

A Mathematical Procedure for Time-Space Diagrams

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INTRODUCTION

This paper will describe a mathematical procedure for constructing a “best fit” time-space diagram for a two-way arterial street. The procedure is an extension of “The Arithmetic of a Balanced Two-way Signal Progression,” an article by Fremont G. Purdy appearing in the January 1967 issue of *Traffic Engineering* magazine. Using the procedure, trial-and-error mathematical manipulation is substituted for graphical manipulation thereby reducing preparation time for time-space diagrams.

Purdy’s method is limited to a two-phase signal operation and a fixed design speed. Building stops into signal progression when bandwidths become very narrow is not easily handled when graphical time-space diagrams are constructed. These problems are overcome in the procedure discussed herein.

This procedure has the following features:

1. Design speed can be varied by direction of travel.
2. Design speed can be varied between signalized intersections.
3. Leading and lagging greens for through movements associated with left turn phases are a part of the maximum bandwidth calculations.
4. Selection of lead-lag, lag-lead or simultaneous left turn phases is identifiable in terms of producing maximum bandwidths for through movements.
5. Fixed phasing arrangements at adjacent signalized intersections, e.g., ramp terminals of a diamond interchange, are easily handled.
6. Alternate plans for “built in” progression stops can be quickly tested when bandwidths become too narrow.
7. Detection is easily made of “double” stops not critical to maximum bandwidths.
8. Adjustment of excess green time can be made to “lead out” stored vehicle prior to the arrival of the progression queue.

9. The mathematical manipulation, properly summarized, allows the graphical plotting of the time-space diagram.

This procedure can also be used for determining progression in a network of signals. Its use in this application is cumbersome, however.

ASSUMPTIONS, DEFINITIONS AND TERMS

To understand this procedure, it is necessary to set forth certain assumptions, definitions and terms. First, a description will be made of the final product, the time-space diagram. Percent of cycle length to the scale of 1 in. = 20% shall be plotted along the vertical axis of the time-space diagram. Distance to the scale of 1 in. = 500 ft. shall be plotted along the horizontal axis. The outbound direction of travel shall be assumed as left to right. The inbound direction of travel shall be assumed as right to left. The Reset A line is an arbitrarily chosen horizontal line. The Reset B line is located $\pm 50\%$ distant from the Reset A line—see Figure 1.

The arrangement of the arterial green signal phasing can be described arithmetically. Yellow time shall be assumed as part of green time.

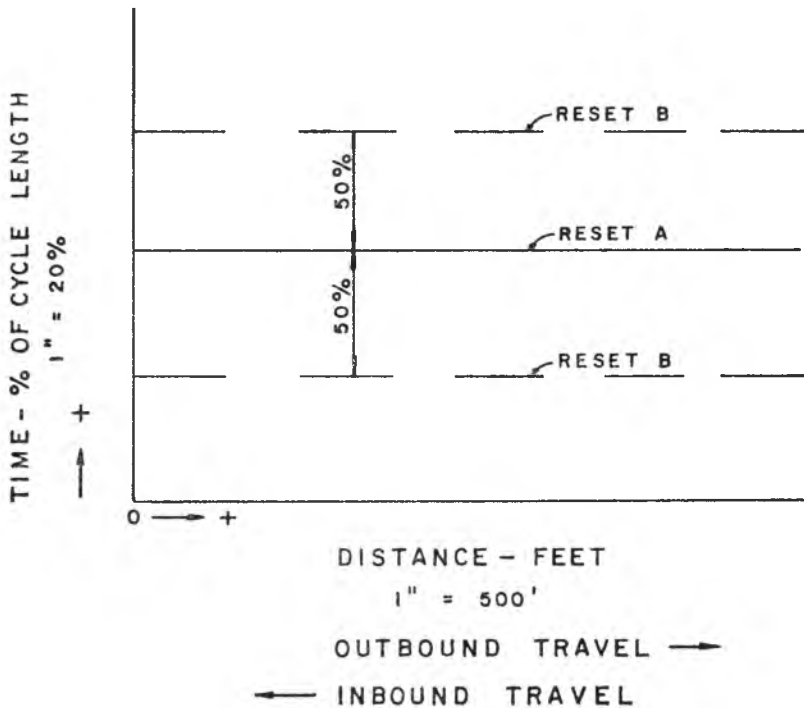


Fig. 1. Time-space diagram relationships.

(The Uniform Vehicle Code considers the yellow indication to be only a *warning* that the related green movement is being terminated.) The "center of green" shall be defined as the mid-point between the green extremities. Only whole percentage numbers shall be used. Consequently, if the difference between the green extremities is an odd number, the "center of green" shall be consistently assumed 1% closer to the lower green extremity than to the upper green extremity.

The outbound (inbound) bandwidth for any given signal is limited by the beginning and end of the green time displayed to outbound (inbound) traffic. These points are defined as follows:

B = beginning of green in outbound direction of travel.

E = end of green in outbound direction of travel.

B' = beginning of green in inbound direction of travel.

E' = end of green in inbound direction of travel.

An arithmetic sign for each of these limiting points is illustrated in Figure 2.

Unless the arterial green signal phasing is highly contorted, the arithmetic signs of B and E' will be plus (+) and the arithmetic signs of B' and E will be minus (-).

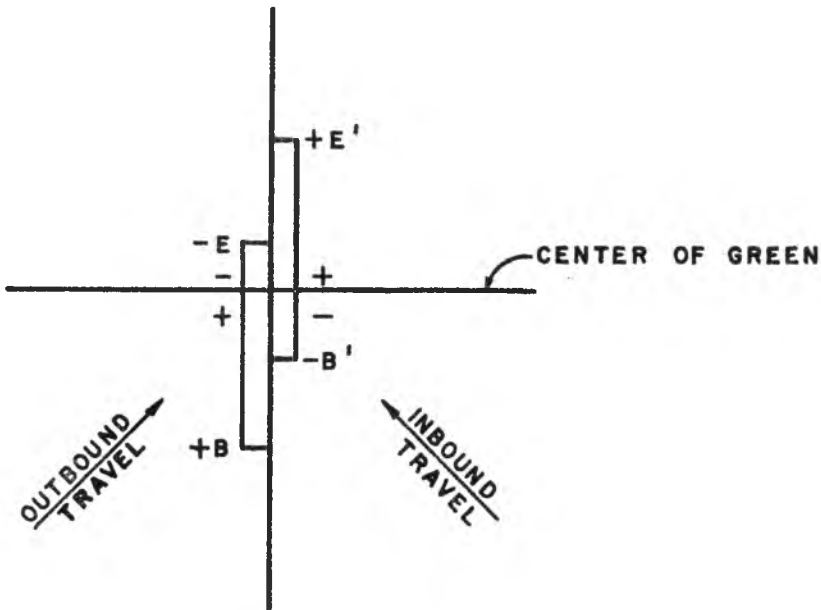


Fig. 2. Arithmetic signs for limiting green time points.

$$(1) \quad Y = \frac{100 D}{f VC} \qquad (2) \quad Y' = \frac{100 D}{f V'C}$$

Y = Outbound offset rounded to nearest whole percent.

Y' = Inbound offset rounded to nearest whole percent.

D = Distance between signalized intersections in feet.

$$f = \frac{88 \text{ ft./sec.}}{60 \text{ mi./hr.}} = 1.467 \text{ ft.-hr./sec.-mi.}$$

V = Outbound design speed between adjacent signalized intersections in miles per hour.

V' = Inbound design speed between adjacent signalized intersections in miles per hour.

C = Cycle length in seconds.

Maximum progression bandwidths for an arterial (or part thereof) are given by the formulae:

$$(3) \quad W = b - e \qquad (4) \quad W' = -b' + e'$$

Where:

W = Maximum outbound bandwidth in %.

W' = Maximum inbound bandwidth in %.

b = Smallest + value (largest - value) of all B values in system.

b' = Smallest - value (largest + value) of all B' values in system.

e = Smallest - value (largest + value) of all E values in system.

e' = Smallest + value (largest - value) of all E' values in system.

THE FOUR-NUMBER FRACTION

The green display at an intersection can be described by four numbers with their appropriate signs: +B, -E, -B' and +E'. It is convenient to array these numbers in "fraction" form:

$$\frac{(+)\text{B} \quad | \quad (-)\text{E}}{(-)\text{B}' \quad | \quad (+)\text{E}'}$$

Note that the two numerator numbers pertain to the outbound band and the two denominator numbers pertain to the inbound band. The vertical line represents the center of green.

Figure 3 illustrates several typical types of phasing and their four-number fraction descriptions.

SAMPLE PROBLEM

To best understand the procedure, the solution of a sample problem will be demonstrated. Assume that the spacing of five signalized intersections along Main Street are known. Assume that travel time runs have been made on Main Street and the design speeds between adjacent

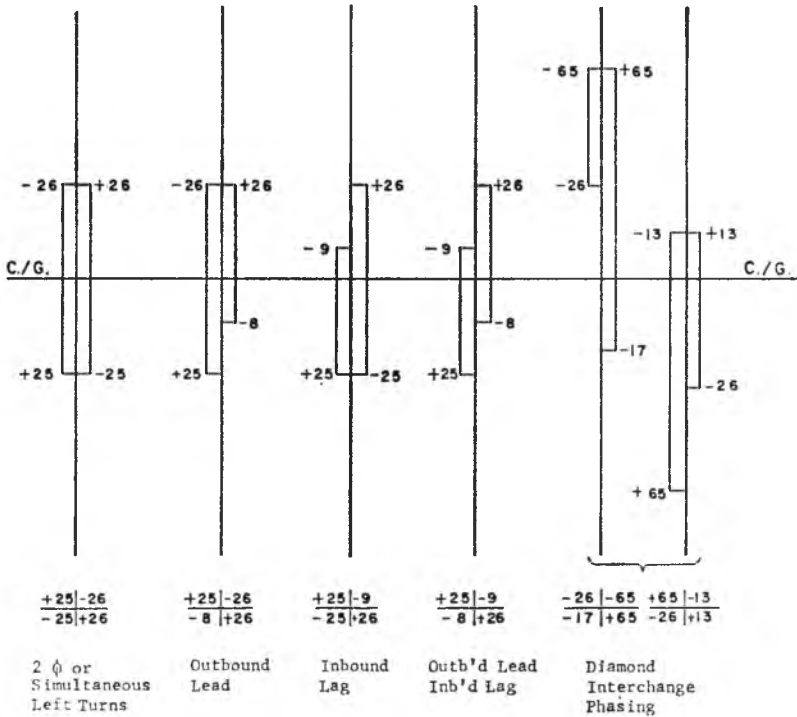


Fig. 3. Typical phasing types.

signalized intersections have been calculated by direction. Assume that turning movement counts have been made and signal split calculations have been completed on the basis of an 80-second cycle.

These data would apply to a street configuration which would schematically appear as follows in Figure 4.

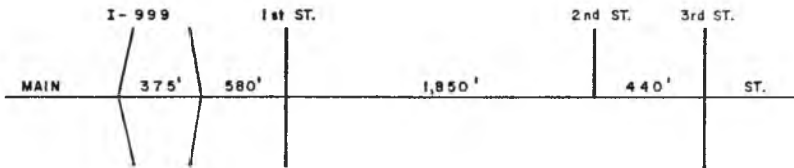


Fig. 4. Main Street—schematic of signalized intersections.

The diamond interchange ramp intersections at I-999 will have a fixed signal arrangement for those two intersections. Fixed signal arrangements generally occur at two or more intersections where travel time is an integral part of the split calculations. The green splits at

TABLE 1. MAIN STREET SIGNAL DATA

Operation = Two-Way		Cycle Length = 80 Seconds					
Signal	Spacing	Station	Design Speed EB WB	Direction of Travel	Green Splits		Lag L&T
					Lead L&T	Thru Only	
I-999, W.		0+00		EB WB 39%	39% 39%
I-999, E.	375'	3+75	35 35	EB WB 43%	39% 39%
1st St.	580'	9+55	33 30	EB WB	60% 60%
2nd St.	1,850'	28+05	25 26	EB WB or EB WB 17%	32% 32%
3rd St.	440'	32+45	27 29	EB WB	32% 32%	17%
				EB WB 19%	24% 24%	17%

I-999 are assumed to have been calculated to produce the fixed signal arrangement as shown in Figure 5.

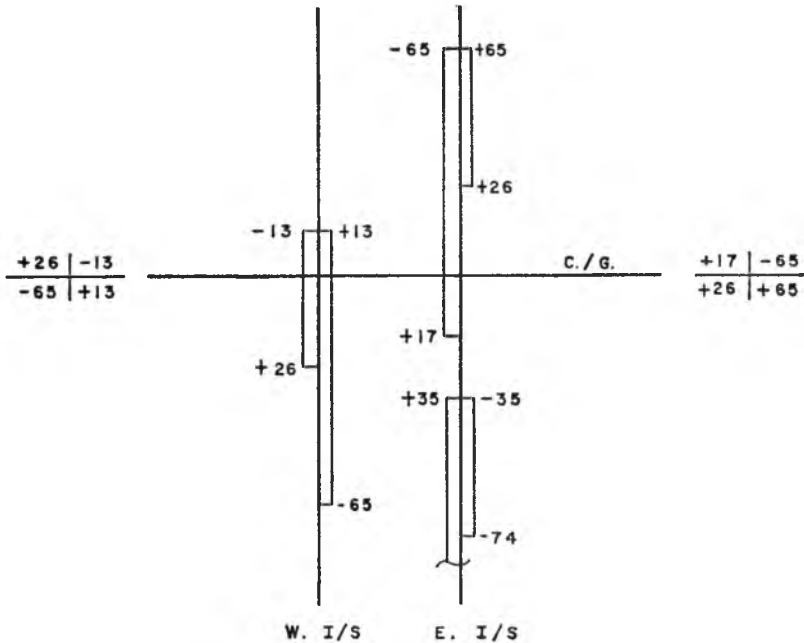


Fig. 5. Fixed signal arrangement at I-999.

Because signal timing is cyclical, it can be seen that $B' = -74$ and $E' = -35$, also said another way, the number 100 can be added or subtracted to *both* numbers in either the numerator or the denominator.

First street is assumed to have a two-phase signal.

At 2nd Street, an eastbound left turn and through phase is assumed warranted. The phase can be either an eastbound lead or an eastbound lag and both possibilities will be examined.

At 3rd Street, westbound lead and eastbound lag phasing will be used.

A convenient form has been developed for entering and recording data used in this procedure. See Figure 6. The west intersection of I-999 is assumed as the origin or starting point. Traffic movement away from this point is assumed as *outbound* and traffic movement toward this point is assumed as *inbound*. The data contained in Table I have been appropriately entered in Figure 6. The two intersections involving I-999 are shown with a bracket to indicate their fixed signal arrangement.

DIAL : 1 LOCATION MAIN STREET: I-999 TO 3RD STREET
 RESET: 1 (+)B (-)E DATE: 3-4-74 BY: CER
 CYCLE: 80 (-)B' (+)E'

SHEET _____ OF _____											
D	V	Y	Σ Y	A	±	CALC.	OFFSET	VERT.	FINAL	W	W'
FT.	V'	Y'	Σ Y'	OR	1/2 C	PHAS.	ADJUST.	SHIFT	ADJUST.	↓	↑
(1)	M.P.H.	(%)	(%)	B	(%)	(%)	(%)	± (%)	(%)	(%)	(%)
(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
I-999, W. I/S											
IB LEAD											
						+26	-13				
						-65	+13				
375'	35										
	35										
I-999, E. I/S											
OB LEAD											
						+17	-65				
						+26	+65				
580'	33										
	30										
1ST STREET											
						+30	-30				
						-30	+30				
1,850'	25										
	26										
2ND STREET											
OB LEAD											
						+24	-25				
						-7	+25				
—	—										
	—										
2ND STREET											
OB LAG											
						+24	-25				
						-24	+8				
440'	27										
	29										
3RD STREET											
IB LEAD-OB LAG											
						+11	-30				
						-30	+13				

Fig. 6.

Using formulae (1) and (2), Column 3 in Figure 7 may be calculated. For the first intersection:

$$Y \text{ (also, } Y') = \frac{100 \times 375}{1,467 \times 35 \times 80} = 9.1\%, \text{ Use } 9\%$$

The accumulated value for both Y and Y' for the first intersection is arbitrarily chosen as 0 percent and entered appropriately in Column 4. The remaining accumulated values for Y and Y' are calculated and entered in Column 4.

If the last two digits of the ΣY value shown in Column 4 above the signal line form a number closer to 00 or 100 than to 50, enter A in Column 5. If those digits form a number closer to 50, enter B in Column 5. This has the effect of placing the center of green for each signal on either the Reset A line of the Reset B line. Referring to Figure 1, these two lines are 50% or a half-cycle apart. This also coincides with the theory that the optimum spacing for two-phase traffic signals located on a two-way street is a half-cycle apart.

Column 6 is the difference between the number represented by last two digits of Column 4 and the reset line shown in Column 5. The appropriate sign is given.

Column 8 is obtained by adding the numerator of Column 6 to the numerators of Column 7.

Referring to formulae (3) and (4), Column 8 is now examined for the values of b, e, b' and e':

$$\begin{aligned} b &= +11, e = -6; \text{ therefore, } W = (+11) - (-6) = 17\% \\ b' &= +35, e' = -6; \text{ therefore, } W' = -(+35) + (-6) = \\ & -41\%. \text{ No band.} \end{aligned}$$

Remembering that the value 100 may either be added to or subtracted from both numbers in either the numerator or denominator of the four-number fraction, 100 is subtracted from the denominator of Column 8, for the I-999 East Intersection. B' becomes -65 and E' becomes -26. Now,

$$\begin{aligned} b' &= -5, e' = -6; \text{ therefore, } W' = -(-5) + (-6) = \\ & -1\%. \text{ No band.} \end{aligned}$$

The center of green can be shifted from one reset line to the other by adding 50 to *both* the numerator and the denominator in Column 8, or by subtracting 50 from *both* the numerator and denominator in Column 8. For 1st Street, we now subtract 50 from both the numerator and denominator in Column 8 and change A to B in Column 5.

The values of W and W' now become:

$$\begin{aligned} W &= (+4) - (-13) = 17\% \\ W' &= -(-21) + (-6) = 15\% \end{aligned}$$

DIAL : I LOCATION MAIN STREET I-999 TO 3RD STREET
 RESET: I (+)B | (-)E DATE: 3-4-74 BY: CER
 CYCLE: 80" (-)B' | (+)E'

D FT. (1)	V V' M.P.H. (2)	Y Y' % (3)	Σ Y Σ Y' % (4)	A OR B (5)	± 1/2 C % (6)	CALC. PHAS. % (7)	OFFSET ADJUST. % (8)	VERT. SHIFT ± % (9)	SHEET		OF	
									FINAL ADJUST. % (10)	W % (11)	W' % (12)	
			0%									
I-999, W. I/S IB LEAD				A	0	+26 -13	+26 -13	-25	(+1)	-38		
			0%		0	-65 +13	-65 +13	+25	-40	+38		
375'	35	9%	9%									
	35	9%	9%									
I-999, E. I/S OB LEAD				A	+9	+17 -65	+26 -56	-25	(+1)	-81		
					+9	+26 +65	+35 +74	+25	-40	(-1)		
580'	33	15%	24%									
	30	16%	25%									
1ST STREET				B	+24	+30 -30	+54 -6	-50/+2	+6	-54		
					+25	-30 +30	-5 +55	-50/-2	-57	+3		
1,850'	25	63%	87%									
	26	61%	86%									
2ND STREET OB LEAD				A	-13	+24 -25	+11 -48					
					-14	-7 +25	-21 +11					
			87%									
			86%									
2ND STREET OB LEAD				A	-13	+24 -25	+11 -38	-5	+6	-43		
					-14	-24 +8	-38 -6	+5	-33	(-1)		
440'	27	14%	101%									
	29	13%	99%									
3RD STREET IB LEAD - OB LAG				A	-1	+11 -30	+12 -29	0	+12	(-29)		
					-3	-30 +13	-31 +12	0	(-31)	+12		

30%
30%

Fig. 7.

By referring to Figure 2, it can be seen that the signal phasing can be shifted vertically downward by adding a number to the numerator and subtracting that same number from the denominator of Column 8. Conversely, an upward vertical shift can be made by subtracting a number from the numerator and adding that same number to the denominator.

Consider the signal phasing at only 2nd Street and 3rd Street. Using the outbound lead alternative at 2nd Street:

$$W = (+11) - (-29) = 40\%$$

$$W' = -(-21) + (+11) = 32\%$$

However, B' is critical at 2nd Street which means that a queue starting inbound at 3rd Street will be stopped at 2nd Street. Therefore, try the outbound lag alternative at 2nd Street:

$$W = (+11) - (-29) = 40\%$$

$$W' = -(-31) + (-6) = 25\%$$

Shifting for nearly equal bandwidths, the fractions become:

	<u>Col. 8</u>	<u>Shift</u>	
2nd St.	$\begin{array}{r l} +11 & -38 \\ \hline -38 & -6 \end{array}$	$\begin{array}{r l} 0 & \\ \hline 0 & \end{array}$	$\begin{array}{r l} +11 & -38 \\ \hline -38 & -6 \end{array}$
3rd St.	$\begin{array}{r l} +12 & -29 \\ \hline -31 & +12 \end{array}$	$\begin{array}{r l} +7 & \\ \hline -7 & \end{array}$	$\begin{array}{r l} +19 & -22 \\ \hline -38 & +5 \end{array}$

$$W = (+11) - (-22) = 33\%$$

$$W' = -(-38) + (-6) = 32\%$$

Now, consider 1st Street, 2nd Street and 3rd Street and make shifts as follows:

		<u>Shift</u>	
1st St.	$\begin{array}{r l} +4 & -56 \\ \hline -55 & +5 \end{array}$	$\begin{array}{r l} 0 & \\ \hline 0 & \end{array}$	$\begin{array}{r l} +4 & -56 \\ \hline -55 & +5 \end{array}$
2nd St.	$\begin{array}{r l} +11 & -38 \\ \hline -38 & -6 \end{array}$	$\begin{array}{r l} -7 & \\ \hline +7 & \end{array}$	$\begin{array}{r l} +4 & -45 \\ \hline -31 & +1 \end{array}$
3rd St.	$\begin{array}{r l} +19 & -22 \\ \hline -38 & +5 \end{array}$	$\begin{array}{r l} -7 & \\ \hline +7 & \end{array}$	$\begin{array}{r l} +12 & -29 \\ \hline -31 & +12 \end{array}$

$$W = (+4) - (-29) = 33\%$$

$$W' = -(-31) + (+1) = 32\%$$

Because the I-999 intersections have a fixed signal arrangement, both signals must be shifted in the same amount unless they are not referenced to same reset line (A or B). In this latter case, a 50 percent shift of one of the signals is necessary.

		1st Shift		2nd Shift	
I-999, W.	$\begin{array}{r l} +26 & -13 \\ \hline -65 & +13 \end{array}$	$\begin{array}{r} -22 \\ \hline +22 \end{array}$	$\begin{array}{r l} +4 & -35 \\ \hline -43 & +35 \end{array}$	$\begin{array}{r} -3 \\ \hline +3 \end{array}$	$\begin{array}{r l} +1 & -38 \\ \hline -40 & +38 \end{array}$
I-999, E.	$\begin{array}{r l} +26 & -56 \\ \hline -65 & -26 \end{array}$	$\begin{array}{r} -22 \\ \hline +22 \end{array}$	$\begin{array}{r l} +4 & -78 \\ \hline -43 & -4 \end{array}$	$\begin{array}{r} -3 \\ \hline +3 \end{array}$	$\begin{array}{r l} +1 & -81 \\ \hline -40 & -1 \end{array}$
1st St.	$\begin{array}{r l} +4 & -56 \\ \hline -55 & +5 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +4 & -56 \\ \hline -55 & +5 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +4 & -56 \\ \hline -55 & +5 \end{array}$
2nd St.	$\begin{array}{r l} +4 & -45 \\ \hline -31 & +1 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +4 & -45 \\ \hline -31 & +1 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +4 & -45 \\ \hline -31 & +1 \end{array}$
3rd St.	$\begin{array}{r l} +12 & -29 \\ \hline -31 & +12 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +12 & -29 \\ \hline -31 & +12 \end{array}$	$\begin{array}{r} 0 \\ \hline 0 \end{array}$	$\begin{array}{r l} +12 & -29 \\ \hline -31 & +12 \end{array}$

$$W = (+4) - (-29) = 33\% \text{ (1st shift)}$$

$$W' = -(-31) + (-4) = 27\% \text{ (1st shift)}$$

$$W = (+1) - (-29) = 30\% \text{ (2nd shift)}$$

$$W' = -(-31) + (-1) = 30\% \text{ (2nd shift)}$$

Balanced bandwidths are now provided. A +2/-2 shift in the 1st Street and 2nd Street signals will provide better "lead out" of stored vehicles without reducing the bandwidths. Columns 9, 10 and 11 of Figure 7 can now be completed.

TIME-SPACE DIAGRAM PLOT

Using Figure 7, the data provided allow the direct plot of the green limiting points on a time-space diagram. Starting with the I-999 West Intersection, Columns 5 and 7 are examined. Before any shift was made, Point B was located 26% below the Reset A line. In the final adjustment, the phasing was shifted upward 25% (see Column 9). Consequently, the beginning of eastbound (outbound) green can be plotted 1% below the Reset A line. The length of the outbound green is 39%. Therefore, the end of the eastbound green is plotted 38% above the Reset A line. Plotting for all other greens in the system is made similarly except for 1st Street.

The beginning of eastbound (outbound) green at 1st Street, after the shift, is plotted 32% below the Reset B line or 82% below the Reset A line since the two reset lines are separated by 50%. The end of the eastbound green is plotted 22% below the Reset A line.

The beginning of the eastbound (outbound) green band is plotted from one of the limiting B values. In this case, the beginning of the eastbound band coincides with the beginning of the eastbound green at

the I-999 West Intersection or 1% below the Reset A line. The travel time to the I-999 East Intersection is 9% (numerator, Column 3). The beginning of the outbound band at the I-999 East Intersection will,

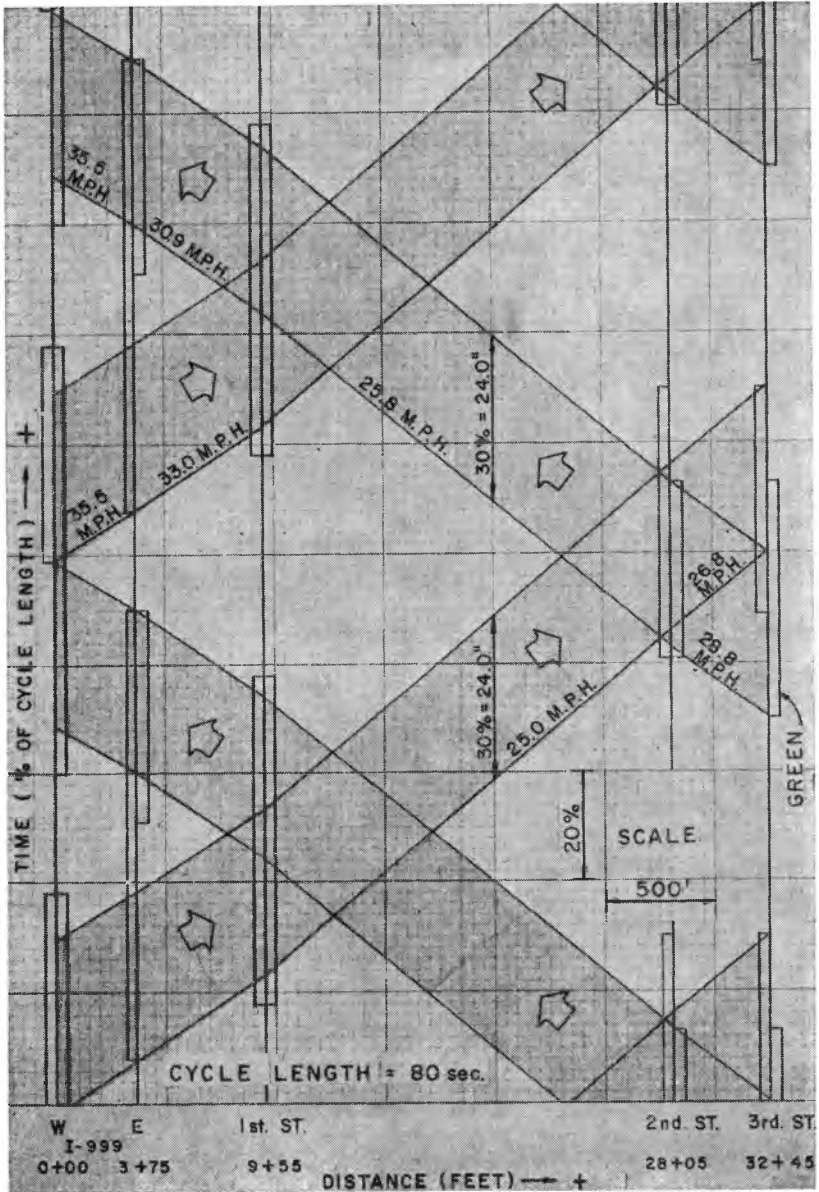


Fig. 8. Plotted time-space diagram.

therefore, be located 8% above the Reset A line. Plotting of the eastbound band is continued similarly using the values shown in Column 3. Obviously, the end of the eastbound band can be plotted 30% (Column 11) above the beginning of the eastbound band. The westbound (inbound) band can be plotted in a similar fashion using the appropriate denominator values.

Plotting of the time-space diagram is precise. See Figure 8. However, because whole percentage numbers have been used for travel time offsets, the progression speed will vary slightly from that shown in Figure 7, Column 2.

COMPUTER ADAPTATION

We have programmed this procedure on an Olivetti P602 desk-top computer equipped with an MLU 600 memory unit which is available in our office. The computer reduces calculation time considerably and allows rapid shifting of the signing phasing.

SUMMARY

This procedure provides a more rapid means of constructing a time-space diagram through mathematical manipulation as opposed to graphical manipulation. An arterial with 27 signalized intersections has been successfully analyzed for signal progression using this procedure.