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Constructing Separating Halfspaces for Plane/Quadric and Quadric/Quadric Intersection*

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Abstract

Connected component decomposition of space induced by a pair of intersecting halfspaces can result in several disjoint components having identical classification with respect to both halfspaces. Therefore, additional halfspaces separating these components are required to classify a given point with respect to components in space decomposition. We consider this problem for general quadrics and give a method to construct separating halfspaces for plane/quadric and quadric/quadric intersection.

1 Introduction

Classification of a point as inside, outside or on the boundary of a solid is a fundamental operation for many applications which repeatedly reason about solids such as collision detection in dynamic simulation systems and discretization of 3D domains for solving partial differential equations. For a solid defined as union of semialgebraic space components, point/solid classification can be done by classifying the given point with respect to components defining the solid.

Binary Space Partition (BSP) tree is a binary tree data structure used to represent volume of a solid. Its interior nodes correspond to oriented halfplanes and leaf nodes represent the regions which are either inside or outside the solid. For details refer to [Vaněček]. BSP tree can also be used to represent exhaustive connected component decomposition of space induced by a set of intersecting halfspaces. If planes are used to construct BSP tree then each leaf node represents a single connected region. If halfspaces of higher degree such as quadrics are used to construct a BSP tree, a leaf node may represent several disjoint components in space which have identical classification with respect to all the halfspaces. If a BSP tree is constructed from the set of halfspaces induced from the faces of the given solid and not all the regions at a leaf node are inside or outside the solid, additional halfspaces separating space components with identical classification are needed such that all the space components corresponding to a leaf node are either inside or outside the solid. Since the space components corresponding to a leaf node can be represented by a semialgebraic expression in terms of

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the halfspaces used at each node in the path from root node to the leaf node, the solid can be represented as union of semialgebraic space components.

This work is motivated by the problem of constructing a sufficient set of separating halfspaces for solids bounded by quadric faces so as to be able to represent them as union of semialgebraic space components. It can also be viewed as a method to convert boundary representation of a solid to volumetric representation as discussed in [ShapiroVossler1, ShapiroVossler2, ShapiroVossler3]. These authors consider this problem for solids bounded by quadric faces with planar edges and give a practical method to construct a set of linear halfspaces separating each face from the rest of halfspace. They prove that it is a sufficient set of separating halfspaces for the solids bounded by planar edges. We approach this problem by constructing separating halfspaces for components with identical classification in space decomposition by halfspaces induced from the faces of the given solid. We consider general solids bounded by planar and quadric faces. In this paper we give a method for construction of separating halfspaces for space components with identical classification resulting from plane/quadric and quadric/quadric intersection. This can be used to construct separating halfspaces for space components resulting from every pair of halfspaces in the set of halfspaces induced from faces of the solid. In our future work, we consider separation of space components resulting from intersection of more than two halfspaces. Subsequently, the minimization techniques described in [ShapiroVossler2] can be used for constructing an efficient CSG representation using the separating halfspaces constructed by our method.

This problem is also of interest from view of decomposing a given solid into simpler components where each component can be described as intersection of halfspaces which define regular semialgebraic subsets of space.

2 Prior Work

Shapiro and Vossler [ShapiroVossler1, ShapiroVossler2, ShapiroVossler3] have considered the problem of constructing separating halfspaces for solids bounded by quadric faces where the edges of solids are planar. They prove the following constraint on the degree of the curve along which two space components with identical classification can touch.

Theorem 1 (Shapiro and Vossler) *Two space components with identical classification resulting from intersection of degree k halfspaces can touch along a curve of degree $k^2/2$ [ShapiroVossler3].*

Hence two space components with identical classification resulting from intersection of a pair of quadrics can touch along a curve of degree at most *two*. If the components touch along a conic, then a plane through the conic separates the two components. If the conic degenerates into a pair of lines, a plane containing the pair of lines separates the two components. When the conic degenerates into a double line such as the case of two parabolic cylinders touching along a line, a pair of planes through double line and parallel to principal planes of the quadrics separate the components with identical classification. When the two components touch along a set of points then a piecewise linear separating surface passing through these points can be constructed. Hence it is always possible to construct a set of linear separators for solids bounded by quadric faces irrespective of whether the the edges are planar or nonplanar.

We use the following result from [Levin] for construction of separating halfspaces for solids bounded by quadric faces.

Theorem 2 (Levin) *The intersection of two quadric surfaces lies in a plane, pair of planes, hyperbolic or parabolic cylinder, or hyperbolic paraboloid [Levin].*

Note that the nonplanar intersection curve of two quadric surfaces always lies in a nonelliptic quadric.

3 Preliminaries

From [ShapiroVossler3], we note the following observations.

1. The number of transversal intersections of a path ab with the boundary ∂h of a halfspace h modulo 2 is called *mod* 2 intersection number and is denoted as $I_2(ab, \partial h)$.

2. Points a, b are strictly separated by a halfspace h if and only if $I_2(ab, \partial h) = 1$, whenever ab and ∂h intersect transversally.

3. All paths ab are equivalent in the sense of having the same intersection number $I_2(ab, \partial h)$.

For formal definitions of these concepts and related discussion, the reader is referred to [GuilleminPollack]. In the rest of paper, we refer to quadric/quadric intersection curve as Quadric Surface Intersection Curve (QSIC).
The self-intersection points of a QSIC divide it into connected components such that any two points in a component are connected by a path lying entirely in the component. Each such component is either delimited by self-intersection points (which may coincide) or it is unbounded (Figure 1).

Similarly adjacent connected components in QSIC can be used as delimiters to define connected components of corresponding quadric surfaces (Figure 1).

4 Development of Method

We make the following observations:

1. Disjoint multiple components and self-intersections in the intersection curve of two halfspaces result in disjoint components in space with identical classification with respect to both intersecting halfspaces.

2. A halfspace separating two components in space also separates the edges of two components, although the reverse is not true in general.

From [ShapiroVossler3], we note that a halfspace h strictly separates points $a, b \notin \partial h$ if and only if $I_2(ab, \partial h) = 1$, where ab is a path joining points a and b. Also, all paths are equivalent in the sense of having same intersection number $I_2(ab, \partial h)$.

These observations suggest a method for constructing a set H_s of separating halfspaces for a given pair of intersecting halfspaces. We begin by constructing halfspaces separating connected components in the intersection curve of two given halfspaces. Then we consider the intersection of these separating halfspaces with the given halfspaces and add more halfspaces to the set until any two points a, b in different components in the space decomposition by given halfspaces are separated i.e, there is a path ab such that $I_2(ab, \partial f) = 1$ for some halfspace f in H_s. We give a constructive method to solve this problem for general quadrics and exploit the geometry of given quadrics to find separating halfspaces which terminate the above construction process.

5 Construction of Separating Halfspaces

Cone, hyperbolic cylinder and hyperboloid of two sheets are the only quadrics which bound disjoint components of space on the same side of the quadric. These components are easily separated by adding principal planes for these quadrics by computing the roots of a characteristic cubic equation [SnyderSisam]. In the first step, we add principal planes separating disjoint space components on same side of given quadrics. Then we construct separating halfspaces for components resulting from plane/quadric and quadric/quadric intersection.

5.1 Plane/Quadric Intersection

The intersection of a plane and a quadric is a conic. There is a need for separating halfspaces only when the intersection conic is a hyperbola or it degenerates into a pair of lines.

5.1.1 Plane/Quadric Intersection is Hyperbola

Plane/Quadric intersection can be a hyperbola only when the quadric is cone, hyperbolic cylinder, hyperbolic paraboloid or hyperboloid of one/two sheet.

In case of cone, hyperbolic cylinder and hyperboloid of two sheets, the principal planes added in first step constitute a sufficient set of separators.

Plane/Hyperbolic Paraboloid Intersection Let the hyperbolic paraboloid in canonical position be given by $x^2/a^2 - y^2/b^2 = 2cz$. Then a plane parallel to z axis given by $c_1x + c_2y + c_3 = 0$ intersects the hyperbolic paraboloid in a parabola.

Claim 1 *If a plane P intersects a hyperbolic paraboloid $Q : x^2/a^2 - y^2/b^2 = 2cz$ in a hyperbola, the two space components having identical classification with respect to plane as well as quadric are separated by a plane P_s through the conjugate principal axis of hyperbola and parallel to z axis.*

Proof: If a plane P intersects the hyperbolic paraboloid $Q : x^2/a^2 - y^2/b^2 = 2cz$ in a hyperbola, the two branches of hyperbola divide the surface of hyperbolic paraboloid into three connected sheets, two of which bound the components to be separated (Figure 2). The separating plane P_s, being parallel to z axis intersects the hyperbolic paraboloid in parabola. It also separates the two branches of hyperbola because it passes through the conjugate principal axis of the hyperbola. Hence a path joining two points on different branches of hyperbola, and lying on the sheet which does not contribute to the boundary of two components to be separated, also intersects P_s in an odd number of points. Since these points must also lie on the parabola, and the parabola does not intersect any branch of hyperbola, therefore the parabola lies completely in the sheet which does not contribute to the boundary of two components to be separated. Hence the separating plane P_s does not intersect the boundary of two components to be separated.

Now for arbitrary points $a \in A$, $b \in B$, the line segment $\overline{ab}$ intersects the boundary of two components in points a' and b' respectively (Figure 2). Since P_s does not intersect the boundary of two components to be separated, it does not intersect the line segments $\overline{aa'}$ and $\overline{b'b}$. Therefore, $I_2(aa', P_s) = I_2(b'b, P_s) = 0$. Let $a'b'$ be a path lying on hyperbolic paraboloid. Since a path joining two points on different branches of hyperbola intersects the separating plane P_s an odd number of times, therefore $I_2(a'b', P_s) = 1$. Consider the path ab formed

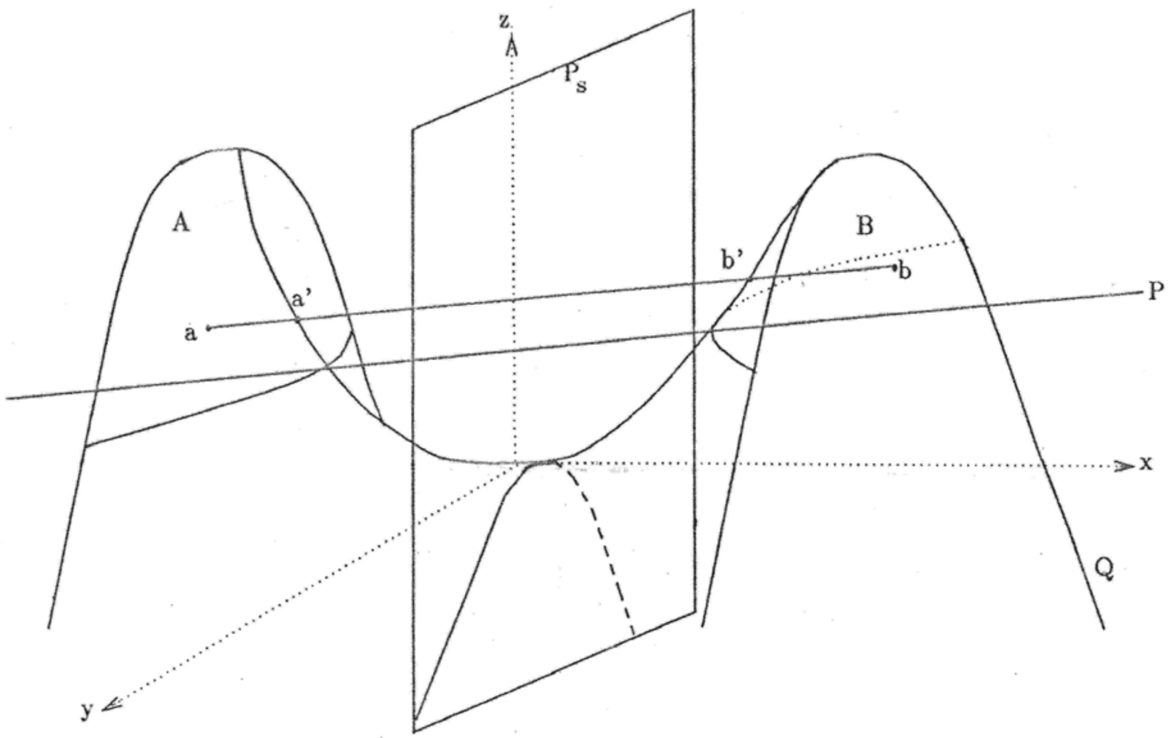

Figure 2: Plane P_s separates the two components A and B resulting from Plane (P)/Hyperbolic paraboloid (Q) Intersection.

by line segment $\overline{aa'}$, path $a'b'$ on the surface of hyperbolic paraboloid and line segment $\overline{b'b}$. Since all paths joining two given points are equivalent in the sense of same *mod* 2 intersection number, hence $I_2(ab, P_s) = 1$. □

Plane/Hyperboloid of Single Sheet Intersection The hyperboloid of single sheet divides the space into two connected components. In the following discussion we call the component containing centre, inside the quadric and other outside the quadric. We make the following observations:

1. A plane intersecting hyperboloid of single sheet in a hyperbola with its conjugate principal axis inside the quadric divides its surface into two connected components.

2. A plane through the origin divides the surface of hyperbolic paraboloid of single sheet into two connected components although the intersection curve in this case may be ellipse, hyperbola or a pair of parallel lines.

In above cases, no two space components have identical classification with respect to both plane and quadric. Hence no separating halfspaces are needed. Note that a plane through origin cannot intersect the hyperboloid of single sheet in a parabola or a pair of intersecting lines. Now we consider the case which results in disjoint space components with identical classification.

Claim 2 *If a plane P intersects hyperboloid of single sheet Q in a hyperbola with its conjugate axis outside the quadric, the two space components having identical classification with respect to plane as well as quadric are separated by a plane P_s through origin and the conjugate axis of hyperbola.*

Proof: If a plane P intersects the hyperboloid of single sheet Q in a hyperbola with its conjugate axis outside the quadric, it divides the surface of quadric into three connected sheets. The components bounded by two sheets of quadric on the same side of plane P need to be separated (Figure 3). The two branches of hyperbola form the edges of the two components to be separated. Therefore, the plane separating two components must separate the two branches of hyperbola. In addition, it must not intersect the boundary of two components to be separated.

Since the separating plane P_s passes through the conjugate principal axis of the hyperbola and passes through origin, it separates the two branches of hyperbola. Now we show that the plane does not intersect the boundary of two components to be separated. Since a plane through origin cannot intersect the hyperboloid of single sheet in a parabola or a pair of intersecting lines, we have only the following cases to be discussed:

1. The conic section of Q and the separating plane P_s cannot be hyperbola because in that case, the conjugate axis of the hyperbola resulting from intersection of P and Q must either intersect this hyperbola and therefore intersect Q, which is not possible or it must lie inside the quadric which is again in contradiction to the assumption made in the claim.

2. When the conic section of the separating plane P_s with the quadric is ellipse, the ellipse lies completely in the sheet which does not contribute to the boundary of two components to be separated. This can be shown to be true by a similar argument as used to prove that the parabola resulting from intersection of separating plane and hyperbolic paraboloid lies in the sheet which does not form the boundary of two components to be separated in proof of Claim 1.

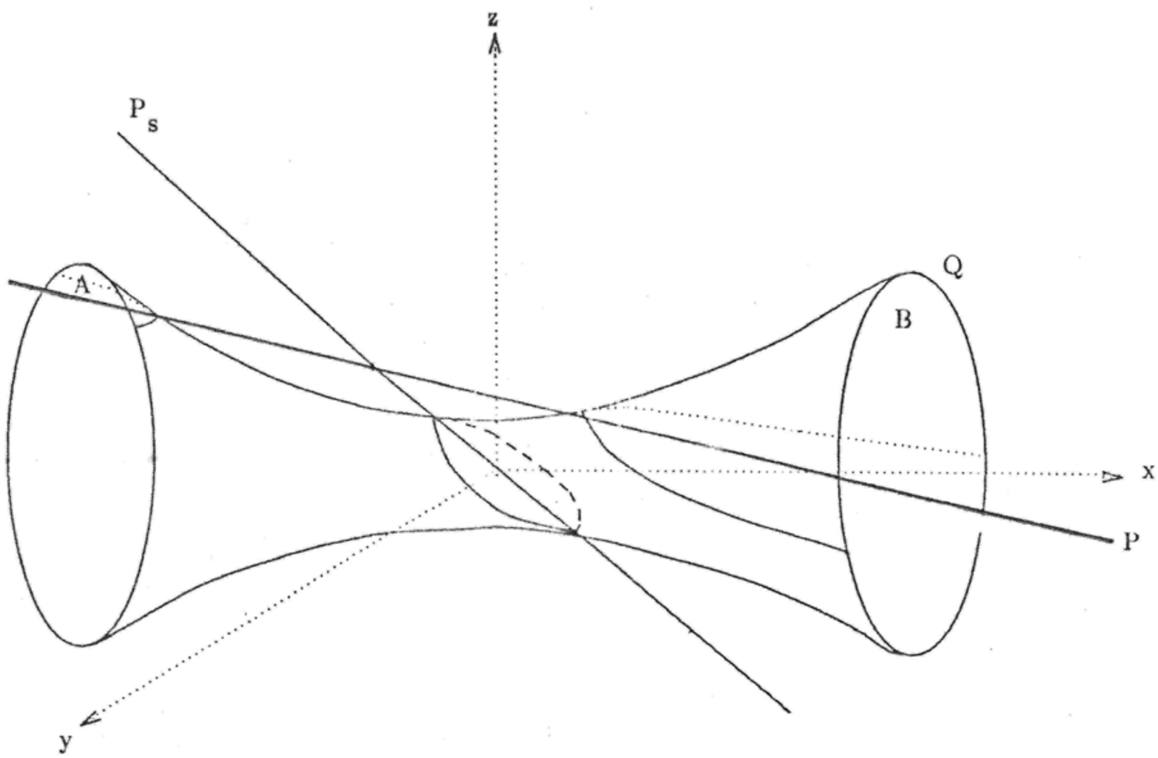

Figure 3: Plane P_s separates the two components A and B resulting from Plane (P)/Hyperboloid of single sheet (Q) Intersection.

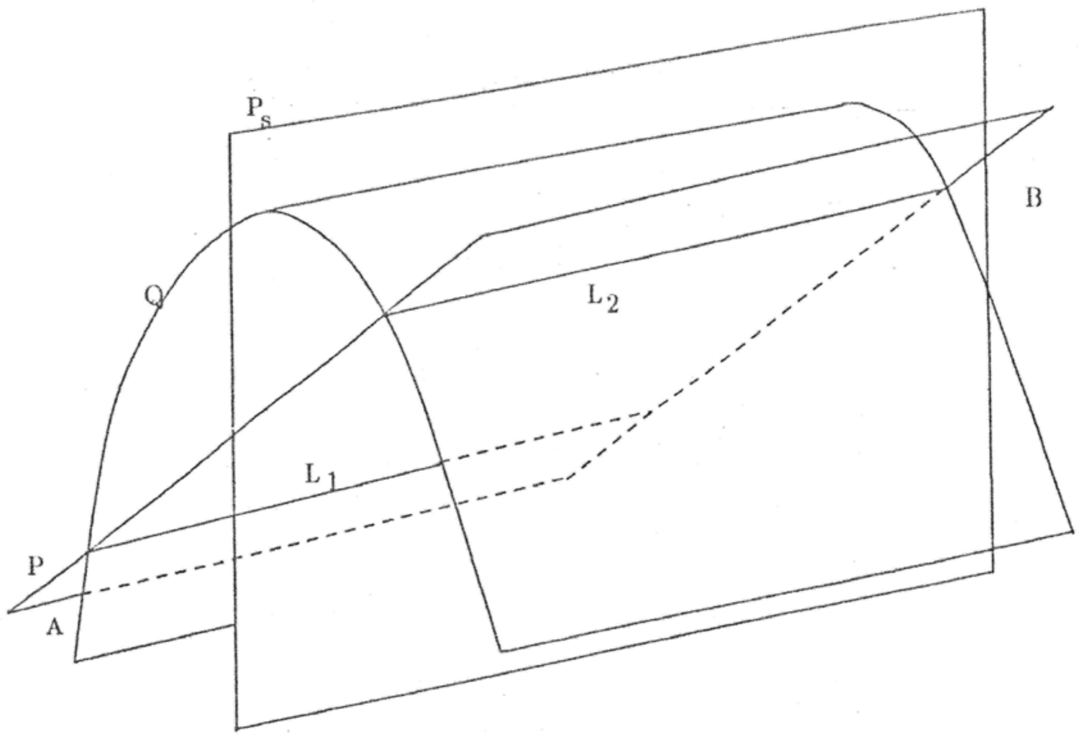

Figure 4a: Plane P and parabolic cylinder Q intersect in a pair of parallel lines L_1 and L_2. Plane P_s passing through bisector of L_1 and L_2 and parallel to principal plane of Q separates two space components A and B with identical classification. When P and Q intersect in a double line, it is handled as a limiting case similar to L_1 coinciding with L_2.

3. When the conic section of P_s with the quadric is a pair of parallel lines, both lines lie on the same side of the given plane P for otherwise the conjugate axis of the hyperbola resulting from intersection of P and Q would have to be inside the quadric which is contrary to the assumption made in the claim. Since P_s separates two branches of hyperbola, at least one point on these lines lies on the sheet which does not contribute to the boundary of two components to be separated. Since both lines are on same side of P, therefore both lie on the sheet which does not contribute to the boundary of two components to be separated. Hence P_s does not intersect the boundary of two components to be separated.

In cases (2) and (3), for any two points $a \in A$ and $b \in B$, a path ab similar to the one in proof of Claim 1 can be constructed to show that $I_2(ab, P_s) = 1$. □

5.1.2 Plane Quadric Intersection is Degenerate

The given plane and quadric may intersect in a double line or a pair of parallel/intersecting lines. A plane can intersect only a singular quadric in double line. There is no need for separation when the plane intersects an elliptic singular quadric in a double line. When the plane intersects a parabolic/hyperbolic cylinder in a double line then a plane through the double line and parallel to principal/transverse principal plane of the respective cylinder separates the two components in space with identical classification. Construction of separating planes when a quadric intersects the given plane in a pair of parallel/intersecting lines is illustrated

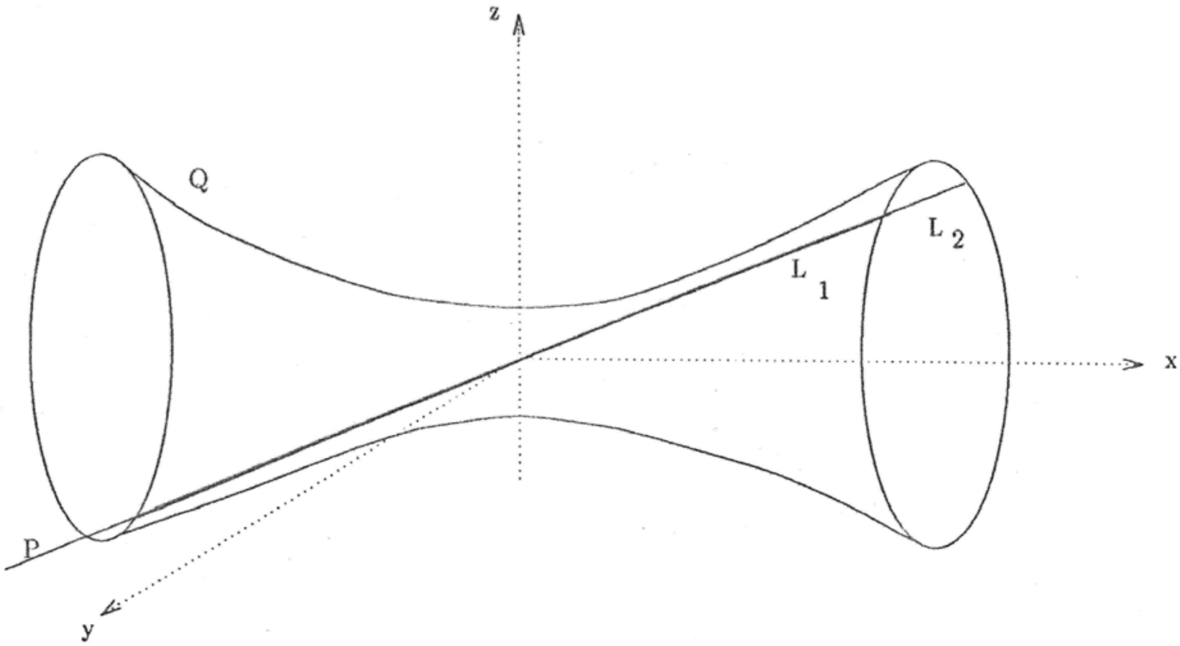

Figure 4b: Plane P and hyperboloid of single sheet Q intersect in a pair of parallel lines L_1 and L_2 dividing the surface of Q in two connected sheets. As a result there are no two components with identical classification.

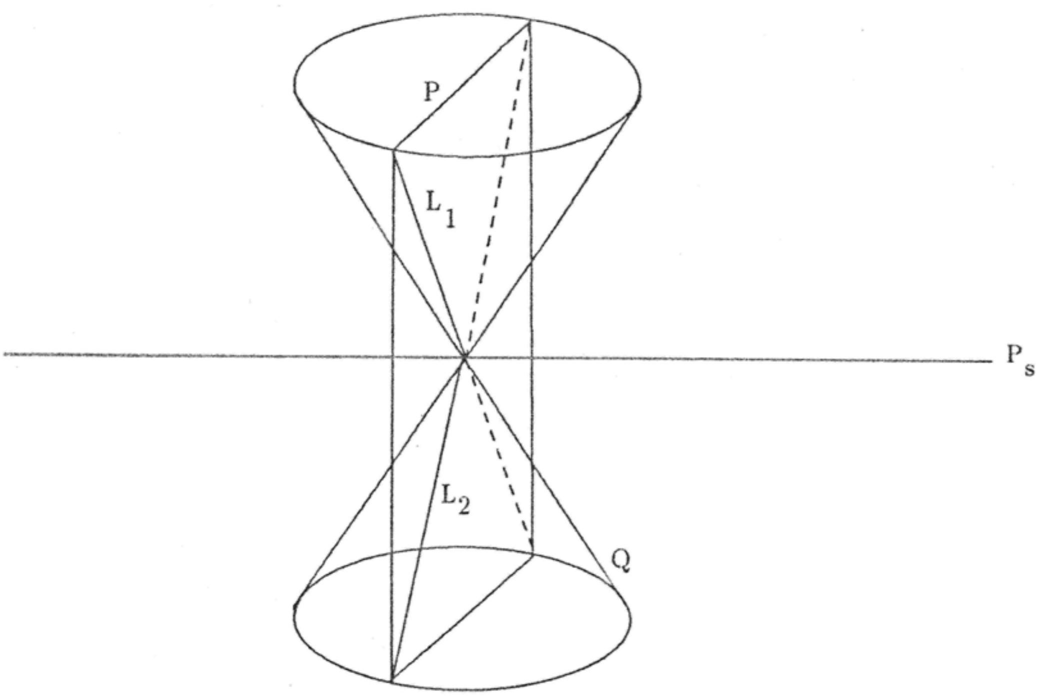

Figure 4c: Plane P and cone Q intersect in a pair of intersecting lines L_1 and L_2. The space components with identical classification resulting from intersection of P and Q are separated by the principal plane P_s.

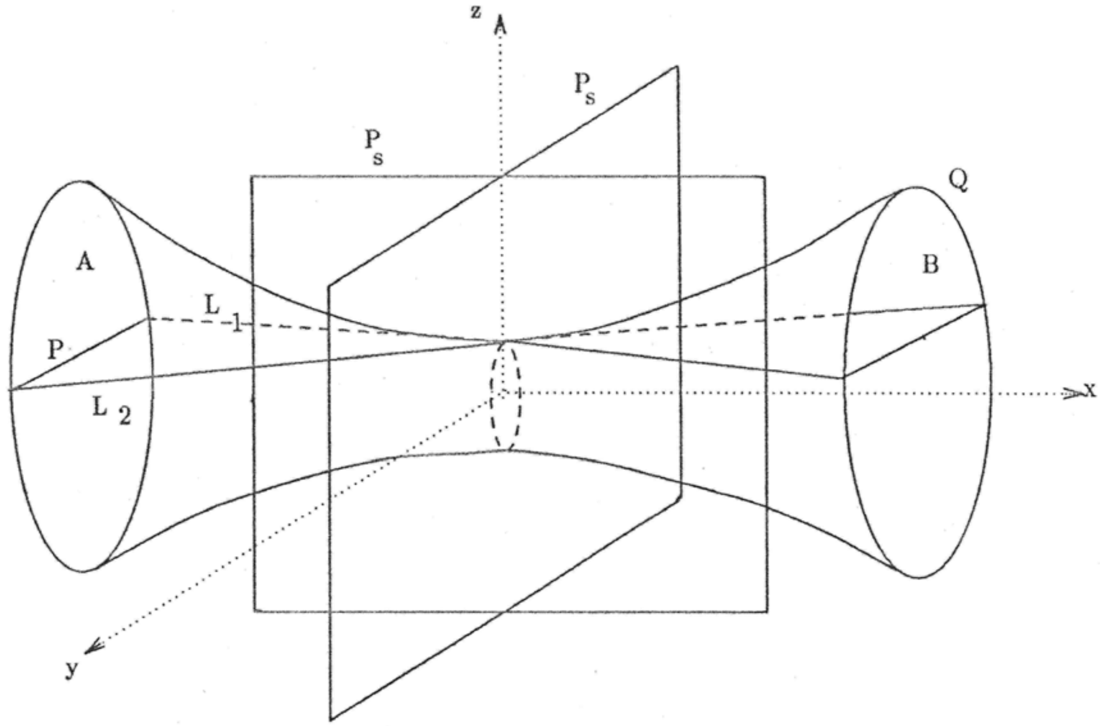

Figure 4d: Plane P and hyperboloid of single sheet Q intersect in a pair of intersecting lines L_1 and L_2. The two space components A and B with identical classification with respect to P and Q are separated by the pair of separating planes P_s passing through the bisectors of L_1 and L_2 and perpendicular to P.

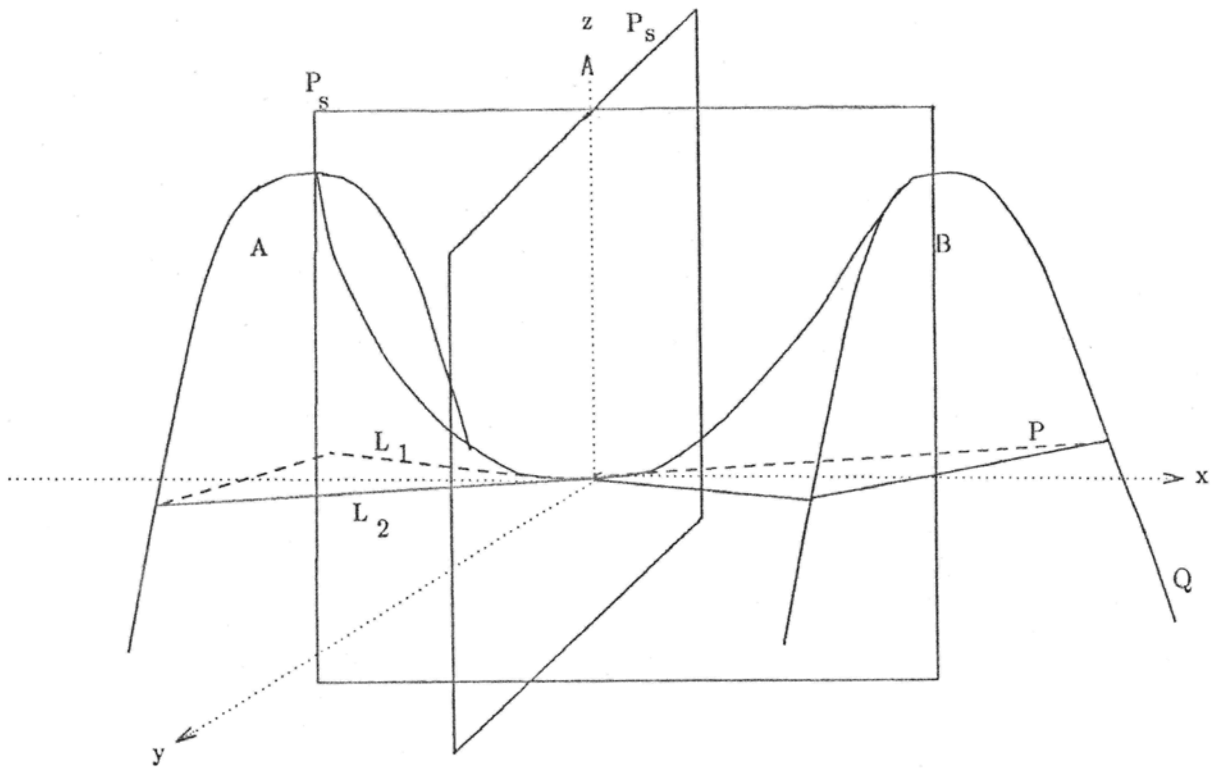

Figure 4e: Plane P and hyperbolic paraboloid Q intersect in a pair of intersecting lines L_1 and L_2. The two space components A and B with identical classification with respect to P and Q are separated by the pair of separating planes P_s passing through the bisectors of L_1 and L_2 and perpendicular to P.

in Figures 4a–4e. Note that although the separating plane P_s parallel to x axis in Figure 4d is redundant whereas both planes marked as P_s in Figure 4e are necessary, this approach allows both cases to be treated similarly as the QSIC is similar in both the cases.

5.2 Quadric/Quadric Intersection

Let a quadric $q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{23}yz + 2q_{31}zx + 2q_{41}x + 2q_{42}y + 2q_{43}z + q_{44} = 0$ be represented by symmetric matrix

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{pmatrix}$$

We consider the pencil $R(\alpha) = F - \alpha G$ of two given quadrics with discriminants F and G. Every point lying on both F and G lies on every quadric of the pencil. Through any point in space not lying on the intersection of F and G, passes one and only one quadric of the pencil. Any two distinct quadrics in the pencil have the same intersection curve. If for some real α, $R(\alpha)$ has rank 1 or 2, then $R(\alpha)$ represents either a plane or a pair of planes and we classify this pencil as planar pencil. A planar pencil satisfies following properties:

1. $|R(\alpha)| = 0$
2. Sum of 3×3 principal minors of $R(\alpha)$ vanishes *i.e.*,

$$\sum_{i=1}^{2} \sum_{j=i+1}^{3} \sum_{k=j+1}^{4} \begin{vmatrix} q_{ii} & q_{ij} & q_{ik} \\ q_{ji} & q_{jj} & q_{jk} \\ q_{ki} & q_{kj} & q_{kk} \end{vmatrix} = 0$$

Failing above, if for some real α, $R(\alpha)$ has rank 3 then pencil is nonplanar singular. If $|R(\alpha)| = 0$ has all four roots complex, *i.e.*, $R(\alpha)$ has rank 4 for every real α then the pencil is classified as nonplanar nonsingular.

Given quadrics F and G, we classify the pencil $F - \alpha G$ as planar, singular or nonsingular. The steps for computing the set H_s of separating halfspaces in each case are described in the following sections.

5.2.1 Planar Pencil

Step 1: Compute the pair of planes in pencil. Add these planes to H_s. Note that in case the two quadrics intersect in a double conic, the two planes coincide.

Step 2: If any of the planes added in the previous step intersect any of given quadrics in a hyperbola or a degenerate conic then for each plane in the pencil and each quadric, add separating planes to H_s as computed in plane/quadric intersection case.

As discussed below, it is easy to prove that the planes constructed in above steps indeed constitute a sufficient set of separating halfspaces when two quadrics intersect in a planar pencil.

Consider the line segment $\overline{ab}$ joining two points $a \in A$ and $b \in B$, where A and B are two disjoint components with identical classification as shown in Figure 5. The line segment $\overline{ab}$ intersects some face of each component an odd number of times even though the components may be unbounded since we are considering the problem in real affine space. Let $\overline{ab}$ intersect the boundary of A in a' and that of B in b'. Now consider the path $a'cdb'$ where path $a'c$ lies on face containing a' ,path cd lies on surface of any of the given quadrics and path db' lies on face containing b'. In Figure 5, this path $a'cdb'$ is shown dotted. It intersects some edge e_a of component A in point c and some edge e_b of component B in point d and hence the planes passing through these edges an odd number of times. If the two planes are different then either of them separates points a and b as in the example shown in Figure 5.

When the two edges e_a and e_b are segments of same conic, same plane passes through them and separates points a and b unless they are on the same side of the plane. If two disjoint components lie on the same side of the plane in a plane quadric intersection then the conic must be either hyperbola or degenerate into a pair of lines. In that case, the two points are separated by the planes constructed in the second step since these planes separate components on the same side of the plane and the quadric when the intersection is hyperbola or it degenerates into a pair of lines as proved in the previous section. Hence $I_2(a'b', p) = 1$ for some plane p in Π_s constructed above. Also, line segments $\overline{aa'}$ and $\overline{bb'}$ do not intersect p since by construction,none of the planes in Π_s intersect the boundary of the components to be separated. When a' or b' lie on an an edge, the proof is trivial since in this case the line segment $\overline{ab}$ intersects the plane through the edge and hence a and b are separated by this plane. Since the path from a to b formed by line segment $\overline{aa'}$, path $a'cdb'$ and line segment $\overline{b'b}$ is equivalent to the path $\overline{ab}$ in terms of *mod* 2 intersection number, $I_2(ab, p) = 1$ for some $p \in P_s$.

Example 1 *(Planar Pencil)*
Consider two cylinders $p : x^2 + y^2 = 1$ and $q : y^2 + z^2 = 1$ represented by the following matrics:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$R(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & \alpha - 1 \end{pmatrix}$$

For $\alpha = 1$, $R(\alpha)$ has rank 2. Hence, the pencil of above two cylinders is planar. The two planes in pencil passing through this planar QSIC are given by the degenerate quadric

$$R(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

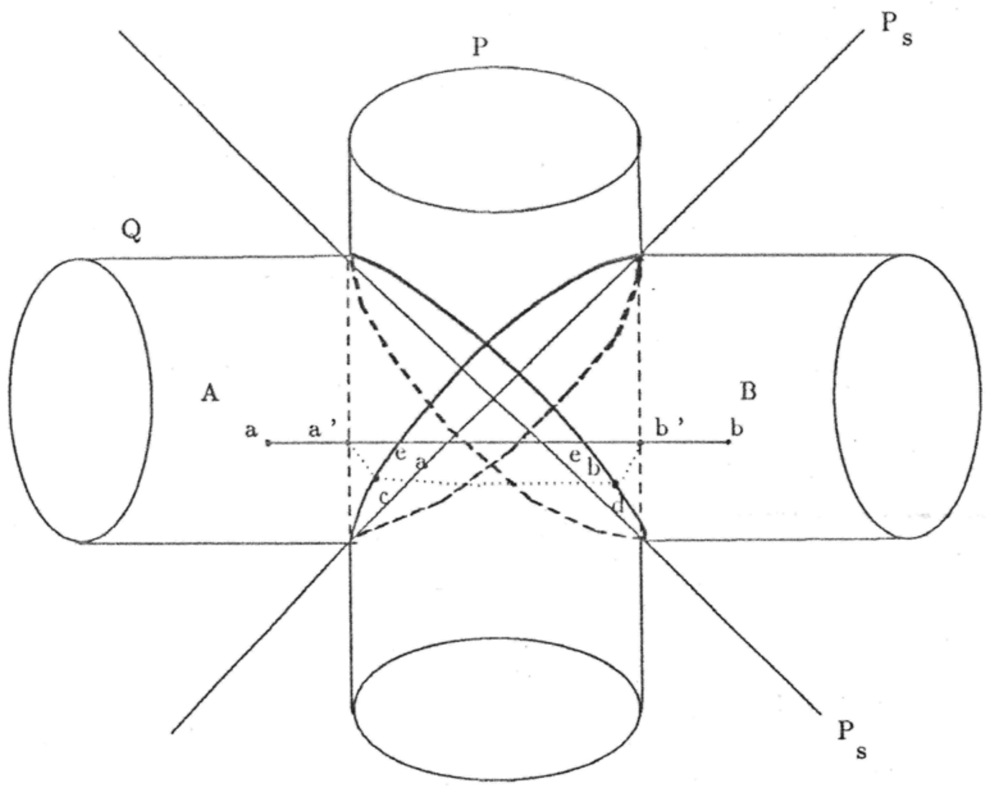

Figure 5: The two cylinders P and Q intersect in a pair of intersecting conics. The planes P_s in the pencil through the conics separate the components with identical classification.

i.e., $r : x^2 - z^2 = 0$
or equivalently
$x - z = 0$ and $x + z = 0$ as shown in Figure 5.

Note that the set of planes so constructed constitutes a sufficient set of separators although only one of the planes in the pencil is necessary.

5.2.2 Singular Pencil

Step 1: Compute α, such that $R(\alpha)$ is a parabolic/hyperbolic cylinder in the pencil of given quadrics. In the following steps we give a method for construction of separating halfspaces when $R(\alpha)$ is a parabolic cylinder. The construction of separating halfspaces when $R(\alpha)$ is hyperbolic cylinder is similar except that the conjugate principal plane of the hyperbolic cylinder is added to H_s.

Step 2: Apply coordinate transformation to F, G and $R(\alpha)$ such that ruled quadric $R(\alpha)$ is in canonical position.

Step 3: Let the ruled quadric be parabolic cylinder $y = c - z^2/2p$ in canonical position, given parametrically by $x = s$, $y = c - t^2/2p$, $z = t$. A fixed value of t specifies a generator parallel to x axis. Intersection of this generator with one of the given quadrics say F has two roots. We compute the values of t corresponding to repeated roots. They represent double points on QSIC as well as turning points (Figure 6). The planes through double/turning points parallel to principal plane of parabolic cylinder separate the multiple components in QSIC. It is easy to classify these repeated roots into turning points and double points. We first sort the double/turning points with respect to t coordinate. Let the sorted sequence of points be (s_1, t_1), (s_2, t_2), (s_3, t_3), (s_4, t_4) as there can be at the most four repeated roots. First we consider a generator on the parabolic cylinder corresponding to a t value smaller than t_1. If this generator does not have any real intersections with F then the point on QSIC corresponding t_1 is a turning point otherwise it is a double point. Similarly we consider a generator corresponding to t value larger than t_4 and classify the point on QSIC corresponding to t_4 as as turning or double point. If generator corresponding to $(t_2 + t_3)/2$ has real intersections with F, then t_2 and t_3 are classified as double points otherwise they are classified as turning points. For each double point we add to H_s the plane through the generator passing through double point and parallel to the principal plane of the parabolic cylinder. For two consequtive turning points say t_2 and t_3, we add to H_s a plane through generator corresponding to $(t_2 + t_3)/2$ and parallel to the principal plane of the parabolic cylinder. If a QSIC degenerates into a cubic and a generator, then in addition to separating halfspaces constructed in previous steps, add a plane through this generator and parallel to the principal plane of the parabolic cylinder in the pencil.

Step 4: Add to H_s the diametrical plane through the midpoints of the two points on QSIC corresponding to each generator. If F is given by

$$ax^2 + by^2 + cz^2 + 2hxy + 2fxz + 2gyz + 2lx + 2my + 2nz + d = 0,$$

then the mid point curve is given by

$$\begin{aligned} x &= -(h(c - t^2/2p) + \ ft + \ l)/a \\ y &= c - t^2/2p \\ z &= t. \end{aligned}$$

The equation of diametrical plane through the midpoint curve is given by $ax + hy + fz + l = 0$.

Step 5: Complete the set $H_s$of separating halfspaces by considering intersection of planes added in previous steps with the given quadrics and adding the separating planes as constructed in Plane/Quadric intersection case.

The planes added in steps 3 and 4 separate the connected components in the QSIC of two given quadrics. In step 5 we consider the intersection of these planes with the given quadrics. If a plane intersects with the quadric in a nondegenerate conic other than hyperbola then the conic completely lies on the connected component of the the quadric surface which does not contribute to the the boundary of two components to be separated. If a plane intersects the quadric in a hyperbola then one branch of the hyperbola lies on the connected component of quadric which does not contribute to the boundary of components to be separated whereas the other branch intersects the boundary of one of the components and hence it does not separate the components with identical classification completely. Therefore, we need to add separating plane as constructed in the case when Plane/Quadric intersection is hyperbola. As discussed below, the separating planes added in step 5 separate these components not separated by planes constructed in steps 3 and 4.

Let a and b be two points in components A and B respectively with identical classification resulting from intersection of two given quadrics. Let B be the component whose boundary is intersected by a branch of hyperbola resulting from intersection of a separating plane constructed in Step 4 with one of the quadrics, thus dividing it into two subcomponents say B_1 and B_2. Let B_1 be the one bounded by a component of QSIC on one side and a branch of the hyperbola on the other side and let B_2 be the other component. If $b \in B_1$, then a path ab similar to the one constructed in proof of separation for planar pencil can be constructed such that it intersects one of the planes separating the connected components in QSIC an odd number of times. Otherwise a path ab can be constructed such that it intersects the plane constructed in step 5 which separates the two branches of hyperbola, an odd number of times. Therefore in either case, $I_2(ab, p) = 1$ for some $p \in H_s$.

Similarly, when a plane constructed in steps 3 and 4 intersects any of the quadrics in a degenerate conic, the planes constructed in step 5 separate the components not separated by planes constructed in steps 3 and 4.

The construction process terminates after step 5 because the separating planes constructed in Plane/Quadric intersection cases as described in section 5.1 do not further result in components which need to be separated.

Example 2 *(Singular Pencil)*
Consider a sphere $p : x^2 + y^2 + z^2 = 16$ and and a cylinder $q : x^2 + (z - 2)^2 = 4$ represented by the following matrics:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -16 \end{pmatrix}$$

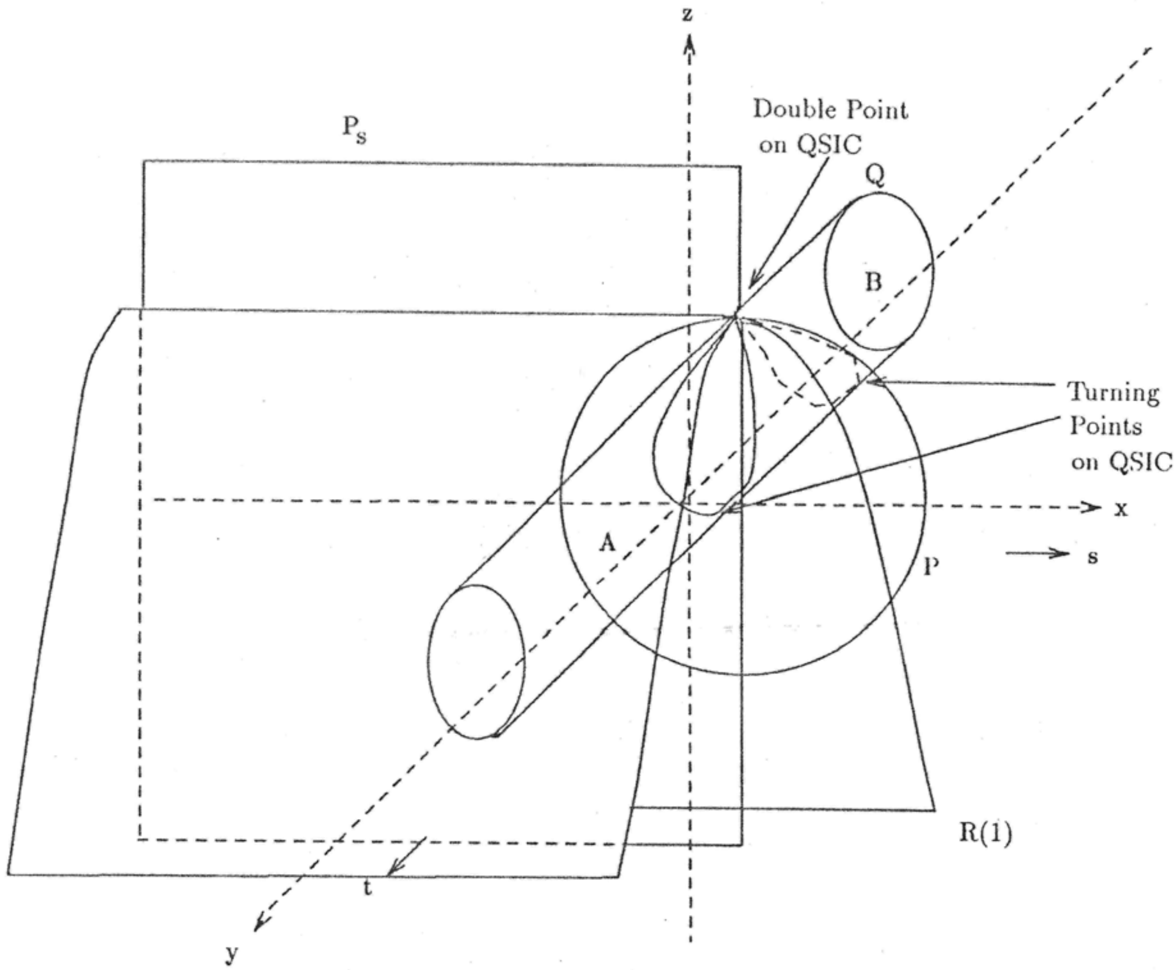

Figure 6: Quadrics P and Q intersect in a singular QSIC resulting in components A and B with identical classification with respect to both the quadrics. The plane P_s parallel to the principal plane of the ruled quadric in the pencil $P - \alpha Q$ and passing through the generator on ruled quadric corresponding to double point on QSIC, separates the components A and B.

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$$R(\alpha) = \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\alpha & 2\alpha \\ 0 & 0 & 2\alpha & -16 \end{pmatrix}$$

For $\alpha = 1$, $R(\alpha)$ has rank 3. For any other value of α, $R(\alpha)$ has rank 4. Hence the pencil of these quadrics is nonplanar singular. The ruled singular quadric in the pencil, a parabolic cylinder is given by

$$R(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -16 \end{pmatrix}$$

or equivalently $r(1) : y^2 = -4(z-4)$ as shown in Figure 6.

The parametric equation of the parabolic cylinder is given by

$$x = s;\ y = t;\ z = 4 - t^2/4$$

Now QSIC can be represented parameterically by two values of s corresponding to each value of t satisfying the following equation

$$s^2 + t^2 + (4 - t^2/4)^2 = 16$$

The double/turning points of QSIC are given for values of t satisfying

$$t^2 - t^4/16 = 0$$

i.e., $t = -4,\ 0,\ 4$.

$t = 0$ corresponds to double point whereas $t = -4$ and $t = 4$ correspond to turning points. Hence the plane through generator on parabolic cylinder corresponding to $t = 0$ and parallel to principal plane of the cylinder is the desired separating plane which in this case is given by $y = 0$.

Since the intersection of given quadrics p and q with the separating plane $y = 0$ is a circle in each case, the construction process terminates with $H_s = \{y = 0\}$.

5.2.3 NonSingular Pencil

Step 1: Compute α, such that $R(\alpha)$ is hyperbolic paraboloid in the pencil of given quadrics.

Step 2: Apply coordinate transformation to F, G and $R(\alpha)$ such that ruled quadric $R(\alpha)$ is in canonical position.

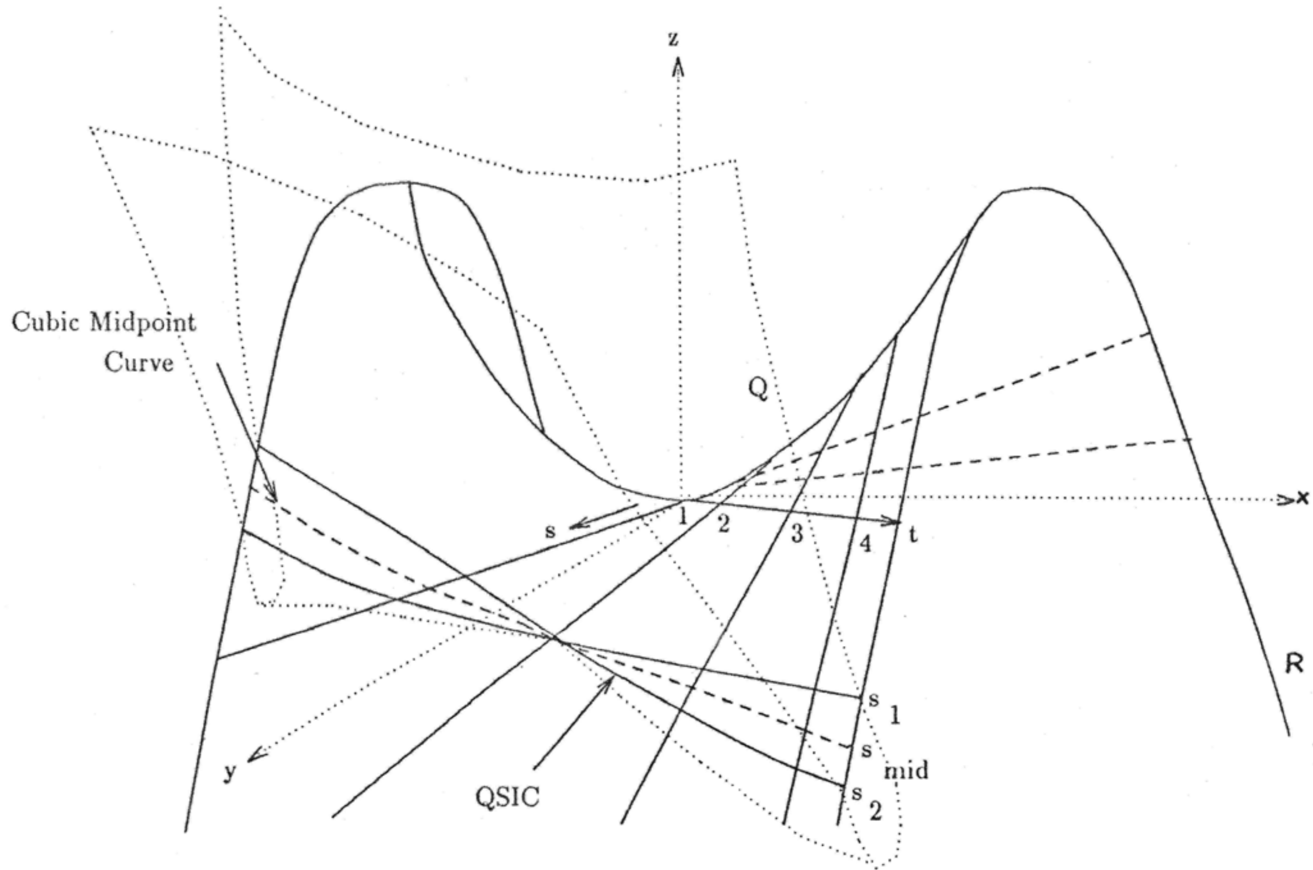

Figure 7: The separating halfspaces needed to separate components resulting from QSIC of quadrics Q and R shown in the Figure are constructed using the generators on the Hyperbolic Paraboloid R.

Step 3: Let the ruled quadric, hyperbolic paraboloid in canonical position be given by $x^2/a^2 - y^2/b^2 = 2cz$.

Hyperbolic paraboloid is a doubly ruled surface. Each family of generators completely defines the surface [Hilbert]. The generators belonging to same family are parallel to a given plane but are mutually skew and each generator of one family intersects every generator of the other family. The surface can be parametrically represented as

$$\begin{aligned} x &= at + as \\ y &= bt - bs \\ z &= 2ts/c \end{aligned}$$

where t varies along a generator through origin given by $s = 0$. Each value of t corresponds to a generator of the other family intersecting this generator as shown in Figure 7.

Different values of t correspond to generators which are mutually skew but are parallel to the plane given by $bx + ay = 0$. Substituting above expressions for x, y and z in implicit equation of one of the quadrics say F, gives two roots for s corresponding to each value of t. Just as in the case of Singular pencil, we compute the values of t corresponding to repeated roots. They represent double points on QSIC as well as turning points. The planes through double/turning points parallel to to the plane $bx + ay = 0$ separate the multiple components in QSIC. The classification of double/turning points and subsequent construction of separating planes is similar to the case of Singular pencil.

Step 4: Consider intersection of planes added in previous steps with the given quadrics and add the separating planes as constructed in Plane/Quadric intersection case to the set H_s.

Step 5: The intersection of a family of parallel planes given by $bx + ay = 2abt$ through generators parallel to s coordinate on surface of hyperbolic paraboloid (Figure 7), with the given quadrics results in similar and similarly placed conics. If conic sections resulting from intersection of above family of planes with either of the quadric are ellipses, then this step should be skipped. Otherwise we need to add the cubic separating halfspaces constructed below to separate the two components resulting from QSIC shown in Figure 7.

The midpoint curve is given by

$$\begin{aligned} x &= at + as_{mid}(t) \\ y &= bt - bs_{mid}(t) \\ z &= (2ts_{mid}(t))/c \end{aligned}$$

where $s_{mid}(t) = (s_1(t) + s_2(t))/2 = N(t)/D(t)$,

s_1 and s_2 being two points of QSIC on a generator specified by a value of t and

N and D being polynomials in t of degree atmost 2.

Since the midpoint curve in this case is a nonplanar cubic curve, we cannot separate these components by a linear halfspace through the curve. To separate these components, we add the following halfspaces

Step 5.1: The diametrical planes through principal axes of similar and similarly placed conics resulting from intersection of family of parallel planes $bx + ay = 2abt$ with the given quadrics F and G. The expressions for these diametrical planes can be computed from the principal axes of the conic sections resulting from intersection with any two planes in the above family of planes.

Step 5.2: The ruled cubic through midpoint curve with generators parallel to z axis given by

$$\begin{aligned} x &= at + as_{mid}(t) \\ y &= bt - bs_{mid}(t) \\ z & \text{ free.} \end{aligned}$$

The implicit equation of the cubic can be obtained by substituting

$$t = (bx + ay)/2ab$$

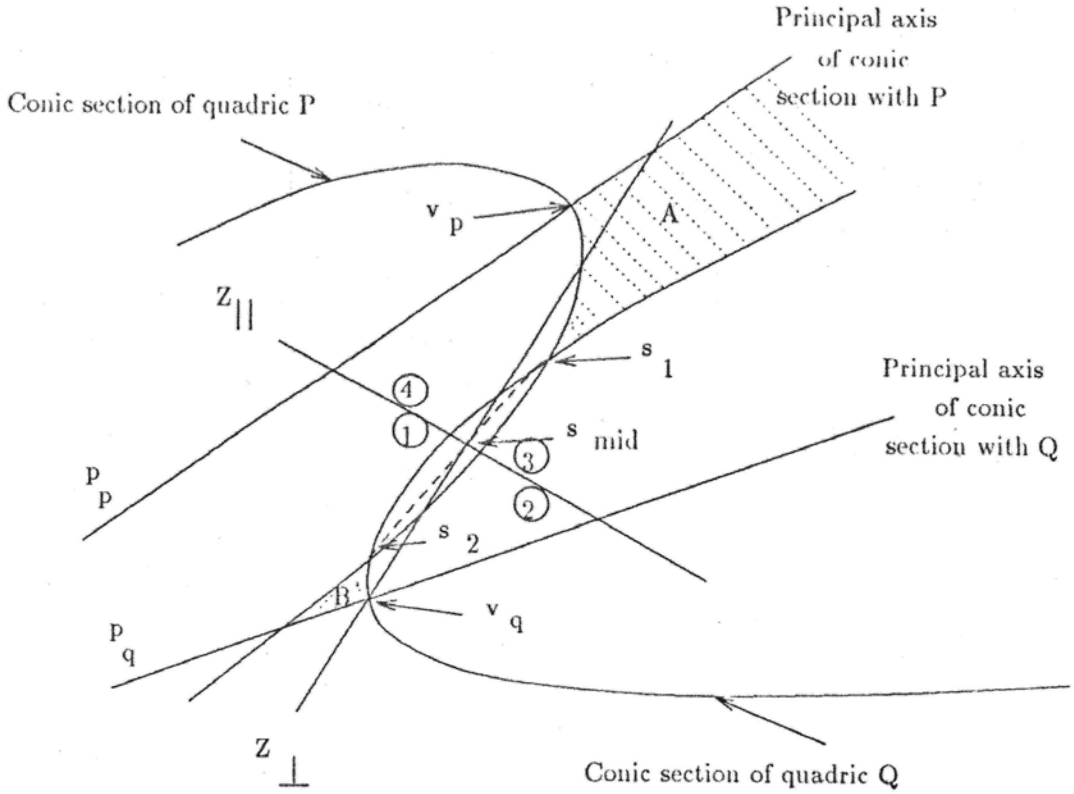

Figure 8a: The principal axes of the conic sections of quadrics P and Q with the plane $bx + ay = 2abt_0$ and the generator of cubic surface perpendicular to z axis shown as $Z_\perp$ separate the points lying in the plane with identical classification with respect to both the quadrics.

in the parametric expression for x.

In Figure ~~7~~, 8a − 8c we denote the generators of this cubic surface by $Z_{||}$.

Step 5.3: The ruled cubic through midpoint curve with generators orthogonal to z axis given by

$$\begin{aligned} x &= at + as_{mid}(t) + ar \\ y &= bt - bs_{mid}(t) - br \\ z &= (2ts_{mid}(t))/c \end{aligned}$$

The implicit equation of this cubic can a be obtained by substituting $t = (bx + ay)/2ab$ in the parametric expression for z. In Figure ~~7~~, we denote the generators of this cubic surface by $Z_\perp$. 8a-8c

Figure 8a shows intersection of a plane $bx + ay = 2abt_0$ for a fixed $t = t_0$ with the given quadrics P and Q, and the two cubics added in Step 5.2 and Step 5.3. The diametrical planes added in Step 5.1 intersect the given plane $bx + ay = 2abt_0$ in the principal axes p_p and p_q of the conic sections of the two quadrics P and Q.

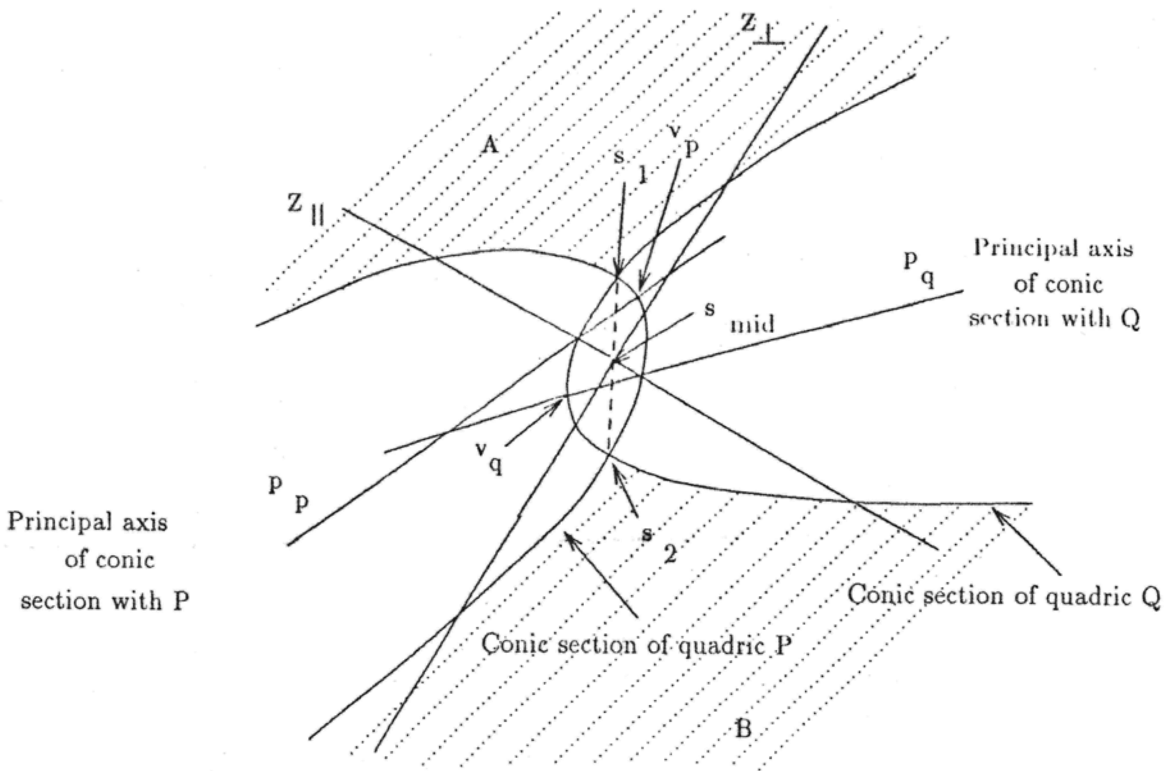

Figure 8b: The principal axes of the conic sections of quadrics P and Q with the plane $bx + ay = 2abt_1$ separate the points lying in the plane with identical classification with respect to both the quadrics when the vertex of one conic section lies inside the other conic section.

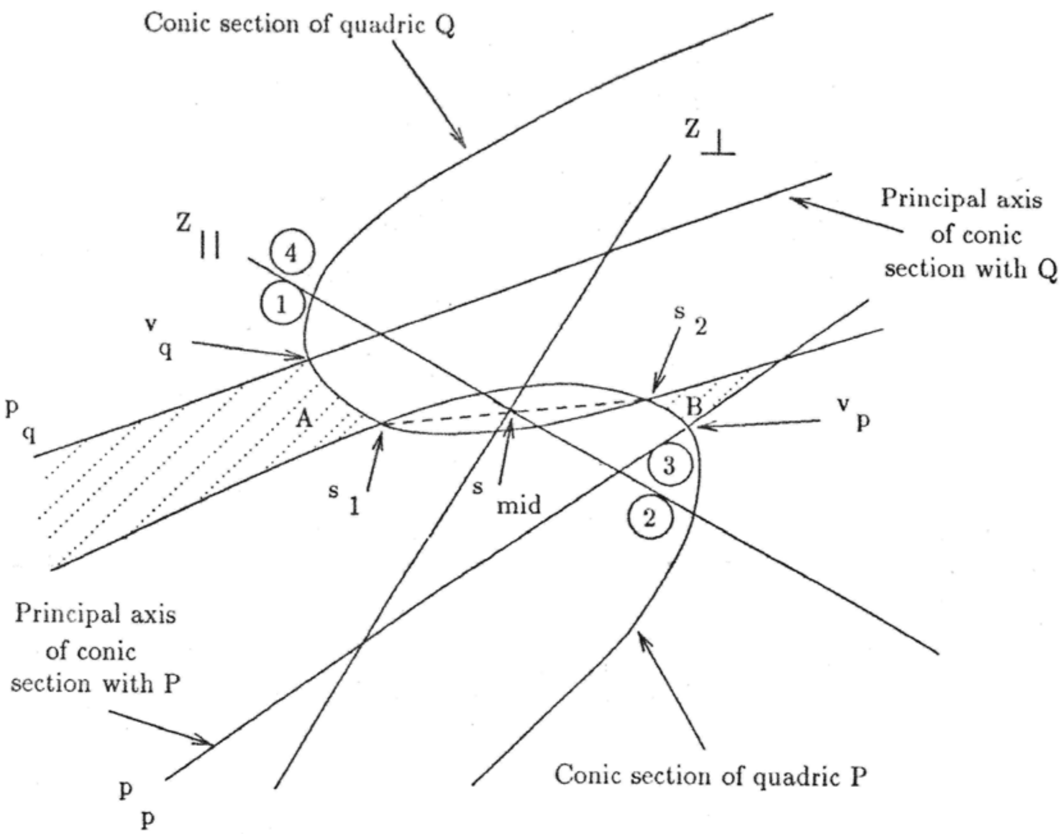

Figure 8c: The principal axes of the conic sections of quadrics P and Q with the plane $bx + ay = 2abt_2$. The generator of cubic surface perpendicular to z axis shown as $Z_{\perp}$ and the generator of cubic surface parallel to z axis shown as $Z_{||}$ separate the points lying in the plane with identical classification with respect to both the quadrics.

The conic sections of the quadrics in Figures 8a–8c are shown as parabolas. As mentioned before, if the conic sections of either of the quadric with the family of planes given by $bx + ay = 2abt$ are elliptic then there is no need for separating halfspaces for the space components resulting from QSIC. If the conic section of either of the quadric is hyperbola, then the conjugate principal axis of the hyperbola separates the two branches and the discussion of this case is similar to the one when the conic sections with both the quadrics are parabolas. The line segment $\overline{s_1 s_2}$ shown in Figure 8a is the the generator of the hyperbolic paraboloid lying in the plane $bx + ay = 2abt_0$ and $s_{mid} = (s_1 + s_2)/2$ is the intersection of the cubic midpoint curve with this plane. By convexity of conics, normal at any point of the conic either intersects its principal axis inside the conic or coincides with it. Note that we define the side containing the focus of the conic as inside the conic. The generators of two cubics shown as $Z_\perp$ and $Z_\|$ in the Figure 8a intersect orthogonally in the point $s_{mid}(t_0)$. Since not both the generators can be parallel to the principal axis of the conic, one of them necessarily intersects the principal axis inside the conic. Hence the points of two components with identical classification with respect to both the quadrics, lying in the plane $bx + ay = 2abt_0$, are separated either by the generators of the cubics or by the principal axis.

We observe that the two cubics are mutually orthogonal along the midpoint curve and hence divide the space into four quadrants. To prove the separation of components in space, we consider various possible configurations of conic sections and cubic sections by the family of planes $bx + ay = 2abt$ which span the space as the quadrant formed by the generators of cubics moves in space along the midpoint curve. In Figure 8a, the two sets of points A and B having identical classification with respect to both conics and not separated by the principal axes p_p and p_q of the conics, are in different quadrants formed by the generators of two cubics. Points of component A are in quadrants marked 3 and 4 whereas the points of component B are in quadrants marked 1 and 2. Hence the two components are separated by the generators of two cubics. The other possible configurations of the conic sections and the generators of the cubics, as the family of planes mentioned above spans the space, are shown in Figures 8b and 8c. Since the principal axes of conic sections of given quadrics by the family of parallel planes $bx + ay = 2abt$ are parallel, only other possible configurations are when the vertex of either of the conics lies inside other conic and when both the points on QSIC are on the opposite side of the principal axis as compared to configuration shown in Figure 8a. Note that the case when both the points on QSIC corresponding to s_1 and s_2 coincide are limiting cases of the above configurations. Also as mentioned before, the side containing the focus of the conic is being considered inside and the other outside the conic.

If vertex of either of the conics lies inside the the other conic, then the principal axes of the conics are sufficient to separate the points with identical classification. In Figure 8b, vertex of each of the conics lies inside the other conic and the principal axes of the conics are sufficient to separate points in component A from those in component B.

In Figure 8c, the points on QSIC corresponding to s_1 and s_2 are on different side of principal axis p_p as compared to Figure 8a. Therefore, even though the space component A in Figure 8c as well as space component B in Figure 8a have points in the same quadrant 1 of the four quadrant space partition by the cubics, the points in A in Figure 8c are separated from points in B in Figure 8a by the diametrical plane through the principal axes of conic sections of quadric P. Therefore, as the family of parallel planes $bx + ay = 2abt$ spans the space, as long as the relative position of points on QSIC with respect to the principal axes of the conic sections of given quadrics by these planes remains unchanged, the cubics separate the points in components with identical classification, otherwise the diametrical planes added in step 5.1 separate the points in components with identical classification.

Hence the points of two components with identical classification are either separated by two cubics or the diametrical planes through the the principal axes of the conic sections of the the quadrics by a family of parallel planes

$$bx + ay = 2abt$$

through the generators of hyperbolic paraboloid.

6 Conclusions

In this paper, we have given a method to construct separating halfspaces for plane/quadric and quadric/quadric intersection. The advantage of our method is that it is a geometric approach based on classification of QSIC of given quadrics and is independent of the position and type of the quadrics involved. We also eliminate square roots in construction of separating halfspaces by taking summation of the two roots of the quadratic equation. The ruled cubics constructed in the method can easily be divided into convex and concave parts by a plane parallel to z axis and passing through the generator of hyperbolic paraboloid through the inflection point of cubic midpoint curve.

As illustrated in Figure 9, separating the components with identical classification resulting from a pair of halfspaces for every pair of halfspaces induced from the faces of the solid need not be sufficient for describing the solid semialgebraically. Figure 9 shows a solid composed of a single component A which has identical classification as B with respect to sphere S, the parabolic cylinders C_1, C_2, C_3 and C_4 and the plane P. Every pair of halfspaces intersect in a single component QSIC, hence no separating halfspaces are added when plane/quadric and quadric/quadric intersection is considered. But the solid A cannot be described semialgebraically using the halfspaces induced from the faces of the component A because component B has same description as A. In our future work, we give a method to separate components resulting from intersection of more than two quadrics. We compute the boundary of the components in exhaustive space decomposition by constructing BSP tree using halfspaces induced from the faces of solid. We classify the components at each leaf node as inside/outside solid by classifying a point inside a component with respect to solid. Then we construct linear halfspaces to separate components inside solid at a leaf node from those outside solid at that leaf node. This set S_1 constitutes a sufficient set of separating planes for describing the solid, but the size of the set may be huge as the boundaries of two components may be arbitrarily close to each other. We also construct the set S_2 of separating halfspaces for plane/quadric and quadric/quadric intersection for every pair of halfspaces induced by the faces of given solid as described in this paper. Set S_2 may not be a sufficient set of separating halfspaces to describe the given solid as discussed above, but the set $S = S_1 \cup S_2$ is a sufficient set of separating halfspaces. Now we use the method described in [ShapiroVossler2], to compute a necessary set of separating halfspaces by eliminating redundant halfspaces from S. We expect that the linear and cubic separating halfspaces added by the method described in this paper will make a large number of planes in S_1 redundant. The remaining steps of construction and minimization of canonical CSG expression for the solid obtained from the set of necessary

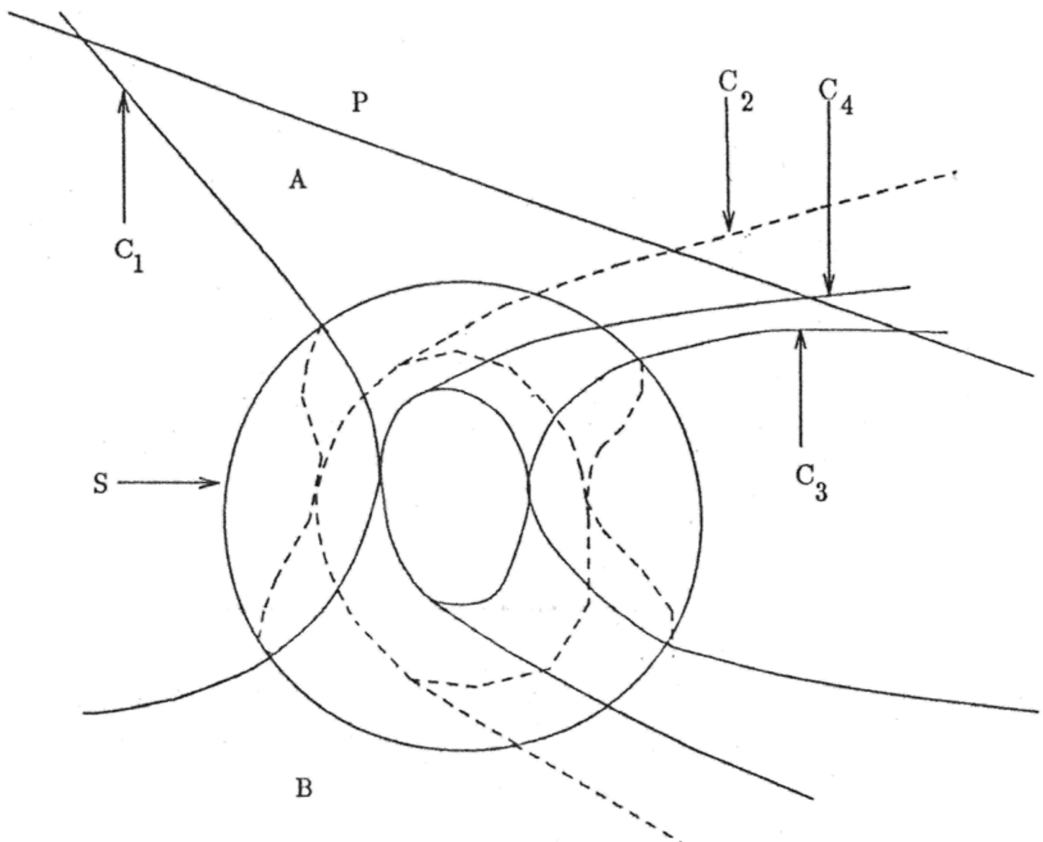

Figure 9: The components A and B have identical classification with respect to sphere S, parabolic cylinders C_1, C_2, C_3 and C_4 and the plane P, even though no separating halfspaces are needed when space components resulting from plane/quadric and quadric/quadric intersection for every pair of halfspaces are considered.

halfspaces and the halfspaces induced from the faces of solid are identical to those described in [ShapiroVossler2, ShapiroVossler3]. Using the separating halfspaces constructed by the method outlined in this paper, the domain of B-Rep→CSG conversion system of Shapiro and Vossler can be extended to solids bounded by quadric faces with nonplanar edges, which currently is restricted to solids with planar edges.

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