

AFFINE AND FUNCTIONAL FORM OF JENSEN'S INEQUALITY FOR 3-CONVEX FUNCTIONS AT A POINT

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ABSTRACT. In this paper, we give the refinement of an extension of Jensen's inequality to affine combinations. Furthermore, we present the functional form of Jensen's inequality for continuous 3-convex functions of one variable at a point.

1. INTRODUCTION

Let \mathcal{X} be a real linear space. A set $\mathcal{A} \subset \mathcal{X}$ is affine if it contains all binomial affine combinations $\alpha a + \beta b$ of points $a, b \in \mathcal{A}$ and coefficient $\alpha, \beta \in \mathbb{R}$ of sum $\alpha + \beta = 1$. The affine hull of a set $\mathcal{S} \subset \mathcal{X}$ as the smallest affine set that contains \mathcal{S} is denoted with $\text{aff}\mathcal{S}$. A function $f : \mathcal{A} \rightarrow \mathbb{R}$ is affine if the equality

$$f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$$

holds for all binomial affine combinations of points of \mathcal{A} .

A set $\mathcal{Y} \subset \mathcal{X}$ is convex if it contains all binomial convex combinations $\alpha a + \beta b$ of points $a, b \in \mathcal{Y}$ and non-negative coefficient $\alpha, \beta \in \mathbb{R}$ of sum $\alpha + \beta = 1$. The convex hull of a set $\mathcal{S} \subset \mathcal{X}$ as the smallest convex set that contains \mathcal{S} is denoted with $\text{conv}\mathcal{S}$. A function $f : \mathcal{Y} \rightarrow \mathbb{R}$ is convex if the inequality

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b)$$

holds for all binomial convex combinations of points of \mathcal{Y} .

Let Ω be a non-empty set, and let \mathbb{X} be a subspace of linear space of all real functions on the domain Ω . Also, assume that the unit function defined by $I(x) = 1$ for every $x \in \Omega$ belongs to \mathbb{X} . Let $\mathcal{I} \subset \mathbb{R}$ be an interval, and let $\mathbb{X}_{\mathcal{I}} \subset \mathbb{X}$ be a subset containing all functions with image in \mathcal{I} . If $\alpha g + \beta h$ is a convex combination of functions $g, h \in \mathbb{X}_{\mathcal{I}}$, then the convex combination $\alpha g(x) + \beta h(x)$ is in \mathcal{I} for every $x \in \Omega$, which indicates that functions set $\mathbb{X}_{\mathcal{I}}$ is convex.

A linear functional $L : \mathbb{X} \rightarrow \mathbb{R}$ is positive (non-negative) if $L(g) \geq 0$ for every non-negative function $g \in \mathbb{X}$, and L is unital (normalized) if $L(1) = 1$. If $g \in \mathbb{X}$, then every unital positive functional L , the number $L(g)$ is in the closed interval of real numbers

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containing the image of the function g .

In 2015, Z. Pavić [3] gave the extension of Jensen's inequality to affine combinations in the following form

Theorem 1. *Let $\alpha_i, \beta_j, \gamma_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^n \alpha_i$, $\beta = \sum_{j=1}^m \beta_j$, $\gamma = \sum_{k=1}^l \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0, 1]$. Let $a_i, b_j, c_k \in \mathbb{R}$ be points such that $c_k \in \text{conv}\{a, b\}$, where*

$$a = \frac{1}{\alpha} \sum_{i=1}^n \alpha_i a_i, \quad b = \frac{1}{\beta} \sum_{j=1}^m \beta_j b_j. \quad (1.1)$$

Then the affine combination

$$\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \quad (1.2)$$

belongs to $\text{conv}\{a, b\}$, and for every convex functions $f : \text{conv}\{a_i, b_j\} \rightarrow \mathbb{R}$ satisfies the inequality

$$f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \leq \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k). \quad (1.3)$$

In 2014, Z. Pavić [4] also gave the functional form of Jensen's inequality for the continuous convex functions of one variable in the following form

Theorem 2. *Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, let $[a, b] \subset \mathcal{I}$, let function $g \in \mathbb{X}_{[a,b]}$ and function $h \in \mathbb{X}_{\mathcal{I} \setminus (a,b)}$. Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous convex function such that $f(g), f(h) \in \mathbb{X}$. If a pair of unital positive linear functionals $L, H : \mathbb{X} \rightarrow \mathbb{R}$ satisfies*

$$L(g) = H(h), \quad (1.4)$$

then

$$L(f(g)) \leq H(f(h)). \quad (1.5)$$

Furthermore, Z. Pavić [4] also gave some consequent results in the form of corollaries and using these corollaries, he gave another important result as follow

Corollary 1. *Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, let function $g \in \mathbb{X}_{[a,b]}$. Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous convex function such that $f(g) \in \mathbb{X}$. If a unital positive linear functional $L : \mathbb{X} \rightarrow \mathbb{R}$ satisfies the implication (1.4) \Rightarrow (1.5) of Theorem 2 for $L = H$, then*

$$f(L(g)) \leq L(f(g)). \quad (1.6)$$

Corollary 2. *Let $[a_1, b_1] \subseteq \dots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathcal{I}$. Let function $g_1 \in \mathbb{X}_{[a_1, b_1]}$, let functions $g_k \in \mathbb{X}_{[a_k, b_k] \setminus (a_{k-1}, b_{k-1})}$ for $k = 2, \dots, n-1$, and let function $g_n \in \mathbb{X}_{\mathcal{I} \setminus (a_{n-1}, b_{n-1})}$. Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous convex function such that $f(g_i) \in \mathbb{X}$. If an n -tuple of unital positive linear functionals $L_i : \mathbb{X} \rightarrow \mathbb{R}$ satisfies*

$$L_i(g_i) = L_{i+1}(g_{i+1}) \quad \text{for } i = 1, \dots, n-1, \quad (1.7)$$

then

$$L_i(f(g_i)) \leq L_{i+1}((g_{i+1})) \text{ for } i = 1, \dots, n-1. \quad (1.8)$$

Corollary 3. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, and let functions $g_1, \dots, g_n \in \mathbb{X}_{\mathcal{I}}$. Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous convex function such that $f(g_i) \in \mathbb{X}$.

Then every n -tuple of positive linear functionals $L_i : \mathbb{X} \rightarrow \mathbb{R}$ with $\sum_{i=1}^n L_i(1) = 1$ satisfies the inclusion

$$\sum_{i=1}^n L_i(g_i) \in \mathcal{I} \quad (1.9)$$

and the inequality

$$f\left(\sum_{i=1}^n L_i(g_i)\right) \leq \sum_{i=1}^n L_i(f(g_i)). \quad (1.10)$$

Theorem 3. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, and let $[a, b] \subset \mathcal{I}$. Let functions $g_1, \dots, g_n \in \mathbb{X}_{[a, b]}$ and $h_1, \dots, h_m \in \mathbb{X}_{\mathcal{I} \setminus (a, b)}$. Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous convex function such that $f(g_i), f(h_j) \in \mathbb{X}$.

If a pair of n -tuple of positive linear functionals $L_i, H_j : \mathbb{X} \rightarrow \mathbb{R}$ with $\sum_{i=1}^n L_i(1) = \sum_{j=1}^m H_j(1) = 1$ satisfies

$$\sum_{i=1}^n L_i(g_i) = \sum_{j=1}^m H_j(h_j), \quad (1.11)$$

then

$$\sum_{i=1}^n L_i(f(g_i)) \leq \sum_{j=1}^m H_j(f(h_j)). \quad (1.12)$$

2. RESULTS

In [1], I. A. Baloch, J. Pečarič, M. Praljak defined a new class of functions which is defined as follow

Definition 1. Let $c \in I^\circ$, where I is an arbitrary interval (open, closed or semi-open in either direction) in \mathbb{R} and I° is its interior. We say that $f : I \rightarrow \mathbb{R}$ is 3-convex function in point c (respectively 3-concave function in point c) if there exists a constant A such that the function $F(x) = f(x) - \frac{A}{2}x^2$ is concave (resp. convex) on $I \cap (-\infty, c]$ and convex (resp. concave) on $I \cap [c, \infty)$. A f is 3-concave function in point c if $-f$ is 3-convex function in point c .

A property that explains the name of the class is the fact that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval (see [1]). Note that $K_1^c(I)$ and $K_2^c(I)$ denote the class of all 3-convex functions in point c and the class of all 3-concave functions in point c respectively.

Theorem 4. Let $\alpha_i, \beta_j, \gamma_k \geq 0$ and $\lambda_i, \mu_j, \nu_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^n \alpha_i$, $\beta = \sum_{j=1}^m \beta_j$, $\gamma = \sum_{k=1}^l \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0, 1]$;

$\lambda = \sum_{i=1}^n \lambda_i$, $\mu = \sum_{j=1}^m \mu_j$, $\nu = \sum_{k=1}^l \nu_k$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0, 1]$. Let $a_i, b_j, c_k \in [a, c]$ be points such that $c_k \in \text{conv}\{a_i, b_j\}$ and $r_i, s_j, t_k \in [c, b]$ be points such that $t_k \in \text{conv}\{r_i, s_j\}$, where

$$a = \frac{1}{\alpha} \sum_{i=1}^n \alpha_i a_i, \quad b = \frac{1}{\beta} \sum_{j=1}^m \beta_j b_j, \quad r = \frac{1}{\lambda} \sum_{i=1}^n \lambda_i r_i, \quad s = \frac{1}{\mu} \sum_{j=1}^m \mu_j s_j.$$

Now, if

$$\begin{aligned} & \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \\ &= \sum_{i=1}^n \lambda_i (r_i)^2 + \sum_{j=1}^m \mu_j (s_j)^2 - \sum_{k=1}^l \nu_k (t_k)^2 - \left(\sum_{i=1}^n \lambda_i r_i + \sum_{j=1}^m \mu_j s_j - \sum_{k=1}^l \nu_k t_k \right)^2 \quad (2.1) \end{aligned}$$

and also there exists $c \in I^\circ$ ($I = [a, b]$) such that

$$\max\{\max_i\{a_i\}, \max_j\{b_j\}, \max_k\{c_k\}\} \leq c \leq \min\{\min_i\{r_i\}, \min_j\{s_j\}, \min_k\{t_k\}\}. \quad (2.2)$$

Then for every $f \in K_1^c(I)$, the following inequality holds

$$\begin{aligned} & \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ & \leq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \quad (2.3) \end{aligned}$$

Proof. Since $f \in K_1^c(I)$, then there exists a constant A such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $I \cap (-\infty, c]$ and for $a_i, b_j, c_k \in [a, c]$ be points such that $c_k \in \text{conv}\{a_i, b_j\}$, so by using inequality (1.3) we have

$$\begin{aligned} 0 & \geq \sum_{i=1}^n \alpha_i F(a_i) + \sum_{j=1}^m \beta_j F(b_j) - \sum_{k=1}^l \gamma_k F(c_k) - F\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ & = \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ & \quad - \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \right\} \end{aligned}$$

Also, since $f \in K_1^c(I)$ is convex on $I \cap [c, \infty)$, hence for $r_i, s_j, t_k \in [c, b]$ be points such that $t_k \in \text{conv}\{r_i, s_j\}$, so by using inequality (1.3) we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^n \alpha_i F(r_i) + \sum_{j=1}^m \beta_j F(s_j) - \sum_{k=1}^l \gamma_k F(t_k) - F\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \\ &= \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \\ &\quad - \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \end{aligned}$$

From above, we have

$$\begin{aligned} &\sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ &- \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right)^2 \right\} \\ &\leq 0 \leq \\ &\sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \\ &- \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \quad (2.4) \end{aligned}$$

So

$$\begin{aligned} &\sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ &- \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right)^2 \right\} \\ &\leq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \\ &- \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \end{aligned}$$

by using (2.1), we get (2.3). \square

Remark 1. From the proof of Theorem 4, we have

$$\begin{aligned} & \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ & \leq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right)^2 \right\} \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \\ & \geq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \end{aligned} \quad (2.6)$$

So under assumption (2.1), we can get a improvement of (2.3) as follows

$$\begin{aligned} & \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\ & \leq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right)^2 \right\} \\ & \left(= \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \right) \\ & \leq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \end{aligned} \quad (2.7)$$

Assume that $\tilde{a} = \max_i \{a_i\}$, $\tilde{b} = \max_j \{b_j\}$, $\tilde{c} = \max_k \{c_k\}$ and $\tilde{r} = \min_i \{r_i\}$, $\tilde{s} = \min_j \{s_j\}$, $\tilde{t} = \min_k \{t_k\}$. Also, let $\tilde{a} = \max\{\tilde{a}, \tilde{b}, \tilde{c}\}$ and $\tilde{r} = \min\{\tilde{r}, \tilde{s}, \tilde{t}\}$. Now, we give the next result which weakens the assumption (2.1) such that inequality (2.5) also holds.

Theorem 5. Let $\alpha_i, \beta_j, \gamma_k \geq 0$ and $\lambda_i, \mu_j, \nu_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^n \alpha_i$, $\beta = \sum_{j=1}^m \beta_j$, $\gamma = \sum_{k=1}^l \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0, 1]$; $\lambda = \sum_{i=1}^n \lambda_i$, $\mu = \sum_{j=1}^m \mu_j$, $\nu = \sum_{k=1}^l \nu_k$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0, 1]$. Let $a_i, b_j, c_k \in [a, c]$ be points such that $c_k \in \text{conv}\{a_i, b_j\}$ and $r_i, s_j, t_k \in [c, b]$ be points such that $t_k \in \text{conv}\{r_i, s_j\}$, where

$$a = \frac{1}{\alpha} \sum_{i=1}^n \alpha_i a_i, \quad b = \frac{1}{\beta} \sum_{j=1}^m \beta_j b_j, \quad r = \frac{1}{\lambda} \sum_{i=1}^n \lambda_i r_i, \quad s = \frac{1}{\mu} \sum_{j=1}^m \mu_j s_j.$$

such that

$$\tilde{a} \leq \tilde{r} \quad (2.8)$$

and $f \in K_1^c(I)$ for some $c \in [\tilde{a}, \tilde{r}]$. Then if

(a)

$$f''(\tilde{a}) \geq 0$$

and

$$\begin{aligned} & \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \\ & \leq \sum_{i=1}^n \lambda_i (r_i)^2 + \sum_{j=1}^m \mu_j (s_j)^2 - \sum_{k=1}^l \nu_k (t_k)^2 - \left(\sum_{i=1}^n \lambda_i r_i + \sum_{j=1}^m \mu_j s_j - \sum_{k=1}^l \nu_k t_k \right)^2 \end{aligned}$$

or

(b)

$$f''_+(\tilde{r}) \leq 0$$

and

$$\begin{aligned} & \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \\ & \geq \sum_{i=1}^n \lambda_i (r_i)^2 + \sum_{j=1}^m \mu_j (s_j)^2 - \sum_{k=1}^l \nu_k (t_k)^2 - \left(\sum_{i=1}^n \lambda_i r_i + \sum_{j=1}^m \mu_j s_j - \sum_{k=1}^l \nu_k t_k \right)^2 \end{aligned}$$

or

(c)

$$f''_-(\tilde{a}) < 0 < f''_+(\tilde{r}) \text{ and } f \text{ is } 3\text{-convex,}$$

then (2.3) holds.

Proof. The idea of proof is similar to proof of Theorem 4. Hence, by proceeding as in the proof of Theorem 4. From the inequality 2.4, we have

$$\begin{aligned} & \frac{A}{2} \left[\sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k \right)^2 \right. \\ & \left. - \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \right\} \right] \\ & \leq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k \right) \\ & \left. - \left\{ \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right) \right\} \right] \end{aligned}$$

Now, due to the concavity of F on $[a, c]$ and convexity of F on $[c, b]$, for every distinct points $a_j \in [a, \tilde{a}]$ and $r_j \in [\tilde{r}, b]$, $j = 1, 2, 3$, we have

$$[a_1, a_2, a_3]f \leq A \leq [r_1, r_2, r_3]f$$

Letting $a_j \nearrow \tilde{a}$ and $r_j \searrow \tilde{r}$, we get (if exists)

$$f''_-(\tilde{a}) \leq A \leq f''_+(\tilde{r})$$

Therefore, if assumptions (a) or (b) holds, then

$$\begin{aligned} & \frac{A}{2} \left[\sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k \right)^2 \right. \\ & \left. - \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \right\} \right] \end{aligned}$$

is positive and we conclude the result. If the assumption (c) holds, the f''_- is left continuous, f''_+ is right continuous, they are both non-decreasing and $f''_- \leq f''_+$. Therefore, there exists $\tilde{c} \in [\tilde{a}, \tilde{r}]$ such that $f \in K_1^c(I)$ with associated constant $\tilde{A} = 0$ and again, we can deduce the result. \square

Remark 2. Again from the proof of Theorem 5, we obtain the inequalities (2.5) and (2.6). Now, under assumption (a), (b) or (c) of Theorem 5, A is positive or negative or zero respectively due to argument discussed in the proof. Therefore, we get a better improvement of (2.3) then (2.7). in this case as follow

$$\begin{aligned} & \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right) \\ & \leq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \right\} \\ & \leq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k \right)^2 \right\} \\ & \leq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k \right) \quad (2.9) \end{aligned}$$

Under the assumption of Theorem 4 with $f \in K_2^c(I)$, the reverse of inequality (2.3) holds. Now, we give only the statement of Theorem with weaker condition under which the reverse of inequality (2.3) also holds for $f \in K_2^c(I)$.

Theorem 6. Let $\alpha_i, \beta_j, \gamma_k \geq 0$ and $\lambda_i, \mu_j, \nu_k \geq 0$ be coefficients such that their sum $\alpha = \sum_{i=1}^n \alpha_i$, $\beta = \sum_{j=1}^m \beta_j$, $\gamma = \sum_{k=1}^l \gamma_k$ satisfy $\alpha + \beta - \gamma = 1$ and $\alpha, \beta \in (0, 1]$; $\lambda = \sum_{i=1}^n \lambda_i$, $\mu = \sum_{j=1}^m \mu_j$, $\nu = \sum_{k=1}^l \nu_k$ satisfy $\lambda + \mu - \nu = 1$ and $\lambda, \mu \in (0, 1]$. Let

$a_i, b_j, c_k \in [a, c]$ be points such that $c_k \in \text{conv}\{a_i, b_j\}$ and $r_i, s_j, t_k \in [c, b]$ be points such that $t_k \in \text{conv}\{r_i, s_j\}$, where

$$a = \frac{1}{\alpha} \sum_{i=1}^n \alpha_i a_i, \quad b = \frac{1}{\beta} \sum_{j=1}^m \beta_j b_j, \quad r = \frac{1}{\alpha} \sum_{i=1}^n \lambda_i r_i, \quad s = \frac{1}{\mu} \sum_{j=1}^m \mu_j s_j.$$

such that

$$\tilde{a} \leq \tilde{r} \tag{2.10}$$

and $f \in K_2^c(I)$ for some $c \in [\tilde{a}, \tilde{r}]$. Then if

(a)

$$f''_-(\tilde{a}) \leq 0$$

and

$$\begin{aligned} & \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \\ & \geq \sum_{i=1}^n \lambda_i (r_i)^2 + \sum_{j=1}^m \mu_j (s_j)^2 - \sum_{k=1}^l \nu_k (t_k)^2 - \left(\sum_{i=1}^n \lambda_i r_i + \sum_{j=1}^m \mu_j s_j - \sum_{k=1}^l \nu_k t_k \right)^2 \end{aligned}$$

or

(b)

$$f''_+(\tilde{r}) \geq 0$$

and

$$\begin{aligned} & \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k \right)^2 \\ & \leq \sum_{i=1}^n \lambda_i (r_i)^2 + \sum_{j=1}^m \mu_j (s_j)^2 - \sum_{k=1}^l \nu_k (t_k)^2 - \left(\sum_{i=1}^n \lambda_i r_i + \sum_{j=1}^m \mu_j s_j - \sum_{k=1}^l \nu_k t_k \right)^2 \end{aligned}$$

or

(c)

$$f''_-(\tilde{a}) < 0 < f''_+(\tilde{r}) \text{ and } f \text{ is 3-concave,}$$

then reverse of (2.3) holds.

Remark 3. From the proof of the Theorem 6, we obtain the reverse of inequalities (2.5) and (2.6). Now, due to the convexity of F on $[a, c]$ and concavity of F on $[c, b]$, for every distinct points $a_j \in [a, \tilde{a}]$ and $r_j \in [\tilde{r}, b]$, $j = 1, 2, 3$, we have

$$[a_1, a_2, a_3]f \geq A \geq [r_1, r_2, r_3]f$$

Letting $a_j \nearrow \tilde{a}$ and $r_j \searrow \tilde{r}$, we get (if exists)

$$f''_-(\tilde{a}) \geq A \geq f''_+(\tilde{r})$$

Now, under assumption (a), (b) or (c) of Theorem 5, A is negative or positive or zero respectively due to argument discussed above. Therefore, we get a better improvement in this case as follow

$$\begin{aligned}
& \sum_{i=1}^n \alpha_i f(a_i) + \sum_{j=1}^m \beta_j f(b_j) - \sum_{k=1}^l \gamma_k f(c_k) - f\left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right) \\
& \geq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (a_i)^2 + \sum_{j=1}^m \beta_j (b_j)^2 - \sum_{k=1}^l \gamma_k (c_k)^2 - \left(\sum_{i=1}^n \alpha_i a_i + \sum_{j=1}^m \beta_j b_j - \sum_{k=1}^l \gamma_k c_k\right)^2 \right\} \\
& \geq \frac{A}{2} \left\{ \sum_{i=1}^n \alpha_i (r_i)^2 + \sum_{j=1}^m \beta_j (s_j)^2 - \sum_{k=1}^l \gamma_k (t_k)^2 - \left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right)^2 \right\} \\
& \geq \sum_{i=1}^n \alpha_i f(r_i) + \sum_{j=1}^m \beta_j f(s_j) - \sum_{k=1}^l \gamma_k f(t_k) - f\left(\sum_{i=1}^n \alpha_i r_i + \sum_{j=1}^m \beta_j s_j - \sum_{k=1}^l \gamma_k t_k\right) \quad (2.11)
\end{aligned}$$

Theorem 7. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, let $[a, b] \subset \mathcal{I}$, let function $g_i \in \mathbb{X}_{[a, b]}$ and function $h_i \in \mathbb{X}_{\mathcal{I} \setminus (a, b)}$ for $i = 1, 2$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_i), f(h_i) \in \mathbb{X}$. If a pair of unital positive linear functionals $L, H : \mathbb{X} \rightarrow \mathbb{R}$ satisfies

$$L(g_i) = H(h_i) \text{ and } H(h_1^2) - L(g_1^2) = H(h_2^2) - L(g_2^2), \quad i = 1, 2, \quad (2.12)$$

then inequality

$$H(f(h_1)) - L(f(g_1)) \leq H(f(h_2)) - L(f(g_2)) \quad (2.13)$$

holds.

Proof. Since $f \in K_1^c(\mathcal{I})$, there exists a constant A such that $F(x) = f(x) - \frac{A}{2}x^2$ is concave on $\mathcal{I} \cap (-\infty, c]$, therefore by reverse of (1.5) for F on $\mathcal{I} \cap (-\infty, c]$, we get

$$\begin{aligned}
0 & \geq H(F(h_1)) - L(F(g_1)) \\
& = H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2))
\end{aligned}$$

Also, since $F(x) = f(x) - \frac{A}{2}x^2$ is convex on $\mathcal{I} \cap [c, \infty)$, therefore by (1.5) for F on $\mathcal{I} \cap [c, \infty)$, we get

$$\begin{aligned}
0 & \leq H(F(h_2)) - L(F(g_2)) \\
& = H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2))
\end{aligned}$$

From above, we have

$$\begin{aligned} H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \\ \leq 0 \leq \\ H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)). \end{aligned}$$

So

$$\begin{aligned} H(f(h_1)) - L(f(g_1)) - \frac{A}{2}(H(h_1^2) - L(g_1^2)) \\ \leq H(f(h_2)) - L(f(g_2)) - \frac{A}{2}(H(h_2^2) - L(g_2^2)), \end{aligned}$$

therefore, by the use of (2.12), we get (2.13). \square

Remark 4. From the proof of the Theorem 7, we have

$$H(f(h_1)) - L(f(g_1)) \leq \frac{A}{2}(H(h_1^2) - L(g_1^2)) \quad (2.14)$$

and

$$H(f(h_2)) - L(f(g_2)) \geq \frac{A}{2}(H(h_2^2) - L(g_2^2)) \quad (2.15)$$

So, under assumption (2.12), we can get a better improvement of (2.13) as follow

$$\begin{aligned} H(f(h_1)) - L(f(g_1)) &\leq \\ &\frac{A}{2}(H(h_1^2) - L(g_1^2)) (= \frac{A}{2}(H(h_2^2) - L(g_2^2))) \\ &\leq H(f(h_2)) - L(f(g_2)) \end{aligned} \quad (2.16)$$

Corollary 4. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, let $[a, b] \subset \mathcal{I}$, let function $g_i \in \mathbb{X}_{[a, b]}$ for $i = 1, 2$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_i) \in \mathbb{X}$. If a unital positive linear functionals $L : \mathbb{X} \rightarrow \mathbb{R}$ satisfies implication (2.12) \Rightarrow (2.13) for $L = H$ such that

$$L(g_1^2) - (L(g_1))^2 = L(g_2^2) - (L(g_2))^2 \quad (2.17)$$

then following inequality holds

$$L(f(g_1)) - f(L(g_1)) \leq L(f(g_2)) - f(L(g_2)) \quad (2.18)$$

Corollary 5. Let $[a_1, b_1] \subseteq \dots \subseteq [a_{n-1}, b_{n-1}] \subseteq \mathcal{I}$. Let function $g_1, h_1 \in \mathbb{X}_{[a_1, b_1]}$, let $g_k, h_k \in \mathbb{X}_{[a_k, b_k] \setminus (a_{k-1}, b_{k-1})}$ for $k = 2, \dots, n-1$, and let function $g_n, h_n \in \mathbb{X}_{\mathcal{I} \setminus (a_{n-1}, b_{n-1})}$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_i) \in \mathbb{X}$.

If an n -tuple of unital positive linear functionals $L_i : \mathbb{X} \rightarrow \mathbb{R}$ satisfies

$$L_i(g_i) = L_{i+1}(g_{i+1}) \text{ and } L_i(h_i) = L_{i+1}(h_{i+1}) \text{ for } i = 1, \dots, n-1, \quad (2.19)$$

such that

$$L_{i+1}(g_{i+1}^2) - L_i(g_i^2) = L_{i+1}(h_{i+1}^2) - L_i(h_i^2), \quad (2.20)$$

then

$$L_{i+1}f(g_{i+1}) - L_i f(g_i) \leq L_{i+1}f(h_{i+1}) - L_i f(h_i) \text{ for } i = 1, \dots, n-1. \quad (2.21)$$

Corollary 6. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, and let functions $g_i, h_i \in \mathbb{X}_{\mathcal{I}}$ for $i = 1, \dots, n$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_i), f(h_i) \in \mathbb{X}$. Then every n -tuple of positive linear functionals $L_i : \mathbb{X} \rightarrow \mathbb{R}$ with $\sum_{i=1}^n L_i(1) = 1$ such that

$$\sum_{i=1}^n L_i((g_i)^2) - \left(\sum_{i=1}^n L_i(g_i) \right)^2 = \sum_{i=1}^n L_i((h_i)^2) - \left(\sum_{i=1}^n L_i(h_i) \right)^2 \quad (2.22)$$

satisfies the inclusion

$$\sum_{i=1}^n L_i(g_i), \sum_{i=1}^n L_i(h_i) \in \mathcal{I} \quad (2.23)$$

and the inequality

$$\sum_{i=1}^n L_i(f(g_i)) - f\left(\sum_{i=1}^n L_i(g_i)\right) \leq \sum_{i=1}^n L_i(f(h_i)) - f\left(\sum_{i=1}^n L_i(h_i)\right) \quad (2.24)$$

Theorem 8. Let $\mathcal{I} \subset \mathbb{R}$ be a closed interval, let $[a, b] \subset \mathcal{I}$, let function $g_i, g_i^* \in \mathbb{X}_{[a, b]}$ for $i = 1, \dots, n$ and $h_j, h_j^* \in \mathbb{X}_{\mathcal{I} \setminus (a, b)}$ for $j = 1, \dots, m$. Let $f \in K_1^c(\mathcal{I})$ be continuous function such that $f(g_i), f(g_i^*), f(h_j), f(h_j^*) \in \mathbb{X}$.

If two pair of n -tuple of positive linear functionals $L_i, L_i^*, H_j, H_j^* : \mathbb{X} \rightarrow \mathbb{R}$ with

$$\sum_{i=1}^n L_i(1) = \sum_{i=1}^n L_i^*(1) = \sum_{j=1}^m H_j(1) = \sum_{j=1}^m H_j^*(1) = 1$$

satisfy

$$\sum_{j=1}^m H_j(h_j) = \sum_{i=1}^n L_i(g_i) \text{ and } \sum_{j=1}^m H_j^*(h_j^*) = \sum_{i=1}^n L_i^*(g_i^*) \quad (2.25)$$

and

$$\sum_{j=1}^m H_j((h_j)^2) - \sum_{i=1}^n L_i((g_i)^2) = \sum_{j=1}^m H_j^*((h_j^*)^2) - \sum_{i=1}^n L_i^*((g_i^*)^2). \quad (2.26)$$

Then

$$\sum_{j=1}^m H_j f(h_j) - \sum_{i=1}^n L_i f(g_i) \leq \sum_{j=1}^m H_j^* f(h_j^*) - \sum_{i=1}^n L_i^* f(g_i^*) \quad (2.27)$$

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