

HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce the concept of operator geometrically convex functions for positive linear operators and prove some Hermite-Hadamard type inequalities for these functions. As applications, we obtain trace inequalities for operators which give some refinements of previous results.

1. Introduction and preliminaries

Let \mathcal{A} be a sub-algebra of $B(H)$ stand for the commutative C^* -algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in \mathcal{A}$ is positive and write $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. Let \mathcal{A}^+ stand for all strictly positive operators in \mathcal{A} .

Let A be a self-adjoint operator in \mathcal{A} . The Gelfand map establishes a $*$ -isometrically isomorphism Φ between the set $C(\text{Sp}(A))$ of all continuous functions defined on the spectrum of A , denoted $\text{Sp}(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows:

For any $f, g \in C(\text{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t \in \text{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \text{Sp}(A)$.

with this notation we define

$$f(A) = \Phi(f) \text{ for all } f \in C(\text{Sp}(A))$$

and we call it the continuous functional calculus for a self-adjoint operator A .

If A is a self-adjoint operator and f is a real valued continuous function on $\text{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \text{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H . Moreover, if both f and g are real valued functions on

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$\text{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \text{Sp}(A) \text{ implies that } f(A) \geq g(A), \quad (1.1)$$

in the operator order of $B(H)$, see [18].

Let I be an interval in \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$ is said to be convex function if

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$$

for $a, b \in I$ and $\lambda \in [0, 1]$.

The following inequality holds for any convex function f defined on \mathbb{R}

$$(b - a)f\left(\frac{a + b}{2}\right) \leq \int_a^b f(x)dx \leq (b - a)\frac{f(a) + f(b)}{2}, \quad a, b \in \mathbb{R}. \quad (1.2)$$

It was firstly discovered by Hermite in 1881 in the journal *Mathesis* (see [10]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result [14].

Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by Hadamard in 1893 [1]. In 1974, Mitrinović found Hermite's note in *Mathesis* [10]. Since (1.2) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality [14].

Definition 1.1. [12] A continuous function $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be geometrically convex function (or multiplicatively convex function) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda}$$

for $a, b \in I$ and $\lambda \in [0, 1]$.

The author of [8] established the Hermite-Hadamard type inequalities for geometrically convex functions as follows:

Theorem 1.2. Let $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a geometrically convex function and $a, b \in I$ with $a < b$. If $f \in L^1[a, b]$, then

$$\begin{aligned} f(\sqrt{ab}) &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} \sqrt{f(t)f\left(\frac{ab}{t}\right)} dt \\ &\leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt \\ &\leq \frac{f(b) - f(a)}{\ln f(b) - \ln f(a)} \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned}$$

By changing variables $t = a^\lambda b^{1-\lambda}$ we have

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(t)}{t} dt = \int_0^1 f(a^\lambda b^{1-\lambda}) d\lambda.$$

Remark 1. It is well-known that for positive numbers a and b

$$\min\{a, b\} \leq G(a, b) = \sqrt{ab} \leq L(a, b) = \frac{b-a}{\ln b - \ln a} \leq A(a, b) = \frac{a+b}{2} \leq \max\{a, b\}.$$

The author of [9] mentioned the following inequality, but here we provide a short proof which gives a refinement for above theorem.

Theorem 1.3. *Let f be a geometrically convex function defined on I a subinterval of \mathbb{R}^+ . Then, we have*

$$f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} \sqrt{f(t)f\left(\frac{ab}{t}\right)} dt \leq \sqrt{f(a)f(b)}$$

for $a, b \in I$.

Proof. Since f is geometrically convex function, we can write

$$\begin{aligned} f(\sqrt{ab}) &= f\left(\sqrt{(a^\lambda b^{1-\lambda})(a^{1-\lambda} b^\lambda)}\right) \\ &\leq \sqrt{f(a^\lambda b^{1-\lambda})f(a^{1-\lambda} b^\lambda)} \\ &\leq \sqrt{f(a)^\lambda f(b)^{1-\lambda} f(a)^{1-\lambda} f(b)^\lambda} \\ &= \sqrt{f(a)f(b)}. \end{aligned}$$

for all $\lambda \in [0, 1]$.

So, we have

$$f(\sqrt{ab}) \leq \sqrt{f(a^\lambda b^{1-\lambda})f(a^{1-\lambda} b^\lambda)} \leq \sqrt{f(a)f(b)}. \quad (1.3)$$

Integrate (1.3) over $[0, 1]$, we have

$$f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{1}{t} \sqrt{f(t)f\left(\frac{ab}{t}\right)} dt \leq \sqrt{f(a)f(b)}.$$

□

Lemma 1.4. [12, Page. 156] *Suppose that I is a subinterval of \mathbb{R}^+ and $f : I \rightarrow (0, \infty)$ is a geometrically convex function. Then*

$$F = \log \circ f \circ \exp : \log(I) \rightarrow \mathbb{R}$$

is a convex function. Conversely, if J is an interval for which $\exp(J)$ is a subinterval of \mathbb{R}^+ and $F : J \rightarrow \mathbb{R}$ is a convex function, then

$$f = \exp \circ F \circ \log : \exp(J) \rightarrow \mathbb{R}^+$$

is geometrically convex function.

Theorem 1.5. Let f be a geometrically convex function defined on $[a, b]$ such that $0 < a < b$. Then, we have

$$\begin{aligned} f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\ &\leq \exp\left(\frac{1}{\log b - \log a} \int_a^b \frac{\log f(t)}{t} dt\right) \\ &\leq \sqrt{f(\sqrt{ab})} \cdot \sqrt[4]{f(a)} \cdot \sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)} \end{aligned}$$

for $a, b \in I$.

Proof. Let $f : [a, b] \rightarrow \mathbb{R}$ be a geometrically convex function. So, by Lemma 1.4 we have

$$F(x) = \log \circ f \circ \exp(x) : [\log a, \log b] \rightarrow \mathbb{R}$$

is convex.

Then, by [13, Remark 1.9.3]

$$\begin{aligned} F\left(\frac{\log a + \log b}{2}\right) &\leq \frac{1}{2} \left(F\left(\frac{3 \log a + \log b}{4}\right) + F\left(\frac{\log a + 3 \log b}{4}\right) \right) \\ &\leq \frac{1}{\log b - \log a} \int_{\log a}^{\log b} F(x) dx \\ &\leq \frac{1}{2} \left(F\left(\frac{\log a + \log b}{2}\right) + \frac{F(\log a) + F(\log b)}{2} \right) \\ &\leq \frac{F(\log a) + F(\log b)}{2}. \end{aligned}$$

By definition of F , we obtain

$$\begin{aligned} \log \circ f \circ \exp(\log \sqrt{ab}) &\leq \frac{1}{2} \left(\log \circ f \circ \exp\left(\log a^{\frac{3}{4}}b^{\frac{1}{4}}\right) + \log \circ f \circ \exp\left(\log a^{\frac{1}{4}}b^{\frac{3}{4}}\right) \right) \\ &\leq \frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log \circ f \circ \exp(x) dx \\ &\leq \frac{1}{2} \left(\log \circ f \circ \exp\left(\log a^{\frac{1}{2}}b^{\frac{1}{2}}\right) + \frac{\log \circ f \circ \exp(\log a) + \log \circ f \circ \exp(\log b)}{2} \right) \\ &\leq \frac{\log \circ f \circ \exp(\log a) + \log \circ f \circ \exp(\log b)}{2}. \end{aligned}$$

It follows that

$$\begin{aligned}
\log f(\sqrt{ab}) &\leq \frac{1}{2} \left(\log f(a^{\frac{3}{4}}b^{\frac{1}{4}}) + \log f(a^{\frac{1}{4}}b^{\frac{3}{4}}) \right) \\
&\leq \frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log \circ f \circ \exp(x) dx \\
&\leq \frac{1}{2} \left(\log f\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right) + \frac{\log f(a) + \log f(b)}{2} \right) \\
&\leq \frac{\log f(a) + \log f(b)}{2}.
\end{aligned}$$

Since $\exp(x)$ is increasing, we have

$$\begin{aligned}
f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\
&\leq \exp\left(\frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log f(\exp(x)) dx\right) \\
&\leq \sqrt{f(\sqrt{ab})} \cdot \sqrt[4]{f(a)} \cdot \sqrt[4]{f(b)} \\
&\leq \sqrt{f(a)f(b)}.
\end{aligned}$$

Using change of variable $t = \exp(x)$ to obtain the desired result. \square

The author of [12, p. 158] showed that every polynomial $P(x)$ with non-negative coefficients is a geometrically convex function on $[0, \infty)$. More generally, every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is geometrically convex function on $(0, R)$ where R denotes the radius of convergence. This gives some different examples of geometrically convex function. It is easy to show that $\exp(x)$ is geometrically convex function.

In this paper, we introduce the concept of operator geometrically convex functions and prove the Hermite-Hadamard type inequalities for these class of functions. These results lead us to obtain some inequalities for trace functional of operators.

2. Inequalities for operator geometrically convex functions

In this section, we prove Hermite-Hadamard type inequality for operator geometrically convex function.

In [5] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f : I \rightarrow \mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with

spectra in I , the following inequalities holds

$$f\left(\frac{A+B}{2}\right) \leq 2 \int_{\frac{1}{4}}^{\frac{3}{4}} f(tA + (1-t)B) dt \quad (2.1)$$

$$\leq \frac{1}{2} \left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right) \right] \quad (2.2)$$

$$\leq \int_0^1 f((1-t)A + tB) dt$$

$$\leq \frac{1}{2} \left[f\left(\frac{A+B}{2}\right) + \frac{f(A) + f(B)}{2} \right] \quad (2.3)$$

$$\leq \frac{f(A) + f(B)}{2}, \quad (2.4)$$

for the first inequality in above, see [16].

To give operator geometrically convex function definition, we need following lemmas.

Lemma 2.1. [11, Lemma 3] *Let A and B be two operators in \mathcal{A}^+ , and f a continuous function on $\text{Sp}(A)$. Then, $AB = BA$ implies that $f(A)B = Bf(A)$.*

Since $f(t) = t^\lambda$ is continuous function for $\lambda \in [0, 1]$ and \mathcal{A} is a commutative C^* -algebra, we have $A^\lambda B = BA^\lambda$. Moreover, by applying above lemma for $f(t) = t^{1-\lambda}$ again, we have $A^\lambda B^{1-\lambda} = B^{1-\lambda} A^\lambda$, for operators A and B in \mathcal{A}^+ . It means A^λ and $B^{1-\lambda}$ commute together whenever A and B commute.

Lemma 2.2. *Let A and B be two operators in \mathcal{A}^+ . Then*

$$\{A^\lambda B^{1-\lambda} : 0 \leq \lambda \leq 1\}$$

is convex.

Proof. We know that $\{\lambda A + (1-\lambda)B : 0 \leq \lambda \leq 1\}$ is convex for arbitrary operator A and B . So, $\{\lambda \log A + (1-\lambda) \log B : 0 \leq \lambda \leq 1\}$ is convex. Since A and B are commutative and knowing that e^f is convex when f is convex, we have

$$\begin{aligned} e^{(\lambda \log A + (1-\lambda) \log B)} &= e^{\lambda \log A} e^{(1-\lambda) \log B} \\ &= A^\lambda B^{1-\lambda}. \end{aligned}$$

So, $A^\lambda B^{1-\lambda}$ is convex for $0 \leq \lambda \leq 1$. □

Lemma 2.3. [18, Theorem 5.3] *Let A and B be in a Banach algebra such that $AB = BA$. Then*

$$\text{Sp}(AB) \subset \text{Sp}(A) \text{Sp}(B).$$

Let A and B be two positive operators in \mathcal{A} with spectra in I . Now, Lemma 2.1 and functional calculus [18, Theorem 10.3 (c)] imply that

$$\mathrm{Sp}(A^\lambda B^{1-\lambda}) \subset \mathrm{Sp}(A^\lambda) \mathrm{Sp}(B^{1-\lambda}) = \mathrm{Sp}(A)^\lambda \mathrm{Sp}(B)^{1-\lambda} \subseteq I$$

for $0 \leq \lambda \leq 1$.

Definition 2.4. A continuous function $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be operator geometrically convex if

$$f(A^\lambda B^{1-\lambda}) \leq f(A)^\lambda f(B)^{1-\lambda}$$

for $A, B \in \mathcal{A}^+$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Now, we are ready to prove Hermite-Hadamard type inequality for operator geometrically convex functions.

Theorem 2.5. Let f be an operator geometrically convex function. Then, we have

$$\log f(\sqrt{AB}) \leq \int_0^1 \log f(A^t B^{1-t}) dt \leq \log \sqrt{f(A)f(B)} \quad (2.5)$$

for $0 \leq t \leq 1$ and $A, B \in \mathcal{A}^+$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Proof. Since f is operator geometrically convex function, we have $f(\sqrt{AB}) \leq \sqrt{f(A)f(B)}$. Let replace A and B by $A^t B^{1-t}$ and $A^{1-t} B^t$ respectively, we obtain

$$f(\sqrt{AB}) \leq \sqrt{f(A^t B^{1-t})f(A^{1-t} B^t)}. \quad (2.6)$$

It is well-known that $\log t$ is operator monotone function on $(0, \infty)$ (see [17]), i.e., $\log t$ is operator monotone function if $\log A \leq \log B$ when $A \leq B$. So, by above inequality, we have

$$\begin{aligned} \log f(\sqrt{AB}) &\leq \log \sqrt{f(A^t B^{1-t})f(A^{1-t} B^t)} \\ &= \frac{1}{2} \log (f(A^t B^{1-t})f(A^{1-t} B^t)) \\ &= \frac{1}{2} (\log f(A^t B^{1-t}) + \log f(A^{1-t} B^t)). \end{aligned}$$

Therefore,

$$\log f(\sqrt{AB}) \leq \frac{1}{2} (\log f(A^t B^{1-t}) + \log f(A^{1-t} B^t)).$$

Integrate above inequality over $[0, 1]$, we can write the following

$$\begin{aligned} \int_0^1 \log f(\sqrt{AB}) dt &\leq \frac{1}{2} \left(\int_0^1 \log f(A^t B^{1-t}) dt + \int_0^1 \log f(A^{1-t} B^t) dt \right) \\ &= \int_0^1 \log f(A^t B^{1-t}) dt. \end{aligned} \quad (2.7)$$

The last above equality follows by knowing that

$$\int_0^1 \log f(A^t B^{1-t}) dt = \int_0^1 \log f(A^{1-t} B^t) dt.$$

Hence, from (2.7), we have

$$\log f(\sqrt{AB}) \leq \int_0^1 \log f(A^t B^{1-t}) dt.$$

This proved left inequality of (2.5).

On the other hand, we have $f(A^t B^{1-t}) \leq f(A)^t f(B)^{1-t}$. It follows that

$$\begin{aligned} \log f(A^t B^{1-t}) &\leq \log f(A)^t f(B)^{1-t} \\ &= \log f(A)^t + \log f(B)^{1-t} \\ &= t \log f(A) + (1-t) \log f(B). \end{aligned}$$

So,

$$\log f(A^t B^{1-t}) \leq t \log f(A) + (1-t) \log f(B). \quad (2.8)$$

Now, integrate of (2.8) on $[0, 1]$, we have

$$\begin{aligned} \int_0^1 \log f(A^t B^{1-t}) dt &\leq \int_0^1 t \log f(A) dt + \int_0^1 (1-t) \log f(B) dt \\ &= \log f(A) \int_0^1 t dt + \log f(B) \int_0^1 (1-t) dt \\ &= \frac{1}{2} (\log f(A) + \log f(B)) \\ &= \log \sqrt{f(A)f(B)}. \end{aligned}$$

This completes the proof. \square

We should mention, when f is operator geometrically convex function, then we have

$$\begin{aligned} f(\sqrt{AB}) &= f(\sqrt{A^t B^{1-t} A^{1-t} B^t}) \\ &\leq \sqrt{f(A^t B^{1-t}) f(A^{1-t} B^t)} \\ &\leq \sqrt{f(A)^t f(B)^{1-t} f(A)^{1-t} f(B)^t} \\ &= \sqrt{f(A) f(B)}. \end{aligned}$$

So, we have

$$f(\sqrt{AB}) \leq \sqrt{f(A^t B^{1-t}) f(A^{1-t} B^t)} \leq \sqrt{f(A) f(B)}.$$

Integrate above inequality over $[0, 1]$, we obtain

$$f(\sqrt{AB}) \leq \int_0^1 \sqrt{f(A^t B^{1-t}) f(A^{1-t} B^t)} dt \leq \sqrt{f(A) f(B)},$$

for $0 \leq t \leq 1$ and $A, B \in \mathcal{A}^+$ such that $\text{Sp}(A), \text{Sp}(B) \subseteq I$.

Let $A, B \in \mathcal{A}$ and $A \leq B$, by continuous functional calculus [18, Theorem 10.3 (b)], we can easily obtain $\exp(A) \leq \exp(B)$. This means $\exp(t)$ is operator monotone on $[0, \infty)$ for $A, B \in \mathcal{A}$.

On the other hand, like the classical case, the arithmetic-geometric mean inequality holds for operators as following

$$A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{\nu} A^{\frac{1}{2}} \leq (1 - \nu)A + \nu B, \quad \nu \in [0, 1] \quad (2.9)$$

with respect to operator order for positive non-commutative operator in $B(H)$. Whenever, A and B commute together, then inequality (2.9) reduces to

$$A^{1-\nu} B^{\nu} \leq (1 - \nu)A + \nu B, \quad \nu \in [0, 1]. \quad (2.10)$$

Since $\exp(t)$ is an operator monotone function, by above inequality we have

$$\begin{aligned} \exp(A^{1-\nu} B^{\nu}) &\leq \exp((1 - \nu)A + \nu B) \\ &= \exp((1 - \nu)A) \exp(\nu B) \\ &= \exp(A)^{1-\nu} \exp(B)^{\nu}, \end{aligned}$$

for $A, B \in \mathcal{A}^+$ and $\nu \in [0, 1]$. So, in this case $\exp(t)$ is an operator geometrically convex function on $[0, \infty)$.

Let replace f in Theorem 2.5 by $\exp(t)$ as an operator geometrically convex function, we have

$$\begin{aligned} \log \exp(\sqrt{AB}) &\leq \int_0^1 \log \exp(A^t B^{1-t}) dt \leq \log \sqrt{\exp(A) \exp(B)} \\ &= \frac{1}{2} \log (\exp(A) \exp(B)) \\ &= \frac{1}{2} (\log \exp(A) + \log \exp(B)). \end{aligned}$$

So,

$$\sqrt{AB} \leq \int_0^1 A^t B^{1-t} dt \leq \frac{A+B}{2}, \quad (2.11)$$

for $A, B \in \mathcal{A}^+$.

Here, we mention some remarks for operator geometrically convex functions.

Remark 2. $f(x) = \|x\|$ is geometrically convex function for usual operator norms since the following hold

$$f(A^{\alpha} B^{1-\alpha}) = \|A^{\alpha} B^{1-\alpha}\| \leq \|A\|^{\alpha} \|B\|^{1-\alpha} = f(A)^{\alpha} f(B)^{1-\alpha}.$$

Above inequality is a special case of McIntosh inequality.

Remark 3. If $f(t)$ is an operator geometrically convex function, then so is $g(t) = tf(t)$

$$\begin{aligned} g(A^\alpha B^{1-\alpha}) &= A^\alpha B^{1-\alpha} f(A^\alpha B^{1-\alpha}) \\ &\leq A^\alpha B^{1-\alpha} f(A)^\alpha f(B)^{1-\alpha} \\ &\leq A^\alpha f(A)^\alpha B^{1-\alpha} f(B)^{1-\alpha} \\ &= g(A)^\alpha g(B)^{1-\alpha} \end{aligned}$$

for $\alpha \in [0, 1]$ and $A, B \in \mathcal{A}^+$.

Remark 4. Operator geometrically convex functions is an algebra with some complication of operators spectra. To see this we make use of the following inequality

$$A^\alpha B^{1-\alpha} + C^\alpha D^{1-\alpha} \leq (A + C)^\alpha + (B + D)^{1-\alpha} \quad (2.12)$$

for $A, B, C, D \in \mathcal{A}^+$.

Let f and g be operator geometrically convex functions.

First, we prove that $f + g$ is an operator geometrically convex function

$$\begin{aligned} (f + g)(A^\alpha B^{1-\alpha}) &= f(A^\alpha B^{1-\alpha}) + g(A^\alpha B^{1-\alpha}) \\ &\leq f(A)^\alpha f(B)^{1-\alpha} + g(A)^\alpha g(B)^{1-\alpha} \\ &\leq (f(A) + g(A))^\alpha + (f(B) + g(B))^{1-\alpha} \\ &= ((f + g)(A))^\alpha + ((f + g)(B))^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$. In the last inequality above we applied (2.12).

Second, we show that mf is an operator geometrically convex function for a scalar m

$$\begin{aligned} (mf)(A^\alpha B^{1-\alpha}) &\leq mf(A)^\alpha f(B)^{1-\alpha} \\ &= (mf(A))^\alpha (mf(B))^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$.

Third, $h = fg$ is an operator geometrically convex function

$$\begin{aligned} h(A^\alpha B^{1-\alpha}) &= f(A^\alpha B^{1-\alpha})g(A^\alpha B^{1-\alpha}) \\ &\leq f(A)^\alpha f(B)^{1-\alpha} g(A)^\alpha g(B)^{1-\alpha} \\ &= f(A)^\alpha g(A)^\alpha f(B)^{1-\alpha} g(B)^{1-\alpha} \\ &= h(A)^\alpha h(B)^{1-\alpha} \end{aligned}$$

for $A, B \in \mathcal{A}^+$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of H , we say that $A \in B(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A|e_i, e_i \rangle < \infty. \quad (2.13)$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $B_1(H)$ the set of trace class operators in $B(H)$.

We define the *trace* of a trace class operator $A \in B_1(H)$ to be

$$\mathrm{Tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \quad (2.14)$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H .

Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.14) converges absolutely.

The following result collects some properties of the trace:

Theorem 2.6. *We have*

(i) *If $A \in B_1(H)$ then $A^* \in B_1(H)$ and*

$$\mathrm{Tr}(A^*) = \overline{\mathrm{Tr}(A)}; \quad (2.15)$$

(ii) *If $A \in B_1(H)$ and $T \in B(H)$, then $AT, TA \in B_1(H)$ and*

$$\mathrm{Tr}(AT) = \mathrm{Tr}(TA) \quad \text{and} \quad |\mathrm{Tr}(AT)| \leq \|A\|_1 \|T\|; \quad (2.16)$$

(iii) *$\mathrm{Tr}(\cdot)$ is a bounded linear functional on $B_1(H)$ with $\|\mathrm{Tr}\| = 1$;*

(iv) *If $A, B \in B_1(H)$ then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.*

For the theory of trace functionals and their applications the reader is referred to [15].

For $A, B \geq 0$ we have $\mathrm{Tr}(AB) \leq \mathrm{Tr}(A) \mathrm{Tr}(B)$. Also, since $f(t) = t^{\frac{1}{2}}$ is monotone we have

$$\sqrt{\mathrm{Tr}(AB)} \leq \sqrt{\mathrm{Tr}(A) \mathrm{Tr}(B)} \quad (2.17)$$

for positive operator A and B in $B(H)$.

We know that $f(t) = \mathrm{Tr}(t)$ is operator geometrically convex function [7, p.513], i.e.

$$\mathrm{Tr}(A^t B^{1-t}) \leq \mathrm{Tr}(A)^t \mathrm{Tr}(B)^{1-t}$$

for $0 \leq t \leq 1$ and positive operators $A, B \in B_1(H)$.

For commutative case, we have

$$\sqrt{\mathrm{Tr}(AB)} \leq \mathrm{Tr}(\sqrt{AB}) \leq \sqrt{\mathrm{Tr}(A) \mathrm{Tr}(B)},$$

since $(\mathrm{Tr}(AB))^{\frac{1}{2}} \leq \mathrm{Tr}(AB)^{\frac{1}{2}}$.

Moreover, by Theorem 2.5 we can write

$$\begin{aligned} \log \operatorname{Tr}(\sqrt{AB}) &\leq \int_0^1 \log \operatorname{Tr}(A^t B^{1-t}) dt \\ &\leq \log \sqrt{\operatorname{Tr}(A) \operatorname{Tr}(B)} \\ &= \frac{1}{2}(\log \operatorname{Tr}(A) + \log \operatorname{Tr}(B)). \end{aligned}$$

Let replace A and B by A^2 and B^2 in above inequality, respectively. By applying commutativity of algebra and knowing that $\operatorname{Tr}(A)^2 \leq (\operatorname{Tr} A^2)$ for positive operator A , we have

$$\log \operatorname{Tr}(AB) \leq \int_0^1 \log \operatorname{Tr}(A^{2t} B^{2(1-t)}) dt \leq \log (\operatorname{Tr}(A) \operatorname{Tr}(B)).$$

3. More results on trace functional class for product of operators

In this section we prove some trace functional class inequalities for operators which are not necessarily commutative.

We consider the wide class of unitarily invariant norms $||| \cdot |||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $||| \cdot |||$. Each unitarily invariant norm $||| \cdot |||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $||| \cdot |||$ and for all unitary operators U and V in $B(H)$. For $1 \leq p < \infty$, the Schatten p -norm of an operator $A \in B_1(H)$ defined by $\|A\|_p = (\operatorname{Tr} |A|^p)^{1/p}$. These Schatten p -norms are unitarily invariant.

In [2], Bhatia and Davis proved the following inequality

$$|||A^*XB|^r|||^2 \leq |||AA^*X|^r||| \cdot |||XBB^*|^r||| \quad (3.1)$$

for all operators A, B, X and $r \geq 0$.

As we know, $\|A\|_1 = \operatorname{Tr} |A|$. From (3.1) for $p = 1$, we have

$$|||A^*XB|^r|||_1^2 \leq |||AA^*X|^r|||_1 \cdot |||XBB^*|^r|||_1. \quad (3.2)$$

So, by inequality (3.2), we can write

$$(\operatorname{Tr} |A^*XB|^r)^2 \leq \operatorname{Tr}(|AA^*X|^r) \operatorname{Tr}(|XBB^*|^r), \quad (3.3)$$

for all operators $A, B \in B_1(H)$, $X \in B(H)$ and $r \geq 0$.

Let, $X = I$ in above inequality, we have

$$(\operatorname{Tr} |A^*B|^r)^2 \leq \operatorname{Tr}(|AA^*|^r) \operatorname{Tr}(|BB^*|^r).$$

Moreover, let $r = 1$ in inequality (3.3), we have

$$|\operatorname{Tr}(A^*XB)|^2 \leq (\operatorname{Tr} |A^*XB|)^2 \leq \operatorname{Tr}(|AA^*X|) \operatorname{Tr}(|XBB^*|).$$

Put X^* instead of X and applying the property of trace we have

$$|\operatorname{Tr}(AB^*X)|^2 \leq \operatorname{Tr}(|AA^*X^*|) \operatorname{Tr}(|X^*BB^*|), \quad (3.4)$$

for all $A, B \in B_1(H)$ and $X \in B(H)$.

Let $X = I$ in (3.4).

Corollary 3.1. *Let $A, B \in B_1(H)$. Then*

$$|\operatorname{Tr}(AB^*)|^2 \leq \operatorname{Tr}(AA^*) \operatorname{Tr}(BB^*). \quad (3.5)$$

In [6, Theorem 5], Dragomir proved the following inequality for $X \in B(H)$, $A, B \in B_1(H)$ and $\alpha \in [0, 1]$

$$|\operatorname{Tr}(AB^*X)|^2 \leq \operatorname{Tr}(|A^*|^2|X|^{2\alpha}) \operatorname{Tr}(|B^*|^2|X^*|^{2(1-\alpha)}).$$

Here, we give a generalization for above inequality when $\alpha \in \mathbb{R}$.

Theorem 3.2. *Let $X \in B_1(H)$, $A, B \in B(H)$ and $\alpha \in \mathbb{R}$. Then*

$$|\operatorname{Tr}(AB^*|X|)|^2 \leq \operatorname{Tr}(|A^*|^2|X|^{2\alpha}) \operatorname{Tr}(|B^*|^2|X|^{2(1-\alpha)}). \quad (3.6)$$

Proof. Let replace A and B in Corollary 3.1 with $|X|^\alpha A$ and $|X|^{(1-\alpha)}B$, where $\alpha \in \mathbb{R}$. It follows that

$$\begin{aligned} |\operatorname{Tr}(AB^*|X|)|^2 &\leq \operatorname{Tr}(|X|^\alpha AA^*|X|^\alpha) \operatorname{Tr}(|X|^{(1-\alpha)}BB^*|X|^{(1-\alpha)}) \\ &= \operatorname{Tr}(AA^*|X|^\alpha|X|^\alpha) \operatorname{Tr}(BB^*|X|^{(1-\alpha)}|X|^{(1-\alpha)}) \\ &= \operatorname{Tr}(|A^*|^2|X|^{2\alpha}) \operatorname{Tr}(|B^*|^2|X|^{2(1-\alpha)}). \end{aligned}$$

So, we have

$$|\operatorname{Tr}(AB^*|X|)|^2 \leq \operatorname{Tr}(|A^*|^2|X|^{2\alpha}) \operatorname{Tr}(|B^*|^2|X|^{2(1-\alpha)}).$$

□

Let $A = B = I$ in Theorem 3.2, we have

$$|\operatorname{Tr}(X)|^2 \leq \operatorname{Tr}(|X|^{2\alpha}) \operatorname{Tr}(|X|^{2(1-\alpha)}),$$

for $X \in B_1(H)$ and $\alpha \in \mathbb{R}$. Above inequality is a refinement for [6, Inequality (3.1)].

Also, let $X \in B_1(H)$ and normal operators $A, B \in B(H)$. For $\alpha \in \mathbb{R}$, we have

$$|\operatorname{Tr}(AB^*|X|)|^2 \leq \operatorname{Tr}(|A|^2|X|^{2\alpha}) \operatorname{Tr}(|B|^2|X|^{2(1-\alpha)}).$$

In [4, Theorem 2.3], F. M. Dannan proved that if S_i and T_i ($i = 1, 2, \dots, n$) are positive definite matrices, then we have

$$\left(\operatorname{Tr} \sum_{i=1}^n S_i T_i \right)^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right). \quad (3.7)$$

Moreover, if $S_i T_i \geq 0$, ($i = 1, 2, \dots, n$). Then

$$\begin{aligned} \operatorname{Tr} \left(\sum_{i=1}^n S_i T_i \right)^2 &\leq \left(\operatorname{Tr} \sum_{i=1}^n S_i T_i \right)^2 \\ &\leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right). \end{aligned}$$

So,

$$\left(\operatorname{Tr} \sum_{i=1}^n S_i T_i \right)^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right).$$

Here, we prove inequality (3.7) for arbitrary operators.

Theorem 3.3. *Let S_i and T_i ($i = 1, 2, \dots, n$) be arbitrary operators in $B_1(H)$. Then,*

$$\left| \operatorname{Tr} \left(\sum_{i=1}^n S_i T_i^* \right) \right|^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i S_i^* \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i T_i^* \right). \quad (3.8)$$

Proof. Let $A = \begin{pmatrix} S_1 & S_2 & \dots & S_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} T_1 & T_2 & \dots & T_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So,

we have

$$\begin{aligned} AB^* &= \begin{pmatrix} \sum_{i=1}^n S_i T_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ AA^* &= \begin{pmatrix} \sum_{i=1}^n S_i S_i^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

$$BB^* = \begin{pmatrix} \sum_{i=1}^n T_i T_i^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put A and B in inequality (3.5), by property of trace, we obtain the desired result. \square

Corollary 3.4. *Let S_i and T_i ($i = 1, 2, \dots, n$) be positive operators in $B_1(H)$. Then, we have*

$$\left(\operatorname{Tr} \sum_{i=1}^n S_i T_i \right)^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right).$$

Proof. By Theorem 3.3 for positive operators S_i and T_i , we obtain

$$\left| \operatorname{Tr} \left(\sum_{i=1}^n S_i T_i \right) \right|^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right).$$

Since S_i and T_i are positive operators, we have $\operatorname{Tr}(S_i T_i) \geq 0$. It follows that $\operatorname{Tr}(\sum_{i=1}^n S_i T_i) \geq 0$ because $\operatorname{Tr}(\sum_{i=1}^n S_i T_i) = \sum_{i=1}^n \operatorname{Tr}(S_i T_i)$. So,

$$\left(\operatorname{Tr} \sum_{i=1}^n S_i T_i \right)^2 \leq \operatorname{Tr} \left(\sum_{i=1}^n S_i^2 \right) \operatorname{Tr} \left(\sum_{i=1}^n T_i^2 \right).$$

\square

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