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Abstract

The goal of this paper is to derive a modified formulation of the finite-horizon LQR problem, which can be cast as semidefinite programming problems (SDPs). In addition, based on the the Lagrangian duality, its dual problem is studied. We establish connections between the proposed primal-dual conditions with existing results. As an application of the proposed results, the decentralized LQR analysis and design problems are addressed. Especially, using the structure of the derived LQR formulations, a sufficient but simple and convex surrogate problem is developed for solving decentralized LQR design problems.

1 Introduction

In this paper, the finite-horizon linear quadratic regulator (LQR) problem is considered. The goal is to investigate a semidefinite programming (SDP) formulation of the finite-horizon LQR problem. It is well known that both finite-horizon and infinite-horizon LQR problems can be transformed into SPD problems (see for example, [1–4]). Recently, another SDP formulation of the finite-horizon LQR problem was proposed in [5]. This method is especially attractive because it converts the LQR problem into the optimal covariance matrix selection problem, and it can be also interpreted as a dual problem of the standard LQR approaches based on the Riccati equations or the Lyapunov methods. The first main result of this paper is a proposition of a modified SDP problem for the infinite-horizon LQR. Compared to the SDP problem in [5], the proposed SDP problem includes explicitly the static feedback gain parameters, and it may enjoy some properties that make it especially useful when special structures are imposed on the feedback gain over the finite time-horizon.

On the other hand, we study a dual counterpart of the proposed LQR formulation by using the Lagrangian duality in optimization theories [6]. There are several duality relations in systems and control theory, which have attracted much attention during the last decades. For instance, a new proof of Lyapunov's matrix inequality was presented in [7] based on the standard semidefinite programming (SDP) duality [8]. In addition, SDP formulations of the LQR problem and their dual formulations were developed in [3] and [4]. Comprehensive studies on the SDP dualities in systems

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and control theory, such as the Kalman-Yakubovich-Popov (KYP) lemma, the LQR problem, and the H_∞-norm computation, were provided in [9]. More recent results include the state-feedback solution to the LQR problem [5], the generalized KYP lemma and H_∞ analysis [10, 11] derived using the Lagrangian duality. In this paper, we derive a dual SDP problem of the proposed LQR formulation and establish a connection between the proposed dual problem and the Q-function approach to the LQR problem considered in [12,13]. In addition, some equivalence relations between the proposed primal and dual formulations and those in [5] are addressed.

Finally, it is proved that the proposed primal LQR formulation can be applied to a more general class of problems, the structured static state-feedback LQR designs including the distributed and decentralized LQR design problems. A sufficient but simple SDP relaxation of the decentralized LQR design problem is developed based on the methods developed in [2].

This paper is organized as follows. Section II presents the standard finite-horizon LQR problem and the proposed formulation consisting of an optimization problem subject to convex matrix inequalities. Section III provides its dual problems, and Section IV discusses connections between the proposed formulations and existing results. In Section V, a convex approximation of the decentralized LQR problem is addressed, and finally, Section VI concludes the paper.

Notation: The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative and positive integers, respectively; $\mathbb{R}$: set of real numbers; $\mathbb{R}_+$: set of nonnegative real numbers; $\mathbb{R}_{++}$: set of positive real numbers; $\mathbb{R}^n$: n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$: set of all $n\times m$ real matrices; A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; I_n: $n\times n$ identity matrix; $\mathbb{S}^n$: symmetric $n\times n$ matrices; $\mathbb{S}^n_+$: cone of symmetric $n\times n$ positive semi-definite matrices; $\mathbb{S}^n_{++}$: symmetric $n\times n$ positive definite matrices; $\mathbf{Tr}(A)$: trace of matrix A.

2 Finite-horizon LQR problem

Consider the stochastic LTI system

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{1}$$

where $k \in \mathbb{N}$, $x(k) \in \mathbb{R}^n$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input vector, $x(0) \sim \mathcal{N}(0, W_f)$ and $w(k) \sim \mathcal{N}(0, W)$ are mutually independent Gaussian random vectors. In this section, the finite-horizon stochastic LQR problem will be considered.

Problem 1 (Finite-horizon LQR problem)**.** *Solve*

$$\begin{aligned}&\min_{F_0,\ldots,F_{N-1}\in\mathbb{R}^{m\times n}} \mathbf{E}(x(k)^T Q_f x(k)) + \sum_{k=0}^{N-1} \mathbf{E}\left(\begin{bmatrix} x(k)\\ u(k)\end{bmatrix}^T \begin{bmatrix} Q & 0\\ 0 & R\end{bmatrix}\begin{bmatrix} x(k)\\ u(k)\end{bmatrix}\right)\\ &\text{subject to}\quad x(k+1) = Ax(k) + Bu(k) + w(k)\\ &u(k) = F_k x(k)\end{aligned}$$

A collection of assumptions that will be used throughout the paper is summarized below.

Assumption 1. *The following assumptions are made:*

1. $Q_f \succeq 0$, $Q \succeq 0$, $R \succ 0$, $W_f \succ 0$, *and* $W \succ 0$*;*

2. (A, B) is stabilizable and (A, Q) is detectable.

If we define the covariance of the augmented vector $[x(k)^T, u(k)^T]^T \in \mathbb{R}^{n\times m}$

$$\mathbf{S}_k = \mathbf{E}\left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T\right),\ k \in \{0,\, 1, \dots,\, N\}$$

then, Problem 1 can be equivalently converted to the matrix equality constrained optimization problem.

Problem 2. *Solve*

$$\begin{aligned} J_p^* := &\min_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_{N-1}\in\mathbb{S}^{n+m} \\ F_0,\dots,F_{N-1}\in\mathbb{R}^{m\times n}}} J_p(\{\mathbf{S}_k\}_{k=0}^{N-1}) \\ &\text{subject to} \\ &\Phi(F_k,\, \mathbf{S}_{k-1}) = \mathbf{S}_k \quad k \in \{1,\, 2, \dots,\, N-1\} \\ &\begin{bmatrix} I_n \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I_n \\ F_0 \end{bmatrix}^T = \mathbf{S}_0 \end{aligned} \tag{2}$$

where

$$J_p(\{\mathbf{S}_k\}_{k=0}^{N-1}) := \mathbf{Tr}\left(Q_f\left(\begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{N-1}\begin{bmatrix} A^T \\ B^T \end{bmatrix} + W\right)\right) + \sum_{k=0}^{N-1}\mathbf{Tr}\left(\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\mathbf{S}_k\right)$$

$$\Phi(F,\, \mathbf{S}) := \begin{bmatrix} I_n \\ F \end{bmatrix}\left(\begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}\begin{bmatrix} A^T \\ B^T \end{bmatrix} + W\right)\begin{bmatrix} I_n \\ F \end{bmatrix}^T$$

In Problem 2, the matrix equality constraints represent the covariance updates. Note that the formulation of Problem 2 is a modified version of the problem in [5, Propisition 1]. The difference is that the gain parameters explicitly appear in the covariance update equations of Problem 2, while this is not the case for [5, Propisition 1]. Later, a relation between Problem 2 and [5, Propisition 1] will be shown.

Since the left-hand side of the matrix equalities are not linear, it is not clear whether or not the optimization in Problem 2 is convex. Instead of dealing with Problem 2 in its present form involving the matrix equality constraints, we will consider the modified problem by replacing the matrix equalities in Problem 2 by inequalities.

Problem 3. *Solve*

$$\begin{aligned} p_{\mathbf{opt}} := &\min_{\substack{\mathbf{S}_0,\dots,\mathbf{S}_{N-1}\in\mathbb{S}^{n+m} \\ F_0,\dots,F_{N-1}\in\mathbb{R}^{m\times n}}} J_p(\{\mathbf{S}_k\}_{k=0}^{N-1}) \\ &\text{subject to} \\ &\Phi(F_k,\, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k, \quad k \in \{1,\, 2, \dots,\, N-1\} \\ &\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \preceq \mathbf{S}_0 \end{aligned} \tag{3}$$

The goal of this section is to study properties of Problem 3 and find relations between Problem 2 and Problem 3. The following results can be established first.

Proposition 1. *The following statements are true:*

1. *The optimization (2) is convex;*
2. *The optimization (2) is strictly feasible.*

Proof. Proof of statement 1): It is enough to prove that the constraints can be equivalently converted to linear matrix inequality (LMI) constraints so that the optimization is a convex semidefinite programming problem (SDP). It can be readily done by using the modified Schur complement in [14, Theorem 1]. More precisely, we claim that $\Phi(F_k, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k$ holds if and only if there exists $G_k \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} \mathbf{S}_k & * \\ \begin{bmatrix} G_k & G_k F_k^T \end{bmatrix} & \left(G_k + G_k^T - \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{k-1} \begin{bmatrix} A^T \\ B^T \end{bmatrix} - W \right) \end{bmatrix} \succeq 0 \tag{4}$$

To prove the necessity of the claim, suppose that $\Phi(F_k, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k$ holds. Since $D_k := \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_k \begin{bmatrix} A^T \\ B^T \end{bmatrix} + W \succ 0$, by the Schur complement argument, $\Phi(F_k, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k$ holds if and only if $\begin{bmatrix} \mathbf{S}_k & * \\ \begin{bmatrix} D_k & D_k F_k^T \end{bmatrix} & D_k \end{bmatrix} \succeq 0$, which equivalent to (4) with $G_k = D_k$, proving the necessary part. To prove the sufficiency, suppose that (4) is satisfied. Then, by algebraic manipulations, it can be proved that pre- and post-multiplying both sides of (4) by $\begin{bmatrix} -I & 0 \\ 0 & -I \\ I & F_k^T \end{bmatrix}^T$ and its transpose yield $\Phi(F_k, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k$. Thus, the claim is proved. Lastly, with the change of variables $G_k F_k^T = L_k$, (4) is equivalent to an LMI. Therefore, the optimization is equivalent to the convex SDP, completing the proof of the statement 1).

Proof of the statement 2): With $F_k = 0$, $\forall k \in \{0, 1, \dots, N-1\}$ and any $\varepsilon > 0$, construct matrices $\{\mathbf{S}_k\}_{k=0}^{N-1}$ as follows:

$$\begin{bmatrix} I_n \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I_n \\ F_0 \end{bmatrix}^T + \varepsilon I_n = \mathbf{S}_0$$
$$\Phi(F_k, \mathbf{S}_{k-1}) + \varepsilon I_n = \mathbf{S}_k$$

The set $\{F_k, \mathbf{S}_k\}_{k=0}^{N-1}$ satisfies the constraints of (2) with strict inequalities. The complete the proof of the statement 2). □

To proceed, denote by $\mathcal{S}$ the set of all optimal solutions of the form $\{(F_k, \mathbf{S}_k)\}_{k=0}^{N-1}$ of (3). In addition, define the projection mapping $\mathcal{F} := \{\{F_k\}_{k=0}^{N-1} : \{(F_k, \mathbf{S}_k)\}_{k=0}^{N-1} \in \mathcal{S}\}$. It is not clear whether or not the optimal solution of (3) is unique. Even if the optimal solution of (3) is not unique, then it is also not clear whether or not all the optimal solutions take equalities in the constraint of (3). However, we can draw a conclusion that if $\{F_k\}_{k=0}^{N-1} \in \mathcal{F}$, then it is also optimal for Problem 2.

Proposition 2. *If $\{F_k\}_{k=0}^{N-1} \in \mathcal{F}$, then it is also optimal for Problem 2.*

Proof. Let $\{F_k\}_{k=0}^{N-1} \in \mathcal{F}$ and construct $\{\bar{\mathbf{S}}_k\}_{k=0}^{N-1}$ such that

$$\Phi(F_k, \bar{\mathbf{S}}_{k-1}) = \bar{\mathbf{S}}_k, \quad k \in \{1, 2, \dots, N-1\}$$
$$\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T = \bar{\mathbf{S}}_0$$

Clearly, $\bar{\mathbf{S}}_k \preceq \mathbf{S}_k, \forall k \in \{0, 1, \dots, N-1\}$ and hence, $p_{\mathbf{opt}} \geq J_p(\{\bar{\mathbf{S}}_k\}_{k=0}^{N-1})$. However, since $\{F_k, \bar{\mathbf{S}}_k\}_{k=0}^{N-1}$ is also a feasible point of (3), and thus, $J_p(\{\bar{\mathbf{S}}_k\}_{k=0}^{N-1}) \geq p_{\mathbf{opt}}$. Therefore, $J_p(\{\bar{\mathbf{S}}_k\}_{k=0}^{N-1}) = p_{\mathbf{opt}}$ and $\{F_k, \bar{\mathbf{S}}_k\}_{k=0}^{N-1}$ is an optimal solution of (3). Since the problem (2) has a feasible set included by the feasible set of (3), and the optimal solution $\{F_k, \bar{\mathbf{S}}_k\}_{k=0}^{N-1}$ of (3) takes equalities in the constraints of (3), $\{F_k, \bar{\mathbf{S}}_k\}_{k=0}^{N-1}$ is also optimal solution of (2). This completes the proof. □

Corollary 1. *$J_p^* = p_{\mathbf{opt}}$ holds.*

Proof. Straightforward from Proposition 2. □

3 Dual to the finite-horizon LQR problem

The aim of this section is to find a dual formulation of Problem 3 using the Lagrangian duality [6, chapter 5]. For any $\mathbf{P}_0, \dots, \mathbf{P}_{N-1} \in \mathbb{S}_+^{n+m}$ and $\bar{\mathbf{P}}_0, \dots, \bar{\mathbf{P}}_{N-1} \in \mathbb{S}_+^{n+m}$, define the Lagrangian of the problem (3)

$$\mathcal{L}(\{(\mathbf{S}_k, F_k, \mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) := J_p(\{\mathbf{S}_k\}_{k=0}^{N-1}) + \sum_{k=1}^{N-1} \mathbf{Tr}((\Phi(F_k, \mathbf{S}_{k-1}) - \mathbf{S}_k)\mathbf{P}_k)$$
$$+ \mathbf{Tr}\left(\left(\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T - \mathbf{S}_0\right)\mathbf{P}_0\right) - \sum_{k=0}^{N-1} \mathbf{Tr}(\mathbf{S}_k \bar{\mathbf{P}}_k)$$

Rearranging some terms, it can be represented by

$$\begin{aligned} &\mathcal{L}(\{(\mathbf{S}_k, F_k, \mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) = J_d(\{\mathbf{P}_k, F_k\}_{k=0}^{N-1}) \\ &+ \mathbf{Tr}\left(\left(\begin{bmatrix} A^T \\ B^T \end{bmatrix} Q_f \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T - \mathbf{P}_{N-1} + \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} - \bar{\mathbf{P}}_{N-1}\right)\mathbf{S}_{N-1}\right) \\ &+ \sum_{k=1}^{N-1} \mathbf{Tr}((\Gamma(F_k, \mathbf{P}_k) - \mathbf{P}_{k-1} - \bar{\mathbf{P}}_{k-1})\mathbf{S}_{k-1}) \end{aligned} \tag{5}$$

Consider the dual problem

$$\begin{aligned} d_{\mathbf{opt}} &:= \sup_{\substack{\mathbf{P}_k \succeq 0, \bar{\mathbf{P}}_k \succeq 0 \\ k \in \{0, \dots, N-1\}}} \mathcal{D}(\{(\mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) \\ &= \sup_{\substack{\mathbf{P}_k \succeq 0, \bar{\mathbf{P}}_k \succeq 0 \\ k \in \{0, \dots, N-1\}}} \inf_{\{\mathbf{S}_k, F_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k, F_k, \mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) \end{aligned} \tag{6}$$

where

$$\mathcal{D}(\{(\mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) := \inf_{\{\mathbf{S}_k, F_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k, F_k, \mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1})$$

Theorem 1. *The following statements are true:*

1. *The strong duality holds, i.e.,* $p_{\mathbf{opt}} = d_{\mathbf{opt}}$*;*
2. *Consider the Riccati equation*

$$A^T X_{k+1} A - A^T X_{k+1} B (R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A + Q = X_k \tag{7}$$

for all $k \in \{0, 1, \ldots, N-1\}$ *with* $X_N = Q_f$*, and define* $\{(\mathbf{S}_k, F_k, \mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}$ *with*

$$\begin{gathered} F_k = -(R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A \\ \mathbf{S}_k = \Phi(F_k, \mathbf{S}_{k-1}), \quad \mathbf{S}_0 = \begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \\ \mathbf{P}_k = \begin{bmatrix} Q + A^T X_{k+1} A & A^T X_{k+1} B \\ B^T X_{k+1} A & R + B^T X_{k+1} B \end{bmatrix} \\ \bar{\mathbf{P}}_k = 0, \quad k \in \{0, 1, \ldots, N-1\} \end{gathered} \tag{8}$$

Then, $\{(\mathbf{S}_k, F_k)\}_{k=0}^{N-1}$ *is an primal optimal point of* (3) *and* $\{(\mathbf{P}_k, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}$ *is the corresponding dual optimal point of* (3)*.*

Proof. Proof of the statement 1): By Proposition 1, the optimization is convex and strictly feasible. By the Slater's condition [6], the strong duality holds.

Proof of the statement 2): Since the primal feasible point of (3) for $\{\mathbf{S}_0\}_{k=0}^{N-1} \subset \mathbb{S}^{n+m}$ are guaranteed to be positive semidefinite, the constraints $\mathbf{S}_k \succeq 0$, $k \in \{0, 1, \ldots, N-1\}$ can be added to (3) without changing the optimal solution set as well as the feasible set. From the KTT condition of the generalized inequality constrained optimization in [6, chapter 5.9.2], its KKT condition can be summarized as the primal feasibility condition

$$\begin{aligned} & \begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \preceq \mathbf{S}_0, \quad \Phi(F_k, \mathbf{S}_{k-1}) \preceq \mathbf{S}_k \\ & k \in \{1, 2, \ldots, N-1\} \\ & \mathbf{S}_k \succeq 0, \quad k \in \{0, 1, \ldots, N-1\} \end{aligned}$$

the complementary slackness condition

$$\begin{gathered} \mathbf{Tr}\left(\left(\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T - \mathbf{S}_0\right) \mathbf{P}_0\right) = 0 \\ \mathbf{Tr}((\Phi(F_k, \mathbf{S}_{k-1}) - \mathbf{S}_k)\mathbf{P}_k) = 0 \\ k \in \{1, 2, \ldots, N-1\} \\ \mathbf{Tr}(\mathbf{S}_k \bar{\mathbf{P}}_k) = 0, \quad k \in \{0, 1, \ldots, N-1\} \end{gathered} \tag{9}$$

and the dual feasibility condition

$$\begin{aligned} & \mathbf{P}_N = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma(0, \mathbf{P}_N) - \bar{\mathbf{P}}_{N-1} = \mathbf{P}_{N-1} \\ & \Gamma(F_k, \mathbf{P}_k) - \bar{\mathbf{P}}_{k-1} = \mathbf{P}_{k-1}, \quad k \in \{1, 2, \ldots N-1\} \\ & W_f(P_{0,12} + F_0^T P_{0,22}) + (P_{0,12}^T + P_{0,22} F_0) W_f = 0 \end{aligned}$$

$$
\begin{aligned}
& M_k(P_{k+1,\,12} + F_{k+1}^T P_{k+1,\,22}) + (P_{k+1,\,12}^T + P_{k+1,\,22}F_{k+1})M_k = 0 \\
& k \in \{1,\, 2, \ldots,\, N-1\} \\
& \mathbf{P}_k \succeq 0, \quad \bar{\mathbf{P}}_k \succeq 0,\, k \in \{0,\, 1, \ldots N-1\}
\end{aligned}
$$

where $M_k = \begin{bmatrix} A & B \end{bmatrix} \mathbf{S}_k \begin{bmatrix} A & B \end{bmatrix}^T + W$. By plugging (8) into the KKT condition, it can be proved that they satisfy the KKT. Since the problem (3) is convex, the point (8) is the primal and dual optimal points of (3). This completes the proof. □

In what follows, we introduce an explicit dual formulation of Problem 3.

Problem 4. *Solve*

$$
\begin{aligned}
& \tilde{d}_{\mathbf{opt}} := \max_{\mathbf{P}_0,\ldots,\,\mathbf{P}_{N-1} \in \mathbb{S}_+^{n+m}} \; J_d(\{\mathbf{P}_k,\, F_k\}_{k=0}^{N-1}) \qquad (10) \\
& \text{subject to} \\
& \Gamma(F_k,\, \mathbf{P}_k) \succeq \mathbf{P}_{k-1}, \quad k \in \{1,\, 2, \ldots N-1\} \\
& \Gamma(0,\, \mathbf{P}_N) \succeq \mathbf{P}_{N-1} \\
& \begin{bmatrix} 0 \\ I \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} 0 \\ I \end{bmatrix} \succ 0, \quad F_k = -P_{k,\,22}^{-1} P_{k,\,12}^T \\
& k \in \{0,\, 1, \ldots N-1\}
\end{aligned}
$$

where

$$
\begin{aligned}
& J_d(\{\mathbf{P}_k,\, F_k\}_{k=0}^{N-1}) := \mathbf{Tr}\left(\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \mathbf{P}_0 \right) + \sum_{k=1}^{N} \mathbf{Tr}\left(\begin{bmatrix} I \\ F_k \end{bmatrix} W \begin{bmatrix} I \\ F_k \end{bmatrix}^T \mathbf{P}_k \right) \\
& \Gamma(F,\, \mathbf{P}) := \begin{bmatrix} A^T \\ B^T \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix}^T \mathbf{P} \begin{bmatrix} I \\ F \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T + \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}
\end{aligned}
$$

and

$$
\mathbf{P}_k = \begin{bmatrix} P_{k,\,11} & P_{k,\,12} \\ P_{k,\,12}^T & P_{k,\,22} \end{bmatrix}, \quad \mathbf{P}_N = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}
$$

Theorem 2. *The following statements are true:*

1. $d_{\mathbf{opt}} \geq \tilde{d}_{\mathbf{opt}}$ *holds;*
2. *The lower bound is tight, i.e.,* $d_{\mathbf{opt}} = \tilde{d}_{\mathbf{opt}}$*;*
3. *Consider the Riccati equation (7). An optimal point of (10) is* $\{\mathbf{P}_k\}_{k=0}^{N-1}$ *with*

$$
\mathbf{P}_k = \begin{bmatrix} Q + A^T X_{k+1} A & A^T X_{k+1} B \\ B^T X_{k+1} A & R + B^T X_{k+1} B \end{bmatrix} \qquad (11)
$$

Proof. Proof of statement 1): Define the set

$$
\mathcal{F} := \left\{ \mathbf{P} \in \mathbb{S}_+^{n+m} : \begin{bmatrix} 0 \\ I_m \end{bmatrix}^T \mathbf{P} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \succ 0 \right\}
$$

Then, it is clear that the dual optimal objective function value is lower bounded as follows:

$$\begin{aligned}
&\sup_{\substack{\mathbf{P}_k \succeq 0,\, \bar{\mathbf{P}}_k \succeq 0 \\ k\in\{0,\ldots,\, N-1\}}} \inf_{\{\mathbf{S}_k,\, F_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k,\, F_k,\, \mathbf{P}_k,\, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) \\
&\geq \sup_{\substack{\mathbf{P}_k \in \mathcal{F},\, \bar{\mathbf{P}}_k \succeq 0 \\ k\in\{0,\ldots,\, N-1\}}} \inf_{\{\mathbf{S}_k,\, F_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k,\, F_k,\, \mathbf{P}_k,\, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1})
\end{aligned} \tag{12}$$

Now, let us focus on the term in the Lagrangian (5) $\sum_{k=1}^{N-1} \mathbf{Tr}(\Gamma(F_k,\, \mathbf{P}_k)\mathbf{S}_{k-1})$, which can be represented by

$$\sum_{k=1}^{N-1} \mathbf{Tr}(\Gamma(F_k,\, \mathbf{P}_k)\mathbf{S}_{k-1}) = \sum_{k=1}^{N-1} \mathbf{E}\left(\begin{bmatrix} z \\ F_k z \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} z \\ F_k z \end{bmatrix} \right) + \sum_{k=1}^{N-1} \mathbf{Tr}\left(\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \mathbf{S}_{k-1} \right)$$

where $z := Ax(k-1) + Bu(k-1)$, and is lower bounded as follows:

$$\sum_{k=1}^{N-1} \mathbf{Tr}(\Gamma(F_k,\, \mathbf{P}_k)\mathbf{S}_{k-1}) \geq \sum_{k=1}^{N-1} \mathbf{E}\left(\min_{u\in\mathbb{R}^m} \left(\begin{bmatrix} z \\ u \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} z \\ u \end{bmatrix} \right) \right) + \sum_{k=1}^{N-1} \mathbf{Tr}\left(\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \mathbf{S}_{k-1} \right)$$

The function $\begin{bmatrix} z \\ u \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} z \\ u \end{bmatrix}$ inside the bracket is a convex quadratic function and has a unique optimizer $u^* = -P_{22,\,k}^{-1} P_{12,\,k}^T z$ if $\begin{bmatrix} 0 \\ I_m \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} 0 \\ I_m \end{bmatrix} \succ 0$. Therefore, the dual optimal objective function value (12) has the lower bound

$$\begin{aligned}
&\sup_{\substack{\mathbf{P}_k \succeq 0,\, \bar{\mathbf{P}}_k \succeq 0 \\ k\in\{0,\ldots,\, N-1\}}} \inf_{\{\mathbf{S}_k,\, F_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k,\, F_k,\, \mathbf{P}_k,\, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1}) \\
&\geq \sup_{\substack{\mathbf{P}_k \in \mathcal{F},\, \bar{\mathbf{P}}_k \succeq 0 \\ k\in\{0,\ldots,\, N-1\}}} \inf_{\{\mathbf{S}_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k,\, \bar{F}_k,\, \mathbf{P}_k,\, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1})
\end{aligned}$$

where $\bar{F}_k := -P_{22,\,k}^{-1} P_{12,\,k}^T$. Since $\inf_{\{\mathbf{S}_k\}_{k=0}^{N-1}} \mathcal{L}(\{(\mathbf{S}_k,\, \bar{F}_k,\, \mathbf{P}_k,\, \bar{\mathbf{P}}_k)\}_{k=0}^{N-1})$ has a finite value only when $\Gamma(F_k^*,\, \mathbf{P}_k) - \bar{\mathbf{P}}_{k-1} = \mathbf{P}_{k-1}$, $k \in \{1,\, 2, \ldots N-1\}$ and $\Gamma(0,\, \mathbf{P}_N) - \bar{\mathbf{P}}_{N-1} = \mathbf{P}_{N-1}$, the problem (12) can be formulated as

$$\begin{aligned}
&\max_{\substack{\mathbf{P}_0,\ldots,\mathbf{P}_{N-1}\in\mathbb{S}_+^n \\ \bar{\mathbf{P}}_0,\ldots,\bar{\mathbf{P}}_{N-1}\in\mathbb{S}_+^n}} J_d(\{\mathbf{P}_k,\, \bar{F}_k\}_{k=0}^{N-1}) \\
&\text{subject to} \\
&\Gamma(\bar{F}_k,\, \mathbf{P}_k) - \bar{\mathbf{P}}_{k-1} = \mathbf{P}_{k-1}, \quad k \in \{1,\, 2, \ldots N-1\} \\
&\Gamma(0,\, \mathbf{P}_N) - \bar{\mathbf{P}}_{N-1} = \mathbf{P}_{N-1}
\end{aligned}$$

or equivalently,

$$\max_{\mathbf{P}_0,\ldots,\mathbf{P}_{N-1}\in\mathbb{S}_+^n} J_d(\{\mathbf{P}_k,\, \bar{F}_k\}_{k=0}^{N-1})$$

$$\text{subject to}$$
$$\Gamma(\bar{F}_k,\, \mathbf{P}_k) \succeq \mathbf{P}_{k-1}, \quad k \in \{1,\, 2, \ldots N-1\}$$
$$\Gamma(0,\, \mathbf{P}_N) \succeq \mathbf{P}_{N-1}$$

which proves the first statement.

Proof of the statement 2): Note that from the solution to the KKT condition in Theorem 1, there exists at least one dual optimal point (11), which satisfies $\begin{bmatrix} 0 \\ I_m \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} 0 \\ I_m \end{bmatrix} \succ 0$. This ensures that the optimal objective function value of the dual problem (6) is not changed when the constraints $\begin{bmatrix} 0 \\ I_m \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} 0 \\ I_m \end{bmatrix} \succ 0$, $k \in \{0,\, 1, \ldots,\, N-1\}$ is added. Therefore, the optimal objective function value of (12) is identical to the optimal objective function value of the dual problem (6). This completes the proof. □

4 Relations with previous results

In this section, equivalence relations between the problem (3) and the problem in [5, Proposition 1] are established. To do so, the SDP problem in [5, Proposition 1] is introduced below.

Problem 5. *([5, Proposition 1]) Solve*

$$\bar{p}_{\mathbf{opt}} := \min_{\mathbf{S}_0,\ldots,\mathbf{S}_{N-1} \in \mathbb{S}_+^{n+m}} J_p(\{\mathbf{S}_k\}_{k=0}^{N-1}) \tag{13}$$
$$\text{subject to}$$
$$W_f = \Pi^T \mathbf{S}_0 \Pi, \quad \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{k-1} \begin{bmatrix} A^T \\ B^T \end{bmatrix} + W = \Pi^T \mathbf{S}_k \Pi$$
$$k \in \{1,\, 2, \ldots N-1\}$$

where $\Pi := \begin{bmatrix} I \\ 0 \end{bmatrix}$.

In the following proposition, it is proved that Problem 5 and the problem (3) is equivalent in some sense.

Proposition 3. *The following statements are true:*

1. $\bar{p}_{\mathbf{opt}} = p_{\mathbf{opt}}$ *holds;*
2. *As before, let $\mathcal{S}$ be the set of all optimal solutions of the form $\{(F_k,\, \mathbf{S}_k)\}_{k=0}^{N-1}$ of the problem (3). Define a subset $\mathcal{G} \subseteq \mathcal{S}$ of $\mathcal{S}$ such that $\{(F_k,\, \mathbf{S}_k)\}_{k=0}^{N-1} \in \mathcal{G}$ is an optimal solution to the problem (3) that takes equalities in the constraints of (3). Moreover, define the projection mapping $\mathcal{F} := \{\{\mathbf{S}_k\}_{k=0}^{N-1} : \{(F_k,\, \mathbf{S}_k)\}_{k=0}^{N-1} \in \mathcal{G}\}$. Then, the set of all optimal solutions of (13) is identical to $\mathcal{F}$.*

Proof. Proof of the statement 1): The proof will be completed by showing both $\bar{p}_{\mathbf{opt}} \le p_{\mathbf{opt}}$ and $p_{\mathbf{opt}} \le \bar{p}_{\mathbf{opt}}$. To prove $\bar{p}_{\mathbf{opt}} \le p_{\mathbf{opt}}$, let $\{(\mathbf{S}_k,\, F_k)\}_{k=0}^{N-1} \in \mathcal{F}$. This optimal point satisfies the equality constraints

$$\Phi(F_k,\, \mathbf{S}_{k-1}) = \mathbf{S}_k, \quad k \in \{1,\, 2, \ldots,\, N-1\}$$

$$\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T = \mathbf{S}_0$$

By pre- and post-multiplying the constraints by Π^T and Π, respectively, we obtain

$$\Pi^T \Phi(F_k,\, \mathbf{S}_{k-1})\Pi = \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{k-1} \begin{bmatrix} A^T \\ B^T \end{bmatrix} + W = \Pi^T \mathbf{S}_k \Pi$$

and $\Pi^T \begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \Pi = W_f = \Pi^T \mathbf{S}_0 \Pi$. This means that $\{\mathbf{S}_k\}_{k=0}^{N-1}$ is guaranteed to be a feasible point of (13). Therefore, $\bar{p}_{\mathbf{opt}} \leq p_{\mathbf{opt}}$ holds. To prove the opposite inequality, suppose that $\{\mathbf{S}_k\}_{k=0}^{N-1}$ is an optimal point of (13). Let

$$\mathbf{S}_k = \begin{bmatrix} S_{k,\,11} & S_{k,\,12} \\ S_{k,\,12}^T & S_{k,\,22} \end{bmatrix}, \quad k \in \{0,\, 1, \ldots N-1\}$$

Since the equality constraints of (13) ensure $S_{k,\,11} \succ 0$ for all $k \in \{0,\, 1, \ldots N-1\}$, applying the Schur complement to $\mathbf{S}_k \succeq 0$ leads to

$$S_{k,\,12}^T S_{k,\,11}^{-1} S_{k,\,12} \preceq S_{k,\,22}, \quad k \in \{0,\, 1, \ldots N-1\}$$

and

$$\begin{bmatrix} S_{k,\,11} & S_{k,\,12} \\ S_{k,\,12}^T & S_{k,\,22} \end{bmatrix} \succeq \begin{bmatrix} S_{k,\,11} & S_{k,\,12} \\ S_{k,\,12}^T & S_{k,\,12}^T S_{k,\,11}^{-1} S_{k,\,12} \end{bmatrix} = \begin{bmatrix} I \\ S_{k,\,12}^T S_{k,\,11}^{-1} \end{bmatrix} S_{k,\,11} \begin{bmatrix} I \\ S_{k,\,12}^T S_{k,\,11}^{-1} \end{bmatrix}^T \tag{14}$$

for all $k \in \{0,\, 1, \ldots N-1\}$.

On the other hand, by pre- and post-multiplying the equality constraints of (13) by $\begin{bmatrix} I & F_k \end{bmatrix}^T$ and its transpose, where $F_k = S_{k,\,12}^T S_{k,\,11}^{-1}$, and using the above inequalities, we have

$$\begin{bmatrix} I \\ F_k \end{bmatrix} \left(\begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{k-1} \begin{bmatrix} A^T \\ B^T \end{bmatrix} + W \right) \begin{bmatrix} I \\ F_k \end{bmatrix}^T \preceq \begin{bmatrix} S_{k,\,11} & S_{k,\,12} \\ S_{k,\,12}^T & S_{k,\,22} \end{bmatrix}, \quad k \in \{1,\, 2, \ldots N-1\}$$

implying that $\{(\mathbf{S}_k,\, F_k)\}_{k=0}^{N-1}$ is a feasible point of the problem (3). Therefore, $p_{\mathbf{opt}} \leq \bar{p}_{\mathbf{opt}}$, and one concludes $\bar{p}_{\mathbf{opt}} = p_{\mathbf{opt}}$.

Proof of the statement 2): From the above proof, if $\{\mathbf{S}_k\}_{k=0}^{N-1}$ is an optimal point of (13), then $\{(\mathbf{S}_k,\, F_k)\}_{k=0}^{N-1}$ with $F_k = S_{k,\,12}^T S_{k,\,11}^{-1}$ is also an optimal point of (3). Therefore, $\mathcal{F}$ includes the set of all the optimal solutions of (13). Conversely, if $\{(\mathbf{S}_k,\, F_k)\}_{k=0}^{N-1} \in \mathcal{G}$, then following the same line of the proof of the statement 1), $\{\mathbf{S}_k\}_{k=0}^{N-1}$ is an optimal point of (13). Therefore, the set of all the optimal solutions of (13) includes $\mathcal{F}$. Therefore, the desired result is obtained. □

Next, an equivalence between the dual problem (10) and the problem in [5, Theorem 1] is proved in the sense that the optimal objective function values of both problems are identical.

Problem 6. *([5, Theorem 1]) Solve*

$$\bar{d}_{\text{opt}} := \max_{X_0, \ldots,\, X_{N-1} \in \mathbb{S}^n} \mathbf{Tr}(W_f X_0) + \sum_{k=1}^{N} \mathbf{Tr}(W X_k) \tag{15}$$

subject to

$$\begin{bmatrix} Q + A^T X_{k+1} A - X_k & A^T X_{k+1} B \\ B^T X_{k+1} A & R + B^T X_{k+1} B \end{bmatrix} \succeq 0, \quad k \in \{0, 1, \dots N-1\}$$

where $X_N = Q_f$*.*

Proposition 4. $\bar{d}_{\mathbf{opt}} = d_{\mathbf{opt}}$ *holds.*

Proof. The proof will be completed by showing both $\bar{d}_{\mathbf{opt}} \leq d_{\mathbf{opt}}$ and $d_{\mathbf{opt}} \leq \bar{d}_{\mathbf{opt}}$. To prove $\bar{d}_{\mathbf{opt}} \leq d_{\mathbf{opt}}$, let $\{X_k\}_{k=0}^{N-1}$ be an optimal solution to (15). Since $R + B^T X_{k+1} B \succ 0$, the Schur complement can be applied to have

$$\begin{aligned} & Q + A^T X_{k+1} A - A^T X_{k+1} B (R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A \\ & \succeq X_k, \quad k \in \{0, 1, \dots N-1\} \end{aligned} \tag{16}$$

Multiplying the last inequality by $[A,\, B]^T$ from the left and by $[A,\, B]$ from the right and adding $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$ to both sides of the inequality yield

$$\begin{aligned} & \begin{bmatrix} A^T \\ B^T \end{bmatrix} \begin{bmatrix} I \\ F_k \end{bmatrix}^T \begin{bmatrix} Q + A^T X_{k+1} A & A^T X_{k+1} B \\ B^T X_{k+1} A & R + B^T X_{k+1} B \end{bmatrix} \begin{bmatrix} I \\ F_k \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \\ & + \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \succeq \begin{bmatrix} Q + A^T X_k A & B^T X_k A \\ A^T X_k B & R + B^T X_k B \end{bmatrix} \end{aligned} \tag{17}$$

where $F_k := -(R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A$. From the result, it is easy to prove that

$$\mathbf{P}_k = \begin{bmatrix} Q + A^T X_{k+1} A & B^T X_{k+1} A \\ A^T X_{k+1} B & R + B^T X_{k+1} B \end{bmatrix}, \quad k \in \{0, 1, \dots, N-1\}$$

is a feasible solution of (10). On the other hand, using the inequalities (16), an upper bound on the objective function (15) can be obtained as

$$\begin{aligned} & \mathbf{Tr}(W_f X_0) + \sum_{k=1}^{N} \mathbf{Tr}(W X_k) \\ & \leq \mathbf{Tr}(W_f (Q + A^T X_1 A - A^T X_1 B (R + B^T X_1 B)^{-1} B^T X_1 A)) \\ & + \sum_{k=1}^{N} \mathbf{Tr} \left(W \left(\begin{array}{l} Q + A^T X_{k+1} A \\ -A^T X_{k+1} B (R + B^T X_{k+1} B)^{-1} \\ \times B^T X_{k+1} A \end{array} \right) \right) \\ & = \mathbf{Tr} \left(W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \mathbf{P}_0 \begin{bmatrix} I \\ F_0 \end{bmatrix} \right) + \sum_{k=1}^{N} \mathbf{Tr} \left(W \begin{bmatrix} I \\ F_k \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} I \\ F_k \end{bmatrix} \right) \end{aligned} \tag{18}$$

which is identical to the formulation of the objective function of (10). This implies that we can always find a feasible point of (10) such that the corresponding objective function value is larger than or equal to $\bar{d}_{\mathbf{opt}}$. Therefore, one concludes $\bar{d}_{\mathbf{opt}} \leq d_{\mathbf{opt}}$.

Next, we will prove the opposite inequality $d_{\mathbf{opt}} \leq \bar{d}_{\mathbf{opt}}$. Consider the optimal solution (11)

$$\left\{ \mathbf{P}_k = \begin{bmatrix} Q + A^T X_{k+1} A & B^T X_{k+1} A \\ A^T X_{k+1} B & R + B^T X_{k+1} B \end{bmatrix} \right\}_{k=0}^{N-1}$$

to the problem (10), where $\{X_k\}_{k=0}^{N-1}$ solves the Riccati equation (7). Using the Riccati equation (7) and following similar lines to derive (18), one gets

$$\begin{aligned}
d_{\mathbf{opt}} &= \mathbf{Tr}\left(W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T \mathbf{P}_0 \begin{bmatrix} I \\ F_0 \end{bmatrix}\right) + \sum_{k=1}^{N} \mathbf{Tr}\left(W \begin{bmatrix} I \\ F_k \end{bmatrix}^T \mathbf{P}_k \begin{bmatrix} I \\ F_k \end{bmatrix}\right) \\
&= \mathbf{Tr}(W_f X_0) + \sum_{k=1}^{N} \mathbf{Tr}(W X_k)
\end{aligned} \tag{19}$$

Moreover, substituting $\{\mathbf{P}_k\}_{k=0}^{N-1}$ into the constraints of (10) leads to (17). Since $(A,\, B)$ is controllable, there exists a state-feedback gain $H \in \mathbb{R}^{m\times n}$ such that $A + BH$ is nonsingular. Pre- and post-multiplying (17) by $[I_n,\, F_k^T]$ from the left and by $[I_n,\, F_k^T]^T$ from the right, one gets

$$\begin{aligned}
&(A+BH)^T \left(Q + A^T X_{k+1} A - A^T X_{k+1} B (R + B^T X_{k+1} B)^{-1} B^T X_{k+1} A\right)(A+BH) \\
&\succeq (A+BH)^T X_k (A+BH), \quad k \in \{0,\, 1, \ldots N-1\}
\end{aligned}$$

Since $A+BH$ is nonsingular, (16) is satisfied. By the Schur complement, it is proved that $\{X_k\}_{k=0}^{N-1}$ is a feasible point of (15). Therefore, it is proved that there always exists a feasible point of (15) such that (19) holds. This implies $d_{\mathbf{opt}} \geq \bar{d}_{\mathbf{opt}}$. Combining $\bar{d}_{\mathbf{opt}} \leq d_{\mathbf{opt}}$ and $d_{\mathbf{opt}} \leq \bar{d}_{\mathbf{opt}}$, one concludes $\bar{d}_{\mathbf{opt}} = d_{\mathbf{opt}}$. □

Solving the KKT condition in the proof of Theorem 1 gives the primal and dual optimal points. It can be proved that under a certain condition, it is possible to solve the KKT condition without the knowledge of the system matrices $[A,\, B]$, which provides a way to adaptively implement the LQR problem. In this respect, the following result will be useful.

Proposition 5. *Assume that $W = 0$, and $\{F_k,\, \mathbf{S}_k\}_{k=0}^{N-1}$ is a feasible point of the problem (3) which takes the equalities*

$$\begin{aligned}
&\Phi(F_k,\, \mathbf{S}_{k-1}) = \mathbf{S}_k, \quad k \in \{1,\, 2, \ldots,\, N-1\} \\
&\begin{bmatrix} I \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I \\ F_0 \end{bmatrix}^T = \mathbf{S}_0
\end{aligned} \tag{20}$$

Then, the following statements are true:

1. *The complementary slackness condition (9) is satisfied;*
2. *Suppose that $\{\mathbf{P}_k\}_{k=0}^{N-1}$ satisfies $\Gamma(F_k,\, \mathbf{P}_k) = \mathbf{P}_{k-1}$, $k \in \{1,\, 2, \ldots N-1\}$, $\Gamma(0,\, \mathbf{P}_N) = \mathbf{P}_{N-1}$ with $\mathbf{P}_N = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}$. Then*

$$\mathbf{Tr}\left(\mathbf{P}_k \mathbf{S}_k - \mathbf{P}_{k-1}\mathbf{S}_{k-1} + \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \mathbf{S}_{k-1}\right) = 0, \quad \forall k \in \{1,\, 2, \ldots,\, N-1\} \tag{21}$$

Proof. Proof of the statement 1): It can be readily proved from (20).

Proof of the statement 2): From $\Gamma(F_k,\, \mathbf{P}_k) = \mathbf{P}_{k-1}$, $k \in \{1,\, 2, \ldots N-1\}$, it follows that

$$\mathbf{Tr}\left(\mathbf{P}_k \mathbf{S}_k - \mathbf{P}_{k-1}\mathbf{S}_{k-1} + \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \mathbf{S}_{k-1}\right)$$

$$
\begin{aligned}
&= \mathbf{Tr}\left(\mathbf{P}_k\mathbf{S}_k + \left(\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} - \mathbf{P}_{k-1}\right)\mathbf{S}_{k-1}\right) \\
&= \mathbf{Tr}\left(\left(\mathbf{S}_k - \begin{bmatrix} A & B \\ F_k A & F_k B \end{bmatrix}\mathbf{S}_{k-1}\begin{bmatrix} A & B \\ F_k A & F_k B \end{bmatrix}^T\right)\mathbf{P}_k\right) \\
&= 0, \quad \forall k \in \{1, 2, \ldots, N-1\}
\end{aligned}
$$

where the last equality follows from $\Phi(F_k, \mathbf{S}_{k-1}) = \mathbf{S}_k, \quad k \in \{1, 2, \ldots, N-1\}$. This completes the proof. □

Remark 1. *The result of Proposition 5 is related to the so-called the Q-learning in [12, 13], which is an adaptive LQR approach. If we solve the finite-horizon LQR problem for a sufficiently large N, then the dual variable converges as follows:*

$$
\lim_{k\to\infty} \mathbf{P}_k = \mathbf{P}^* = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} = \begin{bmatrix} Q + A^T PA & A^T PB \\ B^T PA & R + B^T PB \end{bmatrix}
$$

where P solves the algebraic Riccati equation (ARE) $A^TPA - A^TPB(R+B^TPB)^{-1}B^TPA + Q = P$. Then, $Q(x, u) = \begin{bmatrix} x \\ u \end{bmatrix}^T \mathbf{P}^* \begin{bmatrix} x \\ u \end{bmatrix}$ is called the Q-function [12, 13], and the optimal policy can be deduced by $u(k) = \arg\min_u Q(x(k), u)$. Equivalently, the state-feedback gain of the optimal policy is computed as $F = -P_{22}^{-1}P_{12}^T$. The Q-function provides a way to adaptively implement the optimal control without the knowledge of the system matrices $[A, B]$. In [12], the Q-function is computed from the input and state measurements using the least-square method. The approach can be interpreted as solving the condition (21). Very roughly speaking, for given stabilizing state-feedback gain F and $\mathbf{P}_k$, the input and state vectors are collected during a certain time interval $k \in [k_1, k_2 + 1]$, and construct the matrices

$$
\mathbf{S}_{k-1} = \sum_{k=k_1}^{k_2} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T, \quad \mathbf{S}_k = \sum_{k=k_1+1}^{k_2+1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T
$$

which still satisfy the constraint $\Phi(F, \mathbf{S}_{k-1}) = \mathbf{S}_k$. By solving the equality (21), the dual feasible point $\mathbf{P}_{k-1}$ which satisfies the necessary condition for the dual feasibility. Then, the state-feedback gain F can be appropriately updated by using the dual feasible point. Because a rigorous theoretical analysis is beyond the scope of this paper, we will not discuss this issue in detail.

5 Decentralized LQR performance analysis and design

In this section, we study the decentralized LQR problem by combining the developments of the previous sections and the decentralized controller design technique developed in [2]. The decentralized control problem is a special class of more general structured control design problems. The proposed approach can be extended to the general structured control design problems including the distributed controller design. In particular, the structure of the optimization in Theorem 1 allows us to derive a sufficient but simple convex relaxation for designing a decentralized LQR controller. Consider the stochastic LTI system composed of M interconnected subsystems

$$
x_i(k+1) = \sum_{j=1}^{M} A_{ij}x_j(k) + B_i u_i(k) + w_i(k) \tag{22}
$$

for $i \in \{1, 2, \ldots, M\}$, where $k \in \mathbb{N}$, $x_i(k) \in \mathbb{R}^{n_i}$ is the state vector, $u_i(k) \in \mathbb{R}^{m_i}$ is the control vector, $x_i(0) \sim \mathcal{N}(0, W_f)$ and $w_i(k) \sim \mathcal{N}(0, W)$ are mutually independent Gaussian random vectors. Let us define

$$x(k) = \begin{bmatrix} x_1(k) \\ \vdots \\ x_M(k) \end{bmatrix} \; u(k) = \begin{bmatrix} u_1(k) \\ \vdots \\ u_M(k) \end{bmatrix} \; w(k) = \begin{bmatrix} w_1(k) \\ \vdots \\ w_M(k) \end{bmatrix} \tag{23}$$

Then, the system dynamics (22) can be written as

$$x(k+1) = Ax(k) + Bu(k) + w(k)$$

where $A = \begin{bmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & \ddots & \vdots \\ A_{M1} & \cdots & A_{MM} \end{bmatrix} \in \mathbb{R}^{n \times n}$, $B = \mathbf{diag}(B_1, \ldots, B_M) \in \mathbb{R}^{n \times m}$, $n = n_1 + \cdots + n_M$, and $m = m_1 + \cdots + m_M$. We consider the decentralized static state-feedback LQR problem.

Problem 7 (Decentralized stochastic LQR problem)**.** *Solve*

$$J_{\mathcal{K}}^* := \min_{\substack{F_k \in \mathbb{R}^{m \times n} \\ k \in \{0, 1, \ldots, N-1\}}} \mathbf{E}(x(k)^T Q_f x(k)) + \sum_{k=0}^{N-1} \mathbf{E}\left(\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \right)$$

$$\text{subject to} \quad x(k+1) = Ax(k) + Bu(k) + w(k)$$

$$u(k) = F_k x(k) \quad F_k \in \mathcal{K}$$

where $\mathcal{K}$ is a linear subspace defined as $\mathcal{K} := \{K \in \mathbb{R}^{m \times n} : K = \mathbf{diag}(F_1, F_2, \ldots, F_M), F_i \in \mathbb{R}^{m_i \times n_i}, i \in \{1, \ldots, M\}\}$.

Equivalently, the problem can be converted into (2) and (3) with the additional constraint $F_k \in \mathcal{K}$, $k \in \{0, 1, \ldots, N-1\}$. The problem is a non-convex structured static state-feedback design problem. When $F_k \in \mathcal{K}$, $k \in \{0, 1, \ldots, N-1\}$ is given, then its exact cost can be evaluated as follows.

Proposition 6. *Let $F_k \in \mathcal{K}$, $k \in \{0, 1, \ldots, N-1\}$ be given. The cost corresponding to the given structured static state-feedback gain is $J^*(F_0, \ldots, F_{N-1}) := J_p(\{\mathbf{S}_k\}_{k=0}^{N-1})$ where $\mathbf{S}_k = \Phi(F_k, \mathbf{S}_{k-1})$, $k \in \{1, \ldots, N-1\}$ with $\mathbf{S}_0 = \begin{bmatrix} I_n \\ F_0 \end{bmatrix} W_f \begin{bmatrix} I_n \\ F_0 \end{bmatrix}^T$.*

Remark 2. *The cost can be also evaluated using Problem 3, which is simply a SDP if $F_k \in \mathcal{K}$, $k \in \{0, 1, \ldots, N-1\}$ are constants.*

Next, motivated by the LMI-based decentralized control design method in [2], we suggest a simple convex relaxation of the problem.

Problem 8. *Solve*

$$(\mathbf{S}_k^*, L_k^*, G_k^*)_{k=0}^{N-1} := \operatorname*{argmin}_{\substack{\mathbf{S}_k \in \mathbb{S}^{n+m}, L_k \in \mathbb{R}^{n \times m}, G_k \in \mathbb{R}^{n \times n} \\ k \in \{0, 1, \ldots, N-1\}}} f^p(\{\mathbf{S}_k\}_{k=0}^{N-1}) \tag{24}$$

subject to

$$\begin{bmatrix} \mathbf{S}_k & * \\ \begin{bmatrix} G_k & L_k \end{bmatrix} & \left(G_k + G_k^T - \begin{bmatrix} A^T \\ B^T \end{bmatrix}^T \mathbf{S}_{k-1} \begin{bmatrix} A^T \\ B^T \end{bmatrix} - W \right) \end{bmatrix} \succeq 0, \quad \forall k \in \{0, 1, \ldots, N-1\} \tag{25}$$

$$G_k = \operatorname{diag}(G_{k,1}, \ldots, G_{k,M})$$
$$L_k = \operatorname{diag}(L_{k,1}, \ldots, L_{k,M})$$
$$L_{k,i} \in \mathbb{R}^{n_i \times m_i}, \quad G_{k,i} \in \mathbb{R}^{n_i \times n_i}$$

where $\mathbf{S}_{-1} = 0$.

Proposition 7. *Let* $(\mathbf{S}_k^*, L_k^*, G_k^*)_{k=0}^{N-1}$ *be an optimal point of the problem* (24) *and let* $\tilde{J}_{\mathcal{K}}^*$ *be the optimal objective function value. Then,* $J_{\mathcal{K}}^* \le \tilde{J}_{\mathcal{K}}^*$ *is satisfied under the decentralized control policy* $u_i(k) = (L_{k,i}^*)^T (G_{k,i}^*)^{-T} x_i(k)$ *for all* $k \in \{0, 1, \ldots, N-1\}$ *and* $i \in \{1, 2, \ldots, M\}$.

Proof. The proof follows similar lines as in the proof of Proposition 1. Pre- and post-multiplying both sides of (25) by

$$\begin{bmatrix} -I & 0 \\ 0 & -I \\ I & (G_k^*)^{-1} L_k^* \end{bmatrix}^T, \; k \in \{0, 1, \ldots, N-1\}$$

and its transpose yield

$$\Phi(F_k^*, \mathbf{S}_{k-1}^*) \preceq \mathbf{S}_k^* \quad k \in \{1, 2, \ldots, N-1\}$$
$$\begin{bmatrix} I \\ F_0^* \end{bmatrix} W_f \begin{bmatrix} I \\ F_0^* \end{bmatrix}^T \preceq \mathbf{S}_0^*$$

with $F_k^* = (L_k^*)^T (G_k^*)^{-T}$, $k \in \{0, 1, \ldots, N-1\}$. By using Theorem 1, one concludes that $J_{\mathcal{K}}^* \le \tilde{J}_{\mathcal{K}}^*$ is satisfied under the policy $u(k) = F_k^* x(k)$, $k \in \{0, 1, \ldots, N-1\}$. Since F_k^* has a block diagonal structure according to the state and input partitions in (23), the desired result can be obtained. □

Remark 3. *It can be readily proved that* $J_P^* \le J_{\mathcal{K}}^* \le J^*(F_0^*, \ldots, F_{N-1}^*) \le \tilde{J}_{\mathcal{K}}^*$ *holds, where* $J^*(F_0^*, \ldots, F_{N-1}^*)$ *is the exact cost evaluated using* $F_0^*, \ldots, F_{N-1}^*$ *obtained from* (24).

Example 1. *Consider the interconnected system*

$$x_1(k+1) = A_{11} x_1(k) + A_{12} x_2(k) + B_1 u_1(k) + w_1(k)$$
$$x_2(k+1) = A_{21} x_1(k) + A_{22} x_2(k) + B_2 u_2(k) + w_2(k)$$

where

$$A_{11} = \begin{bmatrix} 0.8220 & -0.0898 \\ -0.2389 & 0.9358 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.4860 & -0.1820 \\ 0.1680 & -0.3143 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} 0.1891 & -0.3195 \\ 0.2067 & -0.6610 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.6404 & 1.4540 \\ 0.2067 & -0.6610 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} -0.3505 \\ -1.9788 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.4901 \\ -0.0515 \end{bmatrix}$$

Solving Problem 8 with $Q = Q_f = I_n$, $R = I_n$, $W = 0.01 I_n$, $W_f = I_n$, *and* $N = 30$ *yields* $\tilde{J}_{\mathcal{K}}^* = 19.6799$ *and* $J^*(F_0^*, \ldots, F_{N-1}^*) = 18.0598$. *On the other hand, the optimal cost corresponding to the centralized LQR is* $J_p^* = 16.2610$. *Therefore, one concludes* $J_p^* = 16.2610 \le J_{\mathcal{K}}^* \le$

$J^*(F_0^*, \ldots, F_{N-1}^*) = 18.0598$. *The time histories of the state under the obtained decentralized control policy is shown in Fig. 1 and the histogram of the cost of 3000 simulations is plotted in Fig. 2.*

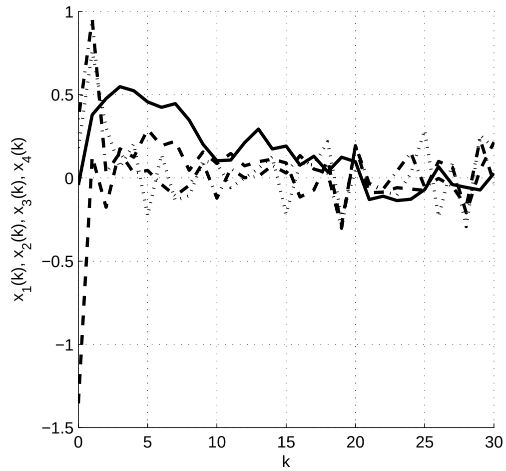

Figure 1: Example 1. The time histories of the state under the obtained decentralized control policy.

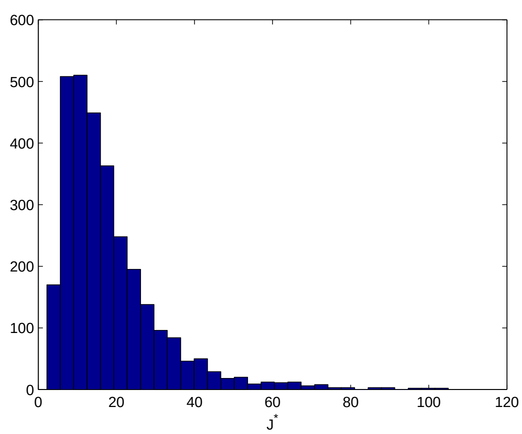

Figure 2: Example 1. The cost histogram of 3000 simulations

6 Conclusion

We presented a new convex formulation of the finite-horizon LQR problem and its dual problem. Connections between the proposed formulations and the existing ones were established. The proposed formulation was also applied to the decentralized LQR design problem.

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