Fall 2014

Epistemic Uncertainty Quantification in Scientific Models

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Graduate School Form 30
(Revised 08/14)

PURDUE UNIVERSITY
GRADUATE SCHOOL
Thesis/Dissertation Acceptance

This is to certify that the thesis/dissertation prepared

By Xiaoxiao Chen

Entitled
EPISTEMIC UNCERTAINTY QUANTIFICATION IN SCIENTIFIC MODELS

For the degree of Doctor of Philosophy

Is approved by the final examining committee:

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Approved by: David Goldberg 11/21/2014

Head of the Department Graduate Program Date

EPISTEMIC UNCERTAINTY QUANTIFICATION IN SCIENTIFIC MODELS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Xiaoxiao Chen

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

December 2014

Purdue University

West Lafayette, Indiana

ACKNOWLEDGMENTS

First of all, I would like to express my utmost gratitude to my Ph.D. advisor, Prof. Dongbin Xiu. I appreciate all his contributions of time, ideas and funding that result in a stimulating Ph.D. experience for me. The enthusiasm and creativity he has for his research is motivational for me.

I would also like to deeply appreciate to Dr. Yanyan He, who helps me to have a good insight of the fuzzy set theory, which the important topic of my research. Meanwhile, I would like to offer my special thanks to my committee members: Prof. Guang Lin, Prof. Suchuan Dong and Prof. Greg Buzzard for their time, helpful comments and insightful questions. Their guidance has served me well and greatly improved the quality of my dissertation.

Lastly, I would like to thank my family for all their love and encouragement. Without their support, I could not complete this work.

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ABSTRACT

Chen, Xiaoxiao Ph.D., Purdue University, December 2014. Epistemic Uncertainty Quantification in Scientific Models. Major Professor: Dongbin Xiu.

In the field of uncertainty quantification (UQ), epistemic uncertainty often refers to the kind of uncertainty whose complete probabilistic description is not available, largely due to our lack of knowledge about the uncertainty. Quantification of the impacts of epistemic uncertainty is naturally difficult, because most of the existing stochastic tools rely on the specification of the probability distributions and thus do not readily apply to epistemic uncertainty. And there have been few studies and methods to deal with epistemic uncertainty. A recent work can be found in [J. Jakeman, M. Eldred, D. Xiu, Numerical approach for quantification of epistemic uncertainty, J. Comput. Phys. 229 (2010) 46484663], where a framework for numerical treatment of epistemic uncertainty was proposed. In this paper, firstly, we present a new method, similar to that of Jakeman et al. but significantly extending its capabilities. Most notably, the new method (1) does not require the encapsulation problem to be in a bounded domain such as a hypercube; (2) does not require the solution of the encapsulation problem to converge point-wise. In the current formulation, the encapsulation problem could reside in an unbounded domain, and more importantly, its numerical approximation could be sought in L^p norm. These features thus make the new approach more flexible and amicable to practical implementation. Both the mathematical framework and numerical analysis are presented to demonstrate the effectiveness of the new approach. And then, we apply this methods to work with one of the more restrictive uncertainty models, i.e., the fuzzy logic, where the p-distance, the weighted expected value and variance are defined to assess the accuracy of the

solutions. At last, we give a brief introduction to our future work, which is epistemic uncertainty quantification using evidence theory.

1. INTRODUCTION

1.1 Epistemic Uncertainty

Mathematical models are used to simulate a wide range of systems and processes in many sciences, such as engineering, physics, biology, chemistry and environmental sciences. However, these systems are subject to a wide range of uncertainties. In order to thoroughly understand the system, the uncertainties involved in the underlying physics need to be explored. There are various sources of uncertainties, for example, random inputs of the system, randomness in the property of the material, unknown structural properties of the physics, observation errors, etc. The treatment (quantification) of the uncertainties is essential for the precise modeling of the real system.

Oberkampf and Roy [1] divide these uncertainties into two categories: aleatory and epistemic. Aleatory uncertainty arises from the intrinsic variability associated with the physical system and is irreducible. Epistemic uncertainty represents any lack of knowledge in any phase or activity of the modeling process and is reducible when adding enough information.

Aleatory uncertainty is also referred to in the literature as irreducible uncertainty, inherent uncertainty, variability and stochastic uncertainty, which is usually characterized with probability distributions because of their randomness. Consequently, we can use the systematic probability theory and mature statistical tools to deal with quantification of aleatory uncertainties.

On the other hand, epistemic uncertainty is referred to as subjective uncertainty, reducible uncertainty, and model form uncertainty. The sources of epistemic uncertainties in the parameter input process, which are interpreted as subjective uncertainty and reducible uncertainty, are not amenable to interpretation in terms of classical probability theory. To represent such uncertainties, Zadeh [2] introduced the concept of a fuzzy set (with fuzzy or imprecise boundary). Unlike in the classical set theory where, if an object is a member of a set A, it cannot be a member of the disjoint set of A. It allows an object or element to be a member of more than one set with different grade of membership. This work began the modern revolution in new ways of thinking about uncertainty that stems from sources other than random processes. Fuzzy sets have been applied in diverse fields such as social science [3], linguistics [4], and pattern recognition [5,6]. In 1978, Zadeh [7] extended the fuzzy set theory to possibility theory, where he interpreted membership functions of fuzzy sets as possibility distributions encoding flexible constraints induced by natural language statements. Here, the application of it in uncertainty analysis is the purpose of the dissertation.

Other sources of epistemic uncertainty resulting from limited understanding or misrepresentation in the modeled process, are known commonly as model form uncertainty. Inclusion of enough additional information can lead to a reduction in the predicted uncertainty of a model output. Consequently, we can consider epistemic uncertainty as providing bounds on an underlying aleatory uncertainty. To represent these uncertainties, Shafer [8] developed what he called a mathematical theory of evidence based on the ideas of Dempster [9, 10]. This theory was immediately espoused by the artificial intelligence community who called it the Dempster-Shafer theory of evidence. It is considered as a generation of probability theory, where cumulative belief and cumulative plausibility functions are defined to represent the uncertainty in the output metrics. Here, belief constructs a measure of the amount of information that supports an event being true and plausibility measures the absence of information that supports the event being false. For decades, the Dempster-Shafer (DS) theory of evidence (also called evidence theory) is well developed in the application context [6, 11–14], where reduction and convergence to the true aleatory uncertainty can be obtained given sufficient additional information, that is, the evidence theory representation of uncertainty approaches the probabilistic representation as the amount of information about the input data increases. In this way, application of DS theory to study uncertainties in modeling and simulations is our future work.

1.2 Recent Work on Epistemic Uncertainty Quantification

Until recently, most uncertainty analysis has focused on aleatory uncertainty. Numerous stochastic methods have been developed to provide accurate and efficient simulations for this form of uncertainty. For example, the stochastic methods based on generalized polynomial chaos (gPC) [15], an extension of the traditional polynomial chaos [16], have demonstrated their efficiency in practice and are widely used for many problems. Their implementations typically follow either stochastic Galerkin (SG) [15–17] or stochastic collocation (SC) [18–22], often termed as intrusive methods or non-intrusive methods, respectively. For a detailed review on the methods, see [23, 24].

Numerical study of epistemic uncertainty is more difficult because of the lack of sufficient probabilistic information. And one naturally can not readily adopt probabilistic approaches. Some of the existing approaches include evidence theory, possibility theory [25] and interval analysis [11,26]. These method have their own advantages, though most do not address efficient numerical implementations. More recent studies can be found in [11, 27].

Our methodology is based on the recent work of [28], where a general numerical approach was proposed for epistemic uncertainty analysis. The method consists of three components: (1) encapsulation of the epistemic uncertain inputs by a bounded domain, i.e., a hypercube; (2) solution of the encapsulation problem, which is the original governing equations defined in the encapsulation domain; and (3) post-processing of the resulting numerical solution for its statistics, whenever the probability distribution of the epistemic inputs is known, or assumed, a posterior. A notable feature of the approach is that the numerical solution of the encapsulation problem needs to have error control in L^∞ norm, i.e., point-wise, in the entire encapsulation domain.

Though effective and mathematically sound, the requirements of using bounded encapsulation domain and controlling errors in L^∞ norm can be difficult to achieve in practical simulations.

1.3 Research Outline

The primary method of this work is presented in Chapter 2, where the method is proposed and both the numerical algorithms and the error analysis are demonstrated. We extend the capabilities of the methodology proposed in [28]. The new approach achieves two notable extensions. One is in the modeling of the epistemic inputs, where the encapsulation domain is not required to be bounded anymore. Whenever appropriate, unbounded domains can be used to encapsulate the inputs. In practice this could impose less constraints on modeling the inputs, because in many problems there may not be obvious bounds for the epistemic variables. The other extension of the new approach is that the numerical approximation of the encapsulation problem can now be measured in $L^p, p > 1$, norm. Naturally this is much easier to achieve in practice than the L^∞ norm used in the earlier work of [28]. (In fact for semi-bounded and unbounded variables, approximation in L^∞ norm is not possible.) The ability to use unbounded domain to model the epistemic inputs and to approximate the encapsulation problem in L^p norm makes the new method more flexible in practical computations, as it applies to a much larger set of problems (bounded variables vs. unbounded variables) and allows more flexible numerical treatment. In fact, the new framework incorporates the earlier one from [28] as a special case, i.e., when the epistemic variables are modeled as bounded variables and a numerical method with point-wise error control is utilized, the new method becomes identical to the earlier one. In term of the analysis of the new method, we present convergence analysis in both weak form and strong form. This is another extension of the earlier work of [28], which only contains convergence result in a weak form.

The application of this method on epistemic uncertainty quantification using Fuzzy set theory is shown in Chapter 3. Firstly, the basics of the fuzzy set theory are introduced, where the fuzzy set and fuzzy number are defined and the property and operation of fuzzy sets are described. And the most notable parts for this chapter are: (1) To assess of the accuracy of a numerical fuzzy equation, unlike using the supreme distance [29], which needs the accuracy of the numerical solution in L^∞-norm, we define the p-Hausdoreff distance, which naturally leads to the requirement of the accuracy over the parameter domain of the numerical solution to the parametric PDE in L^p-norm. (2) Associated with the possibility theory, we define the weighted expected value and variance value of a function of a fuzzy set using the concepts of average values of the function on the α-cut of a fuzzy set. (3) The convergence of the solution in p-Hausdoreff distance, the weighted expected value and the variance are theoretically proved and numerically illustrated.

In Chapter 4, we briefly present the future work of our research, which is epistemic uncertainty quantification using Evidence theory. Similar to the work of chapter 3, firstly, a brief description of the mathematical basics of the DS theory is given, where we define the distance of two m-functions based on p-Hausdorff distance to assess the accuracy of the solution. And the convergence of the m-function can be achieved under the assumption of the L_∞ norm convergence of numerical solution. Some numerical examples are shown there.

At last, final conclusions are drawn in Chapter 5.

The text of this dissertation includes the reprints of the following papers, either accepted or submitted for consideration at the time of publication.

X. Chen, E.-J. Park, and D. Xiu. *A flexible numerical approach for quantification of epistemic uncertainty.* J. Comput. Phys., 240 (2013) 211224.

X. Chen, Y. He and D. Xiu. *Fuzzy Partial Differential Equation.* J. Comput. Phys., Submitted.

2. A FLEXIBLE NUMERICAL APPROACH FOR QUANTIFICATION OF EPISTEMIC UNCERTAINTY

In this chapter, we present a new method, similar to that of [28] but significantly extending its capabilities. Most notably, the new method (1) does not require the encapsulation problem to be in a bounded domain such as a hypercube; (2) does not require the solution of the encapsulation problem to converge point-wise. In the current formulation, the encapsulation problem could reside in an unbounded domain, and more importantly, its numerical approximation could be sought in L^p norm. These features thus make the new approach more flexible and amicable to practical implementation. And the framework is as follows: In Section 2.1, we present the necessary mathematical framework for quantifying epistemic uncertainty. In section 2.2, we discuss the construction and solution of the encapsulation problem, and establish the convergence results when the probability distribution of the variables becomes known a posterior. In section 2.3, we present numerical analysis to demonstrate the effectiveness of the new approach.

2.1 Problem Setup

Let $D \subset \mathbb{R}^l, l = 1, 2, 3$, be a physical domain with coordinates $x = (x_1, \ldots, x_l)$, let $(0, T]$ be a time domain with $T > 0$, and let $I_Z \subset \mathbb{R}^n$ be a parameter domain for uncertain inputs. We consider a general partial differential equation (PDE) as

$$\begin{cases} v_t(x,t,Z) = L(v), & D \times (0,T] \times I_Z, \\ \mathcal{B}(v) = 0, & \partial D \times [0,T] \times I_Z, \\ v = v_0, & D \times \{t = 0\} \times I_Z, \end{cases} \tag{2.1}$$

where L is a (linear or nonlinear) differential operator, $\mathcal{B}$ is the boundary condition operator, v_0 is the initial condition, and $Z = (Z_1, \ldots, Z_n)$ is a set of uncertain parameters characterizing the uncertainty in the inputs of the governing equation. We assume that the problem (2.1) is well-posed in Z and let $v(x, t, Z)$: $D \times (0, T] \times I_Z \to \mathbb{R}^{n_v}$ denote its solution. For simplicity, we further assume that the output of (2.1) is in one dimension, i.e., $n_v = 1$. We also make a fundamental assumption that the problem (2.1) is well posed in I_Z.

Most of the existing studies adopt a probabilistic formulation to quantify aleatory uncertainty the type of uncertainty whose complete probabilistic specification is available. This translates into the availability of the knowledge of the function $F_Z(s) = Prob(Z \in s)$ for real number $s \in \mathbb{R}^d$. (In many practical applications, this is often accomplished by assuming the marginal distribution of each Z_i and then assuming mutual independence among all the components.) In this paper, however, we consider the case where the uncertainty is epistemic. That is, the distribution functions of $F_Z(s)$ is not completely known, primarily due to our lack of knowledge and characterization of the physical system governed by the system of equations (2.1). Since the focus of our study is on the dependence of the solution on the uncertain inputs Z, hereafter we will suppress the notions of x and t, with the understanding that all of the statements are meant for any fixed locations in x and t.

2.2 Methodology

We now present a method for solving system (2.1) subject to epistemic uncertain inputs. The proposed methodology is a three-step procedure that involves identifying the ranges of the uncertain inputs, generating an accurate numerical approximation of the solution to (2.1) within estimated ranges, and post-processing the results. *Note that no probability distribution information will be utilized in the solution procedure.*

2.2.1 Range Estimate

The first task is to identify a range, or bound, that is sufficiently large such that the true, and yet unknown, range of the input uncertainty is *mostly* incorporated. For each variable $Z_i, \quad i = 1, ..., d$, let

$$I_{Z_i} = [\alpha_i, \beta_i], \qquad \alpha_i \leq \beta_i, \tag{2.2}$$

be its (unknown) range. Note that the range can be bounded, semi-bounded, or unbounded. The goal of range estimation is to identify an interval

$$I_{X_i} = [a_i, b_i], \tag{2.3}$$

such that I_{X_i} and I_{Z_i} overlap each other with sufficiently large probability. *Note that the interval can be semi-bounded or unbounded.* This is fundamentally different from the work of [28], where the estimated range is required to be bounded. The estimated range I_{X_i} must be sufficiently large such that it encapsulates the true range I_{Z_i} either completely or, if there is any truncation, the truncation is sufficiently small. In [28], this requirement is termed the overwhelming probability condition. In the current setting, the ability to use unbounded interval I_{X_i} removes the need for highly accurate estimate of the unknown range I_{Z_i}, as one can, in principle, always use an unbounded interval to completely encapsulate the unknown range. This could help alleviate the difficulty in estimating the input ranges, a task that sometimes can be challenging.

It should be noted that when estimating the range of each epistemic variables, it is natural to ensure the governing Eq. (2.1) remains well-posed in the estimated range I_{X_i} . That is, it certainly makes sense not to let I_{X_i} intersect with any parameter values that may violate the physical laws or solvability of the governing equations.

For the collection of all the inputs, let I_Z be the range of the uncertain epistemic variables $Z \in \mathbb{R}^d$. Naturally,

$$I_Z \subset \times_{i=1}^{d} I_{Z_i}. \tag{2.4}$$

Note there may be dependence among some of the components of Z, and the real range of Z may be smaller than the tensor product of each range. To encapsulate the range I_Z, we define an encapsulation set

$$I_X = \times_{i=1}^d I_{X_i} = \times_{i=1}^d [a_i, b_i], \tag{2.5}$$

which is the Cartesian product of (2.3). Now let

$$I^+ = I_Z \cap I_X, \qquad I^0 = I_Z \setminus I_X, \tag{2.6}$$

denoted as super set and common set, respectively, and

$$I^- = I_Z \triangle I_X = I^+ \setminus I^0, \tag{2.7}$$

be the symmetric difference of I_Z and I_X, denoted as difference set. We now require that the range estimation procedure is satisfactory when

$$P(Z \in I) \leq \delta, \tag{2.8}$$

for a sufficiently small non-negative real number $\delta \geq 0$. Therefore, the encapsulation set I_X encapsulates I_Z, the true and unknown support of Z, with probability at least $1-\delta$, where $\delta \geq 0$ can be made small by enlarging the size of I_X. The parameter δ can be zero, i.e., I_X encapsulates I_Z with probability one. One easy way to accomplish this is to ensure $I_Z \subset I_X$, which is relatively easier to enforce in the current framework because I_X can be defined as the entire space $\mathbb{R}^d$.

2.2.2 Encapsulation Problem

We now define the following *encapsulation problem*

$$\begin{cases} u_t(x,t,X) = L(u), & D \times (0,T] \times I_X, \\ \mathcal{B}(u) = 0, & \partial D \times [0,T] \times I_X, \\ u = u_0, & D \times \{t=0\} \times I_X, \end{cases} \tag{2.9}$$

where I_X is the bounded hypercube defined in (2.5). This is effectively the same problem (2.1) defined now on the encapsulation set I_X that covers the original random parameter set I_Z with probability at least $1-\delta$. The new problem is well defined in I_X, because we have assumed the estimated range of each I_{X_i} stays in the range of well-posedness allowed by the governing equation. Since problem (2.1) and (2.9) are exactly the same in the common domain I^0, we have the following trivial result,

$$u(\cdot, s) = v(\cdot, s), \qquad \forall s \in I^0. \tag{2.10}$$

We remark that for the encapsulation problem (2.9) we do not assign any probability information to the variables X. For the solution of the encapsulation problem (2.9), we focus only on the dependence on the variables X, which now resides in $I_X \subset \mathbb{R}^d$, i.e.,

$$u(X) : I_X \longrightarrow \mathbb{R}. \tag{2.11}$$

We equip I_X with an inner product

$$(f, g)_w = \int_{I_X} f(s)g(s)w(s)ds, \tag{2.12}$$

where $w(s) \geq 0$ is a weight function, non-vanishing in the interior of I_X. We also introduce the weighted L^p norm

$$\|f\|_{L^p_w(I_X)} = (\int_{I_X} |f(s)|^p w(s)ds)^{1/p}, \qquad 1 \leq p < +\infty. \tag{2.13}$$

We remark that the choice of the weight function w is entirely a numerical issue. It is used to measure the errors of the numerical approximation and unrelated to the probability information of the uncertain inputs. Let $u_n(X)$ be a numerical solution to the encapsulation problem (2.9), where n is a discretization parameter that indicates a finer discretization for larger values of n. For example, n can be the highest degree of a polynomial approximation, the total number of discretization elements, etc. A critical requirement for the proposed methodology is the need for the numerical approximation of (2.9) to be accurate in the weighted L^p norm. That is, we require

$$\epsilon_n \triangleq \|u - u_n\|_{L^p_w(I_X)} \ll 1. \tag{2.14}$$

Ideally the error should converge, i.e., $\epsilon_n \to 0$ as $n \to \infty$, though convergence is only a mathematical preference, not a practical necessity.

This requirement represents another fundamental difference from the method proposed in [28], where the approximate solution u_n is required to be accurate point-wise in I_X, i.e., in L^∞ norm. The current requirement in the weighted L^p norm is obviously weaker mathematically and easier to accomplish in practice. There exist many numerical approaches that can deliver accurate approximation in the weighted L^p norm. And the choices are problem dependent. Here we will not engage in discussions on the numerical strategies. We assume a satisfactory numerical strategy can always be devised for (2.9) such that the numerical solution is accurate in the weighted L^p norm. We now discuss the way to use u_n in epistemic uncertainty analysis.

2.2.3 Epistemic Uncertainty Analysis

The nature of epistemic uncertainty is that it is primarily due to lack of knowledge. And in principle, the true nature of the epistemic uncertainty can be inferred via added information, typically through more measurement and deeper understanding of the system.

When $u_n(X)$, the polynomial approximation of the true solution $u(X)$, is obtained for (2.9) and accurate in the L^p-norm (2.18), it can serve as an accurate surrogate model. We can then apply various operations on u_n, instead of u. This is particularly useful when the probability distribution of the variables becomes known a posterior. We can then evaluate the solution statistics of u_n based on the posterior distribution on Z. Note the operations on u_n do not require us to solve the governing equations anymore they can be treated as post-processing steps. And one can repeat this step as much as possible for any kind of posterior distributions. Let $\rho_Z(s) = \frac{dF_Z(s)}{ds}, s \in I_Z$ be the posterior probability distribution of the epistemic uncertain input Z. Then, for example, the mean of the true solution $v(Z)$

$$m \triangleq \mathbb{E}[v(Z)] = \int_{I_Z} v(s)\rho_Z(s)ds, \tag{2.15}$$

can be approximated by

$$m_n = \int_{I^0} u_n(s)\rho_Z(s)ds. \tag{2.16}$$

Let us define

$$r(s) = \mathbf{1}_{I^0}\frac{\rho(s)}{w(s)}, \qquad s \in I^+, \tag{2.17}$$

where $\mathbf{1}$ is the indicator function satisfying, for any set A, $\mathbf{1}_A(s) = 1$ for $s \in A$ and $\mathbf{1}_A(s) = 0$ otherwise. Like in [28], where the convergence in mean was established, similar result holds here.

Lemma 2.1. *(Mean convergence). Assume the solution of* (2.1)*, $v(Z)$, is bounded and let $C_v = \|v\|_{L^\infty}$. Let $u_n(X)$ be an approximation to the solution $u(X)$ of* (2.9) *in weighted L^p_w norm and denote*

$$\epsilon_n = \|u - u_n\|_{L^p_w(I_X)}, \qquad p \geq 1. \tag{2.18}$$

Let q be a positive real number satisfying $1/p + 1/q = 1$, and assume the function r defined in (2.17) *satisfies, for $q < 1 (p > 1)$,*

$$C_r = \mathbb{E}[r^{q-1}] = \int_{I^0} r^{q-1}(s)\rho_z(s)ds < \infty. \tag{2.19}$$

Then the mean of v in (2.15) *and the mean of u_n in* (2.16) *satisfy*

$$|m - m_n| \leq C_r^{1/q}\epsilon_n + C_v\delta, \tag{2.20}$$

where for the case of $q = \infty (p = 1)$, we define $C_r^{1/q} = 1$.

Proof. We first extend the domains of the definitions for v, q, and u_n to I^+, by following the definitions in (2.6), and define, for $s \in I^+$,

$$\begin{aligned} v^+(s) &= \mathbf{1}_{I_Z}(s)v(s), \\ \rho^+(s) &= \mathbf{1}_{I_Z}(s)\rho_Z(s), \\ u_n^+(s) &= \mathbf{1}_{I_X}(s)u_n(s). \end{aligned} \tag{2.21}$$

Naturally, ρ^+ is a probability density function on I^+. Then (2.15) can be expressed as

$$m = \int_{I_Z} v(s)\rho_Z(s)ds = \int_{I^+} v^+(s)\rho^+(s)ds, \tag{2.22}$$

which can be split into two parts

$$m = \int_{I^0} v^+(s)\rho^+(s)ds + \int_{I^-} v^+(s)\rho^+(s)ds \tag{2.23}$$

$$= \int_{I^0} v(s)\rho_Z(s)ds + \int_{I^-} v^+(s)\rho^+(s)ds \tag{2.24}$$

$$= \int_{I^0} v(s)\rho_Z(s)ds + \int_{I^-\cap I_Z} v(s)\rho_Z(s)ds. \tag{2.25}$$

By using (2.16), we have

$$m - m_n = \int_{I^0} (v(s) - u_n(s))\rho_Z(s)ds + \int_{I^-\cap I_Z} v(s)\rho_Z(s)ds \tag{2.26}$$

$$= \int_{I^0} (v(s) - u_n(s))(w(s))^{1/p} \frac{\rho_Z(s)}{w(s)^{1/p}} ds + \int_{I^-\cap I_Z} v(s)\rho_Z(s)ds, \tag{2.27}$$

where the property (2.10) has been used. The special case of $p = 1$ immediately leads to the conclusion. And for $p > 1$, an exercise of the Holder's inequality leads to

$$|m - m_n| \leq (\int_{I^0} |u(s) - u_n(s)|^p w(s))^{1/p} (\int_{I^0} \frac{\rho_Z^q(s)}{w^{q/p}(s)})^{1/q} + \int_{I^-\cap I_Z} v(s)\rho_Z(s)ds. \tag{2.28}$$

Utilizing (2.8) and (2.19) and the condition of $1/p + 1/q = 1$, the main result (2.20) is established. □

It is worthwhile to compare this result to that of [28], which states that the error estimate satisfies $|m - m_n| \leq \eta_n + C_v\delta$, where η_n is the error of u_n in L^∞-norm. In both results, the second term is the same, indicating the error induced by truncating the true range of the variables. In the current approach, the domain I_X can be unbounded, therefore it is easier to avoid this truncation error because one does not need to truncate the range. On the other hand, the solution of the encapsulation problem u_n is required to have a less strict approximation property, in the weighted L^p norm, as opposed to the L^∞ norm required by [28]. The weight w in the approximation, however, needs to satisfy the condition posed by (2.19), and this manifests itself as the constant in the first term of the error bound (2.20). Additionally, convergence in a strong form can be established.

Theorem 2.2. *(Strong convergence). Assume the function* r*, defined in* (2.17)*, satisfies*

$$C_{r,0} = \max_{s} r(s) < \infty, \tag{2.29}$$

and the solution of the encapsulation problem u_n *has error in* L^p_w *norm in the form of* (2.18)*. Then*

$$\mathbb{E}(|v - u_n|^p)^{1/p} = \|v - u_n\|_{L^p_\rho(I^+)} \le C_{r,0}^{1/p}\epsilon_n + C_v\delta^{1/p}. \tag{2.30}$$

Proof.

$$\begin{aligned}
\mathbb{E}(|v - u_n|^p) &= \int_{I^+} |v^+ - u_n^+(s)|^p \rho^+(s)ds && (2.31)\\
&= \int_{I^0} |v^+ - u_n^+(s)|^p \rho^+(s)ds + \int_{I^-} |v^+ - u_n^+(s)|^p \rho^+(s)ds && (2.32)\\
&= \int_{I^0} |v - u_n(s)|^p \rho^+(s)ds + \int_{I^-\cap I_Z} |v^+(s)|^p \rho^+(s)ds && (2.33)\\
&= \int_{I^0} |v - u_n(s)|^p w(s)\frac{\rho_Z(s)}{w(s)}ds + \int_{I^-\cap I_Z} |v^+(s)|^p \rho^+(s)ds && (2.34)\\
&\le \int_{I^0} |v - u_n(s)|^p w(s)ds \cdot \max_{s\in I^0} \frac{\rho_Z(s)}{w(s)} + C_v^p\delta && (2.35)\\
&\le C_{r,0}\epsilon_n^p + C_v^p\delta \le (C_{r,0}^{1/p}\epsilon_n + C_v\delta^{1/p})^p && (2.36)
\end{aligned}$$

And this completes the proof. □

With such a strong convergence in $L^p_\rho(I^+)$, we immediately obtain the following trivial result.

Corollary. *(Weak convergence). Suppose that* ϵ_n *and* δ *can be made arbitrarily small. Then, following the same conditions in Theorem* 2.1*,* u_n *converges tov in probability, and consequently also in distribution, with respect to the probability measure* ρ^+ *defined in* (2.21) *on the super set* I^+*.*

2.2.4 Implementation Procedure

We now present a quick summary to illustrate the procedure of applying the method to practical problems. Again, we use the setup for the general stochastic system (2.1).

- Identify the epistemic variables. In (2.1), the variables are denoted as $Z = (Z_1, \ldots, Z_d), d \geq 1$. (We do not discuss the case of aleatory variables because their treatment is well studied.)

- For each epistemic variable $Z_i, i = 1, \ldots, d$, identify a reasonable interval I_{X_i}, (2.9), as an estimate of its range. Note this step requires one to utilize any available information, data, or even experience.

- Define the encapsulation set I_X as the tensor products of each interval I_{X_i}, as in (3.4). Consequently, this defines the encapsulation problem (2.9). Note that if the intervals I_{X_i} are all strictly bounded, then is a hypercube and the encapsulation I_X problem becomes identical to that in the earlier work of [28].

- Solve the encapsulation problem (2.9), which is the same governing Eq. (2.1) defined in the encapsulated parameter space I_X. We now specify a weight function w in IX and seek a numerical solution u_n that approximates the true solution u in $L^p_w, p \geq 1$ norm in I_X. This is a standard approximation approach, e.g., finite element methods, and the choices of the weight and the norm vary from problem to problem.

- Post-process the numerical solution u_n. Since u_n is an accurate approximation in a strong norm L^p_w, it can be used as a surrogate for the unknown solution u. One can now apply various operations on u_n to analyze approximately the properties of u.

To solve the encapsulation problem (2.9) in the parameter space I_X, one can readily borrow several well developed methods such as those based on generalized polynomial chaos, stochastic collocation, etc., all of which offer numerical approximations in a strong norm similar to L^p. Since this is not the focus of the current paper, we refer the interested readers to the book [24] and its large collection of references.

Naturally, this result implies that the statistical moments of u_n converges to those of the true solution v, whenever the errors induced by ϵ_n and δ can be arbitrarily small.

2.3 Numerical Examples

In this section, we present numerical examples to support the theoretical analysis. The focus is on examination of the error behavior. We employ three sets of tests.

The first one utilizes a scalar equation with a single epistemic variable. Though simple, this example allows us to thoroughly examine the convergence properties of the numerical implementations, where we employ different variations of encapsulation strategies. The second example is a homogeneous random diffusion problem with multiple epistemic variables in the diffusivity field. This example allows us to examine the case when the dependence of the variables is unknown. The third example is a time-dependent stochastic diffusion problem with multiple epistemic variables in the diffusivity field, this example allow us to examine the case when weighed function is different.

We remark that even though the first and the second problems resemble those in [28], they are now quite different because of the different modeling assumptions on the input epistemic variables. Here we focus on the cases when the epistemic variables are modeled as semi-bounded and unbounded, in order to demonstrate the flexibility of the new framework. Such assumptions are on the epistemic variables are outside the applicability of the framework of [28].

2.3.1 Ordinary Differential Equation

Let us consider

$$\frac{dv}{dt} = -Zv, \qquad v(0) = 1, \tag{2.37}$$

where the parameter $Z > 0$ is a variable representing the input uncertainty. It is an epistemic uncertainty - its probability distribution is unknown a priori. The exact solution is

$$v(t, Z) = \exp(-Zt) \tag{2.38}$$

We will use this to examine the error behavior of the numerical methods, by assuming different types of *true* distributions on Z. The encapsulation problem takes the following form

$$\frac{du}{dt} = -Xu, \qquad u(0) = 1, \tag{2.39}$$

where the range of X, I_X, is to be determined based on ones modeling assumptions. Here we consider three difference cases of I_X for all possible situations, i.e., I_X can be a bounded interval (a, b), a semi-bounded domain $(0, +\infty)$, or an unbounded domain $(-\infty, +\infty)$. Even though this problem admits a simple analytical solution $u(t, X) = \exp(-Xt)$, we still resort to its numerical solution so that the proposed framework in this paper can be fully examined. Our numerical procedure is a Galerkin method using orthogonal polynomial basis functions. (One is free to use any other suitable numerical methods.) We seek

$$u_n(t, X) = \sum_{j=0}^{n} \hat{u}_j(t)\phi_j(X), \tag{2.40}$$

where $\phi_j(X)$ are Legendre polynomial satisfying

$$\int_{I_X} \phi_i(X)\phi_j(X)w(X)dX = \delta_{i,j}, \tag{2.41}$$

where $\delta_{i,j}$ is the Kronecker delta function. Depending on the range of I_X, we utilize different types of orthogonal polynomials. For bounded interval $I_X = (a, b)$, we use Legendre polynomials with $w \equiv \mathit{const}$; for semi-bounded interval $I_X = (0, +\infty)$, we use Laguerre polynomials with $w(s) \propto \exp(-s)$; and for unbounded interval $I_X = (-\infty, +\infty)$, we use Hermite polynomials with $w(s) \propto \exp(-s^2)$.

Upon substituting (2.40) into (2.39) and enforcing the residue to be orthogonal to the linear polynomial space of degree up to n, we obtain the following Galerkin system

$$\frac{d\hat{u}_j}{dt} = -\sum_{i=0}^{n} e_{i,j}\hat{u}_i, \qquad \hat{u}_j(0) = \delta_{0,j}, \tag{2.42}$$

where $e_{i,j} = \frac{1}{2}\int_{I_X} s\phi_i(s)\phi_j(s)w(s)ds$. And the resulting numerical solution u_n converges to u in L^2_w norm. (See Appendix for proof.) We then assign various distributions to Z a posterior, use these distributions to approximate the statistics of the solution via u_n directly (without resorting to the numerical simulations), and examine the errors induced by this procedure.

Bounded encapsulation

We first assume the exact (and unknown) distribution of Z is an exponential distribution, i.e., $\rho \sim \exp(-s)$. For the encapsulation problem (2.39) we use $X \in (0, b)$ with various length of b. This is the similar case considered in [28], where the convergence in mean, a weak error measure, was discussed. Here we study the convergence in L^2_ρ norm, a strong norm, and examine its behavior based on Theorem 2.1. The results are depicted in Fig. 2.1. It is obvious that the errors converge (exponentially fast) as the order of expansions is increased, before they saturate at certain level. The error saturation level depends on the value of $b-a$, larger value of b induces a smaller error because it causes a smaller truncation in the distribution of Z and hence a smaller value δ in the second error term in (2.30) of Theorem 2.2. Weak error measure, e.g., error in mean value, behaves similarly. It has been documented in [28] and is not presented here.

We further examine the error dependence on b, which is a direct indication of the truncation error δ. In Fig. 2.2, the strong and weak errors are examined against the parameter b, at a fixed and sufficiently high polynomial order of $n = 18$. Numerical

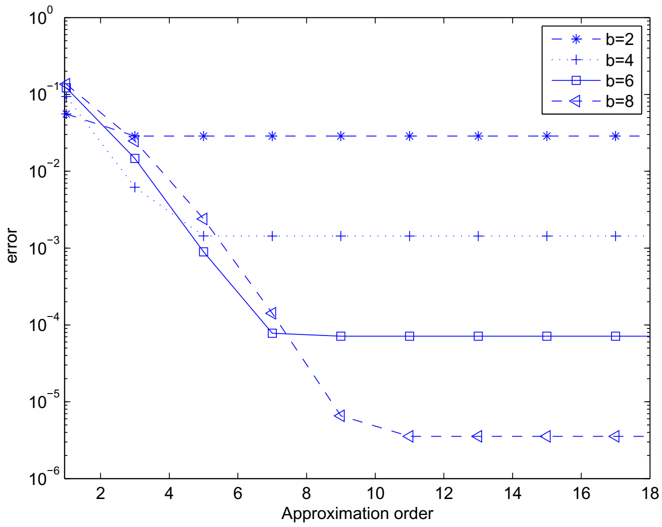


Figure 2.1. Convergence of the errors in strong L^2_ρ norm for linear ODE (2.37) at $t = 1$, where the encapsulation set is $X = (0, b)$ and the exact distribution is exponential $\rho_Z(s) \sim \exp(-s)$.

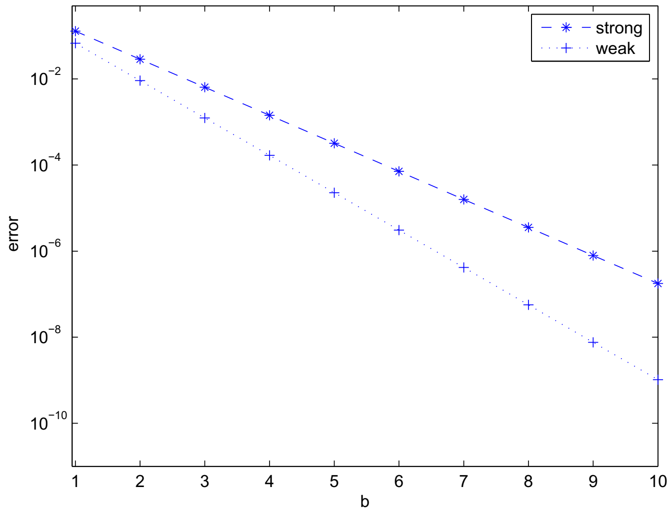


Figure 2.2. Convergence of the errors in strong norm and weak norm for linear ODE (2.37) at $t = 1$, where the encapsulation set is $X = (0, b)$ and the exact distribution is exponential $\rho_Z(s) \sim \exp(-s)$. The results are for fixed polynomial order of $n = 18$ and various values of b.

errors in the Galerkin solver are thus negligible. It can be seen that the errors decrease when b increases, as they should. It is also evident that the strong error converges slower than the weak error with respect to b. This is consistent with the estimate from (2.20) and (2.30), as the strong error scales as $\delta^{1/p}$ ($p = 2$ in this case) but weak error scales as δ.

Semi-bounded encapsulation

We now employ $X \in (0, +\infty)$ as the encapsulation variable. This corresponds to the practical case where one does not have a reliable estimate of the upper bound of the epistemic variable. We then use Laguerre polynomial with weight $w(s) \propto \exp(-s)$ to approximate the encapsulation problem (2.39).

We first assume the *true* distribution of Z is uniform in $(0, 1)$ with $\rho_Z(s) = 1$. In this case there is no truncation error because $I_X = (0, +\infty)$ completely encapsulates

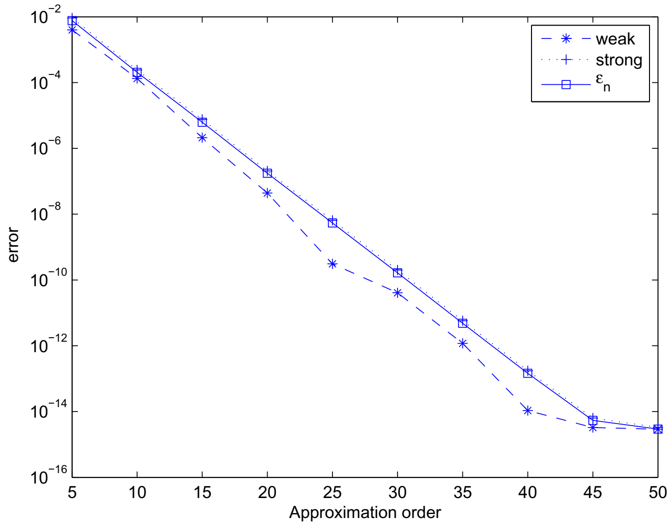


Figure 2.3. Convergence of errors in the strong form of L^2_ρ norm and the weak form of mean, along with the L^2_w error (e_n) of the Galerkin solutions of (2.39), at different polynomial approximation orders n. Here $X \in (0, \infty)$ and the true a posterior distribution of Z is uniform in $(0, 1)$.

the real variable Z. Error convergence, in both the strong form of L^2_ρ and the weak form in mean, are plotted in Fig. 2.3, for different orders of polynomial approximations at a fixed time $t = 1$. It can be seen that in this case both the strong error and the weak error converge at the same rate –exponential rate– as the L^2_w polynomial approximation error of (2.39). This is consistent with the error bounds from Theorems 2.2, as in this case the complete encapsulation of Z by X ensures $\delta = 0$.

We now assume the true distribution of Z is uniform, but resides in different intervals of (a, b). In Fig. 2.4, the errors are presented from intervals of $(0, 1)$, $(5, 6)$, and $(10, 11)$, in both the L^2_ρ form (left) and mean (right). It is obvious that all errors converge (exponentially fast). It also should be remarked that although all intervals have the same size of one, as the interval becomes further away from the origin the errors become larger. This is because in the Laguerre approximation the weight is $w(s) \propto exp(-s)$ it achieves the best accuracy close to zero and becomes progressively less accurate away from zero. Consequently, we should emphasize that even though the ability to use semi-bounded and unbounded encapsulation variables in the current framework does alleviate the modeling effort, sometimes significantly, to estimate the ranges of the epistemic variables, it does not imply this can completely eliminate the need to estimate the epistemic variables. When using unbounded domains, the accuracy of the numerical approximation will inevitably be less satisfactory in the region where the weight function approaches zero. It is thus important to have a good, or even just a vague, idea of the true range of the epistemic variables so that one can put the estimated range close to the peak of the weight function to ensure more accurate solutions.

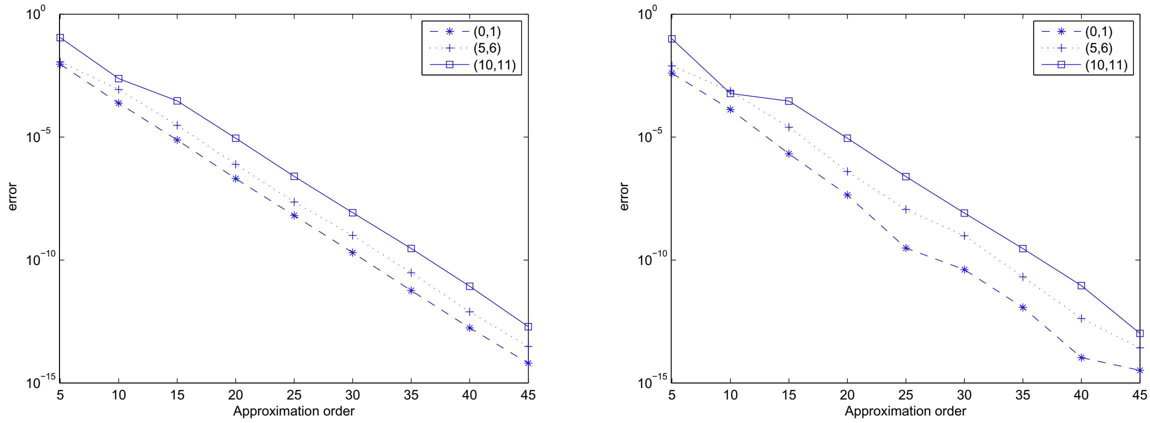


Figure 2.4. Convergence of errors for uniformly distribution Z in different intervals of (a, b), using encapsulation variable $X \in (0, \infty)$. Left: L^2_ρ error; Right: error in mean.

Unbounded encapsulation

Finally we examine a case where the encapsulation variable is assumed to be $X \in (-\infty, +\infty)$. Correspondingly we employ Hermite polynomial to solve (2.39) numerically. This is a simplified case of the practical situation where one does not

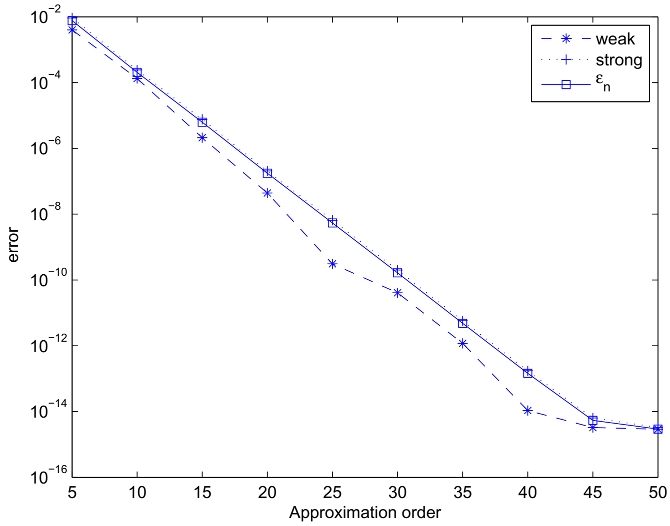


Figure 2.5. Convergence of errors in the strong form of L^2_ρ norm and the weak form of mean, along with the L^2_w error (e_n) of the Galerkin solutions of (2.39), at different polynomial approximation orders n. Here $X \in (-\infty, +\infty)$ and the true a posterior distribution of Z is uniform in $(0, 1)$.

have a good understanding of either the upper bound or the lower bound of the epistemic variables. Using an unbounded encapsulation variable would serve as one way to circumvent the difficulty. Both the errors in the strong form (L^2_ρ) and the weak form (mean) are shown in Fig. 2.5, along with the L^2_w convergence error of the numerical solution of (2.39). Once again we emphasize that in practice it is desirable that one has at least a vague idea of where the true range of the epistemic variable is, and can then center the weight function w of the Hermite polynomial approximation around that range to achieve more accurate numerical results.

2.3.2 Homogenous Diffusion Equation

We consider the following problem in one spatial dimension ($n = 1$) and ($d > 1$) random dimensions.

$$-\frac{\partial}{\partial x}[k(x, Z)\frac{\partial u}{\partial x}(x, Z)] = f(x, Z), \qquad (x, y) \in (0, 1) \times \mathbb{R}^d, \tag{2.43}$$

with forcing term $f = 2$ and the boundary conditions

$$u(0, Z) = 0, \qquad u(1, Z) = 0, \qquad Z \in \mathbb{R}^d.$$

Furthermore assume that the random diffusivity has the form

$$k(x, Z) = 1 + \sigma \sum_{k=1}^{d} \frac{1}{k^2\pi^2}(1 + cos(2\pi kx))Z_k.$$

where $\sigma > 0$ is a parameter to control the variation level of the diffusivity field. Here we choose this fixed form representation to focus on the treatment of the random variables Z_k, which are assumed to be epistemic. Note that usually the expansion is obtained by certain decomposition procedure, e.g., KarhunenLoeve expansion. For non-Gaussian processes, the decomposition usually can not fully determine the probability distributions of Z_k.

Since the detailed examination of using different encapsulation strategies has been carried out in the previous example, here we mostly focus on dealing with the unknown dependence structure of the epistemic random variables Z_k.

We set $d = 6$ and $\sigma = 1$ in the expansion (2.3.2). (These choices are rather arbitrary.) We assume that the posterior distribution (unknown now) of Z is $Z_1 \in beta(0, 1, 3, 2)$, and $Z_3 \in beta(-1, 0, 1, 1)$ and $Z_5 \in beta(-0.5, 0.5, 0, 0)$, $Z_2 = Z_1Z_5$, $Z_4 = (Z_1^2 + 1)Z_3$, and $Z_6 = -Z_5$. Clearly all the variables are bounded, $Z_1 \in (0, 1)$, $Z_2 \in (-0.5, 0.5)$, $Z_3 \in (-1, 0)$, $Z_4 \in (-2, 0)$, $Z_5 \in (-0.5, 0.5)$ and $Z_6 \in (-0.5, 0.5)$. Also by construction the variables are dependent. What appears to be a six-dimensional problem is in fact a three-dimensional one, for Z_2, Z_4 and Z_6 are functions of the other three variables. Let

$$\begin{aligned} Z_1 &= \quad Z_1 \\ Z_2 &= \quad Z_2 + 0.5 \\ Z_3 &= \quad -Z_3 \\ Z_4 &= \quad -Z_4 \\ Z_5 &= \quad Z_5 + 0.5 \\ Z_6 &= \quad Z_6 + 0.5. \end{aligned} \quad , \tag{2.44}$$

Then,

$$\begin{aligned} k(x,Z) =& 1 - 0.5 \sum_{k=2,5,6} \sigma(\frac{1}{k^2\pi^2}(1+cos(k^2\pi x))) \\ & - \sigma \sum_{k=3,4} \frac{1}{k^2\pi^2}(1+cos(2\pi kx))Z_k) \\ & + \sigma \sum_{k=1,k\neq 3,4}^{6} \frac{1}{k^2\pi^2}(1+cos(2\pi kx))Z_k. \end{aligned}$$

In the following epistemic analysis we assume that none of the above information regarding the distributions is unavailable. The only information we have is a conservative estimate of the bounds of the variables. And based on the bound estimation, we employ a straightforward linear transformation (2.44) and define $X \in I_X = (0,\infty)$ to completely encapsulate the epistemic variables Z. Our encapsulation problem is therefore

$$-\frac{\partial}{\partial x}[k(x,X)\frac{\partial u}{\partial x}(x,X)] = f(x,X), \qquad (x,y) \in (0,1)\times(0,\infty)^6, \tag{2.45}$$

where same boundary conditions and source term f hold, and the diffusivity field follows the similar construction of (2.3.2) with the encapsulation variables X in place of Z. Clearly the encapsulation problem is six-dimensional in the parameter space of X. We employ the six-dimensional Laguerre polynomials, in conjunction with a Galerkin procedure, with weight function $w(s) = (\exp(-s))^6$ to solve the problem.

Due to lack of analytical solution, here we employ a sufficiently high-order polynomials solution at $n = 8$ order as the numerical exact solution, against which we

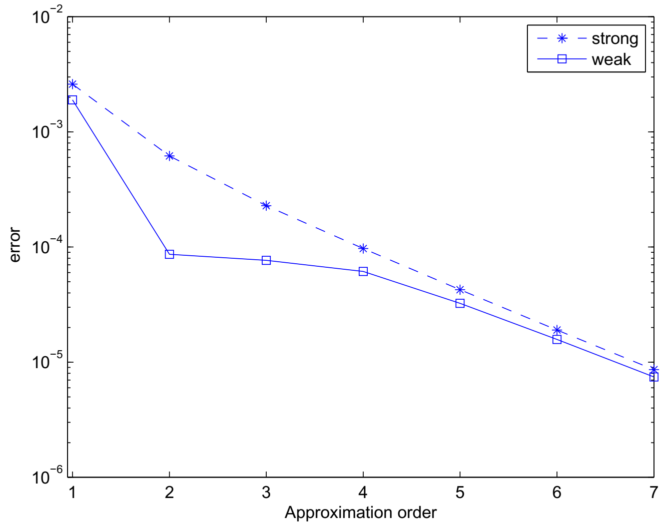


Figure 2.6. Convergence of the errors of in both strong form (L^2 weighted norm) and in a weak form (in mean) for the diffusion problem. Errors are computed against the high-order ($n = 8$) Galerkin solution at a fixed spatial location $x = 0.5257$

compare the solutions obtained by lower-order expansions. Both the strong errors (in L^2_ρ) and weak errors (in mean value) are computed and presented in Fig. 2.6. The errors are computed at a fixed spatial location ($x = 0.5257$), where the solution is non-trivial. (One can of course take a normed error in the spatial dimension as well.) The usual (exponentially) fast convergence with respect to the approximation order can be clearly observed. No error saturation is present because the encapsulation variables can fully encapsulate the epistemic variables.

2.3.3 Time-dependent Diffusion Equation

we present numerical examples to suppose the theoretical analysis. The focus is on the examination of the error behavior. Since we are more interested in the convergence of the methods (The proof is in the appendix). In order to simplify the calculation, we denote that the dimension of the spatial space is $d = 1$, and we consider the equation below:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}[k(x,y)\frac{\partial u}{\partial x}(t,x,y)] = f(t,x,y), \qquad (t,x,y) \in (0,T] \times (0,1) \times \mathbb{R}^d, \quad (2.46)$$

with forcing term $f = 2$, the boundary conditions

$$u(t,0,y) = 0, \qquad u(t,1,y) = 0, \qquad t \in [0,T], y \in \mathbb{R}^d,$$

and initial condition

$$u(0,x,y) = x - x^2, \qquad (x,y) \in [0,1] \times \mathbb{R}^d.$$

Furthermore assume that the random diffusivity has the form

$$k(x,y) = 1 + \sigma \sum_{k=1}^{d} \frac{1}{k^2\pi^2}(1 + cos(2\pi kx))y_k.$$

where $\sigma > 0$ is a parameter to control the variation level of the diffusive field. Here we choose this fixed form representation so that we can focus on the errors induced by solving the equation. So, in the tests below, we assume that $\sigma = 1$.

We consider the case where the dimension of the random variable is $d = 2$ and the *true* distribution of $Z = (Z_1, Z_2)$ is $Z_1, Z_2 \sim^{iid} Beta(0,1,1,1)$. The numerical strategy and the convergence of it are in the appendix B.

Firstly, we employ $X \in (0,1)^2$ as the encapsulation variable and assume that the posterior distribution for $Z = (Z_1, Z_2)$, Z_1 and Z_2 are independent $beta(0,1,1,1)$. Since $I_X = (0,1)^2$ and $I_Z = (0,1)^2$, it is obviously that there is no truncation at the "tails". Numerical tests on the convergence are produced for both the strong form and the weak form. The error are plotted in Figure 2.7.

Secondly, we employ $X \in (0,\infty)^2$ as the encapsulation variable and assume that the posterior distribution for $Z = (Z_1, Z_2)$, Z_1 and Z_2 are independent $beta(0,1,1,1)$. Since $I_y = (0,1)^2$ and $I_X = (0,\infty)^2$, it is obviously that there is no truncation at the "tails". Numerical tests on the convergence are produced for both the strong form and the weak form. The error are plotted in Figure 2.8.

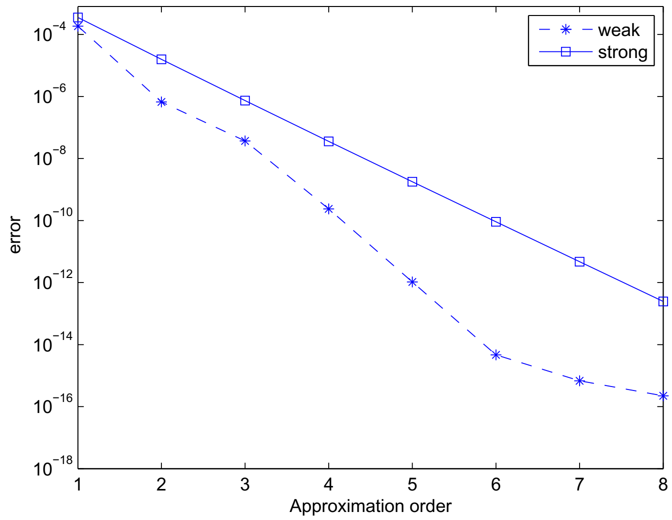


Figure 2.7. Convergence of errors of strong norm, weak norm, where the encapsulation set is $(0,1)^2$ and the exact pdf are $Z_1, Z_2, beta(0,1,1,1)$, and $T = 5.0$. Convergence is with respect to the order of the polynomial expansion.

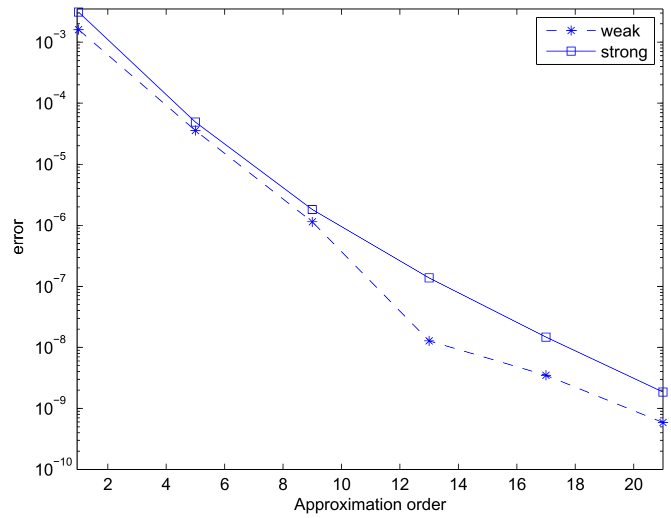


Figure 2.8. Convergence of errors of strong norm, weak norm, where the encapsulation set is $(0,\infty)^2$ and the exact pdf are $Z_1, Z_2, beta(0,1,1,1)$, and $T = 5.0$. Convergence is with respect to the order of the polynomial expansion.

It can be seen that in both cases, both the strong error and the weak error converge at the same rate –exponential rate– as the L_w^2 polynomial approximation error, which are consistent with the error bounds from Theorems 2.2, as in this case the complete encapsulation of Z by X ensures $\delta = 0$. Also, we can see that the convergence rate in the first case is much faster than the second one, because different w result in different constant $C_{r,0}$ in Theorems 2.2.

3. EPISTEMIC UNCERTAINTY QUANTIFICATION USING FUZZY SET THEORY

Here, we will work with one of the more restrictive uncertainty models, i.e., the fuzzy logic. We construct a general numerical approach was proposed for UQ with this uncertainty model. The method consists of three components: (1) range estimation, find the domains of the fuzzy numbers; (2) parameterized representation of the problem and get the numerical solution and (3) fuzzy analysis, for assessment of the accuracy of a numerical fuzzy equation, we define the p-Hausdoreff distance, which naturally leads to the requirement of the accuracy over the parameter domain of the numerical solution to the parametric PDE in L^p-norm. Unlike using the supremum distance [29], which needs the accuracy of the numerical solution in L^∞-norm. This method is more flexible, since it is much easier to get the accuracy in L^p-norm by the existing numerical method, such as Galerkin methods and Finite Element methods.

In section 3.1, we present the basic knowledge of fuzzy set theory and setup the problem we are interested in. In the section 3.2, a numerical method is stated and some proper analysis of the convergence of the method is given. Then, in section 3.3, some examples are presented to numerically demonstrate the convergence of the method.

3.1 Fuzzy Set Theory and Problem Setup

Concepts of vagueness and fuzziness have been contemplated in mathematics and science for quite a long time. For example, in 1923, Bertrand Russell stated that "All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial existence." As for the mathematical basis for formal fuzzy logic, it was first studied by the Polish logician Jan Lukasiewicz in the 1920s, where he constructed a series of multi-valued logical systems, generalizing from small finite numbers of truth-values to those containing infinite sets of truth values. His work and calculation formula are ingrained in modern fuzzy set theory and fuzzy logic. then, in 1937, the philosopher Max Black, who was concerned with vagueness and imprecision in language, and the effect of these concepts on logic, believed that all terms whose application involves using our senses are vague and came up with the concept what called "membership functions". He even conducted a cognitive psychological experiment with a group of people that effectively constructed membership functions exemplifying vagueness of certain words. However, most people considered the beginning of fuzzy set theory to be Zadehs 1965 paper [2], where the concept of fuzzy sets was proposed by L. A. Zadeh [2] in 1965 to represent data and information possessing non-statistical uncertainties. Later in [7], the theory of possibility is related to the theory of fuzzy sets by simply illustrating the concept of a possibility distribution as a fuzzy restriction which acts as an elastic constraint on the values that may be assigned to a variable. An notable achievement of fuzzy sets and fuzzy logic can be found in [30]. In 2005, The journal Fuzzy Sets and Systems published a 40th Anniversary of Fuzzy Sets in December, which contains 14 position papers covering various aspects of the role and future prospects of fuzzy sets.

Next, a brief description of the mathematical basics of the fuzzy set theory is given.

3.1.1 Mathematical Basics of Fuzzy Set theory

In Classical set theory, we learned that all objects can be divided into sets. If a particular object does not lie within a set, then it belongs to the compliment of this set. For example, Consider a set A which contains all the integers, one can affirm without doubt that the number 5 belongs to the natural numbers set N, which is a subset of A, and it can not belong to th set of non-positive integers, which is the compliment of N in A. However, in real life, a high degree of uncertainty exists, and we cannot simply dump objects into discrete classification bins. For example, for the same set A that we defined in the example of the classical set theory, what to say about whether number 3 is in the set of the numbers around 4? Clearly, the answer in this case will depend on the context. We then have various linguistic values of one linguistic variable, which are true to some degree. This degree, subjective as it may be, varies from 0 to 1. The main difference between classical and fuzzy sets is that in the former there is a dichotomy notion that should necessarily be preserved.

In Classical set theory, any classical set could be represented by its characteristic function, that is,

Definition 3.1. Let U be a non-empty set and A a subset of U. The characteristic function of A is given by:

$$A(x) = \begin{cases} 1 & if x \in A \\ 0 & if x \notin A. \end{cases} \tag{3.1}$$

Here, $A(x)$ is a function whose domain is U and the image $\{0, 1\}$. if $A(x) = 1$, this means the element x belongs to subset A, and if $A(x) = 0$, we say that the element x does not belong to A. In this sense, the characteristic function $A : U \to \{0, 1\}$ describes completely what the subset A is. As a generation of the classical set theory, the fuzzy set is defined as follows:

Definition 3.2. Let U be a classical non-empty set and $x \in U$ be an element. A fuzzy set $\tilde{A}$ is defined as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in U\}, \;\; \mu_{\tilde{A}}(x) : U \to [0, 1], \tag{3.2}$$

where $\mu_{\tilde{A}}(x)$ is called membership function.

When $\mu_{\tilde{A}}(x) = 1$, x is considered as a full member of the fuzzy set $\tilde{A}$; when $\mu_{\tilde{A}}(x) = 0$, x is considered as not a member of the fuzzy set $\tilde{A}$.

The membership function $\mu_{\tilde{A}}(x)$ (or denoted as $\tilde{A}(x)$) describes the degree of membership of element x in fuzzy set $\tilde{A}$ for each $x \in U$. While, in classical(crisp) set theory, the membership function only takes values 0 or 1. In this case, one can only know whether an element of the set have or not a particular characteristic, but a ranking of membership is not possible. Here, we use Fig. 3.1 to illustrate the difference between the concept of fuzzy sets and the concept of classical sets.

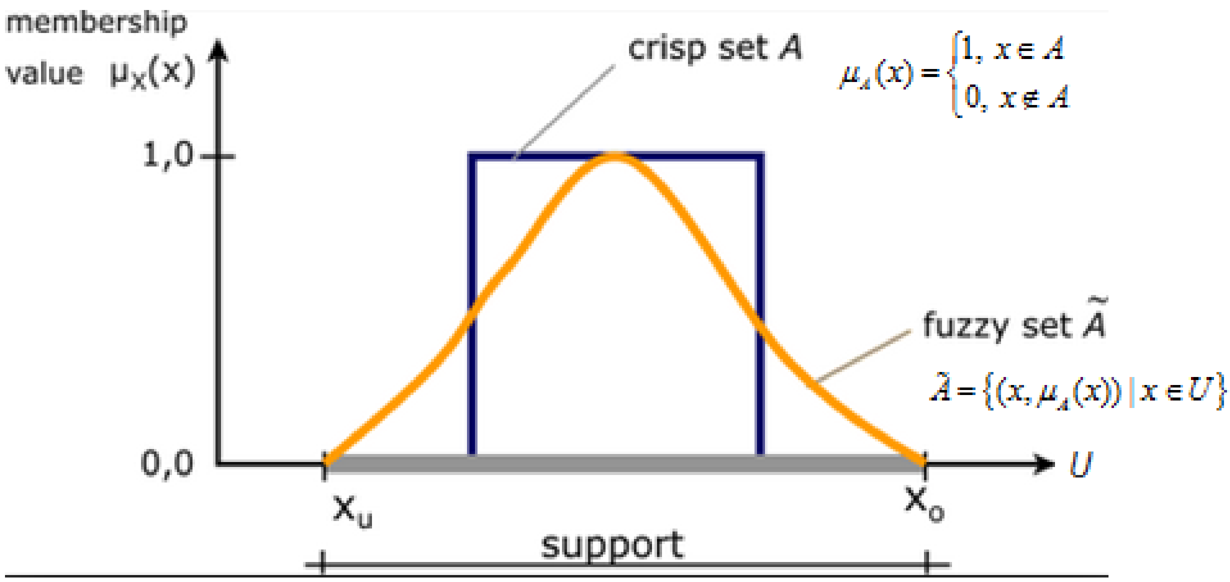


Figure 3.1. Fuzzy and Crisp sets

Conversely, we can interpret the information from a fuzzy set by converting the fuzzy set into classical sets, where we define support, core, α-cut and strong α-cut of a fuzzy set.

Definition 3.3. The support of a fuzzy set $\tilde{A}$, denoted as $Supp(\tilde{A})$, is a classic set whose elements have nonzero membership grade in $\tilde{A}$, i.e.,

$$Supp(\tilde{A}) = \{x \in U | \tilde{A}(x) > 0\}. \tag{3.3}$$

And the core of a fuzzy set $\tilde{A}$, denoted as $Core(\tilde{A})$, is a classic set whose elements have membership grade equal to 1 in $\tilde{A}$, that is

$$Core(\tilde{A}) = \{x \in U | \tilde{A}(x) = 1\}. \tag{3.4}$$

Obviously, if A is a subnormal fuzzy set, then its core is the empty set, $Core(A) = \emptyset$. Core and support definitions are related concepts, since they identify elements belonging to the fuzzy set and $Core(A) \subset Supp(A)$. In order to have a good view of the relations, we use Fig. 3.2 to illustrates the support and core concepts of a fuzzy set.

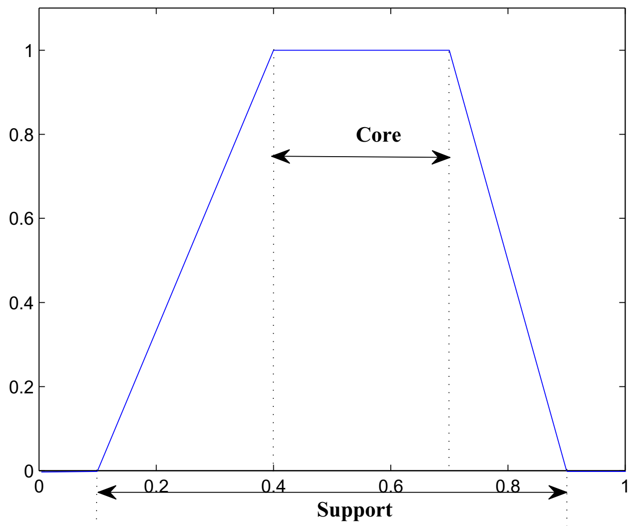


Figure 3.2. Support and core of a fuzzy set

Similarly, we can define the α-cut of a fuzzy set, denoted as $[\tilde{A}]_\alpha$ (or strong α-cut, denoted as $[\tilde{A}]_{\alpha^+}$) consists of elements, which belong to a fuzzy set $\tilde{A}$ with at least (or more than) α degree, i.e.,

$$[\tilde{A}]_\alpha = \{x \in U | \tilde{A}(x) \geq \alpha\},$$
$$[\tilde{A}]_{\alpha^+} = \{x \in U | \tilde{A}(x) > \alpha\}.$$

Furthermore, we have that

Theorem 3.1. *(Second Decomposition Theorem) A fuzzy set $\tilde{A}$ can be decomposed as [31]:*

$$\tilde{A} = \cup_{\alpha \in [0,1]} \alpha \cdot [\tilde{A}]_{\alpha^+}. \tag{3.5}$$

Properties and Operations of Fuzzy sets

Fuzzy sets have several characteristics and properties whose explanation facilitates the understanding and the development of fuzzy models. Bellow, we list those most important and common ones. If you want to know more about it, you can read the book [31].

Definition 3.4. A fuzzy set $\tilde{A}$ is called normal when $\sup_{x \in U} \mu_{\tilde{A}}(x) = 1$. Otherwise, $\tilde{A}$ is called subnormal.

And Fig. 3.3 shows an example of normal and subnormal fuzzy sets.

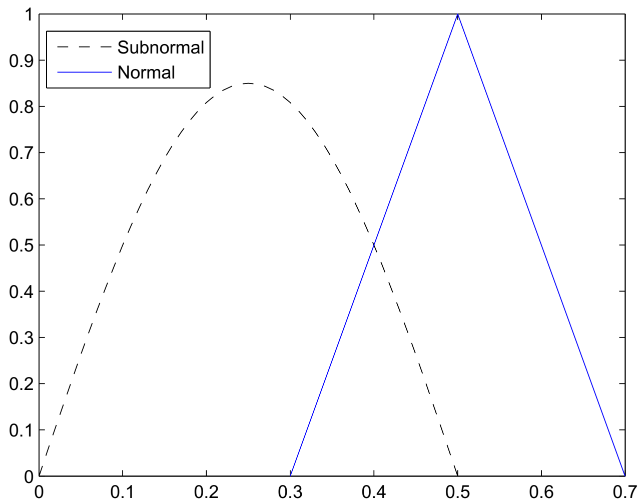


Figure 3.3. Normal fuzzy set and subnormal fuzzy set

Definition 3.5. A fuzzy set $\tilde{A}$ is convex if for any λ in $[0, 1]$, x, y in the support of $\tilde{A}$,

$$\mu_{\tilde{A}}(\lambda x + (1 - \lambda)y) \leq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)). \tag{3.6}$$

Alternatively, $\tilde{A}$ is convex if all its α-cuts are convex.

Analogous to the binary relation between two sets and the operations on sets in classic set theory, the subset, union, intersection, and complement of fuzzy sets are defined as

- Subset $\tilde{A} \subseteq \tilde{B}$ if and only if $\tilde{A}(x) \leq \tilde{B}(x)$ for all $x \in U$.

- Union $(\tilde{A} \cup \tilde{B})(x) = max[\tilde{A}(x), \tilde{B}(x)], \forall x \in U$.

- Intersection $(\tilde{A} \cap \tilde{B})(x) = min[\tilde{A}(x), \tilde{B}(x)], \forall x \in U$.

- Complement $\tilde{A}^c(x) = 1 - \tilde{A}(x), \forall x \in U$.

Let $\tilde{A}$ and $\tilde{B}$ are fuzzy sets with membership functions shown in the left of Fig. 3.1.1, the union, intersection and complement of $\tilde{A}$ and $\tilde{B}$ are illustrated in the right of Fig. 3.1.1.

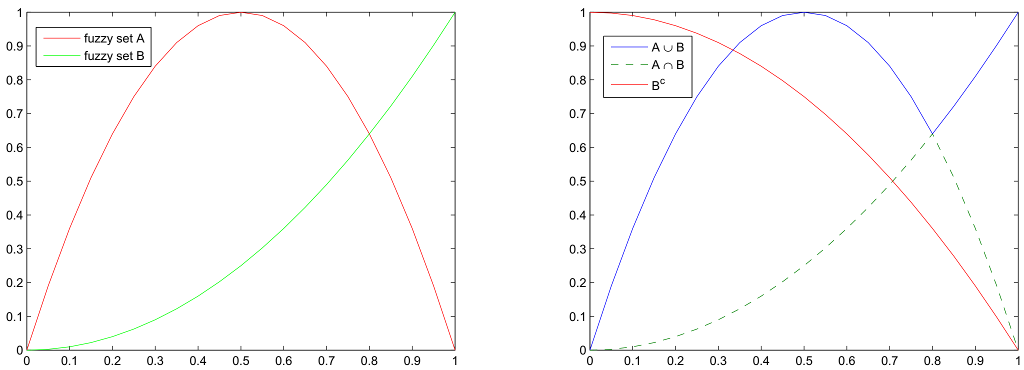


Figure 3.4. Union and intersection of fuzzy sets A and B,complement of fuzzy set B.

Note: The definitions of subset, union, intersection and complement in fuzzy set theory perform exactly as the corresponding ones for classical sets where the values of the membership functions are restricted to either 0 or 1. However, ***the law of excluded middle*** *$(A \cup A^c = U)$ and* ***the law of noncontradiciton*** *$(A \cap A^c = \emptyset)$ hold in classical set theory but not in fuzzy logic.*

Similarly, we can define the Cartesian product of the fuzzy sets, that is,

Definition 3.6. Let $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_N$ be the fuzzy sets with the corresponding membership functions $\mu_{\tilde{A}_1}, \mu_{\tilde{A}_2}, \ldots, \mu_{\tilde{A}_N}$ defined on $U_1, U_2, \ldots, U_N$ respectivly. Let $\tilde{A}$ be the

Cartesian product $\tilde{A}_1 \times \tilde{A}_2, \ldots \times \tilde{A}_N$ and we have $\tilde{A} \subseteq \mathbf{U} = U_1 \times U_2 \times \ldots \times U_N$. The high-dimensional fuzzy set $\tilde{A}$ is characterized by a membership function $\mu_{\tilde{A}}(x_1, x_2, \ldots, x_N)$, for all $x_i \in \tilde{A}_i$, $i = 1, 2, ..., N$. From Zadeh [32], the membership function of $\tilde{A}$ is expressed as

$$\mu_{\tilde{A}} = \min\{\mu_{\tilde{A}_1}, \mu_{\tilde{A}_2}, \ldots, \mu_{\tilde{A}_N}\}. \tag{3.7}$$

Note: Due to the numerical equivalence between the membership function in fuzzy set theory and the possibility distribution function in possibility theory, the expression of the membership function $\mu_{\tilde{A}}$ is the same as joint possibility distribution, which can be calculated using t-norm with the properties of commutativity, monotonicity, associativity [33]. There are various t-norm definitions in the literature. They mainly differ from each other in the way they associate. The minimum operator (3.7) *is one of them and called the min t-norm. The choice of a specific t-norm affects the shape of the joint possibility distribution [33]. Therefore different expressions of $\mu_{\tilde{A}}$ can be obtained. In the current work, the min t-norm is simply used and the choice of t-norm will be explored in future work.*

Extension Principle and Fuzzy Numbers

A special class of fuzzy sets – fuzzy numbers – defined on the set $\mathbb{R}^1$ of real numbers, which captures our intuitive conceptions of approximate numbers, such as "numbers that are close to a given real number," satisfies [31]

- The fuzzy set must be normal;
- The fuzzy number must be convex;
- The support of the fuzzy set must be bounded.

Some examples of fuzzy numbers are show in Fig. 3.5: a triangular, trapezoidal and crisp number respectively, where the latter is actually the fuzzy representation of a classical number. A function of crisp numbers can be extended to the function

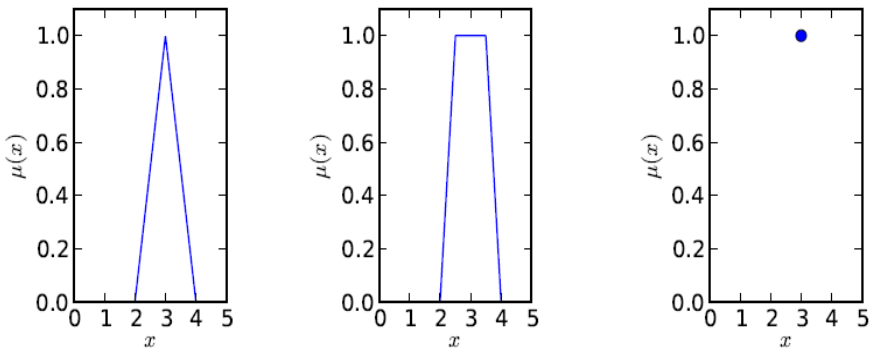


Figure 3.5. Triangular, trapezoidal and crisp number

of fuzzy numbers or more generally, fuzzy sets using Zadeh's extension principle, in other words, a classical map $f : X \to Y$ operating on elements $\vec{x} \in X$ can be extended towards a map $f : \mathcal{F}(X) \to \mathcal{F}(Y)$, operating on fuzzy sets $\tilde{A} \in \mathcal{F}(X)$, where $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are the collections of all the fuzzy sets defined on X and Y respectively.

Definition 3.7. Let $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_N$ be the fuzzy sets with the corresponding membership functions $\mu_{\tilde{A}_1}, \mu_{\tilde{A}_2}, \dots, \mu_{\tilde{A}_N}$ defined on $X_1, X_2, \dots, X_N$ respectively, and X be the Cartesian product $X = X_1 \times X_2, \dots, \times X_N$. If f is a mapping from X to Y, i.e., $y = f(x_1, x_2, ..., x_N)$, then the extension principle allow us to define a fuzzy set $\tilde{B}$ in $\mathcal{F}(Y)$ by [34]

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x_1, x_2, ..., x_N), (x_1, x_2, ..., x_N) \in X\},$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,x_2,\dots,x_N) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), \mu_{\tilde{A}_2}(x_2), ..., \mu_{\tilde{A}_N}(x_N)\} & f^{-1}(y) \neq \emptyset, \\ 0 & f^{-1}(y) = \emptyset, \end{cases} \tag{3.8}$$

where f^{-1} is the inverse of f.

For $N = 1$, the extension principle reduces to

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\},$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x) \in f^{-1}(y)} \mu_{\tilde{A}}(x) & f^{-1}(y) \neq \emptyset, \\ 0 & f^{-1}(y) = \emptyset, \end{cases} \tag{3.9}$$

Theorem 3.2. *If $f : X \to Y$ be an arbitrary crisp function, then for any $\tilde{A} \in \mathcal{F}(X)$, the following property of f fuzzified by the extension principle holds [31]:*

$$[f(\tilde{A})]_{\alpha^+} = f([\tilde{A}]_{\alpha^+}), \ \forall \alpha \in [0, 1]. \tag{3.10}$$

Distance between Two Fuzzy Sets

In order to assess the numerical error of the system of partial differential equations involving fuzzy sets, a distance measure for fuzzy sets is required. We first define a modified Hausdorff distance to measure the difference between two classical sets.

Definition 3.8. Let $\tilde{A}$ be a fuzzy set with membership function $\mu_{\tilde{A}}(s)$, and $X = f([\tilde{A}]_{\alpha^+})$, $Y = g([\tilde{A}]_{\alpha^+})$ be two non-empty classical sets, the p-Hausdorff distance d_{H^p} between X and Y is defined as

$$\begin{aligned} &d_{H^p}(X,Y)\\ &= \max\{(\int_{s\in[\tilde{A}]_{\alpha^+}} \operatorname*{essinf}_{r\in[\tilde{A}]_{\alpha^+}} d^p(f(s),g(r))ds)^{\frac{1}{p}}, (\int_{r\in[\tilde{A}]_{\alpha^+}} \operatorname*{essinf}_{s\in[\tilde{A}]_{\alpha^+}} d^p(f(s),g(r))dr)^{\frac{1}{p}}\}, \end{aligned} \tag{3.11}$$

where $d^p(f(s), g(r))$ is a classical distance measure. We define $d_{H^p}(X, Y) = 0$ when $X = \emptyset$ or $Y = \emptyset$.

Theorem 3.3. *Let $\tilde{A}$ be a fuzzy set with membership function $\mu_{\tilde{A}}(s)$ and with support S, if $\tilde{U} = f(\tilde{A})$ and $\tilde{V} = g(\tilde{A})$, the p-Hausdorff distance d_{H^p} between any strong α-cut of $\tilde{U}$ and $\tilde{V}$ is bounded from above as*

$$d_{H^p}([\tilde{U}]_{\alpha^+}, [\tilde{V}]_{\alpha^+}) \le \left(\int_{s\in S} d^p(f(s),g(s))ds\right)^{\frac{1}{p}}. \tag{3.12}$$

Proof.

$$\begin{aligned} &d_{H^p}([\tilde{U}]_{\alpha^+}, [\tilde{V}]_{\alpha^+}),\\ &= d_{H^p}([f(\tilde{A})]_{\alpha^+}, [g(\tilde{A})]_{\alpha^+}),\\ &= d_{H^p}(f([\tilde{A}]_{\alpha^+}), g([\tilde{A}]_{\alpha^+})), \quad \text{using Thm 3.2}\\ &= \max\{(\int_{s\in[\tilde{A}]_{\alpha}^{+}} \operatorname*{essinf}_{r\in[\tilde{A}]_{\alpha}^{+}} d^p(f(s),g(r))ds)^{\frac{1}{p}}, (\int_{r\in[\tilde{A}]_{\alpha^+}} \operatorname*{essinf}_{s\in[\tilde{A}]_{\alpha^+}} d^p(f(s),g(r))dr)^{\frac{1}{p}}\},\\ &\le \max\{(\int_{s\in[\tilde{A}]_{\alpha^+}\setminus A_1} d^p(f(s),g(s))ds)^{\frac{1}{p}}, (\int_{r\in[\tilde{A}]_{\alpha^+}\setminus A_2} d^p(f(r),g(r))dr)^{\frac{1}{p}}\},\\ &\le (\int_{s\in[\tilde{S}]_{\alpha^+}} d^p(f(s),g(s))ds)^{\frac{1}{p}},\\ &\le (\int_{s\in S} d^p(f(s),g(s))ds)^{\frac{1}{p}}, \end{aligned} \tag{3.13}$$

where A_1 and A_2 are measure 0 subsets in $[\tilde{A}]_{\alpha^+}$. □

Corollary. *If the classical distance measure is defined as* $d^p(f(s), g(s)) = \|f(s) - g(s)\|_{L^p_w}$*, then*

$$d_{H^p}([\tilde{U}]_{\alpha^+}, [\tilde{V}]_{\alpha^+}) \leq C^{1/p}\|f(s) - g(s)\|_{L^p_w}, \tag{3.14}$$

where $C = \max_{I_\xi} \frac{1}{w(\xi)}$ *is a positive real number.*

Proof.

$$\begin{aligned}
&d_{H^p}([\tilde{U}]_{\alpha^+}, [\tilde{V}]_{\alpha^+}) \\
&\leq \left(\int_{s\in S} d^p(f(s), g(s))ds\right)^{\frac{1}{p}} \\
&= \left(\int_{s\in S} d^p(f(s), g(s))w(s)\frac{1}{w(s)}ds\right)^{\frac{1}{p}} \\
&\leq \left(\max_{s\in S} \frac{1}{w(s)}\right)^{\frac{1}{p}}\left(\int_{s\in S} d^p(f(s), g(s))ds\right)^{\frac{1}{p}} \\
&= C^{1/p}\|f(s) - g(s)\|_{L^p_w}.
\end{aligned}$$

□

Weighted Expected Value and Variance

The concept of membership function in the fuzzy set theory can be interpreted as the possibility distribution in possibility theory [7]. In possibility theory, the variable x, analogous to a random variable in probability theory, takes on values from the set U containing the true value. The true value of x is unknown and we have a possibility distribution $\mu_{\tilde{A}}(x)$, which describes the degree to which it is possible that the elements $x \in U$ is the true value of x. In this section, we will use the concept of average values of well-chosen real-valued function on α-level sets of the memebership function to obtain the expected value and the variance of a function of a fuzzy set.

Let $\tilde{B}$ be a joint possibility distribution in $\mathbb{R}^n$, let $\alpha \in [0, 1]$ and $g : \mathbb{R}^n \to \mathbb{R}$ be a function. It is well-known from analysis that the average value of function g on $[\tilde{B}]_\alpha$ is defined by

$$\begin{aligned}
\mathcal{C}_{[\tilde{B}]_\alpha}(g) &= \frac{1}{\int_{[\tilde{B}]_\alpha} dx} \int_{[\tilde{B}]_\alpha} g(x)dx \\
&= \frac{1}{\int_{[\tilde{B}]_\alpha} dx_1 \ldots dx_n} \int_{[\tilde{B}]_\alpha} g(x_1, \ldots, x_n)dx_1 \ldots dx_n
\end{aligned}. \tag{3.15}$$

Following [35], we call $\mathcal{C}$ as the central value operator.

Note: If $[\tilde{B}_\alpha]$ *is degenerated for some* $\alpha \in [0,1]$, *that is*

$$\int_{[\tilde{B}]_\alpha} dx = 0, \tag{3.16}$$

then we approximate $[\tilde{B}]_\alpha$ *with non-degenerated sets and consider the limit case. Namely, let* $\{S_k\}$ *be a sequence of sets such that* $[\tilde{B}]_\alpha \subset S_k \subset \mathbb{R}^n$, $0 < \int_{S_k} dx < \infty$, *and*

$$\lim_{k\to\infty} sup\{\|x-s\| | x \in [\tilde{B}]_\alpha, s \in S_k\} = 0.$$

then,

$$\mathcal{C}_{[\tilde{B}]_\alpha}(g) = \lim_{k\to\infty} \mathcal{C}_{S_k}(g) = \lim_{k\to\infty} \frac{\int_{S_k} g(x)dx}{\int_{S_k} dx}.$$

Definition 3.9. A function $f : [0,1] \to \mathbb{R}$ is said to be a weighting function if f is non-negative, monotone increasing and satisfies the following normalization condition

$$\int_0^1 f(\alpha)d\alpha = 1. \tag{3.17}$$

Different weighting functions can give different (case-dependent) importances to α-levels sets of fuzzy numbers. We can notice that f in monotone increasing, in this way, this gives less importance to the lower levels of fuzzy sets, which is coincided with our sense to acknowledgment.

And the expected value of function g on fuzzy set $\tilde{A}$ with respect to a weighting function f was defined by

$$\begin{aligned} \mathbb{E}_f(g;\tilde{A}) &= \int_0^1 \mathcal{C}_{[\tilde{A}]_\alpha}(g) f(\alpha)d\alpha \\ &= \int_0^1 \frac{1}{\int_{[\tilde{A}]_\alpha} dx} \int_{[\tilde{A}]_\alpha} g(x)dx f(\alpha)d\alpha \end{aligned} . \tag{3.18}$$

Especially, if $g(x) = x$, for all $x \in \mathbb{R}$ is the identity function then we get

$$\mathbb{E}_f(id;\tilde{A}) = \mathbb{E}_f(id;\tilde{A}) = \int_0^1 \frac{a_1(\alpha) + a_2(\alpha)}{2} f(\alpha)d\alpha, \tag{3.19}$$

which is the f-weighted possibilistic mean value of $\tilde{A}$ introduced in [36].

Note: The expected value of a function on a fuzzy number $\tilde{A}$ is nothing else but the expected value of its average values on all α level sets of $\tilde{A}$.

Similarly, the variance of function g on fuzzy set $\tilde{A}$ with respect to a weighting function f can be considered as the expected value of function $gg(x) = (g(x) - \mathcal{C}_{[\tilde{A}_\alpha]}(g))^2$, that is

$$\begin{aligned} Var_f(g;\tilde{A}) &= \mathbb{E}_f(gg;\tilde{A}) \\ &= \int_0^1 \mathcal{C}_{[\tilde{A}]_\alpha}(gg)f(\alpha)d\alpha \\ &= \int_0^1 \frac{1}{\int_{[\tilde{A}]_\alpha} dA} \int_{[\tilde{A}]_\alpha}(g(x) - \mathcal{C}_{[\tilde{A}]_\alpha}(g))^2 dx f(\alpha)d\alpha \end{aligned} \quad . \tag{3.20}$$

Again, if $g(x) = x$, for all $x \in \mathbb{R}$ is the identity function then we get the f-weighted possibilistic variance of the fuzzy number $\tilde{A}$ in [36],

$$Var_f(\tilde{A}) = \int_0^1 \frac{(a_1(\alpha) - a_2(\alpha))^2}{2} f(\alpha)d\alpha. \tag{3.21}$$

3.1.2 Problem Setup

Let $D \subset \mathbb{R}^l, l = 1,2,3,$ be a physical domain with coordinates $x = (x_1,\ldots,x_l)$, let $(0,T]$ be a time domain with $T > 0$, and let $\Xi \subset \mathbb{R}^n$ be a parameter domain for uncertain inputs. We consider a general partial differential equation (PDE) as

$$\begin{cases} u_t(x,t,\xi) = L(u), & D \times (0,T] \times \Xi, \\ \mathcal{B}(u) = 0, & \partial D \times [0,T], \\ u = u_0, & D \times \{t=0\} \times \Xi, \end{cases} \tag{3.22}$$

where L is a (linear or nonlinear) differential operator, $\mathcal{B}$ is the boundary condition operator, u_0 is the initial condition, and $\xi =$
$xi_1,\ldots,\xi_n\}$ is a set of uncertain parameters characterizing the uncertainty in the inputs of the governing equation. We assume that the problem (4.13) is well-posed in Ξ and let $u(x,t,\tilde{\xi})$: $D \times (0,T] \times \Xi \to \mathbb{R}^{n_u}$ denote its solution. For simplicity, we further assume that the output of (4.13) is in one dimension, i.e., $n_u = 1$.

In the situation where the uncertain parameter ξ is random and the associated probability density function (PDF: $P_\xi(s) = P(\xi \leq s)$) is known, the standard probabilistic formulation is adopted and the statistics of the stochastic quantity $u(x, t, \xi)$ are calculated. In the situation where ξ is random however the PDF is not completely known, one can adopt a three-step procedure [37] – estimating the range of ξ, solving the system of equations (4.13) within the estimated range numerically, and then post-processing the obtained solution when the PDF becomes available – to calculate the statistics of $u(x, t, \xi)$. In this paper, we extend the three-step procedure to the situation where the uncertain parameter ξ or some components of ξ are not random but associated with vague and imprecise information. In such case, the epistemic or mixed types of uncertainty in the parameter ξ and the propagated uncertainty in the output or the statistics of the output $u(x, t, \xi)$ are represented mathematically in the framework of fuzzy set theory.

3.2 Methodology

Similar to the work of [37], we use a three-step procedure to quantify the uncertainty in the output of the system (4.13), where the input parameters are random variable associated with probability density functions or fuzzy numbers associated with membership functions. The procedure comprises identifying the ranges of the uncertain inputs, generating the accurate numerical solution for the system (4.13) within the estimated ranges, and analyzing the uncertainty in output using Zadeh's extensive principle in the framework of fuzzy set theory. Hereafter, to be clear, we use $\tilde{\xi}$, $\tilde{u}$ to denote fuzzy sets while ξ, u are classical variables.

3.2.1 Range Estimation

The first task is to identify a range, or a bound for the uncertain parameters, which is sufficiently large such that it can cover all the possible cases we are interested in.

For the uncertain parameters characterized by fuzzy sets, their supports serve the purpose. For each variable $\tilde{\xi}_i$, $i = 1, \ldots, d$, let $Supp(\tilde{\xi}_i) = [a_i, b_i]$ be its range. Denote $I_\xi = \times_{i=1}^{d}[a_i, b_i]$, we have $I_\xi \subseteq \Xi$. According to the definition of the fuzzy number, I_ξ is closed and bounded.

3.2.2 Numerical Solution of System of PDE

Once the ranges of the parameters are estimated, traditional numerical methods can be used to solve the system of equations (4.13) in the domain I_ξ and the approximated solution is denoted as $u_n(x, t, \xi)$, where the index n is associated with the discretization parameters in the approximation. The error of the numerical solution defined as

$$\epsilon_n = \|u_n(x, t, \xi) - u(x, t, \xi)\|_{L^p_w(I_\xi)}, \qquad p \geq 1, \tag{3.23}$$

approaches zero as $n \to \infty$, where $u(x, t, \xi)$ is the exact solution of Eqs. (4.13), $\omega = \omega(\xi)$ is a weight function. In the current work, we use Galerkin method to approximate the solution of (4.13) in polynomial space.

3.2.3 Epistemic Uncertainty Quantification in Output

In this section, we take into account the uncertainty in the inputs $\tilde{\xi}$ and quantify the consequent uncertainty in output u_n using fuzzy set theory, i.e., we obtain the membership function for the fuzzy output $\tilde{u}_n(x, t, \tilde{\xi})$. With any fixed x and t, for simplicity, we write $u_n(x, t, \xi)$ as $u_n(\xi) : I_\xi \to \mathbb{R}$. Using Zadeh's extension principle, one can then calculate the membership function of $u_n(\tilde{\xi})$ using Eq. (3.8). We then denote $u_n(\tilde{\xi})$ at x and t as $\tilde{u}_n(x, t, \tilde{\xi})$.

To show that the true quantity of our interest $\tilde{u}(x, t, \tilde{\xi})$ from the Eq. (4.13) can be approximated by $\tilde{u}_n(x, t, \tilde{\xi})$, according to the second decomposition theorm of fuzzy sets Thm. 3.1 (i.e., $\tilde{u} = \cup_{\alpha\in[0,1]}\alpha\cdot[\tilde{u}]_{\alpha^+}$, $\tilde{u_n} = \cup_{\alpha\in[0,1]}\alpha\cdot[\tilde{u}_n]_{\alpha^+}$), we only need to show for all $\alpha\in[0,1]$, $[\tilde{u}_n]_{\alpha^+}$ approaches $[\tilde{u}]_{\alpha^+}$ as $n\to\infty$. According to the Cor. 3.1.1,

$$d_{H^p}([\tilde{u}]_{\alpha^+},[\tilde{u}_n]_{\alpha^+}) \le C^{1/p}\|u(\xi)-u_n(\xi)\|_{L^p_w} = C^{1/p}\epsilon_n. \tag{3.24}$$

Similarly, we can achieve the convergence of the weighted expected value and variance once we have the L^p convergence. And the proof are as follows:

Theorem 3.4. *(Convergence of Expected Value) Let $u_n(\tilde{\xi})$ be an approximation to the solution $u(\tilde{\xi})$ of equation* (4.13)*, and we have that*

$$\epsilon_n = \|u_n(\xi)-u(\xi)\|_{L^p_w(I_\xi)}, \qquad p\ge 1, \tag{3.25}$$

approaches to 0 *as $n\to\infty$, where $w(\xi)>0$ is a weighted function. Let q be a real positive number satisfied that $1/p+1/q=1$, denote that*

$$C_1 = \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p}d\xi)^{1/q}}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha)d\alpha, \tag{3.26}$$

and if C_1 satisfies $C_1<\infty$, then, the expected value of u and u_n satisfy

$$|\mathbb{E}_f(u-u_n;\tilde{\xi})| = |\mathbb{E}_f(u;\tilde{\xi})-\mathbb{E}_f(u_n;\tilde{\xi})| \le C_1\epsilon_n. \tag{3.27}$$

Proof. By the definition of the expected value in equation (3.18),

$$\begin{aligned} |\mathbb{E}_f(u;\tilde{\xi})-\mathbb{E}_f(u_n;\tilde{\xi})| &= |\int_0^1 \mathcal{C}_{[\tilde{\xi}]_\alpha}(u)f(\alpha)d\alpha - \int_0^1 \mathcal{C}_{[\tilde{\xi}]_\alpha}(u_n)f(\alpha)d\alpha| \\ &\le \int_0^1 |\mathcal{C}_{[\tilde{\xi}]_\alpha}(u)\mathcal{C}_{[\tilde{\xi}]_\alpha}(u_n)|f(\alpha)d\alpha \\ &= \int_0^1 \frac{1}{\int_{[\tilde{\xi}]_\alpha}d\xi}\int_{[\tilde{\xi}]_\alpha}|u-u_n|d\xi f(\alpha)d\alpha \end{aligned}. \tag{3.28}$$

Apply Holder's inequality, we have that

$$\int_{[\tilde{\xi}]_\alpha}|u-u_n|d\xi \le (\int_{[\tilde{\xi}]_\alpha}|u-u_n|^p w(\xi)d\xi)^{1/p}(\int_{[\tilde{\xi}]_\alpha} w^{-q/p}d\xi)^{1/q}. \tag{3.29}$$

Substitute (3.29) into (3.28),

$$\begin{aligned} |\mathbb{E}_f(u;\tilde{\xi})-\mathbb{E}_f(u_n;\tilde{\xi})| &\le \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p}d\xi)^{1/q}}{\int_{[\tilde{\xi}]_\alpha}d\xi}(\int_{[\tilde{\xi}]_\alpha}(|u-u_n|)^p w(\xi)d\xi)^{1/p}f(\alpha)d\alpha \\ &\le \epsilon_n\int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p}d\xi)^{1/q}}{\int_{[\tilde{\xi}]_\alpha}d\xi}f(\alpha)d\alpha \\ &= C_1\epsilon_n \end{aligned}. \tag{3.30}$$

□

Theorem 3.5. *(Convergence of Variance) Let u and u_n be the same as in theorem 3.4 and $p \geq 2$ denote*

$$C_2 = \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q^*/p} d\xi)^{1/q^*}}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha) d\alpha, \tag{3.31}$$

where q^* satisfies $\frac{2}{p} + \frac{1}{q^*} = 1$. And

$$C_3 = \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p} d\xi)^{2/q}}{(\int_{[\tilde{\xi}]_\alpha} d\xi)^2} f(\alpha) d\alpha, \tag{3.32}$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $C_2 < \infty$, $C_3 < \infty$, then, we have

$$Var_f(u - u_n; \tilde{\xi}) \leq 2(C_2 \epsilon_n + C_3 \epsilon_n^2). \tag{3.33}$$

Note: if $p = 2$, C_2 is denoted as $\max_{\xi \in I_\xi} \frac{1}{w(\xi)} \int_0^1 \frac{f(\alpha)}{\int_{[\tilde{\xi}]_\alpha} d\xi} d\alpha$.

Proof. By the definition (3.20) and triangle inequality, we can separate the left hand side of the equation (3.33) into two parts, that is

$$\begin{aligned} & Var_f(u - u_n; \tilde{\xi}) \\ & = \int_0^1 \frac{1}{\int_{[\tilde{\xi}]_\alpha} d\xi} \int_{[\tilde{\xi}]_\alpha} (u(\xi) - u_n(\xi) - \mathcal{C}_{[\tilde{\xi}]_\alpha}(u - u_n))^2 d\xi f(\alpha) d\alpha | \\ & \leq 2 \int_0^1 \frac{\int_{[\tilde{\xi}]_\alpha} (u - u_n)^2 d\xi}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha) d\alpha + 2 \int_0^1 \frac{\int_{[\tilde{\xi}]_\alpha} (\mathcal{C}_{[\tilde{\xi}]_\alpha}(u) - \mathcal{C}_{[\tilde{\xi}]_\alpha}(u_n))^2 d\xi}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha) d\alpha \\ & = 2 \int_0^1 \frac{\int_{[\tilde{\xi}]_\alpha} (u - u_n)^2 d\xi}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha) d\alpha + 2 \int_0^1 (\mathcal{C}_{[\tilde{\xi}]_\alpha}(u) - \mathcal{C}_{[\tilde{\xi}]_\alpha}(u_n))^2 f(\alpha) d\alpha \end{aligned} . \tag{3.34}$$

For the first part, we have

$$\begin{aligned} & \int_0^1 \frac{1}{\int_{[\tilde{\xi}]_\alpha} d\xi} \int_{[\tilde{\xi}]_\alpha} (u - u_n)^2 d\xi f(\alpha) d\alpha \\ & \leq \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q^*/p} d\xi)^{1/q^*}}{\int_{[\tilde{\xi}]_\alpha} d\xi} (\int_{[\tilde{\xi}]_\alpha} (|u - u_n|)^p w(\xi) d\xi)^{1/p} f(\alpha) d\alpha \\ & \leq \epsilon_n \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q^*/p} d\xi)^{1/q^*}}{\int_{[\tilde{\xi}]_\alpha} d\xi} f(\alpha) d\alpha \\ & = C_3 \epsilon_n \end{aligned} , \tag{3.35}$$

and for the second part, we can obtain

$$\begin{aligned}
&\int_0^1 (\mathcal{C}_{[\tilde{\xi}]_\alpha}(u) - \mathcal{C}_{[\tilde{\xi}]_\alpha}(u_n))^2 f(\alpha) d\alpha \\
&\leq \int_0^1 \frac{1}{(\int_{[\tilde{\xi}]_\alpha} d\xi)^2} \int_{[\tilde{\xi}]_\alpha} (\int_{[\tilde{\xi}]_\alpha} (u - u_n) d\xi)^2 d\xi f(\alpha) d\alpha \\
&\leq \int_0^1 \frac{1}{(\int_{[\tilde{\xi}]_\alpha} d\xi)^2} (\int_{[\tilde{\xi}]_\alpha} (|u - u_n|)^p w(\xi) d\xi)^{2/p} (\int_{[\tilde{\xi}]_\alpha} w^{-q/p} d\xi)^{2/q} f(\alpha) d\alpha \\
&\leq \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p} d\xi)^{2/q}}{(\int_{[\tilde{\xi}]_\alpha} d\xi)^2} (\int_{[\tilde{\xi}]_\alpha} (|u - u_n|)^p w(\xi) d\xi)^{2/p} f(\alpha) d\alpha \\
&\leq \epsilon_n^2 \int_0^1 \frac{(\int_{[\tilde{\xi}]_\alpha} w^{-q/p} d\xi)^{2/q}}{(\int_{[\tilde{\xi}]_\alpha} d\xi)^2} f(\alpha) d\alpha \\
&= C_3 \epsilon_n^2
\end{aligned} . \tag{3.36}$$

Adding the two parts, the conclusion holds. □

3.2.4 Mixed Aleatory and Epistemic Uncertainty Quantification in Output

In reality, there might be situations where both aleatory and epistemic uncertainty involve in the model inputs, therefore it is necessary to develop approaches for mixed types of uncertainty quantification in output. Suppose probability density functions (PDFs) are known for the set of random variables $Z = \{Z_1, Z_2, ..., Z_m\}$ ($Z \in I_Z \subseteq \mathcal{R}^m$, $m \geq 1$, $\int_{I_Z} \rho_Z dZ = 1$), while membership functions are associated with the set of non-probabilistic uncertain parameters $\tilde{\xi} = \{\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_d\}$. The proposed approach can be easily extended to such a case involving mixed aleatory and epistemic uncertainty.

Let us consider the following system

$$\begin{cases} u_t(x, t, Z, \tilde{\xi}) = L(u), & D \times (0, T] \times I_Z \times \Xi, \\ \mathcal{B}(u) = 0, & \partial D \times [0, T], \\ u = u_0, & D \times \{t = 0\} \times I_Z \times \Xi. \end{cases} \tag{3.37}$$

For a fixed x and t, the system of equations can be solved numerically in the range $I_Z \times I_\xi$, where I_ξ is the Cartesian product of the supports of the fuzzy numbers $\tilde{\xi}_i$s.

Let u_n denote this numerical solution where n is the approximation order. The error of the numerical solution is defined as

$$\begin{aligned}\epsilon_n &= \|u_n(Z,\xi) - u(Z,\xi)\|_{L^p_\rho(I_Z)\otimes L^p_w(I_\xi)}, \qquad p \geq 1, \\ &= \left(\int_{I_\xi}\left(\int_{Z\in I_Z} |u(Z,\xi) - u_n(Z,\xi)|^p \rho_Z dZ\right)\omega d\xi\right)^{(1/p)},\end{aligned}$$

where $u(Z,\xi)$ is the exact solution of the Eq. (3.37) in the range $I_Z \times I_\xi$.

Proposition 3.6. *Define the conditional expectations as*

$$\begin{aligned}\mu(u(Z,\xi)|\xi) &= \int_{Z\in I_Z} u(Z,\xi)\rho_Z dZ; \\ \mu(u_n(Z,\xi)|\xi) &= \int_{Z\in I_Z} u_n(Z,\xi)\rho_Z dZ.\end{aligned}$$

Then

$$\|\mu(u(Z,\xi)|\xi) - \mu(u_n(Z,\xi)|\xi)\|_{L^p_w(I_\xi)} \leq \epsilon_n.$$

Proof.

$$\begin{aligned}&\|\mu(u|\xi) - \mu(u_n|\xi)\|_{L^p_w(I_\xi)}, \\ &= \left(\textstyle\int_{I_\xi} |\mu(u|\xi) - \mu(u_n|\xi)|^p \omega d\xi\right)^{(1/p)}, \\ &= \left(\textstyle\int_{I_\xi} |\int_{Z\in I_Z}(u(Z,\xi) - u_n(Z,\xi))\rho_Z dZ|^p \omega d\xi\right)^{(1/p)}, \\ &= \left(\textstyle\int_{I_\xi} |\int_{Z\in I_Z}(u(Z,\xi) - u_n(Z,\xi))(\rho_Z)^{1/p}(\rho_Z)^{(p-1)/p} dZ|^p \omega d\xi\right)^{(1/p)}, \\ &\leq \left(\textstyle\int_{I_\xi}(\int_{Z\in I_Z} |u(Z,\xi) - u_n(Z,\xi)|^p \rho_Z dZ)(\int_{Z\in I_Z}(\rho_Z^{(p-1)/p})^q dZ)^{p/q} \omega d\xi\right)^{(1/p)}, \\ &= \left(\textstyle\int_{Z\in I_Z}(\rho_Z^{(p-1)/p})^q dZ\right)^{1/q}\left(\textstyle\int_{I_\xi}(\int_{Z\in I_Z} |u(Z,\xi) - u_n(Z,\xi)|^p \rho_Z dZ)\omega d\xi\right)^{(1/p)},\end{aligned}$$

where $1/p + 1/q = 1$.

We have $\left(\int_{Z\in I_Z}(\rho_Z^{(p-1)/p})^q dZ\right)^{1/q} = 1$ since $(p-1)/p = 1/q$ and $\int_{Z\in I_Z} \rho_Z dZ = 1$. Therefore the following inequality holds,

$$\|\mu(u|\xi) - \mu(u_n|\xi)\|_{L^p_w(I_\xi)} \leq \epsilon_n.$$

□

Due to the epistemic uncertainty in the fuzzy number $\tilde{\xi}$, the conditional expectations $\mu(u(Z,\tilde{\xi})|\tilde{\xi})$ and $\mu(u_n(Z,\tilde{\xi})|\tilde{\xi})$ are not deterministic, but we can calculate the associated membership functions using Zadeh's extension principle. Denote $\tilde{\mu}(\tilde{\xi}) = \mu(u(Z,\tilde{\xi})|\tilde{\xi})$ and $\tilde{\mu}_n(\tilde{\xi}) = \mu(u_n(Z,\tilde{\xi})|\tilde{\xi})$, the error in the conditional expectation introduced by the numerical procedure is measured by the p-Hausdorff distance between the strong α-cut of the two fuzzy sets $\tilde{\mu}(\tilde{\xi})$ and $\tilde{\mu}_n(\tilde{\xi})$ for any $\alpha \in [0,1]$. Using the Cor. 3.1.1 and the , we have

$$d_{H^p}([\tilde{\mu}]_{\alpha^+},[\tilde{\mu}_n]_{\alpha^+}) \leq C^{1/p}\|\mu(u(Z,\xi)|\xi) - \mu(u_n(Z,\xi)|\xi)\|_{L^p_w(I_\xi)} \leq C^{1/p}\epsilon_n.$$

Naturally, we can get the convergence of the weighted expected value and variance of μ and μ_n by using Theorem 3.4, 3.5 and Prop. 3.6.

$$|\mathbb{E}_f(\mu-\mu_n;\tilde{\xi})| = |\mathbb{E}_f(\mu;\tilde{\xi}) - \mathbb{E}_f(\mu_n;\tilde{\xi})| \leq C_1\epsilon_n, \tag{3.38}$$

$$Var_f(\mu-\mu_n;\tilde{\xi}) \leq 2(C_2\epsilon_n + C_3\epsilon_n^2), \tag{3.39}$$

where C_1, C_2 and C_3 have the same value as their definition in Theorem 3.4 and 3.5. Since the proof process is the same as in the proof process of Theorem 3.4, 3.5, which only needs to substitute u by μ, and u_n by μ_n, respectively. We will not repeat it here.

3.3 Numerical Examples

The proposed procedure of analyzing the uncertainty in the output to differential equations is illustrated in numerical examples. The focus are on examination of the membership function behavior and error behavior. When associated with possibility theory, where the membership function defined here can be interpreted as a possibility distribution. the behavior of weighted expected value and variance and their error behavior are naturally become our interest. All the behaviors are tested here.

We first test our procedure on a linear ordinary differential equation with single fuzzy variable. Though simple, this examples allows us to thoroughly examine the the membership function, the weighted expected value and variance behaviors and convergence properties of the numerical implementations. The second example is a non-linear equation with two fuzzy numbers as uncertain inputs. The membership function, the weighted expected value and variance of the uncertain output are provided and the convergence rate with respect to the approximation order is analyzed. Then we test the proposed numerical approach for mixed types of uncertainty quantification in a diffusion problem with both random variables and fuzzy numbers as uncertain inputs.

3.3.1 Ordinary Differential Equation

Let us consider

$$\frac{\partial u(t,\tilde{\xi})}{\partial t} = (1+\sigma\tilde{\xi})u, \qquad u(0,\tilde{\xi}) = 1, \tag{3.40}$$

where $\sigma = 0.1$ and $\tilde{\xi}$ is a one-dimensional fuzzy variable. Let $\tilde{u}(t)$ denote the unknown solution. We consider three different membership functions (see Fig. 3.6) for $\tilde{\xi}$:

$$\begin{aligned}
\mu_1(\xi) &= 0.5*(1+\xi); \\
\mu_2(\xi) &= (0.5*(1+\xi))^{0.5}; \\
\mu_3(\xi) &= \begin{cases} 1+\xi & -1 \le \xi \le 0, \\ 1-\xi & 0 \le \xi \le 1. \end{cases}
\end{aligned} \tag{3.41}$$

The proposed three-step procedure is applied for uncertainty propagation as follows. Firstly, we estimate the range of the uncertain parameter. Due to the definition, the support of $\tilde{\xi}$ is a compact set ($Supp(\tilde{\xi}) = [-1,1]$ in the example) and it is used as the estimated range.

Next, we replace the fuzzy set $\tilde{\xi}$ by the parameter $\xi \in I_\xi = [-1,1]$ and solve the Eq. (3.40) numerically in the range I_ξ. Equation (3.40) with $\xi \in I_\xi$ also yields a simple analytical solution $u(t,\xi) = e^{(1+\sigma\xi)t}$, which can be used to compare the membership function behavior to the numerical solution and examine the error behavior later.

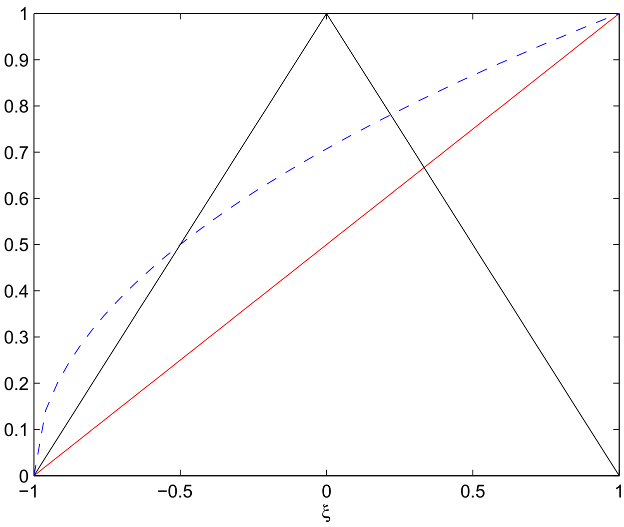


Figure 3.6. Membership functions of the fuzzy number $\tilde{\xi}$.

To solve the Eq. (3.40) numerically, we use Galerkin method with Legendre polynomial basis and scale the equation as

$$\frac{\partial u(t,\xi)}{\partial t} = (1+\sigma\xi)u, \qquad u(0,\xi) = 1, \tag{3.42}$$

where $\xi \in I_\xi = [-1, 1]$.

We seek the solution in the following form:

$$u_n(t,\xi) = \sum_{j=0}^{n} \hat{u}_j(t)\phi_j(\xi), \tag{3.43}$$

where $\phi_j(\xi)$ are Legendre polynomial satisfying

$$\int_{I_\xi} \phi_i(\xi)\phi_j(\xi)w(\xi)d\xi = \delta_{i,j}, \tag{3.44}$$

where $\delta_{i,j}$ is the Kronecker delta function and $w(\xi) = 0.5$.

Substituting (3.43) into (3.42) and forcing the residue to be orthogonal to the linear polynomial space of degree up to n, we obtain the following Galerkin system

$$\frac{d\hat{u}_j}{dt} = \sum_{i=0}^{n} e_{i,j}\hat{u}_i, \qquad \hat{u}_j(0) = \delta_{0.j}, \tag{3.45}$$

where $e_{i,j} = \frac{1}{2}\int_{I_\xi}(1+\sigma\xi)\phi_i(\xi)\phi_j(\xi)w(\xi)d\xi$. This system of ODE yields the solution u_n, which converges to u in L^2_w norm.

Using Zadeh's extension principle, the uncertainty in $u_n(t)$ propagated from the uncertainty in $\tilde{\xi}$ can be represented by membership functions. The membership functions of $\tilde{u}_n(t)$ with $n = 10$ are shown in Figs. 3.7- 3.9, with different membership functions associated with the input fuzzy number $\tilde{\xi}$. We also study the convergence of p-Hausdorff distance with respect to the approximation order n. The p-Hausdorff distance (solide lines), L^2 error (dashed lines) error are computed at time $t = 1$, where the solution is non-trivial. And the figures show that a) the p-Hausdorff distance between $\tilde{u}$ and $\tilde{u}_n$ is independent on the membership function of the input fuzzy number in the example, and b) the p-Hausdorff distance has spectral convergence with respect to the approximation order n.

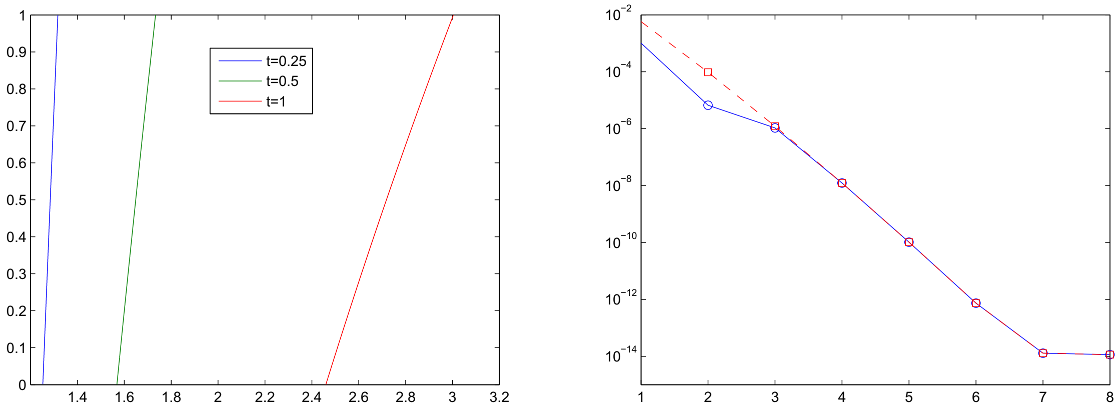


Figure 3.7. Membership function of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at $t = 1$ (right).

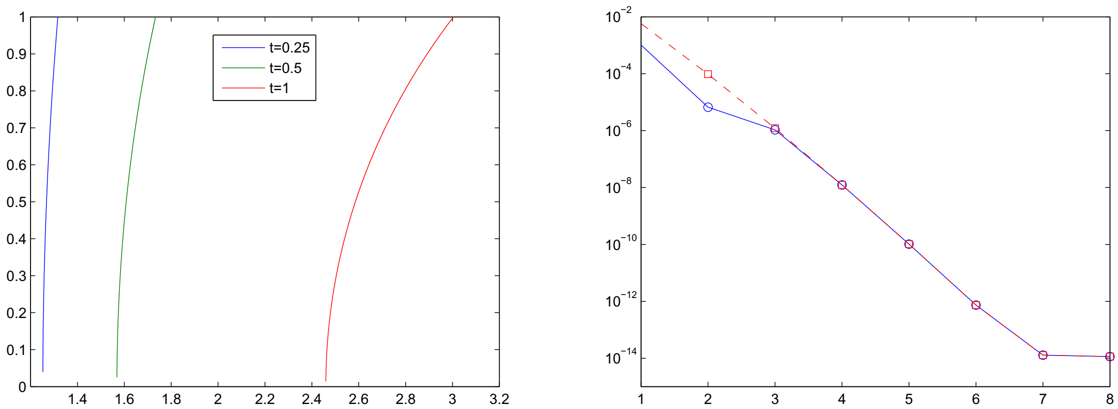
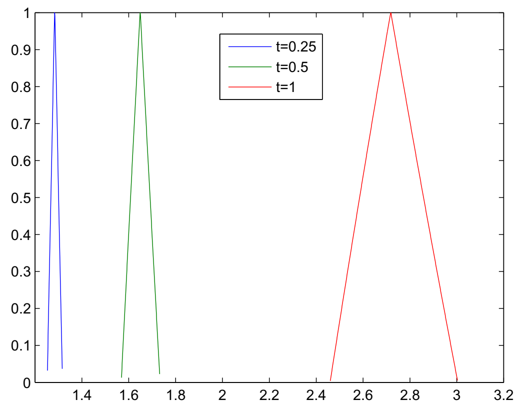


Figure 3.8. Membership function of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approxiamtion order at $t = 1$ (right).



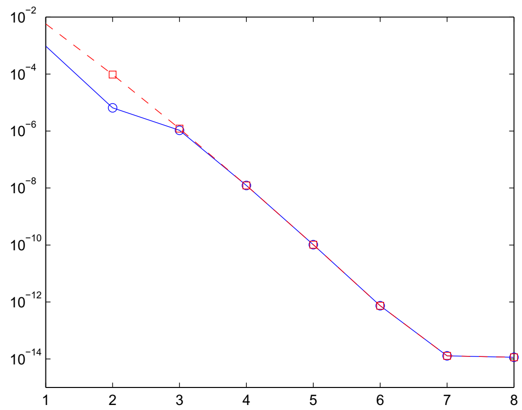


Figure 3.9. Membership function of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approxiamtion order at $t = 1$ (right).

By the definition of the weighted expected value and the variance of the function of a fuzzy set, here, we let $f(\alpha) = 1$. The expected value and variance of $\tilde{u}_n(t)$ with $n = 10$ are shown in left of Figs. 3.10- 3.11, with different membership functions associated with the input fuzzy number $\tilde{\xi}$. We also study the convergence of them with respect to the approximation order n. The error between the expected value under different fuzzy sets, L^2_w error (line with tag) error are computed at time $t = 1$, where the solution is non-trivial. This is consistent with the error bounds from Theorems 3.4 and 3.5.

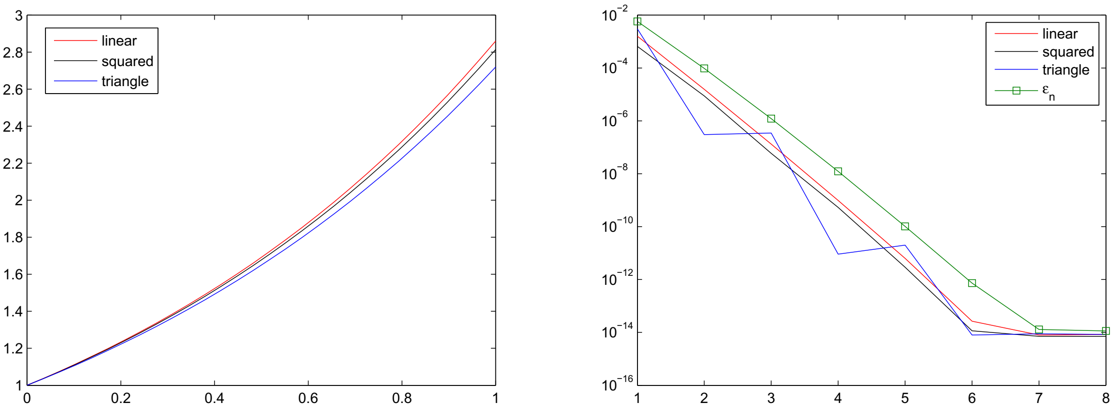


Figure 3.10. Expected value of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at $t = 1$ (right).

3.3.2 Non-linear Ordinary Differential Equation

In this section, we test the proposed approach on the following non-linear equation,

$$\frac{du}{dt} = -\tilde{\xi}_1 u(1 - \frac{u}{A}), \tag{3.46}$$

and the initial condition is

$$u(0, \tilde{\xi}_1, \tilde{\xi}_2) = \tilde{\xi}_2. \tag{3.47}$$

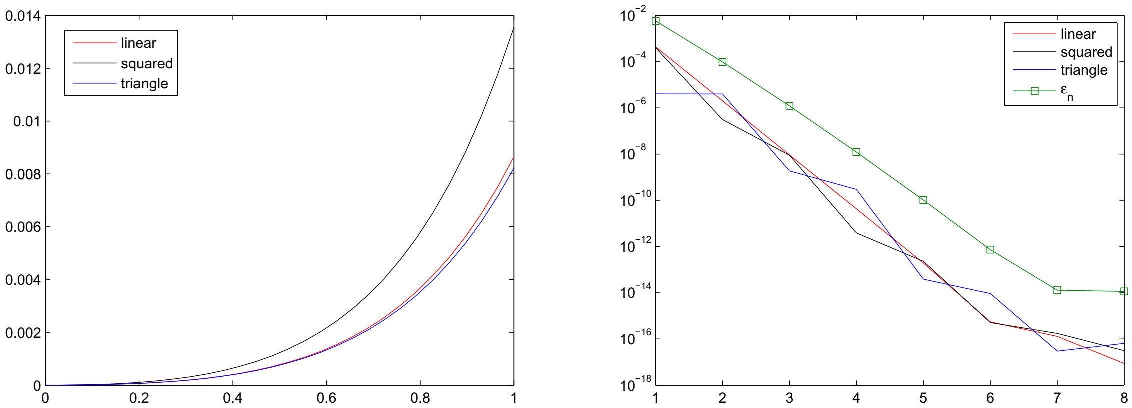


Figure 3.11. Variance value of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at $t = 1$ (right).

Let $A = 2.0$ and $\tilde{\xi}_1, \tilde{\xi}_2$ be fuzzy numbers with membership functions

$$\mu(\xi_1) = \begin{cases} 10\xi_1 - 3 & 0.3 \le \xi_1 \le 0.4, \\ 1 & 0.4 \le \xi_1 \le 0.5, \\ 6 - 10\xi_1 & 0.5 \le \xi_1 \le 0.6; \end{cases}$$

$$\mu_(\xi_2) = \begin{cases} 100\xi_2 - 84 & 0.84 \le \xi \le 0.85, \\ 86 - 100\xi_2 & 0.85 \le \xi \le 0.86. \end{cases} \tag{3.48}$$

The membership function of the two-dimensional fuzzy set $\tilde{\xi} = \tilde{\xi}_1 \times \tilde{\xi}_2$ is shown in Fig. 3.12.

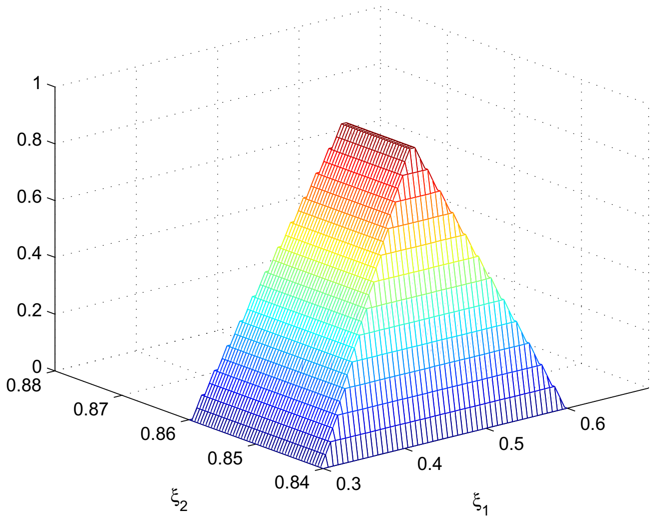


Figure 3.12. Membership function of fuzzy number $\tilde{\xi}_1 \times \tilde{\xi}_2$.

Replacing the fuzzy number $\tilde{\xi}$ by parameter $\xi = [\xi_1, \xi_2]$, where $\xi_1 \in [0.3, 0.6]$ and $\xi_2 \in [0.84, 0.86]$, the analytic solution of Eqs. (3.46)-(3.47) is

$$u(t, \xi_1, \xi_2) = \frac{A\xi_2 e^{-\xi_1 t}}{\xi_2 e^{-\xi_1 t} - \xi_2 + A}. \tag{3.49}$$

To solve the Eqs. (3.46)-(3.47) numerically using Galerkin method with Legendre polynomials, we transfer the support $[0.3, 0.6] \times [0.84, 0.86]$ to $[-1, 1]^2$ (i.e., $\xi \in [-1, 1]^2$) and the Eq. (3.46) is rewritten as:

$$\frac{du}{dt} = -(0.45 + 0.15\xi_1)u(1 - \frac{u}{A}), \qquad \xi \in [-1, 1]^2. \tag{3.50}$$

The initial condition becomes

$$u(0, \xi_1, \xi_2) = 0.85 + 0.01\xi_2. \tag{3.51}$$

Similarly to the linear example, we solve the equations numerically and obtain the approximated solution u_n, where n is the approximation order. Then we can calculate the membership functions using Zadeh's extension principle. The membership function of approximated solution $\tilde{u}_{10}(t)$ at different time are presented on the left hand side of Fig. 3.13. We can see that the membership functions have the same pattern. On the right hand side, we have showed p-Hausdorff distance behavior (solid line) and L^2 error (dashed line) at time $t = 2$, with respect to approximation order n. In Figs. 3.14- 3.15, the weighted expected value and variance of $\tilde{u}_n(t)$ with $n = 10$ and $f(\alpha) = 1$ are shown on the left, and the convergence of them with respect to the approximation order n are studied. The error between the expected value and variance, L^2_w error (line with tag) error are computed at time $t = 2$.

Diffusion Equation with Both Aleatory and Epistemic Uncertainty In this section, we test the proposed numerical approach on diffusion equation with mixed types of uncertainty involved. The diffusion equation is specified as

$$-\frac{\partial}{\partial x}[k(x, Z, \tilde{\xi})\frac{\partial u}{\partial x}(x, Z, \tilde{\xi})] = f(x, Z, \tilde{\xi}), \qquad x \in (0, 1), \tag{3.52}$$

with forcing term $f = 2$ and the boundary conditions

$$u(0, Z, \tilde{\xi}) = 0, \qquad u(1, Z, \tilde{\xi}) = 0.$$

Furthermore, we assume that the uncertain diffusivity has the form

$$k(x, Z, \tilde{\xi}) = 1 + \sigma \sum_{k \in D_1} \frac{1}{k^2\pi^2} cos(2\pi kx)\tilde{\xi}_k + \sigma \sum_{k \in D_2} \frac{1}{k^2\pi^2} cos(2\pi kx) Z_k,$$

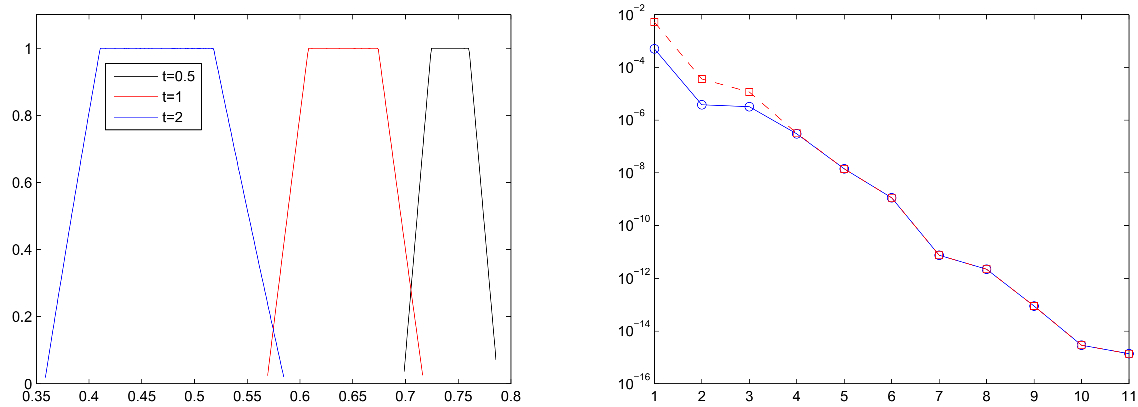


Figure 3.13. Membership function of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at time $t = 2$ (right).

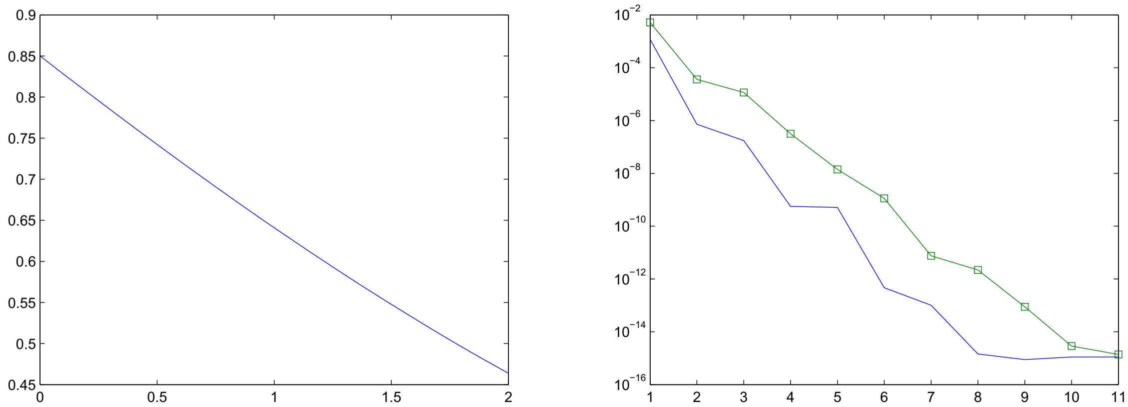


Figure 3.14. Expected value of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at $t = 2$ (right).

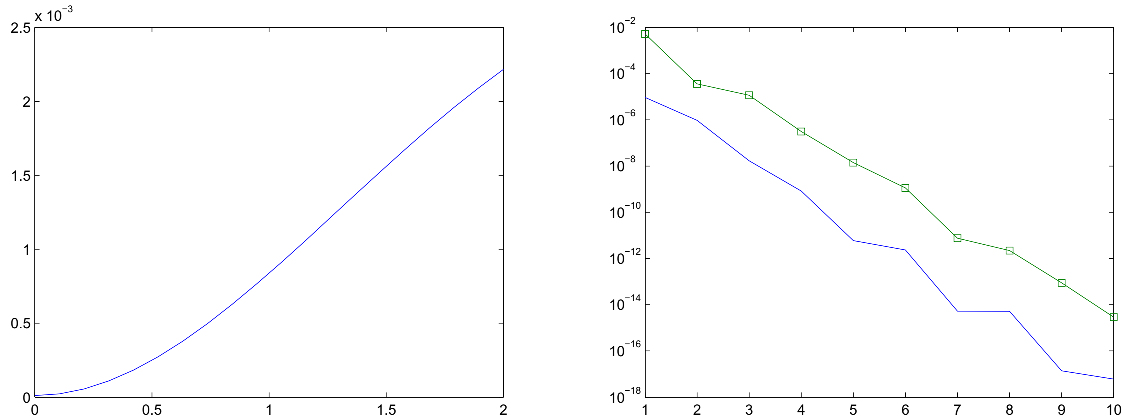


Figure 3.15. Variance of $\tilde{u}_{10}(t)$ (left) and error plot with respect to approximation order at $t = 2$ (right).

where $\sigma = 0.1$ is a parameter to control the variation level of the diffusivity field, $\tilde{\xi}_k$ with $k \in D_1 = \{1, 3, 5, 7\}$ are fuzzy numbers represented with the same membership functions shown in Fig. 3.16, Z_k with $k \in D_2 = \{2, 4, 6, 8, 9, 10\}$ are assumed to be i.i.d. random variables with distribution $Beta(-1, 1, 1, 1)$, we denote the density function as ρ_Z. Then the uncertain input parameters are

$$\tilde{\xi} = \{\tilde{\xi}_1, \tilde{\xi}_3, \tilde{\xi}_5, \tilde{\xi}_7\}, \qquad Z = \{Z_2, Z_4, Z_6, Z_8, Z_9, Z_{10}\}. \tag{3.53}$$

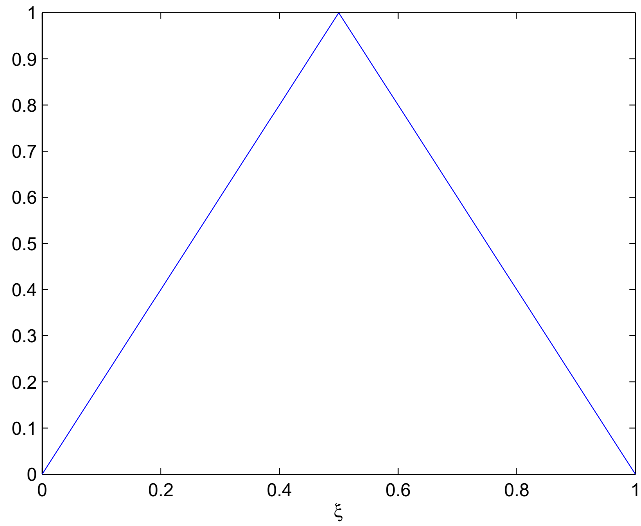


Figure 3.16. Membership function of fuzzy number.

Firstly, we estimate the ranges of the uncertain parameters $\tilde{\xi}$ and Z as $I_\xi = Supp(\tilde{\xi}) = [0, 1]^4$ and $I_Z = [-1, 1]^6$. Then, we solve the Eq. (3.52) numerically in the domain $I_\xi \times I_Z$ using Galerkin method with n-th order Jacobi polynomials with parameters $\alpha = 1$ and $\beta = 1$. Let $u_n(x, Z, \xi)$ be the approximated solution and $u(x, Z, \xi)$ be the exact solution to the Eq. (3.52) in the domain $I_\xi \times I_Z$.

Define ϵ_n as the error of the numerical solution at a fixed point,

$$\epsilon_n = \|u_n(x, Z, \xi) - u(x, Z, \xi)\|_{L^p_\rho([-1,1]^6) \otimes L^p_\omega([-1,1]^4)},$$

and define the conditional expectations given uncertain input parameter $\tilde{\xi}$ as

$$\tilde{\mu}(\tilde{\xi}) = \mu(\tilde{u}(Z,\tilde{\xi})|\tilde{\xi}) = \int_{Z\in[-1,1]^6} \tilde{u}(x,z,\tilde{\xi})\rho(Z)dZ,$$

$$\tilde{\mu}_n(\tilde{\xi}) = \mu(\tilde{u}_n(Z,\tilde{\xi})|\tilde{\xi}) = \int_{Z\in[-1,1]^6} \tilde{u}_n(x,Z,\tilde{\xi})\rho(Z)dZ.$$

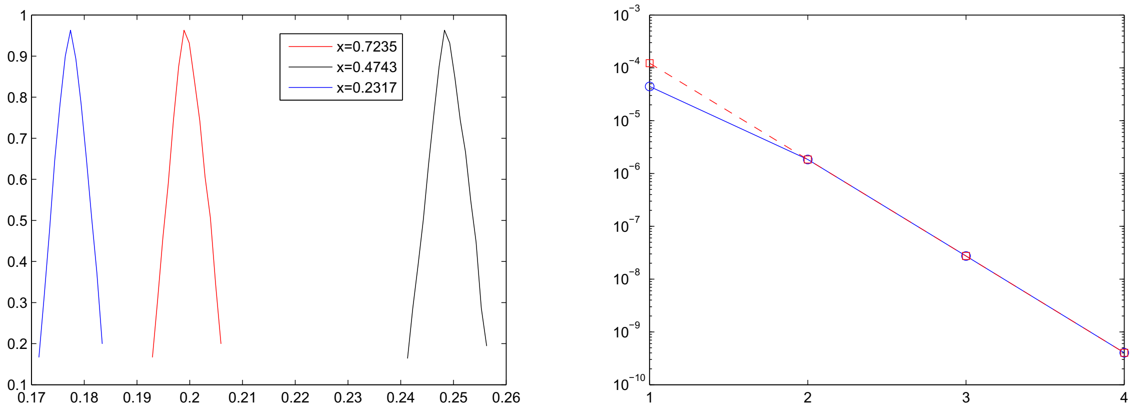


Figure 3.17. Membership function of $\tilde{\mu}_3(x)$ (left) and error plot with respect to approximation order at space $x = 0.5257$ (right).

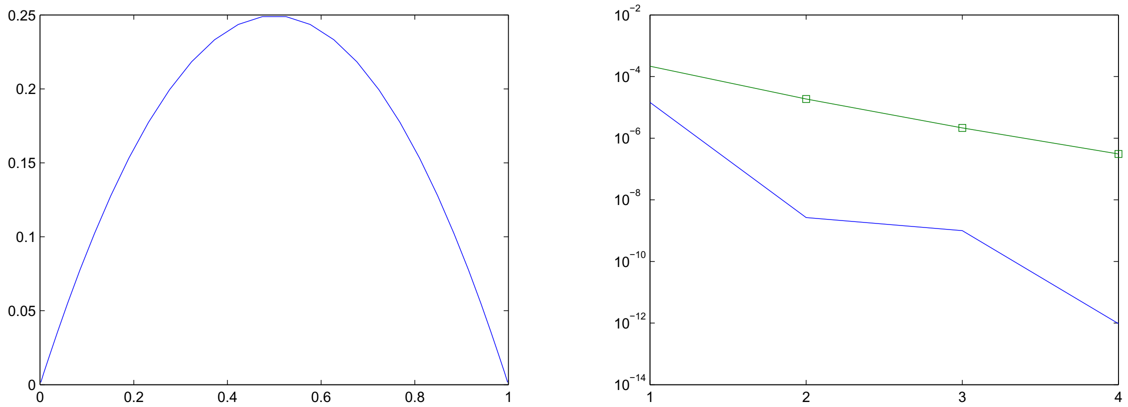


Figure 3.18. Expected value of $\tilde{\mu}_3(x)$ (left) and error plot with respect to approximation order at $x = 0.5257$ (right).

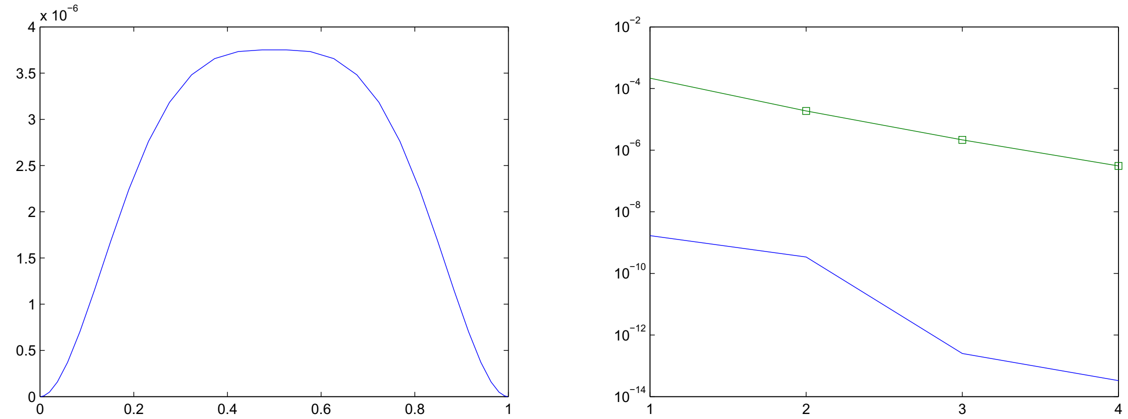


Figure 3.19. Variance of $\tilde{\mu}_3(x)$ (left) and error plot with respect to approximation order at $x = 0.5257$ (right).

4. EPISTEMIC UNCERTAINTY QUANTIFICATION USING EVIDENCE THEORY

Until now, our focus is on epistemic uncertainty quantification using the fuzzy set theory, which have been had may good results. However, as to epistemic uncertainty represented by Dempster-Shafer Theory, where the uncertainties are characterized by the so called "m-function", we have not do much work on it. In order to have a good interpreting of epistemic uncertainty, epistemic Uncertainty quantification using Dempster-Shafer Theory becomes the topic of our future work.

Here, first of all, we have a brief introduction of the Dempster-Shafer Theory, and, similar to the work in chapter 3, we define a distance based on p-hausdorffe distance to measure the dissimilarity of two m-functions. Then, we propose the methodology to approximate the exact solution. At last, we implement two simple examples to illustrate the convergence of the method.

4.1 Dempster-Shafer Theory of Evidence

The Dempster-Shafer (DS) theory of evidence (also called evidence theory)was first introduced by P. Dempster in 1976 [8] as a new approach to the representation of uncertainty. The theory is originated from the earlier works of Dempster [9, 10] in the 1960s in the context of statistical inference. In [10], Dempster considered a multivalued mapping, from a space E of sample observations regarded as sources of evidence/information to the target space X(such as a parameter space or a product of a parameter space and a space of future observations). Here E is carrying a probability measure over its subset. He inferred that the corresponding measure in X would be non-unique and one could at best consider just upper and lower probabilities. Furthermore, in practice, different sources of evidence E_i may provide information about the same uncertain outcome in X. under the assumption of the independence of evidence sources, he proposed a rule to combine them (now known as Dempsters rule of combination) for the purpose of statistical inference.

Based on these fundamental ideas, Shafer modeled ones intuitive perception of belief/opinion and developed what he called a mathematical theory of evidence, where X is called the frame of discernment, and the lower and upper probabilities are called belief and plausibility functions, respectively. Shafer defined a belief function as a measure with the important property of super additivity and laid the foundation for a comprehensive theory of belief functions. This fuzzy character of the belief measure makes the theory naturally suitable for representing and treating epistemic uncertainty stemming from imprecise information or incomplete knowledge.

Since the publication of Shafers work, the DS theory is well developed by many people. For example, Yager proposed a new rule of combination what is called Yagers rule [38, 39] to avoid the counterintuitive results when combining highly conflicted evidence under the Dempsters rule of combination. Moreover, in 2004, Yager [40] proposed the concepts of entropy and specificity of a belief function, which provide information regarding the represented by the belief function. For more work of Yager, please see [41, 42].

Next, a brief description of the mathematical basics of the DS theory is given.

4.1.1 Measures in the DS Theory

Let X be the quantity of interest with a collection of possible values
$\mathbf{X} = \{X_1, \ldots, X_n\}$, then, the set $\mathbf{X}$ is called the universal set or the frame of discernment. For any subset A of $\mathbf{X}$, if we know exactly the true value of X, then our knowledge is complete and there is no uncertainty. However, we may have only partial knowledge and the true value of X is unknown. In such a case, propositions in the form of "the true value of X is in A" are considered. Here, we use a basics

belief assignment to represent the strength of evidence supporting the proposition A, that is

Definition 4.1. Let $\mathbf{X}$ be a universal set and A be any subset of $\mathbf{X}$, a basic belief assignment (BBA, also called an m-function) is a mapping m from power set $2^{\mathbf{X}}$ (which is the set of all subsets of $\mathbf{X}$ including $\mathbf{X}$ and $\emptyset$) to the unit interval $[0, 1]$ to, which satisfies:

$$m(\emptyset) = 0, \tag{4.1}$$

$$\sum_{A \subseteq \mathbf{X}} m(A) = 1. \tag{4.2}$$

The equation (4.1) states that no belief mass should be assigned to the null set, and it implicitly assumes that the true value of X is included in the universal set $\mathbf{X}$, which is called the closed-world assumption. The equation (4.2) states that the sum of the belief masses over all the subsets is unity, which implies that one's total belief has measure one. Hereafter we use m-function and BBA interchangeably.

Remark: The reasons why a m-function may not be called a probability density function (PDF) are listed, below,

1. *First, an m-function assigns mass to sets while the traditional probability theory assigns probability to single points.*
2. *Secondly, $m(A) \leq m(B)$ may not hold for an m-function if $A \subseteq B$.*
3. *At last, the additivity property $m(A_1) + m(A_2) = m(A_1 \cup A_2)$ $(A_1, A_2 \subseteq \mathbf{X})$ may not hold for an m-function while it holds for the probability measure.*

The element $A \subseteq \mathbf{X}$ is called a focal element if $m(A) \neq 0$. And the union of all the focal elements is called the core of an m-function, denoted by ζ.

Foe any subset A of $\mathbf{X}$, since the true value lying in the subset of A implies that the true value also lies in A, the total support (total degree of belief) for A should also include the support for subsets of A. As a result, adding $m(B)$ for all $B \subseteq A$ together gives the total belief committed to A: $Bel(A)$.

Definition 4.2. *Bel* is a belief measure that assigns a number in the unit interval $[0, 1]$ to the power set $2^{\mathbf{X}}$. And it is defined as follows:

$$Bel(A) = \sum_{B \subseteq A} m(B). \tag{4.3}$$

The definition of the belief function is illustrated by the Venn diagram (Fig. 4.1(a)) showing the set A and its subsets B_1, B_2, and B_3. $m(B_1)$, $m(B_2)$, and $m(B_3)$ also support that the true value of the quantity of interest is in A.

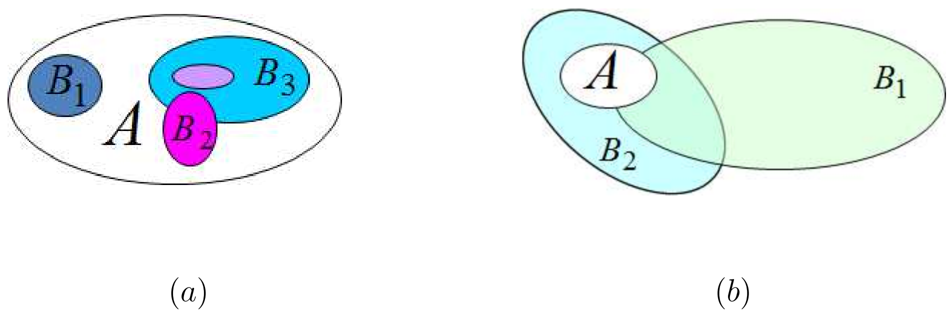


Figure 4.1. Venn diagrams: (a) The set A with subsets, (b) The set A with intersecting sets.

Belief functions have the following properties [8].

1. Boundary conditions: $Bel(\emptyset) = 0$, $Bel(\mathbf{X}) = 1$.

2. Superadditivity: for any $A, B \subseteq \mathbf{X}$, we have:

$$Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B) \tag{4.4}$$

Specifically, $Bel(A) + Bel(\overline{A}) \leq 1$, where $\overline{A}$ is the complement of A.

Definition 4.3. A dual measure of belief related to what extent one believes its negation $\overline{A}$ is defined as

$$Pl(A) = 1 - Bel(\overline{A}) \tag{4.5}$$

This dual measure is called plausibility function ($Pl : 2^{\mathbf{X}} \to [0,1]$), which measures the maximum possible strength of evidence supporting a proposition A. Alternatively, it can also expressed by the adding $m(B)$ for all the $B \cap A \neq \emptyset$.

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B). \tag{4.6}$$

The definition of plausibility is graphically explained with Fig. 4.1(b), which shows that if B_1 and B_2 are two sets intersecting A, $m(B_1)$ and $m(B_2)$ also support that it is possible for the true value of the quantity of interest to fall in A. Belief, plausibility and m-functions are equivalent in the sense that any two can be deduced from the third.

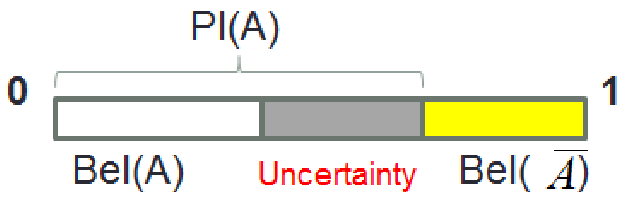


Figure 4.2. Belief and Plausibility of a proposition A.

The formulas 4.3 and 4.6 indicate that $Bel(A) \leq Pl(A)$. We can interpret that $Bel(A) \leq Pl(A)$, $Bel(A) + Bel(\overline{A}) \leq 1$ from Figure 4.2. The gray area describes the epistemic uncertainty in the proposition A, and $Bel(A)$, $Pl(A)$ can be interpret as the lower bound and upper bound of possible strength of support for the proposition A. Unlike super-additivity of the belief function, probability theory has strict additivity: $P(A) + P(\bar{A}) = 1$, i.e., the epistemic uncertainty represented by the gray area of Fig. 4.2 disappears. Thus probability theory is natural for quantifying aleatory uncertainty, in this sense, DS theory can be used to quantify both aleatory and epistemic uncertainty naturally.

In the current work, the universal set is assumed to be a finite interval with a finite number of subintervals as the focal elements. For example, let the interval $\mathbf{X} = [a, b]$

be the universal set. The focal elements are ξ_1, ξ_2 and ξ_3, where the m-function is defined as follows: $m(\xi_1) = 0.4$, $m(\xi_2) = 0.3$ and $m(\xi_3) = 0.2$, and $m(\mathbf{X}) = 0.1$. Here, we illustrate it in Fig. 4.3:

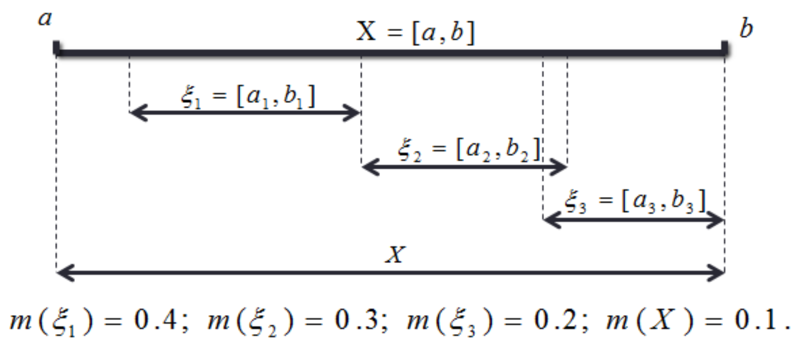


Figure 4.3. An m-function and the focal elements.

By the definition in (4.3), the degrees of belief associated with focal elements are obtained:

$$Bel(\xi_1) = 0.4,\ \ Bel(\xi_2) = 0.3,\ \ Bel(\xi_3) = 0.2,\ \ Bel(X) = 1. \qquad (4.7)$$

Analogous to the cumulative distribution function in probability theory, Oberkampf [13] introduced the concepts of cumulative belief function (CBF) and cumulative plausibility function (CPF) or a complementary cumulative belief function (CCBF) and a complementary cumulative plausibility function (CCPF). The concepts extend probabilistic risk assessment where the probability of high-end risk range is displayed. And the definitions are as follows:

$$CBF(x) = Bel(X \leq x), \qquad (4.8)$$

$$CPF(x) = Pl(X \leq x), \qquad (4.9)$$

$$CCBF(x) = Bel(X > x), \qquad (4.10)$$

$$CCPF(x) = Pl(X > x). \qquad (4.11)$$

Note: Given the belief function, all the CBF, CPF, CCBF, CCPF are obtained uniquely. However, the inverse does not hold, i.e., given a CBF and a CPF, we can not get a unique belief function [42].

4.1.2 Distance between Two *m*-functions

There are a number of measures in the literature to measure the dissimilarity between two m-functions defined over the same finite set [43–50], however, in the case with a finite interval as universal set and with finite number of subintervals as focal elements, the distance between two m-functions is barely studied. Here we define a distance measure based on p-Hausdorff distance to measure the dissimilarity between m-functions defined in such situation.

Recall the definition of p-Hausdorff distance in chapter 3 as follows:

Definition 4.4. Let $X, Y \subset \mathbb{R}^n$ be two non-empty classical sets, the p-Hausdorff distance $d_{\hat{H}^p}$ is defined as

$$d_{\hat{H}^p}(X,Y) = \max\{(\int_{x\in X} \operatorname*{essinf}_{y\in Y} d^p(x,y)dx)^{\frac{1}{p}}, (\int_{y\in Y} \operatorname*{essinf}_{x\in X} d^p(x,y)dy)^{\frac{1}{p}}\}, \quad (4.12)$$

where $d^p(x,y)$ is the L^p-norm of the difference between two vectors.

Next we define a distance measure to measure the dissimilarity between two m-functions in a special case.

Assume m_1, m_2 are two m-functions with the same number of focal elements $A_1, ...A_l$ and $B_1, ...B_l$ respectively, and the belief masses assigned to the focal elements are the same for A_i and B_i, i.e., $m_1(A_i) = m_2(B_i) = a_i$ for $1 \le i \le l$. Then the distance between m_1 and m_2 is defined as

$$Dis(m_1, m_2) = \sum_i d_{\hat{H}}(A_i, B_i) * a_i.$$

4.2 Uncertainty Quantification Using Evidence Theory

Let $D \subset \mathbb{R}^l, l = 1, 2, 3$, be a physical domain with coordinates $x = (x_1, \ldots, x_l)$, let $(0, T]$ be a time domain with $T > 0$, and let $\Xi \subset \mathbb{R}^n$ be a parameter domain for uncertain inputs. We consider a general partial differential equation (PDE) as

$$\begin{cases} u_t(x, t, \xi) = L(u), & D \times (0, T] \times \Xi, \\ \mathcal{B}(u) = 0, & \partial D \times [0, T], \\ u = u_0, & D \times \{t = 0\} \times \Xi, \end{cases} \tag{4.13}$$

where L is a (linear or nonlinear) differential operator, $\mathcal{B}$ is the boundary condition operator, u_0 is the initial condition, and $\xi = \{\xi_1, \ldots \xi_n\}$ is a set of uncertain parameters characterizing the uncertainty in the inputs of the governing equation. We assume that the problem (4.13) is well-posed in Ξ and let $u(x, t, \xi)$: $D \times (0, T] \times \Xi \to \mathbb{R}^{n_u}$ denote its solution. For simplicity, we further assume that the output of (4.13) is in one dimension, i.e., $n_u = 1$.

In the situation where the uncertain parameter ξ is random and the associated probability density function (PDF: $P_\xi(s) = P(\xi \leq s)$) is known, the standard probabilistic formulation is adopted and the statistics of the stochastic quantity $u(x, t, \xi)$ are calculated. In the situation where ξ is random however the PDF is not completely known, one can adopt a three-step procedure [37] – estimating the range of ξ, solving the system of equations (4.13) within the estimated range numerically, and then post-processing the obtained solution when the PDF becomes available – to calculate the statistics of $u(x, t, \xi)$. Here, we extend the three-step procedure to the situation where the information associated with the uncertain variable ξ is insufficient for a probabilistic treatment. In such case, the epistemic or mixed types of uncertainty in the parameter ξ and the propagated uncertainty in the output or the statistics of the output $u(x, t, \xi)$ are represented mathematically in the framework of evidence theory.

Similar to the work of chapter 2 and chapter 3, we use a three-step procedure to quantify the uncertainty in the output of the system (4.13), where the input parameters are random variable associated with probability density functions or uncertain variables associated with m-functions. The procedure comprises identifying the ranges of the uncertain inputs, generating the accurate numerical solution for the system (4.13) within the estimated ranges, and analyzing the uncertainty in output in the framework of evidence theory.

4.2.1 Range Estimation

The first task is to identify a range, or a bound for the uncertain parameters, which is sufficiently large such that it can cover all the possible cases we are interested in.

For the uncertain parameters characterized by m-functions, the universal sets (i.e., finite intervals) serve the purpose. For each variable ξ_i, $i = 1, \ldots, d$, let its universal set $\mathbf{X_i} = [a_i, b_i]$ be its range. And denote $I_\xi = X_1 \times \cdots \times X_d$ to be the domain.

4.2.2 Numerical Solution of System of PDE

Once the ranges of the parameters are estimated, traditional numerical methods can be used to solve the system of equations (4.13) in the domain I_ξ and the approximated solution is denoted as $u_n(x, t, \xi)$, where the index n is associated with the discretization parameters in the approximation. The error of the numerical solution defined as

$$\epsilon_n \triangleq \|u_n(x, t, \xi) - u(x, t, \xi)\|_{L_\infty(I_\xi)}, \tag{4.14}$$

approaches zero as $n \to \infty$, where $u(x, t, \xi)$ is the exact solution of Eqs. (4.13).

4.2.3 Epistemic Uncertainty Quantification in Output

In this section, we take into account the uncertainty in the inputs ξ and quantify the consequent uncertainty in output u_n using evidence theory, i.e., we obtain the m-function for the uncertain output $u_n(x, t, \xi)$. With any fixed x and t, for simplicity, we write $u_n(x, t, \xi)$ as $u_n(\xi) : I_\xi \to \mathbf{R}$. Suppose the m-functions m_i $(1 \leq i \leq d)$ are given for each input parameter ξ_i. Specifically, a universal set $\mathbf{X}_i = [a_i, b_i]$ $(1 \leq i \leq d)$

includes all the possible values of ξ_i. The focal elements are a finite number of subintervals $\xi_{ij} = [a_{ij}, b_{ij}]$, where $1 \leq j \leq C_i$ and $C_i < \infty$ is the number of focal elements of the m-function m_i. Uncertainty propagation using the DS theory attempts to find the m-function for the output u_n. The procedure includes constructing an d-dimensional belief structure for $\vec{X}$, obtaining the focal elements for $u_n(\xi)$, and assigning the belief masses to the focal elements according to the body of evidence.

1. Construct the m-function for ξ.

   The d-dimensional belief structure can be constructed by taking the Cartesian product over all the directions of ξ where the universal set is $\vec{\mathbf{X}} = \mathbf{X}_1 \times \mathbf{X}_2 \times ... \times \mathbf{X}_d$. The focal elements are $\vec{\xi_n} = \xi_{1n_1} \times \xi_{2n_2} \times ... \times \xi_{dn_d}$, $1 \leq n \leq \prod_{i=1}^{d} C_i$, where $\vec{\xi_n}$ is an d-dimensional hypercube and ξ_{in_i} is a focal element of the i-th m-function m_i. The m-function is defined as $m(\vec{\xi_n}) = \prod_{i=1}^{d} m_i(\xi_{in_i})$.

2. Obtain the focal elements for $u_n(\xi)$.

   The uncertainty in ξ, represented by the d-dimensional belief structure, is propagated through the system $u_n = u_n(\xi)$ and accumulated in the uncertainty of the output u_n. To construct an m-function to represent the uncertainty in u_n, we calculate the lower and upper bounds of u_n in each hypercube $\xi \in \vec{\xi_n}$ using (4.15)-(4.16), and the interval between lower and upper bounds constitutes one focal element for u_n.

$$y_{\min} = \min_{\xi \in \vec{\xi_n}} u_n(\xi), \tag{4.15}$$

$$y_{\max} = \max_{\xi \in \vec{\xi_n}} u_n(\xi). \tag{4.16}$$

   There are many algorithms available to solve the constrained optimization problems [11,19], such as Newton's methods, which are very basic iterative methods to find stationary points of differentiable functions. However, this gradient-based local optimization solver cannot guarantee the global optima. In order to find the global optima, multi-start implementations of local optimization methods or global optimizers are considered. These methods search the entire domain to find the global optima, in this way, the computational cost would be a big problem [11]. An alternative approach is to estimate the bounds $y_{\min}$ and $y_{\max}$ of the output by a sampling method, where the uniform distribution is assumed over each input hypercube. The sampling method is easy to implement, but its accuracy depends on the number of samples.

3. Assign the m-function to focal elements for u_n.

   The m-function associated with $\vec{\xi_n}$ would be transfered to the corresponding focal element $[y_{\min}, y_{\max}]$ for u_n, i.e., $m_{u_n}([y_{\min}, y_{\max}]) = m_{\xi}(\vec{\xi_n})$. Obviously, the output u_n should have the same number of focal elements as ξ unless there are two or more hypercubes corresponding to the same focal element for u_n.

4.3 Numerical Examples

4.3.1 Ordinary Differential Equation

Firstly, let us consider an ordinary differential equation, which is defined as follows:

$$\frac{\partial u(t,\xi)}{\partial t} = -\xi u, \qquad u(0,x) = 1, \tag{4.17}$$

where the m-function of ξ, with the universal inteval $\mathbf{X} = [0,1]$ and focal elements $X_1 = [0, \frac{1}{3}]$, $X_2 = [\frac{1}{3}, \frac{2}{3}]$ and $X_3 = [\frac{2}{3}, 1]$ is defined as follows: $m(X_1) = 0.3$, $m(X_2) = 0.5$, $m(X_3) = 0.15$ and $m(\mathbf{X}) = 0.05$. The proposed three-step procedure is applied for uncertainty propagation as follows. Firstly, we estimate the range of the uncertain parameter. Due to the definition, its universal set $\mathbf{X} = [0,1]$ is its range. Then, solve the Eq. (4.17) numerically in the range $\mathbf{X}$. Also, we know that equation (4.17) yields a simple analytical solution $u(t,\xi) = e^{-\xi t}$, which can be used to calculate the distance (4.13) between the m-functions later.

To solve the equations numerically, we apply the same strategy in chapter 3, and get the numerical solution u_n, where n is the approximation order. Then, we can calculate the distance (4.13) between the m-functions of u_n and u, And the distance behavior is show in figure 4.4 at time $t = 1$, with respect to approximation order n.

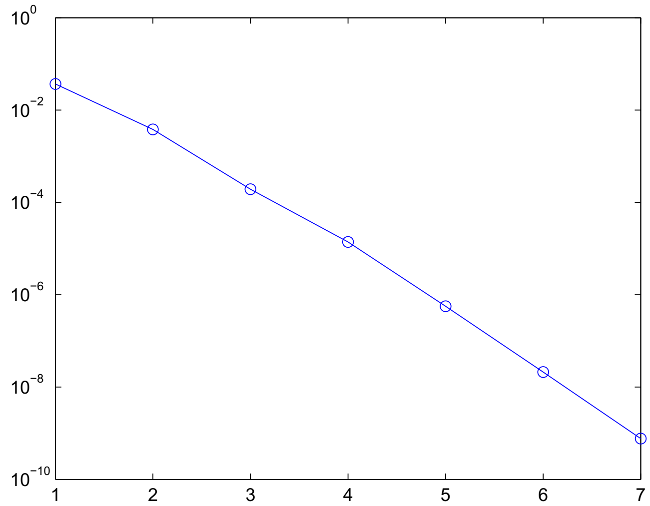


Figure 4.4. The distance between m-functions of u_n and u, at time $t = 2$, with respect to approximation order n

Non-linear Ordinary Differential Equation In this section, we test the proposed approach on the following non-linear equation,

$$\frac{du}{dt} = -\xi_1 u(1 - \frac{u}{A}), \tag{4.18}$$

and the initial condition is

$$u(0, \xi_1, \xi_2) = \xi_2. \tag{4.19}$$

Let $A = 2.0$. The m-function of ξ_1, with the universal inteval $\mathbf{X} = [0.3, 0.6]$ and focal elements $X_1 = [0.3, 0.4]$, $X_2 = [0.4, 0.5]$ and $X_3 = [0.5, 0.6]$ is defined as follows: $m(X_1) = 0.3$, $m(X_2) = 0.5$, $m(X_3) = 0.15$ and $m(\mathbf{X}) = 0.05$. And the m-function of ξ_2, with the universal inteval $\mathbf{Y} = [0.83, 0.86]$ and focal elements $Y_1 = [0.83, 0.84]$, $Y_2 = [0.84, 0.0.85]$ and $Y_3 = [0.85, 0.86]$ is defined as follows: $m(Y_1) = 0.3$, $m(Y_2) = 0.5$, $m(Y_3) = 0.15$ and $m(\mathbf{X}) = 0.05$. Then, the 2-dimensional belief structure $\xi = (\xi_1, \xi_2)$ is defined as $m(Z_{i,j}) = m(X_i)m(Y_j)$, where the universal set is $\mathbf{Z} = X \times Y$ and focal elements are $Z_{ij} = X_i \times Y_j$, $i, j = 1, 2, 3$.

Obviously, the analytic solution of Eqs. (4.18)-(4.19) is

$$u(t, \xi_1, \xi_2) = \frac{A\xi_2 e^{-\xi_1 t}}{\xi_2 e^{-\xi_1 t} - \xi_2 + A}. \tag{4.20}$$

To solve the equations numerically, we apply the same strategy in chapter 3, and get the numerical solution u_n, where n is the approximation order. Similarly to the linear example, we can calculate the distance (4.13) between the m-functions of u_n and u. And the distance behavior is show in figure 4.5 at time $t = 2$, with respect to approximation order n.

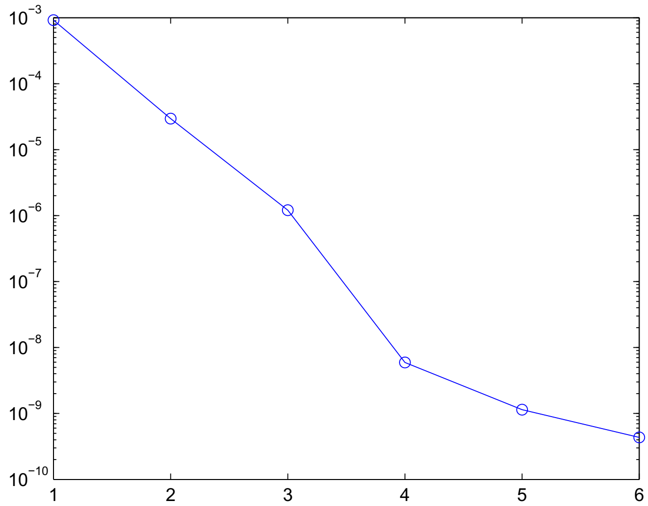


Figure 4.5. The distance between m-functions of u_n and u, at time $t = 2$, with respect to approximation order n

5. SUMMARY

In this thesis, what we do are as follows:

Firstly, we present a computational strategy for epistemic uncertainty analysis. The new method is a notable extension of the work of [28] , in the sense that it allows one to use unbounded intervals to encapsulate the true (and unknown) ranges of the epistemic variables. Consequently the new method can employ numerical solutions that are accurate in L^p weighted norms. These feature makes the method more flexible in practice. Theoretical analysis of the method is conducted, where errors in both strong form and weak form are analyzed. Numerical tests are conducted to verify the theory. Though more extensive work is required, and is ongoing, to further examine the methodology, the new method appears to be a fairly effective tool for epistemic uncertainty analysis.

Secondly, we proposed a framework for quantifying a kind of epistemic uncertainty, which is represented by fuzzy theory. A numerical study and analysis of this approach for solving fuzzy partial differential equations are presented, which can also be applied very effectively for solving fuzzy PDEs. The approach, which intuitively rise from the work in chapter 2, is based on solution of an parametric problem which generates a solution to the governing equations in the support of the fuzzy numbers. No distributional information about any of the variables needs be assumed, only estimates of the support of the variables are needed. Once the domains have been specified, a polynomial approximation can be constructed in the domain. The polynomial approximation to the parametric problem is chosen to converge in L^p weighted norm throughout the input space. As long as this convergence is obtained, the polynomial solution of the parametric problem can be used as an effective model for fuzzy PDEs.

Finally, we briefly introduce the future work of the research, whose topic is epistemic uncertainty quantification using evidence theory. Here, we use the similar methodology as we did in the previous work. And the convergence of the method can be achieved once we have the L_∞ convergence of the approximated solution, while in chapter 3, we only need the L^p weighted norm convergence. In this way, the motivation of finding a more flexible method is our future work.

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APPENDICES

A. CONVERGENCE OF GALERKIN SOLUTION FOR ODE

We now present the convergence proof for the orthogonal polynomial Galerkin solution for the ordinary differential equation (2.39). In fact, we consider a slightly more general form of the problem

$$\frac{du}{dt} = -\alpha(X)u, \qquad u(0) = 1, \tag{A.1}$$

where the coefficient α is bounded away from infinity

$$-\infty < \alpha_{min} \leq \alpha(x) \leq \alpha_{max}. \tag{A.2}$$

Let I_X be the range of X and $\{\Phi_j\}$ be a set of orthogonal polynomials on I_X

$$\int_{I_X} \Phi_i(s)\Phi_j(s)w(s)ds = \delta_{i,j}, \tag{A.3}$$

where w is a weighted function in with inner product defined as

$$(f, g)_w = \int_{I_X} f(s)g(s)w(s)ds,$$

and norm $\|f\|_w = \sqrt{(f, f)_w}$.

In Gelerkin method we seek an expansion

$$v_N(t, X) = \sum_{j=0}^{N} \hat{v}_j(t)\Phi_i(X), \tag{A.4}$$

such that (A.1) is satisfied in a weak from, which is

$$(\frac{dv_N}{dt} + \alpha v_N, \Phi_k(X))_w = 0, \qquad \forall k = 1, \ldots, N. \tag{A.5}$$

Straightforward derivation then gives us, for all $k = 0, \ldots, N$,

$$\frac{d\hat{v}_k}{dt} = -\sum_{j=0}^{N} a_{jk}\hat{v}_j, \tag{A.6}$$

where

$$a_{jk} = \int_{I_X} \alpha(s)\Phi_j(s)\Phi_k(s)w(s)ds, \tag{A.7}$$

and the initial condition becomes $\hat{v}_k(0) = \delta_{0,k}$ One the other hand, for the exact solution $u(t, X) \in L^2_w(I_X)$, then we can define the project of u in to $L^2_w(I_X)$ as

$$P_N u(t, X) = \sum_{j=0}^{N} \hat{u}_j(t)\Phi_i(X), \tag{A.8}$$

where $\hat{u}_j = (u, \Phi_j(X))_w$. Obviously, $P_N u$ is the optimal in the $\| \cdot \|_w$ norm and $\lim_{N\to\infty} P_N u = u$. Assume that for some $m > 1/2$,

$$\|(P_N u - u)(t)\|_w \leq CN^{-m}\|u(t)\|_{H^m_w(I_X)} \qquad \forall t > 0, \tag{A.9}$$

where the constant C does not depend on N. Note that the rate of convergence depends on the regularity of u and the type of orthogonal polynomials $\{\Phi_j\}$.

Theorem A.1. *(Convergence). Suppose that the assumption* (A.9) *holds. The Galerkin solution v_N converges to the solution u of the* (A.1)*. Moreover, if the solution $u(t, X)$ belongs to the weighted Sobolev space $H^m_w(I_X)$ for any t, there exists a constant C, independent of N, such that*

$$\max_{0\leq t\leq T} \|(v_N - u)(t)\|_w \leq CN^{-m+1/2} \max_{0\leq t\leq T} \|u(t)\|_{H^m_w(I_X)}. \tag{A.10}$$

It is easy to verify that the coefficients $\hat{u}_k, \quad k = 1, \dots, N$ satisfy

$$\frac{d\hat{u}_k}{dt} = -\sum_{j=0}^{\infty} a_{jk}\hat{v}_j, \qquad k = 1, \dots, \infty. \tag{A.11}$$

Proof. Let us define that

$$\hat{e}_k = \hat{v}_k - \hat{u}_k, \qquad k = 1, \dots, N.$$

We then have

$$\frac{d\hat{e}_k}{dt} = -\sum_{j=0}^{N} a_{jk}\hat{e}_j + \sum_{j>N} a_{jk}\hat{u}_j. \tag{A.12}$$

Upon multiplying both sides by $\hat{e}_k$ and taking a summation over k, we obtain

$$\sum_{k=0}^{N} \hat{e}_k \frac{d\hat{e}_k}{dt} = -\sum_{k=0}^{N}\sum_{j=0}^{N} \hat{e}_k a_{jk} \hat{e}_j + \sum_{k=0}^{N}\sum_{j>N} \hat{e}_k a_{jk} \hat{u}_j. \tag{A.13}$$

Let $e_N(t,x) = v_N - P_N u$. Naturally, we have $\|e_N\|_w = \sum_{k=0}^{N} \hat{e}_k^2$. We now adopt vector-matrix notation by defining

$$\mathrm{e} = (\hat{e}_1, \ldots, \hat{e}_N)^T, \qquad \mathrm{A} = (a_{jk})_{0\le j,k\le N}. \tag{A.14}$$

Then, (A.13) can be written as

$$\frac{1}{2}\frac{d}{dt}\|\mathrm{e}\|^2 = -\mathrm{e}^{\mathrm{T}}\mathrm{A}\mathrm{e} + \mathrm{e}^{\mathrm{T}}\mathrm{r}, \tag{A.15}$$

where the norm is the vector 2-norm and $\|\mathrm{e}\| = \|e_N\|_w$, and $\mathrm{r} = (\hat{r}_0, \ldots, \hat{r}_N)^T$ with the entries

$$\hat{r}_k = \sum_{j>N} a_{jk}\hat{u}_j, \qquad k = 0, \ldots, N. \tag{A.16}$$

Lemma A.2. *Following the definitions of* e *and* A *and the condition* (A.2)*,*

$$\mathrm{e}^{\mathrm{T}}\mathrm{A}\mathrm{e} \ge \alpha_{min}\|\mathrm{e}\|^2. \tag{A.17}$$

Proof. Consider an arbitrary vector e and construct a function

$$q(s) = \hat{e}_0 + \hat{e}_1\Phi_1(s) + \cdots + \hat{e}_N\Phi_N(s). \tag{A.18}$$

Obviously, $q(s)$ is a polynomial and satisfies $\|q\_w| = \|\mathrm{e}\|$.

$$\begin{aligned} \mathrm{e}^{\mathrm{T}}\mathrm{A}\mathrm{e} &= \textstyle\sum_{j=0}^{N}\sum_{k=0}^{N} \hat{e}_j a_{jk}\hat{e}_k \\ &= \textstyle\sum_{j=0}^{N}\sum_{k=0}^{N} \int \alpha(s)\hat{e}_j\hat{e}_k\Phi_j(s)\Phi_k(s)ds \\ &= \textstyle\int \alpha(s)q^2(s)w(s)ds \\ &\ge \alpha_{min}\|q\|_w^2. \end{aligned} \tag{A.19}$$

□

Substituting this result into (A.15), we obtain

$$\frac{d}{dt}\|\mathrm{e}\| \le -\alpha_{min}\|\mathrm{e}\| + \|\mathrm{r}\|. \tag{A.20}$$

We now analyze the term $\|\mathrm{r}\|$. Using the definition (A.16), we derive

$$\begin{aligned}\hat{r}_k &= \sum_{j>N}^{\infty} a_{jk}\hat{u}_j = \sum_{j>N}^{\infty} \int \alpha(s)\Phi_j(s)\Phi_k(s)w(s)ds\hat{u}_j \\ &= \int \alpha(s)\sum_{j>N}^{\infty} \hat{u}_j\Phi_j(s)\Phi_k(s)w(s)ds \\ &= \int \alpha(s)(u-P_N u)\Phi_k(s)w(s)ds\end{aligned} \quad . \tag{A.21}$$

Then, it follows that

$$\begin{aligned}\|\mathrm{r}\| &= (\textstyle\sum_{k=0}^{N} \hat{r}_k^2)^{1/2} = (\sum_{k=0}^{N}(\int \alpha(s)(u-P_N u)\Phi_k(s)w(s)ds)^2)^{1/2} \\ &\le (\textstyle\sum_{k=0}^{N} \|u-P_N u\|_w^2 \|\alpha\Phi_k\|_w^2)^{1/2} \\ &\alpha_{max}\sqrt{N+1}\|u-P_N u\|_w \triangleq \beta(N,t)\end{aligned} \quad , \tag{A.22}$$

where the Cauchy-Schwartz inequality and the orthonormal property of the basis are used. Note that for $m > 1/2$, $\beta(N,t) \to 0$ as $N \to \infty$ for all t due to (A.8). And from (A.20 and A.21), we obtain

$$\frac{d}{dt}\|\mathrm{e}\| \le -\alpha_{min}\|\mathrm{e}\| + \beta(N,t). \tag{A.23}$$

The Gronwall inequality then leads to

$$\|\mathrm{e}(t)\| \le \exp^{-\alpha_{min} t}\|\mathrm{e}(0)\| + \int_0^t \exp^{-\alpha_{min}(t-s)}\beta(N,s)ds. \tag{A.24}$$

An application of the triangle inequality implies

$$w\|v_N - u\|_w \le \|v_N - P_N u\|_w + \|P_N u - u\|_w. \tag{A.25}$$

and the convergence theorem is then established for $N \to \infty$ by using (A.8), (A.21) and $\|\mathrm{e}(0)\| = 0$. □

It is trivial to see that the same convergence result holds true for the case of $\|\mathrm{e}(0)\| \neq 0$ but converges at the same rate of the projection error $\|u - P_N u\|_w$. This corresponds to the case of (A.1) with a random initial condition $u_0(X)$ that is approximated by its orthogonal projection $P_N u_0$. Correspondingly the initial conditions become $\hat{v}_k(0) = (u_0, \Phi_k)_w, \quad fork = 0, \ldots, N$. This is the typical procedure in practice.

B. CONVERGENCE OF GALERKIN SOLUTION FOR TIME-DEPENDENT DIFFUSION EQUATION

Let D is a bounded and convex domain with piecewise smooth boundary and y is an uncertainty in Ω, assume that $\Omega = I_z$, where I_z is the range of the random variables $y \in \mathbb{R}^d$. We consider the following partial differential equations modeling flows in porous media, which is the stochastic diffusion equation, where v is a stochastic function.($v : \bar{D} \times [0,T] \times \Omega \to \mathbb{R}$).

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (k_d(x,z)\nabla u) = f(x,t,z), & x \in D \subset \mathbb{R}^n, t \in [0,T], z \in I_z, \\ u(x,t,z) \mid_{\partial D} = 0, & t \in [0,T], z \in I_z, \\ u(x,0,z) = u_0(x,z), & x \in \bar{D} \subset \mathbb{R}^n, z \in I_z. \end{cases} \tag{B.1}$$

Assume that the random diffusive coefficient has the form

$$k(x,z) = k_0(x) + \sum_{i=1}^{d} k_i(x) z_i, \qquad \text{for all } x \in D, a.e. \tag{B.2}$$

where $\{k_i\}_{i=0}^d$ are fixed functions with $k_0 > 0$. Alternatively (B.2) can be written as

$$k(x,z) = \sum_{i=0}^{d} k_i(x) z_i, \qquad \text{for all } x \in D, a.e. \tag{B.3}$$

where $z_0 = 1$.

For well-posedness we require

$$k_{max} \geq k(x,z) \geq k_{min} > 0, \qquad \text{for all } x \in D, y \in I_z, \tag{B.4}$$

and

$$\int_{[0,T]\times D} f^2(x,t,s)dxdt < +\infty \qquad \text{for all } s \in I_z. \tag{B.5}$$

Such a requirement obviously excludes random vector y that is unbounded below, e.g. Gaussian distribution. Throughout the paper we will only consider random variables that are bounded from below.

And also, for any fixed random $z \in I_z$, (B.1) is a deterministic partial differential equation, and we recall the notion of the weak solution for the deterministic partial differential equation: we say that $v(x,t,y)$ is a weak solution if it satisfies the initial condition, $u(x,0,z) = u_0$ at $t = 0$, and $u \in L^2(0,T;H_0^1(D)), \partial_t(v) \in L^2(0,T;H_0^1(D))$, and a.e. in $[0,T]$

$$\int_D \partial_t vw dx + \int_D (k(x,y)\nabla u \cdot \nabla w) dx = \int_D fw dx, \qquad \text{for all } w \in H_0^1(D). \tag{B.6}$$

By means of energy estimation, assumptions (B.4) and (B.5), there exists a unique solution $v \in L^2(0,T,H_0^1(D))$, and the following estimate holds

$$\|v(T)\|^2_{L^2(D)} + k_{min}\|u\|^2_{L^2(0,T;H_0^1(D))} \leq \frac{C_D^2}{k_{min}}\|f(x,t,y)\|^2_{L^2([0,T]\times D)} + \|u_0\|^2_{L^2(D)}, \tag{B.7}$$

where C_D is the poincare constant satisfying $\|u\|_{L^2(D)} \leq C_D\|\nabla u\|_{L^2(D)}$, for all $v \in H_0^1(D)$.

Let $\{\phi_i\}_1^\infty$ be d-variate orthonormal Polynomials with weight function $\rho(\xi)$. Define the tensor product Hilbert Spaces

$$H^k(D) \bigotimes L^2(I_z, \mathcal{F}, \rho) := \{v = v(x,\xi); \int_{I_z} \|v(x,\xi)\|^2_{H^k(D)}\rho(\xi)d(\xi) < \infty\}$$

equipped with the norm

$$|||v|||_k := \sqrt{\int_{I_z} \|v(x,\xi)\|^2_{H^k(D)}\rho(\xi)d_\xi}.$$

Since $\{\Phi_m(Z)\}_{m=1}^\infty$ be an orthonormal basis with weight $\rho(\xi)$ in space I_z, we can write $v(x,\xi) = \sum_{i=1}^\infty v_i(x)\Phi_i(\xi)$, and $v \in H^k(D) \otimes L^2(I_z, \mathcal{F}, \rho)$ if and only if

$$\sum_{i=1}^\infty \|v_i\|^2_{H^k(D)} < \infty, \text{ and } \quad |||v|||_{k,D,I_z} = \sqrt{\sum_{i=1}^\infty \|v_i\|^2_{H^k(D)}} < \infty. \tag{B.8}$$

Thus we can identify $v \in H^k(D) \otimes L^2(I_z, \mathcal{F}, \rho)$ with $V = (v_1, v_2, \cdots) \in \mathbb{H}^k(D)$, where

$$\mathbb{H}^k(D) := \{V = (v_1, v_2, \cdots), \ \sum_{i=1}^\infty \|v_i\|^2_{H^k(D)} < \infty\},$$

with $|||V|||_{k,D,I_z}$ defined the same as in $|||v(x,\xi)|||_k$. In what follows, we shall not distinguish $H^k(D) \otimes L^2(I_z, \mathcal{F}, \rho)$ from $\mathbb{H}^k(D)$.

Note: In order to simplify the notation, we denote that $|||V||| = \sqrt{\sum_{i=1}^{\infty} \|v_i\|^2_{L^2(D)}}$.

Assume that u, u_0 and f have the following GPC expansions.

$$u(x,t,z) = \sum_{i=1}^{\infty} u_i(x,t)\phi_i(z). \tag{B.9}$$

$$u_0(x,z) = \sum_{i=1}^{\infty} u_i^0(x)\phi_i(z). \tag{B.10}$$

$$f(x,t,z) = \sum_{i=1}^{\infty} f_i(x,t)\phi_i(z). \tag{B.11}$$

Then (B.1) is equivalent to

$$\begin{cases} \frac{\partial u_i(x,t)}{\partial t} - \sum_{i=0}^{\infty} \nabla \cdot (a_{ij}\nabla u_i) = f_j, & x \in D \subset \mathbb{R}^n, t \in [0,T], j = 1, 2, \cdots, \\ u_j(x,t) \mid_{\partial D} = 0, & t \in [0,T], \\ u_j(x,0) = u_j^0, & x \in \bar{D} \subset \mathbb{R}^n, \end{cases} \tag{B.12}$$

where $a_{ij}(x) = \sum_{l=0}^{d} k_l(x)\mathbb{E}(z_l\phi_i\phi_j)$.

Denote

$$\begin{aligned} &A = (a_{ij}(x)), \\ &U = (u_1(x,t), u_2(x,t), \cdots), \\ &U_0 = (u_1^0, u_2^0, \cdots), \\ &F = (f_1(x,t), f_2(x,t), \cdots). \end{aligned}$$

Alternatively, (B.12) can be written as

$$\begin{cases} \frac{\partial U}{\partial t} - \nabla \cdot (A\nabla U) = F, & x \in D \subset \mathbb{R}^n, t \in [0,T], \\ U(x,t) \mid_{\partial D} = 0, & t \in [0,T], \\ U(x,0) = U_0, & x \in \bar{D} \subset \mathbb{R}^n. \end{cases} \tag{B.13}$$

On the other hand, the Pth-order, gpc approximations of $u(x,t,z)$, $u_0(x,z)$ and $f(x,t,z)$ are

$$u(x,t,z) \approx \sum_{i=1}^{N} \hat{u}_i(x,t)\phi_i(z), \quad u_0(x,z) \approx \sum_{i=1}^{N} u_j^0\phi_i(z), \quad f(x,t,z) \approx \sum_{i=1}^{N} f_i(x,t)\phi_i(z). \tag{B.14}$$

Substituting (B.14) in to (B.1) and projecting the result equation onto the subspace spanned by the first N gPC basis polynomials, we obtain for

$$\begin{cases} \frac{\partial \hat{u}_i(x,t)}{\partial t} - \sum_{i=0}^{N} \nabla \cdot (a_{ij} \nabla \hat{u}_i) = f_j, & x \in D \subset \mathbb{R}^n, t \in [0,T], j = 1, 2, \cdots, N \\ \hat{u}_j(x,t) \mid_{\partial D} = 0, & t \in [0,T], \\ \hat{u}_j(x,0) = u_j^0, & x \in \bar{D} \subset \mathbb{R}^n, \end{cases} \tag{B.15}$$

where $a_{ij}(x) = \sum_{l=0}^{d} k_l(x) \mathbb{E}(z_l \phi_i \phi_j) \qquad 1 \leq i, j \leq N.$

Definite finite dimensional subspaces $P_N(D)$, as follow.

$$\mathbb{H}^k(P_N(D)) := \{V = (v_1, v_2, \cdots), \sum_{i=1}^{N} \|v_i\|^2_{H^k(D)} < \infty\},$$

where

$$v_{\mathbf{m}} = 0, \quad \forall \ \ |\mathbf{m}| > N.$$

Alternatively, the GPC approximation to the solution u is to find $U_N \in \mathbb{H}^k(P_N(D))$ such that

$$\begin{cases} \frac{\partial U^N}{\partial t} - \nabla \cdot (A_N(x) \nabla U_N(x)) = F_N(x,t), & x \in D \subset R^n, t \in [0,T], \\ U_N(x,t) \mid_{\partial D} = 0, & t \in [0,T], \\ U_N(x,0) = U_0^N, & x \in \bar{D} \subset \mathbb{R}^n, \end{cases} \tag{B.16}$$

where

$$A_N = (a_{\mathbf{ij}}),$$
$$F_N = (f_1, \cdots, f_N, \cdots),$$
$$U_0^N = (u_1^0, \cdots, u_N^0, \cdots).$$

Here $a_{\mathbf{ij}} = 0 \qquad \forall \quad |i| > N, or |j| > N$, and $f_{\mathbf{m}} = 0, \qquad \forall \quad |m| > N.$

The following proposition is refer from [51].

Lemma B.1. *Assume that z_i are independent beta distributions or exponential distributions. Then A is a diagonally dominant matrix. In fact, we have that*

$$a_{ii} \geq k_{min} + \sum_{j=1, j \neq i}^{\infty} |a_{ij}|, \quad i = 1, 2, \cdots.$$

Lemma B.2. *Assume that a symmetric $M \times M$ matrix $A = (a_{i,j})$ satisfies*

$$a_{ii} \geq k + \sum_{j=1, j\neq i}^{\infty} |a_{ij}|, \quad i = 1, 2, \cdots, M \tag{B.17}$$

and set

$$\hat{D} = diag(A), \qquad A = \hat{D} + S. \tag{B.18}$$

Then,

$$2|u^t S v| \leq u^t(\hat{D} - kI)u + v^t(\hat{D} - kI)v, \qquad \text{for all } u, v \in \mathcal{R}^M \tag{B.19}$$

and

$$k_{min} u^t u \leq u^t A u \leq k_{max} u^t u, \qquad \text{for all } u \in \mathcal{R}^M. \tag{B.20}$$

Theorem B.3. *Let U be the solution to* (B.13) *and U_N the solution the the gpc approximation* (B.16). *Denote that that W be the projection of U in to the space $\mathbb{H}^k(P_N(D))$, then we can get*

$$\begin{aligned} &\tfrac{1}{2}|||(U - U_N)(T)|||^2 + \tfrac{k_{min}}{2}\int_0^T |||\nabla(U - U_N)|||^2 dt \\ &\leq |||(U - W)(T)|||^2 + (\tfrac{k_{max}^2}{k_{min}} + k_{min})\int_0^T |||\nabla(U - W)|||^2 dt \\ &+ \tfrac{C_D^2}{k_{min}}\int_0^T |||\partial_t(W - U)|||^2 dt. \end{aligned} \tag{B.21}$$

Proof. Taking the inner product, with respect to x in $L^2(D)$, of (B.13) with $W - U^N$ and integration by parts.

we have

$$(\frac{\partial U}{\partial t}, (W - U_N)) + (A(x)\nabla U, \nabla(W - U_N)) = (F, W - U_N). \tag{B.22}$$

Similarly, apply it to (B.16),

$$(\frac{\partial U_N}{\partial t}, W - U_N) + (A_N(x)\nabla U_N, \nabla(W - U_N)) = (F^N, W - U_N), \tag{B.23}$$

we know that

$$(A_N(x)\nabla U_N, \nabla(W - U_N)) = (A(x)\nabla U_N, \nabla(W - U_N)), then, \tag{B.24}$$

$$(\frac{\partial U_N}{\partial t}, W - U_N) + (A(x)\nabla U_N, \nabla(W - U_N)) = (F^N, W - U_N). \tag{B.25}$$

Subtracted (B.22) by (B.25) , we get

$$(\frac{\partial U - U_N}{\partial t}, W - U_N) + (A(x)\nabla(U - U_N), \nabla(W - U_N)) = 0. \tag{B.26}$$

Also, we have

$$\begin{aligned}
&(\tfrac{\partial(W-U_N)}{\partial t}, W - U_N) + k_{min}|||\nabla(W - U_N)|||^2 \\
&= (\tfrac{\partial(W-U_N)}{\partial t}, W - U_N) + k_{min}(\nabla(W - U_N), \nabla(W - U_N)) \\
&= (\tfrac{\partial(W-U)}{\partial t,} W - U_N) + (\tfrac{\partial(U-U_N)}{\partial t}, W - U_N) \\
&+k_{min}(\nabla(W - U_N), \nabla(W - U_N)) \\
&\leq (\tfrac{\partial(W-U)}{\partial t}, W - U_N) + (\tfrac{\partial(U-U_N)}{\partial t}, W - U_N) \\
&+(A\nabla(W - U_N), \nabla(W - U_N))) \\
&= (\tfrac{\partial(W-U)}{\partial t}, W - U_N) + (\tfrac{\partial(U-U_N)}{\partial t}, W - U_N) \\
&+(A\nabla(U - U_N), \nabla(W - U_N))) + (A\nabla(W - U), \nabla(W - U_N))).
\end{aligned} \tag{B.27}$$

by (B.26), we have

$$\begin{aligned}
&(\tfrac{\partial(W-U_N)}{\partial t}, W - U_N) + k_{min}|||\nabla(W - U_N)|||^2 \\
&\leq (\tfrac{\partial(W-U)}{\partial t}, W - U_N) + (A\nabla(W - U), \nabla(W - U_N))).
\end{aligned} \tag{B.28}$$

Now estimate the terms in above

$$\begin{aligned}
(\tfrac{\partial(W-U)}{\partial t}, W - U_N) &\leq |||\partial(W - U)||||||W - U_N||| \\
&\leq C_D|||\partial_t(W - U)||||||\nabla(W - U_N)|||,
\end{aligned}$$

and

$$(A\nabla(W - U), \nabla(W - U_N))) \leq k_{max}|||\nabla(W - U)||||||\nabla(W - U_N)|||.$$

Then apply the inequality $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ and integral from 0 to T, we have

$$\begin{aligned}
&\tfrac{1}{2}|||(W - U_N)(T)|||^2 + \tfrac{k_{min}}{2}\int_0^T |||\nabla(U_N - W)|||^2 dt \\
&\leq \tfrac{C_D^2}{k_{min}}\int_0^T |||\partial_t(W - U)|||^2 dt + \tfrac{k_{max}^2}{k_{min}}\int_0^T |||\nabla(W - U)|||^2 dt.
\end{aligned} \tag{B.29}$$

So, by triangle inequality

$$\begin{aligned}
&\tfrac{1}{2}(|||(U - U_N)(T)|||^2 + k_{min}\int_0^T |||\nabla(U - U_N)|||^2 dt) \\
&\leq |||(W - U_N)(T)|||^2 + k_{min}\int_0^T |||\nabla(W - U_N)|||^2 dt \\
&+|||(U - W)(T)|||^2 + k_{min}\int_0^T |||\nabla(U - W)|||^2 dt.
\end{aligned} \tag{B.30}$$

In this way, we have

$$\begin{aligned}
&\tfrac{1}{2}\||(U-U_N)(T)\||^2 + \tfrac{k_{min}}{2}\int_0^T \||\nabla(U-U_N)\||^2 dt \\
&\leq \||(U-W)(T)\||^2 + \left(\tfrac{k_{max}^2}{k_{min}} + k_{min}\right)\int_0^T \||\nabla(U-W)\||^2 dt \\
&+\tfrac{C_D^2}{k_{min}}\int_0^T \||\partial_t(W-U)\||^2 dt.
\end{aligned} \tag{B.31}$$

□

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