GROWTH OF PLANETARY EMBRYOS

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Mentors

David Minton's research interests include understanding the origin and early evolution of the solar system, including the formation of the planets and the Late Heavy Bombardment. Minton also studies the physics and dynamics of small bodies and their satellites, as well as using the impact records of solar system bodies to understand the history of small body populations.

Andrew Hesselbrock received his BS and MS in physics at Miami University. During his studies, he developed computational models to study the rotational dynamics of Neried, a satellite of Neptune, which resulted in two publications. Hesselbrock entered the physics department at Purdue University in 2012 and is currently developing a disk model to simulate the formation of the Martian satellites.

Abstract

In order to fine-tune our current understanding of the formation of planets, we update the classic model of oligarchic growth to include mass conservation. In the early stages of planet formation, the protoplanetary disk contained only a swarm of planetesimals, rocky bodies with diameters of at most about 100 km, that collided together to form embryos greater than 1,000 km in diameter. Because planetesimals accrete, or accumulate, onto embryos, the surface mass density of the planetesimal swarm decreases with time. Here we describe surface mass density of the planetesimal swarm as the total initial surface mass density of the protoplanetary disk minus the surface mass density of the embryos. However, as the mass of the individual embryos increases, the average spacing between them must also change. Therefore, the parameter that is related to the characteristic spacing between embryos, b, must also change. We incorporate the changing eccentricity and surface mass density into a model that describes the growth of any given planetary embryo in the protoplanetary disk, which includes the new formulation for mass conservation. Finally, we test how the presence of a circum-embryo debris disk affects the growth rate of a planet. Because a debris disk in orbit about an embryo will increase the collisional cross-section of the planetary embryo, the time needed to fully grow the embryo decreases. The accelerated growth rate due to the circum-embryo disk holds implications for describing the formation of gas giant cores within the necessary timescale to capture the gas from the surrounding protoplanetary disk.

Keywords

mass conservation, oligarchic growth, planet formation, planetary embryo, circum-embryo disk

GROWTH OF PLANETARY EMBRYOS:
Conserving Mass During Planet Formation in the Oligarchic Growth Stage

Jennifer Larson, Physics and Astronomy

INTRODUCTION

When the sun formed, it was surrounded by a nebular disk containing gas and a swarm of small rocky bodies called *planetesimals*. The terrestrial planets (Mercury, Venus, Earth, and Mars) formed from this swarm of planetesimals. Terrestrial planets typically undergo three stages of growth: runaway growth, oligarchic growth, and late-stage accretion. The first stage, runaway growth, occurs when the protoplanetary disk contains only small planetesimals, which are solid bodies smaller than about 100 km in diameter. These planetesimals collide and coalesce with each other and grow larger. During this stage, the growth rate of any given planetesimal depends strongly on its mass, and therefore, a planetesimal with even a slightly larger mass than its neighbors will "run away" and become the dominant body in its neighborhood.

When these runaway planetesimals grow to a diameter of about 1,000 km, their growth rate slows down, and they are reclassified as a planetary embryo. Figure 1 depicts the growing embryo orbiting the sun within the planetesimal swarm. As the embryo travels through the planetesimal swarm, the embryo collides with planetesimals, adding the material from the planetesimals to the embryo. The growth rate of embryos is not as strongly dependent on their mass, and rather than exhibiting the *divergent* growth characteristic of the runaway phase, they experience *convergent* growth. That is, embryos typically have a similar mass to neighboring embryos, though much larger than the still-present planetesimals. This oligarchic growth eventually halts as each embryo depletes all of the remaining planetesimals from the feeding zone, which is the region surrounding the embryo that is affected by its gravitational force. Runaway and oligarchic growth are thought to have ceased in less than 10 million years (My) after the sun formed.
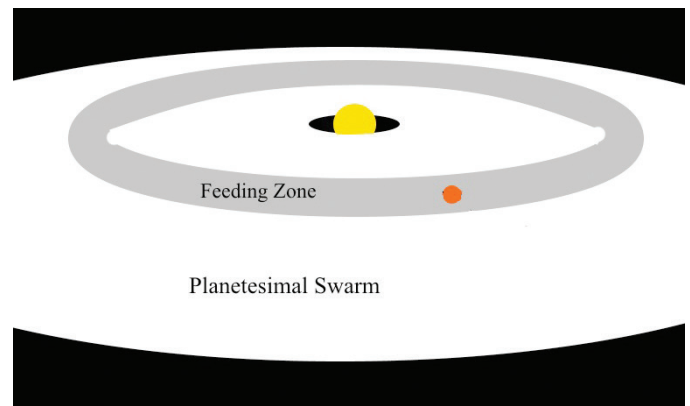

Figure 1. The embryo, depicted by the red-orange circle, orbits the sun (yellow) with many smaller rocky planetesimals with diameters of approximately 100 km. Collisions between the planetesimals and the embryo cause the embryo to obtain more mass.

Finally, the embryos go through late-stage accretion. During this stage, the majority of the planetesimals have accreted into planetary embryos. Late-stage accretion is much slower than the previous stages, lasting as much as 100 My or more, as it is the stage in which embryos merge together to eventually form planets. Due to its small mass and young formation age (Dauphas & Pourmand, 2011), Mars only experienced runaway growth and oligarchic growth, and apparently did not go through late-stage accretion (Minton & Levison, 2014). Since we only study the growth of embryos during the oligarchic growth stage in this study, we apply our model to the case of the formation of Mars.

In this study, we modify the classic oligarchic growth model for a rocky body to include mass conservation. That is, we account for the decrease in the number of

planetesimals surrounding the embryo due to the accretion of planetesimals onto the embryo. Previous models by Kokubo and Ida (1996) and Thommes, Duncan, and Levison (2003) assumed that the number of planetesimals surrounding the embryo is constant despite planetesimals accreting onto the embryo. However, in these models, since the amount of material available to accrete onto the embryo never decreases, the growth of the embryo never converges to the final mass of the planet, as shown in Figure 2. Instead, the growth of the embryo is arbitrarily cut off at approximately the planet's final mass. To increase the accuracy of the classic oligarchic growth model, we input mass conservation into the original oligarchic growth model.
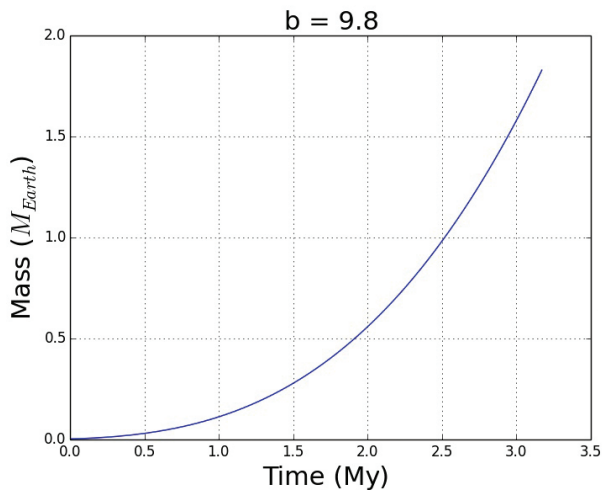
Figure 2. Thommes et al. (2003) describe a model for oligarchic growth in which the number of planetesimals remains constant although planetesimals accrete onto the embryo. As a result, the growth of the embryo never converges to the isolation mass.

In this study, we first affirm the expression given by Kokubo and Ida (1993) for the changing eccentricity of the planetesimal swarm. The average eccentricity of planetesimal orbits is an important quantity, because it strongly influences the rate of growth of the embryo. Next, we update the previous oligarchic growth models by decreasing the number of planetesimals available to add mass to an embryo in order to conserve mass as the embryo grows. To do this, we change the surface density of the planetesimal swarm so that as material accretes onto the embryo, the planetesimal swarm becomes less dense. Finally, we examine the effects of a disk surrounding an embryo on the growth rate of the embryo. We propose that embryos with protoplanetary disks formed several million years faster than embryos with no disks.

Changing Eccentricity

When the largest planetesimals form into embryos and the swarm enters the oligarchic growth phase, the larger embryos control the eccentricity and inclination of the planetesimal swarm (Kokubo & Ida, 1998). This is in contrast to the runaway phase where the eccentricity and inclination of the planetesimals are controlled collectively by the planetesimals (Greenberg, Hartmann, Chapman, & Wacker, 1978; Kokubo & Ida, 1996). It is this change in the control of eccentricity from the planetesimals to the embryos that defines the transition from runaway to oligarchic growth. In the oligarchic growth phase, the eccentricity of the planetesimals within a given region of the disk is determined by the mass of the embryos in that region. Planetesimals are gravitationally scattered during close encounters with embryos and are higher than they would be without the embryos. This is called *viscous stirring*. We also can approximate the inclination, i, to be half the eccentricity of the planetesimal swarm. Eccentricity and inclination growth is opposed by damping from the surrounding gas component of the protoplanetary nebula. In order to determine an expression for the eccentricity of the planetesimals as a function of the mass of the planetary embryo, Thommes and others (2003) equate the timescale for the gas drag,

$$T_{gas}^{em} \simeq \frac{1}{e_m} \frac{m}{(C_D/2)\pi r_m^2 \rho_{gas} a\Omega}$$

to the equation for viscous stirring,

$$T_{VS}^{M-m} \simeq \frac{1}{40} \left(\frac{\Omega^2 a^3}{GM} \right)^2 \frac{e_m^4}{n_{sM} a^2 \Omega}$$

(Ida & Makino, 1993). Then, by solving for the eccentricity, e, the following expression is obtained:

$$e_m^{eq} \simeq \frac{1.7 \, m^{\frac{1}{15}} M^{\frac{1}{3}} \rho_m^{\frac{2}{15}}}{b^{\frac{1}{5}} C_D^{\frac{1}{5}} \rho_{gas}^{\frac{1}{5}} M_\odot^{\frac{1}{3}} a^{\frac{1}{5}}}$$

(Thommes et al., 2003). This expression, which is dependent on the growing mass of the embryo, shows that the oligarchic eccentricity of the planetesimals increases as the embryo grows large enough to disturb the planetesimals in the region surrounding the embryo.

Decreasing Surface Density of Planetesimals

In our calculations, we conserve the amount of mass in an isolated system containing planetesimals and the growing embryo. Therefore, we use $\Sigma = \Sigma_m + \Sigma_M$ to express the total surface density of the planetesimal swarm, where Σ_m is the surface density of the solids in the planetesimal swarm and Σ_M is the surface density of solids that impact the embryo. It is common to use a concept called the Minimum Mass Solar Nebula (MMSN) to define the initial conditions of the protoplanetary disk required to make the observed planets in their present-day orbits. A common MMSN total surface density is given by

$$\Sigma = \Sigma_0 \left(\frac{a}{a_0}\right)^p \text{ g cm}^{-2}$$

where the constant Σ_0 is 8 g cm^{-2} and a_0 is the distance from the earth to the sun at 1 AU. The variable a describes the distance from the sun to the embryo being studied. In this model, we investigate the case of Mars's growth, so we set a equal to 1.5 AU. We also set $p = -\frac{3}{2}$ using the standard MMSN formulation of Weidenschilling (1977). We use the equation describing the total surface density as the initial surface density of solids in the planetesimal swarm. Farther away from the embryo there are fewer planetesimals, so the density of the swarm decreases with distance. A fraction of the total material in the planetesimal swarm will impact the embryo, causing it to grow. We denote the surface density of the embryos by the expression

$$\Sigma_M = \frac{M}{2\pi a \Delta a}$$

where $\Delta a = bR_H$ (Thommes et al., 2003) in which R_H is the radius of the Hill sphere of the embryo being studied and b is the characteristic spacing parameter describing the spacing between bodies in the swarm. Using the definition of the Hill radius $R_H = a\left(\frac{M}{3M_\odot}\right)^{\frac{1}{3}}$, we simplify the terms in Σ_M to examine the dependency on the mass of the embryo.

$$\Sigma_M = \frac{(3M_\odot)^{\frac{1}{3}}}{2\pi a^2 b} M^{\frac{2}{3}}$$

As more planetesimals from the swarm impact the embryo causing the embryo to grow, Σ_M increases until it is equal to the total surface of the protoplanetary disk, as is seen in Figure 3. Therefore, when the embryo has grown to its final mass and has consumed all the planetesimals in the surrounding region, we find that $\Sigma = \Sigma_M$.

The surface density of planetesimals is $\Sigma_m = \Sigma - \Sigma_M$. Therefore, by using the definitions of Σ and Σ_M, the surface density of planetesimals is

$$\Sigma_m = \Sigma_0 \left(\frac{a}{a_0}\right)^p - \frac{(3M_\odot)^{\frac{1}{3}}}{2\pi a^2 b} M^{\frac{2}{3}}$$

As the planetary embryo obtains more mass, the surface density of the planetesimals in the swarm decreases. We implement this equation into the classic model for the growth of planetary embryos described in Thommes and others (2003) in order to conserve mass as planetesimals are accreted onto an embryo.

Initial State of Planetesimal Swarm Around Embryo

State of Planetesimal Swarm Around Embryo After Several Million Years
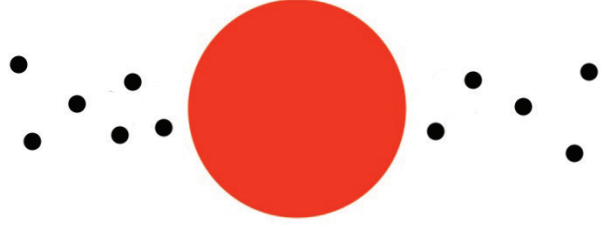
Figure 3. As planetesimals accrete onto the surface of the embryo, the number of planetesimals in the swarm decreases, resulting in a decrease in the surface density of the planetesimals.

General Growth Equation

To derive an equation for the growth of a terrestrial planet such as Mars, we begin with the differential equation describing the change in mass over time given in Thommes and others (2003):

$$\frac{dM}{dt} \simeq \frac{C\Sigma_m M^{\frac{4}{3}}}{e_m^2 a^{\frac{1}{2}}}$$

where $C = 6\pi^{\frac{2}{3}} \left[\frac{3}{4\rho_M}\right]^{\frac{1}{3}} \left[\frac{G}{M_\odot}\right]^{\frac{1}{2}}$ as is given in Thommes and others (2003). The density of the planetary embryo is given by ρ_M in units of g cm^{-3}. In the previous section, we found an expression for the surface density of planetesimals that depends on the mass of the growing embryo. This expression is inserted into the differential equation describing the change of mass over time in order to describe the equation's dependency on the embryo's mass. Therefore, the following expression is obtained:

$$\frac{dM}{dt} \simeq \left[\frac{C}{e_m^2 a^{\frac{1}{2}}}\right]\left[\Sigma_0 \left(\frac{a}{a_0}\right)^p M^{\frac{4}{3}} - \left(\frac{(3M_\odot)^{\frac{1}{3}}}{2\pi a^2 b}\right) M^2\right]$$

However, the eccentricity of the planetesimal swarm also depends on the mass of the embryo. In order to solve the differential equation, we insert the expression for the eccentricity of the planetesimal swarm into the differential equation. Therefore, the differential equation simplifies to the following:

$$\frac{dM}{dt} \simeq AM^{\frac{2}{3}} - BM^{\frac{4}{3}}$$

where A and B are the constants

$$A = \frac{\Sigma_0 \left(\frac{a}{a_0}\right)^p C\left(bC_D\rho_{gas}\right)^{\frac{2}{5}} M_{\odot}^{\frac{2}{3}}}{2.89m^{\frac{2}{15}}a^{\frac{1}{10}}\rho_m^{\frac{4}{15}}}$$

$$B = \left[\frac{3^{\frac{1}{3}}}{2\pi}\right]^{\frac{1}{3}} \left[\frac{C\left(C_D\rho_{gas}\right)^{\frac{2}{5}} M_{\odot}}{2.89b^{\frac{3}{5}}m^{\frac{2}{15}}a^{\frac{21}{10}}\rho_m^{\frac{4}{15}}}\right]$$

This equation describes the change of mass over time while conserving the amount of mass in the embryo-planetesimal swarm system. Using a fourth-order Runge-Kutta integrator to solve this differential equation, we calculate the mass of the embryo at various time steps. Table 1 contains the initial conditions implemented in this study in order to produce a planet approximately the size of Mars. We assume that the planetesimals have an average diameter of 100 km. Therefore, assuming a planetesimal density, ρ_m, of 3.0 g cm^{-3}, we estimate the mass of a planetesimal, m, to be approximately 7.85×10^{20} g. We start the growing embryo off at a mass of 1.68×10^{25} g, which is approximately 10% of the mass of Mars, or approximately the size of Earth's moon. In addition, because the surface density of the planetesimal swarm decreases as more mass accretes onto the embryo, the distance between the bodies in the planetesimal swarm increases. As a result, the unitless characteristic spacing parameter, b, which describes the amount of spacing between bodies in the planetesimal swarm, should change slightly. In this model, we assume that the change in b is infinitesimally small, and therefore b is treated like a constant value in this study. After examining the growth of an embryo at various values of b between 0.1 and 15.0, we find that the ideal range for b required to produce a Mars-sized body is 1.0 to 10.0. Therefore, when solving for the mass of a body at various times, we calculate the mass for various values of b. Figure 4 shows that a characteristic spacing of 9.8 provides the most accurate final mass for Mars.

Finally, we study the effects of a disk on the growth rate of a planetary embryo. The disk, modeled by Andrew Hesselbrock, is made up of particles that orbit a planetary embryo as seen in Figure 5. We modify the equation that describes the growth of an embryo to also add in the mass from the disk that gets deposited on the embryo. As a result, the new growth rate expression depends not only on the mass of the embryo, but also on the mass of the disk surrounding the embryo:

$$\frac{dM}{dt} \simeq AM^{\frac{2}{3}}\left(1 + \frac{M_{disk}}{M}\right)^{\frac{2}{3}} - BM^{\frac{4}{3}}$$

where A and B are the same constants that were previously defined. The main difference between this

Quantity	Initial Condition
Initial mass of embryo, M_0	1.68×10^{25} g
Mass of 100 km planetesimals, m	7.85×10^{20} g
Mass of Sun	2×10^{22} g
Density of embryo, ρ_M	4 g cm^{-3}
Desity of a planetesimal, ρ_m	3.0 g cm^{-3}
Gas density of planetesimal swarm, ρ_{gas}	1.4×10^{-9} cm^{-3}
Gravitational constant, G	6.67×10^{-8} cm^3 g^{-1} s^{-2}
Semi-major axis of orbit of embryo, a	1.5 AU

Table 1. The initial conditions used in this study produce an embryo the size of Mars with a final isolation mass of 6.42×10^{26} grams.

expression and the previous growth equation is that this model adds the mass of the disk to the term that describes the total conditions of the system. As a result, when $M_{disk} = 0$, we obtain the expression for an embryo without a disk that we found previously.

RESULTS

We find that we are able to produce a Mars-sized object from the model we created. As shown in Figure 6, the embryo with no disk grows slowly at first but soon transitions to a rapid growth rate before finally slowing down and reaching its isolation mass of approximately 10% of Earth's mass. Based on these initial conditions, we produce a Mars-sized object in 5.5 million years. In addition, we add a disk around the growing embryo and calculate the embryo's mass for various values of γ, a parameter describing the percentage of material that remains in the disk after planetesimals and other particles collide with the protoplanetary disk. For example, if 80% of the material impacting the disk becomes a part of the disk while the other 20% simply passes through the disk, then we choose $\gamma = 0.8$. Figure 6 displays the growth curve of an embryo using the initial conditions given in Figure 4 and various values of γ. When a disk surrounds the growing embryo, the embryo reaches isolation mass in approximately 1 million years. Therefore, adding a disk to a planetary embryo increases the embryo's growth rate by approximately 5.5 times.

DISCUSSION

By allowing the surface density of planetesimals to change, we conserve mass in the isolated system of the planetary embryo growing amongst planetesimals. This model allows an embryo to stop growing at the isolation
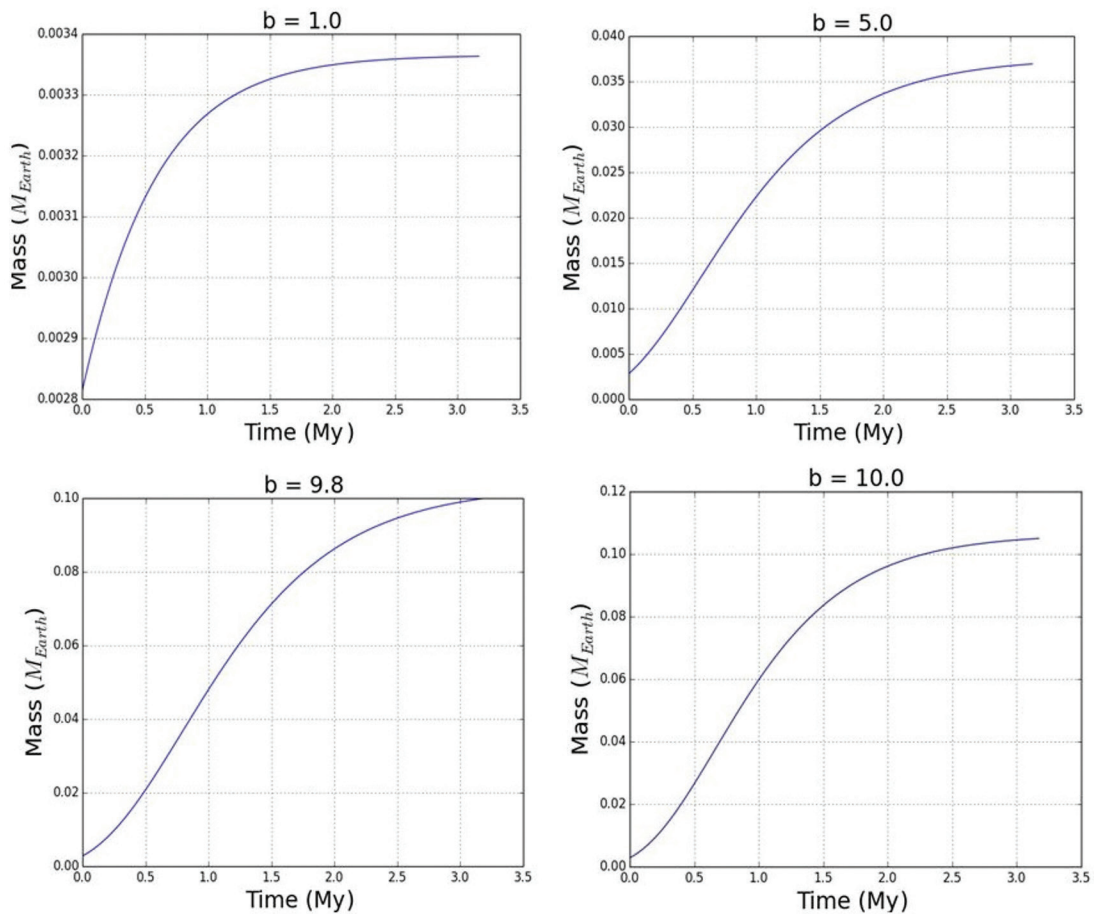
Figure 4. We plot the mass of the embryo over time for values of *b* ranging from 1 to 10. When *b* = 9.8, the isolation mass of the embryo is closest to the mass of Mars.
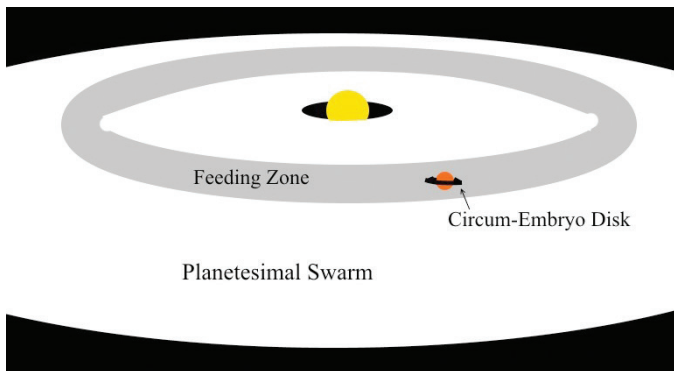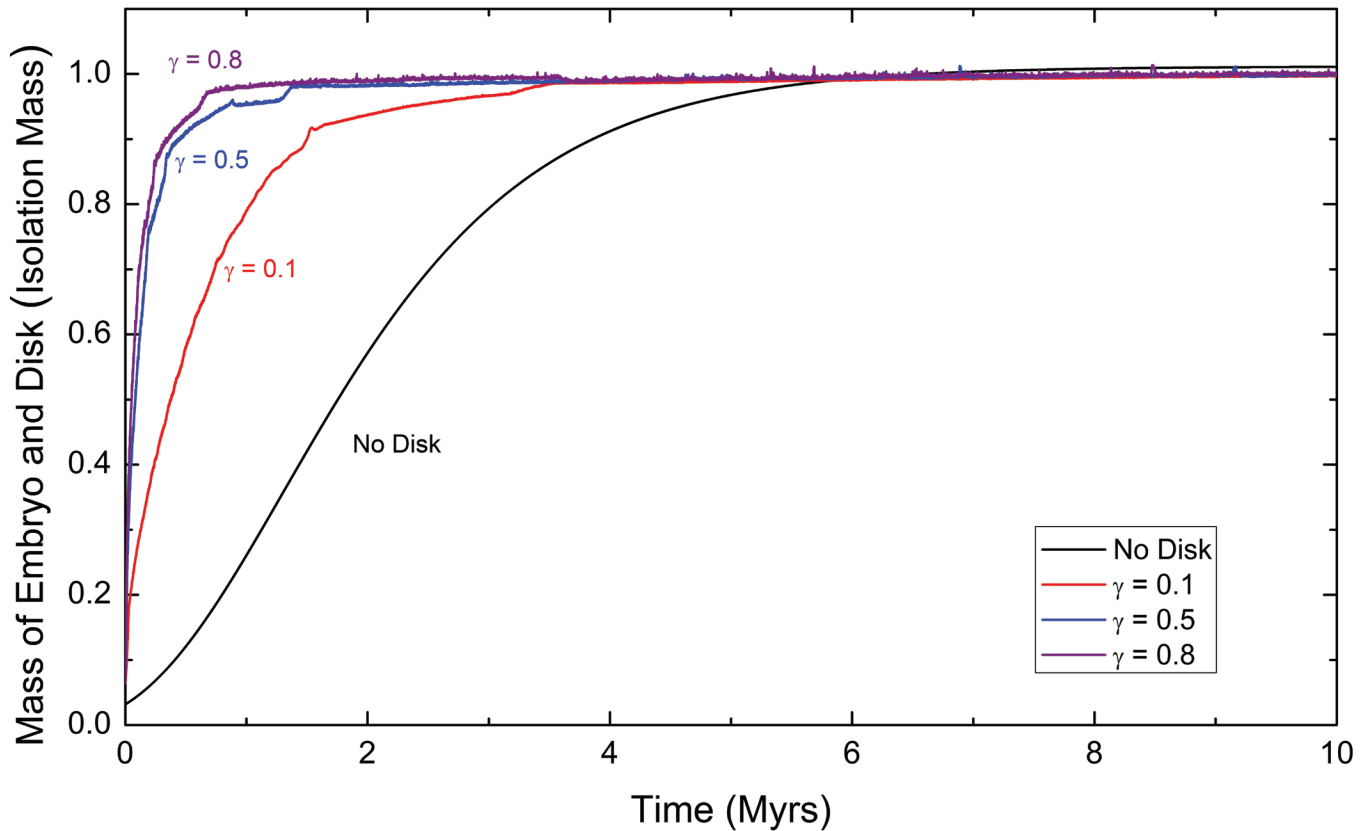

Figure 5. As a second part of this study, we consider the effects of adding a disk around the growing embryo. Again, the embryo orbits the sun with the swarm of planetesimals. However, a disk of particles surrounds the embryo in a manner similar to how Saturn's rings orbit Saturn.

characteristic spacing parameter, *b*, should change as the number of planetesimals decreases. As planetesimals accrete onto the embryo, thus decreasing the total number of planetesimals in the system, the spacing between embryos in the planetesimal swarm increases. Therefore, the parameter describing the spacing between the bodies in the planetesimal swarm should change as the spacing between bodies in the planetesimal swarm increases. Future studies should constrain the characteristic spacing parameter in order to improve the accuracy of this model.

By adding a disk around the growing embryo, we increase the rate at which material accretes onto the embryo. As shown in Figure 7, planetesimals that collide with the disk add material to the disk. Over time, some of the material from the disk falls inward onto the embryo, adding to the mass of the embryo. Therefore, the disk increases the effective cross-sectional area of the growing embryo, allowing for more mass to be captured by the disk and the embryo. This allows the embryo to grow much faster when surrounded by a disk than it would without a disk. Without a disk, the embryo reaches its final isolation mass in approximately 5 million years.

mass due to a lack of material available to accrete onto the embryo. While previous models maintain a constant amount of mass surrounding the embryo, our model decreases the amount of mass surrounding the embryo as mass is added onto the embryo. One issue encountered while formulating this model is that the
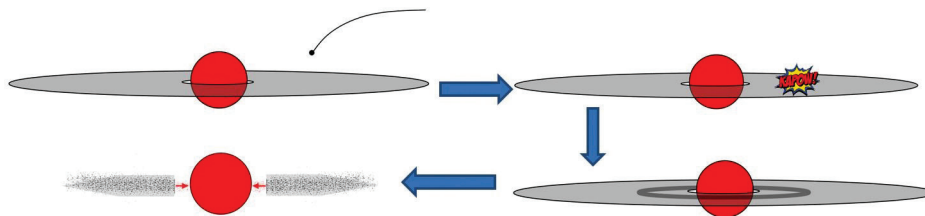
System Mass at 1.5 AU

Figure 6. When surrounded by a disk, an embryo forms on a much shorter timescale. In addition, the growth rate is much quicker for an embryo surrounded by a disk that can capture more particles during collisions.

Figure 7. Material that impacts with the disk adds mass to the disk. This material is distributed throughout the disk. A portion of the material in the disk falls inward onto the embryo.

However, when a disk surrounds the embryo, the embryo only takes 1 million years to reach its final isolation mass. Because the gas in the early solar system dissipated within 3 million years, the gas giant planets must have formed their cores within the first 3 million years in order for the embryos to capture some of the gas. If the growing cores of the gas giants had been surrounded by disks, then the cores would have been able to form fast enough to capture the gas in the protoplanetary disk before it dissipated. Therefore, disks may have surrounded the gas giant planets as they formed their cores.

REFERENCES

Dauphas, N., & Pourmand, A. (2011). Hf-W-Th evidence for rapid growth of Mars and its status as a planetary embryo. *Nature, 473*(7348), 489–492. http://dx.doi.org/10.1038/nature10077

Greenberg, R., Hartmann, W. K., Chapman, C. R., & Wacker J. F. (1978). Planetesimals to planets—Numerical simulation of collisional evolution. *Icarus, 35*(1), 1–26. http://dx.doi.org/10.1016/0019-1035(78)90057-X

Ida, S., & Makino, J. (1993). Scattering of planetesimals by a protoplanet—Slowing down of runaway growth. *Icarus, 106*(1), 210. http://dx.doi.org/10.1006/icar.1993.1167

Kokubo, E., & Ida, S. (1993). Velocity evolution of disk stars due to gravitational scattering by giant molecular clouds. *AIP Conference Proceedings*, 278, 380–383.

Kokubo, E., & Ida, S. (1996). On runaway growth of planetesimals. *Icarus, 123*(1), 180–191. http://dx.doi.org/10.1006/icar.1996.0148

Kokubo, E., & Ida, S. (1998). Oligarchic growth of protoplanets. *Icarus, 131*(1), 171–178. http://dx.doi.org/10.1006/icar.1997.5840

Minton, D. A., & Levison, H. F. (2014). Planetesimal-driven migration of terrestrial planet embryos. *Icarus, 232,* 118–132. http://dx.doi.org/10.1016/j.icarus.2014.01.001

Thommes, E. W., Duncan, M. J., & Levison, H. F. (2003). Oligarchic growth of giant planets. *Icarus, 161*(2), 431–455. http://dx.doi.org/10.1016/S0019-1035(02)00043-X

Weidenschilling, S. J. (1977). The distribution of mass in the planetary system and solar nebula. *Astrophysics and Space Science, 51*(1), 153–158. http://dx.doi.org/10.1007/BF00642464