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ASYMMETRIES IN CYLINDRICAL WAVEGUIDES

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ABSTRACT

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Cylindrical geometries with axial symmetry are often used to model physical systems. Although such a model materially simplifies calculations it may ignore significant effects arising from small asymmetries in the system. In order to demonstrate this and to examine the nature of such effects a model was analyzed consisting of a cylindrical waveguide with perfectly reflecting walls and an isolated point source displaced a small distance from the axis of the hole. The subsequent analysis shows two distinct types of arrivals associated with the geometry. Both of these produce strikingly large asymmetries in the motion of the system. These effects are clearly displayed in the closed form solutions obtained for the problem and the light which they shed on the nature of the reflection process at a cylindrical interface may well be of significant value for ray theory methods in diffraction.

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INTRODUCTION

Cylindrical geometries with axial symmetry are often used to model physical systems. The symmetrical problem for wave propagation in a fluid filled elastic cylinder has been considered by Ingram[1,2,3] and Biot[4]. These treatments discuss the complications arising from the interaction of the fluid and elastic surfaces, but they may neglect significant effects arising from small asymmetries in the system. In order to demonstrate this and to examine the nature of such effects a model was analyzed consisting of an infinite cylindrical waveguide with perfectly reflecting walls and an isolated point source displaced a small distance from the axis of the hole. The geometry of the intended situation is shown in Fig. 1.

The analysis is begun by taking Laplace and Fourier transforms of the acoustic wave equation and applying the boundary conditions, first for a free boundary and then for a rigid wall. In each case, the inversion of the Fourier transform is carried out by contour integration, and the Laplace inversion is known. For small asymmetry in source placement, approximations permit use of generalized function theory to yield a closed form solution.

The closed form solution shows large asymmetries produced by two distinct types of arrivals associated with the geometry, one arrival dependent on the source function itself, and the other on the derivative of the source function. Only the lengths of the time intervals during which the assymetric arrivals are present, and not the magnitudes of the arrivals, are dependent on the asymmetry of the source. That is, an observer somewhere in the cylinder measuring the asymmetric effects of a pulse source would sense repeated bursts of asymmetric motion. These bursts would be the same strength regardless of the degree of asymmetry of the source, but the percentage of time they were present would be small if the source asymmetry were small. The light which these effects shed on the nature of the reflection process at a cylindrical interface may well be of significant value for ray theory methods in diffraction.

II. FREE BOUNDARY

If we define a displacement potential φ such that the displacement $(u,v,w) = \nabla\varphi$, then the acoustic wave equation may be written

$$\nabla^2\varphi = \frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2}$$

or, in cylindrical coordinates,

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2} \tag{1}$$

where α = velocity of sound in the fluid. Taking a Laplace transform in t and a Fourier transform in z gives

$$\frac{\partial^2\bar{\bar{\varphi}}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{\bar{\varphi}}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\bar{\bar{\varphi}}}{\partial\theta^2} - \bar{\bar{\varphi}}\left(p^2 + \frac{s^2}{\alpha^2}\right) = 0 \quad . \tag{2}$$

If $\bar{\bar{\varphi}}$ is now written as a Fourier series in θ given by

$$\bar{\bar{\varphi}} = \frac{a_o(r)}{2} + \sum_{m=-\infty}^{\infty} a_m(r)\, e^{im\theta} \quad , \tag{3}$$

The coefficients a_m must have the form

$$a_m(r) = c_m I_m \left(\sqrt{p^2 + s^2/\alpha^2}\, r\right) \quad . \tag{4}$$

Any wave in the cavity must have the form given by (3) and (4). The potential due to the source may be found by taking the Laplace transform of (1), given by

$$\nabla^2 \bar{\varphi}_s = k^2 \bar{\varphi}_s \quad , \qquad \text{where} \quad k^2 = -\frac{s^2}{c^2} = \frac{\omega^2}{c^2} \quad .$$

For a point source located at (a, θ, 0) this equation has the solution

$$\bar{\varphi}_s = \frac{1}{Z} e^{ikZ} \qquad \text{where} \quad Z = \sqrt{r^2 + a^2 - 2ar\cos\theta + z^2}$$

Identities for Bessel functions yield

$$\bar{\varphi}_s = \begin{cases} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K_m \left(r\sqrt{p^2-k^2}\right) I_m \left(a\sqrt{p^2-k^2}\right) e^{(ipz + im\theta)}\, dp & \text{for } r > a \\ \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K_m \left(a\sqrt{p^2-k^2}\right) I_m \left(r\sqrt{p^2-k^2}\right) e^{(ipz + im\theta)}\, dp & \text{for } r < a \end{cases} \tag{5}$$

The potential in the fluid consists of the sum of source potential given in (5) and a reflected potential having the form given in (3) and (4). If the boundary of the fluid column is to be free, the pressure at its surface must vanish, i.e., p(R, θ, z) = 0. Since $p = -\rho \frac{\partial^2 \varphi}{\partial t^2}$, the sum of the potentials must satisfy

$$\bar{\bar{\varphi}}_R (R) + \bar{\bar{\varphi}}_s(R) = 0 \quad . \tag{6}$$

By applying (6) individually to each term in the series (4), we solve for each c_m and obtain

$$\varphi = \frac{1}{2\pi} \iint \sum_{m=-\infty}^{\infty} \left[K_m\left(r\sqrt{p^2-k^2}\right) I_m\left(a\sqrt{p^2-k^2}\right) - \frac{K_m(R\sqrt{p^2-k^2})\, I_m(a\sqrt{p^2-k^2})}{I_m(R\sqrt{p^2-k^2})} \right.$$

$$\left. \cdot\, I_m\, (r\sqrt{p^2-k^2}) \right] e^{im\theta}\, e^{i(pz + \omega t)}\, dp\, d\omega \quad \text{for } r > a$$

$$\varphi = \frac{1}{2\pi} \iint \sum_{m=-\infty}^{\infty} \left[K_m(a\sqrt{p^2-k^2})\, I_m(r\sqrt{p^2-k^2}) - \frac{K_m(R\sqrt{p^2-k^2})\, I_m(a\sqrt{p^2-k^2})}{I_m(R\sqrt{p^2-k^2})} \right.$$

$$\left. \cdot\, I_m\, (r\sqrt{p^2-k^2}) \right] e^{im\theta}\, e^{i(pz + \omega t)}\, dp\, d\omega \quad \text{for } r < a \tag{7}$$

The integration with respect to p in (7) is carried out by contour integration. For all r, $0 \leq r \leq R$, the only poles of (7) are located at the zeros of $I_m(R\sqrt{p^2-k^2})$, but since I_m has no real zeros, it is best to convert to Bessel functions of the first and second kind. If we denote the zeros of J_m by the sequence $\gamma^m{}_i$, the poles are then located by

$$\left(\frac{\gamma^m{}_i}{R\, p_o}\right)^2 = \frac{\omega^2}{\alpha^2\, p_o^{\,2}} - 1 \tag{8}$$

which gives a set of dispersion relations for the frequencies and phase velocities in the problem.

The residue theorem applied to (7) yields

$$\varphi = -\frac{\pi}{2} \int \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} \frac{Y_m(\gamma^m{}_i)\, J_m(\frac{a}{R}\gamma^m{}_i)\, J_m(\frac{r}{R}\gamma^m{}_i)}{J'(\gamma^m{}_i)}\, e^{im\theta}$$

$$\cdot \left[\frac{R^2}{\gamma^m{}_i} \sqrt{\frac{\omega^2}{\alpha^2} - \left(\frac{\gamma^m{}_i}{R}\right)^2}\, \right]^{-1} \exp i \left[z\sqrt{\frac{\omega^2}{\alpha^2} - \left(\frac{\gamma^m{}_i}{R}\right)^2} + \omega t \right] d\omega \tag{9}$$

If we define $B^m{}_i = -\, Y_m\, (\gamma^m{}_i)/J_m{}'(\gamma^m{}_i)$, we may compute

$B^o{}_1 = .96$, $B^1{}_1 = 1.02$, $B^o{}_2 = 1.00$, and indeed, asymptotic forms given in Watson[5]

$$J_{\varkappa}(z) \approx \left(\frac{2}{\pi z}\right)^{1/2} \left[\cos\left(z - \frac{1}{2}\varkappa\pi - \frac{\pi}{4}\right) \cdot \frac{\Sigma(-1)^m(\varkappa, zm)}{(2z)^{2m}}\right.$$

$$\left. - \sin\left(z - \frac{1}{2}\varkappa\pi - \frac{\pi}{4}\right) \cdot \frac{\Sigma(-1)^m(\varkappa, 2m+1)}{(2z)^{2m+1}}\right]$$

$$Y_{\varkappa}(x) \approx \left(\frac{2}{\pi z}\right)^{1/2} \left[\sin\left(z - \frac{1}{2}\varkappa\pi - \frac{\pi}{4}\right) \frac{\Sigma(-1)^m(\varkappa, zm)}{(2z)^{2m}}\right.$$

$$\left. - \sin\left(z - \frac{1}{2}\varkappa\pi - \frac{\pi}{4}\right) \frac{\Sigma(-1)^m(\varkappa, zm+1)}{(2z)^{2m+1}}\right]$$

show $J_{\varkappa}'(z) \rightarrow -Y_{\varkappa}(z)$, or $B^m_i \rightarrow +1$.

The inverse Laplace transform for (9) is found in tables, and yields the result

$$\varphi = \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} B^m_i \, J_m\left(\frac{a}{R}\gamma^m_i\right) J_m\left(\frac{r}{R}\gamma^m_i\right) \frac{\alpha\gamma^m_i}{R^2} e^{im\theta}$$

$$\cdot J_o\left(\frac{\alpha\gamma^m_i}{R^2}\sqrt{t^2 - z^2/\alpha^2}\right) \quad \text{for } t \geq \frac{z}{\alpha} \tag{10}$$

$$0 \qquad \text{for } t < \frac{z}{\alpha}$$

This result will be studied further in Section IV.

III. PERFECTLY RIGID BOUNDARY

The potential in a fluid column enclosed by a perfectly rigid boundary also consists of a source potential and a reflected potential as given in (3), (5) and (6). In this case the boundary condition is that there should be no outward motion at the boundary r = R, i.e.,

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = 0$$

If we note the relations

$$I'_m(z) = \frac{1}{2}\left[I_{m-1}(z) + I_{m+1}(z)\right]$$

$$K'_m(z) = -\frac{1}{2}\left[K_{m-1}(z) + K_{m+1}(z)\right]$$

The boundary condition becomes

$$\sum_{m=-\infty}^{\infty} e^{im\theta}\left\{\frac{-\sqrt{p^2-k^2}}{2}\left[K_{m-1}\left(R\sqrt{p^2-k^2}\right) + K_{m+1}\left(R\sqrt{p^2-k^2}\right)\right]\right. \tag{11}$$

$$\left.\cdot\, I_m\left(a\sqrt{p^2-k^2}\right) + c_m\frac{\sqrt{p^2-k^2}}{2}\left[I_{m-1}\left(R\sqrt{p^2-k^2}\right) + I_{m+1}\left(R\sqrt{p^2-k^2}\right)\right]\right\} = 0 \ .$$

Solving for c_m in (11), we obtain

$$\bar{\bar{\varphi}} = \sum_{m=-\infty}^{\infty} e^{im\theta}\left\{K_m\left(r\sqrt{p^2-k^2}\right) I_m\left(a\sqrt{p^2-k^2}\right) + \right.$$

$$\left. + \frac{K_{m-1}\left(R\sqrt{p^2-k^2}\right) + K_{m+1}\left(R\sqrt{p^2-k^2}\right)}{I_{m-1}\left(R\sqrt{p^2-k^2}\right) + I_{m+1}\left(R\sqrt{p^2-k^2}\right)} I_m\left(r\sqrt{p^2-k^2}\right) I_m\left(a\sqrt{p^2-k^2}\right)\right\} \ . \tag{12}$$

To find the inverse Fourier transform of (12) by contour integration we note that its poles occur at the zeros of $I_{m-1}\left(R\sqrt{p^2-k^2}\right) + I_{m+1}\left(R\sqrt{p^2-k^2}\right)$. (13)

Since (13) has no real zeros, we must change to Bessel functions of the first and second kind. The poles then occur at the zeros of

$$J_{m+1}\left(R\sqrt{k^2-p^2}\right) - J_{m-1}\left(R\sqrt{k^2-p^2}\right) = 0 \ . \tag{14}$$

If we denote these zeros by δ^m_i we obtain the dispersion relations

$$\left(\frac{\delta^m_i}{R\rho}\right)^2 = \frac{\omega 2}{\alpha^2 p^2} - 1 \tag{15}$$

The residue theorem gives

$$\varphi = \frac{-\pi}{2} \int \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} e^{im\theta} \frac{Y_{m+1}(\delta^m_i) - Y_{m-1}(\delta^m_i)}{J'_{m+1}(\delta^m_i) - J'_{m-1}(\delta^m_i)} \cdot J_m\left(\frac{a}{R}\gamma^m_i\right) J_m\left(\frac{r}{R}\delta^m_i\right)$$

$$\cdot \left[\frac{R^2}{\delta^m_i} \sqrt{\frac{\omega^2}{\alpha^2} - \left(\frac{\delta^m_i}{R}\right)^2} \right]^{-1} \exp\left[z \sqrt{\frac{\omega^2}{\alpha^2} - \left(\frac{\delta^m_i}{R}\right)^2} + \omega t \right] d\omega \ .$$

If we let $C^m_i = - \left\{ \dfrac{Y_{m+1}(\delta^m_i) - Y_{m-1}(\delta^m_i)}{J'_{m+1}(\delta^m_i) - J_{m-1}(\delta^m_i)} \right\}$

and recall $J'_m(z) \to - Y_m(z)$, we obtain $C^m_i \to 1$. Then, the inverse Laplace transform of (16) gives

$$\varphi = \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \sum_{i=1}^{\infty} e^{im\theta} \; C^m_i \; J_m\left(\frac{a}{R}\delta^m_i\right) J_m\left(\frac{r}{R}\delta^m_i\right)$$

$$\cdot \left(\frac{\alpha\,\delta^m_i}{R^2}\right) J_o\left(\frac{\alpha\,\delta^m_i}{R^2}\sqrt{t^2 - \frac{z^2}{\alpha^2}}\right) \quad t \geq \frac{z}{\alpha}$$

$$\varphi = 0 \quad \text{for} \quad t < \frac{z}{\alpha} \ . \tag{17}$$

IV. SIMPLIFICATION FOR SMALL ASYMMETRY IN SOURCE PLACEMENT

If the asymmetry of the source is small ($a/R \ll 1$), asymptotic expansions for Bessel functions may be used to effect considerable simplification in formulas (10) and (17). We will consider (10) in some detail, defining

$$\sigma_m = \sum_{i=1} B^m_i \; J_m\left(\frac{a}{R}\gamma^m_i\right) J_m\left(\frac{r}{R}\gamma^m_i\right) \frac{\alpha\,\gamma^m_i}{R^2} \, e^{im\theta} \, J_o\left(\frac{\alpha\,\gamma^m_i}{R}\sqrt{t^2 - \frac{z^2}{\alpha^2}}\right) \ . \tag{18}$$

First, note that σ_o is symmetrical in θ, and, in fact, differs from the solution for symmetrical source placement only by the factor $J_o\left(\frac{a}{R}\gamma^m_i\right)$.

For $m \neq 0$ and $\frac{a}{R} \ll 1$, we may use the asymptotic forms:

$$J_m(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{2}\right)$$

$$\gamma_k^m \approx \frac{m\pi}{2} + \frac{\pi}{4} + \left(\frac{2\ k-1}{2}\right)\pi \quad .$$

The accuracy of these forms is such that the error in using them for $J_m(z\gamma^m_k)$ is of the order $(z\gamma^m_k)^{-2}\ J_m(z\gamma^m_k)$.

We now write the asymmetrical part of the response:

$$\sigma_m = \hat{\sigma}_m + (\sigma_m - \hat{\sigma}_m)$$

where

$$\hat{\sigma}_m = \frac{1}{\pi} \int_o^{\pi} \frac{1}{R}\sqrt{\frac{\alpha}{rt}}\ e^{im\theta} \sum_{k=1}^{\infty} \cos \pi \left\{\frac{r}{R}\left(\frac{m}{2} - \frac{1}{4} + k\right) - \frac{m}{2} - \frac{1}{4}\right\}$$

$$\cos \pi \left\{\frac{\alpha\bar{t}}{R}\left(\frac{m}{2} - \frac{1}{4} + k\right) - \frac{1}{4}\right\} \cos\left[m\omega - \frac{a\pi}{R}\left(\frac{m}{4} - \frac{1}{4} + k\right) \sin\omega\right] d\omega$$

and

$$\bar{t} = \sqrt{t^2 - \frac{z^2}{\alpha^2}} \quad . \qquad (19)$$

In writing (19), we have used the approximations above and the relation

$$J_m(z) = \frac{1}{\pi}\int_o^{\pi} \cos\ (m\omega - z \sin\omega)\ d\omega \quad .$$

The order of magnitude of the error term, $(\sigma_m - \hat{\sigma}_m)$, is seen to be

$$\max\left\{\left(\frac{r}{R}\ \gamma^m_i\right)^{-2} \left(\frac{\alpha\ \gamma^m_i t}{R}\right)^{-2}\right\} \sigma_m \quad .$$

Trigonometric identies yield:

$$\hat{\sigma}_m = \frac{1}{4\pi}\int_o^{\pi} \frac{1}{R}\sqrt{\frac{\alpha}{r\bar{t}}}\ e^{im\theta} \quad .$$

$$\Big\{\cos\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r+\alpha\bar{t}-a\sin\omega\Big)-\frac{m\pi}{2}-\frac{\pi}{2}+m\omega\Big]\sum_{k=1}^{\infty}\cos\frac{\pi k}{R}\Big(r+\alpha\bar{t}-a\sin\omega\Big)$$

$$-\sin\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r+\alpha\bar{t}-a\sin\omega\Big)-\frac{m\pi}{2}-\frac{\pi}{2}+m\omega\Big]\sum\sin\frac{\pi k}{R}\Big(r+\alpha\bar{t}-a\sin\omega\Big)$$

$$+\cos\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r-\alpha\bar{t}+a\sin\omega\Big)-\frac{m\pi}{2}-m\omega\Big]\sum\cos\frac{\pi k}{R}\Big(r-\alpha\bar{t}+a\sin\omega\Big)$$

$$-\sin\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r-\alpha\bar{t}+a\sin\omega\Big)-\frac{m\pi}{2}-m\omega\Big]\sum\sin\frac{\pi k}{R}\Big(r-\alpha\bar{t}+a\sin\omega\Big)$$

$$+\cos\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r+\alpha\bar{t}+a\sin\omega\Big)-\frac{m\pi}{2}-\frac{\pi}{2}-m\omega\Big]\sum\cos\frac{\pi k}{R}\Big(r+\alpha\bar{t}+a\sin\omega\Big)$$

$$-\sin\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r+\alpha\bar{t}+a\sin\omega\Big)-\frac{m\pi}{2}-\frac{\pi}{2}-m\omega\Big]\sum\sin\frac{\pi k}{R}\Big(r+\alpha\bar{t}+a\sin\omega\Big)$$

$$+\cos\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r-\alpha\bar{t}-a\sin\omega\Big)-\frac{m\pi}{2}+m\omega\Big]\sum\cos\frac{\pi k}{R}\Big(r-\alpha\bar{t}-a\sin\omega\Big)$$

$$-\sin\Big[\frac{\pi}{R}\Big(\frac{m}{2}-\frac{1}{4}\Big)\Big(r-\alpha\bar{t}-a\sin\omega\Big)-\frac{m\pi}{2}+m\omega\Big]\sum\sin\frac{\pi k}{R}\Big(r-\alpha\bar{t}-a\sin\omega\Big)\,d\omega\Big\}\;. \tag{20}$$

The sine and cosine terms in (20) must be summed using different formulas, so we define

$$\sigma_m^{\ 1} = \text{sum of sine terms in } \sigma_m$$

$$\hat{\sigma}_m = \sigma_m^{\ 1} + \sigma_m^{\ 2} \tag{21}$$

$$\sigma_m^{\ 2} = \text{sum of cosine terms in } \sigma_m$$

The sum of the sine terms may be found by using[6]

$$\sum_{k=1}^{\infty} \sin kx = \frac{\sin x}{2(1-\cos x)} \qquad , \quad x \neq 2\pi$$

and various trigonometric identities, along with[7]

$$\sum_{k=-\infty}^{\infty} e^{i\pi kx/L} = L\sum_{m=-\infty}^{\infty} \delta(x-2mL) \quad .$$

The sums given above must be treated as generalized functions in the sense given by Lighthill in Fourier Analysis and Generalized Functions. This is permissible since we are not interested in the function itself but in its convolution with a suitably continuous source function.

In particular[8],

$$\int_{-\infty}^{\infty} \delta\,(x)\,F(x)\,dx = F(0)$$

and

$$\int_{-\infty}^{\infty} f'(x)\,F(x)\,dx == -\int_{-\infty}^{\infty} f(x)\,F'(x)\,dx$$

where f(x) is a generalized function

Then since $\varphi' = \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \sigma_m 1$, we have

$$\varphi' = \sum_{m=-\infty}^{\infty} \frac{1}{16R} \sqrt{\frac{\alpha}{r\bar{t}}} \left\{ \frac{\cos\left[\frac{\pi}{2}\left(\frac{1}{2} - \frac{\omega_b}{\pi} - 2m + \frac{\theta}{\pi}\right)\right] \sin 2\pi\left(\frac{1}{2} - \frac{\omega_b}{\pi} - 2m + \frac{\theta}{\pi}\right)}{1 - \cos 2\pi\left(\frac{1}{2} - \frac{\omega_b}{\pi} - 2m + \frac{\theta}{\pi}\right)} \right.$$

$$+ \frac{\cos\left[\frac{\pi}{2}\left(\frac{1}{2} + \frac{\omega_d}{\pi} - 2m + \frac{\theta}{\pi}\right) - \frac{\pi}{2}\right] \sin 2\pi\left(\frac{1}{2} + \frac{\omega_d}{\pi} - 2m + \frac{\theta}{\pi}\right)}{1 - \cos 2\pi\left(\frac{1}{2} + \frac{\omega_d}{\pi} - 2m + \frac{\theta}{\pi}\right)}$$

$$+ \frac{\cos\left[\frac{\pi}{2}\left(\frac{1}{2} + \frac{\omega_f}{\pi} - 2m + \frac{\theta}{\pi}\right)\right] \sin 2\pi\left(\frac{1}{2} + \frac{\omega_f}{\pi} - 2m + \frac{\theta}{\pi}\right)}{1 - \cos 2\pi\left(\frac{1}{2} + \frac{\omega_f}{\pi} - 2m + \frac{\theta}{\pi}\right)}$$

$$\left. + \frac{\cos\left[\frac{\pi}{2}\left(\frac{1}{2} - \frac{\omega_h}{\pi} - 2m + \frac{\theta}{\pi}\right) - \frac{\pi}{2}\right] \sin 2\pi\left(\frac{1}{2} - \frac{\omega_h}{\pi} - 2m - \frac{\theta}{\pi}\right)}{1 - \cos 2\pi\left(\frac{1}{2} - \frac{\omega_h}{\pi} - 2m + \frac{\theta}{\pi}\right)} \right\} . \quad (22)$$

where the subscripted ω's satisfy:

$$\alpha\bar{t} + r - a \sin \omega_b = 2R\left(\frac{1}{2} - \frac{\omega_b}{\pi} - 2m + \frac{\theta}{\pi}\right)$$

$$r - \alpha\bar{t} + a \sin \omega_d = 2R\left(\frac{1}{2} - \frac{\omega_d}{\pi} - 2m + \frac{\theta}{\pi}\right)$$

$$r + \alpha\bar{t} + a \sin \omega_f = 2R\left(\frac{1}{2} + \frac{\omega_f}{\pi} - 2m \pm \frac{\theta}{\pi}\right) \quad (23)$$

$$r - \alpha\bar{t} - a \sin \omega_h = 2R\left(\frac{1}{2} - \frac{\omega_h}{\pi} - 2m \pm \frac{\theta}{\pi}\right) .$$

Equation (22) has a singular point when any of its cosine functions is unity, and it is important to check the integrability of these singularities. If the source function to the system is F(t), the response will have the form of the convolution:

$$\int_{-\infty}^{\infty} F(t) \frac{\sin t \cos t}{1-\cos t} dt .$$

which is a generalized integral equivalent to

$$\int_{-\infty}^{\infty} \left[F'(t) \cos t - F(t) \sin t\right] \log (1-\cos t)\, dt$$

But log(1-cost) is integrable[9], so the response will be finite except in the physically impossible case where the source function is infinite or has infinite derivative.

The largest contributions from (22) of course come when one of the cosines is unity, and this can only happen for scattered ranges of $\alpha\bar{t}$. A representative plot of the locations of these maximum points is given in Fig. 2, where the peaks are seen to describe sections of spirals for any given radius r.

The cosine terms in (20) are summed by use of the formulas

$$\sum_{k=1}^{\infty} \cos \pi kx/L = L \sum_{m=-\infty}^{\infty} \delta(x-2mL) - \frac{1}{2}$$

$$\sum_{k=-\infty}^{\infty} e^{i\pi kx/L} = L \sum_{m=-\infty}^{\infty} \delta(x-2mL)$$

Thus $$\hat{\sigma}_m^{\,2} = \frac{1}{4\pi} \int_0^{\pi} \sqrt{\frac{\alpha}{r\bar{t}}}\, e^{im\theta}$$

$$\cdot \left\{ \cos \left[\frac{\pi}{R}\left(\frac{m}{2} - \frac{1}{4}\right)\left(r + \alpha\bar{t} - a \sin \omega\right) - \frac{m\pi}{2} - \frac{\pi}{2} + m\omega\right] \sum_{k=-\infty}^{\infty} \delta(r + \alpha\bar{t} - a \sin \omega - 2kR)\right.$$

$$+ \cos \left[\frac{\pi}{R}\left(\frac{m}{2} - \frac{1}{4}\right)\left(r - \alpha\bar{t} + a \sin \omega\right) - \frac{m\pi}{2} - m\omega\right] \sum_{k=-\infty}^{\infty} \delta(r - \alpha\bar{t} + a \sin \omega - 2kR)$$

$$+ \cos\left[\frac{\pi}{R}\left(\frac{m}{2} - \frac{1}{4}\right)\left(r + \alpha\bar{t} + a\sin\omega\right) - \frac{m\pi}{2} - \frac{\pi}{2} - m\omega\right] \sum_{k=-\infty} \delta(r + \alpha\bar{t} + a\sin\omega - 2kR) \tag{24}$$

$$+ \cos\left[\frac{\pi}{R}\left(\frac{m}{2} - \frac{1}{4}\right)\left(r - \alpha\bar{t} - a\sin\omega\right) - \frac{m\pi}{2} + m\omega\right] \sum_{k=-\infty}^{\infty} \delta(r - \alpha\bar{t} - a\sin\omega - 2kR)\Big\} d\omega$$

Equation (24) is non-zero only for scattered ranges of $\alpha\bar{t}$. For $-r-a+2kR \leq \alpha\bar{t} \leq -r + a \pm 2kR$, k any integer,

$$\varphi^2 = \frac{\pi}{2} \sum_{m=-\infty}^{\infty} \hat{\sigma}_m^2 = \frac{1}{4}\sqrt{\frac{\alpha}{r\bar{t}}}\frac{1}{\sqrt{1-\xi}} \cos\left(\frac{\pi}{2}(k+1)\right)$$

$$\left\{\delta\left(\frac{\theta}{\pi} + k - \frac{1}{2} + \frac{\sin^{-1}\xi}{\pi} - 2\bar{m}\right) + \delta\left(\frac{\theta}{\pi} - k + \frac{1}{2} - \frac{\sin^{-1}\xi}{\pi} - 2\bar{m}\right)\right\} \tag{25}$$

where $\xi = \sin\omega = \dfrac{\alpha\bar{t} + r - 2kR}{a}$, and $\bar{m}$ is an integer selected so that the delta function will be non-zero for some $\theta\epsilon[0,2\pi]$.

The other range where φ^2 is non-zero is given by $2kR + r-a \leq \alpha\bar{t} \leq 2kR \pm r + a$, k some integer, and for $\alpha\bar{t}$ in this range,

$$\varphi^2 = \frac{1}{4}\sqrt{\frac{\alpha}{rt(1-\xi^2)}} \cos\left(\frac{k\pi}{2}\right)$$

$$\left\{\delta\left(\frac{\theta}{\pi} + k - \frac{1}{2} - \frac{\sin^{-1}\xi}{\pi} - 2\bar{m}\right) + \delta\left(\frac{\theta}{\pi} - k + \frac{1}{2} + \frac{\sin^{-1}\xi}{\pi} - 2\bar{m}\right)\right\} \tag{26}$$

where $\xi = \sin\omega = \dfrac{2kR + \alpha\bar{t} - r}{a}$, and $\bar{m}$ is an integer.

The locations of the delta functions of (25) and (26) are plotted in Fig. 3.

We now have the complete asymmetrical response, $\varphi^1 + \varphi^2$, and figures 1 and 2 show that the asymmetry remains no matter how small the source asymmetry a. The only effect of reducing the source asymmetry is to shorten the time intervals when it is present.

V. CONCLUSIONS

The asymetric response to an off-center source in a fluid cylinder consists of two types of arrivals, one associated with the source function and the other associated with the derivative of the source function. For a short pulse source function, both arrivals are characterized by repeated bursts of asymmetric motion whose duration but not magnitude is dependent on the source asymmetry. This is a strikingly large effect, and since any physical problem must include at least small asymmetries, we may expect the effects to be significant in physical applications.

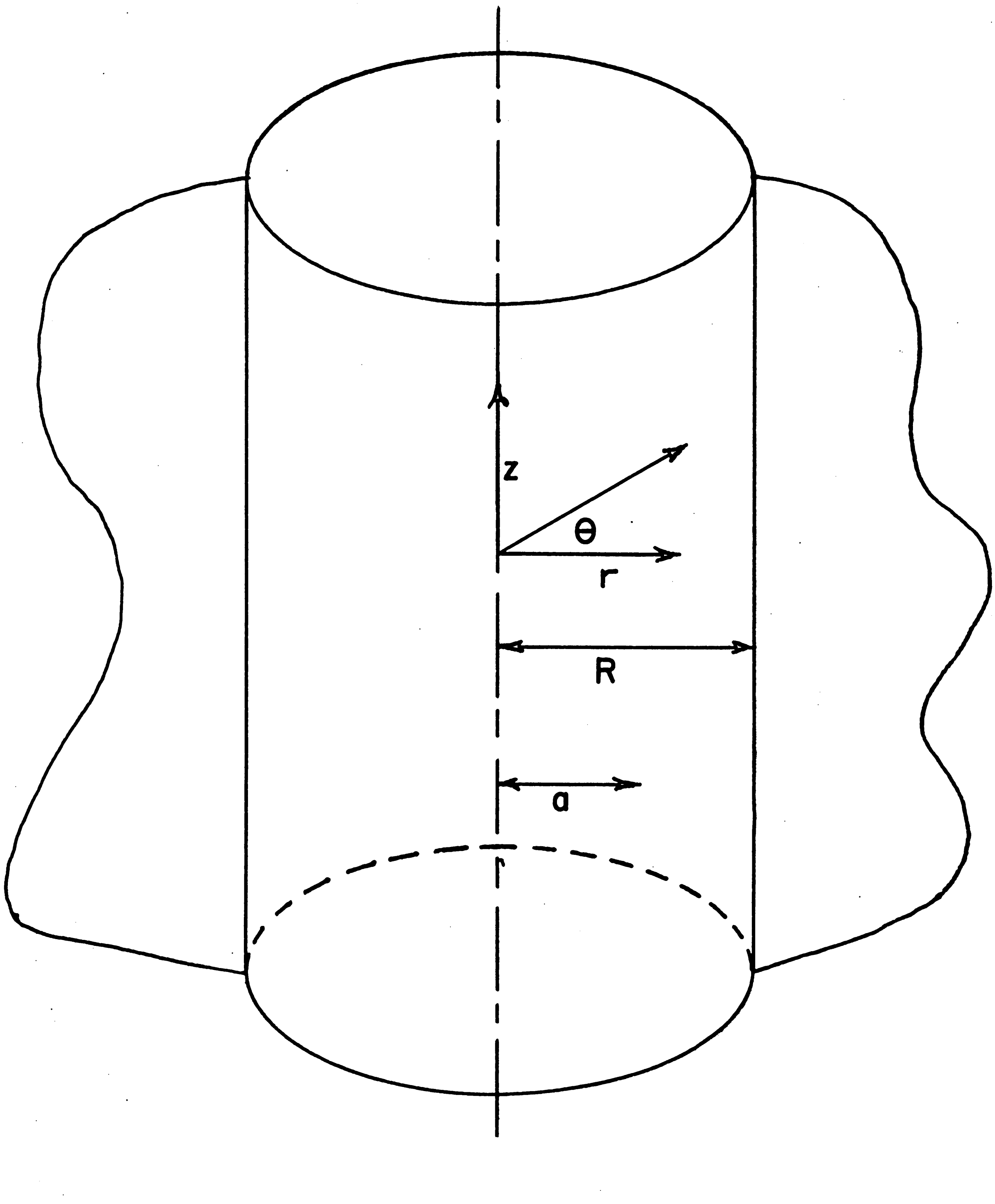

Fig. 1. Geometry

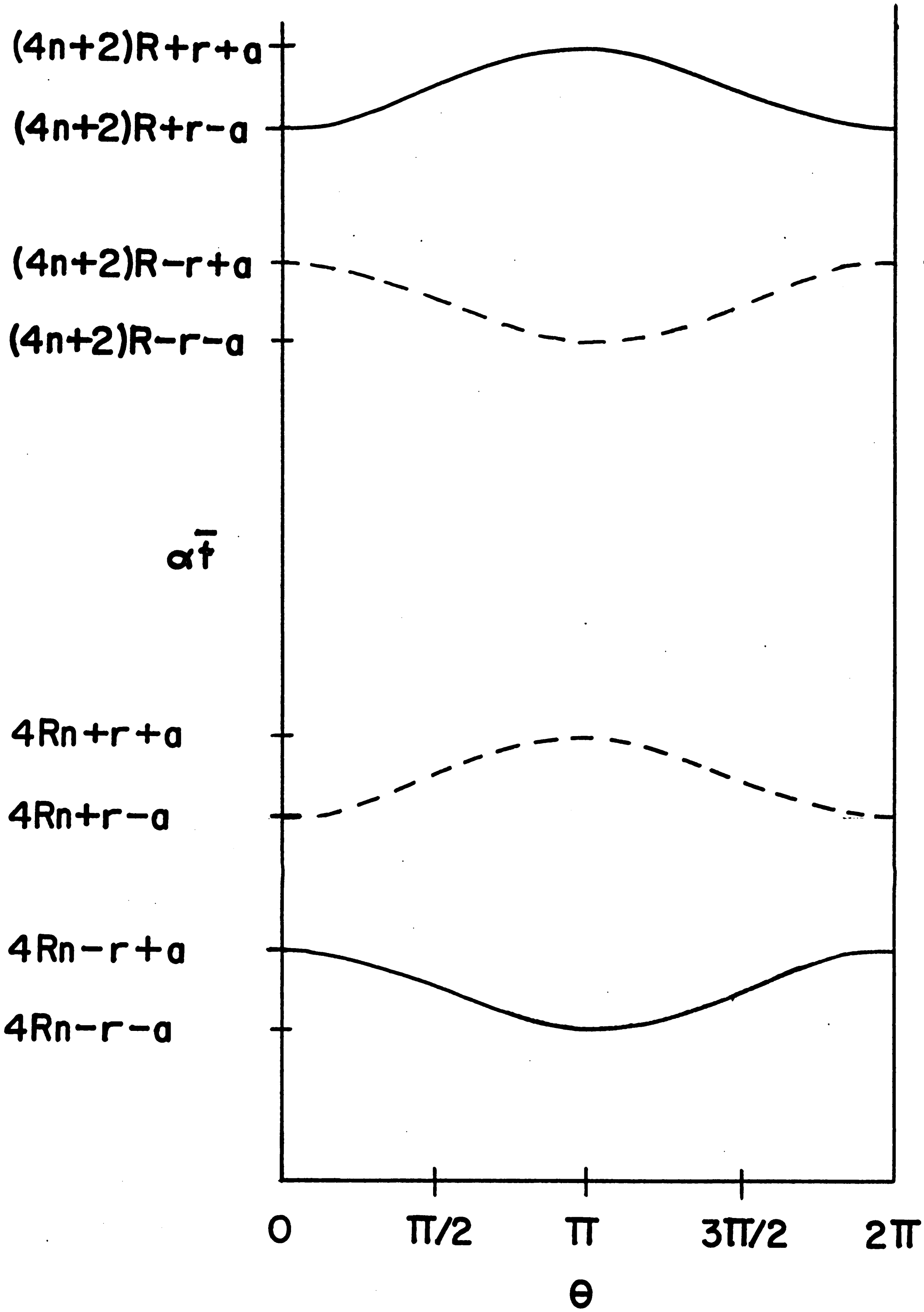

Fig. 2. Representative locations of peaks of sine contributions, for an integer n, showing contribution from source function (———), and contribution from derivative of source function (-----).

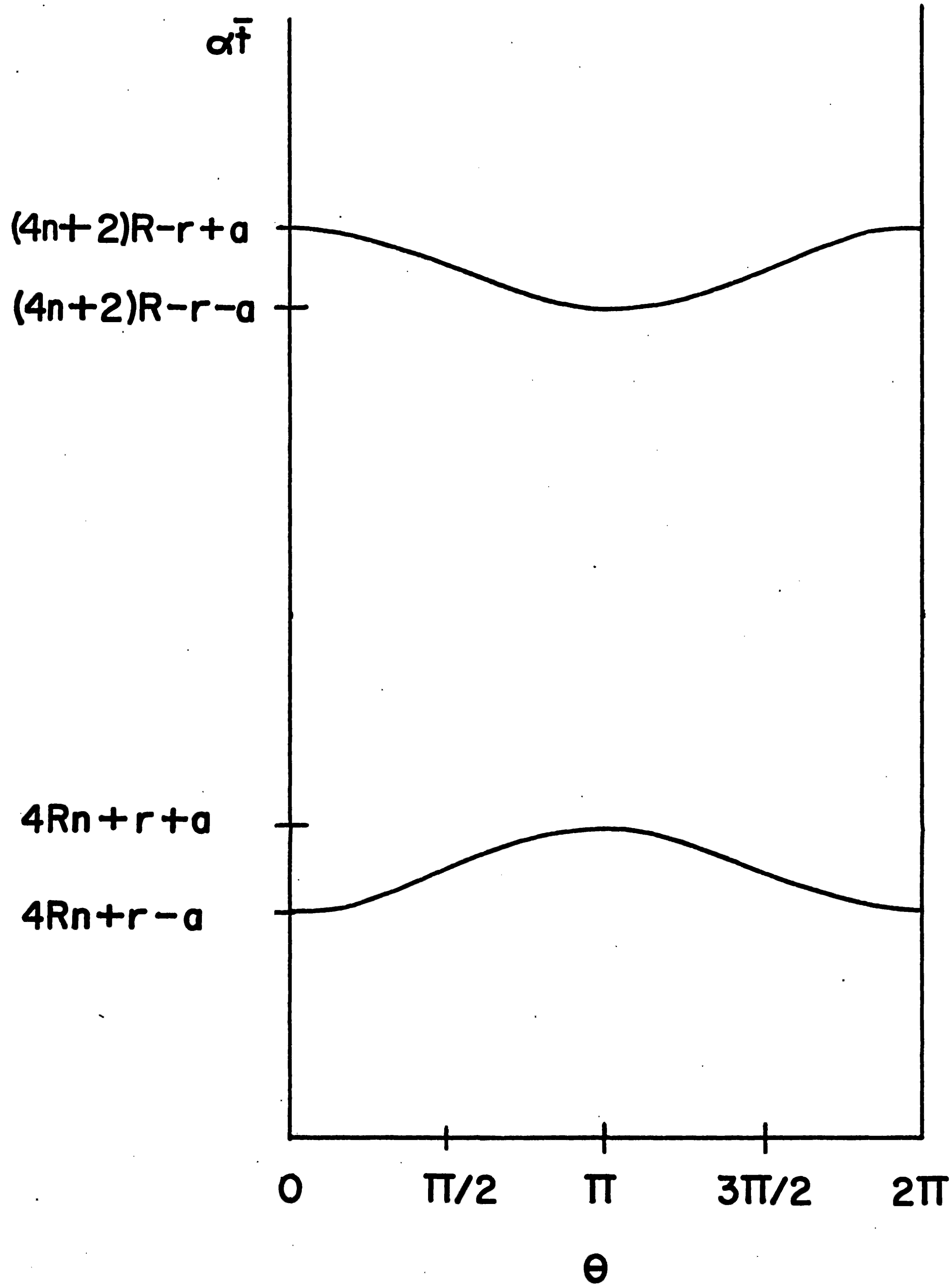

Fig. 3. Representative locations of peaks of cosine contributions, for any integer n.

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