Spring 2015

Heat trace and heat content asymptotics for Schrodinger Operators of stable processes/ fractional Laplacians

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Graduate School Form 30
Updated 1/15/2015

PURDUE UNIVERSITY
GRADUATE SCHOOL
Thesis/Dissertation Acceptance

This is to certify that the thesis/dissertation prepared

By Luis Acuna Valverde

Entitled
HEAT TRACE AND HEAT CONTENT ASYMPTOTICS FOR SCHRODINGER OPERATORS OF STABLE PROCESSES / FRACTIONAL LAPLACIANS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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Approved by Major Professor(s): Rodrigo Banuelos

Approved by: David Goldberg 4/6/2015

Head of the Departmental Graduate Program Date

HEAT TRACE AND HEAT CONTENT ASYMPTOTICS FOR SCHRÖDINGER OPERATORS OF STABLE PROCESSES/FRACTIONAL LAPLACIANS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Luis Acuña Valverde

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2015

Purdue University

West Lafayette, Indiana

To my parents.

ACKNOWLEDGMENTS

I would like to express my appreciation to my advisor, Professor Rodrigo Bañuelos, for the guidance, encouragement and advice he provided throughout my time as his student at Purdue. I would also like to thank Professors Fabrice Baudoin, Burgess Davis and Antônio Sá Barreto for serving in my dissertation committee and for their contributions to my academic training.

I am grateful to my mother and siblings for their support which made this work possible.

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ABSTRACT

Acuña Valverde, Luis. PhD, Purdue University, May 2015. Heat trace and heat content asymptotics for Schrödinger Operators of stable processes/fractional Laplacians. Major Professor: Rodrigo Bañuelos.

Let V be a bounded and integrable potential over $\mathbb{R}^d$ and $0 < \alpha \leq 2$. We show the existence of an asymptotic expansion by means of Fourier Transform techniques and probabilistic methods for the following quantities

$$\mathcal{T}_V^{(\alpha)}(t) = \frac{1}{p_t^{(\alpha)}(0)} \int_{\mathbb{R}^d} \left(p_t^{H_V}(x,x) - p_t^{(\alpha)}(x,x) \right) dx$$

and

$$Q_V^{(\alpha)}(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(p_t^{H_V}(x,y) - p_t^{(\alpha)}(x,y) \right) dxdy$$

as $t \downarrow 0$. These quantities are called the *heat trace* and *heat content* in $\mathbb{R}^d$ with respect to V, respectively. Here, $p_t^{(\alpha)}(x,y)$ and $p_t^{H_V}(x,y)$ denote, respectively, the heat kernels of the heat semigroups with infinitesimal generators given by $(-\Delta)^{\frac{\alpha}{2}}$ and $H_V = (-\Delta)^{\frac{\alpha}{2}} + V$. The former operator is known as the fractional Laplacian whereas the latter one is known as the fractional Schrödinger Operator.

The study of the small time behaviour of the above quantities is motivated by the asymptotic expansion as $t \downarrow 0$ of the following spectral functions for smooth bounded domains $\Omega \subset \mathbb{R}^d$,

$$\mathcal{Z}_\Omega^{(\alpha)}(t) = \frac{1}{p_t^{(\alpha)}(0)} \int_\Omega p_t^{\Omega,\alpha}(x,x)dx,$$
$$Q_\Omega^{(\alpha)}(t) = \int_\Omega \int_\Omega p_t^{\Omega,\alpha}(x,y)dxdy,$$

where $p_t^{\Omega,\alpha}(x,y)$ is the transition density of a stable process killed upon exiting Ω.

The function $\mathcal{Z}_\Omega^{(\alpha)}(t)$ is known as the *heat trace* and a second order expansion is provided in [6] for all $0 < \alpha \leq 2$ for R-smooth boundary domains. In [5] the result is extended

to bounded domains with Lipschitz boundary. As for the spectral function $Q_{\Omega}^{(\alpha)}(t)$, it is called the *spectral heat content* and has only been widely studied for the Brownian motion case. In fact, a third order asymptotic expansion is provided in [12] for $\alpha = 2$. In this work, we will state a conjecture about the second order small time expansion. These expansions differ accordingly to the ranges $1 < \alpha < 2$, $\alpha = 1$ and $0 < \alpha < 1$.

1. INTRODUCTION

A d–dimensional stochastic process $\mathbf{X} = \{X_t\}_{t\geq 0}$ defined on a probability space $(\mathcal{N}, \mathbb{P}, \mathcal{F})$ is said to be a Lévy process started at $x \in \mathbb{R}^d$ if

(i) $X_0 = x$ a.s.

(ii) X has independent and stationary increments. That is, for $0 < s < t$, the random variable $X_{t+s} - X_s$ is independent of the σ-algebra $\sigma(X_u, 0 \leq u \leq s)$ and has the same law as X_t.

(iii) X is stochastically continuous. Namely, for all $\epsilon > 0$ and for all $s > 0$,

$$\lim_{t\to s} \mathbb{P}(|X_t - X_s| > \epsilon) = 0.$$

We now proceed to provide some examples of Lévy processes that we are particularly interested in together with additional properties about their transition densities. We recall that a Lévy process is completely determined by the Fourier transform of its transition densities. This is the celebrated *Lévy-Khintchine Theorem*; see [16] for further details. Henceforth, $\mathbb{P}$ and $\mathbb{E}$ will denote the probability and expectation, respectively, of any Lévy process started at 0.

Example 1.0.1 (rotationally invariant α–stable Processes)

$\mathbf{X}$ *is said to be an α–stable process,* $0 < \alpha \leq 2$ *if the Fourier transform (or characteristic function) of its transition densities, denoted throughout this work by* $p_t^{(\alpha)}(x, y) = p_t^{(\alpha)}(x - y)$*,* $t > 0$*,* $x, y \in \mathbb{R}^d$ *satisfies*

$$e^{-t|\xi|^\alpha} = \mathbb{E}[e^{-i<\xi, X_t>}] = \int_{\mathbb{R}^d} e^{-i<y,\xi>} p_t^{(\alpha)}(y)dy, \tag{1.1}$$

for all $t > 0$*,* $\xi \in \mathbb{R}^d$*.*

The transition densities $p_t^{(\alpha)}(x,y)$ *are only explicit for the cases* $\alpha = 2$ *and* $\alpha = 1$. *In fact, when* $\alpha = 2$, $\mathbf{X}$ *is a Brownian motion at twice speed. The transition density of a Brownian motion is also called Gaussian kernel and is given by*

$$p_t^{(2)}(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}, \;\; x, y \in \mathbb{R}^d, \;\; t > 0.$$

As for $\alpha = 1$, $\mathbf{X}$ *is called Cauchy process. Its transition density is known as either Cauchy or Poisson kernel and is given by*

$$p_t^{(1)}(x,y) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \frac{t}{\left(t^2 + |x-y|^2\right)^{\frac{d+1}{2}}}, \;\; x, y \in \mathbb{R}^d, \;\; t > 0. \tag{1.2}$$

Example 1.0.2 **(** $\alpha/2$ **subordinators)**

An $\alpha/2$*–subordinator is a.s non-decreasing* $[0,\infty)$*-valued Lévy process* $\mathbf{S} = \{S_t\}_{t\geq 0}$ *starting at* 0 *and uniquely determined by its Laplace transform*

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\lambda^{\alpha/2}}, \;\; t > 0, \lambda > 0.$$

Notice that the last equality implies that for all $\xi \in \mathbb{R}^d$,

$$\mathbb{E}\left[e^{-|\xi|^2 S_t}\right] = e^{-t|\xi|^\alpha}. \tag{1.3}$$

The reader may consult [19] for additional examples of subordinators.

Example 1.0.3 **(Relativistic stable processes)**

Let $0 < \alpha \leq 2$ *and* $m > 0$. *A relativistic* α*-stable process* $\mathbf{X}^m$ *on* $\mathbb{R}^d$ *with mass* m *is a Lévy process with characteristic function given by*

$$\mathbb{E}\left[e^{i<X_t^m,\xi>}\right] = e^{-t\left(\left(\lambda + m^{2/\alpha}\right)^{\alpha/2} - m\right)}, \;\; t > 0, \;\; \xi \in \mathbb{R}^d.$$

These processes $\mathbf{X}^m = \{X_t^m\}_{t\geq 0}$ *are of great interest because they behave as a Brownian motion for* t *large and as a rotationally invariant* α*-stable process for small* t. *The interested reader may consult [23] for additional details about heat kernel estimates associated to the stochastic process* $\mathbf{X}^m$.

Example 1.0.4 **(Mixed stable processes)**

Take $0 < \beta < \alpha < 2$ *and* $a > 0$*. The process* $\mathbf{Z}^a = \{Z_t^a\}_{t\geq 0}$ *with* $Z_t^a = X_t + aY_t$*, where* $\mathbf{X} = \{X_t\}_{t\geq 0}$ *and* $\mathbf{Y} = \{Y_t\}_{t\geq 0}$ *are independent rotational invariant* α*-stable and* β*-stable processes, respectively is called the independent sum of the symmetric* α*-stable process* $\mathbf{X}$ *and the symmetric* β*-stable process* $\mathbf{Y}$ *with weight* a*. This is a Lévy process with characteristic function satisfying*

$$\mathbb{E}\left[e^{i<Z_t^a,\xi>}\right] = e^{-t\left(|\xi|^\alpha + a^\beta|\xi|^\beta\right)},\ t > 0,\ \xi \in \mathbb{R}^d.$$

We refer the reader to [21, 22] for further results concerning heat kernel estimates for the stochastic process $\mathbf{Z}^a$*.*

We point out that the previous examples of Lévy processes can be constructed by a subordinated time change of the Brownian motion(see [19] for further details). Since we are mostly interested in α-stable processes we proceed to illustrate such construction with them.

Let $\mathbf{B} = \{B_t\}_{t\geq 0}$ denote a d–dimensional Brownian motion in a probability space $(\mathcal{N}_1, \{\mathcal{F}_t^{\mathbf{B}}\}_{t\geq 0}, \mathbb{P}_{\mathbf{B}}^x)$ and let $\mathbf{S} = \{S_t\}_{t\geq 0}$ be an $\alpha/2$–subordinator started at zero with probability space $(\mathcal{N}_2, \mathcal{G}, \mathbb{P}_{\mathbf{S}})$ and $0 < \alpha \leq 2$. When $\alpha = 2$, we adopt the convention $S_t = t$. We will consider both processes on the product space $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$. In addition, we set $\mathcal{F}_t = \mathcal{F}_t^{\mathbf{B}} \times \mathcal{G}$ and $\mathbb{P}^x = \mathbb{P}_{\mathbf{B}}^x \times \mathbb{P}_{\mathbf{S}}$. Hence, $\mathbf{B}$ is a d-dimensional $\mathcal{F}_t$–Brownian motion and $\mathbf{S}$ is an $\alpha/2$–subordinator independent of $\mathbf{B}$ when they are regarded as stochastic processes defined in $(\mathcal{N}, \mathbb{P}^x)$(see [52] for details). From now on, every process and every random variable will be defined over $\mathcal{N}$.

The Lévy process $\mathbf{X} = \{X_t\}_{t\geq 0}$ defined as $X_t = B_{2S_t}$ is a rotationally invariant α-stable process in $\mathbb{R}^d$ and it has been constructed as a subordinated time change of the Brownian motion. For the rest of the thesis, $\mathbb{E}^x$ and $\mathbb{P}^x$ will denote the expectation and the probability of any process started at x, respectively. We also write $Z \overset{\mathcal{D}}{=} Y$ for two random variables Z, Y with values in $\mathbb{R}^d$ to mean that they are equal in distribution or have the same law. Throughout this work, $\eta_t^{(\alpha/2)}(s)$ will denote the transition density of the random variable S_t.

We remark that for $0 < \alpha < 2$ the transition densities $p_t^{(\alpha)}(x,y)$ can be written as subordination of the Gaussian kernel. That is,

$$p_t^{(\alpha)}(x,y) = \mathbb{E}\left[p_{S_t}^{(2)}(x,y)\right] = \int_0^\infty ds\, \eta_t^{(\alpha/2)}(s)\, p_s^{(2)}(x,y). \tag{1.4}$$

It follows from (1.4) that $p_t^{(\alpha)}(x)$ is radial, symmetric and decreasing in x. Moreover, these functions satisfy the following scaling property and inequality:

$$p_t^{(\alpha)}(x) = t^{-d/\alpha} p_1^{(\alpha)}(t^{-1/\alpha}x) \le t^{-d/\alpha} p_1^{(\alpha)}(0), \tag{1.5}$$

where $p_1^{(\alpha)}(0) = (2\pi)^{-d}\alpha^{-1}\, w_d\, \Gamma(d/\alpha)$ with w_d the surface area of the unit sphere in $\mathbb{R}^d$.

From equation (1.4), we also conclude that

$$(4\pi)^{d/2} p_1^{(\alpha)}(0) = \mathbb{E}\left[S_1^{-d/2}\right]. \tag{1.6}$$

Moreover, we claim that for all $-\infty < \beta < \frac{\alpha}{2}$,

$$\mathbb{E}\left[S_1^{\beta}\right] = \frac{\Gamma(1-\frac{2\beta}{\alpha})}{\Gamma(1-\beta)}. \tag{1.7}$$

To see this, we observe that $\left(ZS_1^{-1}\right)^{\alpha/2} \overset{\mathcal{D}}{=} Z$, where $Z = \exp(1)$ is an exponential random variable with parameter 1 independent of S_1. In fact, by independence we have

$$\begin{aligned}\mathbb{P}\left(\left(ZS_1^{-1}\right)^{\alpha/2} \le \lambda\right) &= \int_0^\infty \left(\int_0^{\lambda^{2/\alpha}s} e^{-u}du\right) \eta_1^{(\alpha/2)}(s)ds \\ &= 1 - \mathbb{E}\left[e^{-\lambda^{2/\alpha}S_1}\right] \\ &= 1 - e^{-\lambda} = \mathbb{P}(Z \le \lambda).\end{aligned}$$

Hence, it also follows by independence that

$$\mathbb{E}\left[Z^{-\beta}\right]\mathbb{E}\left[S_1^{\beta}\right] = \mathbb{E}\left[Z^{-\frac{2\beta}{\alpha}}\right],$$

provided $\mathbb{E}\left[Z^{-\beta}\right]$ and $\mathbb{E}\left[Z^{\frac{-2\beta}{\alpha}}\right]$ are both finite. But, this only holds when $-\infty < \beta < \frac{\alpha}{2}$, since

$$0 < \mathbb{E}\left[Z^{\gamma}\right] = \int_0^\infty s^{(\gamma+1)-1}e^{-s}ds = \Gamma(\gamma+1) < \infty,$$

when $\gamma + 1 > 0$.

1.1 Infinitesimal Generator of a Stable Process and Schrödinger Operators

Let $0 < \alpha \leq 2$ and consider $\mathbf{X} = \{X_t\}_{t \geq 0}$ a rotationally invariant α-stable process as defined in example 1.0.1 . In order to state our results, we need to take into consideration both the spectral and integral definition for the infinitesimal generator associated with the process $\mathbf{X}$, denoted as before by $H_\alpha = (-\Delta)^{\frac{\alpha}{2}}$. In the spectral theoretic sense, $(-\Delta)^{\frac{\alpha}{2}}$ is a positive and self–adjoint linear operator with domain $\left\{ f \in L^2(\mathbb{R}^d) : |\xi|^\alpha \widehat{f}(\xi) \in L^2(\mathbb{R}^d) \right\}$ satisfying

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} f}(\xi) = |\xi|^\alpha \widehat{f}(\xi), \tag{1.8}$$

where $\widehat{f}$ denotes the Fourier transform of f. Moreover, for $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the set of rapidly decreasing smooth functions, we have

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \frac{d}{dt} e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(x) \bigg|_{t=0},$$

where

$$e^{-t(-\Delta)^{\frac{\alpha}{2}}} f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p_t^{(\alpha)}(x,y) f(y) dy$$

is the heat semigroup generated by $\mathbf{X}$. On the other hand, for $0 < \alpha < 2$, $(-\Delta)^{\frac{\alpha}{2}}$ can also be expressed in the integral form

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = \tilde{c}_{d,\alpha} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy,$$

where $\tilde{c}_{d,\alpha} > 0$ is a normalizing constant and the integral is understood in the principal value sense. The last expression allows us to rewrite the Dirichlet form related to $(-\Delta)^{\frac{\alpha}{2}}$ (see [30] for further details)

$$\mathcal{E}_\alpha(f) = \langle (-\Delta)^{\frac{\alpha}{2}} f, f \rangle = \int_{\mathbb{R}^d} (-\Delta)^{\frac{\alpha}{2}} f(x) f(x) dx \tag{1.9}$$

as

$$\mathcal{E}_\alpha(f) = \frac{\tilde{c}_{d,\alpha}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x-y|^{d+\alpha}} dx dy. \tag{1.10}$$

As for the case $\alpha = 2$, due to integration by parts, we have

$$\mathcal{E}_2(f) = \int_{\mathbb{R}^d} (-\Delta f)(x) f(x) dx = \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx, \tag{1.11}$$

which is the classical Dirichlet form of the Laplacian.

Let $V \in L^{\infty}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. The linear operator $H_V = (-\Delta)^{\frac{\alpha}{2}} + V$, known as the *fractional Schrödinger operator*, is self-adjoint and defined similarly as the infinitesimal generator of the heat semigroup,

$$e^{-tH_V} f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} f(X_t)],$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. The heat kernel of e^{-tH_V} is given by the Feynman-Kac formula (see [33], [36] and [51])

$$p_t^{H_V}(x,y) = p_t^{(\alpha)}(x,y)\mathbb{E}_{x,y}^t\left[e^{-\int_0^t V(X_s)ds}\right], \tag{1.12}$$

where $\mathbb{E}_{x,y}^t$ denotes the expectation with respect to the stable process (bridge) starting at x and conditioned to be at y at time t. It is worth mentioning here that the formula (1.12) is also well defined for a wide class of unbounded potentials V(see [36]). In fact, if $\Omega \subset \mathbb{R}^d$ is a bounded open measurable set and we apply formula (1.12) with

$$V_\Omega(x) = \begin{cases} +\infty & \text{if } x \in \Omega^c, \\ 0 & \text{otherwise,} \end{cases}$$

then the resulting function $p_t^{H_{V_\Omega}}(x,y)$, which we denote simply by $p_t^{\Omega,\alpha}(x,y)$, is the transition density for the stable process killed upon exiting Ω. In other words, this is the heat kernel for the Dirichlet fractional Laplacian. An explicit expression for this is

$$p_t^{\Omega,\alpha}(x,y) = p_t^{(\alpha)}(x,y)\mathbb{P}\left(\tau_\Omega^{(\alpha)} > t \middle| X_0 = x, X_t = y\right), \tag{1.13}$$

where $\tau_\Omega^{(\alpha)} = \inf\{s \geq 0 : X_s \in \Omega^c\}$ is the first exit time from Ω. The vast literature and relevant results concerning asymptotic expansions of quantities involving the classical heat kernel $p_t^{\Omega,2}(x,y)$ for the Laplacian motivate the many results in this thesis.

With H_α, H_V and their heat kernels properly introduced, we now proceed to consider the main quantities studied here. Namely, the *heat trace* and the *heat content* for Schrödinger operators.

1.2 Heat Trace for Schrödinger Operators on $\mathbb{R}^d$, statement of results

Heat asymptotic results have been widely used in areas of spectral theory and in applications to scattering theory, statistical and quantum mechanics and in several areas in geometry. We refer the reader to van den Berg [8] for the computation of the first two terms in the asymptotic expansion of the trace of the heat kernel of the Schrödinger operator $-\Delta + V$ under Hölder continuity of the potential and to Bañuelos and Sá Barreto [3] for a more general computation with an explicit formula for all the coefficients for potentials $V \in \mathcal{S}(\mathbb{R}^d)$, and for applications to scattering theory. For applications in statistical mechanics and quantum theory, we refer the reader to the articles of Lieb [38] and Penrose and Stell [47] about the second viral coefficient of a hard–sphere gas at low temperature and sticky spheres, respectively. Heat trace asymptotics for the Laplacian have been of interest for many years for domains in the Euclidean space $\mathbb{R}^d$ and on manifolds where the coefficients reveal many geometric quantities such as volume, surface area, convexity, number of holes, etc. For more on this large literature as well as some historical perspective, we refer the reader to Arendt and Schleich [2, pp 1-71], Bañuelos, Kulczycki and Siudeja [5, 6], Datchev and Hezari's [26], Donelly [28], McKean and Moerbeke [41], and Colin De Verdière [25].

Consider $H_2 = -\Delta$ and $H_V = -\Delta + V$, $V \in \mathcal{S}(\mathbb{R}^d)$. In [3], the existence of an asymptotic expansion of the heat trace of the operator $e^{-tH_V} - e^{-tH_2}$, as $t \downarrow 0$, is proved. To make the connection to the fractional Laplacian more clear, consider the heat kernel for $-\Delta$ given by

$$p_t^{(2)}(x) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}},$$

so that $p_t^{(2)}(0) = (4\pi t)^{-d/2}$.

Before continuing, we introduce some notations. Throughout the work, we will say that $f(t) = \mathcal{O}(g(t))$, as $t \downarrow 0$, to mean that there exist $C, \delta > 0$ such that $|f(t)| \le C|g(t)|$, for $0 < t < \delta$. Also $\widehat{V}$ will stand for the Fourier transform of $V \in \mathcal{S}(\mathbb{R}^d)$.

Set

$$I_j = \left\{ \lambda = (\lambda_1, ..., \lambda_j) \in [0,1]^j : 0 < \lambda_j < \lambda_{j-1} < ... < \lambda_1 < 1 \right\}. \tag{1.14}$$

With this notation the result in [3] can be stated as follows. For any integer $J \geq 1$,

$$\frac{Tr(e^{-tH_V} - e^{-tH_2})}{p_t^{(2)}(0)} = \sum_{\ell=1}^{J} c_\ell(V) t^\ell + \mathcal{O}(t^{J+1}), \tag{1.15}$$

as $t \downarrow 0$, with

$$c_1(V) = -\int_{\mathbb{R}^d} V(\theta) d\theta, \;\; c_\ell(V) = (-1)^\ell \sum_{\substack{j+n=\ell \\ j\geq 2}} C_{n,j}^{(2)}(V), \;\; C_{d,2} = (2\pi)^d,$$

$$C_{n,j}^{(2)}(V) = \frac{C_{d,2}}{(2\pi)^{jd} n!} \int_{I_j} \int_{\mathbb{R}^{(j-1)d}} \left\{ L_j^{(2)}(\lambda, \theta) \right\}^n \widehat{V}(-\sum_{i=1}^{j-1} \theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j, \text{ and}$$

$$L_j^{(2)}(\lambda, \theta) = \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \left| \sum_{i=1}^{k} \theta_i \right|^2 - \left| \sum_{k=1}^{j-1} (\lambda_k - \lambda_{k+1}) \sum_{i=1}^{k} \theta_i \right|^2.$$

In particular, for $J = 2$, the formula gives

$$\frac{Tr(e^{-tH_V} - e^{-tH_2})}{p_t^{(2)}(0)} + t \int_{\mathbb{R}^d} V(\theta) d\theta - \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta = \mathcal{O}(t^3), \tag{1.16}$$

as $t \downarrow 0$, which is the van den Berg [8] results under our assumption on V. For $J = 3$, the formula gives

$$\frac{Tr(e^{-tH_V} - e^{-tH_2})}{p_t^{(2)}(0)} + t \int_{\mathbb{R}^d} V(\theta) d\theta - \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta) d\theta$$
$$+ \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta) d\theta + \frac{t^3}{12} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta = \mathcal{O}(t^4), \tag{1.17}$$

as $t \downarrow 0$.

For $d = 1$, a recurrent formula for the general coefficients in the expansion was obtained in the seminal paper by McKean-Moerbeke [41] using KdV methods. Using these techniques, and the symmetry of certain integrals, Colin De Verdière [25] computed the first four coefficients in $\mathbb{R}^3$. The results in this work are motivated by [3] where (1.15) is proved by Fourier transform methods for all $d \geq 1$. Our proof is a combination of probabilistic arguments and Fourier transform techniques and unfortunately is much more technical than [3]. These results are also motivated by [4], where an analogue of van den Berg's results [8] (the computation of the first two terms) is proved for the fractional Laplacian and other related non-local operators. It is interesting to observe here that according to (1.11)

$$\int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta = \int_{\mathbb{R}^d} -\Delta V(\theta) V(\theta) d\theta = \mathcal{E}_2(V),$$

is the Dirichlet form of V with respect to the Laplacian. Based on this, it is natural to conjecture that the third term in the expansion for the fractional Laplacian should involve the Dirichlet form of V for the operator $(-\Delta)^{\frac{\alpha}{2}}$. But this is not the case, as we shall see later, which is somewhat surprising. Throughout this work, when required, we will write $S_{1,\alpha/2}$ for S_1 to emphasize the α dependence.

As of now, the function

$$\mathcal{T}_V^{(\alpha)}(t) = \frac{1}{p_t^{(\alpha)}(0)} \int_{\mathbb{R}^d} \left(p_t^{H_V}(x,x) - p_t^{(\alpha)}(x,x)\right) dx = \frac{Tr(e^{-tH_V} - e^{-tH_\alpha})}{p_t^{(\alpha)}(0)} \tag{1.18}$$

will be called the *heat trace* in $\mathbb{R}^d$ with respect to V.

Our result, analogue to (1.15), for the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is provided by the following theorem.

Theorem 1.2.1 *Let $0 < \alpha < 2$ be given. Suppose $V \in \mathcal{S}(\mathbb{R}^d)$ and denote the fractional Laplacian and its associated fractional Schrödinger operator by $H_\alpha = (-\Delta)^{\frac{\alpha}{2}}$ and $H_V = H_\alpha + V$, respectively. Denote the heat kernel for $(-\Delta)^{\frac{\alpha}{2}}$ by $p_t^{(\alpha)}(x)$; see (1.5). Assume that $M \geq 1$ is an integer satisfying $M < \frac{d+\alpha}{2}$. Then*

(a) Given $J \geq 2$, for $0 < t < 1$ we have the following expansion

$$\mathcal{T}_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{j=2}^{J} \sum_{n=0}^{M-1} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j} + R_{J+1}^{(\alpha)}(t), \tag{1.19}$$

where

$$R_{J+1}^{(\alpha)}(t) = \mathcal{O}\left(t^{\Phi_{J+1}^{(\alpha)}(M)}\right), \quad \text{as } t \downarrow 0,$$

with

$$\Phi_{J+1}^{(\alpha)}(M) = \min\left\{J+1, 2+\frac{2M}{\alpha}\right\},$$

and the constants $C_{n,j}^{(\alpha)}(V)$ are given by

$$C_{n,j}^{(\alpha)}(V) = \frac{C_{d,\alpha}}{(2\pi)^{jd} n!} \int_{I_j} \int_{\mathbb{R}^{(j-1)d}} \mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2} \left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^n\right] \widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j,$$

$$L_j^{(\alpha)}(\lambda,\theta) = \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} \left|\sum_{i=1}^{k}\theta_i\right|^2 - \frac{1}{S_{1,\frac{\alpha}{2}}} \left|\sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} \sum_{i=1}^{k}\theta_i\right|^2, \text{ and } C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)},$$

where the $\lambda_k's$ *are as in* (1.14)*. Moreover, the random variables* $S^*_{\lambda_1-\lambda_2}$, $S^*_{\lambda_2-\lambda_3}$,...,$S^*_{\lambda_{j-1}-\lambda_j}$, $S^*_{1-(\lambda_1-\lambda_j)}$ *are independent and satisfy*

$$S^*_{1-(\lambda_1-\lambda_j)} + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} = S_{1,\frac{\alpha}{2}}$$

and $S^*_l \overset{\mathcal{D}}{=} S_l$*, for any* $l \in \{1-(\lambda_1-\lambda_j), \lambda_k - \lambda_{k+1}\}_{k=1}^{j-1}$.

(b) For any $j \geq 2$ *and* $1 \leq n \leq M$*,*

$$\lim_{\alpha \uparrow 2} C^{(\alpha)}_{n,j}(V) = C^{(2)}_{n,j}(V).$$

We note that when $\alpha = 2$ the last Theorem remains true and $S^*_{\lambda_k-\lambda_{k+1}} = \lambda_k - \lambda_{k+1}$ and the condition on d and M is not needed. The reason for this is that in the later case, $S_t = t$ and then $S_{1,1} = 1$. What part (b) in the theorem proves is that our results are robust in the sense that not only do we recover the Bañuelos and Sá Barreto result when $\alpha = 2$ but also as $\alpha \to 2$ we recover the coefficients for $\alpha = 2$.

To see the connection to the Bañuelos and Sá Barreto result more clearly, we state the following theorem which is an immediate consequence of Theorem 1.2.1 and which resembles (1.15) more closely.

Theorem 1.2.2 *Under the same conditions of Theorem 1.2.1, we have*

$$\begin{aligned} \mathcal{T}^{(\alpha)}_V(t) &= -t\int_{\mathbb{R}^d} V(\theta)d\theta + \sum_{\substack{\frac{2n}{\alpha}+j<\Phi^{(\alpha)}_{J+1}(M) \\ 2\leq j\leq J,\ 0\leq n\leq M-1}} (-1)^{n+j} C^{(\alpha)}_{n,j}(V) t^{\frac{2n}{\alpha}+j} \\ &+ \mathcal{O}(t^{\Phi^{(\alpha)}_{J+1}(M)}), \end{aligned} \tag{1.20}$$

as $t \downarrow 0$ *with* $\mathcal{T}^{(\alpha)}_V(t)$ *as defined in* (1.18)*.*

Now, to obtain (1.15) from the last theorem we note again that for $\alpha = 2$ we have no restrictions on J and M other than $J \geq 2$ and $M \geq 1$. Also observe that $\Phi^{(2)}_{J+1}(M) = \min\{J+1, M+2\}$. Then, by taking $M = J-1$ we conclude $\Phi^{(2)}_{J+1}(J-1) = J+1$. As a consequence of (1.20), we arrive at

$$\mathcal{T}^{(2)}_V(t) = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \sum_{\substack{n+j<J+1 \\ 2\leq j\leq J,\ 0\leq n\leq J-2}} (-1)^{n+j} C^{(2)}_{n,j}(V) t^{n+j} + \mathcal{O}(t^{J+1}),$$

as $t \downarrow 0$. But, notice that in this case,

$$\sum_{\substack{n+j<J+1\\2\le j\le J,0\le n\le J-2}} (-1)^{n+j} C_{n,j}^{(2)}(V) t^{n+j} = \sum_{\ell=2}^{J} c_\ell(V) t^\ell,$$

and (1.15) follows.

In §2.9, we provide more specific expansion formulas for $\alpha's$ of the form $2/k$, where k is a positive integer. These examples are the only cases where $\frac{2n}{\alpha}+j$ are integers for all n, j, because for the particular case $n=1$ and $j=2$ there exists an integer $m_0 \geq 3$ such that $\frac{2}{\alpha}+2=m_0$, which implies that $\alpha=\frac{2}{m_0-2}$.

The assumption $M<\frac{d+\alpha}{2}$ in our theorem is sufficient to prove two crucial facts needed in our expansion. Namely, (1) that the coefficients in Theorem 1.2.1 are finite and (2) that the remainders that appear in the definition of $R_{J+1}^{(\alpha)}(t)$ (see §2.2 below) are bounded for $t\in(0,1)$. We also observe that the condition $M<\frac{d+\alpha}{2}$ determines, for a given d, the range of α's for which Theorem 1.2.1 holds. Thus, for example when $M=1$ and $d=1$, Theorem 1.2.1 only permits the range $1<\alpha<2$. In §2.5, we will show how a modified version of this condition (namely $\frac{M}{2}-\frac{d}{4}<\frac{\alpha}{2}$) can widen the range of α's for which Theorem 1.2.1 remains true when $d=1,2,3$ and $M=1,2$.

A particular case of Theorem 1.2.1 and our results in §2.5 is the following corollary which extends the results in [4] where the second coefficient is computed.

Corollary 1.2.1

(i) *For* $d=1$,

$$\mathcal{T}_V^{(\alpha)}(t)+t\int_{\mathbb{R}} V(\theta)d\theta-\frac{t^2}{2!}\int_{\mathbb{R}} V^2(\theta)d\theta+\frac{t^3}{3!}\int_{\mathbb{R}} V^3(\theta)d\theta = \begin{cases}\mathcal{O}(t^{2+\frac{2}{\alpha}}), & \text{if } \alpha\in(1,2),\\ \mathcal{O}(t^4), & \text{if } \alpha\in(1/2,1],\end{cases}$$

as $t\downarrow 0$.

(ii) *For $d = 1$ and $\frac{3}{2} < \alpha < 2$, we have*

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}} V^3(\theta)d\theta + \mathcal{L}_{1,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}} |\nabla V(\theta)|^2\, d\theta = \mathcal{O}(t^4),\ t\downarrow 0.$$

(iii) *For $d \geq 2$,*

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta = \begin{cases} \mathcal{O}(t^{2+\frac{2}{\alpha}}), & \text{if } \alpha \in (1,2), \\ \mathcal{O}(t^4), & \text{if } \alpha \in (0,1], \end{cases}$$

as $t \downarrow 0$.

(iv) *For $d \geq 2$ and $1 < \alpha < 2$,*

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2\, d\theta = \mathcal{O}(t^4),\ t\downarrow 0.$$

(v) *For $d \geq 3$, $\frac{2}{3} < \alpha \leq 1$,*

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta - \frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta + \mathcal{L}_{d,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2\, d\theta = \mathcal{O}(t^5),\ t\downarrow 0.$$

Also for $d \geq 3$ and $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$,

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta - \frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta = \mathcal{O}(t^5),\ t\downarrow 0.$$

(vi) *For* $d \geq 4$ *and* $0 < \alpha < \frac{1}{2}$,

$$\begin{aligned}\mathcal{T}_V^{(\alpha)}(t) &+ t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta \\ &- \frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta = \mathcal{O}(t^5),\ t \downarrow 0.\end{aligned}$$

The constants $\mathcal{L}_{d,\alpha}$ *are defined as follows:*

$$\mathcal{L}_{d,\alpha} = \frac{C_{d,\alpha}K_1(d,\alpha)}{(2\pi)^d}, \quad C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)},$$

with

$$K_1(d,\alpha) = \int_0^1\int_0^{\lambda_1} \mathbb{E}\left[\frac{S_{1-w}^* S_w^*}{(S_{1-w}^* + S_w^*)^{1+\frac{d}{2}}}\right] dw d\lambda_1.$$

The question of whether our result holds regardless of the choice of d and M as in (1.15) remains an interesting open problem which reduces to verifying that the expectations in the formula for $C_{n,j}^{(\alpha)}(V)$ are finite for all n and d.

To gain a better understanding of the applications of the robustness result, part (b), in Theorem 1.2.1, which is proved by means of weak convergence, consider the following special case of Corollary 1.2.1. For all $d \geq 1$ and $\frac{3}{2} < \alpha < 2$, we have

$$\begin{aligned}\mathcal{T}_V^{(\alpha)}(t) &+ t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta \\ &+ \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha}\, t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2\, d\theta = \mathcal{O}(t^4),\end{aligned}$$

as $t \downarrow 0$.

Interestingly, due to part (b) we see that $\mathcal{L}_{d,\alpha} \to \frac{1}{12}$ as $\alpha \uparrow 2$, despite of the fact that thus far we are only able to provide a representation which enables us to conclude that the values of $\mathcal{L}_{d,\alpha}$ are finite and strictly positive with no other explicit knowledge for this quantity.

1.2.1 Extensions to other non–local operators, statement of results

By mimicking the techniques employed in Chapter 2 to establish the heat trace asymptotics of α-stable processes we obtain the following results associated to the relativistic and mixed stable processes defined in examples 1.0.3 and 1.0.4, respectively. The proof of these results are omitted.

Theorem 1.2.3 *Let* $H_{V,m} = H_{\alpha,m} + V$ *with* $H_{\alpha,m} = (-\Delta + m^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} - m$. *Denote by* $p_t^{(\alpha,m)}(x,y)$ *the transition density associated to the relativistic* α*-stable process with index* m, $\mathbf{X}^m$. *Then*

$$\frac{Tr(e^{-H_{V,m}t} - e^{-tH_{\alpha,m}})}{p_t^{(\alpha,m)}(0)} + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta =$$

$$\begin{cases} \mathcal{O}(t^{2+\frac{2}{\alpha}}), & \text{if } \alpha \in (1,2) \text{ and } d \geq 1, \\ \mathcal{O}(t^4), & \text{if } \alpha \in (\frac{1}{2},1] \text{ and } d \geq 1, \\ \mathcal{O}(t^4), & \text{if } \alpha \in (0,\frac{1}{2}] \text{ and } d \geq 2, \end{cases}$$

as $t \downarrow 0$.

Theorem 1.2.4 *Let* $H_{V,a} = H^a_{\alpha,\beta} + V$ *with* $H^a_{\alpha,\beta} = (-\Delta)^{\frac{\alpha}{2}} + a\,(-\Delta)^{\frac{\beta}{2}}$, $0 < \alpha < \beta < 2$ *and* $a > 0$. *Denote by* $p_t^{(a)}(x,y)$ *the transition density associated to the process* $\mathbf{Z}^a$. *Then,*

(i) *Assume d=1 and* $1 < \alpha < \beta < 2$. *Then,*

$$\frac{Tr(e^{-tH_{V,a}} - e^{-t\,H^a_{\alpha,\beta}})}{p_t^{(a)}(0)} + t\int_{\mathbb{R}} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}} V^3(\theta)d\theta = \mathcal{O}(t^{2+\frac{2}{\beta}}),$$

as $t \downarrow 0$.

(ii) *Assume* $d \geq 2$, $0 < \alpha < \beta < 2$ *and* $2 + \frac{2}{\alpha} - d\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \geq 0$. *Then,*

$$\frac{Tr(e^{-tH_{V,a}} - e^{-t\,H^a_{\alpha,\beta}})}{p_t^{(a)}(0)} + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta = R(t),$$

where $|R(t)| \leq Ct^{\Phi_d(\alpha,\beta)}$, *for* $t \in \left(0, \min\left\{a^{\frac{\alpha}{\beta-\alpha}}, 1\right\}\right)$ *and some* $C > 0$. *Here,*

$$\Phi_d(\alpha,\beta) = \min\left\{4, 2 + \frac{2}{\alpha} - d\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\right\}.$$

In particular, for d=2, we obtain

$$\frac{Tr(e^{-tH_{V,a}} - e^{-t\,H^a_{\alpha,\beta}})}{p_t^{(a)}(0)} + t\int_{\mathbb{R}^2} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^2} V^2(\theta)d\theta + \frac{t^2}{3!}\int_{\mathbb{R}^2} V^3(\theta)d\theta =$$

$$\begin{cases} \mathcal{O}(t^{2+\frac{2}{\beta}}), & \text{if } 1 \leq \beta \text{ and } \alpha < \beta, \\ \mathcal{O}(t^4), & \text{if } 0 < \alpha < \beta < 1\,. \end{cases}$$

as $t \downarrow 0$.

1.3 Heat content for Schrödinger operators on $\mathbb{R}^d$, statement of results

The above mentioned results on the heat trace of Schrödinger operators on $\mathbb{R}^d$ has motivated the study of what we will call *"the heat content for Schrödinger semigroups"* and which we define by

$$Q_V^{(\alpha)}(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(p_t^{H_V}(x,y) - p_t^{(\alpha)}(x,y)\right) dxdy \tag{1.21}$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_t^{(\alpha)}(x,y)\mathbb{E}_{x,y}^t \left[e^{-\int_0^t V(X_s)ds} - 1\right] dxdy.$$

Notice that the second equality comes from (1.12). To the best of our knowledge, this quantity has not been studied in the literature before even in the case of the Laplacian.

Based on the first expression in (1.21), we point out that the function $Q_V^{(\alpha)}(t)$ is introduced to describe, in terms of the potential V, the average difference between $p_t^{H_V}(x,y)$ and $p_t^{(\alpha)}(x,y)$, for every $t > 0$. The second form in (1.21) suggests that a probabilistic approach involving the paths of the stable bridge should yield a better understanding on this function. In fact, we use the probabilistic representation and the finite dimensional distributions of the stable bridge to obtain the small asymptotic expansion for $Q_V^{(\alpha)}(t)$. To gain further insight into the above expression, consider $V(x) = -\mathbb{1}_\Omega(x)$, where Ω is an open set with finite volume. The stochastic integral in (1.21) becomes

$$\int_0^t \mathbb{1}_\Omega(X_s)ds,$$

which is just the total time spent in Ω up to time t by the stable process. This random variable appears in many problems in probability related to occupation measures, local times, conditional gauge theorems and more. See for example [16] and [19]. Therefore, it should be possible to describe the asymptotic behaviour of $Q_{-\mathbb{1}_\Omega}^{(\alpha)}(t)$ by means of the fluctuations of the paths of the stable process in Ω and Ω^c and we expect that geometric features of the domain Ω, such as volume, surface area of the boundary, capacity, curvature, etc., should appear in the asymptotics. For instance, Theorem 1.3.1 below shows that

$$\lim_{t\downarrow 0} t^{-1}Q_{-\mathbb{1}_\Omega}^{(\alpha)}(t) = |\Omega|.$$

With this potential V, we also have the following inequality

$$\begin{aligned} Q^{(\alpha)}_{-\mathbb{1}_\Omega}(t) &\geq \int_\Omega \int_\Omega p_t^{(\alpha)}(x,y)\mathbb{E}^t_{x,y}\left[e^{\int_0^t \mathbb{1}_\Omega(X_s)ds} - 1, \tau_\Omega^{(\alpha)} > t\right] dxdy \\ &= (e^t - 1)\int_\Omega \int_\Omega p_t^{(\alpha)}(x,y)\mathbb{P}\left(\tau_\Omega^{(\alpha)} > t \middle| X_0 = x, X_t = y\right) dxdy \\ &= (e^t - 1)Q_\Omega^{(\alpha)}(t), \end{aligned}$$

which gives some additional information on the relationship between $Q^{(\alpha)}_{-\mathbb{1}_\Omega}(t)$ and $Q_\Omega^{(\alpha)}(t)$. Here $Q_\Omega^{(\alpha)}(t)$ denotes the spectral heat content in Ω for the α–stable process. Namely,

$$Q_\Omega^{(\alpha)}(t) = \int_\Omega \int_\Omega p_t^{\Omega,\alpha}(x,y)dxdy.$$

We proceed to state our main results. The first two theorems correspond to the results for the heat trace proved in [8] for $\alpha = 2$ and in [4] for $0 < \alpha < 2$. The first theorem provides the first term whereas the second theorem yields a second order expansions under the assumption of a Hölder continuity on the potential V. Both theorems provide uniform bounds for the remainder term for all positive times.

Theorem 1.3.1

(i) For $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we obtain for all $t > 0$

$$\left| Q_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(x)dx \right| \leq t^2 ||V||_1 ||V||_\infty e^{t||V||_\infty}.$$

(ii) Assume $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then if $V : \mathbb{R}^d \to (-\infty, 0]$, we have for all $t > 0$ that

$$-t\int_{\mathbb{R}^d} V(x)dx \leq Q_V^{(\alpha)}(t) \leq -t\int_{\mathbb{R}^d} V(x)dx \left(1 + \frac{1}{2}t||V||_\infty e^{t||V||_\infty}\right).$$

In particular, under both (i) and (ii),

$$Q_V^{(\alpha)}(t) = -t\int_{\mathbb{R}^d} V(x)dx + \mathcal{O}(t^2),\ \ t \downarrow 0.$$

Theorem 1.3.2 *Suppose $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Assume that V is also uniformly Hölder continuous of order γ. That is, there exists a positive constant M such that*

$$|V(x) - V(y)| \leq M|x-y|^\gamma,$$

for all $x, y \in \mathbb{R}^d$, with $0 < \gamma < \min\{1, \alpha\}$ if $0 < \alpha < 2$ and $0 < \gamma \leq 1$ in the case $\alpha = 2$. Then, for all $t > 0$

$$\left| Q_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(x)dx - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(x)dx \right| \leq C(\gamma,\alpha)||V||_1 \left(||V||_\infty^2 e^{t||V||_\infty} t^3 + t^{\frac{\gamma}{\alpha}+2} \right).$$

In particular,

$$Q_V^{(\alpha)}(t) = -t\int_{\mathbb{R}^d} V(x)dx + \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(x)dx + \mathcal{O}(t^{\frac{\gamma}{\alpha}+2}),\ t \downarrow 0.$$

It is interesting to note here that in [4], it is shown that

$$\mathcal{T}_V^{(\alpha)}(t) = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \mathcal{O}(t^{\frac{\gamma}{\alpha}+2}),$$

as $t \downarrow 0$ under the same conditions of Theorem 1.3.2. Thus, under the assumption of Hölder continuity we cannot distinguish between $Q_V^{(\alpha)}(t)$ and $\mathcal{T}_V^{(\alpha)}(t)$, as $t \downarrow 0$ at the second order asymptotic expansion. In order to see the difference in these quantities for $t \downarrow 0$, we need to assume extra regularity conditions on V and go further in the expansion.

Our third result in this section is a general asymptotic expansion in powers of t for potentials $V \in \mathcal{S}(\mathbb{R}^d)$ with an explicit form for the coefficients. In order to avoid the introduction of more complicated notation at this point, we postpone the result to Theorem 3.1.1 in §3.1. A special case of Theorem 3.1.1 where we can compute quite explicitly all the coefficients is the following theorem.

Theorem 1.3.3 *Let $V \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \alpha \leq 2$. Then*

$$\begin{aligned} Q_V^{(\alpha)}(t) = & -t\int_{\mathbb{R}^d} V(\theta)d\theta + \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta - \frac{t^3}{3!}\left(\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{E}_\alpha(V)\right) \\ & + \frac{t^4}{4!}\left(\int_{\mathbb{R}^d} V^4(\theta)d\theta + 2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta + \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta\right) \\ & - \frac{t^5}{5!}\Bigg(\int_{\mathbb{R}^d} V^5(\theta)d\theta + 2\int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta + 2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)_2^{\frac{\alpha}{2}}V(\theta)d\theta \\ & + \int_{\mathbb{R}^d} V(\theta)\left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta + \mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right) + \mathcal{E}_\alpha\left(V^2\right)\Bigg) + \mathcal{O}(t^6), \end{aligned}$$

as $t \downarrow 0$. Here, $\mathcal{E}_\alpha$ is the Dirichlet form as defined in (1.10) and (1.11) whereas $(-\Delta)_2^{\frac{\alpha}{2}}$ is defined to be $(-\Delta)^{\frac{\alpha}{2}} \circ (-\Delta)^{\frac{\alpha}{2}}$.

It is worth mentioning that the techniques employed in Chapter 3 may be used to extend the foregoing theorems in this section to obtain the small time behaviour of the heat content for Schrödinger semigroups related to the relativistic and mixed stable processes.

1.4 Heat content for bounded domains, statement of results

As we have previously mentioned, the study of the small time behaviour of the quantities $Q_V^{(\alpha)}(t)$ and $\mathcal{T}_V^{(\alpha)}(t)$ has been motivated by the asymptotic expansion as $t \downarrow 0$ of the following spectral functions for smooth open bounded domains $\Omega \subset \mathbb{R}^d$,

$$\mathcal{Z}_\Omega^{(2)}(t) = \frac{1}{p_t^{(2)}(0)} \int_\Omega p_t^{\Omega,2}(x,x)dx \quad \text{and} \quad Q_\Omega^{(2)}(t) = \int_\Omega \int_\Omega p_t^{\Omega,2}(x,y)dxdy,$$

where $p_t^{\Omega,2}(x,y)$ is the transition density of a Brownian Motion process killed upon exiting Ω.

The purpose of this section is to provide an insight of the possible small time behavior of the spectral function $Q_\Omega^{(\alpha)}(t)$ for the cases $0 < \alpha < 2$, whose second order expansion is still unknown. To be more precisely, we show the following.

Theorem 1.4.1 *Assume $\Omega \subset \mathbb{R}^d$, $d \geq 2$ is a uniformly $C^{1,1}$-regular bounded domain.*

(i) Let $1 < \alpha < 2$. Then, we have

$$\frac{1}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right)\mathcal{H}^{d-1}(\partial\Omega) \leq \varliminf_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t^{\frac{1}{\alpha}}}$$
$$\leq \varlimsup_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t^{\frac{1}{\alpha}}} \leq 2^{(3d+1)/2}\,\Gamma\left(1 - \frac{1}{\alpha}\right)\mathcal{H}^{d-1}(\partial\Omega).$$

(ii) For $\alpha = 1$, we obtain

$$\frac{1}{\pi}\,\mathcal{H}^{d-1}(\partial\Omega) \leq \varliminf_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(1)}(t)}{t\,\ln\left(\frac{1}{t}\right)}$$
$$\leq \varlimsup_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(1)}(t)}{t\,\ln\left(\frac{1}{t}\right)} \leq 2^{(3d+1)/2}\,\mathcal{H}^{d-1}(\partial\Omega).$$

(iii) *For $0<\alpha<1$, there exists a positive constant $C_{d,\alpha}$ such that*

$$A_{d,\alpha}\,\mathcal{P}_\alpha(\Omega) \leq \varliminf_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t}$$
$$\leq \varlimsup_{t\downarrow 0} \frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t} \leq C_{d,\alpha} \int_\Omega dx\, \rho_\Omega^{-\alpha}(x),$$

where $\rho_\Omega(x) = \inf\{|\sigma - x| : \sigma \in \partial\Omega\}$. Moreover, if Ω satisfies a uniform exterior volume condition, the quantity $\int_\Omega dx\, \rho_\Omega^{-\alpha}(x)$ can be replaced up to some positive constant by $\mathcal{P}_\alpha(\Omega)$. See (4.3) and (4.13) for the definitions of $A_{\alpha,d}$ and $\mathcal{P}_\alpha(\Omega)$, respectively.

The lower bounds established in the previous theorem are obtained first by investigating the small time behavior of the heat content Ω in $\mathbb{R}^d$ (see Theorem 4.0.2) which is denoted by $\mathbb{H}_\Omega^{(\alpha)}(t)$ and defined as

$$\mathbb{H}_\Omega^{(\alpha)}(t) = \int_\Omega dy \int_\Omega dx\ p_t^{(\alpha)}(x,y)$$

and secondly, by considering the following inequality between $\mathbb{H}_\Omega^{(\alpha)}(t)$ and the spectral heat content $Q_\Omega^{(\alpha)}(t)$ which holds for all $t>0$,

$$Q_\Omega^{(\alpha)}(t) \leq \mathbb{H}_\Omega^{(\alpha)}(t) = |\Omega| - \int_\Omega dy \int_{\Omega^c} dx\ p_t^{(\alpha)}(x,y).$$

As for the uppers bounds, we require a more delicate treatment where the $\alpha/2$-subordinator **S** plays a relevant role.

Based on the estimates provided in Chapter 4 below and the above theorem, we state the following conjecture about the small time behavior of the spectral heat content of Ω.

Conjecture

(i) For $1<\alpha<2$, there exists $C_{d,\alpha}>0$ such that

$$Q_\Omega^{(\alpha)}(t) = |\Omega| - C_{d,\alpha}\,\mathcal{H}^{d-1}(\partial\Omega)\, t^{\frac{1}{\alpha}} + \mathcal{O}(t),\ \ t\downarrow 0.$$

(ii) For $\alpha = 1$, there exists $C_d>0$ such that

$$Q_\Omega^{(\alpha)}(t) = |\Omega| - C_d\,\mathcal{H}^{d-1}(\partial\Omega)\, t\, \ln\left(\frac{1}{t}\right) + \mathcal{O}(t),\ \ t\downarrow 0.$$

(iii) For $0 < \alpha < 1$, there exists $C_{d,\alpha} > 0$ such that

$$Q_{\Omega}^{(\alpha)}(t) = |\Omega| - C_{d,\alpha}\mathcal{P}_{\alpha}(\Omega)\, t + o(t),\ \ t \downarrow 0.$$

2. HEAT TRACE FOR SCHRÖDINGER OPERATORS, PROOFS

2.1 Heat trace in terms of Fourier transform.

Let $\widehat{V}$ denote the Fourier transform of $V \in \mathcal{S}(\mathbb{R}^d)$ with the normalization

$$\widehat{V}(\xi) = \int_{\mathbb{R}^d} e^{-i<x,\xi>} V(x)dx. \tag{2.1}$$

We note that because of our definition of $\widehat{V}$, we have

(i) (Inversion formula)

$$V(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i<x,\xi>} \widehat{V}(\xi)d\xi.$$

(ii) For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} e^{-i<x,\xi>} f(x)g(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\theta)\widehat{g}(\xi - \theta)d\theta. \tag{2.2}$$

Our goal now is to derive a formula for $Tr(e^{-tH_V} - e^{-tH_\alpha})$ for the fractional Laplacian similar to the one in [3] for the Laplacian.

Proposition 2.1.1 *Let $V \in \mathcal{S}(\mathbb{R}^d)$, then*

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\widehat{p_t^{H_V}}(\xi, -\xi) - \widehat{p_t^{(\alpha)}}(\xi, -\xi) \right) d\xi.$$

Proof For all $t > 0$ and $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned} \partial_t p_t^{(\alpha)}(x, y) &= -(-\Delta)_x^{\frac{\alpha}{2}} p_t^{(\alpha)}(x, y) \\ p_0^{(\alpha)}(x, y) &= \delta(x - y) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \partial_t p_t^{H_V}(x, y) &= -[(-\Delta)_x^{\frac{\alpha}{2}} + V(x)] p_t^{H_V}(x, y) \\ p_0^{H_V}(x, y) &= \delta(x - y). \end{aligned} \tag{2.4}$$

By taking Fourier transform on $\mathbb{R}^{2d}$, we deduce that

$$\begin{aligned}(\partial_t + |\xi|^\alpha)\widehat{p_t^{(\alpha)}}(\xi,\eta) &= 0 \\ \widehat{p_0^{(\alpha)}}(\xi,\eta) &= (2\pi)^d\delta(\xi+\eta)\end{aligned} \tag{2.5}$$

and that

$$\begin{aligned}(\partial_t + |\xi|^\alpha)\widehat{p_t^{H_V}}(\xi,\eta) &= -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{V}(\theta)\widehat{p_t^{H_V}}(\xi-\theta,\eta)d\theta \\ \widehat{p_0^{H_V}}(\xi,\eta) &= (2\pi)^d\delta(\eta+\xi).\end{aligned} \tag{2.6}$$

Now, by directly solving (2.5) and (2.6) we find that

$$\widehat{p_t^{(\alpha)}}(\xi,\eta) = (2\pi)^d\delta(\eta+\xi)e^{-t|\xi|^\alpha}$$

and that

$$\widehat{p_t^{H_V}}(\xi,\eta) - \widehat{p_t^{(\alpha)}}(\xi,\eta) = -\frac{1}{(2\pi)^d}\int_0^t\int_{\mathbb{R}^d} e^{-(t-s)|\xi|^\alpha}\widehat{V}(\theta)\widehat{p_s^{H_V}}(\xi-\theta,\eta)d\theta ds. \tag{2.7}$$

On the other hand, from (2.3), (2.4) and Duhamel's Principle we see that

$$p_t^{H_V}(x,y) - p_t^{(\alpha)}(x,y) = -\int_0^t\int_{\mathbb{R}^d} p_{t-s}^{(\alpha)}(x,z)p_s^{H_V}(z,y)V(z)dzds.$$

Since $Tr(e^{-tH_V} - e^{-tH_\alpha})$ is by definition equal to $\int_{\mathbb{R}^d}\left(p_t^{H_V}(x,x) - p_t^{(\alpha)}(x,x)\right)dx$, we obtain from the above equality that

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) \;=\; -\int_0^t\int_{\mathbb{R}^{2d}} p_{t-s}^{(\alpha)}(x,z)p_s^{H_V}(z,x)V(z)dzdxds. \tag{2.8}$$

Expressing the right hand side of (2.8) in terms of Fourier transform we arrive at

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) = -\frac{1}{(2\pi)^{2d}}\int_0^t\int_{\mathbb{R}^{3d}}\widehat{V}(\theta)\widehat{p_s^{H_V}}(\mu,\tau)\widehat{p_{t-s}^{(\alpha)}}(-\mu-\theta,-\tau)d\mu d\tau d\theta ds.$$

Since

$$\widehat{p_{t-s}^{(\alpha)}}(-\mu-\theta,-\tau) = (2\pi)^d\delta(\tau+\theta+\mu)e^{-(t-s)|\theta+\mu|^\alpha},$$

we see that

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) = -\frac{1}{(2\pi)^{2d}}\int_0^t\int_{\mathbb{R}^{2d}} e^{-(t-s)|\tau|^\alpha}\widehat{p_s^{H_V}}(-\tau-\theta,\tau)\widehat{V}(\theta)d\tau d\theta ds.$$

The conclusion of the proposition follows by setting $\xi = -\tau$ in the last equation, $\eta = \xi$ in (2.7) and integrating with respect to ξ. ∎

If we now iterate the equation (2.7) J-times, we obtain

Corollary 2.1.1 *Let $V \in \mathcal{S}(\mathbb{R}^d)$, $0 < \alpha < 2$ and set*

$$F_j^{(\alpha)}(s,\xi,\theta) = e^{-(t-s_1)|\xi|^\alpha - \sum\limits_{k=1}^{j-1}(s_k - s_{k+1})|\xi - \sum\limits_{i=1}^{k}\theta_i|^\alpha},$$

where $s = (s_1, ..., s_j)$, $s_{k+1} < s_k$ and $\theta = (\theta_1, ..., \theta_{j-1})$. Then for $J \geq 2$,

$$\widehat{p_t^{H_V}}(\xi,\eta) - \widehat{p_t^{(\alpha)}}(\xi,\eta) = -\frac{1}{(2\pi)^d}\int_0^t \int_{\mathbb{R}^d} e^{-(t-s_1)|\xi|^\alpha}\widehat{V}(\theta_1)\widehat{p_{s_1}^{(\alpha)}}(\xi - \theta_1, \eta)d\theta_1 ds_1 +$$
$$\sum_{j=2}^{J}\frac{(-1)^j}{(2\pi)^{jd}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_{j-1}}\int_{\mathbb{R}^{jd}} F_j^{(\alpha)}(s,\xi,\theta)\widehat{p_{s_j}^{(\alpha)}}(\xi - \sum_{i=1}^{j}\theta_i, \eta)\prod_{i=1}^{j}\widehat{V}(\theta_i)d\theta_i ds_i +$$
$$\frac{(-1)^{J+1}}{(2\pi)^{(J+1)d}}\int_0^t\int_0^{s_1}\cdots\int_0^{s_J}\int_{\mathbb{R}^{(J+1)d}} F_{J+1}^{(\alpha)}(s,\xi,\theta)\widehat{p_{s_{J+1}}^{H_V}}(\xi - \sum_{i=1}^{J+1}\theta_i, \eta)\prod_{i=1}^{J+1}\widehat{V}(\theta_i)d\theta_i ds_i.$$

Furthermore, we conclude

$$Tr(e^{-tH_V} - e^{-tH_\alpha}) = -tp_t^{(\alpha)}(0)\widehat{V}(0) + \tag{2.9}$$
$$\sum_{j=2}^{J}\frac{(-t)^j}{(2\pi)^{jd}}\int_0^1\int_0^{\lambda_1}\cdots\int_0^{\lambda_{j-1}}\int_{\mathbb{R}^{jd}} F_j^{(\alpha)}(t\lambda,\xi,\theta)e^{-t\lambda_j|\xi|^\alpha}\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)d\theta_i d\lambda_i d\lambda_j d\xi +$$
$$\frac{(-t)^{J+1}}{(2\pi)^{(J+2)d}}\int_0^1\int_0^{\lambda_1}\cdots\int_0^{\lambda_J}\int_{\mathbb{R}^{(J+2)d}} F_{J+1}^{(\alpha)}(t\lambda,\xi,\theta)\widehat{p_{t\lambda_{J+1}}^{H_V}}(\xi - \sum_{i=1}^{J+1}\theta_i, -\xi)\prod_{i=1}^{J+1}\widehat{V}(\theta_i)d\theta_i d\lambda_i d\xi.$$

2.2 Boundedness of the (J+1)–th term.

Our goal in this section is to provide an upper bound for the absolute value of the last expression in (2.9) in terms of $p_t^{(\alpha)}(0)$. The following Lemma is a consequence of (2.2), the inversion formula and induction. Therefore its proof is omitted.

Lemma 2.2.1 *Let $J \geq 1$ and $\{p_i\}_{i=0}^{J} \subset \mathcal{S}(\mathbb{R}^d)$ radial functions. Set $\Psi_J(\xi) = \widehat{p_0}(\xi)\prod_{j=1}^{J}\widehat{p_j}(\gamma_j - \xi)$, where $\gamma_j \in \mathbb{R}^d$ are constant vectors. Then,*

$$\widehat{\Psi_J}(\gamma) = (2\pi)^d\int_{\mathbb{R}^{Jd}} e^{-i\sum\limits_{j=1}^{J}\langle\gamma_j, x_j\rangle} p_0(\sum_{j=1}^{J}x_j - \gamma)\prod_{j=1}^{J}p_j(x_j)dx_j.$$

Remark 2.2.1 *Consider the function $F^{(\alpha)}_{J+1}$ defined as in Corollary 2.1.1. If we set $p_0 = p^{(\alpha)}_{t(1-\lambda_1)}$ and $p_j = p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}$ in the last Lemma, with $\gamma_j = \sum_{k=1}^{j}\theta_k$, we obtain*

$$\frac{\widehat{F^{(\alpha)}_{J+1}}(t\lambda, x-y,\theta)}{(2\pi)^d} = \int_{\mathbb{R}^{Jd}} e^{-i\sum_{j=1}^{J}\langle\gamma_j, x_j\rangle} p^{(\alpha)}_{t(1-\lambda_1)}(\sum_{j=1}^{J} x_j-(x-y))\prod_{j=1}^{J} p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}(x_j)dx_j. \tag{2.10}$$

Next, it is known that the transition density $p^{(\alpha)}_t(x,y)$ satisfies the Chapman-Kolmogorov equation, namely,

$$\int_{\mathbb{R}^d} p^{(\alpha)}_s(a-z)p^{(\alpha)}_t(z)dz = p^{(\alpha)}_{t+s}(a), \tag{2.11}$$

for all $a\in\mathbb{R}^d$ and $t,s>0$. With this equality at hand, it easily follows that

$$\int_{\mathbb{R}^{Jd}} p^{(\alpha)}_{t(1-\lambda_1)}(\sum_{j=1}^{J} x_j-(x-y))\prod_{j=1}^{J} p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}(x_j)dx_j = p^{(\alpha)}_{t(1-\lambda_{J+1})}(x-y). \tag{2.12}$$

It can also be proved by means of the inversion formula that

$$\int_{\mathbb{R}^{(J+1)d}} e^{-i\left\{\langle x,\gamma_{J+1}\rangle+\sum_{j=1}^{J}\langle\gamma_j,x_j\rangle\right\}}\prod_{i=1}^{J+1}\widehat{V}(\theta_i)d\theta_i = (2\pi)^{(J+1)d}V(-x)\prod_{k=1}^{J}V(-\sum_{j=k}^{J}x_j-x). \tag{2.13}$$

Proposition 2.2.1 *Assume $0<t<1$ and define*

$$r_{J+1}(t) = \int_0^1\int_0^{\lambda_1}\cdots\int_0^{\lambda_J}\int_{\mathbb{R}^{(J+2)d}} F^{(\alpha)}_{J+1}(t\lambda,\xi,\theta)\,\widehat{p^{H_V}_{t\lambda_{J+1}}}(\xi-\sum_{i=1}^{J+1}\theta_i,-\xi)\prod_{i=1}^{J+1}\widehat{V}(\theta_i)d\xi d\theta_i d\lambda_i.$$

There exists a positive constant $C = C_{J+1,d,\alpha}(V)$ such that

$$|r_{J+1}(t)| \le Cp^{(\alpha)}_t(0).$$

Proof Set $\gamma_j = \sum_{i=1}^{j}\theta_i$ and

$$p(x-y,\{x_j\}_{j=1}^{J}) = p^{(\alpha)}_{t(1-\lambda_1)}(\sum_{j=1}^{J}x_j-(x-y))\prod_{j=1}^{J}p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}(x_j),$$

$$I_{J+1}(t\lambda,\theta) = \int_{\mathbb{R}^d} F^{(\alpha)}_{J+1}(t\lambda,\xi,\theta)\widehat{p^{H_V}_{t\lambda_{J+1}}}(\xi-\gamma_{J+1},-\xi)d\xi.$$

Then, by the definition of Fourier transform and (2.10), we have that

$$\begin{aligned} I_{J+1}(t\lambda,\theta) &= \int_{\mathbb{R}^d} F^{(\alpha)}_{J+1}(t\lambda,\xi,\theta)\int_{\mathbb{R}^{2d}} e^{-i\{\langle \xi, x-y\rangle + \langle -x,\gamma_{J+1}\rangle\}} p^{H_V}_{t\lambda_{J+1}}(x,y)dxdyd\xi \\ &= \int_{\mathbb{R}^{2d}} p^{H_V}_{t\lambda_{J+1}}(x,y) e^{-i\{\langle -x,\gamma_{J+1}\rangle\}} \widehat{F^{(\alpha)}_{J+1}}(t\lambda, x-y,\theta)dxdy \\ &= (2\pi)^d \int_{\mathbb{R}^{Jd}}\int_{\mathbb{R}^{2d}} e^{-i\left\{\langle -x,\gamma_{J+1}\rangle + \sum\limits_{j=1}^{J}\langle x_j,\gamma_j\rangle\right\}} p^{H_V}_{t\lambda_{J+1}}(x,y)p(x-y,\{x_j\}_{j=1}^J)dxdydx_j. \end{aligned}$$

Now, because of (2.13) with x replaced by $-x$, (2.12) and the fact that $p^{H_V}_{t\lambda_{J+1}}(x,y) \leq e^{t||V||_{L^\infty(\mathbb{R}^d)}} p^{(\alpha)}_{t\lambda_{J+1}}(x,y)$ (which follows from (1.12)), we obtain from the last equality that

$$\left|\int_{\mathbb{R}^{(J+1)d}} I_{J+1}(t\lambda,\theta)\prod_{i=1}^{J+1}\widehat{V}(\theta_i)d\theta_i\right| =$$

$$\left|(2\pi)^{(J+2)d}\int_{\mathbb{R}^{(J+2)d}} V(x)\prod_{i=1}^{J} V(-\sum_{j=i}^{J} x_j + x) p^{H_V}_{t\lambda_{J+1}}(x,y)p(x-y,\{x_j\}_{j=1}^J)dxdydx_j\right| \leq$$

$$(2\pi)^{(J+2)d}||V||^J_{L^\infty(\mathbb{R}^d)} e^{t||V||_{L^\infty(\mathbb{R}^d)}}\int_{\mathbb{R}^d}|V(x)|\int_{\mathbb{R}^d} p^{(\alpha)}_{t(1-\lambda_{J+1})}(x-y)p^{(\alpha)}_{t\lambda_{J+1}}(x-y)dydx.$$

It follows from (2.11) that

$$|r_{J+1}(t)| \leq \frac{(2\pi)^{(J+2)d}p^{(\alpha)}_t(0)}{(J+1)!}||V||^J_{L^\infty(\mathbb{R}^d)} e^{t||V||_{L^\infty(\mathbb{R}^d)}}||V||_{L^1(\mathbb{R}^d)}.$$

■

As a consequence, by setting

$$R_{J+1}(t) = \frac{r_{J+1}(t)}{(2\pi)^{(J+2)d}p^{(\alpha)}_t(0)}, \tag{2.14}$$

we have proved that $R_{J+1}(t)$ is bounded for $t \in (0,1)$.

2.3 Heat trace computation by means of subordinators.

In this section we further investigate formula (2.9) involving $Tr(e^{-tH_V} - e^{-tH_\alpha})$. We start by integrating the function

$$F^{(\alpha)}_j(t\lambda,\xi,\theta)e^{-t\lambda_j|\xi|^\alpha} = e^{-t(1-[\lambda_1-\lambda_j])|\xi|^\alpha - t\sum\limits_{k=1}^{j-1}(\lambda_k-\lambda_{k+1})|\xi-\sum\limits_{i=1}^{k}\theta_i|^\alpha} \tag{2.15}$$

with respect to ξ over $\mathbb{R}^d$. The integral could be easily computed when $\alpha = 2$ using two elementary facts. Namely, for any $\gamma \in \mathbb{R}^d$,

$$|\xi + \gamma|^2 = |\xi|^2 + 2 < \xi, \gamma > + |\gamma|^2 \tag{2.16}$$

and

$$\int_{\mathbb{R}^d} e^{-t|\xi-\gamma|^2} d\xi = \pi^{d/2} t^{-d/2}. \tag{2.17}$$

Unfortunately, we cannot calculate (2.15) in the same way because there is not a close form for $|\xi + \gamma|^\alpha$, when $0 < \alpha < 2$. Instead, we will follow a probabilistic approach by means of $\alpha/2$–subordinators and their Laplace transform given in §1.0.2 that relates $|\cdot|^\alpha$ to $|\cdot|^2$ to find the value of the integral involving the quantity in (2.15). We begin by observing that (1.3) implies that for all $c > 0$ and $t > 0$,

$$e^{-tc|\xi|^\alpha} = \mathbb{E}\left[e^{-t^{\frac{2}{\alpha}} S_c |\xi|^2}\right]. \tag{2.18}$$

In addition, for any sequence of numbers $\{\lambda_k\}_{k=1}^j$, $j \geq 2$, satisfying

$$0 < \lambda_j < \lambda_{j-1} < ... < \lambda_2 < \lambda_1 < 1, \tag{2.19}$$

we have

$$S_1 = S_{(1-(\lambda_1-\lambda_j))+\lambda_1-\lambda_j} - S_{\lambda_1-\lambda_j} + \sum_{k=1}^{j-1} \left(S_{\lambda_k-\lambda_{k+1}+(\lambda_{k+1}-\lambda_j)} - S_{\lambda_{k+1}-\lambda_j}\right).$$

For $1 \leq k \leq j-1$ consider the random variables

$$S^*_{\lambda_k-\lambda_{k+1}} = S_{\lambda_k-\lambda_{k+1}+(\lambda_{k+1}-\lambda_j)} - S_{\lambda_{k+1}-\lambda_j} \quad \text{and}$$
$$S^*_{1-(\lambda_1-\lambda_j)} = S_{1-(\lambda_1-\lambda_j)+\lambda_1-\lambda_j} - S_{\lambda_1-\lambda_j}.$$

Since the process S has independent and stationary increments, we see that the random variables $\left\{S^*_{\lambda_k-\lambda_{k+1}}, S^*_{1-(\lambda_1-\lambda_j)}\right\}_{k=1}^{j-1}$ are independent and furthermore,

$$S^*_{\lambda_k-\lambda_{k+1}} \stackrel{\mathcal{D}}{=} S_{\lambda_k-\lambda_{k+1}}$$
$$S^*_{1-(\lambda_1-\lambda_j)} \stackrel{\mathcal{D}}{=} S_{1-(\lambda_1-\lambda_j)}. \tag{2.20}$$

We also have, of course, that

$$S^*_{1-(\lambda_1-\lambda_j)} + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} = S_1. \tag{2.21}$$

As before let us denote, for simplicity, $\gamma_k = \sum_{i=1}^{k} \theta_i$. It follows from (2.18), (2.20), (2.21) and the independence of $\left\{S^*_{\lambda_k-\lambda_{k+1}}, S^*_{1-(\lambda_1-\lambda_j)}\right\}_{k=1}^{j-1}$ that

$$\begin{aligned}
&e^{-t(1-[\lambda_1-\lambda_j])|\xi|^\alpha} \prod_{k=1}^{j-1} e^{-t(\lambda_k-\lambda_{k+1})|\xi-\gamma_k|^\alpha} = \\
&\mathbb{E}\left[\exp\left(-t^{2/\alpha} S^*_{1-(\lambda_1-\lambda_j)}|\xi|^2\right)\right] \mathbb{E}\left[\exp\left(-t^{2/\alpha} \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}|\xi-\gamma_k|^2\right)\right] = \\
&\mathbb{E}\left[\exp\left(-t^{2/\alpha}\left\{S^*_{1-(\lambda_1-\lambda_j)}|\xi|^2 + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}|\xi-\gamma_k|^2\right\}\right)\right]. \qquad (2.22)
\end{aligned}$$

Next, consider the random variable defined as follows

$$L_j^{(\alpha)}(\lambda, \theta) = \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}|\gamma_k|^2 - \frac{1}{S_1}\left|\sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}\gamma_k\right|^2, \tag{2.23}$$

where $\lambda = (\lambda_1, ..., \lambda_j)$ satisfies (2.19). By (2.16), (2.21) and completing squares we easily get that

$$S^*_{1-(\lambda_1-\lambda_j)}|\xi|^2 + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}|\xi-\gamma_k|^2 = S_1\left|\xi - \frac{1}{S_1}\sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}\gamma_k\right|^2 + L_j^{(\alpha)}(\lambda, \theta).$$

Also, observe that by (2.17) and the scaling property (1.5) we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \exp\left(-t^{2/\alpha} S_1\left|\xi - \frac{1}{S_1}\sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}}\gamma_k\right|^2\right) d\xi &= \pi^{d/2} t^{-d/\alpha} S_1^{-d/2} \\
&= C_{d,\alpha} p_t^{(\alpha)}(0) S_1^{-d/2}, \qquad (2.24)
\end{aligned}$$

over the the set where $0 < S_1 < \infty$. Here,

$$C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)}. \tag{2.25}$$

We now combine these calculations to find the value of the desired integral. More precisely, we have

Lemma 2.3.1 *Let* $\lambda = (\lambda_1, ..., \lambda_j)$ *satisfy (2.19),* $\gamma_k = \sum_{i=1}^{k} \theta_i$ *and* $\theta = (\theta_1, ..., \theta_{j-1})$*. Then,*

$$L_j^{(\alpha)}(\lambda, \theta) \geq 0, \quad a.s.$$

and

$$\int_{\mathbb{R}^d} F_j^{(\alpha)}(t\lambda, \xi, \theta) e^{-t\lambda_j |\xi|^\alpha} d\xi = C_{d,\alpha} p_t^{(\alpha)}(0) \mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2} e^{-t^{2/\alpha} L_j^{(\alpha)}(\lambda,\theta)} \right]. \tag{2.26}$$

Proof Assume $\gamma_k = (b_{1,k}, ..., b_{d,k})$. By the Cauchy-Schwarz inequality and (2.21)

$$\begin{aligned}
\left| \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} \gamma_k \right|^2 &= \sum_{m=1}^{d} \left\{ \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} b_{m,k} \right\}^2 = \sum_{m=1}^{d} \left\{ \sum_{k=1}^{j-1} \left\{ S^*_{\lambda_k - \lambda_{k+1}} \right\}^{\frac{1}{2}} b_{m,k} \left\{ S^*_{\lambda_k - \lambda_{k+1}} \right\}^{\frac{1}{2}} \right\}^2 \\
&\leq \sum_{m=1}^{d} \left\{ \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} b^2_{m,k} \right\} \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} \leq S_1 \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} |\gamma_k|^2.
\end{aligned}$$

In fact, under the convention that $\sum_{r=1}^{0} = 0$, we have for $j \geq 2$,

$$L_j^{(\alpha)}(\lambda, \theta) = S_1^{-1} \left[S^*_{1-(\lambda_1 - \lambda_j)} \sum_{k=1}^{j-1} S^*_{\lambda_k - \lambda_{k+1}} |\gamma_k|^2 + \sum_{r=1}^{j-2} \sum_{s=r+1}^{j-1} S^*_{\lambda_r - \lambda_{r+1}} S^*_{\lambda_s - \lambda_{s+1}} |\gamma_r - \gamma_s|^2 \right]. \tag{2.27}$$

On the other hand, (2.26) follows by integrating (2.22) with respect to ξ, applying Fubini's Theorem to (2.22) and using (2.24). ■

Remark 2.3.1 *We note that from* (4.21) *and the fact that* $0 < S_1 < \infty$*, a.s,*

$$0 < \mathbb{E}\left[S_{1,\alpha/2}^{-d/2} e^{-t^{2/\alpha} L_j^{(\alpha)}(\lambda,\theta)} \right] \leq \mathbb{E}\left[S_{1,\alpha/2}^{-d/2} \right] < \infty. \tag{2.28}$$

In fact, all the above results are true for $\alpha = 2$ *in which case* $S_{1,1} = 1$ *and all our calculation considerably simplify.*

2.4 Bounds for remainders and coefficients.

We observe that the exponential function is involved in (2.26) and that this term is part of the expression for $Tr(e^{-tH_V} - e^{-tH_\alpha})$ in (2.9). Our next step is to use a Taylor expansion of the exponential function with a particular remainder to obtain a finer estimate for the trace. This implies, as the reader may note, that in (2.26) we will have to deal with expectations. Hence our goal in this section is to give conditions to guarantee the finiteness of these expectations. Once this is done, it will follow easily that the coefficients and remainders to appear in (2.9) are also finite and bounded, respectively.

We recall the well known expansion for the exponential function

$$e^{-x} = \sum_{n=0}^{m-1} \frac{(-1)^n}{n!} x^n + \frac{(-1)^m}{m!} x^m e^{-x\beta_m(x)} \tag{2.29}$$

valid for every $x \geq 0$ and $m \geq 1$, where we call $\beta_m(x) \in (0,1)$ the remainder of order m. With this expansion at hand, we now introduce two functions from which we will obtain the desired finer estimates in the trace formula (2.9). For $j \geq 2$,

$$T_d(j,t) = \int_0^1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{j-1}} \int_{\mathbb{R}^{jd}} F_j^{(\alpha)}(t\lambda,\xi,\theta) e^{-t\lambda_j|\xi|^\alpha} \widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\xi d\lambda_j = \tag{2.30}$$

$$C_{d,\alpha} p_t^{(\alpha)}(0) \int_0^1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{j-1}} \int_{\mathbb{R}^{(j-1)d}} \mathbb{E}\left[S_{1,\alpha/2}^{-d/2} e^{-t^{\frac{2}{\alpha}} L_j^{(\alpha)}(\lambda,\theta)}\right] \widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j.$$

The remainder function is

$$R_{j,d}^{(m)}(t) = \frac{C_{d,\alpha}}{m!(2\pi)^{jd}} \int_0^1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{j-1}} \int_{\mathbb{R}^{(j-1)d}} \mathbb{E}\left[S_{1,\alpha/2}^{-d/2} \left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^m e^{-\beta_{m,j}^*(t)}\right]$$
$$\times \widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j, \tag{2.31}$$

where the random functions $\beta_{m,j}^*(t) = t^{2/\alpha} L_j^{(\alpha)}(\lambda,\theta) \beta_m(-t^{2/\alpha} L_j^{(\alpha)}(\lambda,\theta))$ are nonnegative.

Remark 2.4.1 *We note by (2.28) and (2.30) that*

$$\left|\frac{T_d(j,t)}{(2\pi)^{jd} p_t^{(\alpha)}(0)}\right| \leq \frac{C_{d,\alpha} \mathbb{E}\left[S_{1,\alpha/2}^{-d/2}\right]}{j!(2\pi)^{jd}} \int_{\mathbb{R}^{(j-1)d}} \left|\widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i)\right| d\theta_i,$$

for all $j \geq 2$, $d \geq 1$. Observe that the right hand side is finite since $V \in \mathcal{S}(\mathbb{R}^d)$, proving at the same time the finiteness of $T_d(j,t)$, for all $t > 0$.

We now proceed to prove the finiteness of the remainder-functions in (2.31) and define the coefficients given in Theorem 1.2.1.

Lemma 2.4.1 *Assume $M \geq 1$ is an integer satisfying $M < \frac{\alpha+d}{2}$. Then, for all $t \geq 0$ and $j \geq 2$,*

$$\left|R_{j,d}^{(M)}(t)\right| \leq C_{j,d,M} \int_{\mathbb{R}^{(j-1)d}} \left(\sum_{k=1}^{j-1} |\gamma_k|^2\right)^M \left|\widehat{V}(-\sum_{i=1}^{j-1} \theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i)\right| d\theta_i,$$

where

$$C_{j,d,M} = \frac{C_{d,\alpha}\mathbb{E}\left[S_{1,\alpha/2}^{M-d/2}\right]}{j!M!(2\pi)^{jd}}, \quad \gamma_k = \sum_{i=1}^{k} \theta_i.$$

Furthermore,

(a) for all integers $n \leq M$,

$$0 \leq \mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^n\right] < \infty,$$

(b) and for all $j \geq 2$,

$$\frac{T_d(j,t)}{(2\pi)^{jd} p_t^{(\alpha)}(0)} = \sum_{n=0}^{M-1} (-1)^n C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}} + (-1)^M t^{\frac{2M}{\alpha}} R_{j,d}^{(M)}(t),$$

where

$$C_{n,j}^{(\alpha)}(V) = \frac{C_{d,\alpha}}{(2\pi)^{jd} n!} \int_0^1 \int_0^{\lambda_1} \cdots \int_0^{\lambda_{j-1}} \int_{\mathbb{R}^{(j-1)d}} \times$$
$$\mathbb{E}\left[S_{1,\alpha/2}^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^n\right] \widehat{V}(-\sum_{i=1}^{j-1}\theta_i) \prod_{i=1}^{j-1} \widehat{V}(\theta_i) d\theta_i d\lambda_i d\lambda_j.$$

Proof We start by observing that the condition $M < \frac{d+\alpha}{2}$ guarantees that $0 \leq \mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{n-d/2}\right] < \infty$ for all integers $n \leq M$, according to (4.21). From this we proceed to prove (a) as follows. Recall that

$$S^*_{1-(\lambda_1-\lambda_j)} + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} = S_{1,\alpha/2}.$$

It follows from (2.23) and the last equality that

$$0 < L_j^{(\alpha)}(\lambda,\theta) \leq \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} |\gamma_k|^2 \leq S_{1,\alpha/2} \sum_{k=1}^{j-1} |\gamma_k|^2.$$

Then, from the last inequality we conclude that for all integer $n \leq M$,

$$\mathbb{E}\left[S_{1,\alpha/2}^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^n\right] \leq \mathbb{E}\left[S_{1,\alpha/2}^{n-d/2}\right]\left(\sum_{k=1}^{j-1}|\gamma_k|^2\right)^n .$$

Thus (a) now follows easily from the last inequality. On the other hand, (b) follows from the Taylor expansion (3.17) applied to (2.30). We remark that $C_{n,j}^{(\alpha)}(V) = R_{j,d}^{(n)}(0)$. Therefore the last expression is finite, according to (2.4.1). ■

Remark 2.4.2 *Lemma 2.4.1 allows us to bound the remainders by a constant for all* $t \geq 0$ *and shows the finiteness of both the remainders and the coefficients* $C_{n,j}^{(\alpha)}(V)$ *by proving the finiteness of the expectations under the condition* $n \leq M < \frac{d+\alpha}{2}$*. Indeed, this condition is introduced to make sense of the Taylor expansion of order M when it is applied to the function (2.28). As a consequence our results are dimensional dependent. The reason why this does not happen when* $\alpha = 2$ *is that in this case the time change is trivial,* $S_{t,1} = t$*, and* $L_j^{(2)}$ *is nonrandom function. These two facts considerably reduce all the above computations, thereby the dimension only appearing in the integrals involving* $\widehat{V}$*, which are finite since* $V \in \mathcal{S}(\mathbb{R}^d)$*.*

2.5 An improvement for dimension $d = 1, 2, 3$.

We recall the basic inequality

$$\left(\sum_{k=1}^{j-1} a_k\right)^M \leq (j-1)^{M-1}\sum_{k=1}^{j-1} a_k^M,$$

valid for all $j \geq 2$ and positive numbers $\{a_k\}_{k=1}^{j-1}$.

From the last inequality and (2.27), it follows for all integer $M \geq 1$ that there exists a constant $C_{j,M} > 0$ such that

$$C_{j,M}^{-1}\mathbb{E}\left[S_1^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^M\right] \leq \sum_{k=1}^{j-1}\mathbb{E}\left[\left(S^*_{1-(\lambda_1-\lambda_j)}S^*_{\lambda_k-\lambda_{k+1}}\right)^M S_1^{-M-\frac{d}{2}}\right]|\gamma_k|^{2M} + \sum_{r=1}^{j-2}\sum_{s=r+1}^{j-1}\mathbb{E}\left[(S^*_{\lambda_r-\lambda_{r+1}}S^*_{\lambda_s-\lambda_{s+1}})^M S_1^{-M-\frac{d}{2}}\right]|\gamma_r-\gamma_s|^{2M}, \tag{2.32}$$

whenever the expectations involved in the last expression are finite. The purpose of this section is to provide conditions under which these last expectations are finite for dimension $d = 1, 2$ and 3.

We proved in §2.3 that

$$S^*_{1-(\lambda_1-\lambda_j)} + \sum_{k=1}^{j-1} S^*_{\lambda_k-\lambda_{k+1}} = S_1,$$

for $\{\lambda_k\}_{k=1}^J$ satisfying (2.19) and where the random variables on the left hand side of the last equality are independent. In particular, it follows that $S_1 \geq S^*_{l_0} + S^*_{l_1}$ for any distinct $l_0, l_1 \in \{1-(\lambda_1-\lambda_j), \lambda_k - \lambda_{k+1}\}_{k=1}^{j-1}$. Now, observe that each expectation in (2.32) can be written as $\mathbb{E}\left[(S^*_{l_1}S^*_{l_0})^M S_1^{-M-\frac{d}{2}}\right]$, and these expectations satisfy

$$\mathbb{E}\left[\frac{(S^*_{l_1}S^*_{l_0})^M}{S_1^{M+\frac{d}{2}}}\right] \leq \mathbb{E}\left[\frac{(S^*_{l_1}S^*_{l_0})^M}{(S^*_{l_1}+S^*_{l_0})^{M+\frac{d}{2}}}\right].$$

Lemma 2.5.1 *Let $j \geq 2$ and $\{\lambda_k\}_{k=1}^j$ satisfying (2.19). Let l_0, l_1 be two distinct numbers in $\{1-(\lambda_1-\lambda_j), \lambda_k - \lambda_{k+1}\}_{k=1}^{j-1}$. Then,*

$$0 \leq \mathbb{E}\left[\frac{(S^*_{l_0}S^*_{l_1})^M}{(S^*_{l_0}+S^*_{l_1})^{M+d/2}}\right] < \infty,$$

provided that $M/2 - d/4 < \alpha/2$.

In particular, when

i) M=1, if $\frac{2-d}{2} < \alpha$.

ii) M=2, if $\frac{4-d}{2} < \alpha$.

Proof Because of the inequality $2(ab)^{1/2} \leq a+b$, for any $a, b \geq 0$, we have that

$$\mathbb{E}\left[\frac{(S^*_{l_0}S^*_{l_1})^M}{(S^*_{l_0}+S^*_{l_1})^{M+d/2}}\right] \leq 2^{-M-d/2}\mathbb{E}\left[\frac{(S^*_{l_0}S^*_{l_1})^M}{(S^*_{l_0}S^*_{l_1})^{M/2+d/4}}\right].$$

Now, recall that $S^*_{l_0}$ and $S^*_{l_1}$ are independent and $S^*_{l_i} \stackrel{\mathcal{D}}{=} l_i^{2/\alpha} S_{1,\alpha/2}$. Therefore,

$$\mathbb{E}\left[\frac{(S^*_{l_0}S^*_{l_1})^M}{(S^*_{l_0}S^*_{l_1})^{M/2+d/4}}\right] = (l_0 l_1)^{\frac{2}{\alpha}\left\{\frac{M}{2}-\frac{d}{4}\right\}} \left(\mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{\frac{M}{2}-\frac{d}{4}}\right]\right)^2. \tag{2.33}$$

The result follows from the inequality $M/2 - d/4 < \alpha/2$ which guarantees the finiteness of the last expectation. ■

As an application of Lemma 2.5.1, we have

Corollary 2.5.1 *Assume $j \geq 2$.*

(i) For $\frac{1}{2} < \alpha < 2$ and $d = 1$, we have

$$\mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-1/2} L_j^{(\alpha)}(\lambda, \theta)\right] < \infty.$$

(ii) For $d = 1$ and $\frac{3}{2} < \alpha < 2$, $d = 2$ and $1 < \alpha < 2$, $d = 3$ and $\frac{1}{2} < \alpha < 2$, we have

$$\mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2} \left\{L_j^{(\alpha)}(\lambda, \theta)\right\}^2\right] < \infty.$$

The following is a version of Lemma 2.4.1 for dimension $1, 2,$ and 3 where the condition $M < \frac{d+\alpha}{2}$ is replaced by $M/2 - d/4 < \alpha/2$.

Lemma 2.5.2

(i) For d=1, and M=1, we have for all $\frac{1}{2} < \alpha < 2$ and $j \geq 2$ that

$$\frac{T_1(j,t)}{(2\pi)^j p_t^{(\alpha)}(0)} = C_{0,j}^{(\alpha)}(V) - t^{\frac{2}{\alpha}} R_{j,1}^{(1)}(t).$$

(ii) For M=2 and $j \geq 2$, we have

$$\frac{T_d(j,t)}{(2\pi)^{dj} p_t^{(\alpha)}(0)} = \sum_{n=0}^{1} (-1)^n C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}} + t^{\frac{4}{\alpha}} R_{j,d}^{(2)}(t),$$

when $d = 1$ and $\frac{3}{2} < \alpha < 2$, $d = 2$ and $1 < \alpha < 2$, or $d = 3$ and $\frac{1}{2} < \alpha < 2$,

where the remainders $R_{j,d}^{(2)}(t)$ are bounded in absolute value by a constant for all $t \geq 0$, according to Corollary 2.5.1 and the fact that $M/2 - d/4 > -1$ for all M, d as stated above.

Proof We start by recalling that for any $j \geq 2$

$$I_j := \{\lambda = (\lambda_1, ..., \lambda_j) : 0 < \lambda_j < \lambda_{j-1} < ... < \lambda_1 < 1\}.$$

Next, under the notation given in Lemma 2.4.1 and because of (2.31), we have that $\left|C_{1,j}^{(\alpha)}(V)\right|$ and $\left|R_{j,d}^{(M)}(t)\right|$, $M=1,2$ are bounded by

$$\int_{I_j}\int_{\mathbb{R}^{(j-1)d}}\mathbb{E}\left[S_1^{-d/2}\left\{L_j^{(\alpha)}(\lambda,\theta)\right\}^M\right]\left|\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)\right|d\lambda_j d\lambda_i d\theta_i.$$

This last expression is also bounded, based on (2.33) and the facts given at the beginning of this section, up to some positive constant by terms of the form

$$\int_{I_j}(l_0l_1)^{\frac{2}{\alpha}(\frac{M}{2}-\frac{d}{4})}d\lambda_j d\lambda_i\left(\mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{\frac{M}{2}-\frac{d}{4}}\right]\right)^2\int_{\mathbb{R}^{(j-1)d}}\sum_{k=1}^{j-1}|\gamma_k|^{2M}\left|\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)\right|d\theta_i$$

where $l_0=l_0(\lambda)$ and $l_1=l_1(\lambda)$ are two distinct numbers in $\{1-(\lambda_1-\lambda_j),\lambda_k-\lambda_{k+1}\}_{k=1}^{j-1}$.

Now the term

$$\int_{I_j}(l_0l_1)^{\frac{2}{\alpha}(\frac{M}{2}-\frac{d}{4})}d\lambda_j d\lambda_i=\int_0^1\int_0^{\lambda_1}\dots\int_0^{\lambda_{j-1}}(l_0l_1)^{\frac{2}{\alpha}(\frac{M}{2}-\frac{d}{4})}d\lambda_j\dots d\lambda_1$$

is clearly finite when $M/2-d/4\geq 0$, which is the case for $M=1,2$, $d=1,2$ and $M=2$, $d=3$. But, when $M=1$ and $d=3$, we obtain $-1<M/2-d/4=1/2-3/4=-1/4$ and this case deserves special attention.

We observe that for all $1\geq\lambda_i>\lambda_{k+1}$ we have

$$\int_0^{\lambda_i}\lambda_{k+1}(\lambda_i-\lambda_{k+1})^{-\frac{1}{2\alpha}}d\lambda_{k+1}\leq\int_0^{\lambda_i}(\lambda_i-\lambda_{k+1})^{-\frac{1}{2\alpha}}d\lambda_{k+1}\leq\int_0^1 w^{-\frac{1}{2\alpha}}dw=\frac{2\alpha}{2\alpha-1}$$

and

$$\int_0^{\lambda_{j-1}}(1-\lambda_1+\lambda_j)^{-\frac{1}{2\alpha}}(\lambda_{j-1}-\lambda_j)^{-\frac{1}{2\alpha}}d\lambda_j\leq(1-\lambda_1)^{-\frac{1}{2\alpha}}\frac{2\alpha}{2\alpha-1}$$

provided that $\alpha>\frac{1}{2}$. Then, it is not difficult to see that

$$\int_{I_j}(l_0l_1)^{-\frac{1}{2\alpha}}d\lambda_j d\lambda_i\leq\left(\frac{2\alpha}{2\alpha-1}\right)^2.$$

■

2.6 Proof of Theorem 1.2.1 and Theorem 1.2.2.

Proof of part (a): Recall that, for $J\geq 2$, we have defined

$$R_{J+1}(t) = \frac{r_{J+1}(t)}{(2\pi)^{(J+2)d} p_t^{(\alpha)}(0)}$$

and also showed, according to Proposition 2.2.1, that this remainder is bounded by a constant for $0 \leq t < 1$. Also $M < \frac{d+\alpha}{2}$ implies that $R_{2,d}^{(M)}(t), \ldots, R_{J,d}^{(M)}(t)$ are, according to Lemma 2.4.1, bounded by a constant for $0 < t$.

Next, (a) follows by substituting the terms found in Lemma 2.4.1 into (2.9). More precisely,

$$\mathcal{T}_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{j=2}^{J} \sum_{n=0}^{M-1} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j}$$
$$(-t)^{J+1} R_{J+1}(t) + \sum_{j=2}^{J} (-1)^{j+M} t^{j+\frac{2M}{\alpha}} R_{j,d}^{(M)}(t). \tag{2.34}$$

Therefore, by defining

$$R_{J+1}^{(\alpha)}(t) = (-t)^{J+1} R_{J+1}(t) + \sum_{j=2}^{J} (-1)^{j+M} t^{j+\frac{2M}{\alpha}} R_{j,d}^{(M)}(t), \tag{2.35}$$

(which is the sum of all the remainders obtained by applying the expansion found in (2.9) together with those provided by the Taylor expansion of the exponential function), we conclude due to the facts given at the start of this section, that there exists a constant $C > 0$ such that

$$\left| R_{J+1}^{(\alpha)}(t) \right| \leq C t^{\Phi_{J+1}^{(\alpha)}(M)}, \tag{2.36}$$

for $0 \leq t < 1$, where $\Phi_{J+1}^{(\alpha)}(M) = \min\left\{J+1, 2+\frac{2M}{\alpha}\right\}$ and this completes the proof of part (a) in Theorem 1.2.1.

By taking into consideration the equations (2.34), (2.35) and (2.36), we proceed at this point to give the proof of Theorem 1.2.2 which follows by noticing that

$$\sum_{j=2}^{J} \sum_{n=0}^{M-1} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j} = \sum_{\substack{\frac{2n}{\alpha}+j < \Phi_{J+1}^{(\alpha)}(M) \\ 2 \leq j \leq J,\ 0 \leq n \leq M-1}} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j}$$
$$+ \sum_{\substack{\frac{2n}{\alpha}+j \geq \Phi_{J+1}^{(\alpha)}(M) \\ 2 \leq j \leq J,\ 0 \leq n \leq M-1}} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j}$$

and

$$t^{\frac{2n}{\alpha}+j} = \mathcal{O}(t^{\Phi^{(\alpha)}_{J+1}(M)}),$$

as $t \downarrow 0$, provided that $\frac{2n}{\alpha} + j \geq \Phi^{(\alpha)}_{J+1}(M)$.

Proof of part (b): In the proof, we will appeal to the elementary properties of weak convergence that can be found in [17]. Since we are only interested in α's close to 2, it suffices to prove that if $n, d \geq 1$ are positive integers satisfying $n \leq \frac{1+d}{2}$, then for all $j \geq 2$ we have

$$\lim_{r\to\infty} C^{(\alpha_r)}_{n,j}(V) = C^{(2)}_{n,j}(V),$$

for any sequence $\{\alpha_r\}_{r\in\mathbb{N}}$ satisfying $\frac{3}{2} < \alpha_r < 2$ and $\alpha_r \uparrow 2$.

To prove the last statement, we need to introduce some notation. Recall that $I_j \subset \mathbb{R}^j$ has been defined as

$$I_j = \{\lambda = (\lambda_1, ..., \lambda_j) : 0 < \lambda_j < \lambda_{j-1} < ... < \lambda_1 < 1\}.$$

We also set

$$X_r(\lambda) = (S^*_{1-(\lambda_1-\lambda_j),\frac{\alpha_r}{2}}, S^*_{\lambda_{j-1}-\lambda_j,\frac{\alpha_r}{2}}, ..., S^*_{\lambda_1-\lambda_2,\frac{\alpha_r}{2}}),$$

$$X(\lambda) = (1 - (\lambda_1 - \lambda_j), \lambda_{j-1} - \lambda_j, ..., \lambda_1 - \lambda_2),$$

$$\theta = (\theta_1, ..., \theta_{j-1}) \in \mathbb{R}^{(j-1)d},$$

$$\gamma_k = \sum_{i=1}^{k} \theta_i \in \mathbb{R}^d,\ k \in \{1, ..., j-1\},$$

$$h_{n,d}(x_0, x_1, ..., x_j) = \begin{cases} \dfrac{\left(x_0 \sum\limits_{k=1}^{j-1} x_k|\gamma_k|^2 + \sum\limits_{m=1}^{j-2} \sum\limits_{s=m+1}^{j-1} x_m x_s|\gamma_m - \gamma_s|\right)^n}{\left(\sum\limits_{k=0}^{j-1} x_k\right)^{n+\frac{d}{2}}}, & \text{for } x_0 > 0, ..., x_j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

With this notation,

$$h_{n,d}(X_r(\lambda)) = S_{1,\frac{\alpha_r}{2}}^{-d/2} \left\{ L_j^{(\alpha_r)}(\lambda,\theta) \right\}^n,$$

$$h_{n,d}(X(\lambda)) = \left\{ L_j^{(2)}(\lambda,\theta) \right\}^n.$$

We now divide our proof into 5 steps.

Step 1. $X_r(\lambda) \Rightarrow X(\lambda)$.

To see this, we recall that for $t \in \{1-(\lambda_1-\lambda_j), \lambda_k - \lambda_{k+1}\}_{k=1}^{j-1}$ and $\lambda > 0$,

$$\mathbb{E}\left[e^{-\lambda S^*_{t,\alpha_r/2}} \right] = e^{-t\lambda^{\alpha_r/2}}.$$

This expectation corresponds to the Laplace transform of $S^*_{t,\alpha_r/2}$ and uniquely determines its distribution. We conclude that

$$\lim_{r\to+\infty} \mathbb{E}\left[e^{-\lambda S^*_{t,\alpha_r/2}} \right] = e^{-t\lambda}.$$

Thus, $S^*_{t,\alpha_r/2} \Rightarrow t$.

On the other hand, due to the independence of $S^*_{1-(\lambda_1-\lambda_j),\alpha_r/2}$, $S^*_{\lambda_{j-1}-\lambda_j,\alpha_r/2}, \dots,$ $S^*_{\lambda_1-\lambda_2,\alpha_r/2}$, the fact that the real and imaginary part of $f(z) = e^{iz}$ are bounded and continuous functions over $\mathbb{R}$ together with Theorem 25.8 in [17], we also obtain for every $v \in \mathbb{R}^{j+1}$ that

$$\mathbb{E}\left[e^{-i<v,X_r(\lambda)>} \right] \to \mathbb{E}\left[e^{-i<v,X(\lambda)>} \right].$$

as $r \to \infty$. Therefore, the result follows by appealing to Theorem 29.4 in [17].

Step 2. $h_{n,d}(X_r(\lambda)) \Rightarrow h_{n,d}(X(\lambda))$.

We note that each component of the vector $X(\lambda)$ is positive. Thus, by our definition of $h_{n,d}$, it is clear that $X(\lambda)$ belongs to the set of continuity points of $h_{n,d}$. Then, $\mathbb{P}(X(\lambda) \in D_{h_{n,d}}) = 0$ and the result follows from Theorem 29.2 in [17].

Step 3. $\{h_{n,d}(X_r)\}_{r\in\mathbb{N}}$ is uniformly integrable.

We shall show that there exist a $p > 1$ and a function $C(n,d,p,\theta) > 0$ such that

$$\sup_r \mathbb{E}\left[\{h_{n,d}(X_r(\lambda))\}^p\right] \leq C(n,d,p,\theta). \tag{2.37}$$

To do this, we consider two cases to determine a proper p.

Case 1. Suppose $n - \frac{d}{2} \leq 0$. In the proof of Lemma 2.4.1, we proved that

$$h_{n,d}(X_r(\lambda)) \leq S_{1,\frac{\alpha_r}{2}}^{n-\frac{d}{2}} \left(\sum_{k=1}^{j-1} |\gamma_k|^2 \right)^n .$$

To prove (2.37), it suffices to show that $\sup_r \mathbb{E}\left[S_{1,\frac{\alpha_r}{2}}^{p\left(n-\frac{d}{2}\right)} \right]$ is bounded for some $p > 1$. Recall that

$$0 < \mathbb{E}\left[S_{1,\frac{\alpha_r}{2}}^{p\left(n-\frac{d}{2}\right)} \right] = \frac{\Gamma(1 - \frac{2p}{\alpha_r}\left(n - \frac{d}{2}\right))}{\Gamma(1 - p\left(n - \frac{d}{2}\right))} < \infty, \tag{2.38}$$

provided that

$$p\left(n - \frac{d}{2}\right) < \frac{\alpha_r}{2} < 1. \tag{2.39}$$

If $n - \frac{d}{2} = 0$, we take $C(n,d,p,\theta) = \left(\sum_{k=1}^{j-1} |\gamma_k|^2\right)^{pn}$ and any $p > 1$, since clearly in this case $\mathbb{E}\left[S_{1,\frac{\alpha_r}{2}}^{p\left(n-\frac{d}{2}\right)} \right] = 1$.

If $n - \frac{d}{2} < 0$, it is clear that (2.39) is satisfied for any $p > 1$. But, we wish to pick p so that (2.38) is uniformly bounded in r. To do this in a suitable manner, we require the following well-known property of the the gamma function(see [50]). There exists $\mu_0 \in (1,2)$ such that $\Gamma(z)$ is decreasing on $(0, \mu_0]$ and increasing over $(\mu_0, +\infty)$. Now, from the last property and the fact that each α_r satisfies $1 < \frac{2}{\alpha_r} < 2$, it follows that for any $p > \max\left\{1, \frac{1}{\frac{d}{2}-n}\right\}$,

$$\Gamma(2) \leq \Gamma\left(1 + \frac{2p}{\alpha_r}\left(\frac{d}{2} - n\right)\right) \leq \Gamma\left(1 + 2p\left(\frac{d}{2} - n\right)\right). \tag{2.40}$$

Therefore,

$$\sup_r \mathbb{E}\left[\{h_{n,d}(X_r(\lambda))\}^p\right] \leq \frac{\Gamma(1 - 2p\left(n - \frac{d}{2}\right))}{\Gamma(1 - p\left(n - \frac{d}{2}\right))} \left(\sum_{k=1}^{j-1} |\gamma_k|^2 \right)^{pn} = C(n,d,p,\theta).$$

Case 2. Suppose $n - \frac{d}{2} > 0$. Because of the inequality $0 < n - \frac{d}{2} \leq \frac{1}{2}$, which is equivalent to $0 < 2n - d \leq 1$, we conclude that $d = 2n - 1$, since d and n are positive integers. Therefore $n-\frac{d}{2}=\frac{1}{2}$. In this case, from the tools we developed in §2.5 we obtain that $0\leq h_{n,d}(X_r(\lambda))$ is bounded, up to some positive constant, by a finite sum containing terms of the form

$$\sum_{k=1}^{j-1}|\gamma_k|^{2n}\left(S^*_{l_0,\alpha_r/2}S^*_{l_1,\alpha_r/2}\right)^{\left(\frac{n}{2}-\frac{d}{4}\right)}, \tag{2.41}$$

for any two distinct numbers l_0,l_1 in $\{1-(\lambda_1-\lambda_j),\lambda_k-\lambda_{k+1}\}_{k=1}^{j-1}$.

Next, due to the fact that $S^*_{l_i,\frac{\alpha_r}{2}}$, $i=0,1$ are independent and have law $l_i^{\frac{2}{\alpha_r}}S_{1,\alpha_r/2}$, $0<l_i<1$ and $n-\frac{d}{2}=\frac{1}{2}$, we conclude that $\mathbb{E}\left[h_{n,d}(X_r(\lambda))^2\right]$ is bounded up to some positive constant by

$$\left(\mathbb{E}\left[S^{\frac{1}{2}}_{1,\frac{\alpha_r}{2}}\right]\sum_{k=1}^{j-1}|\gamma_k|^{2n}\right)^2.$$

We also know that

$$\mathbb{E}\left[S^{1/2}_{1,\alpha_r/2}\right]=\frac{\Gamma(1-\frac{1}{\alpha_r})}{\Gamma(\frac{1}{2})}.$$

On the other hand, the function $\Gamma(z)$ is decreasing over (0,1). Next, we observe that each α_r satisfies

$$\frac{1}{3}\leq 1-\frac{1}{\alpha_r}\leq\frac{1}{2},$$

which yields

$$\mathbb{E}\left[S^{1/2}_{1,\alpha_r/2}\right]\leq\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{2})}.$$

For this,

$$C(n,d,2,\theta)=C\left(\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{2})}\sum_{k=1}^{j-1}|\gamma_k|^{2n}\right)^2,$$

for some $C>0$.

Step 4. $\lim\limits_{r\to+\infty}\mathbb{E}\left[h_{n,d}(X_r(\lambda))\right]=\mathbb{E}\left[h_{n,d}(X)\right]$.

This is a consequence of Steps 2, 3 and Theorem 25.12 in [17].

Step 5. Notice that by the Hölder's inequality and Step 3, we have proved that for some $p>1$,

$$\sup_r\mathbb{E}\left[h_{n,d}(X_r(\lambda))\right]\leq\sup_r\left(\mathbb{E}\left[h_{n,d}(X_r(\lambda))^p\right]\right)^{\frac{1}{p}}\leq\left\{C(n,d,p,\theta)\right\}^{\frac{1}{p}}, \tag{2.42}$$

where $\{C(n,d,p,\theta)\}^{\frac{1}{p}} > 0$ is, indeed, a polynomial function in the variable θ. Using the fact $V \in \mathcal{S}(\mathbb{R}^d)$ and the bounds in Step 3, we have

$$\int_{I_j}\int_{\mathbb{R}^{(j-1)d}} \{C(n,d,p,\theta)\}^{\frac{1}{p}} \left|\widehat{V}(-\sum_{i=1}^{j-1}\theta_i)\prod_{i=1}^{j-1}\widehat{V}(\theta_i)\right| d\theta_i d\lambda_i d\lambda_j < +\infty.$$

Therefore, by (2.42), Step 4, and the dominated convergence theorem, we arrive at the desired result and this completes the proof of part (b).

2.7 Explicit form of some coefficients.

In this section, we compute some coefficients explicitly and again show their finiteness by applying some basic inequalities arising from Lemmas 2.4.1 and 2.5.1.

We start with the simplest case $n = 0$. Observe that an inducting argument together with Inversion formula (2.2) yield

$$\begin{aligned} C_{0,j+1}^{(\alpha)}(V) &= \frac{C_{d,\alpha}\mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2}\right]}{(2\pi)^{(j+1)d}} \int_0^1\int_0^{\lambda_1}\cdots\int_0^{\lambda_j}\int_{\mathbb{R}^{jd}} \widehat{V}(-\sum_{i=1}^{j}\theta_i)\prod_{i=1}^{j}\widehat{V}(\theta_i)d\theta_i d\lambda_i d\lambda_j \\ &= \frac{1}{(j+1)!}\int_{\mathbb{R}^d} V^{j+1}(\theta)d\theta, \end{aligned}$$

for every $j \geq 1$. Here, we have also used that $(4\pi)^{d/2}p_1^{(\alpha)}(0) = \mathbb{E}\left[S_{1,\frac{\alpha}{2}}^{-d/2}\right]$ and $C_{d,\alpha} = \frac{\pi^{d/2}}{p_1^{(\alpha)}(0)}$.

The following Lemma will be useful to prove that the constants appearing in Corollary 1.2.1 are strictly positive. Part of the following proof can be found in [37].

Lemma 2.7.1 *Given $0 < \alpha < 2$, there exists $N_\alpha > 1$ such that*

$$\frac{1-e^{-v_\alpha}}{2} \leq \mathbb{P}(1 < S_{1,\frac{\alpha}{2}} < N_\alpha),$$

where $v_\alpha = (2-\alpha)\alpha^{\frac{\alpha}{2-\alpha}}2^{\frac{-2}{2-\alpha}}$.

Proof Let $a > 0$ be fixed and observe that $S_{1,\frac{\alpha}{2}} \leq a$ if and only if $e^{-\lambda S_{1,\frac{\alpha}{2}}} \geq e^{-\lambda a}$, for any $\lambda > 0$.

Therefore, Chebyshev inequality tells us that

$$\mathbb{P}(S_{1,\frac{\alpha}{2}} \le a) \le \inf_{\lambda>0} e^{\left(a\lambda - \lambda^{\frac{\alpha}{2}}\right)} = e^{\left(-a^{\frac{\alpha}{\alpha-2}} v_\alpha\right)}.$$

On the other hand, $\lim_{n\to+\infty} \mathbb{P}(S_{1,\frac{\alpha}{2}} < n) = 1$ implies that given $\epsilon > 0$, there exists a positive integer N such that

$$1 - \mathbb{P}(S_{1,\frac{\alpha}{2}} < n) \le \epsilon, \quad \text{for all } n \ge N.$$

It follows then that for $\epsilon = \frac{1-e^{-v_\alpha}}{2}$, there exists $N_\alpha > 1$ such that

$$1 - \mathbb{P}(S_{1,\frac{\alpha}{2}} < N_\alpha) \le \frac{1-e^{-v_\alpha}}{2} \quad \text{or } \mathbb{P}(S_{1,\frac{\alpha}{2}} < N_\alpha) \ge \frac{1+e^{-v_\alpha}}{2}.$$

Now use the above facts with $a = 1$, to obtain that

$$\begin{aligned} \mathbb{P}(1 < S_{1,\frac{\alpha}{2}} < N_\alpha) &= \mathbb{P}(S_{1,\frac{\alpha}{2}} < N_\alpha) - \mathbb{P}(S_{1,\frac{\alpha}{2}} \le 1) \\ &\ge \frac{1+e^{-v_\alpha}}{2} - e^{-v_\alpha} \\ &= \frac{1-e^{-v_\alpha}}{2}, \end{aligned}$$

as desired. ■

Remark 2.7.1 *Before proceeding, we give an explicit expression for N_α when $\alpha = 1$. Observe that $v_1 = -\frac{1}{4}$. The $1/2$–subordinator S can be expressed as the first hitting time for the standard one-dimensional Brownian motion $\{W_t\}_{t\ge 0}$. More precisely,*

$$S_t = \inf\left\{ s > 0 : W_s = \frac{t}{\sqrt{2}} \right\}.$$

It is also known (See [1, pp 23-24] for details) that its density is given by

$$\eta_t^{(1/2)}(s) = \frac{t}{2\sqrt{\pi}} s^{-3/2} e^{-t^2/4s}.$$

Therefore, it is not difficult to see that for any $N > 1$,

$$\mathbb{P}(1 < S_{1,\frac{1}{2}} < N) = \frac{1}{2\sqrt{\pi}} \int_1^N s^{-3/2} e^{-1/4s} ds \ge \frac{e^{-1/4}}{2\sqrt{\pi}} \int_1^N s^{-3/2} ds = \frac{e^{-1/4}}{\sqrt{\pi}} (1 - N^{-1/2}).$$

We can take then

$$\frac{e^{-1/4}}{\sqrt{\pi}} (1 - N_1^{-1/2}) = \frac{1-e^{-1/4}}{2}$$

or equivalently, $N_1 = \left\{ 1 - \frac{\sqrt{\pi}}{2}(e^{1/4} - 1) \right\}^{-2}$ which is approximately 1.786.

Let us now consider the case $n = 1$ and $j = 2$. We recall that

$$S^*_{1-(\lambda_1-\lambda_2)} + S^*_{\lambda_1-\lambda_2} = S_1, \tag{2.43}$$

provided $0 < \lambda_2 < \lambda_1 < 1$. In addition, $S^*_{1-(\lambda_1-\lambda_2)}$ and $S^*_{\lambda_1-\lambda_2}$ are independent random variables. Then, it follows by Lemma 2.4.1 that

$$C^{(\alpha)}_{1,2}(V) = \frac{C_{d,\alpha}K_1(d,\alpha)}{(2\pi)^d}\langle -\Delta V, V\rangle = \frac{C_{d,\alpha}K_1(d,\alpha)}{(2\pi)^d}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2\, d\theta, \tag{2.44}$$

where we have replaced S_1 by the left hand side of (2.43) to obtain that

$$\begin{aligned} K_1(d,\alpha) &= \int_0^1\int_0^{\lambda_1} \mathbb{E}\left[\frac{S^*_{1-(\lambda_1-\lambda_2)}S^*_{\lambda_1-\lambda_2}}{(S^*_{1-(\lambda_1-\lambda_2)} + S^*_{\lambda_1-\lambda_2})^{1+\frac{d}{2}}}\right] d\lambda_2 d\lambda_1 \\ &= \int_0^1\int_0^{\lambda_1} \mathbb{E}\left[\frac{S^*_{1-w}S^*_{w}}{(S^*_{1-w} + S^*_{w})^{1+\frac{d}{2}}}\right] dw d\lambda_1. \end{aligned}$$

We now claim that $K_1(d,\alpha)$ is both finite and strictly positive when either $d = 1$ and $\frac{1}{2} < \alpha < 2$, or $d \geq 2$ and $0 < \alpha < 2$ as follows.

We start with $d = 1$ and $\frac{1}{2} < \alpha < 2$. By Lemma 2.5.1, we obtain in this case that

$$0 \leq K_1(1,\alpha) \leq 2^{-3/2}\mathbb{E}\left[S^{1/4}_{1,\alpha/2}\right]\int_0^1\int_0^{\lambda_1} \{(1-w)w\}^{\frac{1}{2\alpha}}\, dw d\lambda_1.$$

On the other hand, when $d \geq 2$ we have

$$0 \leq K_1(d,\alpha) \leq \frac{1}{2}\int_0^1\int_0^{\lambda_1} \mathbb{E}\left[(S^*_{1-w} + S^*_{w})^{1-d/2}\right] dw d\lambda_1 = \frac{\mathbb{E}\left[S^{1-d/2}_{1,\alpha/2}\right]}{4} \tag{2.45}$$

where we have used the basic inequality

$$\frac{ab}{(a+b)^{d/2+1}} = \frac{ab}{(a+b)^{d/2-1}(a+b)^2} \leq \frac{1}{2}(a+b)^{1-d/2},$$

valid for all $a, b > 0$. The expectation in (2.45) is finite for all α since $1 - \frac{d}{2} \leq 0 < \frac{\alpha}{2}$. Therefore $K_1(d,\alpha)$ is finite in the cases stated above.

Next, we prove that $K_1(d,\alpha)$ is strictly positive. We note that (1.3) implies that $S_t \overset{\mathcal{D}}{=} t^{\frac{2}{\alpha}} S_1$. Therefore, we can write

$$S^*_{1-w} \overset{\mathcal{D}}{=} (1-w)^{\frac{2}{\alpha}} X_1, \quad S^*_w \overset{\mathcal{D}}{=} w^{\frac{2}{\alpha}} X_2,$$

where X_1, X_2 are independent copies of S_1. That is, $X_1 \overset{\mathcal{D}}{=} S_1 \overset{\mathcal{D}}{=} X_2$. Thus, for any $0 < w < 1$, we have

$$\begin{aligned}\mathbb{E}\left[\frac{S^*_{1-w}S^*_w}{(S^*_{1-w}+S^*_w)^{1+\frac{d}{2}}}\right] &= \mathbb{E}\left[\frac{(1-w)^{\frac{2}{\alpha}}w^{2/\alpha}X_1X_2}{((1-w)^{2/\alpha}X_1+w^{2/\alpha}X_2)^{1+\frac{d}{2}}}\right] \\ &\geq (1-w)^{\frac{2}{\alpha}}w^{\frac{2}{\alpha}}\mathbb{E}\left[\frac{X_1X_2}{(X_1+X_2)^{1+\frac{d}{2}}}; 1 < X_1, X_2 \ \leq N_\alpha\right] \\ &\geq \frac{(1-w)^{\frac{2}{\alpha}}w^{\frac{2}{\alpha}}}{(2N_\alpha)^{1+\frac{d}{2}}}\mathbb{P}(1 < S_{1,\frac{\alpha}{2}} \leq N_\alpha)^2.\end{aligned}$$

From Lemma 2.7.1 and the last inequality, we conclude that

$$K_1(d,\alpha) \geq \frac{(1-e^{-v_\alpha})^2}{4(2N_\alpha)^{1+\frac{d}{2}}}\int_0^1\int_0^{\lambda_1}(1-w)^{\frac{2}{\alpha}}w^{\frac{2}{\alpha}}dw\,d\lambda_1 > 0.$$

Similarly, for either all $\alpha \in (1,2)$ and $d \geq 2$, or for $d \geq 4$ and $\alpha \in (0,2)$, we have

$$C^{(\alpha)}_{2,2}(V) = \frac{C_{d,\alpha}K_2(d,\alpha)}{2(2\pi)^d}\int_{\mathbb{R}^d}|\Delta V(\theta)|^2\,d\theta,$$

where

$$K_2(d,\alpha) = \int_0^1\int_0^{\lambda_1}\mathbb{E}\left[\frac{(S^*_{1-(\lambda_1-\lambda_2)}S^*_{\lambda_1-\lambda_2})^2}{(S^*_{1-(\lambda_1-\lambda_2)}+S^*_{\lambda_1-\lambda_2})^{2+\frac{d}{2}}}\right]d\lambda_2\,d\lambda_1.$$

By applying the same argument as above, we have the following

(i) $\alpha \in (1,2)$ and $d \geq 3$ or for $d \geq 4$ and $\alpha \in (0,2)$, we obtain

$$0 < \frac{(1-e^{-v_\alpha})^2}{4(2N_\alpha)^{2+\frac{d}{2}}}\int_0^1\int_0^{\lambda_1}(1-w)^{\frac{4}{\alpha}}w^{\frac{4}{\alpha}}dw\,d\lambda_1 \leq K_2(d,\alpha) \leq \frac{\mathbb{E}\left[S^{2-d/2}_{1,\frac{\alpha}{2}}\right]}{12}.$$

(ii) $\alpha \in (1,2)$ and $d = 2$, we have the same lower bound as in (i), but by Lemma 2.5.2 we obtain

$$K_2(2,\alpha) \leq 2^{-3}\mathbb{E}\left[S^{1/2}_{1,\alpha/2}\right]\int_0^1\int_0^{\lambda_1}\{w(1-w)\}^{\frac{1}{\alpha}}\,dwd\lambda_1.$$

Likewise, it is not hard to prove that

$$C^{(\alpha)}_{1,3}(V) = \frac{C_{d,\alpha}K_3(d,\alpha)}{(2\pi)^d}\int_{\mathbb{R}^d}V(\theta)\,|\nabla V(\theta)|^2\,d\theta,$$

where

$$K_3(d,\alpha)=\int_0^1\int_0^{\lambda_1}\int_0^{\lambda_2}\mathbb{E}\left[\frac{S^*_{1-(\lambda_1-\lambda_3)}S^*_{\lambda_1-\lambda_2}+S^*_{1-(\lambda_1-\lambda_3)}S^*_{\lambda_2-\lambda_3}+S^*_{\lambda_1-\lambda_2}S^*_{\lambda_2-\lambda_3}}{(S^*_{1-(\lambda_1-\lambda_3)}+S^*_{\lambda_1-\lambda_2}+S^*_{\lambda_2-\lambda_3})^{1+d/2}}\right]d\lambda_3 d\lambda_2 d\lambda_1,$$

is positive and finite provided that either $d \geq 2$ and $0 < \alpha < 2$ or $d = 1$ and $\frac{1}{2} < \alpha < 2$.

For the rest of the paper, we will use the following notation for the constants given above,

$$\mathcal{M}_{d,\alpha}=\frac{C_{d,\alpha}K_3(d,\alpha)}{(2\pi)^d} \text{ and } \mathcal{N}_{d,\alpha}=\frac{C_{d,\alpha}K_2(d,\alpha)}{2(2\pi)^d}.$$

Remark 2.7.2 *Based on the computations in [3] and part (b) of Theorem 1.2.1, we have under the conditions stated above that*

$$\lim_{\alpha\uparrow 2}\mathcal{N}_{d,\alpha}=\frac{1}{120} \text{ and } \lim_{\alpha\uparrow 2}\mathcal{M}_{d,\alpha}=\frac{1}{12}.$$

2.8 Proof of Corollary 1.2.1.

The proof uses a combination of Theorem 1.2.1 and Lemma 2.5.2.

Case M=1.

1) When $d \geq 2$, we invoke Theorem 1.2.1 with $J = 3$. We have in this case $1-\frac{d}{2} \leq 0 < \frac{\alpha}{2}$. Therefore, we can consider, according to Lemma 2.4.1, any α on the right hand side of the next expression,

$$\mathcal{T}_V^{(\alpha)}(t)+t\int_{\mathbb{R}^d}V(\theta)d\theta-\frac{t^2}{2}\int_{\mathbb{R}^d}V^2(\theta)d\theta+\frac{t^3}{3!}\int_{\mathbb{R}^d}V^3(\theta)d\theta=R_4^{(\alpha)}(t), \quad (2.46)$$

where

$$R_4^{(\alpha)}(t)=\mathcal{O}(t^{\phi_4^{(\alpha)}(1)}),$$

as $t \downarrow 0$, and

$$\phi_4^{(\alpha)}(1)=\min\left\{4, 2+\frac{2}{\alpha}\right\}.$$

Hence, (iii) follows by noticing that

$$2+\frac{2}{\alpha}<4<3+\frac{2}{\alpha}, \quad when\ \alpha\in(1,2)$$
$$4\leq 2+\frac{2}{\alpha}<3+\frac{2}{\alpha}, \quad when\ \alpha\in(0,1].$$

2) For the case $d = 1$, we use Lemma 2.5.2 which guarantees that (2.46) is still true for $\frac{1}{2} < \alpha < 1$. Thus (i) holds.

Case M=2. We apply Theorem 1.2.1 with $J = 4$ and $2 - \frac{d}{2} < \frac{\alpha}{2}$ to obtain

$$\begin{aligned}\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2}\int_{\mathbb{R}^d} V^2(\theta)d\theta + C_{1,2}^{(\alpha)}(V)t^{2+\frac{2}{\alpha}} + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta \\ - C_{1,3}^{(\alpha)}(V)t^{3+\frac{2}{\alpha}} - \frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta + C_{1,4}^{(\alpha)}(V)t^{4+\frac{2}{\alpha}} = R_5^{(\alpha)}(t),\end{aligned} \tag{2.47}$$

with

$$R_5^{(\alpha)}(t) = \mathcal{O}(t^{\phi_5^{(\alpha)}(2)}),$$

as $t \downarrow 0$, and

$$\phi_5^{(\alpha)}(2) = \min\left\{5, 2 + \frac{2\cdot 2}{\alpha}\right\}.$$

1) (iv), (v) and (vi) when $d \geq 4$.

We have $2 - \frac{d}{2} \leq 0 < \frac{\alpha}{2}$ and then any α can be considered. For $\alpha \in (0, 1]$, we have $2 \leq \frac{2}{\alpha}$, which implies that all the powers of t containing α are larger than 5 except $2 + \frac{2}{\alpha}$. Comparing $2 + \frac{2}{\alpha}$ with 5 yields (v) and (vi). Notice that (iv) also follows since all the power of t containing α in (2.47) are larger than 4 (simply use the fact that $1 < \frac{2}{\alpha}$) except $2 + \frac{2}{\alpha}$ which is less than 4 whenever $1 < \alpha < 2$.

In addition, by Lemma 2.5.2 we obtain that (2.47) remains true in the following cases

2) When $d = 3$ and $\frac{1}{2} < \alpha < 2$. Hence, (iv) and (v) holds by part 1).

3) When $d = 2$, $0 < \alpha < 1$. Then, (iv) also holds for $d = 2$.

4) When $d = 1$, $\frac{3}{2} < \alpha < 2$. Thus, (ii) holds.

This covers all the cases and completes the proof of Corollary 1.2.1.

2.9 Explicit expansion for $\alpha = 2/k$, $k \geq 2$ integer and α close to 2

In this section we want to provide to the reader a better insight of the Theorem 1.2.2 by providing the expansion formula of the trace when $\alpha = \frac{2}{k}$ with $k \geq 2$ an integer and for values of α near 2, of course, under the condition that $2M - d < \alpha$, which is equivalent to $2M - d \leq 0$ when $0 < \alpha \leq 1$. We also give examples as an application to the results given below. We refer to the reader to the end of §2.7 for the definition of the constants $\mathcal{L}_{d,\alpha}$, $\mathcal{M}_{d,\alpha}$ and $\mathcal{N}_{d,\alpha}$.

Theorem 2.9.1 *Let $\alpha = 1$ and $2M - d \leq 0$. Then, for any $2 \leq J \leq 2M$,*

$$\mathcal{T}_V^{(1)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \sum_{\ell=2}^{J} t^\ell \left(\sum_{\substack{2n+j=\ell,\\ j\geq 2}} (-1)^{n+j} C_{n,j}^{(1)}(V) \right) = \mathcal{O}(t^{J+1}), \tag{2.48}$$

as $t \downarrow 0$.

Proof We apply Theorem 1.2.2 with $\alpha = 1$ and $J + 1 \leq 2M + 1$, so that

$$\Phi_{J+1}^{(1)}(M) = \min\{J+1, 2M+2\} = J + 1.$$

Therefore, we obtain as $t \downarrow 0$ that

$$\mathcal{T}_V^{(1)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \sum_{\substack{j+2n\leq J,\\ 2\leq j\leq J,\ 0\leq n\leq M-1}} (-1)^{n+j} C_{n,j}^{(1)}(V) t^{2n+j} = \mathcal{O}(t^{J+1}).$$

Notice that for any $\ell \in \{2, ..., J\}$, the set

$$\{(j,n) \in \{2, ..., J\} \times \{0, ..., M-1\} : 2n + j = \ell\}$$

is not empty since $2 \leq 2n + j \leq 2(M-1) + J$. On the other hand, if $2n + j = \ell$ for some $\ell \in \{2, ..., J\}$ and $j \geq 2$, then

$$n = \frac{\ell - j}{2} \leq \frac{J-2}{2} \leq \frac{2M-2}{2} = M - 1.$$

Thus, we have shown

$$\{(j,n) \in \{2, ..., J\} \times \{0, ..., M-1\} : 2n + j \leq J\} = \bigcup_{\ell=2}^{J} \{(j,n) \in \mathbb{N} \times \mathbb{N} : j \geq 2, 2n + j = \ell\}.$$

Then, we conclude that

$$\sum_{\substack{j+2n\leq J,\\ 2\leq j\leq J,\ 0\leq n\leq M-1}} (-1)^{n+j} C_{n,j}^{(1)}(V) t^{2n+j} = \sum_{\ell=2}^{J} t^{\ell} \left(\sum_{\substack{2n+j=\ell,\\ j\geq 2}} (-1)^{n+j} C_{n,j}^{(1)}(V) \right).$$

■

Example 2.9.1 *When $M = 3$ and $d \geq 6$, we have $2 \cdot 3 - d \leq 0$. Then, Theorem 2.9.1 holds for any $2 \leq J \leq 6$. Therefore, for the particular case $J = 5$, we obtain*

$$\begin{aligned} \mathcal{T}_V^{(1)}(t) &+ t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta \\ &+\frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta - \frac{t^4}{4!}\left(\int_{\mathbb{R}^d} V^4(\theta)d\theta + 4!\mathcal{L}_{d,1}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta\right) \\ &+\frac{t^5}{5!}\left(\int_{\mathbb{R}^d} V^5(\theta)d\theta - 5!\mathcal{M}_{d,1}\int_{\mathbb{R}^d} V(\theta)\left|\nabla V(\theta)\right|^2 d\theta\right) = \mathcal{O}(t^6), \end{aligned} \tag{2.49}$$

as $t \downarrow 0$. Notice that part of this expansion is obtained by applying (v) of Corollary 1.2.1 to the specific case $\alpha = 1$.

By mimicking the proof for the case $\alpha = 1$, we conclude that

Theorem 2.9.2 *Let $\alpha = \frac{2}{k}$ with $k \geq 3$ a positive integer. Assume also $2M - d \leq 0$. Then, for any $2 \leq J \leq k(M-1)+2$, we have as $t \downarrow 0$ that*

$$\mathcal{T}_V^{(\frac{2}{k})}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \sum_{\ell=2}^{J} t^{\ell}\left(\sum_{\substack{kn+j=\ell,\\ j\geq 2}} (-1)^{n+j} C_{n,j}^{(\frac{2}{k})}(V)\right) = \mathcal{O}(t^{J+1}), \tag{2.50}$$

Example 2.9.2 *Consider $k = 3$, $M = 2$ and $d \geq 4$. Then, Theorem 1.2.2 holds for $2 \leq J \leq 5$. The particular case $J = 5$ yields*

$$\begin{aligned} &\mathcal{T}_V^{(\frac{2}{3})}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta \\ &-\frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta + \frac{t^5}{5!}\left(\int_{\mathbb{R}^d} V^5(\theta)d\theta + 5!\mathcal{L}_{d,\frac{2}{3}}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2 d\theta\right) = \mathcal{O}(t^6), \end{aligned} \tag{2.51}$$

as $t \downarrow 0$.

Let us consider for $J \geq 2$ the following $J-1 \times J-1$ matrix which contains all the power of t with the form $\frac{2n}{\alpha} + j$ that may appear in the expansion of the trace, $A_J(\alpha) = (a_{r,s})$ with $1 \leq r \leq J-1$ and

$$a_{r,s} = \begin{cases} r - s + 2 + \frac{2}{\alpha}(s-1) & \text{if } s \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.52}$$

In this matrix, $n = r - s + 2$ and $j = s - 1$. Observe that $n + j = r + 1$ and

$$a_{r,s-1} < a_{r,s},$$

for any $s \leq r$. Thus,

Example 2.9.3 *For $\alpha = 1$,*

$$A_6(\alpha) = \begin{pmatrix} 2 & . & . & . & . \\ 3 & 2+1\cdot\frac{2}{\alpha} & . & . & . \\ 4 & 3+1\cdot\frac{2}{\alpha} & 2+2\cdot\frac{2}{\alpha} & . & . \\ 5 & 4+1\cdot\frac{2}{\alpha} & 3+2\cdot\frac{2}{\alpha} & 2+3\cdot\frac{2}{\alpha} & . \\ 6 & 5+1\cdot\frac{2}{\alpha} & 4+2\cdot\frac{2}{\alpha} & 3+3\cdot\frac{2}{\alpha} & 2+4\cdot\frac{2}{\alpha} \end{pmatrix} = \begin{pmatrix} 2 & . & . & . & . \\ 3 & 4 & . & . & . \\ 4 & 5 & 6 & . & . \\ 5 & 6 & 7 & 8 & . \\ 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

We have set the two matrices together to match entry by entry. As an example, we conclude that there are two coefficients related to t^5. Namely, $C^{(1)}_{0,5}(V)$ and $C^{(1)}_{3,1}(V)$. The reader can verify this conclusion from (2.49).

Likewise, for $\alpha = \frac{2}{3}$ we obtain

$$A_7(2/3) = \begin{pmatrix} 2 & . & . & . & . & . \\ 3 & 5 & . & . & . & . \\ 4 & 6 & 8 & . & . & . \\ 5 & 7 & 9 & 11 & . & . \\ 6 & 8 & 10 & 12 & 14 & . \\ 7 & 9 & 11 & 13 & 15 & 17 \end{pmatrix}$$

We can deduce then that the next two terms in the expansion given in (2.51) *are*

$$-t^6\left\{(-1)^6 C_{0,6}^{(\frac{2}{3})}(V)+(-1)^{3+1}C_{3,1}^{(\frac{2}{3})}(V)\right\}-t^7\left\{(-1)^7 C_{0,7}^{(\frac{2}{3})}(V)+(-1)^{4+1}C_{4,1}^{(\frac{2}{3})}(V)\right\}.$$

We point out that for any $0<\alpha<2$, we always have $2<3<2+\frac{2}{\alpha}$ which implies that the influence of the α in the expansion of the trace is expected to be seen in some place after the term $C_{0,3}^{(\alpha)}(V)t^3$.

Notice that for every $J\geq 2$ we have

$$A_J(2)=\begin{pmatrix} 2 & . & . & . & . & \\ 3 & 3 & . & . & . & \\ 4 & 4 & 4 & . & . & \\ 5 & 5 & 5 & 5 & . & \\ . & . & . & . & & . \\ J & J & J & J & \ldots & J \end{pmatrix} \tag{2.53}$$

which says, for example, that in the expansion (1.15) there are three coefficients associated with t^4. Namely, $C_{0,4}^{(2)}(V), C_{1,3}^{(2)}(V)$ and $C_{2,2}^{(2)}(V)$.

Theorem 2.9.3 *Assume* $J\geq 4$ *and* $2(J-2)-d\leq 1$. *Then, for all* $\alpha\in(\frac{2(J-3)}{J-2},2)$ *we have*

$$\mathcal{T}_V^{(\alpha)}(t)+t\int_{\mathbb{R}^d}V(\theta)d\theta-\sum_{\ell=2}^{J-1}\sum_{\substack{n+j=\ell,\\ j\geq 2}}(-1)^{n+j}C_{n,j}^{(\alpha)}(V)t^{\frac{2n}{\alpha}+j}=\mathcal{O}(t^J),\ t\downarrow 0.$$

Proof In Theorem 1.2.1, we take $M=J-2$ so that

$$\Phi_{J+1}^{(\alpha)}(M)=\Phi_{J+1}^{(\alpha)}(J-2)=\min\left\{J+1,2+\frac{2(J-2)}{\alpha}\right\}>J,$$

since $\frac{2}{\alpha}>1$.

As we observed in (2.52), we know that $a_{r,s-1}\leq a_{r,s}$ for any $s\leq r$. Now we want to choose α such that $a_{r,r}<a_{r+1,1}$ for all $r\in\{2,...,J-2\}$, which is equivalent to choosing $\alpha>\frac{2(r-1)}{r}$. Thus, it suffices to consider $\frac{2(J-3)}{J-2}<\alpha$ since

$$\max\left\{\frac{2(r-1)}{r}:r\in\{2,...,J\}\right\}=\frac{2(J-3)}{J-2},$$

and this in turn implies that all the entries of $A_J(\alpha)$ are increasing for these α's. In other words, we have the following arrangement

$$\begin{aligned} & 2 \\ & \leq 3 \leq 2 + \frac{2}{\alpha} \\ & \leq 4 \leq 3 + \frac{2}{\alpha} \leq 2 + 2 \cdot \frac{2}{\alpha} \\ & .... \\ & \leq J-1 \leq (J-2) + \frac{2}{\alpha} \leq ... \leq 2 + (J-3) \cdot \frac{2}{\alpha} < J. \end{aligned} \tag{2.54}$$

Then by Theorem 1.2.1,

$$\begin{aligned} \mathcal{T}_V^{(\alpha)}(t) + t \int_{\mathbb{R}^d} V(\theta)d\theta - \sum_{\substack{j+\frac{2n}{\alpha}<J \\ 2\leq j\leq J-1,\ 0\leq n\leq J-3}} (-1)^{n+j} C_{n,j}^{(1)}(V) t^{\frac{2n}{\alpha}+j} = \\ R_{J+1}^{(\alpha)}(t) + \sum_{\substack{j+\frac{2n}{\alpha}\geq J \\ 2\leq j\leq J-1,\ 0\leq n\leq J-3}} (-1)^{n+j} C_{n,j}^{(1)}(V) t^{\frac{2n}{\alpha}+j}. \end{aligned} \tag{2.55}$$

We observe that the right hand side of (2.55) is $\mathcal{O}(t^J)$ as $t \downarrow 0$, due to $\Phi_{J+1}(J-2) > J$. On the other hand, it is easy to see by the definition of $A_J(\alpha)$ that the only powers of t satisfying $\frac{2n}{\alpha} + j < J$ are those in the arrangement given in (2.54). Therefore, due to this arrangement we can also rewrite the third term in the left hand side of (2.55) as follows

$$\sum_{\substack{j+\frac{2n}{\alpha}<J \\ 2\leq j\leq J-1,\ 0\leq n\leq J-3}} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j} = \sum_{\ell=2}^{J-1} \sum_{\substack{n+j=\ell, \\ j\geq 2}} (-1)^{n+j} C_{n,j}^{(\alpha)}(V) t^{\frac{2n}{\alpha}+j}. \tag{2.56}$$

■

Example 2.9.4 *We take $J = 4$ and $d \geq 3$ so that $2 \cdot 2 - d \leq 1$. Then, according to the last theorem for all $\alpha \in (1, 2)$ we have*

$$\begin{aligned} \mathcal{T}_V^{(\alpha)}(t) + t \int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!} \int_{\mathbb{R}^d} V^2(\theta)d\theta \\ + \frac{t^3}{3!} \int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha} t^{2+\frac{2}{\alpha}} \int_{\mathbb{R}^d} |\nabla V(\theta)|^2 = \mathcal{O}(t^4), \end{aligned}$$

as $t \downarrow 0$. *Notice that this result is already given in Corollary 1.2.1.*

Let us now consider $J = 5$ *and* $d \geq 5$. *Then, for all* $\alpha \in (\frac{4}{3}, 2)$ *we obtain*

$$\mathcal{T}_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d} V(\theta)d\theta - \frac{t^2}{2!}\int_{\mathbb{R}^d} V^2(\theta)d\theta + \left(\frac{t^3}{3!}\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{L}_{d,\alpha}t^{2+\frac{2}{\alpha}}\int_{\mathbb{R}^d} |\nabla V(\theta)|^2\right)$$
$$-\left(\frac{t^4}{4!}\int_{\mathbb{R}^d} V^4(\theta)d\theta + \mathcal{M}_{d,\alpha}t^{3+\frac{2}{\alpha}}\int_{\mathbb{R}^d} V(\theta)\left|\nabla V(\theta)\right|^2 d\theta + \mathcal{N}_{d,\alpha}t^{2+\frac{2\cdot 2}{\alpha}}\int_{\mathbb{R}^d} |(-\Delta V)(\theta)|^2\, d\theta\right)$$
$$= \mathcal{O}(t^5),\ \ t \downarrow 0.$$

3. HEAT CONTENT FOR SCHRÖDINGER OPERATORS, PROOFS

We start this section by proving that $Q_V^{(\alpha)}(t)$ given by (1.21) is a well defined function for all $t \geq 0$ as long as V is bounded and integrable. We begin by observing that the elementary inequality $|e^z - 1| \leq |z|\, e^{|z|}$ gives

$$\left|Q_V^{(\alpha)}(t)\right| \leq e^{t||V||_\infty} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y)\mathbb{E}_{x,y}^t \left[\int_0^t |V(X_s)|ds\right] dxdy.$$

Next, by Fubini's theorem and the properties of the stable bridge (see [4], [16] and (3.7) below) the integral term in the right hand side of the above inequality equals

$$\int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \left(\int_0^t \int_{\mathbb{R}^d} |V(z)| \frac{p_{t-s}^{(\alpha)}(x,z)p_s^{(\alpha)}(z,y)}{p_t^{(\alpha)}(x,y)} dzds\right) dxdy = \tag{3.1}$$
$$\int_0^t \int_{\mathbb{R}^d} |V(z)| \left(\int_{\mathbb{R}^d} p_{t-s}^{(\alpha)}(x,z)dx \int_{\mathbb{R}^d} p_s^{(\alpha)}(z,y)dy\right) dzds = t||V||_1,$$

where we have used the well known facts that for all $x, z \in \mathbb{R}^d$ and $t > 0$, $p_t^{(\alpha)}(x,z) = p_t^{(\alpha)}(z,x)$ and $\int_{\mathbb{R}^d} p_t^{(\alpha)}(x,z)dx = 1$. Thus, we conclude that the heat content satisfies

$$\left|Q_V^{(\alpha)}(t)\right| \leq t||V||_1 e^{t||V||_\infty}. \tag{3.2}$$

Therefore, $Q_V^{(\alpha)}(t)$ is well defined for all $t \geq 0$ and bounded on any interval $(0, T]$, $T > 0$, provided $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. It is also worth noting here that the previous argument together with the Taylor expansion of the exponential function (see (3.17) below) show that

$$Q_V^{(\alpha)}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y)\mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s)ds\right)^k\right] dxdy, \tag{3.3}$$

where the sum is absolutely convergent for all $t > 0$.

It is advantageous at this point to give a different expression for the equation (3.3) in terms of the stable bridge in order to obtain further formulas for the coefficients and estimates for the remainders in the forthcoming sections. Before proceeding, we introduce some notation to conveniently express our formulas below. For $k \in \mathbb{N}$, we set

$$I_k = \left\{ \lambda^{(k)} = (\lambda_k, \lambda_{k-1}, ...\lambda_1) \in [0,1]^k : 0 < \lambda_k < \lambda_{k-1} < ... < \lambda_1 < 1 \right\}, \quad (3.4)$$

$$d\lambda^{(k)} = d\lambda_k d\lambda_{k-1}...d\lambda_1, \quad z^{(k)} = (z_1, ..., z_k) \in \mathbb{R}^{kd}, \quad dz^{(k)} = dz_k...dz_1,$$

$$V_k\left(z^{(k)}\right) = V_k(z_1, ..., z_k) = \prod_{i=1}^{k} V(z_i), \quad p\left(t, z^{(k)}\right) = \prod_{j=1}^{k-1} p^{(\alpha)}_{t(\lambda_j - \lambda_{j+1})}(z_j, z_{j+1}).$$

It is well known [54] that

$$\left(\int_0^1 \tilde{V}(s)ds\right)^k = k! \int_{I_k} \prod_{i=1}^{k} \tilde{V}(\lambda_i) d\lambda^{(k)}, \quad (3.5)$$

for any $\tilde{V} : [0,1] \to \mathbb{R}$ integrable.

Lemma 3.0.1 *For any $t > 0$ and $J \geq 2$,*

$$Q_V^{(\alpha)}(t) = -t\int_{\mathbb{R}^d} V(\theta)d\theta + \sum_{k=2}^{J} (-t)^k \int_{I_k} \int_{\mathbb{R}^{kd}} V_k\left(z^{(k)}\right) p\left(t, z^{(k)}\right) dz^{(k)} d\lambda^{(k)} + R_{J+1}(t),$$

where

$$|R_{J+1}(t)| \leq t^{J+1} ||V||_1 ||V||_\infty^J e^{t||V||_\infty}. \quad (3.6)$$

Proof Set

$$R_{J+1}(t) = \sum_{k=J+1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \mathbb{E}_{x,y}^t \left[\left(\int_0^t V(X_s)ds \right)^k \right] dxdy.$$

It is clear by (3.1) that

$$\begin{aligned} |R_{J+1}(t)| &\leq \sum_{k=J+1}^{\infty} \frac{(t||V||_\infty)^{k-1}}{k!} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) \mathbb{E}_{x,y}^t \left[\int_0^t |V(X_s)|\, ds \right] dxdy \\ &\leq t||V||_1 \sum_{k=J+1}^{\infty} \frac{(t||V||_\infty)^{k-1}}{k!} \leq t^{J+1} ||V||_1 ||V||_\infty^J e^{t||V||_\infty}. \end{aligned}$$

On the other hand, by making a suitable change of variables and appealing to (3.5) we observe that

$$\mathbb{E}^t_{x,y}\left[\left(\int_0^t V(X_s)ds\right)^k\right] = t^k\,\mathbb{E}^t_{x,y}\left[\left(\int_0^1 V(X_{ts})ds\right)^k\right]$$
$$= k!\,t^k\,\mathbb{E}^t_{x,y}\left[\int_{I_k} V(X_{t\lambda_1})...V(X_{t\lambda_k})d\lambda^{(k)}\right].$$

We recall that the finite dimensional distributions of the stable bridge (see [4], [16] and references therein for details) are given by

$$\mathbb{P}^t_{x,y}\left(X_{t\lambda_1}\in dz_1, X_{t\lambda_2}\in dz_2, ..., X_{t\lambda_k}\in dz_k\right) \tag{3.7}$$
$$= \frac{1}{p_t^{(\alpha)}(x,y)}\prod_{j=0}^{k} p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}(z_j,z_{j+1})dz^{(k)},$$

where $z_0 = x$, $z_{k+1} = y$, $\lambda_0 = 1$, and $\lambda_{k+1} = 0$. Hence, using the fact that

$$\int_{\mathbb{R}^d} p^{(\alpha)}_{t(1-\lambda_1)}(x,z_1)dx = \int_{\mathbb{R}^d} p^{(\alpha)}_{t\lambda_k}(z_k,y)dy = 1,$$

the notation given in (3.4) and the finite distributions for the stable bridge given above, we conclude by Fubini's theorem that

$$\int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y)\mathbb{E}^t_{x,y}\left[\left(\int_0^t V(X_s)ds\right)^k\right]dxdy \tag{3.8}$$
$$= k!t^k\int_{I_k}\int_{\mathbb{R}^{kd}}\prod_{i=1}^{k}V(z_i)\prod_{j=1}^{k-1}p^{(\alpha)}_{t(\lambda_j-\lambda_{j+1})}(z_j,z_{j+1})dz^{(k)}d\lambda^{(k)}$$
$$= k!t^k\int_{I_k}\int_{\mathbb{R}^{kd}} V_k\left(z^{(k)}\right)p\left(t,z^{(k)}\right)dz^{(k)}d\lambda^{(k)}.$$

Therefore, the lemma follows from equation (3.3). ∎

Proof of Theorem 1.3.1: The inequality in (i) is an easy consequence of (3.1) and (3.3), since we have

$$\left|Q_V^{(\alpha)}(t) + t\int_{\mathbb{R}^d}V(x)dx\right| \le \sum_{k=2}^{\infty}\frac{1}{k!}\int_{\mathbb{R}^{2d}}p_t^{(\alpha)}(x,y)\mathbb{E}^t_{x,y}\left[\left(\int_0^t |V(X_s)|\,ds\right)^k\right]dxdy$$
$$\le t||V||_1\sum_{k=2}^{\infty}\frac{1}{k!}t^{k-1}||V||_\infty^{k-1} \le t^2||V||_1||V||_\infty e^{t||V||_\infty}.$$

Next, we proceed to show (ii). Assume $V \leq 0$. By setting $a = \int_0^t V(X_s)ds$ and $b = t||V||_\infty$, we observe that $-b \leq a < 0$. We use the elementary inequality

$$-a \leq e^{-a} - 1 \leq -a\left(1 + \frac{1}{2}be^b\right),$$

to obtain

$$-\int_0^t V(X_s)ds \leq e^{-\int_0^t V(X_s)ds} - 1 \leq -\int_0^t V(X_s)ds\left(1 + \frac{1}{2}t||V||_\infty e^{t||V||_\infty}\right). \quad (3.9)$$

By taking expectations $\mathbb{E}^t_{x,y}$ at both sides of (3.9), multiplying through by $p_t^{(\alpha)}(x,y)$, integrating on $\mathbb{R}^{2d}$ with respect to x and y and appealing to (3.1) where $|V|$ is replaced by $-V \geq 0$, we arrive at

$$-t\int_{\mathbb{R}^d} V(x)dx \leq Q_V^{(\alpha)}(t) \leq -t\int_{\mathbb{R}^d} V(x)dx\left(1 + \frac{1}{2}t||V||_\infty e^{t||V||_\infty}\right),$$

and this completes the proof.

Proof of Theorem 1.3.2: We start by recalling two basic facts about the α-stable process $\mathbf{X}$, $0 < \alpha \leq 2$. First,

$$\mathbb{E}^0\left[|X_1|^\gamma\right] < \infty, \quad (3.10)$$

for all $0 < \gamma < \alpha < 2$. As for the case $\alpha = 2$, the above fact is also true for all $0 < \gamma \leq 1$. Secondly,

$$X_t \stackrel{\mathcal{D}}{=} t^{1/\alpha}X_1, \quad (3.11)$$

as we can see from the characteristic function (1.1).

The Hölder continuity assumption on V, as we shall see in the next lemma, enables us to estimate the second term in (3.3).

Lemma 3.0.2 *Under the same assumptions on the potential V given in Theorem 1.3.2, we have for all $t > 0$ that*

$$\int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y)\mathbb{E}^t_{x,y}\left[\left(\int_0^t V(X_s)ds\right)^2\right] dxdy = t^2\int_{\mathbb{R}^d} V^2(x)dx + R(t),$$

where the remainder $R(t)$ satisfies

$$|R(t)| \leq C_0(\alpha,\gamma)||V||_1 t^{\frac{\gamma}{\alpha}+2}.$$

Proof We start by applying (3.8) with $k = 2$, so that

$$\int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y)\mathbb{E}_{x,y}^t\left[\left(\int_0^t V(X_s)ds\right)^2\right]dxdy =$$
$$2t^2\int_0^1\int_0^{\lambda_1}\int_{\mathbb{R}^{2d}} V(z_1)V(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_2dz_1d\lambda_2d\lambda_1.$$

Now,

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} V(z_1)V(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2 =\\&\int_{\mathbb{R}^{2d}} \left(V(z_1)-V(z_2)\right)V(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2\\&+\int_{\mathbb{R}^{2d}} V^2(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2.\end{aligned} \tag{3.12}$$

The second term on the right hand side of equality (3.12) equals $\int_{\mathbb{R}^d} V^2(x)dx$, whereas for the first term, by using (3.10), (3.11) and the Hölder continuity assumption on V, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2d}} \left(V(z_1)-V(z_2)\right)V(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2\right|\\&\le M\int_{\mathbb{R}^{2d}} |z_1-z_2|^\gamma|V(z_2)|p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2\\&= M\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|z_1-z_2|^\gamma p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1\right)|V(z_2)|\,dz_2\\&= M\ ||V||_1\ \mathbb{E}^0[\,\left|X_{t(\lambda_1-\lambda_2)}\right|^\gamma\,]\\&= M\ ||V||_1\ \left(t(\lambda_1-\lambda_2)\right)^{\frac{\gamma}{\alpha}}\ \mathbb{E}^0[\,|X_1|^\gamma\,].\end{aligned} \tag{3.13}$$

Thus, by using the fact that

$$\int_0^1\int_0^{\lambda_1}(\lambda_1-\lambda_2)^{\gamma/\alpha}\,d\lambda_2d\lambda_1 = \left(\frac{\gamma}{\alpha}+2\right)^{-1}\left(\frac{\gamma}{\alpha}+1\right)^{-1},$$

we obtain that the conclusion of the lemma follows from (3.13), (3.12) by setting

$$R(t) = 2t^2\int_0^1\int_0^{\lambda_1}\int_{\mathbb{R}^{2d}}\left(V(z_1)-V(z_2)\right)V(z_2)p_{t(\lambda_1-\lambda_2)}^{(\alpha)}(z_2,z_1)dz_1dz_2d\lambda_1d\lambda_2.$$

■

Therefore, using that $t^3 \leq t^{2+\frac{\gamma}{\alpha}}$ for $t \in (0,1)$, we have that Theorem 1.3.2 is a consequence of applying Lemma 3.0.2 to the following expression obtained in Lemma 3.0.1 when $J = 2$,

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(x)dx + \frac{1}{2} \int_{\mathbb{R}^{2d}} p_t^{(\alpha)}(x,y) E_{x,y}^t \left[\left(\int_0^t V(X_s)ds \right)^2 \right] dxdy + R_3(t),$$

where we already know that

$$|R_3(t)| \leq t^3 ||V||_1 ||V||_\infty^2 e^{t||V||_\infty}.$$

3.1 General expansion for rapidly decreasing smooth potential

We have already seen in the previous section that by adding an extra regularity condition on the potential V, namely, Hölder continuity and using $X_t \stackrel{\mathcal{D}}{=} t^{1/\alpha} X_1$, we have been able to extract a second term in the expansion of $Q_V^{(\alpha)}(t)$. In this section, we will obtain more terms and find explicit expressions for these which as before will depend on the potential V.

Let $V \in \mathcal{S}(\mathbb{R}^d)$ and denote by $\widehat{V}$ the Fourier transform. In what follows, we set $\bar{d}\xi = (2\pi)^{-d} d\xi$.

The fact that $V \in \mathcal{S}(\mathbb{R}^d)$ will allow us to apply the inversion formula to each summand in Lemma 3.0.1 which in turn will provide the terms obtained in Theorem 1.3.3. To do this, we need the following proposition.

Proposition 3.1.1 *For any $k \geq 2$,*

$$\int_{\mathbb{R}^{kd}} \prod_{i=1}^{k} V(z_i) \prod_{j=1}^{k-1} p_{t(\lambda_j - \lambda_{j+1})}^{(\alpha)}(z_j, z_{j+1}) dz_k ... dz_1 = \tag{3.14}$$

$$\int_{\mathbb{R}^{(k-1)d}} \widehat{V}(-\sum_{i=1}^{k-1} \theta_i) \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \exp\left(-t \sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) \left| \sum_{m=1}^{r} \theta_m \right|^\alpha \right) \bar{d}\theta_{k-1} ... \bar{d}\theta_1.$$

Proof Under the notation given in (3.4) we have

$$V_k\left(z^{(k)}\right) = V_k(z_1, ..., z_k) = \prod_{i=1}^{k} V(z_i);$$

$$p\left(t, z^{(k)}\right) = \prod_{r=1}^{k-1} p_{t(\lambda_r - \lambda_{r+1})}^{(\alpha)}(z_r, z_{r+1}).$$

By applying Fourier transform in $\mathbb{R}^{kd}$ and Plancherel's identity, we obtain

$$\int_{\mathbb{R}^{kd}} V_k\left(z^{(k)}\right) p\left(t, z^{(k)}\right) dz^{(k)} = \int_{\mathbb{R}^{kd}} \widehat{V}_k\left(\theta^{(k)}\right) \widehat{p}\left(t, -\theta^{(k)}\right) \bar{d}\theta^{(k)}. \tag{3.15}$$

Next, it follows easily that if $\theta^{(k)} = (\theta_1, ...\theta_k)$, $\theta_i \in \mathbb{R}^d$, then

$$\widehat{V}_k(\theta^{(k)}) = \prod_{i=1}^{k} \widehat{V}(\theta_i).$$

On the other hand, we claim that

$$\widehat{p}(t, -\theta^{(k)}) = (2\pi)^d \delta\left(\sum_{i=1}^{k} \theta_i\right) \exp\left(-t\sum_{r=1}^{k-1}(\lambda_r - \lambda_{r+1})\left|\sum_{m=1}^{r} \theta_m\right|^\alpha\right). \tag{3.16}$$

To see this, we observe by (2.1) that

$$\widehat{p}(t, -\theta^{(k)}) = \int_{\mathbb{R}^{kd}} \exp\left(i\sum_{j=1}^{k} \xi_j \cdot \theta_j\right) \prod_{r=1}^{k-1} p^{(\alpha)}_{t(\lambda_r - \lambda_{r+1})}(\xi_r - \xi_{r+1}) d\xi^{(k)}.$$

By considering the substitutions $z_r = \xi_r - \xi_{r+1}$, $r \in \{1, ..., k-1\}$, we have for any $j \in \{1, ..., k-1\}$ that

$$\xi_j = \xi_k + \sum_{r=j}^{k-1} z_r.$$

Therefore, we obtain after interchanging the order of summation that

$$\sum_{j=1}^{k-1} \xi_j \cdot \theta_j = \sum_{r=1}^{k-1} z_r \cdot \left(\sum_{m=1}^{r} \theta_m\right) + \xi_k \cdot \sum_{i=1}^{k-1} \theta_i.$$

Thus, (3.16) follows by using that

$$\widehat{p^{(\alpha)}_t}(\xi) = e^{-t|\xi|^\alpha},$$

$$\int_{\mathbb{R}^d} \exp\left(i\xi_k \cdot \sum_{i=1}^{k} \theta_i\right) d\xi_k = (2\pi)^d \delta\left(\sum_{i=1}^{k} \theta_i\right),$$

and

$$\begin{aligned}\widehat{p}(t, -\theta^{(k)}) &= \int_{\mathbb{R}^{kd}} \exp\left(i\sum_{r=1}^{k-1} z_r \cdot \left(\sum_{m=1}^{r} \theta_m\right) + i\xi_k \cdot \sum_{i=1}^{k} \theta_i\right) \\ &\quad \times \prod_{r=1}^{k-1} p^{(\alpha)}_{t(\lambda_r - \lambda_{r+1})}(z_r) dz^{(k-1)} d\xi_k.\end{aligned}$$

Consequently, the conclusion of the proposition follows from (3.15) and (3.16). ■

We next recall the Taylor expansion of the exponential function

$$e^{-x} = \sum_{n=0}^{M} \frac{(-1)^n}{n!} x^n + \frac{(-1)^{M+1}}{(M+1)!} x^{M+1} e^{-x\beta_{M+1}(x)}, \tag{3.17}$$

valid for every $x \geq 0$ and integer $M \geq 0$, where we call $\beta_{M+1}(x) \in (0,1)$ the remainder of order $M+1$.

We also recall that for $k \geq 2$ integer, the Binomial theorem asserts that

$$(x_1 + x_2 + \cdots + x_{k-1})^n = \sum_{\substack{(\ell_1,\dots,\ell_{k-1}) \,\in\, \mathbb{N}^{k-1}, \\ \ell_1+\ell_2+\dots+\ell_{k-1}=n}} \binom{n}{\ell_1, \ell_2, \dots, \ell_{k-1}} x_1^{\ell_1} x_2^{\ell_2} \cdots x_{k-1}^{\ell_{k-1}}.$$

Next, bearing in mind the notation given in (3.4), we set $\gamma_r = \sum_{m=1}^{r} \theta_m$,

$$\ell^{(k-1)} = (\ell_1, \dots, \ell_{k-1}) \in \mathbb{N}^{k-1},$$

$$A(n, \ell^{(k-1)}) = \binom{n}{\ell_1, \ell_2, \dots, \ell_{k-1}} \int_{I_k} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} d\lambda^{(k)},$$

and

$$\begin{aligned} T_k(t) = \int_{I_k} \int_{\mathbb{R}^{(k-1)d}} & \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \widehat{V}(-\sum_{i=1}^{k-1} \theta_i) \\ & \times \exp\left(-t \sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) \left| \sum_{m=1}^{r} \theta_m \right|^\alpha \right) \bar{d}\theta^{(k-1)} d\lambda^{(k)}. \end{aligned} \tag{3.18}$$

Therefore, under this notation, we obtain the following expansion for the term $T_k(t)$.

Corollary 3.1.1 *Let $M \geq 0$ and $k \geq 2$ be integers. Then*

$$T_k(t) = \sum_{n=0}^{M} \frac{(-t)^n}{n!} C_{n,k}(V) + R_{M+1}^{(k)}(t),$$

where

$$\begin{aligned} R_{M+1}^{(k)}(t) \;\; = \;\; & \frac{(-t)^{M+1}}{(M+1)!} \int_{I_k} \int_{\mathbb{R}^{(k-1)d}} \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \widehat{V}(-\sum_{i=1}^{k-1} \theta_i) \\ & \times \left(\sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) |\gamma_r|^\alpha \right)^{M+1} e^{-\Upsilon} \bar{d}\theta^{(k-1)} d\lambda^{(k)}, \end{aligned}$$

for some non-negative function $\Upsilon = \Upsilon(t, \lambda^{(k)}, \theta^{(k-1)}, M+1)$. *The remainder satisfies*

$$R^{(k)}_{M+1}(t) = \mathcal{O}(t^{M+1}), \tag{3.19}$$

as $t \downarrow 0$. *Moreover, the coefficients are given by*

$$\begin{aligned} C_{n,k}(V) &= \sum_{\substack{(\ell_1,\ldots,\ell_{k-1}) \in \mathbb{N}^{k-1}, \\ \ell_1+\ell_2+\ldots+\ell_{k-1}=n}} A(n, \ell^{(k-1)}) \int_{\mathbb{R}^{(k-1)d}} \widehat{V}(-\sum_{i=1}^{k-1}\theta_i) \prod_{i=1}^{k-1} \widehat{V}(\theta_i) \\ &\quad \times \left| \sum_{m=1}^{i} \theta_m \right|^{\alpha \ell_i} \bar{d}\theta^{(k-1)}. \end{aligned}$$

Proof The formula for the coefficients is obtained by applying the Taylor expansion of the exponential function and the Binomial theorem to our expression of $T_k(t)$ in (3.18).

Next, we proceed to show our claim about the remainder. In order to do so, we point out that $V \in \mathcal{S}(\mathbb{R}^d)$ implies that all quantities to appear below are finite. Also, the constant C below will depend on k, M and α and its value may change from line to line. It is easy to observe that for some $C > 0$, we have

$$\left(\sum_{r=1}^{k-1} (\lambda_r - \lambda_{r+1}) |\gamma_r|^{\alpha} \right)^{M+1} \leq C \max_{m=1,..,k-1} |\theta_m|^{\alpha(M+1)}.$$

In particular, if we let $\Lambda_r = \left\{ \theta^{(k-1)} \in \mathbb{R}^{d(k-1)} : \max_{m=1,..,k-1} |\theta_m| = |\theta_r| \right\}$, we arrive at

$$\begin{aligned} \left| R^{(k)}_{M+1}(t) \right| &\leq C t^{M+1} \sum_{r=1}^{k-1} \int_{\Lambda_r} |\widehat{V}|(-\gamma_{k-1}) |\widehat{V}(\theta_r)| \theta_r|^{\alpha(M+1)}| \prod_{i=1, i\neq r}^{k-1} |\widehat{V}|(\theta_i) \bar{d}\theta^{(k-1)} \\ &\leq C t^{M+1} ||\widehat{V}||_{\infty} ||\widehat{(-\Delta)^{\frac{\alpha}{2}}_{M+1} V}||_1 ||\widehat{V}||_1^{k-2}. \end{aligned}$$

Here, $(-\Delta)^{\frac{\alpha}{2}}_{M+1}$ stands for the composition of $(-\Delta)^{\frac{\alpha}{2}}$ with itself $M+1$–times and this completes the proof. ∎

With Corollary 3.1.1 at hand, we carry on showing the existence of a general expansion for $Q^{(\alpha)}_V(t)$ for small time.

Theorem 3.1.1 *For any integer $N \geq 2$,*

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{\ell=2}^{N} (-t)^\ell C_\ell(V) + \mathcal{O}(t^{N+1}), \tag{3.20}$$

as $t \downarrow 0$. Here,

$$C_\ell(V) = \sum_{\substack{n+k=\ell \\ 0 \leq n,\, 2 \leq k}} \frac{1}{n!} C_{n,k}(V),$$

where $C_{n,k}(V)$ is as defined in Corollary 3.1.1.

Proof As a result of Lemma 3.0.1, Corollary 3.1.1 and (3.14), we have for any integers $J \geq 2$ and $M \geq 0$ that

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{k=2}^{J} \sum_{n=0}^{M} \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + R_{M+1,J+1}(t), \tag{3.21}$$

where

$$R_{M+1,J+1}(t) = R_{J+1}(t) + \sum_{k=2}^{J} (-t)^k R_{M+1}^{(k)}.$$

In other words, $R_{J+1,M+1}(t)$ is the sum of all those remainders provided by Lemma 3.0.1 and Corollary 3.1.1. We also point out that due to (3.6) and (3.19), we conclude

$$R_{M+1,J+1}(t) = \mathcal{O}\left(t^{\min\{J+1, M+3\}}\right),$$

as $t \downarrow 0$.

Since M and J are arbitrary, given $N \geq 2$, we may choose M and J as large as we desire so that

$$\min\{J+1, M+3\} \geq N+1$$

and such that formula (3.21) can be decomposed as follows

$$Q_V^{(\alpha)}(t) = -t \int_{\mathbb{R}^d} V(\theta) d\theta + \sum_{\substack{2 \leq n+k \leq N \\ 2 \leq k}} \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + \tilde{R}_{N+1}(t), \tag{3.22}$$

where $\tilde{R}_{N+1}(t)$ is defined to be

$$\sum_{\substack{n+k \geq N+1 \\ 2 \leq k}} \frac{(-t)^{k+n}}{n!} C_{n,k}(V) + R_{M+1,J+1}(t).$$

Thus, it is easy to observe that $\tilde{R}_{N+1}(t) = \mathcal{O}(t^{N+1})$ as $t \downarrow 0$.

The conclusion of the theorem follows by noticing that the second terms on the right hand side of both (3.22) and (3.20) are the same under our definition of $C_\ell(V)$. ■

Before proceeding, we give an application concerning the coefficients $C_\ell(V)$. The corollary roughly says that we can characterize the potential V from the coefficients under some extra assumptions. This corollary should be compared to the result for the trace (case $\alpha = 2$) given in [3, Corollary 2.1].

Corollary 3.1.2 *Let $V \in \mathcal{S}(\mathbb{R}^d)$ be such that $\widehat{V} \geq 0$. If $C_\ell(V) = 0$ for some $\ell \geq 2$, then we must have $V(x) = 0$ for all $x \in \mathbb{R}^d$.*

Proof By Theorem 3.1.1 and Corollary 3.1.1, we see that $C_\ell(V) \geq 0$ for all $\ell \geq 2$ when $\widehat{V} \geq 0$. In particular, the condition $C_\ell(V) = 0$ for some $\ell \geq 2$ implies that

$$C_{\ell-2,2}(V) = \binom{\ell-2}{\ell-2} \int_{I_2} (\lambda_1 - \lambda_2)^{\ell-2} d\lambda^{(2)} \int_{\mathbb{R}^d} \widehat{V}(-\theta_1)\widehat{V}(\theta_1)\, |\theta_1|^{\alpha(\ell-2)}\, \bar{d}\theta_1 = 0.$$

Therefore, we must have $\widehat{V}(-\theta_1)\widehat{V}(\theta_1) = 0$ for all $\theta_1 \in \mathbb{R}^d$. Now, by applying Plancherel's identity, we have

$$\int_{\mathbb{R}^d} |V(x)|^2 dx = \int_{\mathbb{R}^d} \widehat{V}(-\theta_1)\widehat{V}(\theta_1)\bar{d}\theta_1 = 0$$

and this gives the claimed result. ■

3.2 Computation of coefficients

In this section, we write down explicitly the first five coefficients of the asymptotic expansion given in (3.20). This also proves Theorem 1.3.3. All the results in the previous section also hold for $\alpha = 2$. Therefore, we will consider $0 < \alpha \leq 2$.

In order to find the coefficients $C_3(V)$, $C_4(V)$ and $C_5(V)$, we will resort to Lemma 3.2.1 below. We start by observing that by means of the inversion formula, it follows easily that

$$C_{0,k}(V) = \frac{1}{k!} \int_{\mathbb{R}^d} V^k(\theta) d\theta, \tag{3.23}$$

for any integer $k \geq 2$.

Lemma 3.2.1 *Let $k \geq 2$ be an integer. Assume that $\{\ell_i, i \in \{1, ..., k-1\}\}$ is a sequence of non-negative real numbers satisfying*

$$\sum_{i=1}^{k-1} \ell_i = n, \tag{3.24}$$

for some positive real number n. Then

(a) If $k = 2$, we have

$$\int_{I_2} (\lambda_1 - \lambda_2)^n d\lambda^{(2)} = \frac{1}{(1+n)(2+n)}.$$

(b) If $k \geq 3$, we obtain

$$\int_{I_k} \prod_{i=1}^{k-1} (\lambda_i - \lambda_{i+1})^{\ell_i} d\lambda^{(k)} = \frac{1}{(k+n)(\ell_{k-1}+1)} \prod_{i=1}^{k-2} \int_0^1 (1-s)^{\ell_i} s^{k+n-(i+1+\sum_{j=1}^{i} \ell_j)} ds.$$

Proof We only need to prove (b). Let $\lambda_1 \in (0,1)$ be fixed. Consider the following change of variables

$$\lambda_{i+1} = \lambda_i s_i,$$

for $i \in \{1, ..., k-1\}$. Using the fact that $0 < \lambda_{i+1} < \lambda_i$ we must have that $s_i \in (0,1)$. Notice that this change of variables yields

$$\lambda_{i+1} = \lambda_1 \prod_{j=1}^{i} s_j. \tag{3.25}$$

Thus, the Jacobian associated to this change of variables is the determinant of an upper triangular matrix and it is given explicitly by the following formula.

$$\frac{\partial(\lambda_2, ..., \lambda_k)}{\partial(s_1, ..., s_{k-1})} = \lambda_1^{k-1} \prod_{i=1}^{k-2} s_i^{k-(i+1)}.$$

Observe that by (3.25) and (3.24) we have

$$\prod_{i=2}^{k-1} \lambda_i^{\ell_i} = \prod_{i=2}^{k-1} \left(\lambda_1 \prod_{j=1}^{i-1} s_j \right)^{\ell_i} = \lambda_1^{\sum_{j=2}^{k-2} \ell_j} \left(\prod_{i=1}^{k-2} s_i^{\sum_{j=i+1}^{k-1} \ell_j} \right) = \lambda_1^{n-\ell_1} \prod_{i=1}^{k-2} s_i^{\,n-\sum_{j=1}^{i} \ell_j}.$$

From this we conclude that

$$\prod_{i=1}^{k-1}(\lambda_i-\lambda_{i+1})^{\ell_i}=\prod_{i=1}^{k-1}\lambda_i^{\ell_i}(1-s_i)^{\ell_i}=\lambda_1^{\ell_1}(1-s_{k-1})^{\ell_{k-1}}\prod_{i=1}^{k-2}(1-s_i)^{\ell_i}\prod_{i=2}^{k-1}\lambda_i^{\ell_i}$$
$$=\lambda_1^{n}(1-s_{k-1})^{\ell_{k-1}}\prod_{i=1}^{k-2}(1-s_i)^{\ell_i}\prod_{i=1}^{k-2}s_i^{\,n-\sum_{j=1}^{i}\ell_j}.$$

As a result, integrating both sides of the above identity, we see that (b) is a consequence of the following equality.

$$\int_{I_k}\prod_{i=1}^{k-1}(\lambda_i-\lambda_{i+1})^{\ell_i}d\lambda^{(k)} = \int_0^1\lambda_1^{n+k-1}d\lambda_1\int_0^1(1-s_{k-1})^{\ell_{k-1}}ds_{k-1}$$
$$\times\int_{[0,1]^{k-2}}\prod_{i=1}^{k-2}(1-s_i)^{\ell_i}s_i^{\,n+k-(i+1+\sum_{j=1}^{i}\ell_j)}ds^{(k-2)}.$$

■

For the computations to be performed below is worth recalling that

$$\mathcal{E}_\alpha(V)=\int_{\mathbb{R}^d}(-\Delta)^{\frac{\alpha}{2}}V(\theta)V(\theta)d\theta=\int_{\mathbb{R}^d}\widehat{V}(-\theta)\widehat{V}(\theta)|\theta|^\alpha\bar{d}\theta=\int_{\mathbb{R}^d}|\widehat{V}(\theta)|^2|\theta|^\alpha\bar{d}\theta.$$

Lemma 3.2.2

$$C_3(V)=\frac{1}{3!}\left(\int_{\mathbb{R}^d}V^3(\theta)d\theta+\mathcal{E}_\alpha(V)\right).$$

Proof By Theorem 3.1.1, we have

$$C_3(V)=C_{0,3}(V)+C_{1,2}(V).$$

From (3.23), it suffices to compute $C_{1,2}(V)$. Following Corollary 3.1.1, we have, by Plancherel's Theorem and (1.8), that

$$C_{1,2}(V)=A(1,1)\int_{\mathbb{R}^d}\widehat{V}(-\theta_1)\widehat{V}(\theta_1)|\theta_1|^\alpha\bar{d}\theta_1=\frac{1}{6}\int_{\mathbb{R}^d}V(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta,$$

which gives the formula above. ■

Lemma 3.2.3

$$C_4(V)=\frac{1}{4!}\left(\int_{\mathbb{R}^d}V^4(\theta)d\theta+2\int_{\mathbb{R}^d}V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta+\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2d\theta\right).$$

Proof By Theorem 3.1.1,

$$C_4(V) = C_{0,4}(V) + C_{1,3}(V) + \frac{1}{2!}C_{2,2}(V).$$

By Corollary 3.1.1 with $n = 1$ and $k = 3$, we have

$$\begin{aligned}C_{1,3}(V) =& A(1,(1,0)) \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2)\widehat{V}(\theta_1)\widehat{V}(-\theta_1-\theta_2)|\theta_1|^\alpha \bar{d}\theta_2\bar{d}\theta_1 \\ &+A(1,(0,1)) \int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2)\widehat{V}(\theta_1)\widehat{V}(-\theta_1-\theta_2)|\theta_1+\theta_2|^\alpha \bar{d}\theta_2\bar{d}\theta_1.\end{aligned}$$

From Lemma 3.2.1, we obtain

$$\begin{aligned}A\left(1,(1,0)\right) &= \binom{1}{1,0}\frac{1}{4}\int_0^1 (1-s)sds = \frac{1}{4!}, \\ A\left(1,(0,1)\right) &= \binom{1}{0,1}\frac{1}{4\cdot 2}\int_0^1 s^2 ds = \frac{1}{4!}.\end{aligned}$$

On the other hand, due to the basic properties of the Fourier transform,

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2)\widehat{V}(\theta_1)\widehat{V}(-\theta_1-\theta_2)|\theta_1+\theta_2|^\alpha \bar{d}\theta_2\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \widehat{V}(\theta_2)\widehat{(-\Delta)^{\frac{\alpha}{2}}V}(-\theta_1-\theta_2)\bar{d}\theta_2\right)\widehat{V}(\theta_1)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{i\theta\cdot\theta_1}V(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta\right)\widehat{V}(\theta_1)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta.\end{aligned}$$

A similar argument yields

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_2)\widehat{V}(\theta_1)\widehat{V}(-\theta_1-\theta_2)|\theta_1|^\alpha \bar{d}\theta_2\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} |\theta_1|^\alpha\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1-\theta_2)\widehat{V}(\theta_2)\bar{d}\theta_2\right)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} |\theta_1|^\alpha\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d} e^{i\theta_1\cdot\theta}V^2(\theta)d\theta\right)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta.\end{aligned}$$

Thus, we arrive at

$$C_{1,3}(V) = \frac{2}{4!}\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta.$$

Next,

$$C_{2,2}(V) = A(2,2)\int_{\mathbb{R}^d} \widehat{V}(-\theta_1)\widehat{V}(\theta_1)|\theta_1|^{2\alpha}\bar{d}\theta_1 = \frac{2!}{4!}\int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta.$$

Therefore, the announced formula for $C_4(V)$ follows from the above identities. ■

Lemma 3.2.4

$$\begin{aligned} C_5(V) &= \frac{1}{5!}\Bigg(\int_{\mathbb{R}^d} V^5(\theta)d\theta + 2\int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta \\ &+ 2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)_2^{\frac{\alpha}{2}}V(\theta)d\theta \\ &+ \int_{\mathbb{R}^d} V(\theta)\left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta + \mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right) + \mathcal{E}_\alpha\left(V^2\right)\Bigg), \end{aligned}$$

where $(-\Delta)_2^{\frac{\alpha}{2}}$ denotes the composition of $(-\Delta)^{\frac{\alpha}{2}}$ with itself.

Proof Once again, Theorem 3.1.1 gives

$$C_5(V) = C_{0,5}(V) + C_{1,4}(V) + \frac{1}{2!}C_{2,3}(V) + \frac{1}{3!}C_{3,2}(V).$$

The first term $C_{0,5}(V)$ follows from (3.23). From Corollary 3.1.1 with $n = 1$ and $k = 4$, we have

$$C_{1,4}(V) = A(1,\ (1,0,0))\int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^{3}\theta_i)\prod_{i=1}^{3}\widehat{V}(\theta_i)|\theta_1|^{\alpha}\bar{d}\theta^{(3)} \tag{3.26}$$

$$+ A(1,\ (0,1,0))\int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^{3}\theta_i)\prod_{i=1}^{3}\widehat{V}(\theta_i)\left|\sum_{m=1}^{2}\theta_m\right|^{\alpha}\bar{d}\theta^{(3)} \tag{3.27}$$

$$+ A(1,\ (0,0,1))\int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^{3}\theta_i)\prod_{i=1}^{3}\widehat{V}(\theta_i)\left|\sum_{m=1}^{3}\theta_m\right|^{\alpha}\bar{d}\theta^{(3)}. \tag{3.28}$$

The most difficult term to compute in the above equality is the one appearing in (3.27) and we proceed to deal with this one first. By integrating first with respect to θ_3 and applying Plancherel's formula we obtain

$$\int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^{3}\theta_i)\prod_{i=1}^{3}\widehat{V}(\theta_i)\left|\sum_{m=1}^{2}\theta_m\right|^{\alpha}\bar{d}\theta^{(3)} \tag{3.29}$$

$$= \int_{\mathbb{R}^d} V^2(\theta)\left(\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_1)\widehat{V}(\theta_2)\left|\theta_1+\theta_2\right|^{\alpha} e^{i\theta\cdot(\theta_1+\theta_2)}\bar{d}\theta_1\bar{d}\theta_2\right)d\theta.$$

Consider the change of variable $\theta_2 = z - \theta_1$, where the independent variable is θ_2. Then, the integral in (3.29) between parenthesis equals

$$\int_{\mathbb{R}^{2d}} \widehat{V}(\theta_1)\widehat{V}(z-\theta_1)\,|z|^\alpha\, e^{i\theta\cdot z} \bar{d}\theta_1 \bar{d}z.$$

Thus, integrating the last expression with respect to θ_1 gives

$$\int_{\mathbb{R}^d} |z|^\alpha\, e^{i\theta\cdot z}\left(\int_{\mathbb{R}^d} e^{-i\eta\cdot z} V^2(\eta)d\eta\right)\bar{d}z = \int_{\mathbb{R}^d} |z|^\alpha\, e^{i\theta\cdot z}\widehat{V^2}(z)\bar{d}z = (-\Delta)^{\frac{\alpha}{2}} V^2(\theta).$$

In other words, we have shown that

$$\int_{\mathbb{R}^{3d}} \widehat{V}(-\sum_{i=1}^3 \theta_i)\prod_{i=1}^3 \widehat{V}(\theta_i)\left|\sum_{m=1}^2 \theta_m\right|^\alpha \bar{d}\theta^{(3)} = \mathcal{E}_\alpha(V^2).$$

Next, we claim that the other two integral terms in (3.28) equal

$$\int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta)d\theta.$$

To see this, it suffices to consider the following equalities.

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1-\theta_2-\theta_3)\,|\theta_1+\theta_2+\theta_3|^\alpha\, \widehat{V}(\theta_3)\bar{d}\theta_3\right)\widehat{V}(\theta_1)\widehat{V}(\theta_2)\bar{d}\theta_1\bar{d}\theta_2\\
&= \int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d} e^{i\theta\cdot(\theta_1+\theta_2)} V(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta)d\theta\right)\widehat{V}(\theta_1)\widehat{V}(\theta_2)\bar{d}\theta_1\bar{d}\theta_2\\
&= \int_{\mathbb{R}^{d}}\left(\int_{\mathbb{R}^{2d}} e^{i\theta\cdot(\theta_1+\theta_2)} \widehat{V}(\theta_1)\widehat{V}(\theta_2)\bar{d}\theta_1\bar{d}\theta_2\right) V(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta)d\theta\\
&= \int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta)d\theta,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d} \widehat{V}(-\theta_1-\theta_2-\theta_3)\widehat{V}(\theta_3)\bar{d}\theta_3\right)|\theta_1|^\alpha\, \widehat{V}(\theta_1)\widehat{V}(\theta_2)\bar{d}\theta_1\bar{d}\theta_2\\
&= \int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d} e^{i\theta\cdot(\theta_1+\theta_2)} V^2(\theta)d\theta\right)\widehat{(-\Delta)^{\frac{\alpha}{2}}V}(\theta_1)\widehat{V}(\theta_2)\bar{d}\theta_1\bar{d}\theta_2\\
&= \int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}} V(\theta)d\theta.
\end{aligned}$$

As far for the quantities, $A(1,\,(1,0,0))$, $A(1,\,(0,1,0))$ and $A(1,\,(0,0,1))$ we have

$$\begin{aligned}
A(1,\,(1,0,0)) &= \frac{1}{5}\int_0^1 (1-s)s^2 ds \int_0^1 s ds = \frac{1}{5!}\\
A(1,\,(0,1,0)) &= \frac{1}{5}\int_0^1 s^3 ds \int_0^1 (1-s) s ds = \frac{1}{5!}\\
A(1,\,(0,0,1)) &= \frac{1}{5}\cdot\frac{1}{2}\int_0^1 s^3 ds \int_0^1 s^2 ds = \frac{1}{5!}.
\end{aligned}$$

Therefore, we conclude that

$$C_{1,4}(V) = \frac{1}{5!}\left(2\int_{\mathbb{R}^d} V^3(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta + \mathcal{E}_\alpha(V^2)\right). \tag{3.30}$$

Next, we compute $C_{2,3}(V)$. This time we have

$$\begin{aligned} C_{2,3}(V) = A(2,\ (1,1)) &\int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1-\theta_2)\widehat{V}(\theta_1)\widehat{V}(\theta_2)|\theta_1|^\alpha|\theta_1+\theta_2|^\alpha \bar{d}\theta^{(2)} \\ + A(2,\ (2,0)) &\int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1-\theta_2)\widehat{V}(\theta_1)\widehat{V}(\theta_2)|\theta_1|^{2\alpha} \bar{d}\theta^{(2)} \\ + A(2,\ (0,2)) &\int_{\mathbb{R}^{2d}} \widehat{V}(-\theta_1-\theta_2)\widehat{V}(\theta_1)\widehat{V}(\theta_2)|\theta_1+\theta_2|^{2\alpha} \bar{d}\theta^{(2)}. \end{aligned} \tag{3.31}$$

The first integral term in the right hand side of above equality equals

$$\begin{aligned} &\int_{\mathbb{R}^d} |\theta_1|^\alpha\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d}\widehat{V}(-\theta_1-\theta_2)|\theta_1+\theta_2|^\alpha\widehat{V}(\theta_2)\bar{d}\theta_2\right)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} |\theta_1|^\alpha\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d} e^{i\theta\cdot\theta_1}V(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)d\theta\right)\bar{d}\theta_1 \\ &= \int_{\mathbb{R}^d} V(\theta)\left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta. \end{aligned}$$

As for the other two integral terms in (3.31), we claim they both equal

$$\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}_{2}V(\theta)d\theta,$$

since the third integral term equals

$$\begin{aligned} &\int_{\mathbb{R}^d}\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d}\widehat{V}(\theta_2)\widehat{V}(-\theta_1-\theta_2)|\theta_1+\theta_2|^{2\alpha}\bar{d}\theta_2\right)\bar{d}\theta_1 \\ = &\int_{\mathbb{R}^d}\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d} e^{i\theta\cdot\theta_1}V(\theta)(-\Delta)^{\frac{\alpha}{2}}_{2}V(\theta)d\theta\right)\bar{d}\theta_1, \end{aligned}$$

whereas the second one equals

$$\begin{aligned} &\int_{\mathbb{R}^d}|\theta_1|^{2\alpha}\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d}\widehat{V}(\theta_2)\widehat{V}(-\theta_1-\theta_2)\bar{d}\theta_2\right)\bar{d}\theta_1 \\ = &\int_{\mathbb{R}^d}|\theta_1|^{2\alpha}\widehat{V}(\theta_1)\left(\int_{\mathbb{R}^d} e^{i\theta\cdot\theta_1}V^2(\theta)d\theta\right)\bar{d}\theta_1. \end{aligned}$$

As for the coefficients in front of the integral terms, we have

$$\begin{aligned} A(2,\ (1,1)) &= \binom{2}{1,1}\frac{1}{5\cdot 2}\int_0^1(1-s)s^2ds = \frac{2}{5!} \\ A(2,\ (2,0)) &= \binom{2}{2,0}\frac{1}{5}\int_0^1(1-s)^2sds = \frac{2}{5!} \\ A(2,\ (0,2)) &= \binom{2}{0,2}\frac{1}{5\cdot 3}\int_0^1 s^3ds = \frac{2}{5!}. \end{aligned}$$

Therefore,

$$C_{2,3}(V) = \frac{2!}{5!}\int_{\mathbb{R}^d} V(\theta)|(-\Delta)^{\frac{\alpha}{2}}V(\theta)|^2 d\theta + \frac{2\cdot 2!}{5!}\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}_{2}V(\theta)d\theta. \quad (3.32)$$

Likewise, we obtain

$$\begin{aligned} C_{3,2}(V) &= A(3,3)\int_{\mathbb{R}^d}\widehat{V}(-\theta_1)\widehat{V}(\theta_1)|\theta_1|^{3\alpha}\bar{d}\theta_1 \quad (3.33) \\ &= \frac{1}{20}\int_{\mathbb{R}^d}(-\Delta)^{\frac{\alpha}{2}}V(\theta)(-\Delta)^{\frac{\alpha}{2}}_{2}V(\theta)d\theta = \frac{3!}{5!}\mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right). \end{aligned}$$

Combining (3.23), (3.30), (3.32), and (3.33), we obtain our expression for $C_5(V)$. ■

In the case of the Laplacian $\alpha = 2$, the signs of the coefficients can be used to give information on the poles on the meromorphic extension of the resolvent of the operator H_V; see for example [3, Theorem 4.1]. In particular, it is shown in [3] that the first five coefficients in the trace expansion are non-negative provided the potential is non-negative. Our computations above yield a similar result for the first five coefficients of the heat content. More precisely, we have

Corollary 3.2.1 *Suppose* $V \in \mathcal{S}(\mathbb{R}^d)$, $V \geq 0$. *Then* $C_\ell(V) \geq 0$, *for* $1 \leq \ell \leq 5$.

Proof With

$$C_1(V) = \int_{\mathbb{R}^d} V(\theta)d\theta, \;\; C_2(V) = \frac{1}{2}\int_{\mathbb{R}^d} V^2(\theta)d\theta,$$

and

$$C_3(V) = \frac{1}{3!}\left(\int_{\mathbb{R}^d} V^3(\theta)d\theta + \mathcal{E}_\alpha(V)\right),$$

the assertion trivially holds for these coefficients.

We can rewrite the expression in Lemma 3.2.3 for $C_4(V)$ as

$$\begin{aligned} C_4(V) &= \frac{1}{4!}\int_{\mathbb{R}^d}\left(V^4(\theta) + 2V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta) + \left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2\right)d\theta \\ &= \frac{1}{4!}\int_{\mathbb{R}^d}\left|V^2(\theta) + (-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta \end{aligned}$$

and this shows that $C_4(V) \geq 0$.

For $C_5(V)$, we re-group the expression given by Lemma 3.2.4 as follows.

$$\begin{aligned}
C_5(V) &= \frac{1}{5!}\int_{\mathbb{R}^d} V(\theta)\left(V^4(\theta)+2V^2(\theta)(-\Delta)^{\frac{\alpha}{2}}V(\theta)+\left|(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2\right)d\theta \\
&+ \frac{1}{5!}\left(\mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right)+2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\frac{\alpha}{2}}{2}}V(\theta)d\theta+\mathcal{E}_\alpha\left(V^2\right)\right) \\
&= \frac{1}{5!}\int_{\mathbb{R}^d} V(\theta)\left|V^2(\theta)+(-\Delta)^{\frac{\alpha}{2}}V(\theta)\right|^2 d\theta \\
&+ \frac{1}{5!}\left(\mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right)+2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\frac{\alpha}{2}}{2}}V(\theta)d\theta+\mathcal{E}_\alpha\left(V^2\right)\right).
\end{aligned}$$

If V is non-negative the first of the last two terms above is clearly non-negative. We claim the last term is also non-negative. To show this, we use Plancherel's identity for the second term and write the Dirichlet form in terms of the Fourier transform. However, we need to be a little careful here since the Fourier transform of a real valued function may be complex valued. Below we write $Re(z)$ for the real part of the complex number z and use the fact that for real valued functions, $\widehat{V}(-\theta)=\overline{\widehat{V}(\theta)}$. We write

$$\begin{aligned}
2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\frac{\alpha}{2}}{2}}V(\theta)d\theta &= \int_{\mathbb{R}^d}\widehat{V^2}(-\theta)|\theta|^{2\alpha}\widehat{V}(\theta)\bar{d}\theta+\int_{\mathbb{R}^d}\widehat{V^2}(\theta)|\theta|^{2\alpha}\widehat{V}(-\theta)\bar{d}\theta \\
&= \int_{\mathbb{R}^d}\overline{\widehat{V^2}}(\theta)|\theta|^{2\alpha}\widehat{V}(\theta)\bar{d}\theta+\int_{\mathbb{R}^d}\widehat{V^2}(\theta)|\theta|^{2\alpha}\overline{\widehat{V}}(\theta)\bar{d}\theta \\
&= 2\int_{\mathbb{R}^d}|\theta|^{2\alpha}Re\left(\widehat{V^2}(\theta)\widehat{V}(\theta)\right)\bar{d}\theta.
\end{aligned}$$

Similarly,

$$\mathcal{E}_\alpha\left(V^2\right)=\int_{\mathbb{R}^d}|\theta|^\alpha|\widehat{V^2}(\theta)|^2\bar{d}\theta \quad\text{and}\quad \mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right)=\int_{\mathbb{R}^d}|\theta|^{3\alpha}|\widehat{V}(\theta)|^2\bar{d}\theta.$$

Putting these identities together gives

$$\begin{aligned}
&\mathcal{E}_\alpha\left((-\Delta)^{\frac{\alpha}{2}}V\right)+2\int_{\mathbb{R}^d} V^2(\theta)(-\Delta)^{\frac{\frac{\alpha}{2}}{2}}V(\theta)d\theta+\mathcal{E}_\alpha\left(V^2\right) \\
&= \int_{\mathbb{R}^d}\left(|\theta|^{2\alpha}|\widehat{V}(\theta)|^2+2|\theta|^\alpha Re\left(\widehat{V^2}(\theta)\widehat{V}(\theta)\right)+|\widehat{V^2}(\theta)|^2\right)|\theta|^\alpha\bar{d}\theta \\
&= \int_{\mathbb{R}^d}\left||\theta|^\alpha\widehat{V}(\theta)+\widehat{V^2}(\theta)\right|^2|\theta|^\alpha\bar{d}\theta.
\end{aligned}$$

This together with our previous estimates show that $C_5(V)\geq 0$, for $V\geq 0$. ■

Remark 3.2.1 *It is interesting to observe that for all* $V \in \mathcal{S}(\mathbb{R}^d)$ *(regardless of the sign),* $C_2(V)$ *and* $C_4(V)$ *are non-negative. Whether or not this pattern remains as we move up along the even integers is an interesting question. With some patience one may be able to test this for* $C_6(V)$ *and perhaps even* $C_8(V)$ *but the general term is not clear at all.*

4. HEAT CONTENT FOR SMOOTH DOMAINS, PROOFS

For the purposes of this chapter, we need to make use of the following two facts about $p_t^{(\alpha)}(x,y)$ for all $0<\alpha<2$. First, there exists $c_{\alpha,d}>0$ such that

$$c_{\alpha,d}^{-1}\min\left\{t^{-d/\alpha},\frac{t}{|x-y|^{d+\alpha}}\right\}\leq p_t^{(\alpha)}(x-y)\leq c_{\alpha,d}\min\left\{t^{-d/\alpha},\frac{t}{|x-y|^{d+\alpha}}\right\},\quad (4.1)$$

for all $x,y\in\mathbb{R}^d$ and $t>0$. Secondly, in [18], R. M. Blumenthal and R. K. Getoor proved that

$$\lim_{t\downarrow 0}\frac{p_t^{(\alpha)}(x-y)}{t}=\frac{A_{\alpha,d}}{|x-y|^{d+\alpha}},\quad (4.2)$$

for all $x\neq y$. Here,

$$A_{\alpha,d}=\alpha\,2^{\alpha-1}\,\pi^{-1-\frac{d}{2}}\,\sin\left(\frac{\pi\alpha}{2}\right)\,\Gamma\left(\frac{d+\alpha}{2}\right)\,\Gamma\left(\frac{\alpha}{2}\right).\quad (4.3)$$

With all the necessary facts about the transition densities $p_t^{(\alpha)}(x,y)$ being properly recalled, we proceed to introduce the geometric objects where the stable processes will be studied. In order to do so, some additional notation and definitions need to be set.

Let $\Omega\subset\mathbb{R}^d$ satisfy the following assumptions according to the dimension d under consideration. If $d=1$, Ω will be an open interval (a,b), $-\infty<a<b<\infty$ whose length $b-a$ will be denoted by $|\Omega|$. As for $d\geq 2$, the set Ω will represent a uniformly $C^{1,1}$-regular bounded domain where $|\Omega|$ and $\partial\Omega$ stand for the Lebesgue measure of Ω in $\mathbb{R}^d$ and its boundary, respectively. We recall that

Definition 4.0.1 *$\Omega\subset\mathbb{R}^d$, $d\geq 2$ with either finite or infinite Lebesgue measure and non–empty boundary $\partial\Omega$ is said to be a uniformly $C^{1,1}$-regular set if there are constants $r,L>0$ such that for every $\sigma\in\partial\Omega$, the set $\partial\Omega\cap B_r(\sigma)$ is the graph of a $C^{1,1}$ function Λ with $||\nabla\Lambda||_\infty\leq L$. Here and for the remainder of the paper, $B_r(\sigma)$ will represent the open ball about σ with radius r.*

We point out that uniformly $C^{1,1}$–regular bounded domains are also R-smooth boundary domains (see [53, p.350]). That is, for every $\sigma \in \partial\Omega$, there are two open balls B_1 and B_2 with radii R such that $B_1 \subset \Omega$, $B_2 \subset \mathbb{R}^d \setminus \bar{\Omega}$ and $\partial B_1 \cap \partial B_2 = \sigma$. Henceforth, for any $\Omega \subset \mathbb{R}^d$, we set

$$\mathcal{H}^{d-1}(\partial\Omega) = \begin{cases} \text{Hausdorff measure of the boundary of } \Omega, & \text{if } d \geq 2, \\ \#\left\{x \in \mathbb{R} : x \in \partial\Omega\right\}, & \text{if } d = 1. \end{cases} \tag{4.4}$$

Of course, for $C^{1,1}$–domains as above, this is just the surface area of the boundary of the domain.

Let us consider for any Borel measurable sets Ω, Ω_0 in $\mathbb{R}^d$, the following quantity

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega_0}(t) = \int_{\Omega} dx\, \mathbb{P}^x\left(X_t \in \Omega_0\right) = \int_{\Omega} dx \int_{\Omega_0} dy\; p_t^{(\alpha)}(x,y), \tag{4.5}$$

which turns out to be well defined for example when either Ω or Ω_0 has finite Lebesgue measure. When $\Omega = \Omega_0$, we simply denote $\mathbb{H}^{(\alpha)}_{\Omega,\Omega}(t)$ by $\mathbb{H}^{(\alpha)}_{\Omega}(t)$.

One of the goals of this paper is to study the small time behavior of the function $\mathbb{H}^{(\alpha)}_{\Omega}(t)$, which is equivalent to analyzing the behavior of $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ as $t \downarrow 0$ since

$$\mathbb{H}^{(\alpha)}_{\Omega}(t) = |\Omega| - \mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t). \tag{4.6}$$

We note that the function $u(t,x) = \int_{\Omega} dy\, p_t^{(\alpha)}(x,y)$ is the unique weak solution to the initial value problem

$$\begin{aligned} \frac{du}{dt} &= -(-\Delta)^{\frac{\alpha}{2}} u(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= \mathbb{1}_{\Omega}(x). \end{aligned} \tag{4.7}$$

In other words, the initial value problem (4.7) exactly says that $\mathbb{H}^{(\alpha)}_{\Omega}(t)$ represents the amount of heat in Ω, if Ω is at initial temperature 1 and if Ω^c is at initial temperature 0. In [10], M. van den Berg called $\mathbb{H}^{(2)}_{\Omega}(t)$ the heat content of Ω in $\mathbb{R}^d$ and analyzed its behavior when the domain is a horn-shaped domain. Following the terminology introduced by M. van den Berg, we will also call $\mathbb{H}^{(\alpha)}_{\Omega}(t)$ the heat content of Ω in $\mathbb{R}^d$.

We now proceed to interpret $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ and discuss its connections with a spectral function and the heat semi-group. From definition (4.5), we observe that $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ describes how fast in average the underlying stochastic process $\mathbf{X}$, when started at some point inside of Ω, escape from Ω. When $\alpha = 2$, as mentioned previously, the process $\mathbf{X}$ is the Brownian motion at twice speed whose paths are continuous, whereas for $0 < \alpha < 2$, the paths of $\mathbf{X}$ are only càdlàg. Thus, $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$, by definition, is related to the jumps or the fluctuation of the paths up to time t of the corresponding process under consideration.

The interest in studying $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ derives from the results known about $\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t)$ in higher dimensions which we proceed to mention. We consider the heat semi-group acting on $L^2(\mathbb{R}^d)$ associated with the process $\mathbf{X}$. Namely,

$$T_t^{(\alpha)}(f)(x) = \int_{\mathbb{R}^d} dy\ f(y)\ p_t^{(\alpha)}(x-y) = \mathbb{E}\left[f(x - X_t)\right]. \tag{4.8}$$

Therefore, it follows from (4.5) that

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) = \left\langle T_t^{(\alpha)}(\mathbb{1}_\Omega), \mathbb{1}_{\Omega^c}\right\rangle, \tag{4.9}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $L^2(\mathbb{R}^d)$. In [45] and [48], Miranda, Pallora, Paronnetto and Preunkert have investigated for the Brownian motion case $\alpha = 2$ the connections between $\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t)$, functions of bounded variation and the isoperimetric inequality by means of analytic tools when $d \geq 2$ for not only uniformly $C^{1,1}$-regular bounded domains but also bounded Cacciopoli sets. In fact, it is shown in [48, Prop. 8] that

$$\frac{\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t)}{t^{\frac{1}{2}}} \leq \frac{1}{\sqrt{\pi}}\mathcal{H}^{d-1}(\partial\Omega) \tag{4.10}$$

for all $t > 0$, while in [45, Th 2.4] is proved that

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t)}{t^{\frac{1}{2}}} = \frac{1}{\sqrt{\pi}}\mathcal{H}^{d-1}(\partial\Omega). \tag{4.11}$$

Consequently, the preceding limit and (4.6) yield the following asymptotic expansion for such domains,

$$\mathbb{H}^{(2)}_{\Omega}(t) = |\Omega| - \frac{1}{\sqrt{\pi}}\mathcal{H}^{d-1}(\partial\Omega)\, t^{\frac{1}{2}} + o(t^{\frac{1}{2}}),\ \ t \downarrow 0.$$

The main observation here is that we are able to recover a geometry feature of the set Ω in addition to its volume from the small asymptotic expansion of $\mathbb{H}^{(2)}_{\Omega}(t)$, namely, the surface area of its boundary $\partial\Omega$. At this point, it is natural to ask:

Question 4.0.1 *Is there a function $f_\alpha(t)$, $0<\alpha<2$ such that*

$$\lim_{t\downarrow 0}\frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)}{f_\alpha(t)}$$

exists? And, what geometry quantities of Ω can we recover from this limit?

The answer to the first question is affirmative and we have explicit expressions for $f_\alpha(t)$ by investigating the one dimensional case. Regarding the second question, we will see later that we recover the surface area of the boundary if $1\le\alpha<2$ and the fractional α–perimeter when $0<\alpha<1$, (see (4.13) below).

For $d=1$, our main result is the following.

Theorem 4.0.1 *Let $\Omega=(a,b)$, $-\infty<a<b<\infty$ and $|\Omega|=b-a$.*

(i) *For $1<\alpha\le 2$ and all $t>0$, we have*

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)=\frac{2}{\pi}\,\Gamma\left(1-\frac{1}{\alpha}\right)t^{\frac{1}{\alpha}}+R_\alpha(t),$$

with

$$|R_\alpha(t)|\le C\left(t\,\mathbb{1}_{(1,2)}(\alpha)+t^{3/2}\,\mathbb{1}_{\{2\}}(\alpha)\right).$$

(ii) *For $\alpha=1$ and all $t>0$, the following equality holds.*

$$\mathbb{H}^{(1)}_{\Omega,\Omega^c}(t)=\frac{2}{\pi}\,t\ln\left(\frac{1}{t}\right)+\frac{2}{\pi}\left(|\Omega|\,\arctan\left(\frac{t}{|\Omega|}\right)+\frac{1}{2}\,t\,\ln\left(t^2+|\Omega|^2\right)\right),$$

(iii) *Let $0<\alpha<1$ and $0<t<\min\{|\Omega|^\alpha,e^{-1}\}$. We obtain the subsequent expansions according to the following sub-cases.*

(iv) *If $1/\alpha\notin\mathbb{N}$, then there is a constant C_α independent of Ω such that*

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)=\frac{2}{\pi}\sum_{n=1}^{\left[\frac{1}{\alpha}\right]}(-1)^{n-1}\frac{\Gamma(n\alpha)}{(1-n\alpha)n!}\,|\Omega|^{1-n\alpha}\sin\left(\frac{\pi n\alpha}{2}\right)t^n+\;C_\alpha\,t^{\frac{1}{\alpha}}+R_\alpha(t),$$

with $|R_\alpha(t)|\le C\;t^{\left[\frac{1}{\alpha}\right]+1}$.

(v) *If $\alpha = 1/N$, for some $N \in \mathbb{N}$, then there is a constant $C_N(\Omega)$ such that*

$$\mathbb{H}^{(1/N)}_{\Omega,\Omega^c}(t) = \frac{2}{\pi}\sum_{n=1}^{N-1}(-1)^{n-1}\frac{\Gamma(n/N)}{(1-n/N)n!}\,|\Omega|^{1-n/N}\sin\left(\frac{\pi n}{2N}\right)t^n$$
$$+\,(-1)^{N-1}\frac{2}{\pi(N-1)!}\,t^N\ln\left(\frac{1}{t}\right)+C_N(\Omega)\,t^N+R_{1/N}(t),$$

with $\left|R_{1/N}(t)\right| \leq C\; t^{N+1}$.

In all the above statements, $C > 0$ depends only on α and Ω.

We notice that Theorem 4.0.1 ensures on one hand the existence of a non-zero function $h_\alpha(t)$ such that

$$\lim_{t\downarrow 0}\frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)-h_\alpha(t)}{t^{\frac{1}{\alpha}}} \tag{4.12}$$

exists for all $0 < \alpha \leq 1$. On the other hand, for $1 < \alpha \leq 2$ the above limit also exists with $h_\alpha(t) = 0$.

The upcoming Theorem 4.0.2 will show that the preceding limit (4.12) also exists in higher dimensions for $1 < \alpha < 2$ whereas for $0 < \alpha \leq 1$ we are only able to obtain a weaker version of the statements (ii) and (iii) provided in Theorem 4.0.1. For $\alpha = 1$, it is worth noting that Theorems 4.0.1 and 4.0.2 indicate that $h_1(t)$ should be equal to

$$\pi^{-1}\;\mathcal{H}^{d-1}(\partial\Omega)\;t\;\ln\left(\frac{1}{t}\right).$$

The main difficulty here would consist in identifying the limit (4.12).

We now continue to elaborate further in the observation previously made. We point out that the factor 2 which appears in the first term of each expansion in Theorem 4.0.1 comes from the boundary points of the interval (a, b) and by definition (4.4), we have $\mathcal{H}^0(\partial(a, b)) = 2$. With simple observation, we notice that part (i) can be restated as

$$\lim_{t\downarrow 0}\frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)}{t^{\frac{1}{\alpha}}} = \frac{1}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)\mathcal{H}^0(\partial\Omega).$$

For $\alpha = 2$, the above limit is the one-dimensional analogue of (4.11) with the same constant which is not unusual since when dealing with a d-dimensional Brownian motion most of the computations reduce to the one dimensional setting due to the independence of the components. However, for $0 < \alpha < 2$ the components are no longer independent and an approach involving estimates of the heat kernels is required.

Because of the last considerations, we are led to conjecture that in higher dimensions we should expect to recover, with the first term of each expansion, the Hausdorff measure of the boundary. Our Theorem 4.0.2 asserts that the conjecture is correct when $1 < \alpha < 2$ with the same constant as in part (i) of Theorem 4.0.1. As for $\alpha = 1$, we are also able to recover the Hausdorff measure of the boundary but the constant is dimensional dependent, as it is to be expected. For $0 < \alpha < 1$, the fractional α-perimeter $\mathcal{P}_\alpha(\Omega)$, defined to be

$$\mathcal{P}_\alpha(\Omega) = \int_\Omega \int_{\Omega^c} \frac{dx\, dy}{|x-y|^{d+\alpha}}, \tag{4.13}$$

is recovered. The above quantity turns out to be linked with celebrated Hardy inequalities. We refer the reader to the papers of Z.Q Chen, R. Song [24] and R.L. Frank, R. Seiringer [29] for further results involving this quantity. In fact, it is shown in [29] that there exists $C_{d,\alpha} > 0$ such that

$$|\Omega|^{(d-\alpha)/d} \leq C_{d,\alpha}\, \mathcal{P}_\alpha(\Omega)$$

with equality if and only if Ω is a ball. It is also proved in [31] that

$$\lim_{\alpha \downarrow 0} \alpha \mathcal{P}_\alpha(\Omega) = d\, |B_1(0)|\, |\Omega|,$$

$$\lim_{\alpha \uparrow 1} (1-\alpha) \mathcal{P}_\alpha(\Omega) = K_d\, \mathcal{H}^{d-1}(\partial\Omega),$$

for some $K_d > 0$.

It is interesting to notice that the last limit intuitively gives an insight that the surface area of the boundary should be recovered when considering the small time behavior of the function $\mathbb{H}^{(1)}_{\Omega,\Omega^c}(t)$(Cauchy process, $\alpha = 1$) which is exactly what our next result shows.

Theorem 4.0.2 *Assume $\Omega \subset \mathbb{R}^d$, $d \geq 2$ is a uniformly $C^{1,1}$-regular bounded domain.*

(i) *For $1 < \alpha < 2$, we have*

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) \leq \frac{1}{\pi}\Gamma\left(1 - \frac{1}{\alpha}\right) \mathcal{H}^{d-1}(\partial\Omega)\, t^{\frac{1}{\alpha}} \tag{4.14}$$

for all $t > 0$. *Moreover,*

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)}{t^{\frac{1}{\alpha}}} = \frac{1}{\pi}\,\Gamma\left(1-\frac{1}{\alpha}\right)\,\mathcal{H}^{d-1}(\partial\Omega). \tag{4.15}$$

(ii) *For* $\alpha = 1$,

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(1)}_{\Omega,\Omega^c}(t)}{t\ln\left(\frac{1}{t}\right)} = \frac{1}{\pi}\,\mathcal{H}^{d-1}(\partial\Omega).$$

(iii) *For* $0 < \alpha < 1$,

$$\lim_{t\downarrow 0} \frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)}{t} = A_{\alpha,d}\;\mathcal{P}_\alpha(\Omega),$$

with $A_{\alpha,d}$ *and* $\mathcal{P}_\alpha(\Omega)$ *as defined in* (4.3) *and* (4.13)*, respectively.*

The proof of (i) is a consequence of the Lebesgue Dominated Convergence Theorem and subordination techniques. Part (iii) is obtained by combining once again the Lebesgue Dominated Convergence Theorem with (4.2). The case $\alpha = 1$ requires a more elaborate approach.

We next state some connections between $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ and the spectral heat content of Ω which has been widely studied only for the Brownian motion case. We recall that

$$\tau^{(\alpha)}_\Omega = \inf\left\{s \geq 0 : X_s \in \Omega^c\right\}$$

is the first exit time from Ω. The *spectral heat content* of Ω, denoted by $Q^{(\alpha)}_\Omega(t)$, is defined as

$$Q^{(\alpha)}_\Omega(t) = \int_\Omega dx \int_\Omega dy\; p^{\Omega,\alpha}_t(x,y), \tag{4.16}$$

where $p^{\Omega,\alpha}_t(x,y)$ is the transition density for the stable process killed upon exiting Ω. More precisely, this is the heat kernel for the Dirichlet fractional Laplacian. Recall by (1.13) that

$$p^{\Omega,\alpha}_t(x,y) = p^{(\alpha)}_t(x,y)\;\mathbb{P}\left(\tau^{(\alpha)}_\Omega > t \mid X_0 = x,\; X_t = y\right).$$

The name *spectral heat content* given to $Q^{(\alpha)}_\Omega(t)$ comes from the fact that $p^{\Omega,\alpha}_t(x,y)$ can be written in terms of the eigenvalues and eigenfunctions of the domain Ω. That is, when $|\Omega| <$

∞, it is known ([27]) that there exists an orthonormal basis of eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$ for $L^2(\Omega)$ with corresponding eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfying $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ...$ and $\lambda_n \to \infty$ as $n \to \infty$ such that

$$p_t^{\Omega,\alpha}(x,y) = \sum_{n=1}^{\infty} e^{-t\lambda_n}\,\phi_n(x)\,\phi_n(y). \tag{4.17}$$

Notice that due to (4.16) and the last equality, we obtain an expression for $Q_\Omega^{(\alpha)}(t)$ involving both the spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ and eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}}$. Namely,

$$Q_\Omega^{(\alpha)}(t) = \sum_{n=1}^{\infty} e^{-t\lambda_n}\left(\int_\Omega dx\,\phi_n(x)\right)^2.$$

We remark for the sake of completeness that by mimicking the proof provided in [12, Prop 1.4], we have

$$Q_\Omega^{(\alpha)}(t) = e^{-\lambda_1 t}\left(||\phi_1||_1^2 + \mathcal{O}(t^{-d/\alpha})\right),\ t\uparrow\infty.$$

Henceforth, we will only be concerned about the behavior of $Q_\Omega^{(\alpha)}(t)$ as $t\downarrow 0$.

The study of the small time behavior of the spectral heat content $Q_\Omega^{(\alpha)}(t)$ arises from the results associated with the asymptotic expansion of the heat trace for smooth domains. The heat trace of a bounded domain Ω is defined to be

$$\mathcal{Z}_\Omega^{(\alpha)}(t) = \frac{1}{p_t^{(\alpha)}(0)}\int_\Omega dx\,p_t^{\Omega,\alpha}(x,x) = \frac{1}{p_t^{(\alpha)}(0)}\sum_{n=1}^{\infty} e^{-\lambda_n t},$$

where the second equality is obtained by means of (4.17). In [6], R. Bañuelos and T. Kulczycki provide the following second order expansion of the heat trace for R-smooth boundary domains which holds every $0<\alpha\leq 2$ (the case $\alpha=2$ was proved in [14] by M. van der Berg).

$$\mathcal{Z}_\Omega^{(\alpha)}(t) = |\Omega| - C_{d,\alpha}\,\mathcal{H}^{d-1}(\partial\Omega)\,t^{\frac{1}{\alpha}} + \mathcal{O}(t^{\frac{2}{\alpha}}), \tag{4.18}$$

as $t\downarrow 0$, where $C_{d,\alpha}>0$ admits a probabilistic representation in terms of the exit time from the upper half–plane of the underlying α–stable process. This result was extended by Bañuelos, Kulczycki and Siudeja to domains with Lipschitz boundaries in [7]. It is interesting to note that the above expansion for $0<\alpha<2$ was motivated by scaling and keeping in mind the behavior of the heat trace for the Brownian motion. Based on this, it is natural to predict the second order expansion of $Q_\Omega^{(\alpha)}(t)$ by considering as a model the spectral heat content of the Brownian motion $Q_\Omega^{(2)}(t)$. To our surprise, $Q_\Omega^{(2)}(t)$ only models the behavior of $Q_\Omega^{(\alpha)}(t)$ for the cases $1 < \alpha < 2$.

The small time asymptotic behavior of $Q_\Omega^{(\alpha)}(t)$ is known so far only for $\alpha = 2$. In fact, the following result was proved by van den Berg and Le Gall in [12] for smooth domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$.

$$Q_\Omega^{(2)}(t) = |\Omega| - \frac{2}{\sqrt{\pi}}\mathcal{H}^{d-1}(\partial\Omega)\, t^{\frac{1}{2}} + \left(2^{-1}(d-1)\int_{\partial\Omega} \mathcal{M}(\sigma)d\sigma\right) t + \mathcal{O}(t^{\frac{3}{2}}), \tag{4.19}$$

as $t \downarrow 0$. Here, $\mathcal{M}(\sigma)$ denotes the mean curvature at the point $\sigma \in \partial\Omega$. For more on the heat content asymptotics and its connections to the eigenvalues (spectrum) of the Laplacian in the domain Ω, we direct the reader to Gilkey's monograph [32] and to van den Berg, Dryden and Kappeler [9] and the many references to the literature contained therein. We also refer the reader to [15] for matters related to the spectral heat content and Brownian motion for regions with a fractal boundary.

According to [20, Corollary 1], for Ω a uniformly $C^{1,1}$–regular bounded domain is known that there exists $c > 0$ such that

$$c^{-1} \min\left\{1, \frac{\rho_\Omega^{\alpha/2}(x)}{\sqrt{t}}\right\} \leq \int_\Omega dy\; p_t^{\Omega,\alpha}(x,y) \leq c \min\left\{1, \frac{\rho_\Omega^{\alpha/2}(x)}{\sqrt{t}}\right\}$$

for all $x \in \Omega$ and $0 < t \leq 1$. Here, $\rho_\Omega(x)$ represents the distance from x to the boundary of Ω. Therefore, for bounded domains Ω with smooth boundary $\partial\Omega$, it is possible to prove by using the techniques developed in [12] that

$$\int_\Omega dx\; \min\left\{1, \frac{\rho_\Omega^{\alpha/2}(x)}{\sqrt{t}}\right\} = |\Omega| - C_\alpha \mathcal{H}^{d-1}(\partial\Omega)\; t^{\frac{1}{\alpha}} + \mathcal{O}(t^{\frac{2}{\alpha}}), \tag{4.20}$$

as $t \downarrow 0$ for some $C_\alpha > 0$. Hence, based on the preceding expansion and the small time expansion (4.19) for the Brownian motion, we are led to conjecture that a similar asymptotic expansion to the right hand side of (4.20) should also hold for $Q_\Omega^{(\alpha)}(t)$. However, Theorem 1.4.1 asserts that such a conjecture may only hold for $1 < \alpha < 2$.

4.1 Proof of theorem 4.0.1

We will begin this section by presenting some fundamental properties about the $\alpha/2$–subordinator $\mathbf{S} = \{S_t\}_{t\geq 0}$.

Proposition 4.1.1

(i) *For all* $\lambda, t > 0$,

$$\mathbb{E}\left[\exp\left(-\lambda S_t\right)\right] = \exp\left(-t\lambda^{\alpha/2}\right).$$

(ii) *For all* $-\infty < \beta < \frac{\alpha}{2}$,

$$\mathbb{E}\left[S_1^{\beta}\right] = \int_0^{\infty} ds\, s^{\beta}\, \eta_1^{(\alpha/2)}(s) = \frac{\Gamma(1-\frac{2\beta}{\alpha})}{\Gamma(1-\beta)}. \tag{4.21}$$

(iii) *Let* $\kappa > 0$*. Then, there exists* $C_\alpha > 0$ *such that*

$$\mathbb{E}\left[\exp\left(-\frac{\kappa^2}{S_1}\right)\right] \leq C_\alpha\, \kappa^{-\alpha}. \tag{4.22}$$

Proof (i) and (ii) are standard facts already established in (1.6) and example 1.0.2.

Regarding (iii), it is known (see [19, p 97]) that $\eta_1^{(\alpha/2)}(s) \leq C_0(\alpha)\min\left\{1, s^{-1-\frac{\alpha}{2}}\right\}$ for some $C_0(\alpha) > 0$. Hence, after a suitable change of variables, we arrive at

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\frac{\kappa^2}{S_1}\right)\right] &\leq C_0(\alpha)\int_0^{\infty}\exp\left(-\kappa^2/s\right)\min\left\{1, s^{-1-\frac{\alpha}{2}}\right\} ds \\ &\leq C_0(\alpha)\kappa^2\int_0^{\infty}\exp\left(-w\right)\left\{\frac{w}{\kappa^2}\right\}^{1+\frac{\alpha}{2}}\frac{dw}{w^2} = \frac{C_0(\alpha)\Gamma(\alpha/2)}{\kappa^{\alpha}}.\end{aligned}$$

Thus, the proof is complete by taking $C_\alpha = C_0(\alpha)\Gamma(\alpha/2)$. ■

In what follows, we shall assume that $\Omega = (a,b)$, $a < b$ with length $b - a = |\Omega|$. We start by expressing $\mathbb{H}_{\Omega,\Omega^c}^{(\alpha)}(t)$ in a more convenient form. For this purpose, we require the following two fundamental identities concerning the process $\mathbf{X}$ which can be easily deduced from the characteristic function (1.1).

$$\mathbb{P}^x\left(X_t \in A\right) = \mathbb{P}\left(x - t^{\frac{1}{\alpha}}X_1 \in A\right)$$

$$\mathbb{P}\left(X_t \in A\right) = \mathbb{P}\left(-X_t \in A\right),$$

for all $t > 0$, $x \in \mathbb{R}^d$ and A Borel measurable set in $\mathbb{R}^d$. In particular, when $d = 1$, we obtain

$$\mathbb{P}^x\left(X_t \leq a\right) = \mathbb{P}\left((x-a)t^{-\frac{1}{\alpha}} \leq X_1\right),$$
$$\mathbb{P}^x\left(b \leq X_t\right) = \mathbb{P}\left((b-x)t^{-\frac{1}{\alpha}} \leq X_1\right),$$

for all $x, a, b \in \mathbb{R}$ and $t > 0$. The last identities in turn imply that

$$\begin{aligned}\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) &= \int_a^b dx\,\left[\mathbb{P}^x\left(X_t \leq a\right) + \mathbb{P}^x\left(b \leq X_t\right)\right] \\ &= \int_a^b dx\;\mathbb{P}\left((x-a)t^{-\frac{1}{\alpha}} \leq X_1\right) + \int_a^b dx\;\mathbb{P}\left((b-x)t^{-\frac{1}{\alpha}} \leq X_1\right).\end{aligned}$$

Next, a simple change of variables yields

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) = 2\,t^{\frac{1}{\alpha}} \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dw\;\mathbb{P}\left(w \leq X_1\right),$$

which shows that $\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)$ is related to the tail behavior of the process $\mathbf{X}$.

We set

$$\ell_\alpha(t) = \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dw\;\mathbb{P}\left(w \leq X_1\right). \tag{4.23}$$

Proof of Theorem 4.0.1: Since the tail behavior of the Brownian motion and stable processes have an exponential and an algebraic decay at infinity, respectively, we need to treat the cases $1 < \alpha \leq 2$, $\alpha = 1$ and $0 < \alpha < 1$ separately.

Case $1 < \alpha \leq 2$: We rewrite $\ell_\alpha(t)$ as a double integral as follows.

$$\ell_\alpha(t) = \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dw \int_w^\infty dz\, p_1^{(\alpha)}(z).$$

Thus, by interchanging the order of integration, we arrive at

$$\begin{aligned}\ell_\alpha(t) &= \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dz\, p_1^{(\alpha)}(z) \int_0^z dw + \int_{|\Omega|t^{-\frac{1}{\alpha}}}^\infty dz\, p_1^{(\alpha)}(z) \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dw \\ &= \int_0^{|\Omega|t^{-\frac{1}{\alpha}}} dz\, z\, p_1^{(\alpha)}(z) + |\Omega|\, t^{-\frac{1}{\alpha}} \int_{|\Omega|t^{-\frac{1}{\alpha}}}^\infty dz\, p_1^{(\alpha)}(z).\end{aligned}$$

In probabilistic terms, we have shown that

$$\ell_\alpha(t) = \mathbb{E}\left[X_1, 0 \le X_1 \le |\Omega|\, t^{-\frac{1}{\alpha}}\right] + |\Omega|\, t^{-\frac{1}{\alpha}}\, \mathbb{P}\left(|\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right)$$
$$= \mathbb{E}\left[X_1, 0 \le X_1\right] - \mathbb{E}\left[X_1, |\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right] + |\Omega|\, t^{-\frac{1}{\alpha}}\, \mathbb{P}\left(|\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right).$$

Let us denote

$$j_\alpha(t) = \mathbb{E}\left[X_1, |\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right] \tag{4.24}$$

and observe that

$$j_\alpha(t) \ge |\Omega|\, t^{-\frac{1}{\alpha}}\, \mathbb{P}\left(|\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right).$$

Thus, the remainder function $R_\alpha(t)$ to be defined as follows

$$R_\alpha(t) = 2t^{\frac{1}{\alpha}}\left(-\mathbb{E}\left[X_1, |\Omega|\, t^{-\frac{1}{\alpha}} < X_1\right] + |\Omega|\, t^{-\frac{1}{\alpha}}\, \mathbb{P}\left(|\Omega|\, t^{-\frac{1}{\alpha}} \le X_1\right)\right),$$

satisfies $|R_\alpha(t)| \le 4\, t^{\frac{1}{\alpha}}\, j_\alpha(t)$. Therefore, to finish the proof of part (i) of Theorem 4.0.1, it suffices to obtain upper bounds for the function $j_\alpha(t)$ according to the cases $\alpha = 2$ and $1 < \alpha < 2$.

Case $\alpha = 2$: It is clear from (4.24) that

$$j_2(t) = (4\pi)^{-1/2} \int_{|\Omega| t^{-\frac{1}{2}}}^{\infty} dz\, z \exp\left(-\frac{z^2}{4}\right) = \pi^{-1/2} \exp\left(-\frac{|\Omega|^2}{4t}\right).$$

Next, by applying the elementary inequality

$$\exp(-x) \le x^{-1}, \;\; x > 0,$$

we conclude that $j_2(t) \le 4\pi^{-1/2}\, |\Omega|^{-2}\, t$. Hence, we have shown that

$$\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t) = 2\, \mathbb{E}\left[X_1, 0 \le X_1\right]\, t^{\frac{1}{2}} + R_2(t),$$

with $|R_2(t)| \le C\, t^{\frac{3}{2}}$ for all $t > 0$.

Case $1 < \alpha < 2$: We observe due to (4.1) that for all $z \in \mathbb{R} \setminus \{0\}$ we have

$$p_1^{(\alpha)}(z) \le c_{\alpha,1} \, |z|^{-1-\alpha}$$

so that

$$j_\alpha(t) \le c_{\alpha,1} \int_{|\Omega| t^{-\frac{1}{\alpha}}}^{\infty} dz \, z^{-1-\alpha} \, z = c_{\alpha,1} (\alpha - 1)^{-1} \, |\Omega|^{1-\alpha} \, t^{1-\frac{1}{\alpha}}.$$

Thus, we arrive at

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) = 2 \, \mathbb{E} \left[X_1, 0 \le X_1 \right] \, t^{\frac{1}{\alpha}} + R_\alpha(t),$$

with $|R_\alpha(t)| \le C \, t$ for all $t > 0$.

Remark 4.1.1 *By combining (1.4) and Fubini's Theorem, we obtain for all* $1 < \alpha \le 2$ *that*

$$\begin{aligned}
\mathbb{E} \left[X_1, 0 \le X_1 \right] &= \int_0^\infty dz \, z \, \mathbb{E} \left[p^{(2)}_{S_1}(z) \right] \\
&= \mathbb{E} \left[\int_0^\infty dz \, z \, p^{(2)}_{S_1}(z) \right] = \frac{1}{\sqrt{\pi}} \, \mathbb{E} \left[S_1^{1/2} \right] = \frac{1}{\pi} \, \Gamma \left(1 - \frac{1}{\alpha} \right),
\end{aligned}$$

where in the last equality we have appealed to formula (4.21).

We proceed to deal with Cauchy processes.

Case $\alpha = 1$: We begin by recalling some elementary calculus identities.

$$\arctan(w) + \arctan\left(\frac{1}{w} \right) = \frac{\pi}{2} \tag{4.25}$$

$$\int dw \; \arctan(w) = w \arctan(w) - \frac{1}{2} \ln \left(1 + w^2 \right) + C. \tag{4.26}$$

By appealing to the above identities, the explicit expression of the Cauchy heat kernel (1.2) and (4.23), we have

$$\begin{aligned}
\ell_1(t) &= \int_0^{|\Omega| t^{-1}} dw \int_w^\infty \frac{dz}{\pi (1 + z^2)} \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} \, |\Omega| \, t^{-1} - \int_0^{|\Omega| \, t^{-1}} dw \;\; \arctan(w) \right) \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} \, |\Omega| \, t^{-1} - \left[|\Omega| \; t^{-1} \; \arctan(|\Omega| \, t^{-1}) - \frac{1}{2} \ln \left(1 + \frac{|\Omega|^2}{t^2} \right) \right] \right) \\
&= \frac{1}{\pi} \ln \left(\frac{1}{t} \right) + \frac{1}{\pi} \left(|\Omega| \; t^{-1} \; \arctan \left(\frac{t}{|\Omega|} \right) + \frac{1}{2} \ln \left(t^2 + |\Omega|^2 \right) \right).
\end{aligned}$$

Therefore, it follows from the above expression that

$$\mathbb{H}^{(1)}_{\Omega,\Omega^c}(t) = \frac{2}{\pi}\, t \ln\left(\frac{1}{t}\right) + \frac{2}{\pi}\left(|\Omega| \,\arctan\left(\frac{t}{|\Omega|}\right) + \frac{1}{2}\, t\, \ln\left(t^2 + |\Omega|^2\right)\right),$$

for all $t > 0$ and this completes the proof of part (ii) of Theorem 4.0.1.

Case $0 < \alpha < 1$: Assume $0 < t \leq \min\{|\Omega|^{\alpha}, e^{-1}\}$. In [49, p. 88], the following power series representation is provided for the one dimensional density function $p_1^{(\alpha)}(z)$ for any $z > 0$, $0 < \alpha < 1$.

$$p_1^{(\alpha)}(z) = \sum_{n=1}^{\infty} a_n(\alpha) z^{-1-n\alpha}$$

with

$$a_n(\alpha) = (-1)^{n-1} \frac{\Gamma(n\alpha + 1)}{\pi\, n!} \sin\left(\frac{\pi\, n\, \alpha}{2}\right).$$

Notice that by applying Fubini's Theorem, we obtain for $w > 0$

$$\int_w^{\infty} dz \left(\sum_{n=1}^{\infty} |a_n(\alpha)|\, z^{-1-n\alpha}\right) = \sum_{n=1}^{\infty} \frac{|a_n(\alpha)|}{n\alpha}\left(\frac{1}{w^{\alpha}}\right)^n. \tag{4.27}$$

By appealing to the following estimate

$$\Gamma(t+1) \backsim \sqrt{2\pi t}(te^{-1})^t, \quad t \to \infty, \tag{4.28}$$

we can prove that the series on the right hand side of the (4.27) is well defined for all $w > 0$ since its radius of convergence is infinity. To see this, we note that

$$|a_n(\alpha)| \leq \frac{\Gamma(n\alpha + 1)}{n!}, \tag{4.29}$$

so that by (4.28) we arrive at

$$\overline{\lim_{n\to\infty}} \left(\frac{\Gamma(n\alpha+1)}{n\, n!}\right)^{1/n} = \overline{\lim_{n\to\infty}} \left(\frac{\sqrt{\alpha}}{n}\right)^{1/n} \frac{\alpha^{\alpha} e^{1-\alpha}}{n^{1-\alpha}} = 0, \tag{4.30}$$

whenever $0 < \alpha < 1$. Therefore, by using once more Fubini's Theorem, we have for $w > 0$

$$\int_w^{\infty} dz\, p_1^{(\alpha)}(z) = \sum_{n=1}^{\infty} \frac{a_n(\alpha)}{n\alpha}\left(\frac{1}{w^{\alpha}}\right)^n. \tag{4.31}$$

Next, it is easy to show that

$$\int_1^{|\Omega|t^{-\frac{1}{\alpha}}} dw \int_w^\infty dz\, z^{-1-n\alpha} = \left(n\ln\left(\frac{1}{t}\right) + \ln\left(|\Omega|\right)\right)\cdot \mathbb{1}_{\{n\alpha=1\}} + \left(\frac{|\Omega|^{1-n\alpha}\ t^{n-\frac{1}{\alpha}} - 1}{n\alpha\ (1-n\alpha)}\right)\cdot \mathbb{1}_{\{n\alpha\neq 1\}}. \tag{4.32}$$

Before continuing, let us introduce some notation to simplify the formulas to appear below. For $m \in \mathbb{N}\cup\{\infty\}$, $t > 0$ and $1/\alpha \notin \mathbb{N}$, we set

$$s_m(t) = \sum_{n=1}^{m} \frac{a_n(\alpha)\,|\Omega|^{1-n\alpha}\ t^n}{n\alpha(1-n\alpha)},\quad r_m(t) = \sum_{n=1}^{m}\frac{a_n(\alpha)\,t^n}{n\alpha(1-n\alpha)}, \tag{4.33}$$
$$\tilde{s}_m(t) = \sum_{n=m}^{\infty} \frac{a_n(\alpha)\,|\Omega|^{1-n\alpha}\ t^n}{n\alpha(1-n\alpha)},\quad \tilde{r}_m(t) = \sum_{n=m}^{\infty}\frac{a_n(\alpha)\,t^n}{n\alpha(1-n\alpha)}.$$

These series are well defined for all $t > 0$ since by using (4.29) and (4.30), we obtain that

$$\overline{\lim_{n\to\infty}}\left(\frac{|a_n(\alpha)|\,|\Omega|^{1-n\alpha}}{n\alpha\,|1-n\alpha|}\right)^{1/n} = \overline{\lim_{n\to\infty}}\left(\frac{|a_n(\alpha)|}{n\alpha\,|1-n\alpha|}\right)^{1/n} = 0$$

for all $0 < \alpha < 1$ and $1/\alpha \notin \mathbb{N}$.

As a result of the preceding facts and the elementary tools of calculus, we are allowed to interchange in (4.31) the sum with the integral sign over any compact set contained in $(0,\infty)$. Thus, if $1/\alpha \notin \mathbb{N}$, we conclude by (4.32) and (4.33) that

$$\begin{aligned}\int_1^{|\Omega|t^{-\frac{1}{\alpha}}} dw \int_w^\infty dz\, p_1^{(\alpha)}(z) &= t^{-\frac{1}{\alpha}}\ s_\infty(t) - r_\infty(1) \\ &= t^{-\frac{1}{\alpha}}\ s_{[\frac{1}{\alpha}]}(t) - r_\infty(1) + t^{-\frac{1}{\alpha}}\ \tilde{s}_{[\frac{1}{\alpha}]+1}(t),\end{aligned} \tag{4.34}$$

where $[1/\alpha]$ denotes the integer part of $1/\alpha$. On the other hand, if $\alpha = 1/N$ for some $N \in \mathbb{N}$, we obtain

$$\begin{aligned}\int_1^{|\Omega|t^{-N}} dw \int_w^\infty dz\, p_1^{(1/N)}(z) &= t^{-N}\, s_{N-1}(t) - r_{N-1}(1) + a_N(1/N)N\ \ln\left(\frac{1}{t}\right) \\ &\quad + a_N(1/N)\ln(|\Omega|) + t^{-N}\,\tilde{s}_{N+1}(t) - \tilde{r}_{N+1}(1) \\ &= t^{-N}\, s_{N-1}(t) + a_N(1/N)N\ \ln\left(\frac{1}{t}\right) + C_N^*(\Omega) + t^{-N}\tilde{s}_{N+1}(t)\end{aligned} \tag{4.35}$$

where

$$C_N^*(\Omega) = a_N\,(1/N)\ln(|\Omega|) - r_{N-1}(1) - \tilde{r}_{N+1}(1). \tag{4.36}$$

We rewrite $\ell_\alpha(t)$ given in (4.23) as follows.

$$\ell_\alpha(t) = \int_0^1 dw\, \mathbb{P}\,(w \le X_1) + \int_1^{|\Omega| t^{-\frac{1}{\alpha}}} dw \int_w^\infty dz\, p_1^{(\alpha)}(z).$$

Then, by using the last equality and the identities (4.34) and (4.35), we arrive at

$$\mathbb{H}_{\Omega,\Omega^c}^{(\alpha)}(t) = 2\, s_{[\frac{1}{\alpha}]}(t) + C_\alpha\, t^{\frac{1}{\alpha}} + 2\, \tilde{s}_{[\frac{1}{\alpha}]+1}(t)$$

for $1/\alpha \notin \mathbb{N}$. Here,

$$C_\alpha = 2\left(\int_0^1 dw\ \mathbb{P}\,(w \le X_1) - r_\infty(1)\right).$$

As for the case $\alpha = 1/N$, some $N \in \mathbb{N}$, we have

$$\mathbb{H}_{\Omega,\Omega^c}^{(1/N)}(t) = 2\ s_{N-1}(t) + 2N\, a_N(1/N)\, t^N \ln\left(\frac{1}{t}\right) + C_N(\Omega)\, t^N + 2\, \tilde{s}_{N+1}(t)$$

with

$$C_N(\Omega) = 2\left(\int_0^1 dw\, \mathbb{P}\,(w \le X_1) + C_N^*(\Omega)\right)$$

and $C_N^*(\Omega)$ as defined in (4.36). Hence, the proof of part (iii) in Theorem 4.0.1 is complete by taking $R_\alpha(t) = 2\,\tilde{s}_{[\frac{1}{\alpha}]+1}(t)$ and observing that

$$|R_\alpha(t)| \le C\, t^{[\frac{1}{\alpha}]+1}, \quad 0 < t \le \min\left\{|\Omega|^\alpha, e^{-1}\right\},$$

for some $C > 0$.

4.2 Proof of Theorem 4.0.2

We start by recalling equation (1.4) which allows us to write the transition densities $p_t^{(\alpha)}(x,y)$ by subordination of the Gaussian kernel. Therefore, an application of Fubini's Theorem yields

$$\begin{aligned}\mathbb{H}_{\Omega,\Omega^c}^{(\alpha)}(t) &= \int_\Omega dx \int_{\Omega^c} dy\, p_t^{(\alpha)}(x-y) \\ &= \int_\Omega dx \int_{\Omega^c} dy\, \mathbb{E}\left[p_{S_t}^{(2)}(x-y)\right] = \mathbb{E}\left[\mathbb{H}_{\Omega,\Omega^c}^{(2)}(S_t)\right].\end{aligned} \tag{4.37}$$

Proof of part (i)**:** Assume $1 < \alpha < 2$. With the aid of the inequality (4.10) which is valid for all positive time, equality (4.37), the fact that $S_t \stackrel{\mathcal{D}}{=} t^{2/\alpha} S_1$ and formula (4.21), it easily follows that

$$\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t) \leq \frac{\mathcal{H}^{d-1}(\partial\Omega)}{\sqrt{\pi}} \mathbb{E}\left[S_t^{1/2}\right] = \frac{1}{\pi}\,\Gamma\left(1 - \frac{1}{\alpha}\right)\, \mathcal{H}^{d-1}(\partial\Omega)\; t^{\frac{1}{\alpha}} \tag{4.38}$$

for all $t > 0$ and with this we have proved (4.14).

On the other hand, by using once more (4.37) and $S_t \stackrel{\mathcal{D}}{=} t^{2/\alpha} S_1$, we obtain

$$\begin{aligned} \frac{\mathbb{H}^{(\alpha)}_{\Omega,\Omega^c}(t)}{t^{\frac{1}{\alpha}} \mathcal{H}^{d-1}(\partial\Omega)} &= \int_0^\infty ds \left(\frac{\mathbb{H}^{(2)}_{\Omega,\Omega^c}(t^{2/\alpha} s)}{t^{\frac{1}{\alpha}} \mathcal{H}^{d-1}(\partial\Omega)} \right) \eta_1^{(\alpha/2)}(s) \\ &= \int_0^\infty ds\; G(s,t), \end{aligned} \tag{4.39}$$

with

$$G(s,t) = s^{1/2} \left(\frac{\mathbb{H}^{(2)}_{\Omega,\Omega^c}(s\, t^{2/\alpha})}{(s\, t^{2/\alpha})^{1/2}\, \mathcal{H}^{d-1}(\partial\Omega)} \right) \eta_1^{(\alpha/2)}(s).$$

We now observe two facts. First, by (4.11) we have

$$\lim_{t \downarrow 0} G(s,t) = \frac{1}{\sqrt{\pi}}\; s^{1/2}\; \eta_1^{(\alpha/2)}(s).$$

Secondly, by (4.10)

$$0 \leq G(s,t) \leq \frac{1}{\sqrt{\pi}}\; s^{1/2}\; \eta_1^{(\alpha/2)}(s)$$

for all $t, s > 0$, with $s^{1/2}\; \eta_1^{(\alpha/2)}(s) \in L^1((0,+\infty))$ because of (4.21). Hence, the assertion (4.15) is an easy consequence of the Lebesgue Dominated Convergence Theorem and the identity (4.39).

We now continue with the proof of (ii) of Theorem (4.0.2). This requires a much more delicate approach. In order to make this presentation as clear as possible, we devote the next section to it.

4.3 Cauchy processes in higher dimension

In this section, we will adapt the techniques used in [45] for the Gaussian heat kernel. This requires some additional considerations since as we have already pointed out the Gaussian kernel has an exponential decay whereas the Cauchy heat kernel $p_t^{(1)}(x,y)$ defined in (1.2) has a polynomial decay.

From now on, we write every vector $x \in \mathbb{R}^d$ as $x = (\bar{x}, x_d)$ with $\bar{x} = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}$.

Lemma 4.3.1 *Let $H = \left\{(\bar{x}, x_d) \in \mathbb{R}^d : x_d < 0\right\}$ and $\delta, \varepsilon > 0$. Set $H^\delta = \mathbb{R}^{d-1} \times (0, \delta)$ and $H_\varepsilon = \mathbb{R}^{d-1} \times (-\varepsilon, 0)$. Assume $\varphi \in C_c^1(\mathbb{R}^d)$ and consider the compact set*

$$K = \left\{\bar{x} \in \mathbb{R}^{d-1} : \exists\, x_d \in \mathbb{R} \text{ such that } (\bar{x}, x_d) \in supp(\varphi)\right\}. \tag{4.40}$$

Then, there exists a function $R(t)$ such that

$$\int_{H^\delta} dx\, \varphi(x) \int_{H_\varepsilon} dy\, p_t^{(1)}(x,y) = \frac{1}{\pi}\left(\int_K d\bar{x}\, \varphi(\bar{x}, 0)\right) t \ln\left(\frac{1}{t}\right) + R(t), \tag{4.41}$$

with

$$|R(t)| \leq C_{\varepsilon,\delta,\varphi}\; t \tag{4.42}$$

for all $0 < t < e^{-1}$.

Proof We first note that the integral on the left hand side of (4.41) equals

$$\int_{H^\delta} dx\ \varphi(x) \int_{-\varepsilon}^0 dy_d \int_{\mathbb{R}^{d-1}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \cdot \frac{t\, d\bar{y}}{\left(t^2 + |x_d - y_d|^2 + |\bar{x} - \bar{y}|^2\right)^{(d+1)/2}}.$$

By considering the change of variable

$$\bar{y} = \bar{x} - \sqrt{\left(t^2 + |x_d - y_d|^2\right)}\ w,$$

we reduce the last integral to

$$\gamma_d\, t \int_{\mathbb{R}^{d-1}} d\bar{x} \int_0^\delta dx_d\ \varphi(\bar{x}, x_d) \int_{-\varepsilon}^0 \frac{dy_d}{t^2 + |x_d - y_d|^2}, \tag{4.43}$$

where

$$\gamma_d = \Gamma\left(\frac{d+1}{2}\right) \pi^{-\frac{(d+1)}{2}} \int_{\mathbb{R}^{d-1}} dw\, (1 + |w|^2)^{-\frac{(d+1)}{2}}.$$

Notice that by appealing to spherical coordinates, properties of the Gamma funtion and the following fact

$$\int dr\, r^{d-2}(1 + r^2)^{-\frac{d+1}{2}} = \frac{r^{d-1}}{(d-1)(1+r^2)^{\frac{d-1}{2}}} + C,$$

we have $\gamma_d = \pi^{-1}$ for all $d \geq 2$.

Next, by making the new change of variables $x_d - y_d = tz$ in the integral expression (4.43), we arrive at

$$\int_{H^\delta} dx\, \varphi(x) \int_{H_\varepsilon} dy\, p_t^{(1)}(x,y) = \pi^{-1} \int_{\mathbb{R}^{d-1}} d\bar{x} \int_0^\delta dx_d\ \varphi(\bar{x}, x_d)\ F_t(x_d, \varepsilon) \tag{4.44}$$

with

$$\begin{aligned} F_t(x_d, \varepsilon) &= \arctan\left(\frac{x_d + \varepsilon}{t}\right) - \arctan\left(\frac{x_d}{t}\right) \\ &= \arctan\left(\frac{t}{x_d}\right) - \arctan\left(\frac{t}{x_d + \varepsilon}\right). \end{aligned} \tag{4.45}$$

Let us set at this point

$$h(\bar{x}, x_d) = \varphi(\bar{x}, x_d) - \varphi(\bar{x}, 0). \tag{4.46}$$

Notice that according to (4.40), we have

$$\varphi(\bar{x}, x_d) = h(\bar{x}, x_d) = 0 \tag{4.47}$$

for all $(\bar{x}, x_d) \in K^c \times \mathbb{R}$. Since $\varphi(x)$ is compactly supported with continuous partial derivatives it follows from the Taylor expansion that

$$|h(\bar{x}, x_d)| = \left| \int_0^1 \nabla\varphi((\bar{x}, x_d) - s(\bar{x}, 0)) \cdot (0, x_d) ds \right| \leq ||\nabla\varphi||_\infty |x_d|. \tag{4.48}$$

We next consider the continuous function $\Pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ defined by $\Pi(\bar{x}, x_d) = \bar{x}$. Then

$$K = \{\Pi(\bar{x}, x_d) : (\bar{x}, x_d) \in supp(\varphi)\}.$$

Thus, because of the continuity of Π, K is a compact set in $\mathbb{R}^{d-1}$ whose finite Lebesgue measure will be denoted in what follows by $|K|$.

Now, by appealing to (4.44), (4.46) and (4.47), we find that

$$\int_{H^\delta} dx\, \varphi(x) \int_{H_\varepsilon} dy\, p_t^{(1)}(x,y) = \pi^{-1} \int_K d\bar{x} \int_0^\delta dx_d\, \varphi(\bar{x}, 0)\, F_t(x_d, \varepsilon) + R_2(t) \tag{4.49}$$

with

$$R_2(t) = \pi^{-1} \int_K d\bar{x} \int_0^\delta dx_d\, h(\bar{x}, x_d)\, F_t(x_d, \varepsilon). \tag{4.50}$$

As for the first integral term on the right hand side of the equation (4.49), we have by using the elementary identities (4.25) and (4.26) that it is equal to

$$\pi^{-1}\left(\int_K d\bar{x}\,\varphi(\bar{x},0)\right)\left(t\ln\left(\frac{1}{t}\right)+R_1(t)\right),$$

with

$$R_1(t) = \varepsilon\arctan\left(\frac{t}{\varepsilon}\right)+\delta\arctan\left(\frac{t}{\delta}\right) - (\delta+\varepsilon)\arctan\left(\frac{t}{\delta+\varepsilon}\right)+\frac{t}{2}\ln\left(\frac{(t^2+\varepsilon^2)(t^2+\delta^2)}{t^2+(\delta+\varepsilon)^2}\right).$$

We remark that due to the inequality $\arctan(x) \le x$ for $x > 0$ and the fact that $0 < t < e^{-1}$, we obtain that $|R_1(t)| \le C_{\delta,\varepsilon}\, t$.

As for $R_2(t)$, we first observe due to (4.45) that for $t, x_d > 0$, we have

$$0 \le F_t(x_d,\varepsilon) \le \frac{t}{x_d}. \tag{4.51}$$

Therefore, by combining (4.48), (4.50) and (4.51), we have

$$|R_2(t)| \le \pi^{-1}\int_K d\bar{x}\int_0^\delta dx_d\,|h(\bar{x},x_d)|\,F_t(x_d,\varepsilon) \le \pi^{-1}\,\delta\,||\nabla\varphi||_\infty|K|\,t.$$

Now by setting $R(t) = \pi^{-1}\left(\int_K d\bar{x}\,\varphi(\bar{x},0)\right)R_1(t)+R_2(t)$ and putting together all the estimates given above we conclude (4.42) and this finishes the proof of Lemma 4.3.1. ■

Before proceeding, we comment further on the last result. In probabilistic terms, we have

$$\int_{H^\delta} dx\;\varphi(x)\int_{H_\varepsilon} dy\;p_t^{(1)}(x,y) = \int_{H^\delta} dx\;\varphi(x)\;\mathbb{P}^x\left(X_t \in H_\varepsilon\right).$$

The goal of the last integral is to understand how the paths of the Cauchy process "perceive" the boundary of H. The above lemma says that when $\varphi(x) \in C_c^1(\mathbb{R}^d)$, the process "feels" the influence of the boundary $\partial H = \mathbb{R}^{d-1}\times\{0\}$ by means of the term

$$\int_{K\subset\mathbb{R}^{d-1}} d\bar{x}\,\varphi(\bar{x},0).$$

For a bounded domain with smooth boundary Ω, the paths conditioned to start in Ω and exit at time t should "view" the boundary as a half-plane. Therefore, it is expected that we can replace $\int_K d\bar{x}\, \varphi(\bar{x},0)$ with $\int_{\partial\Omega} \varphi(\sigma)\ d\mathcal{H}^{d-1}(\sigma)$. To this aim, we recall some definitions and geometric properties on uniformly $C^{1,1}$-regular domains. We refer the reader to [45], [55] and references therein for details and further considerations on the topic.

We set $\rho_\Omega(x) = \inf\{|x-\sigma| : \sigma \in \partial\Omega\}$ and for $\delta, \varepsilon > 0$ we define

$$\begin{aligned} \Omega^\delta &= \{x \in \Omega^c : \rho_\Omega(x) < \delta\}, \\ \Omega_\varepsilon &= \{x \in \Omega \ \ : \rho_\Omega(x) < \varepsilon\}. \end{aligned} \tag{4.52}$$

Proposition 4.3.1 *Let $\Omega \subset \mathbb{R}^d$ be a uniformly $C^{1,1}$-regular bounded domain. Then,*

(a) *there exists $\varepsilon, \delta > 0$ such that the maps*

$$J : \partial\Omega \times [0,\delta] \to \Omega^\delta, \ \ J(\sigma, r) = \sigma + r\,\nu(\sigma),$$

$$\tilde{J} : \partial\Omega \times [0,\varepsilon] \to \Omega_\varepsilon, \ \ \tilde{J}(\sigma, r) = \sigma - r\,\nu(\sigma), \tag{4.53}$$

where $\nu(\sigma)$ is the outward unit normal to $\partial\Omega$ at σ, are $C^{1,1}$-diffeomorphisms.

(b) *Given $\eta > 0$, there exists a finite covering $V = \{V_i\}$ of $\partial\Omega$ and $C^{1,1}$–diffeomorphisms $\psi_i : K_i \to V_i$, with K_i open subset of $\mathbb{R}^{d-1}$ such that if we set*

$$\Psi_i(\xi, \rho) = \psi_i(\xi) + \rho\,\nu(\psi_i(\xi)), \quad \xi \in K_i, \ \rho \in (-\varepsilon, \delta),$$

then the family of open sets $U = \{U_i\}$ with $U_i = \Psi_i\left(K_i \times (-\varepsilon, \delta)\right)$ covers $\Omega_\varepsilon \cup \Omega^\delta$ with Jacobians satisfying

$$\begin{aligned} |D\Psi_i(\xi,\rho)| &= 1 + \mathcal{O}(\eta), \ \xi \in K_i, \ \rho \in (-\varepsilon, \delta), \\ |D\Psi_i^{-1}(x)| &= 1 + \mathcal{O}(\eta), \ x \in U_i \\ |D\psi_i^{-1}(x)| &= 1 + \mathcal{O}(\eta), \ x \in V_i. \end{aligned} \tag{4.54}$$

Also

$$|\Psi_i(z,r) - \Psi_i(\xi,\rho)|^2 = |(z,r) - (\xi,\rho)|^2 \left(1 + \mathcal{O}(\eta)\right), \tag{4.55}$$

for all $\xi, z \in K_i$ *and* $\rho, r \in (-\varepsilon, \delta)$. *Here, we use the notation* $\mathcal{O}(\eta)$ *to mean a function which is upper bounded in absolute value by* $C\eta$, *where the constant* C *depends only on* $\Omega, \varepsilon, \delta$.

The main result of this section is the following.

Theorem 4.3.1 *Let* $\Omega \subset \mathbb{R}^d$ *be a uniformly* $C^{1,1}$*-regular bounded domain. Consider* Ω_ε *and* Ω^δ *the inner and outer tubular neighbourhoods of* $\partial\Omega$ *defined in* (4.52). *Then, for every* $\varphi \in C_c^1(\mathbb{R}^d)$ *we have*

$$\lim_{t\downarrow 0} \frac{1}{t\ln\left(\frac{1}{t}\right)} \int_{\Omega^\delta} dx\, \varphi(x) \int_{\Omega_\varepsilon} dy\, p_t^{(1)}(x,y) = \pi^{-1} \int_{\partial\Omega} \varphi(\sigma)\, d\,\mathcal{H}^{d-1}(\sigma). \tag{4.56}$$

Proof Let $\eta > 0$ and consider the finite family of open sets $U = \{U_i\}$ provided by part (b) in the last proposition. Now, let $\{\chi_i\}$ be a smooth partition of the unity subordinated to the covering U (see [55, Th 1.2]). We assume without loss of generality that $supp(\chi_i) \subset U_i$. Therefore, by using the fact that $\sum_i \chi_i(x) = 1$ for every $x \in \cup U_i$, we have

$$\int_{\Omega^\delta} dx\, \varphi(x) \int_{\Omega_\varepsilon} dy\, p_t^{(1)}(x,y) = \sum_i \left(I_i + \tilde{I}_i\right)$$

with

$$I_i = \int_{\Omega^\delta \cap supp(\chi_i)} dx\, \varphi(x)\chi_i(x) \int_{\Omega_\varepsilon \cap U_i} dy\, p_t^{(1)}(x,y),$$
$$\tilde{I}_i = \int_{\Omega^\delta \cap supp(\chi_i)} dx\, \varphi(x)\chi_i(x) \int_{\Omega_\varepsilon \setminus U_i} dy\, p_t^{(1)}(x,y).$$

Observe that $supp(\chi_i) \subset U_i$ is compact and also disjoint from the compact set $\overline{\Omega_\varepsilon \setminus U_i}$, then

$$\inf\left\{|x-y| : x \in supp(\chi_i), y \in \overline{\Omega_\varepsilon \setminus U_i}\right\} = \mu_i > 0.$$

Thus, by appealing to the explicit form of the Cauchy heat kernel and the fact $0 \le \chi_i \le 1$ for every i, we conclude

$$\lim_{t\downarrow 0}\left|\frac{1}{t\ln\left(\frac{1}{t}\right)}\sum_i \tilde{I}_i\right| \le \lim_{t\downarrow 0} \frac{C_d}{\ln\left(\frac{1}{t}\right)} \left(\sum_i \mu_i^{-(d+1)}\right) |\Omega| \int_{\Omega^\delta} dx\ \ |\varphi(x)| = 0.$$

Now, we proceed to deal with the term I_i. We start by expressing every $x \in \Omega^\delta \cap supp(\chi_i)$ and $y \in \Omega_\varepsilon \cap U_i$ under the new variables introduced in Proposition 4.3.1. Namely,

$$y = \Psi(z, r), \quad z \in K_i,\, r \in [-\varepsilon, 0],$$
$$x = \Psi(\xi, \rho), \quad \xi \in K_i,\, \rho \in [0, \delta].$$

Then, using these equalities, we obtain

$$I_i = \int_{K_i \times (0,\delta)} d\xi\, d\rho\, \chi_i\left(\Psi(\xi,\rho)\right) \varphi\left(\Psi(\xi,\rho)\right) \int_{K_i \times (-\varepsilon, 0)} dz\, dr\, p_t((z,r),(\xi,\rho)),$$

where we have set

$$p_t((z,r),(\xi,\rho)) = p_t^{(1)}(\Psi(z,r), \Psi(\xi,\rho))\, |D\Psi(z,r)|\, |D\Psi(\xi,\rho)|\,.$$

Define $g_t(x,y) = \frac{|x-y|^2}{t^2+|x-y|^2}$ with $x, y \in \mathbb{R}^{d+1}$ and $t > 0$. Hence, by using the estimates given in (4.54) and (4.55), we find that

$$p_t((z,r),(\xi,\rho)) = p_t^{(1)}((z,r),(\xi,\rho)) \left[\frac{1+\mathcal{O}(\eta)}{\left(1 + g_t(\xi - z, \rho - r)\mathcal{O}(\eta)\right)^{(d+1)/2}} \right].$$

We now observe by using that $0 \le g_t \le 1$ and the above expression, we can chose η very small but arbitrary such that

$$p_t((z,r),(\xi,\rho)) = p_t^{(1)}((z,r),(\xi,\rho))\left(1 + \mathcal{O}(\eta)\right). \tag{4.57}$$

Therefore, we conclude by Proposition 4.3.1 and (4.54) that

$$\begin{aligned}
\lim_{t \downarrow 0} \frac{1}{t \ln\left(\frac{1}{t}\right)} \sum_i I_i &= \left(1 + \mathcal{O}(\eta)\right) \pi^{-1} \sum_i \int_{K_i} \chi_i\left(\Psi_i(\xi,0)\right) \varphi\left(\Psi_i(\xi,0)\right) d\xi \\
&= \left(1 + \mathcal{O}(\eta)\right) \pi^{-1} \sum_i \int_{V_i} \chi_i\left(\sigma\right) \varphi\left(\sigma\right) \left|D\Psi_i^{-1}(\sigma)\right| d\,\mathcal{H}^{d-1}(\sigma) \\
&= \left(1 + \mathcal{O}(\eta)\right) \pi^{-1} \int_{\partial\Omega} \left(\sum_i \chi_i(\sigma) \mathbb{1}_{V_i \cap supp(\chi_i)}(\sigma) \right) \varphi\left(\sigma\right) d\,\mathcal{H}^{d-1}(\sigma) \\
&= \left(1 + \mathcal{O}(\eta)\right) \pi^{-1} \int_{\partial\Omega} \varphi\left(\sigma\right) d\,\mathcal{H}^{d-1}(\sigma).
\end{aligned}$$

The proof is complete by letting η go to zero. ■

Remark 4.3.1 *Let $\Omega \subset \mathbb{R}^d$ be a uniformly $C^{1,1}$-regular bounded domain and ε, δ as given in Proposition 4.3.1. It is clear because of the boundedness of Ω that $\overline{\Omega^\delta \cup \Omega_\varepsilon}$ is contained in some open ball. Thus, by Corollary 1.2 in [55, p. 8], there exists an infinitely differentiable and compactly supported function φ such that*

$$\overline{\Omega^\delta \cup \Omega_\varepsilon} \subset \{x \in supp(\varphi) : \varphi(x) = 1\}.$$

Therefore, as an application of Theorem 4.3.1, we conclude

$$\lim_{t \downarrow 0} \frac{\mathbb{H}^{(1)}_{\Omega_\varepsilon, \Omega^\delta}(t)}{t \ln\left(\frac{1}{t}\right)} = \pi^{-1}\, \mathcal{H}^{d-1}(\partial\Omega).$$

We observe that for every $\delta, \varepsilon > 0$, we have

$$\mathbb{H}^{(1)}_{\Omega, \Omega^c}(t) = \mathbb{H}^{(1)}_{\Omega \setminus \Omega_\varepsilon, \Omega^c}(t) + \mathbb{H}^{(1)}_{\Omega_\varepsilon, \Omega^\delta}(t) + \mathbb{H}^{(1)}_{\Omega_\varepsilon, \Omega^c \setminus \Omega^\delta}(t),$$

so that in order to prove part (ii) of Theorem 4.0.2, we still need to show the following.

Lemma 4.3.2 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider Ω_ε and Ω^δ the inner and outer tubular neighbourhoods of $\partial\Omega$ defined in (4.52). Then,*

$$\lim_{t \downarrow 0} \frac{\mathbb{H}^{(1)}_{\Omega \setminus \Omega_\varepsilon, \Omega^c}(t)}{t \ln\left(\frac{1}{t}\right)} = \lim_{t \downarrow 0} \frac{\mathbb{H}^{(1)}_{\Omega_\varepsilon, \Omega^c \setminus \Omega^\delta}(t)}{t \ln\left(\frac{1}{t}\right)} = 0.$$

Proof Along the proof, C_d will denote the constant $\pi^{-(d+1)/2}\Gamma\left(\frac{d+1}{2}\right)$. We begin by observing that

$$\begin{aligned}
\mathbb{H}^{(1)}_{\Omega \setminus \Omega_\varepsilon, \Omega^c}(t) &= C_d\ t \int_{\Omega \setminus \Omega_\varepsilon} dx \int_{\Omega^c} \frac{dy}{\left(t^2 + |x-y|^2\right)^{\frac{d+1}{2}}} \\
&\leq C_d\ t \int_{\Omega \setminus \Omega_\varepsilon} dx \int_{\Omega^c} dy\, |x-y|^{-d-1} \\
&\leq C_d\, t \int_{\Omega \setminus \Omega_\varepsilon} dx \int_{B^c_{\rho_\Omega(x)}(x)} dy\ |x-y|^{-d-1} \qquad (4.58) \\
&= C_d\, t\, \mathcal{H}^{d-1}(\partial B_1(0)) \int_{\Omega \setminus \Omega_\varepsilon} dx\ \rho_\Omega^{-1}(x) \\
&\leq C_d\, \mathcal{H}^{d-1}(\partial B_1(0))\, \varepsilon^{-1}\, |\Omega|\, t,
\end{aligned}$$

where in the last inequality we have used that $\Omega \setminus \Omega_\varepsilon = \{x \in \Omega : \rho_\Omega(x) \geq \varepsilon\}$, whereas to compute the integral term in (4.58) we have employed spherical coordinates.

Next, since $\bar{\Omega}$ is compact, we have

$$0 < r_\Omega = \sup_{x\in\Omega} |x| < \infty. \tag{4.59}$$

Choose any $r > r_\Omega$ and notice that $\Omega \subset B_r(0)$. Thus, we find that

$$\begin{aligned}\int_{\Omega_\varepsilon} dx \int_{\Omega^c\setminus\Omega^\delta} dy\, |x-y|^{-d-1} = &\int_{\Omega_\varepsilon} dx \int_{\left(\Omega^c\setminus\Omega^\delta\right)\cap B_r(0)} dy\, |x-y|^{-d-1} \\ &+ \int_{\Omega_\varepsilon} dx \int_{\left(\Omega^c\setminus\Omega^\delta\right)\cap B_r^c(0)} dy\, |x-y|^{-d-1}.\end{aligned}$$

Note that for all $x \in \Omega$ and $y \in \Omega^c \setminus \Omega^\delta$, we have the following inequality $\delta \leq \rho_\Omega(y) \leq |x-y|$. Thus the first integral term on the right hand side of the previous equality is bounded above by $\delta^{-d-1}|\Omega||B_r(0)|$. As far as the second integral is concerned, we have for all $x \in \Omega_\varepsilon$, by (4.59), that

$$|y-x| \geq |y| - |x| \geq |y| - r_\Omega.$$

Thus,

$$\int_{\Omega_\varepsilon} dx \int_{\left(\Omega^c\setminus\Omega^\delta\right)\cap B_r^c(0)} dy\, |x-y|^{-d-1} \leq |\Omega| \int_{B_r^c(0)} dy\, (|y| - r_\Omega)^{-d-1}$$

for all $r > r_\Omega$. By appealing to spherical coordinates, we obtain

$$\begin{aligned}\int_{B_r^c(0)} dy\, (|y| - r_\Omega)^{-d-1} &= \mathcal{H}^{d-1}(\partial B_1(0)) \int_{r-r_\Omega}^{\infty} dp\, (p + r_\Omega)^{d-1}\, p^{-d-1} \\ &= \mathcal{H}^{d-1}(\partial B_1(0)) \sum_{j=0}^{d-1} \binom{d-1}{j} \frac{r_\Omega^j}{(j+1)(r-r_\Omega)^{j+1}} < \infty.\end{aligned}$$

Hence, we have shown that

$$\mathbb{H}^{(1)}_{\Omega_\varepsilon,\Omega^c\setminus\Omega^\delta}(t) \leq C_{\delta,\Omega,\varepsilon}t.$$

Finally, the assertion of the Lemma follows by combining all the estimates given above. ■

Proof of part (iii) of Theorem 4.0.2 As before, we notice that $\Omega^c \subset B^c_{\rho_\Omega(x)}(x)$ for every $x \in \Omega$ so that

$$\int_{\Omega_\varepsilon} dx \int_{\Omega^c} |x-y|^{-d-\alpha} \leq \int_{\Omega_\varepsilon} dx \int_{B^c_{\rho_\Omega(x)}(0)} dz\, |z|^{-d-\alpha}$$
$$= \mathcal{H}^{d-1}(\partial B_1(0))\, \alpha^{-1} \int_{\Omega_\varepsilon} dx\, \rho_\Omega^{-\alpha}(x). \tag{4.60}$$

Since uniformly $C^{1,1}$ bounded domains are also R–smooth boundary domains, we have according to Corollary 2.14 in [6] that there exists $\varepsilon > 0$ (this ε might not be the same provided in Proposition 4.3.1, however we can choose the smaller of them) such that

$$\mathcal{H}^{d-1}(\partial\Omega_r) \leq 2^{d-1}\mathcal{H}^{d-1}(\partial\Omega), \tag{4.61}$$

for all $0 < r < \varepsilon$. Hence, by the co–area formula, we obtain

$$\int_{\Omega_\varepsilon} dx\, \rho_\Omega^{-\alpha}(x) = \int_0^\varepsilon dr\, r^{-\alpha}\, \mathcal{H}^{d-1}(\partial\Omega_r) \leq 2^{d-1}\,(1-\alpha)^{-1}\,\mathcal{H}^{d-1}(\partial\Omega)\,\varepsilon^{1-\alpha} < \infty. \tag{4.62}$$

Likewise, as in (4.60)

$$\int_{\Omega\setminus\Omega_\varepsilon} dx \int_{\Omega^c} dy\, |x-y|^{-d-\alpha} \leq \mathcal{H}^{d-1}(\partial B_1(0))\, \alpha^{-1} \int_{\Omega\setminus\Omega_\varepsilon} dx\, \rho_\Omega^{-\alpha}(x)$$
$$\leq \mathcal{H}^{d-1}(\partial B_1(0))\, \alpha^{-1}\, |\Omega|\, \varepsilon^{-\alpha}. \tag{4.63}$$

We have shown with (4.60) and (4.63) that

$$\mathcal{P}_\alpha(\Omega) = \int_\Omega dx \int_{\Omega^c} dy|x-y|^{-d-\alpha} < \infty$$

provided that $0 < \alpha < 1$. Thus, by combining the finiteness of the last integral with (4.1) and (4.2), we conclude part (iii) of Theorem 4.0.2 by an application of the Lebesgue Dominated Convergence Theorem.

4.4 Upper bounds in Theorem 1.4.1

Let Ω be a bounded domain. Then, it is clear that for every $x \in \Omega$, we have

$$\tau^{(\alpha)}_{B_{\rho_\Omega(x)}(x)} \leq \tau^{(\alpha)}_\Omega$$

which implies

$$\mathbb{P}^x\left(\tau_\Omega^{(\alpha)} < t\right) \leq \mathbb{P}^x\left(\tau^{(\alpha)}_{B_{\rho_\Omega(x)}(x)} < t\right) = \mathbb{P}\left(\tau^{(\alpha)}_{B_{\rho_\Omega(x)}(0)} < t\right) \tag{4.64}$$

for all $t > 0$. Therefore, we conclude that

$$Q_\Omega^{(\alpha)}(t) = \int_\Omega dx\, \mathbb{P}^x\left(\tau_\Omega^{(\alpha)} \geq t\right) = |\Omega| - \int_\Omega dx\, \mathbb{P}^x\left(\tau_\Omega^{(\alpha)} < t\right)$$

satisfies for all $t > 0$ the following inequality

$$|\Omega| - Q_\Omega^{(\alpha)}(t) \leq \int_\Omega dx\, \mathbb{P}\left(\tau^{(\alpha)}_{B_{\rho_\Omega(x)}(0)} < t\right). \tag{4.65}$$

We now turn to the following result whose proof and applications to Subordinate Killed Brownian motion in a domain can be found in [52, Prop. 2.1].

Proposition 4.4.1 *Assume D is a bounded domain satisfying an exterior cone condition. Then, there exists $C \in (0,1)$ such that*

$$(1-C)\,\mathbb{P}^x(\tau_D^{(2)} \leq S_t) \leq \mathbb{P}^x(\tau_D^{(\alpha)} \leq t) \leq \mathbb{P}^x(\tau_D^{(2)} \leq S_t),$$

for all $t > 0$ and $x \in D$.

In particular, by appealing to the last Proposition with $D = B_{\rho_\Omega(x)}(0)$ and (4.65), we find that

$$|\Omega| - Q_\Omega^{(\alpha)}(t) \leq \int_\Omega dx\; \mathbb{P}\left(\tau^{(2)}_{B_{\rho_\Omega(x)}(0)} < S_t\right). \tag{4.66}$$

Next, the independence between the Brownian Motion $\mathbf{B}$ and $\alpha/2$-subordinator $\mathbf{S}$ as stated in the introduction yields

$$\begin{aligned}\mathbb{P}\left(\tau^{(2)}_{B_{\rho_\Omega(x)}(0)} < S_t\right) &= \mathbb{P}\left(\tau^{(2)}_{B_{\rho_\Omega(x)}(0)} < t^{2/\alpha} S_1\right) \\ &= \int_0^\infty ds\, \eta_1^{(\alpha/2)}(s)\, \mathbb{P}_{\mathbf{B}}\left(\tau^{(2)}_{B_{\rho_\Omega(x)}(0)} < t^{2/\alpha} s\right).\end{aligned} \tag{4.67}$$

In [12, Lemma 3.3], it is shown that

$$\mathbb{P}_{\mathbf{B}}\left(\tau^{(2)}_{B_{\rho_\Omega(x)}(0)} < t^{2/\alpha} s\right) \leq 2^{(d+2)/2} \exp\left(-\frac{\rho_\Omega^2(x)}{8\, t^{2/\alpha}\, s}\right)$$

so that by using the last inequality, (4.66) and (4.67), we arrive at

$$|\Omega| - Q_{\Omega}^{(\alpha)}(t) \leq 2^{(d+2)/2} \int_{\Omega} dx\, \mathbb{E}\left[\exp\left(-\frac{\rho_{\Omega}^2(x)}{8\, t^{2/\alpha}\, S_1}\right)\right]. \tag{4.68}$$

We split the foregoing integral as follows

$$\int_{\Omega} dx\, \mathbb{E}\left[\exp\left(-\frac{\rho_{\Omega}^2(x)}{8\, t^{2/\alpha}\, S_1}\right)\right] = I_{\alpha}(t) + II_{\alpha}(t)$$

with

$$I_{\alpha}(t) = \int_{\Omega_{\varepsilon}} dx\, \mathbb{E}\left[\exp\left(-\frac{\rho_{\Omega}^2(x)}{8\, t^{2/\alpha}\, S_1}\right)\right],$$
$$II_{\alpha}(t) = \int_{\Omega\setminus\Omega_{\varepsilon}} dx\, \mathbb{E}\left[\exp\left(-\frac{\rho_{\Omega}^2(x)}{8\, t^{2/\alpha}\, S_1}\right)\right]$$

and observe by (4.22) with $\kappa = \rho_{\Omega}(x)\, 8^{-1/2}\, t^{-\frac{1}{\alpha}}$, we obtain

$$II_{\alpha}(t) \leq C_{\alpha}\, |\Omega|\, 8^{\alpha/2} \varepsilon^{-\alpha}\, t$$

and by (4.61) and co–area formula, we also have

$$\begin{aligned} I_{\alpha}(t) &= \int_0^{\varepsilon} dr\, \mathbb{E}\left[\exp\left(-\frac{r^2}{8\, t^{2/\alpha}\, S_1}\right)\right] \mathcal{H}^{d-1}(\partial\Omega_r) \\ &\leq 2^{d-1}\, \mathcal{H}^{d-1}(\partial\Omega) \int_0^{\varepsilon} dr\, \mathbb{E}\left[\exp\left(-\frac{r^2}{8\, t^{2/\alpha}\, S_1}\right)\right] \\ &= 2^{(2d+1)/2}\, \mathcal{H}^{d-1}(\partial\Omega)\, t^{\frac{1}{\alpha}} \int_0^{\varepsilon\, 8^{-1/2}\, t^{-\frac{1}{\alpha}}} dw\, \mathbb{E}\left[\exp\left(-\frac{w^2}{S_1}\right)\right] \end{aligned} \tag{4.69}$$

for all $0 < \alpha < 2$ and $t > 0$.

In order to obtain upper bounds it suffices to deal with the integral in the above inequality. As before, we divide this into various cases according to α.

Case $1 < \alpha < 2$: By appealing to the identity

$$\int_0^{\infty} dw\, \exp\left(-\frac{w^2}{s}\right) = 2^{-1}\, \pi^{1/2}\, s^{1/2}$$

and Fubini's Theorem, we arrive at

$$\int_0^{\infty} dw\, \mathbb{E}\left[\exp\left(-\frac{w^2}{S_1}\right)\right] = 2^{-1}\, \pi^{1/2}\, \mathbb{E}\left[S_1^{1/2}\right] = 2^{-1}\Gamma\left(1 - \frac{1}{\alpha}\right)$$

so that by (4.69)

$$I_\alpha(t) \le 2^{(2d-1)/2}\,\mathcal{H}^{d-1}\left(\partial\Omega\right) t^{\frac{1}{\alpha}}\,\Gamma\left(1-\frac{1}{\alpha}\right).$$

By putting together the preceding estimates and the inequality (4.68), we obtain for all $t>0$ that

$$|\Omega| - Q_\Omega^{(\alpha)}(t) \le 2^{(d+2)/2}\left(C_\alpha\;|\Omega|\,8^{\alpha/2}\varepsilon^{-\alpha}\,t + 2^{(2d-1)/2}\,\mathcal{H}^{d-1}(\partial\Omega)\,\Gamma\left(1-\frac{1}{\alpha}\right)\,t^{\frac{1}{\alpha}}\right).$$

It easily follows that

$$\overline{\lim_{t\downarrow 0}}\,\frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t^{\frac{1}{\alpha}}} \le\; 2^{(3d+1)/2}\;\Gamma\left(1-\frac{1}{\alpha}\right)\,\mathcal{H}^{d-1}(\partial\Omega).$$

Case $\alpha=1$**:** The $1/2$–subordinator **S** can be expressed as the first hitting time for the standard one-dimensional Brownian motion $\{W_t\}_{t\ge0}$. More precisely, $S_t = \inf\left\{s>0 : W_s = \frac{t}{\sqrt{2}}\right\}$. It is known (see [1] for details) that its transition density is given by

$$\eta_t^{(1/2)}(s) = \frac{t}{2\sqrt{\pi}}s^{-3/2}e^{-t^2/4s}.$$

A simple computation yields

$$\mathbb{E}\left[\exp\left(-\frac{w^2}{S_1}\right)\right] = \frac{1}{\sqrt{4w^2+1}},$$
$$\int_0^{\varepsilon 8^{-1/2}\,t^{-1}}\frac{dw}{\sqrt{4w^2+1}} = \frac{1}{2}\ln\left(\frac{1}{t}\right) + \frac{1}{2}\ln\left(\frac{\varepsilon}{\sqrt{2}}+\sqrt{\frac{\varepsilon^2}{2}+t^2}\right).$$

Therefore, (4.68) and the previous calculations show that $|\Omega| - Q_\Omega^{(1)}(t)$ is bounded above by

$$2^{(d+2)/2}\left(C_\alpha\;|\Omega|\,\sqrt{8}\varepsilon^{-1}\,t + 2^{(2d-1)/2}\,\mathcal{H}^{d-1}(\partial\Omega)\,t\left[\ln\left(\frac{1}{t}\right)+\ln\left(\frac{\varepsilon}{\sqrt{2}}+\sqrt{\frac{\varepsilon^2}{2}+t^2}\right)\right]\right)$$

which in turn implies

$$\overline{\lim_{t\downarrow 0}}\,\frac{|\Omega| - Q_\Omega^{(1)}(t)}{t\ln\left(\frac{1}{t}\right)} \le\; 2^{(3d+1)/2}\;\mathcal{H}^{d-1}(\partial\Omega).$$

Case $0 < \alpha < 1$**:** By applying again to (4.22) with $\kappa = \rho_\Omega(x)\, 8^{-1/2}\, t^{-\frac{1}{\alpha}}$, we have

$$\int_\Omega dx\, \mathbb{E}\left[\exp\left(-\frac{\rho_\Omega^2(x)}{8\, t^{2/\alpha}\, S_1}\right)\right] \le 8^{\alpha/2}\, C_\alpha \left(\int_\Omega dx\ \rho_\Omega^{-\alpha}(x)\right)\ t.$$

The integral term at the right hand side turns out to be finite because of (4.62) and the fact

$$\int_{\Omega\setminus\Omega_\varepsilon} dx\ \rho_\Omega^{-\alpha}(x) \le\ \varepsilon^{-\alpha}\, |\Omega|.$$

Therefore, we find that

$$\overline{\lim_{t\downarrow 0}}\, \frac{|\Omega| - Q_\Omega^{(\alpha)}(t)}{t} \le 2^{(d+2+3\alpha)/2}\, C_\alpha \int_\Omega dx\, \rho_\Omega^{-\alpha}(x).$$

Assume now that Ω also satisfies a uniform exterior volume condition. That is, there exists $c > 0$ such that for any $\sigma \in \partial\Omega$ and any $r > 0$ we have $|B_r(\sigma) \cap \Omega^c| \ge c\, r^d$. Then, we claim that

$$\frac{c}{2^{d+\alpha}}\, \rho_\Omega^{-\alpha}(x) \le \int_{\Omega^c} \frac{dy}{|x-y|^{d+\alpha}},\ \ x \in \Omega. \tag{4.70}$$

To see this, let $x \in \Omega$ and choose $\sigma_x \in \partial\Omega$ such that $\rho_\Omega(x) = |\sigma_x - x|$. Thus, for any y belonging to $B_{\rho_\Omega(x)}(\sigma_x) \cap \Omega^c$, we obtain

$$|x-y| \le |x-\sigma_x| + |\sigma_x - y| \le 2\, \rho_\Omega(x).$$

Thus, it follows from the last inequality that

$$\begin{aligned}\int_{\Omega^c} \frac{dy}{|x-y|^{d+\alpha}} &\ge \int_{B_{\rho_\Omega(x)}(\sigma_x)\cap\Omega^c} \frac{dy}{|x-y|^{d+\alpha}} \\ &\ge \frac{1}{2^{d+\alpha}}\, \left|B_{\rho_\Omega(x)}(\sigma_x) \cap \Omega^c\right| \rho_\Omega(x)^{-d-\alpha} \ge \frac{c}{2^{d+\alpha}}\, \rho_\Omega^{-\alpha}(x).\end{aligned}$$

In other words, for bounded domains Ω with smooth boundary and $0 < \alpha < 1$, the small time behavior of $t^{-1}\left(|\Omega| - Q_\Omega^{(\alpha)}(t)\right)$ and the fractional α-perimeter $\mathcal{P}_\alpha(\Omega)$ defined in (4.13) are related and this completes the proof of Theorem 1.4.1.

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