:

$$\begin{aligned}
f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \\
&\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f(q_1 x_{i_1}+\cdots+q_k x_{i_{k+1}}) \\
&\leq \sum_{i=1}^{n} p_i f(x_i), \qquad (7)
\end{aligned}$$

where $1 \leq k \leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as defined above.

For more refinements and applications related to the generalised triangle inequality, the arithmetic-geometric mean inequality, the f-divergence measures, Ky Fan's inequality, *etc.*, we refer the readers to [6–10] and [4].

2.2 Hermite-Hadamard's inequality

The following inequality also holds for any convex function f defined on $\mathbb{R}$:

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)\,dx \leq (b-a)\frac{f(a)+f(b)}{2}, \quad a,b\in\mathbb{R}. \qquad (8)$$

It was first discovered by Hermite in 1881 in the journal *Mathesis* [11]. However, this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [12].

Beckenbach, a leading expert on the history and the theory of convex functions wrote that this inequality was proven by Hadamard in 1893 [13]. In 1974, Mitrinović found Hermite's note in *Mathesis* [11]. Since (8) was known as Hadamard's inequality, the inequality is now commonly referred to as *Hermite-Hadamard's inequality* [12].

Hermite-Hadamard's inequality has been extended in many different directions. One of the extensions of this inequality is in the vector space settings. Firstly, we start with the following definitions and notation: Let X be a vector space and x, y be two distinct vectors in X. We define the segment generated by x and y to be the set

$$[x,y] := \{(1-t)x + ty, t \in [0,1]\}.$$

For any real-valued function f defined on the segment $[x,y]$, there exists an associated function $g_{x,y} : [0,1] \to \mathbb{R}$ with

$$g_{x,y}(t) = f[(1-t)x + ty].$$

We remark that f is convex on $[x,y]$ if and only if g is convex on $[0,1]$. For any convex function defined on a segment $[x,y] \subset X$, we have the *Hermite-Hadamard integral inequality* (*cf.* Dragomir [14, p.2] and Dragomir [15, p.2]):

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x + ty]\,dt \le \frac{f(x)+f(y)}{2}, \quad x, y \in X, \tag{9}$$

which can be derived by the classical Hermite-Hadamard inequality (8) for the convex function $g_{x,y} : [0,1] \to \mathbb{R}$. Consider the function $f(x) = \|x\|^p$ ($x \in X$ and $1 \le p < \infty$), which is convex on X, then we have the following norm inequality (derived from (9)) [16, p.106]:

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x + ty\|^p\,dt \le \frac{\|x\|^p + \|y\|^p}{2} \tag{10}$$

for any $x, y \in X$.

2.3 Non-commutative generalisation of Jensen's inequality

Hansen [17] discussed Jensen's operator inequality for operator monotone functions. Motivated by Aujla's work [18] on the matrix convexity of functions of two variables, Hansen [19] characterised operator convex functions of two variables in terms of a non-commutative generalisation of Jensen's inequality (*cf.* [19, Theorem 3.1]). A simplified proof of this result formulated for matrices is given in Aujla [20]. The case for several variables is given in Hansen [21]. The case for self-adjoint elements in the algebra M_n of n-square matrices is given in Hansen and Pedersen [22]. Finally, Hansen and Pedersen [1] presented a generalisation of the above results for self-adjoint operators defined on a Hilbert space.

Theorem 1 *We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on the Hilbert space H. If f is a real continuous function on an interval I, and $\mathcal{A}(H)$ is the set of bounded self-adjoint operators on a Hilbert space H with spectra in I, then the following conditions are equivalent*:

(i) *f is operator convex*;

(ii) *$f(\sum_{i=1}^n a_i^* x_i a_i) \le \sum_{i=1}^n a_i^* f(x_i) a_i$ for $x_1, \ldots, x_n \in \mathcal{A}(H)$ and $a_1, \ldots, a_n \in \mathcal{B}(H)$ with $\sum_{i=1}^n a_i^* a_i = \mathbf{1}$*;

(iii) $f(v^*xv) \leq v^*f(x)v$ *for any* $x \in \mathcal{A}(H)$ *and any isometry* $v \in \mathcal{B}(H)$;
(iv) $pf(pxp + s(\mathbf{1} - p))p \leq pf(x)p$ *for* $x \in \mathcal{A}(H)$, *projection* $p \in \mathcal{B}(H)$, *every self-adjoint operator* x *with spectrum in* I *and* $s \in I$.

2.4 Subgradient inequality

Recall the following definition of a subgradient [2].

Definition 2 A vector x^* is said to be a subgradient of a convex function $f : K \subset \mathbb{R}^n \to \mathbb{R}$ at point x if

$$f(x) - f(y) \geq x^* \cdot (x - y) \quad \text{for all } y \in K.$$

The following theorem is a useful characterisation of convexity (*cf.* Eisenberg [3, Theorem 1]).

Theorem 3 *If* U *is a nonempty open subset of* $\mathbb{R}^n$, $f : U \to \mathbb{R}$ *is a differentiable function on* U, *and* K *is a convex subset of* U, *then* f *is convex on* K *if and only if*

$$f(x) - f(y) \geq (x - y)^T f'(y) \quad \text{for all } x, y \in K \tag{11}$$

where $f'(y)$ *denotes the gradient of* f *at* y.

This theorem has been generalised and employed in obtaining optimality conditions of a non-differentiable minimax programming problem in complex spaces (*cf.* Lai and Liu [23]). Note that $(x - y)^T f'(y)$ can be written as $f'(y) \cdot (x - y)$, which can be interpreted as the directional derivative of f at point y in $x - y$ direction.

3 Order generalised gradient

Throughout the paper, we use the following notation. We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, and by $\mathcal{A}(H)$ the linear subspace of all self-adjoint operators on H. We denote by $\mathcal{P}_+(H) \subset \mathcal{A}(H)$ the convex cone of all positive definite operators defined on H, that is, $P \in \mathcal{P}_+(H)$ if and only if $\langle Px, x\rangle \geq 0$, and for all $x \in H$, $\langle Px, x\rangle = 0$ implies $x = 0$. This gives a partial ordering (we refer to it as the operator order) on $\mathcal{A}(H)$, where two elements $A, B \in \mathcal{A}(H)$ satisfy $A \leq B$ if and only if $B - A \in \mathcal{P}_+(H)$.

Definition 4 Let $\mathcal{C}$ be a convex set in $\mathcal{A}(H)$. A function $f : \mathcal{C} \to \mathcal{A}(H)$ has the function $\nabla_f : \mathcal{C} \times \mathcal{A}(H) \to \mathcal{A}(H)$ as an *order generalised gradient* if

$$f(A) - f(B) \geq \nabla_f(B, A - B) \quad \text{for any } A, B \in \mathcal{C} \tag{12}$$

in the operator order of $\mathcal{A}(H)$.

Remark 5 We remark that in (12), if f is a real-valued differentiable function on an open set $U \subset \mathbb{R}^n$, and ∇_f is the gradient of f, then (12) becomes (11). We also note that there is no assumption of convexity at this point. We discuss the convexity case in Section 5.

Proposition 6 *If $Q \in \mathcal{A}(H)$ and $f : \mathcal{A}(H) \to \mathcal{A}(H)$, $f(A) = QA^2Q$, then*

$$\nabla_f(B,X) := Q(BX + XB)Q \tag{13}$$

is an order generalised gradient for f.

Proof Observe that $BX + XB \in \mathcal{A}(H)$ and if $P \in \mathcal{A}(H)$ then $P(BX + XB)P \in \mathcal{A}(H)$. We need to prove that

$$f(A) - f(B) \geq \nabla_f(B, A - B)$$

for any $A, B \in \mathcal{A}(H)$, that is,

$$QA^2Q - QB^2Q \geq Q\big[B(A - B) + (A - B)B\big]Q. \tag{14}$$

Since

$$Q\big[B(A - B) + (A - B)B\big]Q = QBAQ - QB^2Q + QABQ - QB^2Q,$$

hence (14) is equivalent to

$$QA^2Q - QB^2Q \geq QBAQ - QB^2Q + QABQ - QB^2Q$$

which is also equivalent to

$$Q(A - B)^2Q \geq 0$$

which always holds. This completes the proof. □

We denote by $\mathcal{P}(H) \subset \mathcal{A}(H)$ the convex cone of all nonnegative operators defined on H.

Proposition 7 *If $P \in \mathcal{P}(H)$, then the function $f : \mathcal{A}(H) \to \mathcal{A}(H)$, $f(A) = APA$ has*

$$\nabla_f(B,X) := XPB + BPX \tag{15}$$

as an order generalised gradient.

Proof Observe that $XPB + BPX \in \mathcal{A}(H)$. We need to prove that

$$\begin{aligned} APA - BPB &\geq (A - B)PB + BP(A - B) \\ &= APB - BPB + BPA - BPB, \end{aligned}$$

that is,

$$APA - APB - BPA + BPB \geq 0.$$

But $APA - APB - BPA + BPB = (A - B)P(A - B)$ and since $(A - B)P(A - B) \geq 0$, and this completes the proof. □

Recall $\mathcal{P}_+(H) \subset \mathcal{A}(H)$ the convex cone of all positive definite operators defined on H, that is, $P \in \mathcal{P}_+(H)$ if and only if $\langle Px, x\rangle \geq 0$, and for all $x \in H$, $\langle Px, x\rangle = 0$ implies $x = 0$.

Proposition 8 *Let $f : \mathcal{P}_+(H) \to \mathcal{A}(H)$ defined by*

$$f(A) = QA^{-1}Q,$$

where $Q \in \mathcal{A}(H)$. The function $\nabla_f : \mathcal{P}_+(H) \times \mathcal{P}_+(H) \to \mathcal{A}(H)$ with

$$\nabla_f(B, X) = -QB^{-1}XB^{-1}Q$$

is an order generalised gradient for f.

Proof For $B \in \mathcal{P}_+(H)$, $B^{-1} \in \mathcal{P}_+(H)$ then $B^{-1}XB^{-1} \in \mathcal{P}_+(H)$ for any $X \in \mathcal{P}_+(H)$ and thus $QB^{-1}XB^{-1}Q \in \mathcal{P}_+(H)$ showing that $\nabla_f(B, X) \in \mathcal{A}(H)$. We need to prove that

$$QA^{-1}Q - QB^{-1}Q \geq -QB^{-1}(A - B)B^{-1}Q,$$

that is,

$$QA^{-1}(B - A)B^{-1}Q + QB^{-1}(A - B)B^{-1}Q \geq 0$$

or equivalently

$$QA^{-1}(B - A)B^{-1}Q - QB^{-1}(B - A)B^{-1}Q \geq 0$$

or

$$Q\left(A^{-1} - B^{-1}\right)(B - A)B^{-1}Q \geq 0.$$

But

$$\begin{aligned} Q\left(A^{-1} - B^{-1}\right)(B - A)B^{-1}Q &= Q\left(A^{-1} - B^{-1}\right)AA^{-1}(B - A)B^{-1}Q \\ &= Q\left(A^{-1} - B^{-1}\right)A\left(A^{-1} - B^{-1}\right)Q \geq 0, \end{aligned}$$

which is true since for $A \in \mathcal{P}_+(H)$ we have that

$$\left(A^{-1} - B^{-1}\right)A\left(A^{-1} - B^{-1}\right) \geq 0$$

and $Q \in \mathcal{A}(H)$. □

4 Inequalities involving order generalised gradients

We start this section by the following definition.

Definition 9 An order generalised gradient $\nabla_f : \mathcal{C} \times \mathcal{A}(H) \to \mathcal{A}(H)$ is

(i) *operator convex* if

$$\nabla_f(B, \alpha X + \beta Y) \leq \alpha \nabla_f(B, X) + \beta \nabla_f(B, Y)$$

for any $B \in \mathcal{C}$, $X, Y \in \mathcal{A}(H)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(ii) *operator sub-additive* if

$$\nabla_f(B, X + Y) \leq \nabla_f(B, X) + \nabla_f(B, Y)$$

for any $B \in \mathcal{C}$ and $X, Y \in \mathcal{A}(H)$;

(iii) *positive homogeneous* if

$$\nabla_f(B, \alpha X) = \alpha \nabla_f(B, X)$$

for any $B \in \mathcal{C}$, $X \in \mathcal{A}(H)$ and $\alpha \geq 0$;

(iv) *operator linear* if

$$\nabla_f(B, \alpha X + \beta Y) = \alpha \nabla_f(B, X) + \beta \nabla_f(B, Y)$$

for any $B \in \mathcal{C}$, $X, Y \in \mathcal{A}(H)$ and $\alpha, \beta \in \mathbb{R}$.

It can be seen that if $\nabla_f(\cdot,\cdot)$ is operator linear, then it is positive homogeneous and sub-additive. If $\nabla_f(\cdot,\cdot)$ is positive homogeneous and sub-additive, then it is operator convex.

Theorem 10 *Let $f : \mathcal{C} \to \mathcal{A}(H)$ be operator convex and $\nabla_f : \mathcal{C} \times \mathcal{A}(H) \to \mathcal{A}(H)$ be an order generalised gradient for f. Then, for any $A, B, \in \mathcal{C}$ and $t \in [0,1]$, we have the inequalities*

$$\begin{aligned} -(1-t)\nabla_f\big(B, -t(B-A)\big) - t\nabla_f\big(A, (1-t)(B-A)\big) \\ \geq tf(A) + (1-t)f(B) - f\big(tA + (1-t)B\big) \\ \geq \nabla_f\big(tA + (1-t)B, 0\big). \end{aligned} \tag{16}$$

Proof If we write the definition of ∇_f for B instead of A, we get

$$f(B) - f(A) \geq \nabla_f(A, B - A),$$

which is equivalent to

$$-\nabla_f(A, B - A) \geq f(A) - f(B).$$

Therefore, for any $A, B \in \mathcal{C}$, we have the gradient inequalities

$$-\nabla_f(A, B - A) \geq f(A) - f(B) \geq \nabla_f(B, A - B). \tag{17}$$

Since $\mathcal{C}$ is a convex set, hence by (17) we have

$$\begin{aligned} -\nabla_f\bigl(A,(1-t)(B-A)\bigr) &\geq f(A)-f\bigl(tA+(1-t)B\bigr) \\ &\geq \nabla_f\bigl(tA+(1-t)B,-(1-t)(B-A)\bigr) \end{aligned} \tag{18}$$

and

$$\begin{aligned} -\nabla_f\bigl(B,-t(B-A)\bigr) &\geq f(B)-f\bigl(tA+(1-t)B\bigr) \\ &\geq \nabla_f\bigl(tA+(1-t)B,t(B-A)\bigr) \end{aligned} \tag{19}$$

for any $t\in(0,1)$.

If we multiply (18) by t and (19) by $(1-t)$ and add the obtained inequalities, then we get

$$\begin{aligned} &-t\nabla_f\bigl(A,(1-t)(B-A)\bigr)-(1-t)\nabla_f\bigl(B,-t(B-A)\bigr) \\ &\quad\geq tf(A)+(1-t)f(B)-f\bigl(tA+(1-t)B\bigr) \\ &\quad\geq t\nabla_f\bigl(tA+(1-t)B,-(1-t)(B-A)\bigr)+(1-t)\nabla_f\bigl(tA+(1-t)B,t(B-A)\bigr). \end{aligned}$$

Since $\nabla_f(\cdot,\cdot)$ is operator convex, we also know that

$$\begin{aligned} &t\nabla_f\bigl(tA+(1-t)B,-(1-t)(B-A)\bigr)+(1-t)\nabla_f\bigl(tA+(1-t)B,t(B-A)\bigr) \\ &\quad\geq \nabla_f\bigl(tA+(1-t)B,-t(1-t)(B-A)+(1-t)t(B-A)\bigr) \\ &\quad\geq \nabla_f\bigl(tA+(1-t)B,0\bigr), \end{aligned}$$

which completes the proof. □

Corollary 11 *Under the assumptions of Theorem* 10,

(1) *If $\nabla_f(\cdot,\cdot)$ is positive homogeneous, then we have*

$$\begin{aligned} &-t(1-t)\bigl[\nabla_f(B,A-B)+\nabla_f(A,B-A)\bigr] \\ &\quad\geq tf(A)+(1-t)f(B)-f\bigl(tA+(1-t)B\bigr)\geq 0. \end{aligned} \tag{20}$$

(2) *If $\nabla_f(\cdot,\cdot)$ is operator linear, then*

$$\begin{aligned} &t(1-t)\bigl[\nabla_f(B,B-A)-\nabla_f(A,B-A)\bigr] \\ &\quad\geq tf(A)+(1-t)f(B)-f\bigl(tA+(1-t)B\bigr)\geq 0. \end{aligned} \tag{21}$$

4.1 Hermite-Hadamard type operator inequalities

In this subsection, we will state inequalities of Hermite-Hadamard type for order generalised gradients.

Corollary 12 *Under the assumptions of Theorem* 10, *if* ∇_f *is positive homogeneous, then we have the following inequality*:

$$-\frac{1}{6}\left[\nabla_f(B,A-B)+\nabla_f(A,B-A)\right] \geq \frac{f(A)+f(B)}{2}-\int_0^1 f\bigl(tA+(1-t)B\bigr)\,dt \geq 0. \tag{22}$$

We obtain (22) by integrating (20) over $t \in [0,1]$.

Example 13

1. We consider the function $f(A) = QA^2Q$ with $Q \in \mathcal{A}(H)$. We note that the order generalised gradient

   $$\nabla_f(B,X) = Q(BX+XB)Q$$

   is operator linear. Then

   $$\begin{aligned}\nabla_f(B,X)-\nabla_f(A,X) &= Q(BX+XB)Q-Q(AX+XA)Q\\ &= Q\bigl[(B-A)X+X(B-A)\bigr]Q.\end{aligned}$$

   For $X = B-A$, we then get

   $$\nabla_f(B,B-A)-\nabla_f(A,B-A) = 2Q(B-A)^2Q.$$

   Applying inequality (21) we have

   $$2t(1-t)Q(B-A)^2Q \geq Q\bigl[tA^2+(1-t)B^2-\bigl(tA+(1-t)B\bigr)^2\bigr]Q \geq 0 \tag{23}$$

   for any $A, B \in \mathcal{A}(H)$ and $Q \in \mathcal{A}(H)$.
2. We consider the function $f(A) = APA$ with $P \in \mathcal{P}(H)$. We note that the order generalised gradient

   $$\nabla_f(B,X) = XPB+BPX$$

   is operator linear. Then

   $$\begin{aligned}\nabla_f(B,X)-\nabla_f(A,X) &= XPB+BPX-XPA-APX\\ &= XP(B-A)+(B-A)PX.\end{aligned}$$

   If $X = B-A$, we then get

   $$\nabla_f(B,B-A)-\nabla_f(A,B-A) = 2(B-A)P(B-A).$$

Applying inequality (21) we have

$$\begin{aligned}&2t(1-t)(B-A)P(B-A)\\&\quad\geq tAPA+(1-t)BPB-\big(tA+(1-t)B\big)P\big(tA+(1-t)B\big)\geq 0\end{aligned}$$

for any $A, B \in \mathcal{A}(H)$ and $P \in \mathcal{P}(H)$.

3. For $f(A) = QA^{-1}Q$ with $Q \in \mathcal{A}(H)$ and $A \in \mathcal{P}_+(H)$, we note that the order generalised gradient

$$\nabla_f(B, X) = -QB^{-1}XB^{-1}Q$$

is operator linear. Then

$$\nabla_f(B, X) - \nabla_f(A, X) = -QB^{-1}XB^{-1}Q + QA^{-1}XA^{-1}Q.$$

For $X = B - A$, we get

$$\begin{aligned}&\nabla_f(B, B-A) - \nabla_f(A, B-A)\\&\quad= -QB^{-1}(B-A)B^{-1}Q + QA^{-1}(B-A)A^{-1}Q\\&\quad= -Q\big(B^{-1} - B^{-1}AB^{-1}\big)Q + Q\big(A^{-1}BA^{-1} - A^{-1}\big)Q\\&\quad= QA^{-1}BA^{-1}Q + QB^{-1}AB^{-1}Q - QB^{-1}Q - QA^{-1}Q.\end{aligned}$$

By (21) we have the inequality

$$\begin{aligned}&t(1-t)\big[QA^{-1}BA^{-1}Q + QB^{-1}AB^{-1}Q - QB^{-1}Q - QA^{-1}Q\big]\\&\quad\geq tQA^{-1}Q + (1-t)QB^{-1}Q - Q\big(tA + (1-t)B\big)^{-1}Q \geq 0\end{aligned}$$

for any $A, B \in \mathcal{P}_+(H)$ and $Q \in \mathcal{A}(H)$.

4.2 Jensen type operator inequalities

In this subsection, we will state inequalities of Jensen type for order generalised gradients.

Theorem 14 *Let $f : \mathcal{C} \subset \mathcal{A}(H) \to \mathcal{A}(H)$ be a function that possesses $\nabla_f : \mathcal{C} \times \mathcal{A}(H) \to \mathcal{A}(H)$ as an order generalised gradient. Then, for any $A_i \in \mathcal{C}$, $i \in \{1, \ldots, n\}$ and $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$, we have the inequalities*

$$\begin{aligned}&-\frac{1}{P_n}\sum_{j=1}^n p_j\nabla_f\left(A_j, \frac{1}{P_n}\sum_{i=1}^n p_iA_i - A_j\right)\\&\quad\geq \frac{1}{P_n}\sum_{j=1}^n p_jf(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_iA_i\right)\\&\quad\geq \frac{1}{P_n}\sum_{j=1}^n p_j\nabla_f\left(\frac{1}{P_n}\sum_{i=1}^n p_iA_i, A_j - \frac{1}{P_n}\sum_{i=1}^n p_iA_i\right).\end{aligned}\tag{24}$$

Proof From the definition of an order generalised gradient we have

$$-\nabla_f(A, B-A) \geq f(A) - f(B) \geq \nabla_f(B, A-B). \tag{25}$$

Now, if we choose $A = A_j$, $j \in \{1, \dots, n\}$ and $B = (1/P_n)\sum_{i=1}^n p_i A_i$ in (25), then we get

$$\begin{aligned}
&-\nabla_f\left(A_j, \frac{1}{P_n}\sum_{i=1}^n p_i A_i - A_j\right)\\
&\quad\geq f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i\right)\\
&\quad\geq \nabla_f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i, A_j - \frac{1}{P_n}\sum_{i=1}^n p_i A_i\right)
\end{aligned} \tag{26}$$

for any $j \in \{1, \dots, n\}$. We obtain the desired inequalities (24) by multiplying the inequalities in (26) by $p_j \geq 0$ and taking the sum over j from 1 to n; and divide the resulted inequalities by P_n. □

Corollary 15 *Under the assumptions of Theorem* 14, *we have the following results*:

(1) *If* $\nabla_f : \mathcal{C} \times \mathcal{A}(H)$ *is convex, then*

$$\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i\right) \geq \nabla_f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i, 0\right). \tag{27}$$

(2) *If* ∇_f *is linear, then* $\nabla_f(B, 0) = 0$ *for any* B, *and we get the Jensen's inequality*

$$\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i\right) \geq 0. \tag{28}$$

(3) *If* ∇_f *is linear, we have*

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^n p_j \nabla_f\left(A_j, A_j - \frac{1}{P_n}\sum_{i=1}^n p_i A_i\right)\\
&\quad\geq \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i A_i\right) \geq 0.
\end{aligned} \tag{29}$$

Theorem 16 *Under the assumptions of Theorem* 14, *we have the following results*:

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - \frac{1}{P_n}\sum_{j=1}^n p_j \nabla_f(A, A_j - A)\\
&\quad\geq f(A)\\
&\quad\geq \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) + \frac{1}{P_n}\sum_{j=1}^n p_j \nabla_f(A_j, A - A_j).
\end{aligned}$$

Proof From (25) we also have

$$-\nabla_f(A, A_j - A) \geq f(A) - f(A_j) \geq \nabla_f(A_j, A - A_j). \tag{30}$$

If we multiply (30) by $p_j \geq 0$ and take the sum over j from 1 to n and divide the resulted inequalities by P_n, then

$$\begin{aligned} -\frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A, A_j - A) &\geq f(A) - \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) \\ &\geq \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A - A_j), \end{aligned}$$

which completes the proof. □

Remark 17 If ∇_f is linear in Theorem 16, then we get simpler inequalities such as

$$f(A) \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A_j)$$

and

$$f(A) \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A, A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A, A).$$

Therefore, if $A \in \mathcal{A}(H)$ is such that

$$\frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A) \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A_j),$$

then we have the Slater type inequality (*cf.* Slater [24] and Pečarić [25])

$$f(A) \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j).$$

5 Connection with Gâteaux derivatives

In this section, we consider the connection between order generalised gradients and Gâteaux derivatives. We refer the reader to Dragomir [26] for some inequalities of Jensen type, involving Gâteaux derivatives of convex functions in linear spaces.

Let $\mathcal{C} \subset \mathcal{A}(H)$ be a convex set. Then $f : \mathcal{C} \to \mathcal{A}(H)$ is said to be *operator convex* if for all $t \in [0,1]$ and $A, B \in \mathcal{C}$, we have

$$f\big[(1-t)A + tB\big] \leq (1-t)f(A) + tf(B).$$

We start with the following lemmas.

Lemma 18 *Let* $F:\mathbb{R}\to\mathcal{B}(H)$ *be a function such that* $\lim_{t\to 0^\pm} F(t)$ *exists. Then* $\lim_{t\to 0^\pm} F(t)$ *is a bounded linear operator and*

$$\left\langle\left[\lim_{t\to 0^\pm} F(t)\right]x,y\right\rangle = \lim_{t\to 0^\pm}\langle F(t)x,y\rangle$$

for all nonzero $x,y\in H$.

Proof We provide the proof for the right-sided limit, as the proof for the left-sided limit follows similarly. Let $\varepsilon>0$ and for $x,y\in H$, where $x,y\neq 0$, set $\varepsilon_0=\varepsilon/(\|x\|_H\|y\|_H)$. Since $\lim_{t\to 0^+}F(t)=L$, there exists δ_0 such that

$$\|F(t)-L\|_{\mathcal{B}(H)}<\varepsilon_0$$

when $0<t<\delta_0$. Note that $L\in\mathcal{B}(H)$ since $\mathcal{B}(H)$ is a Banach space, hence $F(t)-L$ is also a bounded linear operator. Now, we have

$$\begin{aligned}\left|\langle F(t)x,y\rangle-\langle Lx,y\rangle\right| &\le \left\|\big(F(t)-L\big)x\right\|_H\|y\|_H\\ &\le \left\|F(t)-L\right\|_{\mathcal{B}(H)}\|x\|_H\|y\|_H<\varepsilon_0\|x\|_H\|y\|_H=\varepsilon,\end{aligned}$$

which completes the proof. □

Lemma 19 *Let* $f:\mathcal{A}(H)\to\mathcal{A}(H)$ *be operator convex and* $A\in\mathcal{A}(H)$. *Then, for all* $B\in\mathcal{A}(H)$, *both limits*

$$\big(\nabla_- f(A)\big)(B)=\lim_{t\to 0^-}\frac{f(A+tB)-f(A)}{t}$$

and

$$\big(\nabla_+ f(A)\big)(B)=\lim_{t\to 0^+}\frac{f(A+tB)-f(A)}{t}$$

exist and are bounded self-adjoint operators.

Proof Fix an arbitrary $B\in\mathcal{A}(H)$, and let

$$G(t)=\frac{f(A+tB)-f(A)}{t},\quad t\in\mathbb{R}\setminus\{0\}.$$

We want to show that G is nondecreasing. Let $0<t_1<t_2$, then

$$\begin{aligned}f(A+t_1B)-f(A)&=f\left[\frac{t_1}{t_2}(A+t_2B)+\left(1-\frac{t_1}{t_2}\right)A\right]-f(A)\\ &\le\frac{t_1}{t_2}f(A+t_2B)+\left(1-\frac{t_1}{t_2}\right)f(A)-f(A)\\ &=\frac{t_1}{t_2}\big[f(A+t_2B)-f(A)\big].\end{aligned}$$

Thus,

$$\frac{f(A+t_1B)-f(A)}{t_1} \le \frac{f(A+t_2B)-f(A)}{t_2}.$$

Also,

$$\begin{aligned}\frac{f(A-t_2B)-f(A)}{-t_2} &= -\frac{f[A+t_2(-B)]-f(A)}{t_2}\\ &\le -\frac{f[A+t_1(-B)]-f(A)}{t_1} = \frac{f(A-t_1B)-f(A)}{-t_1}.\end{aligned}$$

Note also that

$$\begin{aligned}f(A) = f\left(\frac{2A+t_1B-t_1B}{2}\right) &= f\left[\frac{1}{2}(A+t_1B)+\frac{1}{2}(A-t_1B)\right]\\ &\le \frac{1}{2}f(A+t_1B)+\frac{1}{2}f(A-t_1B),\end{aligned}$$

which implies that

$$2f(A) \le f(A+t_1B)+f(A-t_1B);$$

and thus

$$f(A+t_1B)-f(A) \ge -\big[f(A-t_1B)-f(A)\big],$$

which implies that

$$\frac{f(A+t_1B)-f(A)}{t_1} \le \frac{f(A-t_1B)-f(A)}{-t_1}.$$

By the above expositions, we conclude that G is nondecreasing on $\mathbb{R}\setminus\{0\}$. This proves that both $(\nabla_- f(A))(B)$ and $(\nabla_+ f(A))(B)$ exist and are bounded linear operators by Lemma 18. Note that for all $t\in\mathbb{R}$, $t\neq 0$ and $A,B\in\mathcal{A}(H)$,

$$\frac{f[B+t(A-B)]-f(B)}{t}$$

is a self-adjoint operator. If $x,y\in H$, then Lemma 18 gives us

$$\begin{aligned}&\left\langle\left[\lim_{t\to0^\pm}\frac{f[B+t(A-B)]-f(B)}{t}\right]x,y\right\rangle\\ &\quad= \lim_{t\to0^\pm}\left\langle\left[\frac{f[B+t(A-B)]-f(B)}{t}\right]x,y\right\rangle\\ &\quad= \lim_{t\to0^\pm}\left\langle x,\left[\frac{f[B+t(A-B)]-f(B)}{t}\right]y\right\rangle\\ &\quad= \left\langle x,\lim_{t\to0^\pm}\left[\frac{f[B+t(A-B)]-f(B)}{t}\right]y\right\rangle,\end{aligned}$$

which completes the proof. □

Theorem 20 *Let $\mathcal{C} \subset \mathcal{A}(H)$ be a convex set and $f : \mathcal{C} \to \mathcal{A}(H)$ be operator convex. Then $\nabla_{\pm} f$ defined by*

$$\big(\nabla_{\pm} f(A)\big)(B) = \lim_{t\to 0^{\pm}} \frac{f(A+tB) - f(A)}{t}, \quad A, B \in \mathcal{C} \tag{31}$$

is an order generalised gradient.

Proof Let $t \in (0,1)$ and $A, B \in \mathcal{C}$. Since f is operator convex, we have

$$\begin{aligned} \frac{f[B + t(A-B)] - f(B)}{t} &= \frac{f[(1-t)B + tA] - f(B)}{t} \\ &\leq \frac{(1-t)f(B) + tf(A) - f(B)}{t} = f(A) - f(B). \end{aligned}$$

This is equivalent to

$$K := f(A) - f(B) - \frac{f[B + t(A-B)] - f(B)}{t} \in \mathcal{P}_{+}(H).$$

Note that for all $x \in H$,

$$\Big\langle \Big[\lim_{t\to 0^{\pm}} K\Big] x, x \Big\rangle = \lim_{t\to 0^{\pm}} \langle Kx, x \rangle$$

by Lemma 18. Since $K \in \mathcal{P}_{+}(H)$, $\langle Kx, x\rangle \geq 0$, hence $\langle [\lim_{t\to 0^{\pm}} K]x, x\rangle \geq 0$, which implies that

$$\lim_{t\to 0^{\pm}} \left[f(A) - f(B) - \frac{f[B + t(A-B)] - f(B)}{t} \right] \in \mathcal{P}_{+}(H).$$

Therefore,

$$\big(\nabla_{+} f(B)\big)(A - B) = \lim_{t\to 0^{\pm}} \frac{f[B + t(A-B)] - f(B)}{t} \leq f(A) - f(B).$$

Lemma 19 gives us

$$\big(\nabla_{-} f(B)\big)(A - B) \leq \big(\nabla_{+} f(B)\big)(A - B),$$

which implies that

$$\big(\nabla_{-} f(B)\big)(A - B) \leq f(A) - f(B).$$

Thus both $\nabla_{+} f$ and $\nabla_{-} f$ are order generalised gradients. □

Proposition 21 *Let $f : \mathcal{A}(H) \to \mathcal{A}(H)$ be operator convex and $A \in \mathcal{A}(H)$. The right Gâteaux derivative of f is sub-additive, i.e.*

$$\big(\nabla_{+} f(A)\big)(B + C) \leq \big(\nabla_{+} f(A)\big)(B) + \big(\nabla_{+} f(A)\big)(C)$$

for any $B, C \in \mathcal{A}(H)$. The left Gâteaux derivative of f is super-additive, i.e.

$$\big(\nabla_- f(A)\big)(B+C) \geq \big(\nabla_- f(A)\big)(B) + \big(\nabla_- f(A)\big)(C)$$

for any $B, C \in \mathcal{A}(H)$.

Proof Since f is operator convex, we have the following for any $B, C \in \mathcal{A}$ and $t > 0$:

$$\begin{aligned}\frac{f[A+t(B+C)]-f(A)}{t} &= \frac{f[\frac{1}{2}(A+2tB)+\frac{1}{2}(A+2tC)]-f(A)}{t} \\ &\leq \frac{f(A+2tB)-f(A)}{2t} + \frac{f(A+2tC)-f(A)}{2t}.\end{aligned}$$

By a similar argument to the proof of Theorem 20, we conclude that

$$\begin{aligned}\big(\nabla_+ f(A)\big)(B+C) &= \lim_{t\to 0^+} \frac{f[A+t(B+C)]-f(A)}{t} \\ &\leq \lim_{t\to 0^+} \frac{f(A+2tB)-f(A)}{2t} + \lim_{t\to 0^+} \frac{f(A+2tC)-f(A)}{2t} \\ &= \big(\nabla_+ f(A)\big)(B) + \big(\nabla_+ f(A)\big)(C)\end{aligned}$$

as desired. The proof for the left Gâteaux derivative of f follows similarly. □

Remark 22 We remark that the Gâteaux (lateral) derivative(s) is always positive homogeneous with respect to the second variable, *i.e.* for any function $f : \mathcal{A}(H) \to \mathcal{A}(H)$ and fixed $A \in \mathcal{A}(H)$,

$$\big(\nabla_{\pm} f(A)\big)(\alpha B) = \alpha\big(\nabla_{\pm} f(A)\big)(B)$$

for all $\alpha \geq 0$ and $B \in \mathcal{A}(H)$. The Gâteaux derivative, on the other hand, is always homogeneous with respect to the second variable, *i.e.* for any function $f : \mathcal{A}(H) \to \mathcal{A}(H)$ and fixed $A \in \mathcal{A}(H)$,

$$\big(\nabla f(A)\big)(\alpha B) = \alpha\big(\nabla f(A)\big)(B)$$

for all $\alpha \in \mathbb{C}$ and $B \in \mathcal{A}(H)$.

The following result restates Theorem 3 in the setting of operator-valued functions.

Corollary 23 *Let $\mathcal{C} \subset \mathcal{A}(H)$ be a convex set and $f : \mathcal{C} \to \mathcal{A}(H)$ be a Gâteaux differentiable function. Then f is operator convex if and only if ∇f defined by*

$$\big(\nabla f(A)\big)(B) = \lim_{t\to 0} \frac{f(A+tB)-f(A)}{t}, \quad A, B \in \mathcal{C} \tag{32}$$

is an order generalised gradient.

Proof For any $A, B \in \mathcal{C}$, if f is operator convex, then by Theorem 20

$$\big(\nabla_{\pm} f(A)\big)(B) = \lim_{t\to 0^{\pm}} \frac{f(A+tB)-f(A)}{t}$$

are order generalised gradients. Since f is assumed to be Gâteaux differentiable, both limits are equal, hence

$$\big(\nabla f(A)\big)(B) = \lim_{t\to 0} \frac{f(A+tB)-f(A)}{t}$$

is an order generalised gradient for any $A, B \in \mathcal{C}$. Conversely, we have the following inequality:

$$\big(\nabla f(B)\big)(A-B) \le f(A) - f(B)$$

for any $A, B \in \mathcal{C}$. Let $C, D \in \mathcal{C}$, $t \in (0,1)$, and choose $A = C$ and $B = tC + (1-t)D$. Then we have

$$(1-t)\big(\nabla f\big[tC + (1-t)D\big]\big)(C-D) \le f(C) - f\big[tC + (1-t)D\big]. \tag{33}$$

Let $A = D$ and $B = tC + (1-t)D$. Then we have

$$(-t)\big(\nabla f\big[tC + (1-t)D\big]\big)(C-D) \le f(D) - f\big[tC + (1-t)D\big]. \tag{34}$$

Multiply (33) by t and (34) by $(1-t)$, and add the resulting inequalities to obtain

$$f\big[tC + (1-t)D\big] \le tf(C) + (1-t)f(D),$$

which completes the proof. □

The following result follows by Corollary 12 and employing the fact that the Gâteaux lateral derivatives are positive homogenous.

Corollary 24 (Hermite-Hadamard type inequality) *Let $f : \mathcal{C} \subset \mathcal{A}(H) \to \mathcal{A}(H)$ be operator convex. The following inequality holds*:

$$\begin{aligned} &-\frac{1}{6}\big[\big(\nabla_{\pm} f(B)\big)(A-B) + \big(\nabla_{\pm} f(A)\big)(B-A)\big] \\ &\quad \ge \frac{f(A)+f(B)}{2} - \int_0^1 f\big(tA + (1-t)B\big)\,dt \ge 0. \end{aligned}$$

The above inequality also holds for ∇f when f is Gâteaux differentiable.

Example 25

(1) We note that the function $f(x) = -\log(x)$ is operator convex. The log function is (operator) Gâteaux differentiable with the following explicit formula for the derivative (*cf.* Pedersen [27, p.155]):

$$\big(\nabla \log(A)\big)(B) = \int_0^{\infty} (sI + A)^{-1} B (sI + A)^{-1}\,ds$$

for $A, B \in \mathcal{A}(H)$ and I the identity operator. Thus, we have the following inequality:

$$\begin{aligned}&\frac{1}{6}\left[\int_0^\infty (sI+B)^{-1}(A-B)(sI+B)^{-1}\,ds\right.\\&\qquad\left.+\int_0^\infty (sI+A)^{-1}(B-A)(sI+A)^{-1}\,ds\right]\\&\quad\ge -\frac{\log(A)+\log(B)}{2}+\int_0^1 \log\bigl(tA+(1-t)B\bigr)\,dt\ge 0.\end{aligned}$$

(2) We consider the operator convex function $f(x) = x\log(x)$, and using the following representation (*cf.* Pedersen [27, p.155])

$$\log(t)=\int_0^\infty \frac{t-1}{(s+t)(s+1)}\,ds,$$

and noting the fact that $\frac{d}{dt}t\log(t) = \log(t)$, we have

$$\bigl(\nabla f(A)\bigr)(B)=\int_0^\infty \frac{1}{s+1}(sI+A)^{-1}(A-I)B\,ds.$$

Then we have the following inequalities:

$$\begin{aligned}&-\frac{1}{6}\left[\int_0^\infty \frac{1}{s+1}(sI+B)^{-1}(B-I)(A-B)\,ds\right.\\&\qquad\left.+\int_0^\infty \frac{1}{s+1}(sI+A)^{-1}(A-I)(B-A)\,ds\right]\\&\quad\ge \frac{A\log(A)+B\log(B)}{2}-\int_0^1 \bigl[tA+(1-t)B\bigr]\log\bigl(tA+(1-t)B\bigr)\,dt\ge 0.\end{aligned}$$

The following results follow by Theorems 14 and 16.

Corollary 26 (Jensen type inequality) *Let $f : \mathcal{C} \subset \mathcal{A}(H) \to \mathcal{A}(H)$ be operator convex. Then, for any $A_i \in \mathcal{C}$, $i \in \{1,\dots,n\}$ and $p_i \ge 0$ with $P_n := \sum_{i=1}^n p_i > 0$, we have the inequalities*

$$\begin{aligned}&-\frac{1}{P_n}\sum_{j=1}^n p_j\bigl(\nabla_{\pm} f(A_j)\bigr)\left(\frac{1}{P_n}\sum_{i=1}^n p_iA_i - A_j\right)\\&\quad\ge \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_iA_i\right)\\&\quad\ge \frac{1}{P_n}\sum_{j=1}^n p_j\left(\nabla_{\pm} f\left(\frac{1}{P_n}\sum_{i=1}^n p_iA_i\right)\right)\left(A_j-\frac{1}{P_n}\sum_{i=1}^n p_iA_i\right).\end{aligned}$$

We also have

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - \frac{1}{P_n}\sum_{j=1}^{n} p_j\big(\nabla_{\pm} f(A)\big)(A_j - A) \\
&\quad \geq f(A) \\
&\quad \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j\big(\nabla_{\pm} f(A_j)\big)(A - A_j).
\end{aligned}$$

The above inequalities also hold for ∇f *when* f *is Gâteaux differentiable.*

Example 27

(1) We have the following inequalities for the operator convex function $f(x) = -\log(x)$:

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty (sI + A_j)^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i - A_j\right)(sI + A_j)^{-1}\, ds \\
&\quad \geq -\frac{1}{P_n}\sum_{j=1}^{n} p_j \log(A_j) + \log\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right) \\
&\quad \geq -\frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty \left(sI + \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right)^{-1} \\
&\qquad \times \left(A_j - \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right)\left(sI + \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right)^{-1} ds
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{P_n}\sum_{j=1}^{n} p_j \log(A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty (sI + A)^{-1}(A_j - A)(sI + A)^{-1}\, ds \\
&\quad \geq -\log(A) \\
&\quad \geq -\frac{1}{P_n}\sum_{j=1}^{n} p_j \log(A_j) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty (sI + A_j)^{-1}(A - A_j)(sI + A_j)^{-1}\, ds.
\end{aligned}$$

(2) We have the following inequalities for the operator convex function $f(x) = x\log(x)$:

$$\begin{aligned}
&-\frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty \frac{1}{s+1}(sI + A_j)^{-1}(A_j - I)\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i - A_j\right) ds \\
&\quad \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \log(A_j) - \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right)\log\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right) \\
&\quad \geq \frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty \frac{1}{s+1}\left(sI + \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right)^{-1} \\
&\qquad \times \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i - I\right)\left(A_j - \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right) ds
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \log(A_j) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty \frac{1}{s+1}(sI+A)^{-1}(A-I)(A_j-A)\,ds \\
&\quad \ge A\log(A) \\
&\quad \ge \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \log(A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j \int_0^\infty \frac{1}{s+1}(sI+A_j)^{-1}(A_j-I)(A-A_j)\,ds.
\end{aligned}$$

6 Conclusions

For a function $f : \mathcal{C} \to \mathcal{A}(H)$ defined on a convex set $\mathcal{C} \subset \mathcal{A}(H)$, the function $\nabla_f : \mathcal{C} \times \mathcal{A}(H) \to \mathcal{A}(H)$ is an *order generalised gradient* if

$$f(A) - f(B) \ge \nabla_f(B, A-B) \quad \text{for any } A, B \in \mathcal{C}$$

in the operator order of $\mathcal{A}(H)$. We have the following operator inequalities.

(1) Operator inequalities of Hermite-Hadamard type:

$$\begin{aligned}
&-\frac{1}{6}\bigl[\nabla_f(B, A-B) + \nabla_f(A, B-A)\bigr] \\
&\quad \ge \frac{f(A)+f(B)}{2} - \int_0^1 f\bigl(tA + (1-t)B\bigr)\,dt \ge 0 \quad \text{for any } A, B \in \mathcal{C}.
\end{aligned}$$

(2) Operator inequalities of Jensen type:

$$\begin{aligned}
&-\frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f\left(A_j, \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i - A_j\right) \\
&\quad \ge \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right) \\
&\quad \ge \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_i, A_j - \frac{1}{P_n}\sum_{i=1}^{n} p_i A_i\right);
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A, A_j - A) \\
&\quad \ge f(A) \\
&\quad \ge \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) + \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A - A_j)
\end{aligned}$$

for any $A \in \mathcal{C}$, $A_i \in \mathcal{C}$, $i \in \{1, \dots, n\}$ and $p_i \ge 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$.

(3) Operator inequalities of Slater type: if ∇_f is linear and $A \in \mathcal{A}(H)$ is such that

$$\frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A) \ge \frac{1}{P_n}\sum_{j=1}^{n} p_j \nabla_f(A_j, A_j),$$

then

$$f(A) \geq \frac{1}{P_n} \sum_{j=1}^{n} p_j f(A_j)$$

for any $A \in \mathcal{C}$, $A_i \in \mathcal{C}$, $i \in \{1, \ldots, n\}$ and $p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$.

Order generalised gradients extend the notion of subgradients, without the assumption of convexity, for operator-valued functions. This notion is also connected to the Gâteaux (lateral) derivatives. If f is operator convex, then $\nabla_{\pm} f$ defined by

$$\big(\nabla_{\pm} f(A)\big)(B) = \lim_{t \to 0^{\pm}} \frac{f(A + tB) - f(A)}{t}, \quad A, B \in \mathcal{C}$$

is an order generalised gradient. Furthermore, if $f : \mathcal{C} \to \mathcal{A}(H)$ is a Gâteaux differentiable function, we have the following characterisation: f is operator convex if and only if ∇f defined by

$$\big(\nabla f(A)\big)(B) = \lim_{t \to 0} \frac{f(A + tB) - f(A)}{t}, \quad A, B \in \mathcal{C}$$

is an order generalised gradient. This characterisation of convexity is a generalised version of Theorem 3 of Section 2 (*cf.* Eisenberg [3, Theorem 1]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SSD and EK contributed equally in all stages of writing the paper. All authors read and approved the final manuscript.

Author details
[1]School of Engineering and Science, Victoria University, PO Box 14428, Melbourne, Victoria 8001, Australia. [2]Department of Pure and Applied Mathematics, University of Johannesburg, PO Box 524, Auckland Park, 2006, South Africa.

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