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Applications of microlocal analysis to some hyperbolic inverse problems

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APPLICATIONS OF MICROLOCAL ANALYSIS TO SOME HYPERBOLIC INVERSE PROBLEMS

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To Michael Grenat for his patience, wisdom and understanding.

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ABSTRACT

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This thesis compiles my work on three inverse problems: ultrasound recovery in thermoacoustic tomography, cancellation of singularities in synthetic aperture radar, and the injectivity and stability of some generalized Radon transforms. Each problem is approached using microlocal methods. In the context of thermoacoustic tomography under the damped wave equation, I show uniqueness and stability of the problem with complete data, provide a reconstruction algorithm for small attenuation with complete data, and obtain stability estimates for visible singularities with partial data. The chapter on synthetic aperture radar constructs microlocally several infinite-dimensional families of ground reflectivity functions which appear microlocally regular when imaged using synthetic aperture radar. Finally, based on a joint work with Hanming Zhou, we show the analytic microlocal regularity of a class of analytic generalized Radon transforms, using this to show injectivity and stability for a generic class of generalized Radon transforms defined on analytic Riemannian manifolds.

1. INTRODUCTION

1.1 Inverse problems

In applied mathematics, one often constructs a model of a physical system by considering a certain class of models (e.g., linear models) and tuning the parameters of the model to predict the system's output given the system's input. This is sometimes called the "forward problem", or the "direct problem." The problem is, "given the input state x, predict the output state $y = Fx$." An inverse problem, on the other hand, takes the input as unknown and attempts to recover them from observations of the system's response. The problem becomes, "given the output state y, predict the input state x such that $y = Fx$." The first inverse problems appeared in the context of seismology, in which case observing the input state (i.e., the material properties and velocity distribution of the interior of the Earth) is impossible.

As an illustration, consider Calderón's problem [9], which is the basis of the medical imaging technique called electrical impedance tomography [10]. Let $\Omega \subset \mathbb{R}^n$ be a smooth, compact region of interest. If $f \in H^{1/2}(\partial\Omega)$ is a voltage density induced on the boundary of the region of interest, then the electrical potential u inside Ω is the solution of the PDE,

$$\left\{ \begin{array}{rcl} \nabla \cdot \gamma(x)\nabla u(x) & = & 0 \text{ in } \Omega, \\ u|_{\partial\Omega}(x) & = & f(x) \text{ on } \partial\Omega. \end{array} \right. \tag{1.1}$$

Here $\gamma \in L^\infty(\Omega)$ is the electrical conductivity of the material. What one measures in this system is the current density at the boundary that is induced by each choice of f. After many experiments with various applied voltage densities, one gains knowledge of the Dirichet-to-Neumann operator

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega} \in H^{-1/2}(\Omega), \tag{1.2}$$

where $\partial/\partial\nu$ is the normal derivative. In this example, the forward problem is to find the current density $\Lambda_\gamma f$ given γ and f by solving the elliptic PDE (1.1). This is an elliptic, second-order PDE, and is solvable by classical techniques. The inverse problem is to determine the conductivity γ, given the operator Λ_γ, which depends on γ in a non-linear manner.

In the study of an inverse problem, there are three themes that one tends to follow.

1. Uniqueness: Does the known data uniquely determine the model parameters?
2. Stability: If the known data is perturbed slightly (e.g., by noise), is the solution stable with respect to the perturbation?
3. Reconstruction: Is there an efficient algorithm for recovering the model parameters from the measured data?

This dissertation is concerned with these questions in the context of the following three applications:

1. Thermoacoustic Tomography (TAT): A hybrid medical imaging technique attempting to image the electromagnetic absorption density of tissue via ultrasound and the thermoacoustic effect. This is also done using near-infrared light, in which case it is known as photoacoustic tomography (PAT). Some results from this work were published in [25].
2. Synthetic Aperture Radar (SAR): An airplane or satellite imaging technique that involves recovering the electromagnetic reflectivity of the ground from the scattering of a signal emitted from an antenna as it traverses a known flight path. Some results from this work were published in [26].
3. Generalized Radon Transform (GRT): An integral operator that associates a function defined on a smooth manifold to the function's integral over a smooth family of codimension one submanifolds. Some work on this problem was done in collaboration with Mr. Hanming Zhou [27].

The first two applications involve a model using the wave equation, and so can be called hyperbolic inverse problems. While the third application belongs more properly to the field of integral geometry, examples of generalized Radon transforms occur frequently in the context of thermoacoustic tomography and other inverse problems involving the wave equation.

1.2 Microlocal analysis

It is a fact of life that many of the inverse problems one encounters in applications are ill-posed. It could be that the problem has no unique solution for some values of the model parameters. Perhaps the forward problem is invertible, but the inverse is unbounded; reconstructions based on such are unstable and can be sensitive to noise. However, in applications it may not be necessary to recover quantitatively all of the image. For example, it may be sufficient to recover, qualitatively, the outline of a patient's lungs and the chambers of their heart in order to diagnose a certain disorder of the cardiopulminary system.

In the context of microlocal analysis, the discontinuities in material parameters at the interface of, say, the lungs and the rest of the thoracic cavity, may be modelled by the wavefront set of the material parameters, considered as a distribution. We define the smooth wavefront set in Definition 1.2.4, and its analytic counterpart in Definition 1.2.10. The calculus of pseudodifferential and Fourier integral operators can then serve as a toolkit for recovering the wavefront set. As an example, Hörmander's theorem on propagation of singularities, reproduced below as Theorem 1.2.1, describes the evolution of the wavefront set of Cauchy data for a hyperbolic PDE by a certain Hamiltonian flow determined by that PDE.

For clarity and self-containment, the basic facts of differential geometry and microlocal analysis that will be needed in the sequel are summarized here.

1.2.1 Riemannian geometry

The principal setting in what follows will be a Riemannian manifold without boundary. This is only a brief summary of results. For a more complete treatment, we refer the reader to [40], for example. Here, and in the sequel, the Einstein summation notation will be assumed.

Let (M, g) be a Riemannian manifold of dimension n without boundary. The tangent bundle TM is identified with the space of derivations on $C^\infty(M)$. Let $(x^i)_{i=1}^n$ be local coordinates for an open set $U \subset M$. There is a local frame $(\partial/\partial x^i)_{i=1}^n$ for TM. Recall the metric g is a non-degenerate 2-form on M. In local coordinates, it is determined by coefficients g_{ij}, where

$$g(\partial/\partial x^i, \partial/\partial x^j) = g_{ij} \tag{1.3}$$

This determines a dual frame $(dx^i)_{i=1}^n$ of the cotangent bundle T^*M, which is the dual bundle of TM. Often we will consider the fiber bundle $T^*M \setminus 0$, in which the zero section has been deleted.

$$dx^i(\partial/\partial x^j) = \delta^i_j. \tag{1.4}$$

The cotangent bundle is endowed with a canonical symplectic structure. In local coordinates, it is given by,

$$\omega = \sum_{i=1}^n dx^i \wedge d\xi^i. \tag{1.5}$$

It is common to write $g^{ij} = (g_{ij})^{-1}$. In local coordinates, the Christoffel symbols (of the first kind) are defined by

$$\Gamma^k_{ij} = g^{k\ell}\frac{1}{2}\left(\frac{\partial}{\partial x^i}(g_{j\ell}) + \frac{\partial}{\partial x^j}(g_{i\ell}) - \frac{\partial}{\partial x^\ell}(g_{ij})\right). \tag{1.6}$$

The metric induces an inner product on each tangent space T_xM, for all $x \in M$. Write $|v|_g = g(v, v)^{1/2}$ for the norm of the tangent spaces. The unit sphere bundle is the bundle of tangent vectors of unit length with respect to this norm,

$$SM = \{(x, v) : x \in M, v \in T_xM, |v|_g = 1\}. \tag{1.7}$$

Naturally there is also a norm induced by an inner product on the various cotangent spaces T_x^*M, which by abuse of notation will also be written $|\xi|_g$. This is used to define the unit cosphere bundle

$$S^*M = \{(x,\xi) : x \in M, \xi \in T_x^*M, |\xi|_g = 1\}. \tag{1.8}$$

Let $\gamma : [t_0, t_1] \to M$ be a smooth curve. Such a curve is a geodesic when it solves the second-order system of ODEs,

$$\ddot{\gamma}^k(t) + \Gamma_{ij}^k \dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0, \tag{1.9}$$

in local coordinates. Here dots indicate derivatives with respect to the parameter. It follows from the ODE that $|\dot{\gamma}(t)|_g$ is constant, determined by its initial value $|\dot{\gamma}(t_0)|_g$. After possibly rescaling the parameter, one may assume without loss of generality that γ is parameterized in such a way that the velocity vector field $\dot{\gamma}$ has unit length.

By the local uniqueness and existence of solutions to ODE, for every choice of initial data $(x, v) \in SM$ there exists an $\epsilon > 0$ such that $\gamma_{x,v} : [0, \epsilon] \to M$ is the unique geodesic with $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. By compactness for every $x \in M$ there exists a uniform $\epsilon > 0$ such that every geodesic with initial data $(x, v) \in S_xM$ is at least defined up to time ϵ. This defines a family of exponential maps

$$\exp_x(tv) = \gamma_{x,v}(t), \tag{1.10}$$

for $v \in S_xM$ and $t \in [0, \epsilon]$. On a complete manifold, the exponential maps define a geodesic flow ϕ^t on SM given by

$$\phi^t(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)). \tag{1.11}$$

This lifts to a cogeodesic flow on the unit cosphere bundle, which we also refer to as ϕ^t.

If in addition (M, g) is orientable, there exists an n-form $d\mathrm{Vol}$, unique up to constant multiple, called the volume form of g. In local coordinates, we have

$$d\mathrm{Vol} = \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n. \tag{1.12}$$

This is the natural measure on (M, g), and it defines the Lebesgue space $L^2(M)$:

$$L^2(M) = \left\{ f : M \to \mathbb{C}, \text{ measurable } : \int |f|^2 \, d\mathrm{Vol} < \infty \right\}. \tag{1.13}$$

This is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int f \overline{g} \, d\mathrm{Vol}, \tag{1.14}$$

and one writes the norm

$$||f||_{L^2(M)} = \langle f, f \rangle^{1/2}. \tag{1.15}$$

For $k \in \mathbb{N}$, the Sobolev spaces $H^k(M)$ are defined as the completion of $C_0^\infty(M)$ under the norm

$$||f||_{H^k(M)} = \sum_{|\alpha| \leq k} ||\partial^\alpha f||_{L^2(M)}. \tag{1.16}$$

Define $H^{-k}(M) = (H^k(M))^*$. The fractional Sobolev spaces $H^s(M), s \in \mathbb{R} \setminus \mathbb{Z}$ are defined by interpolation.

A Riemannian manifold (M, g) is real-analytic (which will be shortened to "analytic" when no confusion results, reserving "holomorphic" for complex-analytic objects) when the underlying manifold is analytic and the coefficients of the metric g_{ij} are analytic in every analytic coordinate chart. It follows from (1.12) that dVol is in this case an analytic n-form.

Let $U \subset M$ be a sufficiently small neighborhood of an analytic Riemannian manifold. Then there exists a complex Riemannian manifold $U_{\mathbb{C}}$ such that U is embedded isometrically into $U_{\mathbb{C}}$. One somewhat natural way to do this is given by the Grauert tube construction, for which see [20]. The intimate details of this construction will not be necessary; we will only use the fact that we can extend analytic local coordinates on U to holomorphic local coordinates on $U_{\mathbb{C}}$, and continue analytic functions defined on U to holomorphic functions on $U_{\mathbb{C}}$.

1.2.2 Pseudodifferential operators

Recall the Fourier transform and its inverse on $\mathbb{R}^n$,

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx, \quad f(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi. \tag{1.17}$$

It is known that if $f \in C^\infty(\mathbb{R}^n)$, the Fourier transform decays faster than any polynomial as $|\xi| \to \infty$. More precisely, if $f \in H^k(\mathbb{R}^n)$, then the Fourier transform $\hat{f} \in L^2(\mathbb{R}^n, \langle\xi\rangle^k\, d\xi)$. (Recall $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.) Therefore, roughly speaking, higher regularity corresponds to faster decay for the Fourier transform. The basic idea of the wavefront set is to use this correspondence to determine the singularities of $f \in \mathcal{D}'(\mathbb{R}^n)$ by observing the decay rate of the Fourier transform of χf, where χ is a cut-off function localizing f near a specific point of interest.

Notice that the decay rate of $\widehat{\chi f}$ may be different along different rays in phase space; this is the intuition behind defining the wavefront set $\mathrm{WF}(f) \subset T^*\mathbb{R}^n \setminus 0$, which we state formally in Definition 1.2.4. To work "microlocally" is to consider a small conic neighborhood $\Gamma \subset T^*\mathbb{R}^n \setminus 0$, and cut-off f in such a way as to restrict its wavefront set to Γ. This is possible through the application of pseudodifferential operators. These operators, which include the ring of differential operators with smooth coefficients as a special case, also contain the pseudo-inverses for all elliptic PDEs with smooth coefficents. This section summarizes the techniques and details of such operators and their effect on the wavefront set, following the general approach of [31, 32, 56, 58].

Symbol classes and mapping properties

For now, we restrict our attention to a smooth domain $\Omega \subset \mathbb{R}^n$. In this case, $T^*\Omega$ is globally diffeomorphic to $\Omega \times \mathbb{R}^n$, and so we may take $(x, \xi) \in \Omega \times \mathbb{R}^n$ as coordinates for the cotangent bundle.

Definition 1.2.1. *Let $a \in C^\infty(T^*\Omega)$. Then a is a symbol of class $S^m(\Omega)$ when for all multi-indices α, β and all $K \Subset \Omega$ compact there exists $C > 0$ such that*

$$\sup_{x\in K} \left|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)\right| \le C\langle\xi\rangle^{m-|\alpha|}. \tag{1.18}$$

To each symbol a of class $S^m(\Omega)$ is associated a pseudodifferential operator $\mathrm{Op}(a)$ *of class $OPS^m(\Omega)$, which acts on $u \in C_0^\infty(\Omega)$ by*

$$\mathrm{Op}(a)u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)u(y)\, dy\, d\xi. \tag{1.19}$$

This definition extends to an operator $\mathrm{Op}(a) : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega)$, *using [30, Theorem 7.8.2] to define* (1.19) *as an oscillating integral.*

Define the class of negligible operators as

$$OPS^{-\infty} = \bigcap_{m=1}^{\infty} OPS^m. \tag{1.20}$$

These operators map $\mathcal{D}'(\Omega) \to C^\infty(\Omega)$, and are sometimes called smoothing operators. However, we refer to any integral operator with a smooth Schwartz kernel as a smoothing operator.

The Schwartz kernel of $\mathrm{Op}(a)$ is the distribution $K_a \in \mathcal{D}'(\Omega \times \Omega)$ given by the oscillating integral

$$K_a(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi)\, d\xi. \tag{1.21}$$

It can be shown that $K_a(x,y)$ is singular at most on the diagonal of $\Omega \times \Omega$. ($\mathrm{Op}(a)$ is negligible iff K_a is smooth.)

A Schwartz kernel is said to be of proper support when for all $K \Subset \Omega$ compact, both $\operatorname{supp} K_a \cap (K \times \Omega)$ and $\operatorname{supp} K_a \cap (\Omega \times K)$ are compact. All pseudodifferential operators can be reduced to such kernels in the following way. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function equal to one in a neighborhood of zero. Then the Schwartz kernel of any pseudodifferential operator may be decomposed as the sum

$$K_a(x,y)\chi(|x-y|^2) + K_a(x,y)(1 - \chi(|x-y|^2)). \tag{1.22}$$

The first term is a properly supported Schwartz kernel; the second is a smooth Schwartz kernel, and therefore the resulting integral operator is smoothing. This shows that every pseudodifferential operator has a properly supported Schwartz kernel, up to a smoothing operator.

Properly supported pseudodifferential operators are continuous $C^\infty(\Omega) \to C^\infty(\Omega)$ and have $L^2(\Omega)$–adjoints that are also pseudodifferential operators of the same class [56, Theorem II.4.1]. They also act naturally on Sobolev spaces.

Lemma 1.2.1 ([32, Theorem 18.1.3])**.** *If $A \in OPS^m(\Omega)$ is properly supported, then for all $s \geq 0$, A is a continuous operator mapping $H^s(\Omega) \to H^{s-m}(\Omega)$.*

Note that each S^m class can be equipped with the structure of a Fréchet space. Let $(K_m)_{m=1}^\infty$ be a sequence of monotonically increasing compact sets exhausting Ω. For each pair of multi-index α, β, take $|a|_{m,\alpha,\beta}$ to be the minimal constant $C(K_m, \alpha, \beta)$ necessary for the corresponding symbol estimate (1.18) to hold for $x \in K_m$. This describes a countable family of norms for $S^m(\Omega)$. The following lemma describes the continuity of Op as a map $S^m(\Omega) \to OPS^m(\Omega)$, and follows from a well-known estimate for the Schwartz kernel of $OPS^m(\Omega)$; see [54, Proposition VI.4.1].

Lemma 1.2.2. *Let $K \Subset \Omega$ be compact, and let $a_1, a_2 \in S^m(\Omega), m \geq 0$. Then for some $N > 0$ and any $\epsilon > 0$ there exists $\delta > 0$ with*

$$\sum_{|\alpha|,|\beta| \leq N} |a_1 - a_2|_{N,\alpha,\beta} < \delta \Longrightarrow ||\mathrm{Op}(a_1) - \mathrm{Op}(a_2)||_{H^m(K) \to L^2_{loc}(\Omega)} < \epsilon. \tag{1.23}$$

The following notation is useful in describing the relationship between symbols and their corresponding operators.

Definition 1.2.2. *Let $A \in OPS^m(\Omega)$. Then we write $\sigma(A) \in S^m(\Omega)$ for the full symbol associated to A, and $\sigma_m(A)$ for any representative of the equivalence class $[\sigma(A)] \in S^m(\Omega)/S^{m-1}(\Omega)$. The latter is referred to as the principal symbol of A.*

Symbol calculus and parametrix construction

One key strength of the symbol calculus is the ability to construct symbols asymptotically. One application of this to the calculus of pseudodifferential operators is the construction of parametrices for elliptic operators, which are inverses up to smoothing error. We will also use this idea to construct geometric optics solutions of the damped wave equation in the second chapter. We begin by defining a notion of asymptotic summation for a series of symbols. The following lemma is a special case of [56, Theorem II.3.1].

Lemma 1.2.3. *Let $a_k \in S^{m_k}(\Omega)$ with m_k a strictly decreasing sequence of integers not bounded from below. Then there exists $a \in S^{m_0}(\Omega)$ such that for all $N > 0$,*

$$a - \sum_{k=0}^{N} a_k \in S^{m_{N+1}}(\Omega). \tag{1.24}$$

The construction of such an asymptotic symbol is based on a classical lemma of Borel showing that for every sequence b_n, there exists a smooth function whose Taylor coefficients at zero are b_n. In this situation, we will write

$$a \sim \sum_{k=0}^{\infty} a_k. \tag{1.25}$$

It is clear that the correspondence between symbols and pseudodifferential operators is linear. The relationship between the two as algebras is more complicated, as the next lemma shows.

Lemma 1.2.4. *Let $A \in OPS^{m_1}(\Omega)$, $B \in OPS^{m_2}(\Omega)$ and let both be properly supported. Then $B \circ A \in OPS^{m_1+m_2}(\Omega)$ and*

$$\sigma(B \circ A) \sim \sum_{\alpha \geq 0} \frac{i^{-|\alpha|}}{\alpha!} [\partial_\xi^\alpha \sigma(B)][\partial_x^\alpha \sigma(A)]. \tag{1.26}$$

In particular,

$$\sigma_{m_1+m_2}(B \circ A) = \sigma(B)\sigma(A). \tag{1.27}$$

By convention the sum is over all multi-indexes α is to be interpreted as

$$\sum_{k=0}^{\infty}\left(\sum_{|\alpha|=k}\frac{i^{-k}}{\alpha!}[\partial_\xi^\alpha\sigma(B)][\partial_x^\alpha\sigma(A)]\right), \tag{1.28}$$

where the interior sums are finite and belong to the symbol class $S^{m_1+m_2-k}(\Omega)$.

We now concentrate on the class of elliptic pseudodifferential operators and their corresponding parametrices.

Definition 1.2.3. *A symbol $a \in S^m(\Omega)$ is elliptic when for all $K \Subset \Omega$ compact, there exists $C > 0$ and $R > 0$ such that for all $x \in K$ and $|\xi| > R$, we have,*

$$|a(x,\xi)| \geq C\langle\xi\rangle^m. \tag{1.29}$$

Such operators can be inverted up to smooth error by a parametrix. The construction below produces a left pseudo-inverse Q to any elliptic pseudodifferential operator P.

Lemma 1.2.5. *Let $P \in OPS^m(\Omega)$ be a properly supported, elliptic pseudodifferential operator. Then there exists $Q \in OPS^{-m}(\Omega)$ such that*

$$PQ = I + R,\ \ R \in OPS^{-\infty}. \tag{1.30}$$

Proof. Define $p(x,\xi) = \sigma(P)$. Let $\chi \in C^\infty(T^*\Omega)$ be a smooth cut-off function that vanishes in a neighborhood of $\{(x,\xi) : p(x,\xi) = 0\}$ and is equal to one for $|\xi| \geq R$, where $R > 0$ is the same constant as in the definition of ellipticity. We will define a symbol $q(x,\xi)$ asymptotically, via Lemma 1.2.3, such that $P \circ \mathrm{Op}(q)$ is the identity modulo smoothing error.

Define

$$q_{-m}(x,\xi) = \begin{cases}\chi(x,\xi)p(x,\xi)^{-1} & p(x,\xi) \neq 0\\ 0 & \text{otherwise}\end{cases} \tag{1.31}$$

By ellipticity, we see that the $S^{-m}(\Omega)$ estimates (1.18) hold with $\alpha, \beta = 0$. The remaining estimates follow from the chain rule and the symbol estimates of p; the contribution of χ can be ignored, as the symbol estimates only depend on the growth rate of the derivatives of the symbol as ξ becomes large. It follows from Lemma 1.2.4 that

$$\sigma_0(P \circ \mathrm{Op}(q_{-m})) = 1. \tag{1.32}$$

Therefore $P \circ \mathrm{Op}(q_{-m}) = I + R_1$, where R_1 is a pseudodifferential operator of class $OPS^{-1}(\Omega)$. Formally, the right hand side has a pseudoinverse via Neumann series,

$$(I + R_1)\left(\sum_{k=0}^{\infty}(-R_1)^k\right) = I. \tag{1.33}$$

Each term of the series is a pseudodifferential operator of class OPS^{-k}, and therefore meaning can be given to the Neumann series by applying Lemma 1.2.3 to the symbols of each term, obtaining a symbol

$$r_1^{\dagger}(x,\xi) \sim \sum_{k=0}^{\infty} \sigma((-R_1)^k). \tag{1.34}$$

The resulting operator $R_1^{\dagger} = \mathrm{Op}(r_1^{\dagger})$ is such that for all k

$$(1 + R_1)R_1^{\dagger} = I \mod OPS^{-k}. \tag{1.35}$$

Therefore

$$P \circ [\mathrm{Op}(q_{-m}) \circ R_1^{\dagger}] = I \mod OPS^{-\infty}, \tag{1.36}$$

and the term in braces is the parametrix Q that we sought. This parametrix is actually two-sided, as is shown in [56, Theorem III.1.3]. $\square$

Our main application of elliptic pseudodifferential operators is their use in obtaining stability estimates, as the following lemma demonstrates.

Lemma 1.2.6. *Let $P \in OPS^m(\Omega)$ be a properly supported, elliptic pseudodifferential operator, and $K \Subset \Omega$ compact. Then for all $u \in H^m(\Omega)$, $\operatorname{supp} u \subset K$, we have a constant $C > 0$ and for all $s > 0$ a constant $C_s > 0$ such that*

$$||u||_{H^m(\Omega)} \leq C||Pu||_{L^2(\Omega)} + C_s||u||_{H^{-s}(\Omega)}. \tag{1.37}$$

If, in addition, it is known that P is injective, then there is a constant $C' > 0$ with

$$||u||_{H^m(\Omega)} \leq C'||Pu||_{L^2(\Omega)}. \tag{1.38}$$

Proof. Let Q be a parametrix for P, so that $QP = I + R$ with R smoothing. Then,

$$\begin{aligned} ||u||_{H^m(\Omega)} &= ||(QP - R)u||_{H^m(\Omega)} \\ &\leq ||QPu||_{H^m(\Omega)} + ||Ru||_{H^m(\Omega)}. \end{aligned}$$

R is smoothing and is therefore continuous $L^2_{\text{loc}}(\Omega) \to H^{m+s}(\Omega)$ for all $s > 0$. By the mapping properties of Q, we have (1.37).

For the final stability estimate (1.38), we apply an argument from functional analysis detailed in [56, Theorem V.3.1]. □

Wavefront set

The last tool we will require from the calculus of pseudodifferential operators is the wavefront set. We recall the definition from [30, Definition 8.1.2].

Definition 1.2.4. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and fix $(x_0, \xi_0) \in T^*\mathbb{R}^n \backslash 0$. We say u is microlocally smooth near (x_0, ξ_0) if there exists a smooth cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x_0) = 1$ and an open conic neighborhood Γ of $\xi_0 \in T^*_{x_0}\mathbb{R}^n$ such that for all $N \in \mathbb{N}$ there exists $C_N > 0$ with*

$$\sup_{\xi \in \Gamma} |\widehat{\chi u}(\xi)| \leq C_N \langle \xi \rangle^{-N}. \tag{1.39}$$

The wavefront set $\mathrm{WF}(u)$ of u is the complement of the set of covectors (x, ξ) at which u is microlocally smooth.

One may characterize differential operators by Peetre's theorem [44], which states that they are the only linear operators which do not increase the support of distributions. In a similar sense, pseudodifferential operators may be characterized as the linear operators which do not increase the wavefront set.

Lemma 1.2.7. *If $u \in \mathcal{E}'(\Omega)$ and $P \in OPS^m(\Omega)$, then*

$$\mathrm{WF}(Pu) \subset \mathrm{WF}(u). \tag{1.40}$$

This lemma will be a special case of a similar lemma for Fourier integral operators in the sequel. Notice that if P is elliptic and Q is a parametrix for it, then

$$\mathrm{WF}(u) = \mathrm{WF}(QPu) \subset \mathrm{WF}(u). \tag{1.41}$$

Therefore elliptic pseudodifferential operators preserve the wavefront set. From the definition it can be shown that $\mathrm{WF}(u)$ is always a closed, conic set of $T^*\Omega$; also, given an arbitrary closed, conic subset $\Gamma \subset T^*\Omega$, there exists a distribution $u \in \mathcal{D}'(\Omega)$ such that $\mathrm{WF}(u) = \Gamma$, see for example the construction of [30, Theorem 8.1.4].

For applications to the damped wave equation in our study of thermoacoustic tomography, we state without proof Hörmander's theorem on propagation of singularities, which describes how the solution operator of a strictly hyperbolic Cauchy problem interacts with the wavefront set of the Cauchy data.

Theorem 1.2.1 ([32, Theorem 23.2.9]). *Let P be a strictly hyperbolic differential operator of order m with smooth coefficients on Ω and principal symbol $p(x,\xi)$. If $Pu = f$, then $\mathrm{WF}(u) \setminus \mathrm{WF}(f)$ is a subset of $\{p = 0\}$ and is invariant under the Hamiltonian flow, which is generated by the vector field H_p defined on $T^*\Omega \setminus 0$ by*

$$H_p = \sum_{i=1}^{n} \frac{\partial p}{\partial \xi_i}\frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i}\frac{\partial}{\partial \xi_i}. \tag{1.42}$$

Pseudodifferential operators on manifolds

It will sometimes be necessary to discuss the results of this section in the context of manifolds; usually, compact Riemannian manfiolds with boundary. While there is a calculus of pseudodifferential operators on manifolds with boundary (and even more exotic singularities), for simplicity we tend to use only the usual calculus with the caveat that we are always only concerned with distributions whose support is located a positive distance away from all boundary components.

For example, in the chapter on thermoacoustic tomography, we will work with distributions on $\partial\Omega \times [0,\infty)$ where $\partial\Omega$ is the boundary of a bounded, smooth, convex domain. In this case, these distributions will be the restriction of the solution of a

Cauchy problem with data at $\{t = 0\}$ to this cylinder of the boundary. In this case we will assume *a priori*, perhaps by increasing the size of Ω slightly, that the region of interest on which the Cauchy data is supported lies a positive distance away from the boundary. By finite speed of propagation, there is a small time $t_{\text{min}} > 0$ such that the solution is zero on the boundary before t_{min}.

With this in mind, we consider a pseudodifferential operator on an open manifold *without* boundary M to be any linear operator whose Schwartz kernel in local coordinates can be written as the kernel of a pseudodifferential operator, as in (1.21). Note however that this definition does not agree with our definition of $OPS^m(\mathbb{R}^n)$, because the symbol estimates associated to the latter operators are stronger than requiring only that these estimates hold locally. In any case, we will always be concerned in applications with a compact region of interest, and so the disagreement between these definitions is not problematic.

If $P \in OPS^m(M)$, then $\sigma_m(P) \in C^\infty(T^*M)$ is invariantly defined. Typically, the lower order terms of any asymptotic expansion for $\sigma(P)$ depend on the choice of coordinates. It is well-known that the wavefront set is also invariantly defined, and transforms under change of coordinates as a subset of the cotangent bundle.

1.2.3 Fourier integral operators

Beyond the calculus of pseudodifferential operators, one may also consider linear operators defined by Schwartz kernels with more general singular support than the diagonal. For the purpose of the applications to come, we require only the local theory of such operators, and so we restrict ourselves to a bounded open domain $\Omega \subset \mathbb{R}^n$ and an open, conic subset $\Gamma \subset \Omega \times (\mathbb{R}^N \setminus 0)$, for some $N > 0$. In this context we define a local symbol $a(x, \theta) \in C^\infty(\Gamma)$ to be a smooth function with support in a closed, conic subset of Γ satisfying some order $m - \frac{N}{2} + \frac{n}{4}$ of symbol estimates (1.18). The odd-looking symbol class is necessary for the following definition to agree with the order of a pseudodifferential operator.

An oscillating integral of order m is a distribution defined by a local symbol $a \in S^{m-\frac{N}{2}+\frac{n}{4}}(\Gamma)$ and a phase function $\varphi(x,\theta) \in C^\infty(\Gamma)$. We assume phase functions are non-degenerate in the following sense:

1. φ is positive homogeneous of degree one in the fiber variable on Γ.
2. The imaginary part of φ is non-negative on Γ.
3. $d_{x,\theta}\varphi \neq 0$ on Γ.
4. If $d_\theta \varphi = 0$, then the forms
$$d_{x,\theta}\left(\frac{\partial \varphi}{\partial \theta_j}\right), \quad j = 1, \ldots, N,$$
are linearly independent.

We define the distribution $I_{a,\varphi}$ for $u \in C_0^\infty(\mathbb{R}^n)$ via
$$\langle I_{a,\varphi}, u \rangle = \iint_\Gamma e^{i\varphi(x,\theta)} a(x,\theta)\, dx\, d\theta. \tag{1.43}$$
If $a \in S^{-N}(\Gamma)$ then this integral is absolutely convergent. There is an extension of $I_{a,\varphi}$ as a continuous linear operator on each $S^m(\Gamma)$, for which see [30, Theorem 7.8.2].

Local FIO as oscillating integrals

A special class of oscillating integrals are the Fourier integral operators (FIO), studied extensively by Duistermaat and Hörmander in [28, 15]. In the rest of this section, we follow the development of [14]. We take $\Omega = X \times Y$, with $X \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$ bounded open domains.

Definition 1.2.5. *A (local) FIO is a linear operator $A : C_0^\infty(Y) \to C^\infty(X)$ defined by the oscillatory integral*
$$Au(x) = \iint e^{i\varphi(x,y,\theta)} a(x,y,\theta) u(y)\, dy\, d\theta. \tag{1.44}$$
Here $a(x,y,\theta) \in S^{m-\frac{N}{2}+\frac{n_1+n_2}{4}}(\Gamma)$ is a local symbol and $\varphi(x,y,\theta) \in C^\infty(\Gamma)$ is a non-degenerate phase function.

Pseudodifferential operators are examples of (global) FIO with the standard phase function $\varphi(x, y, \theta) = (x - y) \cdot \theta$, in which $m - \frac{n}{2} + \frac{2n}{4} = m$, but there are many other examples. While defining local FIO via phase functions is convenient, the phase function is not fundamental to the definition and only serves to parameterize a certain conic, Lagrangian submanifold of $T^*(X \times Y) \setminus 0$, which is the wavefront set of the FIO's Schwartz kernel. Here, we will consider $T^*(X \times Y) \setminus 0$ to have the canonical symplectic form (1.5) given by

$$\omega = \sum_{k=1}^{n_1} dx^k \wedge d\xi^k + \sum_{\ell=1}^{n_2} dy^\ell \wedge d\eta^\ell. \tag{1.45}$$

Definition 1.2.6. *Let A be an FIO defined by amplitude a and phase function φ. Then the characteristic manifold is the submanifold of $X \times Y \times (\mathbb{R}^N \setminus 0)$ defined by*

$$C_\varphi = \{(x, y, \theta) : d_\theta \varphi(x, y, \theta) = 0\}. \tag{1.46}$$

The nondegeneracy assumptions on φ imply that the map

$$T(x, y, \theta) = (x, d_x\varphi(x, y, \theta), y, d_y\varphi(x, y, \theta)) \in T^*(X \times Y) \setminus 0. \tag{1.47}$$

is an immersion from C_φ to $T^*(X \times Y) \setminus 0$; its image is the conic Lagrangian submanifold Λ_φ associated to the phase function φ. There is another submanifold of $T^*(X \times Y) \setminus 0$,

$$\Lambda'_\varphi = \{(x, \xi, y, \eta) : (x, \xi, y, -\eta) \in \Lambda_\varphi\}, \tag{1.48}$$

which is called the canonical relation of A.

If A is a pseudodifferential operator, then $X = Y = \mathbb{R}^n$ and the characteristic submanifold is $\{(x, y, \xi) : x = y\}$. The corresponding Lagrangian submanifold is given by

$$\Lambda_{\Psi\mathrm{DO}} = \{(x, \xi, y, \eta) : (x, \xi, x, -\xi)\}, \tag{1.49}$$

which is sometimes referred to as the "twisted diagonal" of $T^*\mathbb{R}^{2n} \setminus 0$. It is clear that this is a Lagrangian submanifold with respect to ω. The canonical relation is the (untwisted) diagonal of $T^*\mathbb{R}^{2n} \setminus 0$.

The following important lemma shows how the canonical relation encodes all the relevant microlocal information carried by the corresponding Fourier integral operator. This generalizes the observation made earlier that pseudodifferential operators do not decrease the wavefront set of the distributions they act upon. We see that this is a consequence of the fact that all pseudodifferential operators have canonical relation equal to the diagonal of $T^*\mathbb{R}^{2n} \setminus 0$.

Lemma 1.2.8. *Let $u \in \mathcal{E}'(Y)$ and A a Fourier integral operator with canonical relation Λ'_φ, whose phase function and amplitude are defined on Γ. Then*

$$\mathrm{WF}(Au) \subset \Lambda'_\varphi \circ \mathrm{WF}(u) \tag{1.50}$$

where the action of the canonical relation on the wavefront set is the usual image of a set under a relation, that is,

$$\Lambda'_\varphi \circ \mathrm{WF}(u) = \left\{(x,\xi) : \exists (y,\eta) \in T^*Y \setminus 0, (x,\xi,y,\eta) \in \Lambda'_\varphi\right\}. \tag{1.51}$$

Note that A is smoothing on the subspace of distributions

$$\{u \in \mathcal{E}'(Y) : \mathrm{WF}(u) \cap \mathrm{supp}(a) = \emptyset\}, \tag{1.52}$$

where a is the amplitude of A.

FIO of graph type

It is often the case that the forward operators of applied linear inverse problems may be expressed in terms of FIOs, and this is the case with the three particular applications we will consider. If the forward operator A is an FIO, then it is common to attempt a reconstruction of $f \in \mathcal{E}'(Y)$ from knowledge of Af by applying the adjoint to the latter, A^*Af. In this situation, we refer to $N = A^*A$ as the normal operator of A, and in some cases the normal operator is invertible, which yields an inverse $(A^*A)^{-1}A^*$ for A.

A particularly well-behaved class of FIOs, which we will encounter in application to synthetic aperture radar, are FIOs of graph type.

Definition 1.2.7. *A local FIO $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ is said to be of graph type when the associated Lagrangian submanifold is the graph of a bijective symplectomorphism from $T^*(Y) \setminus 0$ to $T^*(X) \setminus 0$.*

We will show that if the amplitude of A is nonvanishing, the normal operator of A is an elliptic pseudodifferential operator, which may be inverted up to smoothing error by a parametrix as constructed in Lemma 1.2.5. First we will present a condition under which a given FIO may be microlocalized into an FIO of graph type. Recall $\Lambda \subset T^*(X \times Y) \setminus 0$. There exist two canonical vector bundle projections

$$\pi_X : T^*(X \times Y) \setminus 0 \to T^*X \setminus 0, \quad \pi_Y : T^*(X \times Y) \setminus 0 \to T^*Y \setminus 0. \tag{1.53}$$

In general the image of Λ under these projections may be quite singular. However, when one can microlocalize away from these singularities, one can obtain a restricted FIO that agrees with A in a small conic neighborhood but that is also of graph type.

Lemma 1.2.9. *Let A be an FIO with respect to the Lagrangian submanifold Λ, and fix two closed, conic subsets $F_1 \subset T^*Y \setminus 0$ and $F_2 \subset T^*X \setminus 0$. Define the restriction*

$$\widetilde{\Lambda}' = \{(x, \xi, y, \eta) \in \Lambda' : (x, \xi) \in \operatorname{int} F_1, (y, \eta) \in \operatorname{int} F_2\}. \tag{1.54}$$

Then there exists an FIO $\widetilde{A}$ such that for all $u \in \mathcal{E}'(Y)$ with $\mathrm{WF}(u) \subset F_1$, $(A - \widetilde{A})u$ is microlocally smooth on F_2.

If it is also the case that the restrictions of the canonical projections to $\widetilde{\Lambda}'$ are both bijective diffeomorphisms, then $\widetilde{A}$ is of graph type.

Proof. Assume $(\Lambda' \circ F_1) \cap F_2 \neq \emptyset$; otherwise, the constructed $\widetilde{A}$ is smoothing and satisfies the lemma trivially. We define two cut-off functions $\chi_1 \in C_0^\infty(T^*Y \setminus 0)$ and $\chi_2 \in C_0^\infty(T^*X \setminus 0)$, both homogeneous of degree zero. We require that $\chi_j(F_j) = 1$, and that $\operatorname{supp} \chi_j \subset \Gamma_j$, where Γ_j is a small conic neighborhood of F_j. We see that,

$$\operatorname{Op}(\chi_j) \in OPS^0(\Gamma_j), \;\; j = 1, 2. \tag{1.55}$$

Define

$$\widetilde{A} = \operatorname{Op}(\chi_2) \circ A \circ \operatorname{Op}(\chi_1). \tag{1.56}$$

Then by linearity we have

$$\begin{aligned} A - \widetilde{A} = {} & \mathrm{Op}(1-\chi_2) \circ A \circ \mathrm{Op}(1-\chi_1) \\ & + \mathrm{Op}(\chi_2) \circ A \circ \mathrm{Op}(1-\chi_1) \\ & + \mathrm{Op}(1-\chi_2) \circ A \circ \mathrm{Op}(\chi_1). \end{aligned}$$

If $u \in \mathcal{E}'(Y)$ has $\mathrm{WF}(u) \subset F_1$, then $\mathrm{Op}(1-\chi_1)u$ is smooth. Therefore the first and two terms above are smoothing on this subspace of distributions. By Lemma 1.2.8,

$$\mathrm{WF}(A \circ \mathrm{Op}(\chi_1)u) \subset \widetilde{\Lambda}' \circ F_1. \tag{1.57}$$

On the other hand, $\mathrm{Op}(1-\chi_2)$ is smoothing on the subspace of compactly supported distributions with wavefront set contained in F_2. Therefore,

$$\mathrm{WF}((A-\widetilde{A})u) \cap F_2 = \emptyset, \tag{1.58}$$

for all $u \in \mathcal{E}'(Y)$ with $\mathrm{WF}(u) \subset F_1$.

If the restriction of both canonical projections are bijective diffeomorphisms, then the map $\rho = \pi_X|_{\widetilde{\Lambda}} \circ \pi_Y|_{\widetilde{\Lambda}}^{-1}$ is a diffeomorphism that has $\widetilde{\Lambda}$ as its graph. Since $\widetilde{\Lambda}$ is both a Lagrangian submanifold and the graph of a bijective diffeomorphism, ρ is a symplectomorphism. □

We will refer to $\widetilde{A}$ as constructed in the previous lemma as the microlocalization of A to $(F_1 \times F_2) \cap \Lambda'$.

There is also a notion of ellipticity for such operators of graph type [33, Definition 25.3.4], but for simplicity we avoid the symbol calculus of Fourier integral operators and instead require the following stronger definition.

Definition 1.2.8. *Let A be an FIO of graph type. We say A is (strongly) elliptic if the amplitude $a(x, y, \theta) \in S^m(\Gamma)$ does not vanish on the interior of its support.*

In this case it follows from [32, Theorem 18.1.24] that the L^2-adjoint A^* has canonical relation Λ'^{-1}, and A^*A is an elliptic pseudodifferential operator. In the sequel this result will be used in conjunction with the previous lemma to construct microlocal parametrices of the forward operator A.

1.2.4 Analytic microlocal analysis

Following the general themes of microlocal analysis, one can also develop a calculus of pseudodifferential operators in the (real) analytic category. The tools of this field tend to be more restrictive and delicate than the smooth microlocal analysis that we have considered so far. This is due in part to the non-existence of analytic cut-off functions, and the difficulties involved with microlocalization while preserving some semblance of analyticity. There is also not, as of yet, a calculus of analytic Fourier integral operators, which further complicates matters. Here we present an abbreviated account of the basic theory, following for the most part Sjöstrand [50].

In exchange for its complexities, the analytic calculus can sometimes be more powerful than its smooth counterpart. We will only refer to it in our work on the generalized Radon transform, and so we make some anticipatory remarks here. We consider a Fourier integral operator R which, under some assumptions, is elliptic and of graph type. Therefore R^*R is an elliptic pseudodifferential operator, which was shown in this case by Guillemin and Sternberg [21]. However, this analysis yields only a parametrix for R; one cannot show that R is injective from this alone.

Instead of studying the normal operator, we consider directly the oscillatory integral defining R. A complex stationary phase lemma due to Sjöstrand can be used to show a relationship between the microlocal analyticity of Rf and that of f, superficially similar in form to Lemma 1.2.8. One weak application of this microlocal regularity result is that if $f \in \mathcal{E}'(\Omega)$ and $Rf = 0$, then f is analytic. But f has compact support, so in fact $f = 0$, and R is therefore injective (and not merely invertible up to smoothing error).

The first obstacle to be overcome in developing an analytic microlocal calculus is the problem, alluded to above, that there are no nontrivial, compactly supported, analytic cut-off functions. As a partial workaround, we use the following quasi-analytic cut-off functions, which are serviceable for the task at hand.

Lemma 1.2.10 ([58, Lemma V.1.1]). *Let $U \subset \mathbb{R}^n$ be an open set, and fix a small parameter $d > 0$. Then there exists a constant $C > 0$ and a sequence of smooth functions $\chi_N(\mathbb{R}^n)$ such that*

1. $0 \leq \chi_N \leq 1$, $\chi_N|_U = 1$, *and vanishing on* $\{x : \mathrm{dist}(x, U) > d\}$.
2. *The following estimate holds for every multi-index α with $|\alpha| \leq N$:*

$$|\partial^\alpha \chi_N| \leq (CN/d)^{|\alpha|}. \tag{1.59}$$

We now define the analytic symbol classes that we will use in the sequel. These are somewhat similar to the classical symbols defined previously, except that we will also allow these symbols to depend on a large parameter $\lambda \gg 1$. In the literature of semiclassical analysis and mathematical physics, it is common to take $\hbar = \lambda^{-1}$ to be a small parameter instead, but we follow Sjöstrand's convention here. Let $\Omega \subset \mathbb{C}^n$ to be an open domain. Let $U \subset \mathbb{C}^n$ be a small neighborhood of zero, and take $\Gamma = \Omega \times U$.

Definition 1.2.9 (Sjöstrand). *A (local) analytic symbol is a smooth function $a(x, \xi, \lambda)$ such that for all $\lambda \gg 1$ the function $a(\cdot, \cdot, \lambda)$ is holomorphic on Γ and for all $K \Subset \Gamma$ compact and all $\epsilon > 0$ there exists $C > 0$ with*

$$\sup_{(x,\xi)\in K} |a(x, \xi, \lambda)| \leq Ce^{\epsilon\lambda}. \tag{1.60}$$

We define the symbol class $S^m_\lambda(\Gamma)$ to be the space of analytic symbols $a(x, \xi, \lambda)$ defined on Γ such that for $K \Subset \Gamma$ compact there exists $C > 0$ with

$$\sup_{(x,\xi)\in K} |a(x, \xi, \lambda)| \leq C\lambda^m. \tag{1.61}$$

By analogy with classical symbols, we may define the principal symbol of an analytic symbol $a \in S^m_\lambda(\Gamma)$ to be any representative of the equivalence class of a in $S^m_\lambda(\Gamma)/S^{m-1}_\lambda(\Gamma)$. However, contrary to the case of classical symbols, an analytic symbol is elliptic when a does not vanish anywhere on Γ; not merely outside a compact set.

Analytic wavefront set

Historically, there were many definitions of the analytic wavefront set of a distribution, including some due to Sato [49], Hörmander [29], and Bros-Iagolnitzer [7]. All of these definitions were shown to be equivalent on distributions by Bony [6]. We will use a characterization in terms of oscillating integrals with complex phase, due to Bros-Iagolnitzer.

Let $\Omega \subset \mathbb{R}^n$ be a real domain. We will work microlocally, on a small neighborhood Γ of a fixed covector $(x_0, \xi_0) \in T^*\Omega \setminus 0$. Define

$$\tilde{\Gamma} = \{(x, y, \xi) : (x, \xi) \in \Gamma, (y, \xi) \in \Gamma\}. \tag{1.62}$$

Definition 1.2.10 ([50, Definition 6.1])**.** *Let $\varphi(x, y, \xi)$ be an analytic function defined on $\tilde{\Gamma}$ satisfying the following:*

1. *For all $(x, \xi) \in \Gamma$, $\varphi(x, x, \xi) = 0$ and $d_x\varphi(x, x, \xi) = \xi$.*
2. *There exists $C > 0$ such that $\Im\varphi(x, y, \xi) \geq C|x - y|^2$ on $\tilde{\Gamma}$.*

Let $a(x, y, \xi, \lambda)$ be an elliptic, analytic symbol defined on $\tilde{\Gamma}$.

Define for $u \in \mathcal{D}'(\Omega)$

$$Au(x, \xi) = \int e^{i\lambda\varphi(x,y,\xi)} a(x, y, \xi, \lambda)\chi(y)\overline{u}(y)\, dy, \tag{1.63}$$

in the sense of oscillating integrals. Here $\chi \in C_0^\infty(\Omega)$ is a cut-off function, $\chi(x_0) = 1$. Then we say u is microlocally analytic near $(x_0, \xi_0) \in \Gamma$ when

$$Au(x, \xi) = O(e^{-\lambda/C}) \text{ as } \lambda \to \infty \tag{1.64}$$

for some constant $C > 0$, uniformly for (x, ξ) in a neighborhood (x_0, ξ_0).

As before, define $\mathrm{WF}_A(u)$ to be the closed conic subset of the cotangent bundle on which u is not microlocally analytic. This definition does not depend on the particular choice of (φ, a) (by [50, Proposition 6.2]), which we will exploit in our study of the generalized Radon transform. As with the smooth wavefront set, the analytic wavefront set is invariant under change of coordinates, but the proof of this is quite involved.

Complex stationary phase lemma

It must be admitted that Definition 1.2.10 is not the most concise characterization of the analytic wavefront set of a distribution. However, it forms the basis of a technique for showing the microlocal regularity of some oscillating integrals.

The idea is to augment the phase function with an additional $2n$ complex variables in such a way that the resulting augmented phase function is stationary along a submanifold of complex codimension n. One then applies a complex stationary phase lemma to show that contributions away from the stationary submanifold are of exponential decay in λ. Near the stationary submanifold, the phase function satisfies the conditions in the definition of the analytic wavefront set. This technique is used more explicitly in Theorem 4.3.1.

The following lemma is a variant of [50, Theorem 2.8] with a parameter, as mentioned but not proved in [50, Remark 2.10].

Lemma 1.2.11. *Let $W \subset \mathbb{C}^k$, $U \subset \mathbb{C}^n$ be two neighborhoods of zero, and let $\varphi(w,z)$ be a holomorphic function on $W \times U$, with $z = 0$ an isolated, nondegenerate critical point of the function $z \mapsto \varphi(0,z)$. Let $V \subset U$ be a proper neighborhood of zero, with $V_{\mathbb{R}}$ its real part, and suppose $\Re\varphi(0,\cdot) \geq 0$ on $V_{\mathbb{R}}$, with strict inequality on the boundary. After perhaps shrinking W, we have*

$$e^{\lambda\varphi(w,z(w))} \int_{V_{\mathbb{R}}} e^{-\lambda\varphi(w,x)} u(x)\, dx = \sum_{0 \leq k \leq \lambda/C} A_k(z(w)) + R(w,\lambda), \tag{1.65}$$

where each term is of the form

$$A_k(z) = \frac{(2\pi)^{n/2}}{k!} \lambda^{-\frac{n}{2}-k} (\tilde{\Delta}_W)^k \left(\frac{u}{\mathcal{I}_W}\right)(z(w)), \tag{1.66}$$

and $\tilde{\Delta}_W, \mathcal{I}_W$ and $z(w)$ all depend holomorphically on w. The remainder term $R(w,\lambda)$ is uniformly exponentially decaying as $\lambda \to \infty$.

Proof. As a first step, notice that $z(w)$ may be defined as the solution $z(w)$ of

$$d_z\varphi(w,z) = 0 \tag{1.67}$$

for w in a neighborhood of zero via the implicit mapping theorem. It follows that $\varphi(w, z(w))$ is a non-degenerate critical point with respect to the second variable for all $w \in W$, perhaps after shrinking W.

Our goal is to reduce (1.65) to the simpler theorem [50, Theorem 2.8], which yields the desired estimate, but is not uniform for $w \in W$. It remains to show that $\widetilde{\Delta}_W$ and $\mathcal{I}_W$ depend holomorphically on w and that the error is uniformly, exponentially decaying. In the context of this lemma, $\widetilde{\Delta}$ is the Laplacian in Morse coordinates $\tilde{z}$ which reduce the phase function to a quadratic form, and

$$\mathcal{I}_W = \pm \det \frac{\partial \tilde{z}}{\partial z} \tag{1.68}$$

is the Jacobian of the Morse coordinates. The sign is chosen so that

$$\mathcal{I}_W(0,0) = (\det \varphi(0,0))^{1/2} \tag{1.69}$$

is the principal branch of the square root.

To show $\widetilde{\Delta}_W$ is holomorphic in w, we use a parameterized, holomorphic variant of the Morse lemma; see [50, Lemma 2.7]. We have, uniformly for $w \in W$, the expansion

$$\varphi(w,z) = \frac{\partial^2 \varphi(w, z(w))}{\partial z^j \partial z^k}(z - z(w))^j (z - z(w))^k + O(|z - z(w)|^3). \tag{1.70}$$

After perhaps shrinking W again, we can fix the signature of the Hessian to be constant. There is a preliminary change of coordinates, which we refer to again as z, such that

$$\varphi(w,z) = \frac{1}{2}|z - z(w)|^2 + O(|z - z(w)|^3). \tag{1.71}$$

We now prove the parameterized version of the Morse lemma needed to reduce each $\varphi(w, \cdot)$ to a quadratic form in new coordinates $\tilde{z}$. By Taylor's theorem,

$$\begin{aligned} \varphi(w,z) &= \int_0^1 (1-t)\frac{\partial^2}{\partial t^2}[\varphi(w, tz + (1-t)(z - z(w)))]\, dt \\ &= \frac{1}{2}(z - z(w))^t Q(w,z)(z - z(w)) \end{aligned}$$

where

$$Q(w,z) = 2\int_0^1 (1-t)\frac{\partial^2 \varphi}{\partial z^j \partial z^k}(w, tz + (1-t)(z - z(w)))\, dt. \tag{1.72}$$

and

$$Q(0,0) = I. \tag{1.73}$$

After shrinking W, we have that $Q(w,z)$ is the identity plus a small perturbation on $W \times U$. We may then define $A(w,z) = Q(w,z)^{1/2}$, and $\tilde{z} = A(w,z)z$. The new coordinates $\tilde{z}(w,z)$ depend holomorphically on (w,z), and so $\tilde{\Delta}_W$ depends holomorphically on w. The Jacobian $\mathcal{I}_W$ is well-defined, as we can choose the same branch of the square root for each w to define

$$\mathcal{I}_W(w, z(w)) = (\det \partial_{z^j} \partial_{z^k} \varphi(w, z(w)))^{1/2}. \tag{1.74}$$

The error depends on $\Re u|_{\partial V_{\mathbb{R}}} > 0$, and is therefore also uniform in w. This concludes the modification of [50, Theorem 2.8] to accept a complex parameter. $\square$

2. THERMOACOUSTIC TOMOGRAPHY

2.1 Historical notes

In this chapter we study an inverse problem related to the multi-wave medical imaging technique thermoacoustic tomography (TAT). In thermoacoustic tomography, the patient is illuminated by weak microwaves, which slightly penetrate the body and heat it. The heating of the interior is not uniform, as various tissues absorb microwaves at varying rates. Once heated, the tissue vibrates via the thermoacoustic effect, which generates ultrasound waves. While the body is mostly opaque to microwaves, it is mostly transparent to ultrasound. These waves then propagate throughout the body, and those that exit are recorded at the boundary of the body, using ultrasound transducers.

Properly speaking, there are two inverse problems that must be solved to recover the microwave absorption coefficient in the interior of the body. First, one must recover the ultrasound source from the transducer data. Next, one must recover the absorption coefficient from the ultrasound data. The second inverse problem is referred to as quantitative thermoacoustic tomography (QTAT) and is not considered here, as it is an elliptic inverse problem [2]. The first problem is also shared with another multi-wave imaging method, called photoacoustic tomography (PAT), which uses near-infrared light instead of microwaves, and the photoacoustic effect instead of the thermoacoustic effect. However, the problem of recovering the ultrasound source is mathematically the same.

Classically, this inverse problem belongs to the field of integral geometry. If one assumes that the sound speed of the human body is nearly constant and neglects attenuation effects, the waves generated by delta-like ultrasound sources are spherical. The heating process occurs on a much shorter time scale than ultrasound propagation, and so approximating the source as a very short pulse is acceptable. In this case the transducer data may be interpreted as the circular Radon transform of the source, with the center of each circle located at the boundary [22]. There were also solutions to this problem via eigenfunction expansions [34, 38].

However, the human body does not have constant sound speed; it varies from the speed of sound in water by up to 20%. In this case, ultrasound waves are no longer spherical and may exhibit complicated behavior, including caustics. The method of time reversal is capable of solving the problem in this case. The original motivation for time reversal was the symmetry of the wave equation under the change of coordinates $t \mapsto -t$. The method assumes formally that measurements are made for all time, obtaining Dirichlet data on the boundary. Assuming some amount of energy decay as $t \to \infty$, one can solve a mixed boundary problem for the wave equation, backward in time, and use the resulting solution at $t = 0$ as an approximation to the ultrasound source. To avoid having to take measurements for large time, and to avoid cutting off the data, a modified method of time reversal was invented that replaced imposing zero Cauchy data at $t = \infty$ with imposing fictious Cauchy data at $t = T < \infty$, compatible with the measurement data [51, 52]. This method has also been studied in the context of the elastic wave equation [57].

My contribution to this area is an extension of the modified time reversal method to a model with some attenuation, which is known to cause artifacts in photoacoustic tomography [11, 13]. The problem of attenuation in heterogenous media is complicated, and there are many competing models of wave attenuation – we refer the reader to [36] for an overview. A kind of regularized time reversal with complete data for large times has been studied for some of these as homogeneous models [1, 35].

The damped wave equation has the advantage of being the simplest linear model, taking attenuation into account as a lower order perturbation of the wave equation. By contrast, many of the proposed models are parabolic, with the effect of attenuation appearing in the higher order terms. It is an oddity that what a physicist might consider a "lower order term" in a model manifests as a PDE with "higher order terms" from the perspective of microlocal analysis.

On the other hand, the damped wave equation does not remain invariant under time reversal, which complicates matters considerably. The global energy of the damped wave equation is non-increasing, and at worst is exponentially decaying. Therefore, when we attempt to solve the wave equation backward, the energy of the solution grows at most exponentially. If we were to attempt an unmodified time reversal in this regime, any data cut-off for large time would induce an exponentially large error in the reconstruction. However, as we attempt only a modified time reversal and measure data up to a fixed time $T < \infty$, the energy growth is at worst $O(e^{T||a||_\infty})$, which is bounded.

Within the inverse problem at hand we consider two separate problems. First, there is the problem of recovering the ultrasound source given complete data for a sufficiently long time (and we will make this assumption clearer in the sequel). Second, there is the same problem, but with partial data given on a subset of the boundary. For the first problem, we show uniqueness, stability, and – for small attenuation – a Neumann-series reconstruction algorithm. For the second problem, we prove a slightly weaker estimate that ensures the stable recovery of singularities.

2.1.1 Model assumptions

We assume the region of interest is contained in the interior of a bounded, strictly convex, smooth region $\Omega \subset \mathbb{R}^n$. The function $f \in H_0^1(\Omega)$ will model the ultrasound source distribution within the region of interest. We assume that the support of f is at least some small distance away from the boundary of Ω, so that the solution of (2.1) is zero on the boundary up to some time $t_{\min} > 0$.

The damped wave equation we introduce in the next section depends on two parameters, $c(x)$ and $a(x)$. The speed of sound $c(x)$ is assumed to be a smooth function, $0 < c_{\min} \leq c(x) \leq c_{\max} < \infty$. We also assume that $c(x) = 1$ in a neighborhood of

$\mathbb{R}^n \setminus \Omega$. The attenuation coefficient $a(x)$ is assumed to be a smooth, non-negative function, supported inside Ω. We assume both are known *a priori*, perhaps via an alternative imaging technique. We note that these model assumptions are consistent with the practice of immersing the patient in either water or another acoustically similar homogeneous medium, which serves as a transition between the boundary of the patient's body and the transducer array.

2.2 Damped wave equation

In this section, we will use the damped wave equation as a model for ultrasound propagation. In accordance with the modelling assumptions specific to thermoacoustic and photoacoustic tomography, we are interested in solutions of the system

$$\begin{cases} (\partial_t^2 + a\partial_t - c^2\Delta)u &= f(x)\delta'(t) \text{ in } \mathbb{R}^{n+1}, \\ u|_{t<0} &= 0, \end{cases} \tag{2.1}$$

in the sense of distributions. For brevity in the sequel we write $\Box_a$ for the operator on the left-hand side of (2.1), so that $\Box_0$ is the usual wave equation with respect to the metric $c^{-2}(x)\,dx^2$.

For concreteness we will work instead with a Cauchy problem equivalent to (2.1) when the ultrasound source $f(x)$ is *a priori* in $H_D(\Omega)$. Recall $H_D(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||f||_{H_D} = \left(\int_\Omega |\nabla f|^2\,dx\right), \tag{2.2}$$

and is (by Poincaré's lemma) equivalent to $H_0^1(\Omega)$.

Lemma 2.2.1. *Assume $f \in H_D(\Omega)$. Then the solution of* (2.1) *agrees with that of the Cauchy problem*

$$\begin{cases} \Box_a u &= 0 \text{ in } (0,\infty) \times \mathbb{R}^n, \\ u|_{t=0} &= f, \\ \partial_t u|_{t=0} &= -af, \end{cases} \tag{2.3}$$

when the latter is extended by zero to a distribution on $\mathbb{R}^{n+1}$.

Proof. To show this, we begin by taking a smooth solution $u \in C^\infty(\mathbb{R}^{n+1})$ that satisfies (2.3) everywhere in $\mathbb{R}^{n+1}$. Take $H(t)$ to be the Heaviside function. Then $H(t)u(t,x)$ is a distribution that is supported on $\{t \geq 0\}$. By the calculus of distributions, we see that

$$\Box_a(Hu) = u\delta' + 2(\partial_t u)\delta + au\delta, \tag{2.4}$$

where $\delta(t)$ is the Dirac delta distribution and $\delta'(t)$ is its weak derivative.

Let $v \in C_0^\infty(\mathbb{R}^{n+1})$ be a test function. Then we may calculate

$$\begin{aligned}\langle \Box_a(Hu), v\rangle &= \int_{\mathbb{R}^n} [-(\partial_t u)v - u(\partial_t v) + 2(\partial_t u)v + auv]|_{t=0}\, dx \\ &= -\int_{\mathbb{R}^n} f\partial_t v|_{t=0}\, dx \\ &= \langle f\delta', v\rangle\,.\end{aligned}$$

By density this extends to $f \in H_D(\Omega)$, as the Cauchy problem (2.3) is well-posed for Cauchy data in $H_D(\Omega) \times L^2(\Omega)$. $\Box$

Measurements are modelled by the trace of the solution u to the lateral boundary $(0,T) \times \partial\Omega$. This defines the foward operator

$$\Lambda f = u|_{(0,T)\times\partial\Omega}. \tag{2.5}$$

The inverse problem of thermoacoustic tomography that we are concerned with in this section is the unique and stable recovery of the ultrasound source f from knowledge of Λf. We first study the uniqueness of the problem for complete data, finding some geometric conditions sufficient for uniqueness. In the case of partial data (e.g., $\Lambda f = u|_{(0,T)\times\Gamma}$, with $\Gamma \subset \partial\Omega$ open), uniqueness remains an open problem. Next we introduce the method of modified time reversal, developed for the undamped inverse problem in [51, 52]. We show that this recovery method is stable with either complete data or some kinds of partial data, using a geometric optics construction.

One of the advantages of modified time reversal in the undamped case is that it also serves as the basis of a reconstruction method, via Neumann series, that may be implemented numerically [45]. We show by perturbation that this Neumann series converges in this model, provided the attenuation is small.

We state an a priori estimate for solutions of the damped wave equation with mixed boundary data, based on similar estimates for non-homogeneous second-order hyperbolic problems due to [39].

Proposition 2.2.1 ([25, Proposition 1])**.** *Let Ω be a smooth domain, and u a solution of*

$$\begin{cases} \Box_a u &= F \text{ in } (0,T)\times\Omega, \\ u|_{t=0} &= f, \\ \partial_t u|_{t=0} &= g, \\ u|_{(0,T)\times\partial\Omega} &= h, \end{cases} \tag{2.6}$$

subject to the compatibility condition $h|_{t=0} = f|_{\partial\Omega}$. Then there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T]} ||(u,\partial_t u)||_{\mathcal{H}} \leq Ce^{T||a||_\infty}\left\{||F||_{L^1(0,T;L^2)} + ||f||_{H^1} + ||g||_{L^2} + ||h||_{H^1}\right\}. \tag{2.7}$$

The proof is contained in Section 5 of [25].

2.3 Uniqueness

As mentioned earlier, we will show in this section that the inverse problem of recovering $f \in H_0^1(\Omega)$ from Λf, subject to the model assumptions, has a unique solution, provided the measurement duration T is sufficiently large. Recall in particular that we have assumed that the distance from the support of f to the boundary is positive, and that the region of interest is strictly convex. The main tool is a special case of a unique continuation theorem due to Tataru [55], which we state below.

Lemma 2.3.1 ([51, Theorem 4])**.** *Let u be the solution of the Cauchy problem* (2.3)*, and assume that there exists a neighborhood U of some $x_0 \in \mathbb{R}^n$ such that $u = 0$ on $(0,T)\times U$, with $T > 0$.*

Then $u = 0$ on the following set, which is the intersection of a forward light cone with vertex at $(0, x_0)$ and a backward light cone with vertex at (T, x_0):

$$\left\{(t, x) \in \mathbb{R}^{n+1} : d(x, x_0) < \frac{T}{2} - \left|\frac{T}{2} - t\right|\right\}.$$

This allows us to prove uniqueness for all measurement durations $T > 2T_0(\Omega)$, where T_0 is the characteristic uniqueness time for the corresponding undamped inverse problem on the same domain, namely

$$T_0(\Omega) = \sup_{x \in \Omega} d(x, \partial\Omega), \tag{2.8}$$

where $d(x, \partial\Omega)$ is the infimum of the lengths (with respect to the metric) of all curves connecting x to the boundary.

Theorem 2.3.1 ([25, Theorem 2])**.** *In addition to the model assumptions, let $\Lambda f = 0$ and $2T_0(\Omega) < T < \infty$. Then $f = 0$.*

Proof. Let u be the solution of the Cauchy problem (2.3) with data $(f, -af)$. Then u is also a solution of the damped wave operator on $(0, T) \times (\mathbb{R}^n \setminus \overline{\Omega})$, with Cauchy data $(0, 0)$ at $t = 0$ and zero Dirichlet data on $(0, T) \times \partial\Omega$. There exists a sufficiently large ball $B \subset \mathbb{R}^n$ such that $u = 0$ on $(0, T) \times \partial B$ by finite speed of propagation. We may apply Proposition 2.2.1 to $(0, T) \times (B \setminus \overline{\Omega})$ to conclude that $u = 0$ there.

For each $x \in (B \setminus \overline{\Omega})$, we may apply Lemma 2.3.1 to conclude that $u = 0$ in the intersection of the forward light cone with vertex $(0, x)$ and the backward light cone with vertex (T, x). The union of all such regions, if $T > 2T_0(\Omega)$, contains a neighborhood of $\{T/2\} \times \Omega$. Using the a priori estimate again, we conclude that $u = 0$ in $[0, T/2] \times \Omega$, and in particular $f = 0$. $\square$

It is also of interest to study the inverse problem with partial data on some $\Gamma \subset \partial\Omega$, as we do for stability below. However, in this case the question of uniqueness remains open for any time $T > 0$. It is known for the undamped problem [51], but there they use the fact that even extensions of solutions to the undamped wave equation to $t < 0$ are also solutions of the undamped wave equation. This is not true of the damped wave equation; such extensions solve a PDE with discontinuous coefficients, and Tataru's theorem no longer applies.

2.4 Modified time reversal

For a candidate pseudo-inverse to the forward operator Λ, we consider the modified time reversal method, which was successful in solving the undamped inverse problem [51, 52, 45]. This technique consists of constructing a pseudo-inverse A to the forward operator Λ by solving the wave equation "backward", using fictious Cauchy data at some time $t = T$ that is compatible with the boundary data $h = \Lambda f$. For the damped wave equation, we consider the following auxillary PDE:

$$\begin{cases} \Box_a v &= 0 \text{ in } (0,T) \times \Omega, \\ v|_{t=T} &= \phi, \\ \partial_t v|_{t=T} &= 0, \\ v|_{(0,T)\times\partial\Omega} &= h. \end{cases} \tag{2.9}$$

Here ϕ is the solution of the Laplace equation,

$$\begin{cases} \Delta\phi &= 0 \text{ in } \Omega, \\ \phi|_{\partial\Omega} &= h|_{t=T}. \end{cases} \tag{2.10}$$

The main problem one expects with this choice of pseudo-inverse is that solutions of the backward damped wave equation may grow in energy exponentially with respect to time. This would pose a problem in usual time reversal techniques, where one formally takes $T = \infty$. However, a modified time reversal technique requires only finite time to obtain unique and stable reconstructions, and so, as we see below, the increase in energy is bounded.

If $h = \Lambda f$, we define the pseudo-inverse in terms of the corresponding solution to (2.9):

$$Ah = v|_{t=0}. \tag{2.11}$$

It is this pseudo-inverse that we will study in the sequel. The error between the modified backward problem and the forward problem is controlled by the solution $w = u - v$ of the following mixed boundary problem:

$$\begin{cases} \Box_a w &= 0 \text{ in } (0,T)\times\Omega, \\ w|_{t=T} &= u|_{t=T} - \phi, \\ \partial_w|_{t=T} &= \partial_t u|_{t=T}, \\ w|_{(0,T)\times\partial\Omega} &= 0. \end{cases} \tag{2.12}$$

2.5 Stability

The stability of the inverse problem depends on a different characteristic measurement time, $T_1(\Omega)$. Intuitively, the problem should be stable when every covector of $\mathrm{WF}(f)$ can be recovered from Λf in a stable manner. By Theorem 1.2.1, the wavefront set of f propagates along the geodesic flow in both forward and backward directions, so that $\mathrm{WF}(u(t,\cdot)) = \phi^t(\mathrm{WF}(f)) \cup \phi^{-t}(\mathrm{WF}(f))$, where ϕ^t is the cogeodesic flow.

Let $(x,\xi) \in S^*\Omega$ parameterize the space of geodesics starting from Ω with unit speed. Recall $\gamma_{x,\xi}(t)$ is the unique geodesic with $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = c(x)\xi$, which is a vector in the tangent space of $x \in \Omega$ of unit length. If $\gamma_{x,\xi}$ is non-trapping, then define the exit time

$$\tau_{x,\xi} = \sup\left\{t \in (0,\infty) : \gamma_{x,\xi}((0,t)) \subset \overline{\Omega}\right\}. \tag{2.13}$$

Otherwise, we take $\tau_{x,\xi} = \infty$. In our model assumptions we assumed that the speed of propagation c is equal to one in a neighborhood of $\partial\Omega$ and also in the exterior of Ω. Further, Ω is strictly convex. Therefore every geodesic starting inside Ω with a finite exit time approaches the boundary non-tangentially and does not re-enter Ω.

We will first prove a simpler stability theorem for complete data, for measurement times in the regime

$$\sup_{(x,\xi)\in S^*\Omega} \tau_{x,\xi} < T < \infty. \tag{2.14}$$

However, this is strictly larger than the characteristic stability time,

$$T_1(\Omega) = \sup_{(x,\xi)\in S^*\Omega} \min\{\tau_{x,\xi}, \tau_{x,-\xi}\}. \tag{2.15}$$

We will show that $T_1(\Omega)$ is sufficient for stability in the proof of stability with partial data.

2.5.1 Complete data

If we have both complete data and a non-trapping metric, stability of the pseudo-inverse A (defined by (2.11)) almost follows directly from propagation of singularities (Theorem 1.2.1).

Theorem 2.5.1 ([25, Theorem 3])**.** *In addition to the model assumptions, assume*

$$T' = \sup_{(x,\xi)\in S^*\Omega} \tau_{x,\xi} < T < \infty, \tag{2.16}$$

i.e., the metric is non-trapping. Let $\chi \in C^\infty((0,\infty)\times\partial\Omega)$ be a cut-off function equal to one on $[0,T']\times\partial\Omega$ and zero in a neighborhood of $\{T\}\times\partial\Omega$. Then,

1. *$A\Lambda : H_D(\Omega) \to H_D(\Omega)$ is Fredholm.*

2. *$A\chi\Lambda = I + R$, where R is smoothing.*

3. *There exists $C > 0$ such that*

$$||f||_{H_D} \le C||\Lambda f||_{H^1}. \tag{2.17}$$

Proof. Define $K = I - A\Lambda$ as an operator $H_D(\Omega) \to H_D(\Omega)$. To show $A\Lambda$ is Fredholm, we show K is compact. Let u be the solution of the forward problem (2.3). By Theorem 1.2.1, $\mathrm{WF}(u(t,\cdot)) = \phi^t\mathrm{WF}(f)$, where ϕ^t is the cogeodesic flow on $(\mathbb{R}^n, c^{-2}(x)\,dx^2)$. Therefore $\mathrm{WF}(u(T,\cdot))$ does not lie over Ω.

Consider the operator

$$f \mapsto (u|_{t=T} - \phi, \partial_t u|_{t=T}) \tag{2.18}$$

which maps f to the Cauchy data of the PDE (2.12), describing the error $w = u - v$. It is a continuous operator from $H_D(\Omega)$ to $\mathcal{H}$. By propagation of singularities, this map is smoothing and therefore compact. The a priori estimate implies that the operator

$$(u|_{t=T} - \phi, \partial_t u|_{t=T}) \mapsto w|_{t=0} \tag{2.19}$$

is bounded as a map from $\mathcal{H}$ to $H_D(\Omega)$. The composition of the two, i.e., K, is therefore compact.

Now consider applying the pseudo-inverse A to cut-off measurement data $h = \chi\Lambda f$. In this case, define $Rf = w|_{t=0}$. By the construction of the cut-off function, the boundary data defining w via (2.12) is smooth and compatible to infinite order; therefore R is smoothing, and $A\chi\Lambda = I + R$.

This implies an estimate of the form

$$||f||_{H_D} \leq ||A\chi\Lambda f||_{H_D} + ||Rf||_{L^2}. \tag{2.20}$$

The pseudo-inverse is a bounded operator $H_1((0,T) \times \partial\Omega) \to H_D(\Omega)$, therefore there exists $C' > 0$ such that

$$||f||_{H_D} \leq C' \left(||\Lambda f||_{H^1}\right) + ||Rf||_{L^2}. \tag{2.21}$$

Recall the inverse problem with complete data is unique by Theorem 2.3.1, and the fact that $T' > T_0(\Omega)$. We apply [56, Proposition V.3.1] to conclude that for some $C > 0$,

$$||f||_{H_D} \leq C||\Lambda f||_{H^1}. \tag{2.22}$$

Therefore the inverse problem with complete data is stable for $T > T'$. $\square$

2.5.2 Partial data

In one respect the requirements of Theorem 2.5.1 are strictly stronger than what one would expect from microlocal considerations. A covector $(x_0, \xi_0) \in \mathrm{WF}(f)$ propagates in both the forward and backward directions, but in order to stably recover

(x_0, ξ_0), we need only recover only one of these propagating singularities as it exits the lateral boundary. Therefore the characteristic stability time that one expects is half the measurement time used in the proof of Theorem 2.5.1,

$$T_1(\Omega) = \sup_{(x,\xi)\in S^*\Omega} \min\{\tau_{x,\xi}, \tau_{x,-\xi}\}. \tag{2.23}$$

In this section we consider measurements made only on an open subset Γ of the boundary. Accordingly, we take a non-negative cut-off function $\chi \in ((0,\infty) \times \partial\Omega)$, supported on $(0,T) \times \Gamma$ and define

$$\Lambda f = \chi \cdot u|_{(0,T)\times\partial\Omega}. \tag{2.24}$$

To establish a criterion for the stable recovery of singularities, we must first describe those subregions of Ω that are visible from the measurement surface Γ.

Definition 2.5.1. *Let $\mathcal{K}$ be an open subset of Ω. A covector $(x,\xi) \in S^*\mathcal{K}$ is visible from Γ if either $\gamma_{x,\xi}$ or $\gamma_{x,-\xi}$ exits Ω nontangentially through Γ. Define $\tau'_{x,\xi}$ to be the exit time of the geodesic exiting through Γ, or the minimum of the two in the case that both exit through Γ, or ∞ if neither exits. Then the characteristic stability time for the partial data inverse problem is*

$$T_1(\mathcal{K}, \Gamma) = \sup_{(x,\xi)\in S^*\mathcal{K}} \tau'_{x,\xi}. \tag{2.25}$$

If we have complete data, we write $T_1(\Omega) = T_1(\Omega, \partial\Omega)$. If every covector of $S^*\mathcal{K}$ is visible from Γ, we say $\mathcal{K}$ is visible from Γ, and in this case $T_1(\mathcal{K}, \Gamma) < \infty$.

Our goal in this section is to prove the following theorem:

Theorem 2.5.2 ([25, Theorem 4])**.** *In addition to the model assumptions, assume $T_1(\mathcal{K}, \Gamma) < T < \infty$ and $\operatorname{supp} f \subset \mathcal{K}$. Let $\chi \in C^\infty((0,\infty) \times \partial\Omega)$ be a nonnegative cut-off function with support $[0, T_1(\mathcal{K}, \Gamma)] \times \overline{\Gamma}$. Then,*

1. *$A\chi\Lambda : H_D(\mathcal{K}) \to H_D(\Omega)$ is Fredholm.*

2. *$A\chi\Lambda : H_D(\Omega) \to H_D(\Omega)$ is a pseudodifferential operator of order zero whose symbol is elliptic on $T^*\mathcal{K} \setminus 0$.*

3. There exists a constant $C > 0$ such that

$$||f||_{H_D} \leq C\left(||\Lambda f||_{H^1} + ||f||_{L^2}\right). \tag{2.26}$$

Proof. The first part is the same as Theorem 2.5.1.

For the second part, we use a geometric optics construction, detailed in Lemma 2.5.1 below. This yields a microlocal parametrix of (2.9) with boundary data given by $h = \chi\Lambda$.

Finally, by elliptic regularity we have

$$||f||_{H_D} \leq C\left(||A\chi\Lambda f||_{H_D} + ||f||_{L^2}\right). \tag{2.27}$$

Then we apply the boundedness of A as a continuous operator $H^1(\Gamma) \to H_D(\Omega)$ to obtain the estimate. □

The geometric optics construction we use in the previous theorem is based on the fact that the principal symbol of $\Box_a$ is hyperbolic, and factors as

$$\sigma_2(\Box_a) = (\tau + c(x)|\eta|)(-\tau + c(x)|\eta|). \tag{2.28}$$

Indeed, this is the full symbol of the undamped wave equation.

We will use (y, η) as coordinates for $T^*\Omega \setminus 0$, and (t, x, τ, ξ) for coordinates on $T^*((0, \infty) \times \partial\Omega)$.

Lemma 2.5.1 ([25, Lemma 3 & 4])**.** *The forward operator Λ is the sum of two Fourier integral operators, Λ^+ and Λ^-, with canonical relations*

$$\{(t, x, \tau, \xi; y, \eta) : t = \tau_{y,\pm\eta}, x = \gamma_{y,\pm\eta}(t), \tau = \mp|\dot{\gamma}_{y,\pm\eta}(t)|, \xi = \pi(\dot{\gamma}_{y,\pm\eta}(t))\}. \tag{2.29}$$

where $\pi : T^*_x\mathbb{R}^n \to T^*_x\partial\Omega$ is the tangential projection.

Further, $A\chi\Lambda$ is a pseudodifferential operator of order zero with principal symbol

$$\sigma_0(A\chi\Lambda)(y, \eta) = \frac{1}{2}\left[\chi(\tau_{y,\eta}, \gamma_{y,\eta}(\tau_{y,\eta})) + \chi(\tau_{y,-\eta}, \gamma_{y,-\eta}(\tau_{y,-\eta}))\right]. \tag{2.30}$$

If $\mathcal{K}$ is visible from Γ, then this symbol is elliptic on $T^*\mathcal{K} \setminus 0$.

Proof. We will construct the two operators $\Lambda^{\pm}$ with local representations given by two amplitudes $A^{\pm}(t,y,\eta)$ and two phase functions $\phi^{\pm}(t,y,\eta)$. For each $t>0$, the representation will be a local Fourier integral operator, as in Definition 1.2.5. Each amplitude will be given by an asymptotic series

$$A^{\pm}(t,y,\eta) \sim \sum_{j\geq 0} A_j^{\pm}(t,y,\eta) \tag{2.31}$$

where each $A_j^{\pm}$ is homogeneous of degree $-j$ in η. After iteratively constructing each $A_j^{\pm}$, we choose $A^{\pm}$ in accordance with Lemma 1.2.3, up to an amplitude of order $-\infty$.

A lengthy calculation shows that

$$\Box_a u = (2\pi)^{-n} \sum_{\sigma=\pm} \int e^{i\phi^{\sigma}} [I_2 + I_1 + I_0] \hat{f}\, d\eta. \tag{2.32}$$

where

$$\begin{aligned}
I_2 &= -A^{\sigma}((\partial_t\phi^{\sigma})^2 - c^2|\nabla_y\phi^{\sigma}|^2),\\
I_1 &= i\left[2(\partial_t\phi^{\sigma})(\partial_t A^{\sigma}) - 2c^2(\nabla_y\phi^{\sigma})\cdot(\nabla_y A^{\sigma}) + A^{\sigma}\Box_a\phi^{\sigma}\right],\\
I_0 &= \Box_a A^{\sigma},
\end{aligned}$$

is an expansion of the amplitude of $\Box_a(e^{i\phi^+}A^+ + e^{i\phi^-}A^-)$ by order of homogeneity in the phase variable. Recall that the phase functions $\phi^{\pm}$ will be homogeneous of degree one in η.

We may impose the eikonal equations

$$\begin{cases} \mp\partial_t\phi^{\pm} &= c|\nabla_y\phi^{\pm}| \\ \phi^{\pm}|_{t=0} &= y\cdot\eta \end{cases} \tag{2.33}$$

on each $\phi^{\pm}$, noting that they are homogeneous of degree one in η. We assume these equations are solvable for $t\in[0,T]$, and remove this assumption later. By imposing these conditions on the phase function, we may ensure that I_2 vanishes.

The first two terms of I_1 may be recognized as the transport operators

$$X^{\pm} = 2(\partial_t\phi^{\pm})\partial_t - 2c^2(\nabla_y\phi^{\pm})\cdot\nabla_y, \tag{2.34}$$

applied to $A^{\pm}$ respectively. To control $I_1 + I_0$, we recursively solve the system of transport equations

$$\begin{cases} X^{\pm} A_0^{\pm} + A_0^{\pm} \Box_a \phi^{\pm} &= 0 \\ X^{\pm} A_j^{\pm} + A_j^{\pm} \Box_a \phi^{\pm} &= -\Box_a A_{j-1}^{\pm}, \quad j \geq 1, \end{cases} \tag{2.35}$$

together with suitable boundary conditions induced by the Cauchy data $(f, -af)$ given by the forward problem. These equations reduce to ordinary differential equations along the geodesics of $(\mathbb{R}^n, c^{-2}\, dx^2)$, whenever the eikonal equations are solvable.

The forward problem requires that $u|_{t=0} = f$. When applied to the local representation, we obtain

$$f(y) = (2\pi)^{-n} \int e^{iy\cdot\eta} \hat{f}(\eta)[A^+ + A^-]|_{t=0}\, d\eta. \tag{2.36}$$

Similarly, since $\partial_t u|_{t=0} = -af$, we have

$$\int e^{iy\cdot\eta} \hat{f}(\eta)(-a(y))\, d\eta = \int e^{iy\cdot\eta} \hat{f}(\eta) \left[ic|\eta|(-A^+ + A^-) + \partial_t(A^+ + A^-)\right]\, d\eta. \tag{2.37}$$

These two relations yield the following system of boundary conditions for the coefficients of the amplitudes.

$$\begin{cases} A_0^+ + A_0^- &= 1, \\ A_1^+ + A_1^- &= 0, \\ A_j^+ + A_j^- &= 0, \quad j \geq 2 \end{cases} \tag{2.38}$$

$$\begin{cases} A_0^+ - A_0^- &= 0, \\ A_1^+ - A_1^- &= -a - \partial_t(A_0^+ + A_0^-), \\ A_j^+ - A_j^- &= -\partial_t(A_{j-1}^+ + A_{j-1}^-), \quad j \geq 2 \end{cases} \tag{2.39}$$

This system may be solved recursively with the transport equations above to obtain each coefficient of each amplitude. In particular, $A_0^{\pm}(0, y, \eta) = \frac{1}{2}$. This completes the construction of u.

Restricted to the boundary, we obtain a global representation

$$\Lambda^{\pm} f(t, x) = (2\pi)^{-n} \int e^{i\phi^{\pm}(t,x,\eta)} A^{\pm}(t, x, \eta) \hat{f}(\eta)\, d\eta, \tag{2.40}$$

up to smoothing error. By Definition 1.2.6, we have that the characteristic submanifold of $\Lambda^{\pm}$ is

$$C_{\phi^{\pm}} = \{(t,x,y,\eta) : y = \partial_\eta \phi^{\pm}(t,x,\eta)\}. \tag{2.41}$$

It is a property of the eikonal equations that $y = \partial_\eta \phi^{\pm}(t,x,\eta)$ iff $\gamma_{y,\pm\eta}(t) = x$. This yields the form of the canonical relations given in the lemma. This concludes the construction of a parametrix for the forward operator.

We now proceed to construct a microlocal parametrix for the pseudo-inverse A, by performing a similar geometric optics construction for the solution of the backward problem, v. Fix a covector $(t_0, x_0, \tau_0, \xi_0) \in T^*_{(t_0,x_0)}((0,T) \times \Gamma)$. We will take $\rho \in C_0^\infty(\mathbb{R}^{n+1})$ to be a cut-off function supported in a small neighborhood U of (t_0, x_0) such that $\rho(t_0, x_0) = 1$. We will assume $\tau_0 < 0$, so that $h = \rho\chi\Lambda^+ f$; otherwise, take $h = \rho\chi\Lambda^- f$ in what follows, the construction is very similar. (Note that $\tau_0 \neq 0$ by the assumption that h is in the image of some $\Lambda^{\pm}$.) With this boundary data, the backward problem we are concerned with is,

$$\begin{cases} \Box_a v &= 0 \text{ in } (0,T) \times \Omega, \\ v|_{t=T} &= 0, \\ \partial_t v|_{t=T} &= 0, \\ v|_{(0,T)\times\partial\Omega} &= \rho\chi\Lambda^+ f. \end{cases} \tag{2.42}$$

We take as an Ansatz for v the local representation

$$v(t,y) = (2\pi)^{-n} \int e^{i\psi(t,y,\eta)} B(t,y,\eta) \hat{f}(\eta)\, d\eta, \tag{2.43}$$

which is similar to the forward problem. Recall $Ah = v|_{t=0}$.

There is an ambiguity in determining which null bicharacterstic the projected covector $(t_0, x_0, \tau_0, \xi_0)$ came from, as there are two (by strict convexity of $\partial\Omega$) whose projection onto $T^*_{(t_0,x_0)}((0,T) \times \Gamma)$ is (τ_0, ξ_0); one pointing inward, and one pointing outward, relative to $\partial\Omega$. Let (y_0, η_0) be related to $(t_0, x_0, \tau_0, \xi_0)$ under the canonical relation of Λ^+. Then the null bicharacteristic connecting the two is pointing outward at the boundary, and it is a small conic neighborhood of this null bicharactertistic that we will work near in the sequel.

As for the inward pointing bicharacteristic, those singularities near it propagate along it, possibly reflecting off the boundary at most finitely many times until the bicharacteristic passes over $\{t = T\}$. Here $\mathrm{WF}(v)$ is empty, and so v is smooth along the entire broken geodesic that this broken bicharacteristic lies over.

We now begin constructing a microlocal parametrix for the backward problem supported in a small neighborhood of the outward pointing null bicharacteristic. Our boundary data is zero outside of U, where it is

$$v|_U = (2\pi)^{-n} \int e^{i\phi^+} \rho\chi A^+ \hat{f}\, d\eta. \tag{2.44}$$

For v to be a parametrix, ψ must satisfy the eikonal equation and agree with ϕ^+ when restricted to U. This holds in particular if we set

$$-\partial_t \psi = c(x)|\nabla_x \psi|, \quad \psi|_U = \phi^+|_U. \tag{2.45}$$

Under the assumption that the eikonal equation for ϕ^+ is solvable for $t \in [0, T]$, this equation is also solvable for the same duration. By the method of characteristics, it agrees with ϕ^+ on its domain of definition. In particular, $\psi(0, y, \eta) = y \cdot \eta$.

Similarly, B can be expanded into an asymptotic series, whose coefficients satisfy the same system of transport equations as A^+. However, the boundary data is different; on U, $B = \rho\chi A^+$. By the homogeneity of the first transport equation, we have in particular that

$$B_0(0, y_0, \eta_0) = \rho(t_0, x_0)\chi(t_0, x_0)A_0^+(0, y_0, \eta_0) = \frac{1}{2}\chi(\tau_{y_0,\eta_0}, \gamma_{y_0,\eta_0}(\tau_{y_0,\eta_0})). \tag{2.46}$$

Restricting (2.43) to $t = 0$ yields a representation of $A\chi\Lambda$ as a pseudodifferential operator, with principal symbol $B_0(0, y, \eta)$.

So far we have assumed that the eikonal equation is solvable globally for $t \in [0, T]$. In general, it is only solvable on some small interval $t \in [0, t_1]$. To continue past $t = t_1$, we use the previous parametrix to obtain Cauchy data on this hyperplane, and repeat the construction with a new pair of phase function $\phi_1^\pm$ equal to $y \cdot \eta$ on $t = t_1$. These eikonal equations will be solvable on another interval, $[t_1, t_2]$. By compactness, the eikonal equations on Ω are always solvable for some small time interval $t \in [0, \epsilon]$. Therefore, after repeating this construction finitely many times (for $T < \infty$), one obtains a global parametrix on $(0, T) \times \Omega$. The backward parametrix is then constructed similarly, on the same intervals.

Finally, note that both terms in $\sigma_0(A\chi\Lambda)$ are nonnegative. If $\mathcal{K}$ is visible from Γ, then at least one term is positive. Therefore $A\chi\Lambda$ is elliptic on $T^*\mathcal{K} \setminus 0$. Even if $\mathcal{K}$ is not visible from Γ, $A\chi\Lambda$ is still a pseudodifferential operator, and is also Fredholm. $\square$

2.6 Neumann series reconstruction

The microlocal analysis of the previous two sections has shown that modified time reversal can be used to stably recover the visible singularities of the ultrasound source with either complete or partial data. However, there is no requirement that the smoothing errors that have accumulated in the analysis are small in either L^2 norm or energy. The practicality of modified time reversal in this model remains open. In this section we hope to bridge this gap by showing that for sufficiently small attenuation coefficients, a Neumann series approximating f still converges.

Let $K = A\Lambda - I$ be the error operator associated to the reconstruction given by modified time reversal for the damped wave equation. If the norm of K as an operator $H_D(\Omega) \to H_D(\Omega)$ is strictly less than one, it follows that the Neumann series

$$f = \sum_{m=0}^{\infty} K^m A h \tag{2.47}$$

converges in $H_D(\Omega)$ sense, where $h = \Lambda f$. Let $K_0 = A_0\Lambda_0 - I$ be the error operator associated to the undamped case, as constructed in [51]. In this case it is known that $||K_0|| < 1$ and the associated Neumann series is effective in recovering f from Λf as was explored in [45]. To show that (2.47) converges for small attenuation, we will treat K as a perturbation of K_0 and use classical PDE estimates to control $||K - K_0||$. We will work under the assumption that we have complete data, and that the measurement time T is large enough for all singularities to escape, as in Theorem 2.5.1.

Recall $Kf = w|_{t=0}$ where w is the solution of the damped wave equation with the following mixed boundary conditions:

$$\begin{cases} \Box_a w &= 0 \text{ in } (0,T)\times\Omega, \\ w|_{t=T} &= u|_{t=T} - \phi, \\ w|_{t=T} &= \partial_t u|_{t=T}, \\ w|_{[0,T]\times\partial\Omega} &= 0, \end{cases} \tag{2.48}$$

Here, u is the solution of the forward problem (2.3) and ϕ is the harmonic extension of $\Lambda f(T,\cdot)$ to the interior of Ω. Similarly, $K_0 f = w_0|_{t=0}$, where w_0 is the solution of the following wave equation:

$$\begin{cases} \Box_0 w_0 &= 0 \text{ in } (0,T)\times\Omega, \\ w_0|_{t=T} &= u_0|_{t=T} - \phi_0, \\ \partial_t w_0|_{t=T} &= \partial_t u_0|_{t=T}, \\ w_0|_{[0,T]\times\partial\Omega} &= 0, \end{cases} \tag{2.49}$$

where accordingly u_0 is the solution of the forward problem with $a = 0$ and ϕ_0 is the harmonic extension of the corresponding measurement data at $t = T$.

To treat w as a small perturbation of w_0, we require some estimates on the energy decay of the damped wave equation. Intuitively speaking, the Dirichlet boundary conditions in both mixed boundary problems preserves energy whether the equation is solved backward or forward in time. However, as a solution of the damped wave equation evolves forward in time, energy decays at most on the order of $e^{||a||_\infty T}$. Therefore, if one reverses the direction of time and solves the same equation backward in time, energy grows at most exponentially at the same rate. This intuition is verified by the subsequent lemma.

Define the energy of a solution on Ω as

$$E_\Omega(u,t) = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 + c^{-2}|\partial_t u(t,x)|^2\, dx. \tag{2.50}$$

The following is a quantitative bound on energy growth for solving the damped wave equation backward.

Lemma 2.6.1 ([25, Lemma 1])**.** *Let w solve* (2.48)*. Then, for $0 \leq t \leq T$, we have*

$$E_\Omega(w, t) \leq e^{2(T-t)||a||_\infty} E_\Omega(w, T). \tag{2.51}$$

Proof. Let $W(t, x) = (w(t, x), \partial_t w(t, x))$. Then (2.48) reduces to the first-order system

$$(\partial_t + Q_a)W = 0, \tag{2.52}$$

where

$$Q_a = \begin{bmatrix} 0 & -I \\ -c^2\Delta & a \end{bmatrix}. \tag{2.53}$$

By [48, Theorem X.48], $Q_a + ||a||_\infty$ generates a continuous semigroup of contractions on the energy space $\mathcal{H}$. Therefore one can define

$$e^{-tQ_a} = e^{-t(Q_a+||a||_\infty I)} e^{t||a||_\infty I}, \tag{2.54}$$

and verify directly that this is the solution operator for the first-order system. Therefore, for $0 \leq t \leq T$,

$$E_\Omega(w, t) \leq ||e^{-(T-t)Q_a} W(T)||^2_{\mathcal{H}} \leq e^{2T||a||_\infty} E_\Omega(w, T). \tag{2.55}$$

We will use this fact significantly in the future. □

We now estimate $||K - K_0||$ using the previous energy estimate.

Lemma 2.6.2 ([25, Proposition 2])**.** *Fix $a_0 > 0$. Then there exists $C > 0$ such that for all a with $||a||_\infty < a_0$,*

$$||K_0 f - Kf||_{H_D} \leq C a_0 (1 + a_0^2)^{1/2} e^{Ta_0} ||f||_{H_D}. \tag{2.56}$$

Proof. We return to the system of PDEs defining K_0 and K. Let w_0, w be the solutions of (2.49) and (2.48), with u_0, u the corresponding solutions of the forward problem and ϕ_0, ϕ the corresponding harmonic extensions. Define $w' = w_0 - w$, $u' = u_0 - u$, and $\phi' = \phi_0 - \phi$.

Then w' satisfies the following PDE:

$$\left\{\begin{array}{rcl} \Box_0 w' & = & a\partial_t w \text{ in } (0,T)\times\Omega, \\ w'|_{t=T} & = & u'|_{t=T} - \phi', \\ \partial_t w'|_{t=T} & = & \partial_t u'|_{t=T}, \\ w'|_{[0,T]\times\partial\Omega} & = & 0. \end{array}\right. \tag{2.57}$$

By the a priori estimate, we have

$$E_\Omega(w',0) \le C\left[||a\partial_t w||^2_{L^1(0,T;L^2(\Omega))} + E_\Omega(u',T)\right]. \tag{2.58}$$

We begin by estimating the first term.

$$||a\partial_t w||^2_{L^1(0,T;L^2(\Omega))} = \left(\int_0^T \left|\int_\Omega |a\partial_t w|^2\,dx\right|^{1/2} dt\right)^2$$

By Jensen's inequality, we have

$$\begin{aligned} &\le \int_0^T \int_\Omega |a\partial_t w|^2\,dx\,dt, \\ &\le ||a||^2_\infty \int_0^T ||\partial_t w(t,\cdot)||^2_{L^2}\,dt. \end{aligned}$$

The integrand is uniformly bounded, for by Lemma 2.6.2,

$$\begin{aligned} ||\partial_t w(t,\cdot)||^2_{L^2} &\le C||W(0)||^2_{\mathcal{H}} \\ &\le Ce^{2T||a||_\infty}||W(T)||^2_{\mathcal{H}}. \end{aligned}$$

As ϕ is harmonic, we can estimate the energy norm of $W(T)$ by that of $u(0)$.

$$||W(T)||^2_{\mathcal{H}} = ||(u|_{t=T} - \phi, \partial_t u|_{t=T})||^2_{\mathcal{H}} \le E_\Omega(u,T) \le E_\Omega(u,0). \tag{2.59}$$

Altogether, we have so far that

$$||a\partial_t w||^2_{L^1(0,T;L^2(\Omega))} \le Ce^{2T||a||_\infty} E_\Omega(u,0). \tag{2.60}$$

By Poincaré's inequality, we may replace the energy norm of the Cauchy data by $C(1+||a||^2_\infty)||f||^2_{H_D}$. This yields an estimate for the first term of (2.58) of the following type.

$$||a\partial_t w||_{L^1(0,T;L^2(\Omega))} \le Ca_0(1+a_0^2)^{1/2}e^{Ta_0}||f||_{H_D}. \tag{2.61}$$

Now we consider the second term, $E_\Omega(u', T)$. We represent u' as an integral via Duhamel's principle.

$$u'(t,x) = \int_0^t u(s,t,x)\,ds, \tag{2.62}$$

where, for $0 \le s \le t$

$$\begin{cases} \Box_0 u(s,t,x) &= 0 \text{ in } [s,t]\times\mathbb{R}^n, \\ u(s,s,x) &= af, \\ \partial_t u(s,s,x) &= a\partial_t u(s,x). \end{cases} \tag{2.63}$$

The wave equation preserves energy, so

$$E_\Omega(u(s,\cdot),t) \le E_\Omega(u(s,\cdot),s) \le ||a||_\infty^2 \left[||f||_{H_D}^2 + E_\Omega(u,s)\right]. \tag{2.64}$$

We can bound the second term by $E_\Omega(u,0)$, which yields an estimate of the form

$$E_\Omega(u',T) \le \int_0^T E_\Omega(u(s,\cdot),T)\,ds \le Ca_0^2(1+a_0^2)e^{2Ta_0}||f||_{H_D}^2. \tag{2.65}$$

Finally, recall $w'(0,\cdot) = (K-K_0)f$. So we have shown,

$$||(K-K_0)f||_{H_D} \le Ca_0(1+a_0^2)^{1/2}e^{Ta_0}||f||_{H_D}, \tag{2.66}$$

which was what we wanted. □

This yields a condition under which the Neumann series given by the pseudo-inverse A – defined via the damped wave equation – converges.

Theorem 2.6.1 ([25, Theorem 1]). *Let $T < \infty$ be large enough that all singularities escape, as in Theorem 2.5.1. Then there exists an $a_0 > 0$ such that for all $a \in C_0^\infty(\Omega)$ with $||a||_\infty < a_0$, the Neumann series*

$$f = \sum_{m=0}^{\infty} K^m A h \tag{2.67}$$

converges, with $h = \Lambda f$.

Proof. By [51, Theorem 1], under these assumptions $||K_0|| < 1$. Then

$$||K|| = ||K-K_0|| + ||K_0|| < ||K_0|| + Ca_0(1+a_0^2)^{1/2}e^{Ta_0}. \tag{2.68}$$

We may choose a_0 so that the right-hand side is strictly less than one, so that K is a strict contraction. □

3. SYNTHETIC APERTURE RADAR

3.1 Historical notes

This chapter is devoted to an inverse problem occurring in synthetic aperture radar (SAR). This is a radar imaging technique in which an airplane or satellite travels along a known path, illuminating a region of the Earth or another planet with an RF signal generated by an onboard antenna. There are many different kinds of SAR adapted to different environmental conditions and different imaging needs, but in general the goal is to recover the reflectivity coefficient of the ground from the scattering of the incident electromagnetic wave. In the specific case we consider here, we assume the scattered electromagnetic wave is received at the same antenna that is generating the incident wave.

If one assumes the Earth is flat and the airplane flies at a constant altitude, this problem reduces to the inversion of a circular Radon transform, much like the problem we studied in connection with thermoacoustic tomography. However, we cannot in general assume that the ground reflectivity function has compact support, so the question of injectivity becomes problematic. Even if the flight path is a straight line, there is a natural "left-right ambiguity" that obstructs uniqueness. In practice, this problem is solved by illuminating only one side of the flight path.

The mathematical model of SAR that we consider is due to Nolan and Cheney [42, 43]. For simplicity they study only one component of the electromagnetic field, which satisfies the usual wave equation. Then the imaging operator can be described under some assumptions as a Fourier integral operator. They consider curved flight paths in [42], and describe a reconstruction algorithm for non-flat topography, noting some kinds of artifacts that may occur in this case.

It was shown in [53] that the circular Radon transform model of SAR with a curved flight path still exhibits left-right ambiguity in a microlocal sense, using the notion of "mirror points", for which see Definition 3.3.1. In the circular Radon transform, mirror points occur in isolated pairs, though they may be related in complicated ways. In both the circular Radon transform and FIO models of SAR, there is always a difficulty involved in imaging directly under the flight path, due to a folding type singularity in the projections of the canonical relation.

Encouraged by the examples of artifacts caused by non-flat terrain in [42], I study the FIO model of SAR with the Earth considered as a smooth surface in $\mathbb{R}^3$. After an analysis of the canonical relation, I find that "mirror point sets" in this setting may occur in families with both discrete and continuous components. Near each finite set of isolated mirror point sets, one can construct an infinite dimensional family of example ground reflectivity distributions that exhibit cancellation of singularities. I also present an example, inspired by [42], of a continuous family of mirror points that also can be made to cancel.

3.1.1 Model assmuptions

We assume the airplane's flight path is modelled by a smooth, embedded curve $\gamma : [s_1, s_2] \to \mathbb{R}^3$. Over each part of the flight path, the waveform on the receiving antenna is recorded for a certain time duration $(t_1, t_2) \subset \mathbb{R}$. In the usual model one takes the surface of the Earth to be flat, but in this case we consider it to be a smooth, embedded surface Ψ given by a function $\psi(u, v)$. We assume the two are separated by some distance.

In dry air, the speed of electromagnetic propagation is reasonably approximated by a constant, which we denote c_0. When electromagnetic radiation hits the earth, the speed of propagation changes. This model of SAR approximates this by taking the speed of sound to be a singular perturbation of c_0, supported on Ψ.

$$c_0^{-2} - c^{-2} = V(u, v)\delta(\psi(u, v) - (x, y, z)). \tag{3.1}$$

Under some assumptions on the antenna geometry and a single-scattering approximation, [43] finds the forward operator is a Fourier integral operator given by

$$FV(s,t) = \int_{\mathbb{R}\times X} A(u,v,s,t,\omega) e^{-i\omega\left(t-\frac{2}{c_0}|\psi(u,v)-\gamma(s)|\right)} V(u,v)\, du\, dv\, d\omega. \tag{3.2}$$

A is an amplitude of order two; it is zero outside of a certain "visible set"

$$X = \left\{(u,v) : \exists s \in (s_1,s_2), \frac{2}{c_0}|\psi(u,v)-\gamma(s)| \in (t_1,t_2)\right\}. \tag{3.3}$$

The measured data is a function over the parameter space

$$Y = \{(s,t) : s \in (s_1,s_2), t \in (t_1,t_2)\} \tag{3.4}$$

We will write

$$R(u,v,s) = \psi(u,v) - \gamma(s) \tag{3.5}$$

for the vector from a point $\gamma(s)$ on the flight path to a given point on the ground. Each tangent plane $T_{(u,v)}\Psi$ can be identified with the affine plane passing through $\psi(u,v)$. There is a natural projection from $\mathbb{R}^3$ onto this plane. Let

$$\pi_{T\Psi} R(u,v,s) = \pi_{T_{(u,v)}\Psi} R(u,v,s) \tag{3.6}$$

be the projection of $R(u,v,s)$ to this tangent plane. We will also write

$$\widehat{R}(u,v,s) = \frac{R(u,v,s)}{|R(u,v,s)|}. \tag{3.7}$$

3.2 Canonical relation of the forward operator

Let the phase function of the forward operator be

$$\phi(s,t,u,v,\omega) = \omega\left(t - \frac{2}{c_0}|R(u,v,s)|\right). \tag{3.8}$$

Take local coordinates $(s,t,\sigma,\tau) \in T^*Y\setminus 0$ and $(u,v,\xi,\eta) \in T^*X\setminus 0$. Using Definition 1.2.6, we can calculate the canonical relation of the forward operator.

Lemma 3.2.1 ([26, Proposition 1]). *The forward operator F is associated to the canonical relation*

$$\begin{aligned}\Lambda' = \Big\{(s,t,\sigma,\tau;u,v,\xi,\eta) : t &= \frac{2}{c_0}|R(u,v,s)|,\\ \sigma &= \frac{2\tau}{c_0}\widehat{R}(u,v,s)\cdot\dot{\gamma}(s),\\ (\xi,\eta) &= \frac{2\tau}{c_0}\pi_{T\Psi}\widehat{R}(u,v,s)\Big\}.\end{aligned}$$

Proof. The characteristic submanifold associated to the phase function is

$$C_\phi = \left\{(s,t,u,v,\omega) : t = \frac{2}{c_0}|R(u,v,s)|\right\}. \tag{3.9}$$

Note that the phase function is smooth only if $|R(u,v,s)| > \epsilon > 0$, but this is assumed in the context of the model.

We may also calculate

$$\begin{aligned}d_{s,t}\phi &= \left(\frac{2\omega}{c_0}\widehat{R}(u,v,s)\cdot\dot{\gamma}(s),\omega\right),\\ d_{u,v}\phi &= \left(-\frac{2\omega}{c_0}\widehat{R}(u,v,s)\partial_u\psi(u,v), -\frac{2\omega}{c_0}\widehat{R}(u,v,s)\partial_v\psi(u,v)\right).\end{aligned}$$

This implies that $\tau = \omega$, and we recognize $d_{u,v}\phi$ as the projection of $-2\tau\widehat{R}(u,v,s)/c_0$ to the tangent plane of Ψ at (u,v).

The image of C_ϕ under the map T of Definition 1.2.6 is the Lagrangian manifold (relative to the canonical symplectic structure on $T^*(X\times Y)\setminus 0$) associated to the forward operator F. To obtain the canonical relation, we multiply the fiber variables over Y by -1. □

Ultimately, our goal is to use Lemma 1.2.9 to construct some microlocal parametrices of the forward operator. However, this is only possible away from a certain degenerate subset Σ of the canonical relation, where the canonical projections to $T^*Y\setminus 0$ and $T^*X\setminus 0$ are not of full rank. There are two contributions; $\Sigma = \Sigma_1\cup\Sigma_2$, where Σ_1 is the contribution of the left projection $\pi_Y : \Lambda' \to T^*Y\setminus 0$ and Σ_2 is the contribution of the right projection $\pi_X : \Lambda' \to T^*X\setminus 0$. From the previous lemma, we can take (s,τ,u,v) to be local coordinates on the canonical relation.

Lemma 3.2.2 ([26, Proposition 2]). *Λ' is a homogeneous canonical relation that is locally of graph type away from the degenerate set $\Sigma = \Sigma_1 \cup \Sigma_2$, where*

$$\Sigma_1 = \left\{\pi_{T\Psi}\widehat{R}(u,v,s) \parallel d_{u,v}\left(\widehat{R}(u,v,s)\cdot\dot{\gamma}(s)\right)\right\} \cap \Lambda'$$
$$\Sigma_2 = \left\{\pi_{T\Psi}\widehat{R}(u,v,s) \parallel \partial_s \pi_{T\Psi}\widehat{R}(u,v,s)\right\} \cap \Lambda'$$

Proof. From the coordinate representations of π_X and π_Y, we may calculate directly the Jacobian of each projection.

For π_Y, we have

$$\pi_Y(s,\tau,u,v) = \left(s, \frac{2}{c_0}|R(u,v,s)|, \frac{2\tau}{c_0}\widehat{R}(u,v,s)\cdot\dot{\gamma}(s), \tau\right) \tag{3.10}$$

and so a coordinate representation of its Jacobian is

$$D\pi_Y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 0 & 2c_0^{-1}\pi_1(\pi_{T\Psi}\widehat{R}(u,v,s)) & 2c_0^{-1}\pi_2(\pi_{T\Psi}\widehat{R}(u,v,s)) \\ * & * & 2\tau c_0^{-1}\partial_u(\widehat{R}(u,v,s)\cdot\dot{\gamma}(s)) & 2\tau c_0^{-1}\partial_v(\widehat{R}(u,v,s)\cdot\dot{\gamma}(s)) \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{3.11}$$

where π_1, π_2 are the projections on the two components of $\pi_{T\Psi}\widehat{R}$, respectively. Evidently this is of full rank, provided $\pi_{T\Psi}\widehat{R}$ is not parallel to $\nabla_{u,v}(\widehat{R}\cdot\dot{\gamma})$. This occurs, for example, when the velocity of the flight path is in the same direction as $\widehat{R}$. In SAR with a flat Earth and constant-altitude flight path, one avoids imaging directly under the flight path [43, Assumption 4]. This is an instance of the degeneracy of this Jacobian. Near these degenerate points, the projected canonical relation can have singularities of folding type, or worse, the Jacobian of either projection could be of rank two. The latter can only happen directly under the flight path, i.e., where $\pi_{T\Psi}\widehat{R} = 0$.

Similarly, for π_X, we have

$$\pi_X(s,\tau,u,v) = \left(u, v, \frac{2\tau}{c_0}\pi_1(\pi_{T\Psi}\widehat{R}(u,v,s)), \frac{2\tau}{c_0}\pi_2(\pi_{T\Psi}\widehat{R}(u,v,s))\right), \tag{3.12}$$

and therefore,

$$D\pi_X = \begin{bmatrix} 0 & 0 & 2\tau c_0^{-1}\pi_1\partial_s(\pi_{T\Psi}\widehat{R}(u,v,s)) & 2\tau c_0^{-1}\pi_2\partial_s(\pi_{T\Psi}\widehat{R}(u,v,s)) \\ 0 & 0 & 2c_0^{-1}\pi_1\pi_{T\Psi}\widehat{R}(u,v,s) & 2c_0^{-1}\pi_2\pi_{T\Psi}\widehat{R}(u,v,s) \\ 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}. \tag{3.13}$$

Note that while Σ_1 and Σ_2 are defined by different expressions, the two are equal as sets by [28, Theorem 4.1.9]. However, the rank of $D\pi_X$ and $D\pi_Y$ may differ. $\square$

This lemma reveals that the correct generalization of "under the flight path" to non-flat terrain is the subset of Ψ with minimal travel time to some point on the flight path. We also provide some intuition for when a covector not under the flight path lies on a singularity of the projected canonical relation. Let $(u, v, \xi, \eta) \in T^*Y \setminus 0$. After embedding the tangent plane into $\mathbb{R}^3$, this covector can be identified with a certain vector $v_1 \in T\mathbb{R}^3$, with basepoint $\psi(u, v)$. Similarly, the velocity of the flight path is also a vector $v_2 \in T\mathbb{R}^3$ with basepoint $\gamma(s)$. Neither v_1 nor v_2 has zero length. There is exactly one affine plane of $\mathbb{R}^3$ containing $\gamma(s)$ and $\psi(u, v)$ such that v_2 is identified with a tangent vector on this plane. If v_1 can also be identified with a tangent vector on this plane, then the projected canonical relation is of folding type here.

In the sequel, we will refer to Σ as the degenerate subset of Λ', and all of our work (e.g., microlocal constructions) will take place away from it, near "non-degenerate covectors."

3.3 Cancellation of singularities

In SAR with flat topography, every nondegenerate covector in $p \in T^*Y \setminus 0$ is the image of two distinct covectors in $T^*X \setminus 0$; see for example [43, 53]. We refer to covectors sharing the same covector in their image under the canonical relation as "mirror points" (though perhaps "mirror covectors" would have been a better term). We will exploit these mirror points to construct distributions with wavefront set near two mirror points, whose image under the forward operator is microlocally smooth near their common image. We formalize this intuition with the definition below.

Definition 3.3.1. *Fix $p \in T^*Y \setminus 0$. Then the mirror point set of p is*

$$M_p = \{q \in T^*X \setminus 0 : (q,p) \in \Lambda'\} \subset T^*X \setminus 0. \tag{3.14}$$

In other words, M_p is the inverse image of p under Λ', similar to the definition of the image under a relation in (1.51)*.*

A covector $q \in M_p$ is degenerate (with respect to p) if $(q,p) \in \Sigma$. Non-degenerate mirror points are isolated [26, Proposition 3], which is clear from the fact that in a neighborhood of each non-degenerate covector, Λ' acts as a bijective diffeomorphism.

There is a related notion of mirror points that are related to multiple covectors, via "multiple scattering." These are discussed in connection with a Radon transform model of SAR in [53], and were recently analyzed in connection with cancellation of singularities in the model we consider here by Caday [8]. While it is clear that some of our results extend naturally to these kinds of mirror points, for simplicity we do not consider them here.

Our main result is the following construction, taking place near a pair of isolated mirror points, of an infinite dimensional subspace of non-smooth distributions whose image under the forward operator is smooth near the common image of the mirror points. This phenomenon, which we refer to as "cancellation of singularities", shows that the inverse problem of recovering V from FV is not stable in any Sobolev space.

For the purposes of this construction, fix $p \in T^*Y \setminus 0$ and two distinct $q_1, q_2 \in M_p \subset T^*X \setminus 0$ that are nondegenerate in the sense of Definition 3.3.1. It is possible to find small conic neighborhoods Γ of p and Γ_1, Γ_2 of q_1, q_2 respectively so that $\Lambda'(\Gamma_1)$ and $\Lambda'(\Gamma_2)$ both properly contain $\overline{\Gamma}$.

Theorem 3.3.1 ([26, Theorem 1])**.** *Let $\Gamma \subset T^*Y \setminus 0$ and $\Gamma_1, \Gamma_2 \subset T^*X \setminus 0$ be small conic neighborhoods associated to distinct, isolated mirror points $q_1, q_2 \in M_p$ as described above. Assume the amplitude A of the forward operator F is nonzero in a* *neighborhood of* $(\pi_X(q_1), \pi_Y(p)) \times (\mathbb{R} \setminus 0)$ *and in a neighborhood of* $(\pi_X(q_2), \pi_Y(p)) \times (\mathbb{R} \setminus 0)$.

Then, for every $V_1 \in \mathcal{E}'(X)$ *such that* $\mathrm{WF}(V_1) \subset \Gamma_1$, *there exists* $V_2 \in \mathcal{E}'(X)$ *with* $\mathrm{WF}(V_2) \subset \Gamma_2$, *related by a Fourier integral operator whose canonical relation is the graph of a bijective diffeomorphism between* Γ_1 *and* Γ_2, *such that*

$$\mathrm{WF}(F(V_1 + V_2)) \cap \Gamma = \emptyset. \tag{3.15}$$

Proof. We apply the construction of Lemma 1.2.9 to both $\overline{\Gamma}_1 \times \overline{\Gamma}$ and $\overline{\Gamma}_2 \times \overline{\Gamma}$. This yields two microlocalizations of the forward operator F, which we refer to as F_1 and F_2. Recall that both F_1 and F_2 are Fourier integral operators of graph type whose canonical relation is the restriction of Λ' to their respective defining neighborhoods.

The condition on the amplitude ensures that, in addition, F_1 and F_2 are elliptic Fourier integral operators of graph type, in accordance with Definition 1.2.8. Let F_1^{-1} and F_2^{-1} be microlocal parametrices for F_1, F_2 respectively. We claim that

$$V_2 = -F_2^{-1} F_1 V_1 \tag{3.16}$$

satisfies the conditions of the theorem. Recall that from Lemma 1.2.8 it follows that $\mathrm{WF}(V_2) \subset \Gamma_2$. Then

$$F(V_1 + V_2) = FV_1 - FF_2^{-1} F_1 V_1. \tag{3.17}$$

Since $\mathrm{WF}(V_1) \subset \Gamma_1$, we may replace in the first term F with F_1, as $(F - F_1)V_1$ is microlocally regular near Γ. Similarly, we may replace in the second term F with F_2. Therefore

$$F(V_1 + V_2) = F_1 V_1 - F_2 F_2^{-1} F_1 V_1. \tag{3.18}$$

By construction $F_2 F_2^{-1} = I + R$, where R is smoothing. Hence

$$\mathrm{WF}(F(V_1 + V_2)) \cap \Gamma = \emptyset. \tag{3.19}$$

This is the sense in which singularities from V_1 cancel singularities in V_2, under F. □

This proof may also be extended to construct microlocally smooth images of distributions whose wavefront set is contained in a small conic neighborhood of any subset of isolated, nondegenerate mirror points in M_p. In particular, given $V_i \in \mathcal{E}'(X), i = 1, \ldots, n-1$, each microlocally supported near some nondegenerate $q_i \in M_p$, there is a distribution

$$V_n = -F_n^{-1} \sum_{i=1}^{n-1} F_i V_i \tag{3.20}$$

microlocally supported near $q_n \in M_p$ that is again microlocally regular near p.

3.3.1 Cancellation of singularities on degenerate mirror points

We conclude this section with the example considered in [26] that shows continuous families of mirror points may also cancel each other out. We consider a cylindrical valley with flight path along its axis of revolution, as in the Figure 3.1.

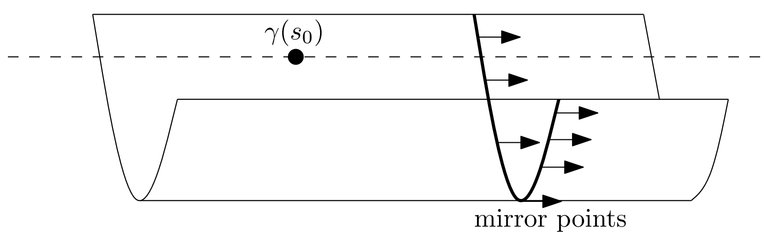

Figure 3.1. A cylindrical valley. From [26, Figure 2], used with permission.

Every covector in this setting is degenerate. The mirror points over any fixed covector have two components, each a continuous curve lying over a cross-section of the half-cylinder.

Let Ψ be the cylinder given by $\psi(u, v) = (\cos u, v, 1-\sin u)$, where $(u, v) \in (0, \pi) \times \mathbb{R}$, and $\gamma(s) = (0, s, 1)$. Let $V = f(u)H(v)$ where $H(v) = \chi_{[0,\infty)}$ is the Heaviside function and $f(u) \in C^\infty((0,\pi))$. Assume $A = 1$ uniformly. When $A = 1$, the forward operator reduces to the Fourier transform of a delta function. We calculate explicitly,

$$\begin{aligned} FV(s,t) &= \int_{\mathbb{R}\times\Psi} e^{-i\omega\left(t-\frac{2}{c_0}\sqrt{(v-s)^2+1}\right)} f(u)H(v)\,du\,dv\,d\omega \\ &= \left\langle \delta\left(t - \frac{2}{c_0}\sqrt{(v-s)^2+1}\right), f(u)H(v)\right\rangle \\ &= \left\langle \delta\left(t - \frac{2}{c_0}\sqrt{(v-s)^2+1}\right), H(v)\right\rangle \int_0^\pi f(u)\,du. \end{aligned}$$

The map

$$w(v) = t - \frac{2}{c_0}\sqrt{(v-s)^2+1}$$

is two-to-one onto the interval $(-\infty, t-2/c_0)$. To calculate the pull-back, we divide the domain of w into two intervals, $(-\infty, s)$ and (s,∞). Then there are two inverses v_- and v_+ with range on each interval, respectively:

$$v_\pm(w) = s \pm \sqrt{\frac{c_0^2}{4}(w-t)^2 - 1}.$$

In either case, the derivative is non-zero on $(-\infty, t-2/c_0)$, and explicitly,

$$\left|\frac{dv_\pm}{dw}(w)\right| = \frac{c_0^2}{4}\frac{t-w}{\sqrt{c_0^2(t-w)^2/4-1}}.$$

Using this, we may calculate the pullback of $\delta(w)$ via $w(v)$. Let

$$\alpha(t) = \sqrt{c_0^2 t^2/4 - 1}.$$

Then:

$$\begin{aligned} \langle \delta(w(v)), H(v)\rangle &= \left\langle \delta(w), \left|\frac{dv_\pm}{dw}(w)\right| [H(v_-(w)) + H(v_+(w))]\right\rangle \\ &= \frac{c_0^2}{4}\frac{t}{\sqrt{c_0^2t^2/4-1}}[H(s+\alpha(t)) + H(s-\alpha(t))] \end{aligned}$$

So, the forward operator reduces to

$$FV(s,t) = \frac{c_0^2}{4}\frac{t[H(s+\alpha(t)) + H(s-\alpha(t))]}{\sqrt{c_0^2t^2/4-1}}\int_0^\pi f(u)\,du,$$

which vanishes whenever the integral of f vanishes. There is a subspace of $C^\infty((0,\pi))$ for which this is true with infinite dimension. Since

$$\mathrm{WF}(f(u)H(v)) = (\{v = 0\} \times \{\xi = 0\}) \setminus 0$$

and $\mathrm{WF}(FV(f(u)H(v))) = \emptyset$, this shows that singularities on degenerate mirror points may also cancel.

4. GENERALIZED RADON TRANSFORM

4.1 Historical notes

In this chapter, we study the injectivity and stability of a certain class of generalized Radon transforms on analytic Riemannian manifolds. Given a smooth family of hypersurfaces Σ on a Riemannian manifold (M, g), there is an operator mapping functions $f \in C_0^\infty(M)$ to a function on $C^\infty(\Sigma)$ whose values are the integrals of f over each hypersurface, with respect to the induced volume form. The simplest example is the Euclidean Radon transform on $\mathbb{R}^n$, in which case Σ is the space of affine hyperplanes. In the previous two chapters, we also saw circular Radon transforms appear as basic models of thermoacoustic tomography and synthetic aperture radar. We refer the reader to [23, 16, 24] for an overview of classical results on the Euclidean Radon transform and its generalization to Lie groups and homogeneous spaces.

When Σ is a smooth manifold, we define the incidence relation $\mathcal{R} \subset M \times \Sigma$ to be the set of ordered pairs (x, σ) such that x is a point on the hypersurface σ. The incidence relation is a double fibration [18] if it is a smooth, embedded submanifold of $M \times \Sigma$, and the two canonical projections on $\mathcal{R}$ are smooth maps giving $\mathcal{R}$ the structure of a fiber bundle over M, Σ respectively. When a generalized Radon transform R has an incidence relation that is also a double fibration, it is known that both R and its adjoint R^* are Fourier integral operators, and the canonical relation of R is the conormal bundle $N^*\mathcal{R}$ [21, 19, 20]. If, in addition, $\mathcal{R}$ satisfies the Bolker condition, then R^*R is an elliptic pseudodifferential operator, and is therefore invertible up to smoothing error. However, this is not enough to show injectivity.

The first half of this chapter considers the injectivity of a class of analytic generalized Radon transforms, satisfying the Bolker condition. Such transforms were studied by Boman and Quinto [4, 5, 46, 47], who showed injectivity and Helgason-type support theorems for several sub-classes of analytic generalized Radon transforms. Both their approach and the one used here relies on analytic microlocal analysis to study the analytic microlocal regularity of the generalized Radon transform. In particular, we show that if Rf is analytic on a neighborhood of hypersurfaces, then f is microlocally analytic near the conormal bundle of that neighborhood. We follow Sjöstrand's development of analytic microlocal analysis, using the techniques set forth in Section 1.2.4.

From this regularity result, injectivity for those $f \in \mathcal{E}'(M)$ whose generalized Radon transform is analytic follows immediately, and Lemma 1.2.6 yields a stability estimate for this analytic class of generalized Radon transforms. We then perturb this estimate using the symbol calculus to obtain a similar estimate for a neighborhood of smooth generalized Radon transforms (see Definition 4.2.3) near the analytic ones. Throughout we assume all transforms satisfy the Bolker condition. This yields injectivity and stability for a generic class of generalized Radon transforms defined on analytic manifolds.

This chapter is based on joint work with Hanming Zhou [27].

4.2 Bolker condition

Let (M, g) be a compact Riemannian manifold with boundary. The generalized Radon transforms that we have in mind are those given by a space of oriented hypersurfaces Σ parametrized as the level sets of a defining function, after the definition of Beylkin [3]. We consider M to be isometrically embedded in a slightly larger manifold M_1, whose metric we also refer to as g. We may identify $L^2(M, d\mathrm{Vol})$ with the subspace of $L^2(M_1, d\mathrm{Vol})$ consisting of those functions supported on M by extending the former to M_1 by zero. We begin by describing the class of defining functions that we will consider.

Definition 4.2.1. *Let $\varphi \in C^\infty(M_1 \times (\mathbb{R}^n \setminus 0))$. We say φ is a defining function when the following criteria are satisfied.*

1. *$\varphi(y,\theta)$ is positive homogeneous of degree one in the fiber variable.*

2. *φ is nondegenerate in the sense that $d_{y,\theta}\varphi(y,\theta) \neq 0$.*

3. *The mixed Hessian of φ is strictly positive, i.e.,*

$$\det\left(\frac{\partial^2\varphi}{\partial y^i \partial \theta^j}\right) > 0. \tag{4.1}$$

Given a fixed defining function, the level sets of φ will be denoted by

$$H_{s,\theta} = \{y \in M_1 : \varphi(y,\theta) = s\}. \tag{4.2}$$

Note that by homogeneity, $H_{s,\theta} = H_{\lambda s,\lambda\theta}$ for all $\lambda > 0$. Therefore we can consider $(s,\theta) \in \mathbb{R}\times S^{n-1}$ as coordinates on Σ. We will also implicitly consider φ as a function on $M \times S^{n-1}$.

The third condition in the definition of a defining function is a local form of the Bolker condition, which ensures that, locally, the incidence relation is a double filtration [18]. It also allows us to locally identify

$$M_1 \times S^{n-1} \ni (y,\theta) \Longleftrightarrow \frac{d_\theta\varphi(y,\theta)}{|d_\theta\varphi(y,\theta)|_g} \in S^*_y M_1. \tag{4.3}$$

However, for our analysis below, we also require a stronger, global Bolker condition.

Definition 4.2.2. *A defining function φ satisfies the global Bolker condition if:*

1. *For each $\theta \in S^{n-1}$, the map $y \mapsto d_\theta\varphi(y,\theta)$ is injective.*

2. *For each $y \in M$, the map $\theta \mapsto d_y\varphi(y,\theta)$ is surjective.*

The first condition is essentially a “no conjugate points” assumption, similar to that used for similar results involving the geodesic ray transform [17, 37]. Without this assumption, at least in dimension two, there are examples of cancellation of singularities [41], which, as we saw in the previous chapter, is an obstruction to stability. The second condition ensures that all singularities are conormal to at least one hypersurface.

We may now define the class of generalized Radon transforms that we will consider in this chapter.

Definition 4.2.3. *Given a Riemannian manifold with boundary (M,g) (realized as a compact submanifold of M_1 as above), a defining function φ satisfying the global Bolker condition (Definition 4.2.2), and a nonvanishing weight $w \in C^\infty(M_1 \times S^{n-1})$, then $R_w : C^\infty(M) \to C^\infty(\Sigma)$ is defined as the integral*

$$R_w f(s,\theta) = \int_{H_{s,\theta}} w(y,\theta) f(y)\, d\mu_{s,\theta}(y) \tag{4.4}$$

where $d\mu_{s,\theta}$ is the volume form on $H_{s,\theta}$ induced by $d\mathrm{Vol}$.

We say R_w is an analytic generalized Radon transform if (M,g) is an analytic manifold, and both the defining function and weight are analytic.

By the assumptions made on the defining function, there exists a smooth, nonvanishing function $J(y,\theta)$ such that

$$J(y,\theta)\, d\mathrm{Vol}(y) = d\mu_{s,\theta}(y) \wedge ds. \tag{4.5}$$

The adjoint of R_w on $L^2(\Sigma, ds \wedge d\theta)$ (sometimes called a "generalized backprojection") is given by

$$R_w^* g(x) = \int_{S^{n-1}} \overline{w}(x,\theta) \overline{J}(x,\theta) g(\varphi(x,\theta),\theta)\, d\theta. \tag{4.6}$$

This is just the usual generalized backprojection with weight $\overline{wJ}$.

4.3 Analytic microlocal regularity

In this section, we consider the class of analytic generalized Radon transforms satisfying the Bolker condition. Recall that (M_1, g) is in this case an analytic manifold. Our first object will be a study of the analytic microlocal regularity of an analytic generalized Radon transform R_w, given by the defining function φ. The analytic microlocal regularity of a distribution $f \in \mathcal{D}'(M)$ is characterized by its analytic wavefront set, $\mathrm{WF}_A(f) \subset T^*M \setminus 0$ as stated in Definition 1.2.10.

Fix a covector $(y_0, \theta_0) \in T^*M_1 \setminus 0$, with $s_0 = \varphi(y_0, \theta_0)$. From now on we will work in a small conic neighborhood of this covector.

Theorem 4.3.1. *Let $R_w f(s,\theta) = 0$ in a neighborhood of (s_0, θ_0). Then f is analytic microlocally near $(y_0, d_y\varphi(y_0, \theta_0))$.*

Proof. First, let us fix coordinate systems. We have (y, θ) as local coordinates on $T^*M_1 \setminus 0$. Without loss of generality we can take $s_0 = 0$ and $|\theta_0|_g = 1$. To simplify the corresponding coordinates on Σ, we perform a stereographic projection of $\theta \in S^*_y M_1$ to the tangent plane of the sphere at θ_0, whose image we refer to as η. This is a bijective analytic diffeomorphism from a neighborhood of $\theta_0 \in S^{n-1}$ to a neighborhood of the origin in $\mathbb{R}^{n-1}$.

Recall (s, θ) form a system of local coordinates for Σ, and so (s, η) also serves as a system of coordinates for Σ. In this case, it is clear that Σ is also an analytic manifold. We will work in a neighborhood of Σ such that $|\xi| < \delta$, $|s| < 2\epsilon$, where $\epsilon, \delta > 0$ are small parameters, small enough that $R_w f(s, \eta) = 0$.

Let $\chi_N(s)$ be the sequence of quasianalytic cut-off functions in Lemma 1.2.10. We assume these are supported in $(-2\epsilon, 2\epsilon)$, and are equal to one on $(-\epsilon, \epsilon)$. Take $\lambda \gg 1$ to be a large parameter, to be fixed later. We integrate $R_w f(s, \eta)$ against $e^{i\lambda s}\chi_N(s)$ to obtain

$$0 = \int e^{i\lambda s}\chi_N(s) \int_{H_{s,\eta}} w(y, \eta) f(y)\, d\mu_{s,\eta}\, ds. \tag{4.7}$$

By (4.5), this reduces to the oscillating integral

$$\int e^{i\lambda\varphi(y,\eta)} a_N(y, \eta) f(y)\, d\mathrm{Vol}(y) = 0. \tag{4.8}$$

Here $a_N(y, \eta)$ is a sequence of local analytic symbols (Definition 1.2.9) defined on the same neighborhood of $(y_0, 0) \in M_1 \times \mathbb{R}^{n-1}$.

The coordinates (y, η) are real-analytic, and so we may extend their domain of definition slightly, by analytic continuation, to a small Grauert tube of a neighborhood of $H_{0,0} \times \{0\} \subset M_1 \times \mathbb{R}^{n-1}$. This continuation in principle depends on the choice of analytic coordinates, but as we are concerned with the analytic wavefront set, which is invariantly defined, the final result will not depend on this choice of coordinates. We choose a perhaps smaller parameter δ such that $\{|\eta| < \delta/2\}$ is contained in this complex neighborhood. Denote the complex coordinate patch of y_0 as $U \subset \mathbb{C}^n$.

At this point we cannot yet apply Definition 1.2.10, as the phase function is not of the form required by that definition. Our next goal will be to augment the phase function with additional variables, in such a way that an application of the complex stationary phase lemma will leave a phase function amenable to the definition of analytic wavefront set.

Take $(x, \xi) \in U \times \mathbb{C}^{n-1}$, with $|\xi| < \delta/2$, and take $\rho(\eta) = 1$ for $|\eta| \leq \delta$ and zero otherwise. We then integrate (4.8) against the function

$$\rho(\eta - \xi) \exp\left(-\frac{\lambda}{2}|\eta - \xi|^2 - i\lambda\varphi(x, \eta)\right), \tag{4.9}$$

with respect to η. This yields a new oscillating integral

$$\int e^{i\lambda\Phi(x,y,\xi,\eta)} b_N(y, \eta, \xi) f(y)\, d\mathrm{Vol}(y)\, d\eta = 0. \tag{4.10}$$

The augmented phase function is

$$\Phi(x, y, \xi, \eta) = \frac{i}{2}|\eta - \xi|^2 + \varphi(y, \eta) - \varphi(x, \eta), \tag{4.11}$$

and the augmented symbol is

$$b_N(y, \eta, \xi) = a_N(y, \eta)\rho(\eta - \xi) = \rho(\eta - \xi)\chi_N(\varphi(y, \eta))w(y, \eta). \tag{4.12}$$

This is a sequence of analytic local symbols in the sense of Definition 1.2.9, all defined on the same neighborhood of $H_{0,0} \times \{0\} \times \{0\}$.

To apply complex stationary phase, we need to have some control over the critical points of $\eta \mapsto \Phi(x, y, \xi, \eta)$. We have

$$\Phi_\eta(x, y, \xi, \eta) = i(\eta - \xi) + \partial_\eta\varphi(y, \eta) - \partial_\eta\varphi(x, \eta). \tag{4.13}$$

If $|\eta| < \delta/2$ is real, then the only critical points are those with $\eta = \xi$ and $y = x$. These critical points are non-degenerate, and therefore extend to a family of complex critical points $\eta_c(x, y, \xi) = \xi + i(x - y) + O(\delta)$.

Consider the case $x \neq y$ and $|\eta| < \delta/2$. The only real critical points in this regime are where $\eta = \xi$ and $\partial_\eta\varphi(y, \eta) = \partial_\eta\varphi(x, \eta)$. However, by the global Bolker condition, the latter condition is never satisfied, as $y \mapsto \partial_\eta \varphi(y,\eta)$ is injective for all η. By non-degeneracy again this extends to y in a small complex neighborhood of $H_{0,0} \subset M_1$, with $|\eta - \xi| < \delta$.

For now, treat $(x,\xi) \in U \times C^{n-1}$ as fixed. Then we can contain the critical points of Φ with respect to η in a neighborhood

$$I_+ = \{(y,\eta) : |x-y| \leq \delta/C_0, |\xi - \eta| < \delta\}, \tag{4.14}$$

with $C_0 > 1$ large enough that $(y,\eta) \in I_+$ implies $|\varphi(y,\eta)| < \epsilon$. We have then excluded these critical points from the neighborhood

$$I_- = \{(y,\eta) : |x-y| > \delta/C_0, |\xi - \eta| < \delta\}. \tag{4.15}$$

Here, $C_0 > 1$ is a positive constant. In I_-, we have $|\partial_\eta \Phi| > 0$. Let

$$L = \frac{\partial_\eta \overline{\Phi} \cdot \partial_\eta}{i\lambda |\partial_\eta \Phi|^2} \tag{4.16}$$

be a first-order differential operator, which is well-defined on I_-.

Using this operator, we may repeatedly integrate by parts with respect to η, obtaining the estimate,

$$\begin{aligned}\left|\int_{I_-} e^{i\lambda\Phi} b_N f \, d\mathrm{Vol}(y)\, d\eta\right| &= \left|\int_{I_-} (L^N e^{i\lambda\Phi}) b_N f \, d\mathrm{Vol}(y)\, d\eta\right| \\ &\leq \left|\int_{I_-} e^{i\lambda\Phi} (L^*)^N [b_N f] \, d\mathrm{Vol}(y)\, d\eta\right| + \sum_{k=1}^{N} |\mathcal{B}_k|.\end{aligned}$$

The terms $\mathcal{B}_k$ are exponentially small in λ, as $\Im\Phi > 0$ for $|\xi - \eta| = O(\delta)$. From (4.12), we see that the worst growth of $(L^*)^N[b_N f]$ occurs when all derivatives are applied to $\chi_N(\varphi(y,\eta))$. In this case, Lemma 1.2.10 yields an estimate of the form

$$\left|\partial_s^{(N)} \chi_N(s)\right| \leq (C_1 N)^N, \tag{4.17}$$

with C uniform in N. (In the sequel, C will stand for various positive constants, all uniform with respect to N.)

As for the integral over I_+, notice that b_N is independent of N here, so we remove the dependence on N. Apply the complex stationary phase lemma of Sjöstrand, in the form Lemma 1.2.11. This yields an estimate of the form

$$\int_{I_+} e^{i\lambda\Phi} bf \, d\mathrm{Vol}(y)\, d\eta = C\lambda^{-n/2} \int_{I_+} e^{i\lambda\psi} Bf \, d\mathrm{Vol}(y) + R(x,\xi), \tag{4.18}$$

where $R(x,\xi)$ is a remainder term of the order

$$R(x,\xi) = O\left((CN/\lambda)^N + Ne^{-\lambda/C}\right). \tag{4.19}$$

On the right-hand side of (4.18), the new phase function is

$$\psi(x,y,\xi) = \frac{i}{2}|\eta_c(x,y,\xi) - \xi|^2 + \varphi(y, \eta_c(x,y,\xi)) - \varphi(x, \eta_c(x,y,\xi)), \tag{4.20}$$

and the new amplitude is

$$B(x,y,\xi) = b(x,y,\eta_c(x,y,\xi),\xi). \tag{4.21}$$

We may now fix $N \le (\lambda/Ce) \le N+1$. Then $CN\lambda^{-1} \le e^{-1}$ and by monotonicity we have

$$\left(\frac{CN}{\lambda}\right)^N \le e^{-N} \le e^{-N-1} \le e^{-\lambda/Ce}. \tag{4.22}$$

Using this in (4.18), we conclude that

$$\int e^{i\lambda\psi(x,y,\xi)} B(x,y,\xi) f(y)\, d\mathrm{Vol}(y) = O(e^{-\lambda/C}). \tag{4.23}$$

This estimate is uniform for (x,ξ) near a small conic neighborhood of $(y_0, \eta(\theta_0))$. However, the new phase function ψ still does not quite satisfy the assumptions of Definition 1.2.10. We will show that it can be made to do so after a final change of coordinates.

Note that, for x real, $\eta_c(x,x,\xi) = \xi$ and therefore $\psi(x,x,\xi) = 0$. In addition,

$$\partial_y \psi(x,x,\xi) = \partial_y \varphi(x,\xi). \tag{4.24}$$

By the global Bolker condition, we may make a change of variables $\xi'(x,\xi)$ so that $\xi' = \partial_y \varphi(x,\xi)$. Finally, it is clear that $\Im\psi(x,y,\xi') \ge C|x-y|^2$ for x,y real. Now Definition 1.2.10 applies and $(y, d_y\varphi(y_0,\theta_0)) \notin \mathrm{WF}_A(f)$. $\square$

From this theorem, we can show that R_w is injective. Let $f \in \mathcal{D}'(M)$ be extended by zero to a distribution on M_1, and assume $R_w(f) = 0$. Then the above theorem shows that $\mathrm{WF}_A(f) = \emptyset$, i.e., f is analytic. Its support is compact, so therefore $f = 0$. Hence R_w is injective on $\mathcal{D}'(M)$.

It is sufficient, in the proof of the theorem, to assume merely that $\mathrm{WF}_A(R_w f) \cap \Gamma = \emptyset$, where Γ is a small conic neighborhood of $T^*\Sigma \setminus 0$. After microlocalizing near the hypersurface H_{s_0,θ_0}, the left-hand side of (4.7) will be exponentially decaying, perhaps after shrinking ϵ and δ further.

This result also yields a support theorem of Helgason type, for analytic generalized Radon transforms. Take f with analytic singular support in a subset of M that is convex with respect to Σ. If the generalized Radon transform of f is analytic in a neighborhood of a fixed hypersurface conormal to the convex set, then it is possible to continue f analytically from the exterior across the hypersurface.

4.4 Stability

We now return to those generalized Radon transforms that are given by a smooth defining function and smooth weight, though we keep the underlying manifold to be analytic. The object of interest in this section is the normal operator $N = R_w^* R_w$, where R_w^* is the $L^2(\Sigma)$-adjoint given in (4.6).

It is known that N is an elliptic pseudodifferential operator [21, Proposition 8.2] mapping $L^2(M)$ to $H^{n-1}(M_1)$ continuously. However, elliptic regularity only yields the following estimate for every $s > 0$:

$$||f||_{L^2(M)} \leq C||Nf||_{H^{n-1}(M_1)} + C_s||f||_{H^{-s}}. \tag{4.25}$$

Here $C > 0$ is independent of s, while $C_s > 0$ does depend on s. To promote this inequality to a stability estimate, we again turn to [56, Propostion V.3.1], which permits this (with different constant) under the assumption that N is known to be injective.

Theorem 4.3.1 proves that N is injective when it is the normal operator of an analytic generalized Radon transform. In this section, our goal is to perturb the resulting stability estimate for analytic generalized Radon transforms to a corresponding stability estimate for a class of smooth generalized Radon transforms (on analytic manifolds). As a consequence, this larger class is injective. Throughout we continue to assume that these Radon transforms are given by a defining function, and satisfy the global Bolker condition.

First, we obtain a representation for the Schwartz kernel of N.

Lemma 4.4.1 ([27, Lemma 1])**.** *The Schwartz kernel $K_N \in \mathcal{D}'(\mathbb{R} \times S^{n-1} \times M_1)$ of N is*

$$K_N(x,y) = (2\pi)^{-1} \int_{S^{n-1}} \int_{\mathbb{R}} e^{is'(\varphi(x,\theta)-\varphi(y,\theta))} \overline{w}(x,\theta)\overline{J}(x,\theta) w(y,\theta) J(y,\theta)\, ds'\, d\theta. \quad (4.26)$$

Recall $J(x,\theta)$ is the smooth, nonvanishing function of (4.5).

Proof. Let $\mathcal{F}_s$ be a partial Fourier transform, taking s to the dual variable s'. If we apply this to $R_w f$, we obtain,

$$\begin{aligned} \mathcal{F}_s R_w f(s',\theta) &= \int_{\mathbb{R}} e^{-iss'} \int_{H_{s,\theta}} w(y,\theta) f(y)\, d\mu_{s,\theta} ds \\ &= \int_{M_1} e^{-is'\varphi(y,\theta)} w(y,\theta) J(y,\theta) f(y)\, d\mathrm{Vol}(y). \end{aligned}$$

Taking the inverse Fourier transform of this with respect to s', we see that the Schwartz kernel of R_w is given by,

$$K_{R_w} = (2\pi)^{-1} \delta(s - \varphi(y,\theta)) w(y,\theta) J(y,\theta). \quad (4.27)$$

The kernel of the adjoint is found in a similar manner, and then the two may be composed, resulting in the oscillating integral of (4.26). $\square$

The form of this kernel is roughly the kernel of the normal operator of the geodesic ray transform, see [12, 17]. We will now use their techniques to calculate the principal symbol of the normal operator.

Lemma 4.4.2 ([27, Lemma 2]). *The principal symbol of N is*

$$\sigma_{1-n}(N)(x,\xi) = (2\pi)^{1-n}|\xi|^{1-n}\left[W(x,x,\xi/|\xi|) + W(x,x,-\xi/|\xi|)\right]. \tag{4.28}$$

where W is the auxillary function

$$W(x,y,\theta) = \overline{w}(x,\theta)\overline{J}(x,\theta)w(y,\theta)J(y,\theta). \tag{4.29}$$

Proof. We divide the representation of K_N given in (4.26) into two terms, K_N^+ corresponding to integration over $\{s' > 0\}$ and K_N^- corresponding to integration over $\{s' < 0\}$. These two kernels yield two operators $N^\pm$, such that $N = N^+ + N^-$. We have,

$$K_N^\pm = (2\pi)^{-1}\int_{S^{n-1}}\int_0^\infty e^{\pm i(\varphi(x,s'\theta)-\varphi(y,s'\theta))}W(x,y,\theta)\,ds'\,d\theta. \tag{4.30}$$

In what follows we will take $\xi = s'\theta$ to be polar coordinates for a new phase variable ξ taking values in $\mathbb{R}^n$. This change of variables is justified when the kernel is applied to a test function; by the proof of [30, Theorem 7.8.2] it is justified for the kernel itself. From this we obtain

$$K_N^\pm = \int_{\mathbb{R}^n} e^{\pm i(\varphi(x,\xi)-\varphi(y,\xi))}W\left(x,y,\pm\frac{\xi}{|\xi|}\right)|\xi|^{1-n}\,d\xi. \tag{4.31}$$

As K_N is the kernel of a pseudodifferential operator, we know that $K_N^+ + K_N^-$ is smooth away from the diagonal of $M_1 \times M_1$.

Fix $x_0 \in M_1$ and take $\chi \in C_0^\infty(M_1)$ to be a smooth cutoff function equal to one in a neighborhood U of x_0. To determine the principal symbol of N, we restrict $K_N^\pm$ to $U \times U$ and rewrite each $\chi N^\pm \chi$ as a pseudodifferential operator. Let (x^i) be a system of local coordinates on U; then take (x^i, y^i) to be a system of local coordinates on $U \times U$, with $x^i = y^i$. In these coordinates, we can expand the phase function near the diagonal as

$$\varphi(x,\xi) - \varphi(y,\xi) = (x-y)\cdot\int_0^1 \partial_x\varphi(x+t(y-x),\xi)\,dt. \tag{4.32}$$

We will make a change of coordinates in the phase variable, given by the map

$$\xi'(x,y,\xi) = \int_0^1 \partial_x(x+t(y-x),\xi)\,dt. \tag{4.33}$$

This map is positive homogeneous of degree one in ξ; therefore, there is a strictly positive function $c(x,y)$ such that $|\xi'(x,y,\xi)| = c(x,y)|\xi|$. Near the diagonal, ξ' yields a smooth change of coordinates whose Jacobian is

$$\det\left(\frac{\partial\xi'}{\partial\xi}\right)(x,x,\xi) = \det\left(\frac{\partial^2\varphi}{\partial\xi^i\partial x^j}(x,\xi)\right). \tag{4.34}$$

This is the mixed Hessian of the defining function, which we assumed was strictly positive. Under this change of variables, we have

$$\chi K_N^{\pm}\chi = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi'} W\left(x,y,\frac{\xi'}{\xi'}\right)\chi(x)\chi(y)|\xi'|^{1-n}c(x,y)^{n-1}\left|\det\frac{\partial\xi'}{\partial\xi}\right| d\xi'. \tag{4.35}$$

This is a pseudodifferential operator of order $1-n$. The principal symbol of N is the sum of the restrictions of the two amplitude to the diagonal; for convenience, we state this symbol in the original phase coordinates ξ. $\square$

We can now give a stability estimate for analytic generalized Radon transforms.

Proposition 4.4.1. *Let R_w be an analytic generalized Radon transform (as in Definition 4.2.3) on the analytic Riemannian manifold (M,g). Then for all $f\in L^2(M)$ there exists $C>0$ such that*

$$||f||_{L^2(M)} \le C||Nf||_{H^{n-1}(M_1)}. \tag{4.36}$$

Proof. Since N is an elliptic pseudodifferential operator of order $1-n$, we have by Lemma 1.2.6 and Theorem 4.3.1 a stability estimate for analytic generalized Radon transforms. $\square$

Finally, we may perturb this stability estimate slightly to smooth generalized Radon transforms (still defined on (M_1,g)). This yields an open subset of generalized Radon transforms on analytic manifolds that are both injective and stable.

Theorem 4.4.1. *Let (M_1,g) be an analytic manifold with M a compact submanifold with boundary. Let R be an analytic generalized Radon transform on M_1 given by defining function φ and weight w. Let $\tilde{R}$ be a smooth generalized Radon transform* *also on M_1 given by defining function $\tilde{\varphi}$ and weight $\tilde{w}$. Then there exists an integer $K \gg n$ and a parameter $0 < \delta \ll 1$ such that if*

$$||\varphi - \tilde{\varphi}||_{C^K(M_1 \times S^{n-1})} + ||w - \tilde{w}||_{C^K(M_1 \times S^{n-1})} < \delta, \tag{4.37}$$

then we have, for $\tilde{N} = \tilde{R}^* \tilde{R}$, a stability estimate of the form

$$||f||_{L^2(M)} \leq C||\tilde{N} f||_{H^{n-1}(M_1)}. \tag{4.38}$$

In addition, $\tilde{R}$ is injective on $L^2(M)$.

Proof. A similar estimate holds for $N = R^* R$ by Proposition 4.4.1. By Lemma 1.2.2, there exists $K > 0$ such that for all $\epsilon > 0$, there exists $\delta' > 0$ so that

$$\sum_{|\alpha|,|\beta| \leq N} |\sigma(N) - \sigma(\tilde{N})|_{N,\alpha,\beta} < \delta \Longrightarrow ||N - \tilde{N}||_{L^2(M) \to H^{n-1}(M_1)} < \epsilon. \tag{4.39}$$

If the C^K–norms of the pair of defining functions and the pair of weights are less than some small δ, then Lemma 4.4.2 shows that the C^{K-2}–norms of the respective symbols are smaller than $C'\delta$, for some $C' > 0$. Given $\epsilon > 0$, take $\delta < \delta'/C^prime$. Then we have, for K large enough,

$$\begin{aligned} ||f||_{L^2(M)} &\leq C_1 ||Nf||_{H^{n-1}(M_1)} \\ &\leq C_1 ||\tilde{N} f||_{H^{n-1}(M_1)} + C_1 \epsilon ||f||_{L^2(M)}. \end{aligned}$$

Let $\epsilon \ll \min\{C^{-1}, 1\}$. Then the second term on the right-hand side can be absorbed into the left. The resulting stability estimate for $\tilde{N}$ implies that $\tilde{R}$ is injective. □

While the theorem holds for K sufficiently large, we recall that [17] showed that $K = 2$ was sufficent for the geodesic ray transform. One would then expect the above theorem to hold for $K = n$.

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