University of Nebraska - Lincoln
DigitalCommons@University of Nebraska - Lincoln

Dissertations, Theses, and Student Research Papers in Mathematics

Mathematics, Department of

8-2016

Management of Invasive Species using Optimal Control Theory

Christina J. Edholm
University of Nebraska-Lincoln, cedholm2@math.unl.edu

MANAGEMENT OF INVASIVE SPECIES USING OPTIMAL CONTROL THEORY

by

Christina J. Edholm

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Richard Rebarber and Brigitte Tenhumberg

Lincoln, Nebraska

August, 2016

MANAGEMENT OF INVASIVE SPECIES USING OPTIMAL CONTROL THEORY

Christina J. Edholm, Ph.D.

University of Nebraska, 2016

Advisers: Richard Rebarber and Brigitte Tenhumberg

In my dissertation I will discuss the use of optimal control theory to determine management strategies for an invasive species. I focus on a Diaprepes Root Weevil, which is an invasive species having a substantial negative impact on citrus tree growth in regions such as Florida and California. At the larva stage of the life cycle Diaprepes Root Weevils cause destruction of citrus trees at the root level resulting in loss of citrus crops. This detrimental effect for farmers motivates research into how to minimize the economic loss due to the Diaprepes Root Weevil. For my work, I use optimal control theory to determine levels of pesticide or biological control to apply to the Diaprepes Root Weevil to reduce the economic loss.

ACKNOWLEDGMENTS

I want to thank my advisors Dr. Richard Rebarber and Dr. Brigitte Tenhumberg for all their help, guidance, input, and assistance. I have learned a tremendous amount from both of them, and am deeply grateful for all the time and effort they've invested in my education and future success. Thank you Dr. Richard Rebarber for teaching me about applied mathematics, working with me through all the mathematical problems I faced during my research, and connecting my mathematics to biology. Thank you Dr. Brigitte Tenhumberg for helping me expand my education by teaching me about biological systems and helping me understand the biological links to my mathematical models. Furthermore, thank you to my other collaborators, Dr. Yu Jin, Dr. Stuart Townley, Dr. Chris Guiver, Stephanie Lloyd, and Dr. Jim Powell, for new ideas and possible paths for our models. Thank you to Dr. Daniel Toundykov and Dr. Yu Jin for both teaching me in my graduate courses, and for being readers on my committee that gave me valuable feedback on my thesis. Also, thank you Dr. Kevin Pope for being on the committee, and giving insight into future avenues for me to research. Additionally, I would like to thank Dr. Benjamin Levy, Stephanie Lloyd, and Dr. David Logan for teaching me MATLAB techniques.

I would like to thank the University of Nebraska-Lincoln mathematics departments for the opportunity to pursue my education. In addition, thank you to my fellow UNL graduate students for their help through various courses, hard work, and late nights. Thank you to my professors who have helped me on my educational journey, both teaching me new concepts and how to teach others. Further, I want to thank the UNL staff who have helped me through forms, teaching, and all issues that have arisen.

Lastly, but not least, I would like to thank some of the people in my life who have offered me support during this journey.Thank you to the Mathletes, my mathematical sport support system, it has been amazing playing with so many different mathematicians. Also, thank you to No Coast Roller Derby, which has brought me a balance in my life and a team of sisters who have encouraged me to be strong and succeed. Thank you Dr. Julia St. Goar for working with me on coursework and also being part of my foundation for our graduate school career and life. To Eli Holmes, thank you for being there to support me in both times of success and when I need a boost. In conclusion thank you to all the friends I have made at UNL; each of you have helped me in your own way and I will always remember our time together.

DEDICATION

To Suzanne Joy Edholm, my mom, thank you for your strength and support.

Table of Contents

Part I

Invasive Species Single Patch Optimal Control

Chapter 1

Background

1.1 Mathematical Background

A useful source for the history of control theory is a paper entitled Control Theory: History, Mathematical Achievements and Perspectives [FCZI03]. The article covers highlights from the development of Control Theory, additionally exploring specific topics and examples. Furthermore, the article considers feedback, optimization, controllability, and optimal control. There is also a look at specific examples utilizing control theory, and possible avenues for future study. As mentioned in the article one of the key development of Optimal Control Theory can be traced to Pontryagin. Specifically, there was a book published in 1962, *The Mathematical Theory of Optimal Process*, by L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelize, and E.F. Mishchenko[Pon87]. An important development was the Pontryagin Maximum principle which established necessary conditions to an optimal control problem and relates this to the Hamiltonian, a useful tool for solving optimal control problems. For a more in-depth look we refer the reader to the original book or the book *Optimal Control Applied to Biological Models* by Lenhart, S. and Workman, J.T. [LW07], a very useful source. The book by Lenhart and Workman covers an introduction to Optimal Control Theory, focusing on a full treatment of continuous time systems, and includes discrete time as well. Additionally, it includes many examples of optimal control applied to biological systems, with both the mathematics and code included.

For specifically Discrete-Time Optimal Control Theory, a good resource is *Optimal Control in Discrete Pest Control Models* by Kathryn Dabbs[Dab10]. The paper gives an overview of how to solve discrete-time optimal control problems and looks at specific models. Another paper on discrete-time optimal control with existence, necessary condition, and uniquness proofs is *Optimal control of gypsy moth populations* by Whittle, Lenhart, and White [WLW08]. In this thesis, we focus on a model which does not fit into this framework, allowing for variations to the mathematical set-up and a full treatment of existence, necessary condition, and uniqueness proofs for the optimal control.

Additionally, there have been many papers linking Optimal Control Theory to biology, a few that we have found useful in our studies: [MS12], [Fil62], [Gra10], [Dab10], [WLW08], [MLW15], [JLPB05], [Ris77], and [Leu93]. Some of these papers also address invasive species as their biological inspiration for implementing Optimal Control. For instance, the Gypsy Moth is a specific invasive species studied in both [WLW08] and [MLW15], which utilize a different model but use Optimal Control Theory to study management, and in [MLW15] include an integrodifference model. In the next section I will explain more about invasive species.

1.2 Biological Background

Our research involves applying control theory techniques to natural resources management, in particular management of invasive species.

Since the beginning of agriculture, people have always had to deal with pests affecting their crops, and developing methods to control the effects. Originally people had to eliminate pests by hand, through picking or mechanical methods, until 2500-1500 B.C. when the Sumerians and Chinese introduced pesticide. Today there is still a great loss of crops to pests. Specifically we consider crops which we use in our daily lives. For instance, there is a loss of approximately 50% of wheat to pests, while cotton loss can exceed 80% [Oer06]. There are various methods applied to combat pests including implementing predators, weeding techniques, biological control agents, and pesticides[Oer06].

Across the world annually there is approximately $40 billion spent on pesticides, while the United States made up a quarter of that cost [PU03, PG97]. Despite attempts to apply pesticide, in the United States there was still a loss 37% of crops, to the ecological pests. Specifically there was 13% lost to insects[Pim05]. Furthermore, even though we have increased pesticide application in the past 50 years by more than a factor or ten, there is still approximately twice as much damage now from insects than then [PMZ+91]. [Pim05]

Another important factor to consider is the human element which affects invasive species. When humans disrupt a territory, the result is a possible response growth in invasive species, with the destruction of the terrain linked to the original species increased chance of eradication [Hob00, Fah02, DL09]. So, humans not only inadvertently encourage the growth of these dangerous invasive species, we also cause the extermination of the preexisting healthy organisms. The resulting inhospitable area becomes an impediment for both the invasive and native species[Fah02, DL09], shaping the landscape. There has already been research looking into humans affects on the landscape linked to increase in invasions [Hob00, Wit02]. However with the evolution of human society changes are constantly occurring that could influence dangerous invasive species. Additionally, with the increase in the human population of around 5000 million people in the last 65 years, there will be more cases of invasive species and more control required to produce enough crops for the population [CAP16].

1.3 Overview

Our plan is to explore management of invasive species using optimal control theory. In part one we will consider a single patch model with no dispersal.

In Chapter 2 we will introduce a basic model which takes into consideration an invasive pest lifecycle and applying a control, for instance a pesticide, a non-persistent short-lived biocontrol agent known as control agent. Furthermore we will prove existence, necessary conditions, and uniqueness for the optimal control. In Chapter 3, we consider what happens when the control persists longer than one time step. Again we will prove existence, necessary conditions, and uniqueness for the optimal control. In Chapter 4, we explore a case study investigating a specific invasive species, Diaprepes abbreviatus, DRW.

1.4 Reference Chart

	Notation	Description
Pest Vector	P_e	Number of eggs
	P_l	Number of larva
	P_p	Number of pupa
	P_a	Number of adult
Pest Matrix	γ_1	Egg survival
	γ_2	Transition rate egg to larva
	θ_1	Fecundity rate of female adults
	θ_2	Adult survival
	ζ_1	Larva survival
	ζ_2	Transition rate larva to pupa
	ν_1	Pupa survival
	ν_2	Transition rate pupa to adult
Initial Pest Vector	ϕ_e	Initial Proportion eggs
	ϕ_l	Initial Proportion larva
	ϕ_p	Initial Proportionpupa
	ϕ_a	Initial Proportionadults
Control	N	Number of control agents
	α	search efficiency/encounter rate of control
Cost Function	β_1	loss of harvest per square meter per time steo
	β_2	cost of control per square meter per time step
Nematodes Persist	μ	mortality/degradation of control agent

Chapter 2

Basic Model

2.1 Parameters

We denote pests by the vector P and control by the vector N. We consider a system where it is possible to apply control every time step, hence we establish a discrete-time model with constant time steps. The pest life cycles and dynamics, we used a 4×4 matrix, A, taking into account the pest eggs (P_e), larva (P_l), pupa (P_p), and adults (P_a). Note this can be generalized and applied to pests with a larger or smaller number of stages; additionally the matrix can characterize different pest stages. Let:

$$A = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 \\ \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & \nu_2 & \theta_2 \end{bmatrix}.$$

The control is applied only to the larva stage P_l, or the second stage of the pest stages. The control search/application efficiency is denoted by α, and accounts for how likely a control agent is to encounter a pest larva. Below is the formulation of the pest dynamics with the control included in the larva stage, where t is a time step.

$$\mathbf{P}(t+1) = A_l\mathbf{P}(t) \tag{2.1}$$

$$\begin{bmatrix} P_e(t+1) \\ P_l(t+1) \\ P_p(t+1) \\ P_a(t+1) \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 \cdot e^{-\alpha N(t)} & 0 & \theta_1 \\ \gamma_2 & \zeta_1 \cdot e^{-\alpha N(t)} & 0 & 0 \\ 0 & \zeta_2 \cdot e^{-\alpha N(t)} & \nu_1 & 0 \\ 0 & 0 \cdot e^{-\alpha N(t)} & \nu_2 & \theta_2 \end{bmatrix} \begin{bmatrix} P_e(t) \\ P_l(t) \\ P_p(t) \\ P_a(t) \end{bmatrix}$$

We denote initial values by

$$\begin{bmatrix} P_e(0) \\ P_l(0) \\ P_p(0) \\ P_a(0) \end{bmatrix} = \begin{bmatrix} \phi_e \\ \phi_l \\ \phi_p \\ \phi_a \end{bmatrix}.$$

2.2 Cost Function

We constructed the cost function by breaking it down into the control and pest components. Specifically, if we look at the cost incurred to an environment by an invasive pest there will be the loss of profit from the pest existing in the environment and the cost to purchase control to apply to the environment to deal with the pest.

Since destruction of the environment is catastrophic we expect a nonlinear term for the cost of pest damage. Specifically, when there is a low density of the pests, we expect the affected specie will not suffer large losses, but at a high density of pests the mortality rate becomes exponentially large. Furthermore, since we don't have a functional term we use the square which ensures mathematical uniqueness. For mathematical convenience we choose to model the cost of pest damage as $\beta_1 P_l(t)^2$.

The exponential increase of damage ensures that control will be applied at some point, which is a desirable feature in the cost function because it prevents plant death as a result of too high pest density.

In addition to the cost related to pest damage, we need to consider the cost of purchasing the control agent which is $\beta_2 N(t)$. So β_2 is the price of a single control unit. So the total cost is cost due to pest damage, $\beta_1 P_l(t)^2$, plus the cost of using control, $\beta_2 N(t)$,

$$Cost = \beta_1 P_l(t)^2 + \beta_2 N(t)$$

where β_1 and β_2 will be determined by the specific invasive species.

2.3 Optimal Control Problem

Realistically, there is going to be a maximum amount of control we can purchase and apply. We denote N_{max} as the maximum amount of control at any time step we can apply to the environment.

The set-up over our Optimal Control Problem is to minimize the objective functional for T time steps

$$J(N) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N(t)$$

subject to

$$P_e(t+1) = \gamma_1 P_e(t) + \theta_1 P_a(t) \qquad P_e(0) = \phi_e$$

$$P_l(t+1) = \gamma_2 P_e(t) + \zeta_1 e^{-\alpha N(t)} P_l(t) \quad P_l(0) = \phi_l$$

$$P_p(t+1) = \zeta_2 e^{-\alpha N(t)} P_l(t) + \nu_1 P_p(t) \quad P_p(0) = \phi_p \tag{2.2}$$

$$P_a(t+1) = \nu_2 P_p(t) + \theta_2 P_a(t) \qquad P_a(0) = \phi_a$$

where $N(t) \geq 0$ for all t and $N \in \mathbf{N} = \{N : \{1, ..., T\} \rightarrow \{x \in \mathbb{R} | 0 \leq x(t) \leq N_{max}, t = 1, 2, ..., T\}\}$.

We will prove the existence and uniqueness of the optimal control, which we denote by $\mathcal{N}$. We will also prove necessary conditions for the optimal control $\mathcal{N}$. The proofs roughly follow the proofs in *Optimal Control of Gypsy Moth Populations* by Whittle, Lenhart, and White [WLW08]. The existence proof roughly follows from *Optimal Control in Discrete Pest Control Models* by Kathryn Dabbs[Dab10]. A useful source for proofs in Optimal Control theory is *Optimal Control Applied to Biological Models* by Lenhart, S. and Workman, J.T. [LW07].

Note in the following proofs each $\mathcal{P}_e, \mathcal{P}_l, \mathcal{P}_p, \mathcal{P}_a$ is a function of $\mathcal{N}$. Similarly each $\mathcal{P}_e^\varepsilon, \mathcal{P}_l^\varepsilon, \mathcal{P}_p^\varepsilon, \mathcal{P}_a^\varepsilon$ is a function of $\mathcal{N} + \eta\varepsilon$.

2.3.1 Existence

Theorem 2.3.1. *There exists $\mathcal{N} \in \boldsymbol{N}$ which minimizes $J(N)$.*

Proof. Each P_e, P_l, P_p, P_a is continuous as a function of N at every time step by Equation 2.2. Define $B^+ = \{(N(1), ...N(T))|N \in \mathbf{N}\}$. We note that there is a natural isomorphism between $\mathbf{N}$ and B^+. Considering $J : \mathbf{N} \to B^+ \to \mathbb{R}$, we see that J is continuous as a function of N. We have that B^+ is a compact subset of $\mathbb{R}^T$ in the standard Euclidean topology. Thus, $\inf_{N \in \mathbf{N}} J(N)$ exists. Hence, we have a sequence $N_k \in \mathbf{N}$ such that $\lim_{k\to\infty} J(N_k) = \inf_{N \in \mathbf{N}} J(N)$, with corresponding $P_{e_k}, P_{l_k}, P_{p_k}, P_{a_k}$ sequences. Thus we can find subsequences $N_{k_j}, P_{e_{k_j}}, P_{l_{k_j}}, P_{p_{k_j}}, P_{a_{k_j}}$, such that $\lim_{j\to\infty} J(N_{k_j}) = \inf_{N \in \mathbf{N}} J(N)$, $N_{k_j} \to \mathcal{N}, P_{e_{k_j}} \to \mathcal{P}_e, P_{l_{k_j}} \to \mathcal{P}_l, P_{p_{k_j}} \to \mathcal{P}_p, P_{a_{k_j}} \to \mathcal{P}_a$. Therefore, there exists $\mathcal{N} \in \mathbf{N}$ which minimizes $J(N)$.

□

2.3.2 Necessary Conditions

Adjoint System: Define the following terminal value system, called an adjoint system:

$$\lambda_e(t) = \lambda_e(t+1)\gamma_1 + \lambda_l(t+1)\gamma_2$$

$$\lambda_l(t) = 2\beta_1\mathcal{P}_l(t) + \lambda_l(t+1)\zeta_1 e^{-\alpha\mathcal{N}(t)} + \lambda_p(t+1)\zeta_2 e^{-\alpha\mathcal{N}(t)}$$

$$\lambda_p(t) = \lambda_p(t+1)\nu_1 + \lambda_a(t+1)\nu_2$$

$$\lambda_a(t) = \lambda_e(t+1)\theta_1 + \lambda_a(t+1)\theta_2$$

$$\lambda_e(T) = 0,\ \lambda_l(T) = 0,\ \lambda_p(T) = 0,\ \lambda_a(T) = 0.$$

These adjoints, λ, are useful in establishing the formulas and necessary conditions for the optimal control. Additionally adjoints are effective for computational purposes, specifically the forward backward sweep discussed later. Note the adjoints are constructed by

$$\lambda_e(t) = [\beta_1 \mathcal{P}_l(t)^2 + \beta_2 \mathcal{N}(t)]_{P_e} + \mathcal{P}_e(t)_{P_e} \lambda_e(t+1) + \mathcal{P}_l(t)_{P_e} \lambda_l(t+1) + \mathcal{P}_p(t)_{P_e} \lambda_p(t+1)$$

$$+\mathcal{P}_a(t)_{P_e} \lambda_a(t+1),$$

similar construction follows for the other adjoints. The adjoints were formulated by Pontryagin and colleagues, the adjoints variables preform a function similar to that of Lagrange multipliers.[LW07]

Theorem 2.3.2. *If there exists an optimal control $\mathcal{N}$, then there exists an adjoint system 2.3.2 and*

$$\mathcal{N}(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi(t) \\ \frac{1}{\alpha} \ln[\frac{\alpha}{\beta_2} \xi(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi(t) \end{cases}$$

where $\xi(t) = \zeta_1 \lambda_l(t+1) \mathcal{P}_l(t) + \zeta_2 \lambda_p(t+1) \mathcal{P}_l(t)$.

Proof. Since we have that $\mathcal{N}$ minimizes $J(N)$; for all sufficiently small $\varepsilon > 0$ and for all $\eta \in \{\eta = (\eta(1), ..., \eta(T)) | \eta(t) \leq 1, t = 1, ..., T\}$ we have that $J(\mathcal{N} + \eta\varepsilon) \geq J(\mathcal{N})$. To determine the structure of the control consider directional derivatives of the cost J, we will take a directional derivative of functional J; for the directional derivative in the direction of η with $\varepsilon > 0$ sufficiently small and $0 \leq \mathcal{N} + \eta\varepsilon = \mathcal{N}^\varepsilon \in \mathbf{N}$ we have that:

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} [J(\mathcal{N} + \eta\varepsilon) - J(\mathcal{N})]$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left[\sum_{t=0}^{T-1} \beta_1 \mathcal{P}_l^\varepsilon(t)^2 + \beta_2 \mathcal{N}^\varepsilon(t) - \sum_{t=0}^{T-1} \beta_1 \mathcal{P}_l(t)^2 + \beta_2 \mathcal{N}(t) \right]$$

$$= \sum_{t=0}^{T-1} \left[\beta_1 \lim_{\varepsilon \to 0^+} \frac{\mathcal{P}_l^\varepsilon(t)^2 - \mathcal{P}_l(t)^2}{\varepsilon} + \beta_2 \lim_{\varepsilon \to 0^+} \frac{\mathcal{N}^\varepsilon(t) - \mathcal{N}(t)}{\varepsilon} \right].$$

We have that $\lim_{\varepsilon \to 0^+} \frac{\mathcal{N}^\varepsilon(t) - \mathcal{N}(t)}{\varepsilon} = \eta(t)$, and we will define the sensitivities, $\psi_e(t), \psi_l(t), \psi_p(t), \psi_a(t)$ as:

$$\psi_e(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_e^\varepsilon(t) - \mathcal{P}_e(t)}{\varepsilon}, \quad \psi_l(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_l^\varepsilon(t) - \mathcal{P}_l(t)}{\varepsilon},$$

$$\psi_p(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_p^\varepsilon(t) - \mathcal{P}_p(t)}{\varepsilon}, \quad \psi_a(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_a^\varepsilon(t) - \mathcal{P}_a(t)}{\varepsilon}$$

where $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$. We have the limits exists from Chapter 23 in Optimal Control Applied to Biological Models [LW07].

Hence, we can write:

$$\psi_e(t+1) = \gamma_1 \psi_e(t) + \theta_1 \psi_a(t)$$

$$\psi_l(t+1) = \gamma_2 \psi_e(t) + \zeta_1 e^{-\alpha \mathcal{N}(t)} \psi_l(t) - \zeta_1 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t)$$

$$\psi_p(t+1) = \nu_1 \psi_p(t) + \zeta_2 e^{-\alpha \mathcal{N}(t)} \psi_l(t) - \zeta_2 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t)$$

$$\psi_a(t+1) = \nu_2 \psi_p(t) + \theta_2 \psi_a(t).$$

Now, returning to

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N} + \eta\varepsilon) - J(\mathcal{N})] = \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) + \beta_2\eta(t).$$

To remove the sensitivity $\psi_l(t)$ we will manipulate the sensitivities and adjoints equations.

We have that:

$$\begin{bmatrix} \psi_e(t+1) \\ \psi_l(t+1) \\ \psi_p(t+1) \\ \psi_a(t+1) \end{bmatrix} - B \begin{bmatrix} \psi_e(t) \\ \psi_l(t) \\ \psi_p(t) \\ \psi_a(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\zeta_1 \alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)\eta(t) \\ -\zeta_2 \alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)\eta(t) \\ 0 \end{bmatrix}$$

$$\text{where } B = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 \\ \gamma_2 & \zeta_1 e^{-\alpha\mathcal{N}(k)} & 0 & 0 \\ 0 & \zeta_2 e^{-\alpha\mathcal{N}(k)} & \nu_1 & 0 \\ 0 & 0 & \nu_2 & \theta_2 \end{bmatrix}.$$

Now we have that:

$$\sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) = \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) \end{bmatrix} \begin{bmatrix} 0 \\ \beta_1 2\mathcal{P}_l(t) \\ 0 \\ 0 \end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) \end{bmatrix} \left(\begin{bmatrix} \lambda_e(t) \\ \lambda_l(t) \\ \lambda_p(t) \\ \lambda_a(t) \end{bmatrix} - B^T \begin{bmatrix} \lambda_e(t+1) \\ \lambda_l(t+1) \\ \lambda_p(t+1) \\ \lambda_a(t+1) \end{bmatrix} \right)$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) \end{bmatrix} \begin{bmatrix} \lambda_e(t) \\ \lambda_l(t) \\ \lambda_p(t) \\ \lambda_a(t) \end{bmatrix}$$

$$- \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) \end{bmatrix} B^T \begin{bmatrix} \lambda_e(t+1) \\ \lambda_l(t+1) \\ \lambda_p(t+1) \\ \lambda_a(t+1) \end{bmatrix}.$$

Recall that $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$ and $\lambda_e(T) = 0$, $\lambda_l(T) = 0$, $\lambda_p(T) = 0$, $\lambda_a(T) = 0$. Therefore we can change the indices, so that:

$$\sum_{t=0}^{T-1}\begin{bmatrix}\psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t)\end{bmatrix}\begin{bmatrix}\lambda_e(t)\\ \lambda_l(t)\\ \lambda_p(t)\\ \lambda_a(t)\end{bmatrix}$$

$$=\begin{bmatrix}\psi_e(0) & \psi_l(0) & \psi_p(0) & \psi_a(0)\end{bmatrix}\begin{bmatrix}\lambda_e(0)\\ \lambda_l(0)\\ \lambda_p(0)\\ \lambda_a(0)\end{bmatrix}+\ldots+\begin{bmatrix}\psi_e(T-1) & \psi_l(T-1) & \psi_p(T-1) & \psi_a(T-1)\end{bmatrix}\begin{bmatrix}\lambda_e(T-1)\\ \lambda_l(T-1)\\ \lambda_p(T-1)\\ \lambda_a(T-1)\end{bmatrix}$$

$$=\begin{bmatrix}\psi_e(1) & \psi_l(1) & \psi_p(1) & \psi_a(1)\end{bmatrix}\begin{bmatrix}\lambda_e(1)\\ \lambda_l(1)\\ \lambda_p(1)\\ \lambda_a(1)\end{bmatrix}+\ldots+\begin{bmatrix}\psi_e(T-1) & \psi_l(T-1) & \psi_p(T-1) & \psi_a(T-1)\end{bmatrix}\begin{bmatrix}\lambda_e(T-1)\\ \lambda_l(T-1)\\ \lambda_p(T-1)\\ \lambda_a(T-1)\end{bmatrix}$$

$$=\begin{bmatrix}\psi_e(1) & \psi_l(1) & \psi_p(1) & \psi_a(1)\end{bmatrix}\begin{bmatrix}\lambda_e(1)\\ \lambda_l(1)\\ \lambda_p(1)\\ \lambda_a(1)\end{bmatrix}+\ldots+\begin{bmatrix}\psi_e(T) & \psi_l(T) & \psi_p(T) & \psi_a(T)\end{bmatrix}\begin{bmatrix}\lambda_e(T)\\ \lambda_l(T)\\ \lambda_p(T)\\ \lambda_a(T)\end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t+1) & \psi_l(t+1) & \psi_p(t+1) & \psi_a(t+1) \end{bmatrix} \begin{bmatrix} \lambda_e(t+1) \\ \lambda_l(t+1) \\ \lambda_p(t+1) \\ \lambda_a(t+1) \end{bmatrix}.$$

Using this reindexing we have that:

$$\sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) =$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t+1) & \psi_l(t+1) & \psi_p(t+1) & \psi_a(t+1) \end{bmatrix} \begin{bmatrix} \lambda_e(t+1) \\ \lambda_l(t+1) \\ \lambda_p(t+1) \\ \lambda_a(t+1) \end{bmatrix}$$

$$- \sum_{t=0}^{T-1} \begin{bmatrix} \lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) \end{bmatrix} B \begin{bmatrix} \psi_e(t) \\ \psi_l(t) \\ \psi_p(t) \\ \psi_a(t) \end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) \end{bmatrix} \left(\begin{bmatrix} \psi_e(t+1) \\ \psi_l(t+1) \\ \psi_p(t+1) \\ \psi_a(t+1) \end{bmatrix} - B \begin{bmatrix} \psi_e(t) \\ \psi_l(t) \\ \psi_p(t) \\ \psi_a(t) \end{bmatrix} \right)$$

$$
\begin{aligned}
&= \sum_{t=0}^{T-1} \begin{bmatrix} \lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) \end{bmatrix} \begin{bmatrix} 0 \\ -\zeta_1 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t) \\ -\zeta_2 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t) \\ 0 \end{bmatrix} \\
&= \sum_{t=0}^{T-1} \lambda_l(t+1)(-\zeta_1 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t)) + \lambda_p(t+1)(-\zeta_2 \alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t)) \\
&= \sum_{t=0}^{T-1} [\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2][-\alpha e^{-\alpha \mathcal{N}(t)} \mathcal{P}_l(t) \eta(t))].
\end{aligned}
$$

Now combining everything we have that:

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N} + \eta\varepsilon) - J(\mathcal{N})] = \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) + \beta_2\eta(t)$$

$$= \sum_{t=0}^{T-1}[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2][-\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)\eta(t))] + \beta_2\eta(t)$$

$$= \sum_{t=0}^{T-1}\eta(t)\left[-\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2\right].$$

Considering the previous equation with equality

$$0 = \sum_{t=0}^{T-1}\eta(t)\left[-\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2\right]$$

for all $\eta \in \{\eta = (\eta(1), ..., \eta(T))|\eta(t) \leq 1, t = 1, ..., T\}$. Then we have that for all t,

$$0 = -\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2.$$

Consider:

$$0 = -\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2 \iff$$

$$e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] = \frac{\beta_2}{\alpha} \iff$$

$$e^{-\alpha\mathcal{N}(t)} = \frac{\beta_2}{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]} \iff$$

$$-\alpha\mathcal{N}(t) = \ln\left[\frac{\beta_2}{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}\right] \iff$$

$$\alpha\mathcal{N}(t) = \ln\left[\frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2}\right].$$

Note that $\alpha > 0$. We need that $\mathcal{N}(t) \geq 0$, so

$$\ln\left[\frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2}\right] \geq 0,$$

meaning

$$\frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2} \geq 1.$$

Hence when

$$\frac{\beta_2}{\alpha} \leq \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2],$$

then we have

$$\mathcal{N}(t) = \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}(\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2])].$$

Now we will consider if

$$\frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2].$$

Then returning to

$$0 = \sum_{t=0}^{T-1}\eta(t)\left[-\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2\right]$$

for all $\eta \in \{\eta = (\eta(1), ..., \eta(T)) | \eta(t) \leq 1, t = 1, ..., T\}$.

$$0 = \sum_{t=0}^{T-1}\eta(t)\left[-\alpha e^{-\alpha\mathcal{N}(t)}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] + \beta_2\right]$$

$$< \sum_{t=0}^{T-1}\eta(t)\left[-\alpha e^{-\alpha\mathcal{N}(t)}\left(\frac{\beta_2}{\alpha}\right) + \beta_2\right] = \sum_{t=0}^{T-1}\eta(t)\left[-\beta_2 e^{-\alpha\mathcal{N}(t)} + \beta_2\right]$$

Hence we have if

$$\frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]$$

then

$$0 < \sum_{t=0}^{T-1} \eta(t)\left[-\beta_2 e^{-\alpha\mathcal{N}(t)} + \beta_2\right] = \sum_{t=0}^{T-1} \eta(t)\beta_2\left[-e^{-\alpha\mathcal{N}(t)} + 1\right].$$

Recall we have that $\mathcal{N}(t) \geq 0$.

If $\mathcal{N}(t) > 0$ we have that $\beta_2(-e^{-\alpha\mathcal{N}(t)} + 1) < 0$ contradiction.

Thus, if

$$\frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]$$

we must have that $\mathcal{N}(t) = 0$. Set

$$\xi(t) = \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]$$

$$\mathcal{N}(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi(t) \end{cases}.$$

Meaning that

$$\mathcal{N}(t) = \max(0, \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}(\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2])]).$$

□

2.3.3 Uniqueness

Theorem 2.3.3. *Uniqueness: If the optimal control $\mathcal{N}$ exists, then it is unique.*

Proof. In order to show $\mathcal{N}$ is unique we will show that $J(N) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N(t)$ is strictly convex. Recall that if a function is strictly convex then there exists a unique minimum such that $J(\mathcal{N}) < J(N)$ for all $N \in \mathbf{N} \setminus \mathcal{N}$. To show that J is strictly convex we will look at J along a line segment from N to η by defining $z(\varepsilon) = J((1-\varepsilon)N + \varepsilon\eta) = J(N + \varepsilon(\eta - N))$ for $N, \eta \in \mathbf{N}$, and $0 < \varepsilon < 1$. Note that if z, a one dimensional function, is convex in every possible direction then J will be convex. To establish convexity we will show that $z''(\varepsilon) > 0$. First take the derivative of z, note that P_l^{ε} is a function of $N + \varepsilon(\eta - N)$ and $P_l^{\tau+\varepsilon}$ is a function of $N + (\tau + \varepsilon)(\eta - N)$.

$$z'(\varepsilon) = \lim_{\tau \to 0} \left(\frac{J(N + (\tau + \varepsilon)(\eta - N)) - J(N + \varepsilon(\eta - N))}{\tau} \right) =$$

$$= \lim_{\tau \to 0} \sum_{t=0}^{T-1} \frac{\beta_1}{\tau} \left[P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2 \right] + \frac{\beta_2}{\tau} \Big([N(t) + (\tau+\varepsilon)(\eta(t) - N(t))] - [N(t) + \varepsilon(\eta(t) - N(t))] \Big)$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2}{\tau} \right] + \beta_2 \left[\lim_{\tau \to 0} \frac{\tau(\eta(t) - N(t))}{\tau} \right]$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2}{\tau} \right] + \beta_2(\eta(t) - N(t)).$$

By The Chain Rule:

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2 P_l^{\varepsilon}(t) \psi_l^{\varepsilon}(t) + \beta_2(\eta(t) - N(t)).$$

Note we define sensitivities similar to in Theorem 2.3.2:

$$\psi_e^{\varepsilon}(t+1) = \gamma_1 \psi_e^{\varepsilon}(t) + \theta_1 \psi_a^{\varepsilon}(t)$$

$$\psi_l^\varepsilon(t+1) = \gamma_2\psi_e^\varepsilon(t) + \zeta_1 e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) - \zeta_1\alpha e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))$$

$$\psi_p^\varepsilon(t+1) = \nu_1\psi_p^\varepsilon(t) + \zeta_2 e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) - \zeta_2\alpha e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))$$

$$\psi_a^\varepsilon(t+1) = \nu_2\psi_p^\varepsilon(t) + \theta_2\psi_a^\varepsilon(t)$$

where $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$.

In order to continue we must define derivatives of the sensitives, $\sigma_e^\varepsilon(t), \sigma_l^\varepsilon(t), \sigma_p^\varepsilon(t), \sigma_a^\varepsilon(t)$ as:

$$\sigma_e^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_e^{\tau+\varepsilon}(t+1) - \psi_e^\varepsilon(t+1)}{\tau}, \quad \sigma_l^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_l^{\tau+\varepsilon}(t+1) - \psi_l^\varepsilon(t+1)}{\tau},$$

$$\sigma_p^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_p^{\tau+\varepsilon}(t+1) - \psi_p^\varepsilon(t+1)}{\tau}, \quad \sigma_a^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_a^{\tau+\varepsilon}(t+1) - \psi_a^\varepsilon(t+1)}{\tau}.$$

Hence, we can write:

$$\sigma_e^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_e^{\tau+\varepsilon}(t+1) - \psi_e^\varepsilon(t+1)}{\tau} = \gamma_1\lim_{\tau\to 0}\frac{\psi_e^{\tau+\varepsilon}(t) - \psi_e^\varepsilon(t)}{\tau} + \theta_1\lim_{\tau\to 0}\frac{\psi_a^{\tau+\varepsilon}(t) - \psi_a^\varepsilon(t)}{\tau}$$

$$= \gamma_1\sigma_e^\varepsilon(t) + \theta_1\sigma_a^\varepsilon(t)$$

$$\sigma_a^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_a^{\tau+\varepsilon}(t+1) - \psi_a^\varepsilon(t+1)}{\tau} = \nu_2\lim_{\tau\to 0}\frac{\psi_p^{\tau+\varepsilon}(t) - \psi_p^\varepsilon(t)}{\tau} + \theta_2\lim_{\tau\to 0}\frac{\psi_a^{\tau+\varepsilon}(t) - \psi_a^\varepsilon(t)}{\tau}$$

$$= \nu_2\sigma_p^\varepsilon(t) + \theta_2\sigma_a^\varepsilon(t).$$

Now, we will compute $\sigma_l^\varepsilon(t+1)$.

$$\sigma_l(t+1) = \lim_{\tau\to 0} \frac{\psi_l^{\tau+\varepsilon}(t+1) - \psi_l^\varepsilon(t+1)}{\tau} =$$

$$= \lim_{\tau\to 0} \frac{\gamma_2\psi_e^{\tau+\varepsilon}(t) + \zeta_1 e^{-\alpha N^{\tau+\varepsilon}(t)}\psi_l^{\tau+\varepsilon}(t) - \zeta_1\alpha e^{-\alpha N^{\tau+\varepsilon}(t)} P_l^{\tau+\varepsilon}(t)(\eta(t) - N(t)) - \gamma_2\psi_e^\varepsilon(t) - \zeta_1 e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) + \zeta_1\alpha e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))}{\tau}$$

$$= \gamma_2 \lim_{\tau\to 0} \frac{\psi_e^{\tau+\varepsilon}(t) - \psi_e^\varepsilon(t)}{\tau} + \zeta_1 \lim_{\tau\to 0} \frac{e^{-\alpha N^{\tau+\varepsilon}(t)}\psi_l^{\tau+\varepsilon}(t) - e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)}{\tau} - \zeta_1\alpha(\eta(t) - N(t)) \lim_{\tau\to 0} \frac{e^{-\alpha N^{\tau+\varepsilon}(t)} P_l^{\tau+\varepsilon}(t) - e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)}{\tau}$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1[e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(v(t) - N(t))] - \zeta_1\alpha(\eta(t) - N(t))[e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))]$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - \zeta_1\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t) - N(t)) - \zeta_1\alpha(v(t) - N(t)) e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) + \zeta_1(\eta(t) - N(t))^2\alpha^2 e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)$$

$$= \gamma_2 \sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha N^\varepsilon(t)} \sigma_l^\varepsilon(t) - 2\zeta_1 \alpha e^{-\alpha N^\varepsilon(t)} \psi_l^\varepsilon(t)(\eta(t) - N(t)) + \zeta_1 \alpha^2 e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))^2.$$

Now, we will compute $\sigma_p^\varepsilon(t+1)$.

$$\sigma_p(t+1) = \lim_{\tau \to 0} \frac{\psi_p^{\tau+\varepsilon}(t+1) - \psi_p^\varepsilon(t+1)}{\tau} =$$

$$= \nu_1 \sigma_p^\varepsilon(t) + \zeta_2 [e^{-\alpha N^\varepsilon(t)} \sigma_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)} \psi_l^\varepsilon(t)(\eta(t) - N(t))] - \zeta_2 \alpha (v(t) - N(t)) [e^{-\alpha N^\varepsilon(t)} \psi_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))]$$

$$= \nu_1 \sigma_p^\varepsilon(t) + \zeta_2 e^{-\alpha N^\varepsilon(t)} \sigma_l^\varepsilon(t) - 2\zeta_2 \alpha e^{-\alpha N^\varepsilon(t)} \psi_l^\varepsilon(t)(\eta(t) - N(t)) + \zeta_2 \alpha^2 e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))^2.$$

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2 P_l^{\varepsilon}(t)\psi_l^{\varepsilon}(t) + \beta_2(\eta(t) - N(t))$$

$$z''(\varepsilon) = \lim_{\tau \to 0} \left(\frac{z'(\tau + \varepsilon) - z'(\varepsilon)}{\tau} \right) =$$

$$= \lim_{\tau \to 0} \sum_{t=0}^{T-1} \beta_1 2 P_l^{\tau+\varepsilon}(t)\psi_l^{\tau+\varepsilon}(t) + \beta_2(\eta(t) - N(t)) - [\beta_1 2 P_l^{\varepsilon}(t)\psi_l^{\varepsilon}(t) + \beta_2(\eta(t) - N(t))]$$

$$= \sum_{t=0}^{T-1} \beta_1 2 \lim_{\tau \to 0} \frac{P_l^{\tau+\varepsilon}(t)\psi_l^{\tau+\varepsilon}(t) - P_l^{\varepsilon}(t)\psi_l^{\varepsilon}(t)}{\tau} = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^{\varepsilon}(t) P_l^{\varepsilon}(t) + \psi_l^{\varepsilon}(t)^2]$$

We now need to show that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^{\varepsilon}(t)P_l^{\varepsilon}(t) + \psi_l^{\varepsilon}(t)^2] > 0$. To bound $z''(\varepsilon) > 0$ we will show that $[\sigma_l^{\varepsilon}(t)P_l^{\varepsilon}(t) + \psi_l^{\varepsilon}(t)^2] > 0$. To do this we will show that $\sigma_l^{\varepsilon}(t) > 0$ for all t. Note we have that $P_l^{\varepsilon}(t) \geq 0$ and $\psi_l^{\varepsilon}(t)^2 > 0$.

Sensitivities We need the equations for the sensitivities for various t values. We have that $\psi_e^{\varepsilon}(0) = 0$, $\psi_l^{\varepsilon}(0) = 0$, $\psi_p^{\varepsilon}(0) = 0$, $\psi_a^{\varepsilon}(0) = 0$, so for $t = 1$:

$$\psi_e^{\varepsilon}(1) = \gamma_1 \psi_e^{\varepsilon}(0) + \theta_1 \psi_a^{\varepsilon}(0) = 0$$

$$\psi_l^{\varepsilon}(1) = \gamma_2 \psi_e^{\varepsilon}(0) + \zeta_1 e^{-\alpha N^{\varepsilon}(0)} \psi_l^{\varepsilon}(0) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}(0)} P_l^{\varepsilon}(0)(\eta(0) - N(0))$$

$$= -\zeta_1 \alpha e^{-\alpha N^{\varepsilon}(0)} P_l^{\varepsilon}(0)(\eta(0) - N(0))$$

$$\psi_p^{\varepsilon}(1) = \nu_1 \psi_p^{\varepsilon}(0) + \zeta_2 e^{-\alpha N^{\varepsilon}(0)} \psi_l^{\varepsilon}(0) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}(0)} P_l^{\varepsilon}(0)(\eta(0) - N(0))$$

$$= -\zeta_2 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))$$

$$\psi_a^\varepsilon(1) = \nu_2 \psi_p^\varepsilon(0) + \theta_2 \psi_a^\varepsilon(0) = 0.$$

Next, for $t = 2$

$$\psi_e^\varepsilon(2) = \gamma_1 \psi_e^\varepsilon(1) + \theta_1 \psi_a^\varepsilon(1) = 0$$

$$\psi_l^\varepsilon(2) = \gamma_2 \psi_e^\varepsilon(1) + \zeta_1 e^{-\alpha N^\varepsilon(1)} \psi_l^\varepsilon(1) - \zeta_1 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$= \zeta_1 e^{-\alpha N^\varepsilon(1)}(-\zeta_1 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))) - \zeta_1 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$= -\zeta_1^2 \alpha e^{-\alpha N^\varepsilon(0) - \alpha N^\varepsilon(1)} P_l^\varepsilon(0)(\eta(0) - N(0))) - \zeta_1 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$\psi_p^\varepsilon(2) = \nu_1 \psi_p^\varepsilon(1) + \zeta_2 e^{-\alpha N^\varepsilon(1)} \psi_l^\varepsilon(1) - \zeta_2 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$= \nu_1(-\zeta_2 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$+\zeta_2 e^{-\alpha N^\varepsilon(1)}(-\zeta_1 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))) - \zeta_2 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$= -\nu_1 \zeta_2 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))$$

$$-\zeta_2 \zeta_1 \alpha e^{-\alpha N^\varepsilon(0) - \alpha N^\varepsilon(1)} P_l^\varepsilon(0)(\eta(0) - N(0))) - \zeta_2 \alpha e^{-\alpha N^\varepsilon(1)} P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$\psi_a^\varepsilon(2) = \nu_2 \psi_p^\varepsilon(1) + \theta_2 \psi_a^\varepsilon(1) = \nu_2(-\zeta_2 \alpha e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))).$$

Then, for $t = 3$

$$\psi_e^\varepsilon(3) = \gamma_1\psi_e^\varepsilon(2) + \theta_1\psi_a^\varepsilon(2) = \theta_1[\nu_2\psi_p^\varepsilon(1) + \theta_2\psi_a^\varepsilon(1)]$$

$$= \theta_1\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$\psi_l^\varepsilon(3) = \gamma_2\psi_e^\varepsilon(2) + \zeta_1 e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2) - \zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$= \zeta_1 e^{-\alpha N^\varepsilon(2)}[-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0)-N(0)))-\zeta_1\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))]$$

$$-\zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$= -\zeta_1^3\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$-\zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$\psi_p^\varepsilon(3) = \nu_1\psi_p^\varepsilon(2) + \zeta_2 e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2) - \zeta_2\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$= \nu_1[-\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0)) - \zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))]+\zeta_2 e^{-\alpha N^\varepsilon(2)}[-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-\zeta_1\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1) - N(1)] - \zeta_2\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$= -\nu_1^2\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0)) - \nu_1\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1) - N(1)) - \zeta_2\zeta_1^2\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1) - N(1) - \zeta_2\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))$$

$$\psi_a^\varepsilon(3) = \nu_2\psi_p^\varepsilon(2) + \theta_2\psi_a^\varepsilon(2) =$$

$$= \nu_2[-\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)) - \zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))] + \theta_2[\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)))]$$

$$= -\nu_2\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)) - \nu_2\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-\nu_2\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1)) + \theta_2\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))).$$

Lastly, when $t = 4$

$$\psi_e^\varepsilon(4) = \gamma_1\psi_e^\varepsilon(3) + \theta_1\psi_a^\varepsilon(3) = \gamma_1[\theta_1\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)))]$$

$$+\theta_1[-\nu_2\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))$$

$$-\nu_2\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0)-N(0))) - \nu_2\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))$$

$$+\theta_2\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)))]$$

$$\psi_l^\varepsilon(4) = \gamma_2\psi_e^\varepsilon(3) + \zeta_1 e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3) - \zeta_1\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))$$

$$= \gamma_2[\theta_1\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)))]$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[-\zeta_1^3\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1)-N(1) - \zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))]$$

$$-\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))$$

$$\psi_p^\varepsilon(4) = \nu_1\psi_p^\varepsilon(3) + \zeta_2 e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3) - \zeta_2\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3) - N(3))$$

$$= \nu_1[-\nu_2\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))$$

$$-\nu_2\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0) - N(0))) - \nu_2\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$+\theta_2\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0)))]$$

$$+\zeta_2 e^{-\alpha N^\varepsilon(3)}[-\zeta_1^3\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1) - N(1)) - \zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))]$$

$$-\zeta_2\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3) - N(3))$$

$$\psi_a^\varepsilon(4) = \nu_2[-\nu_1^2\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))$$

$$-\nu_1\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0) - N(0))) - \nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$-\zeta_2\zeta_1^2\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0) - N(0)))$$

$$-\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1) - N(1) - \zeta_2\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2) - N(2))]$$

$$+\theta_2[-\nu_2\nu_1\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))$$

$$-\nu_2\zeta_2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)}P_l^\varepsilon(0)(\eta(0) - N(0))) - \nu_2\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1) - N(1))$$

$$+\theta_2\nu_2(-\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0)))].$$

We have established values for $\psi_e^\varepsilon(t)$, $\psi_l^\varepsilon(t)$, $\psi_p^\varepsilon(t)$, $\psi_a^\varepsilon(t)$ for $t = 0, 1, 2, 3, 4$.

Values for σ^ε We need $\sigma_l^\varepsilon > 0$, so we will focus on the values of σ_l^ε. However, we must recall that the formulas for $\sigma^\varepsilon, \psi^\varepsilon$, and P^ε are all interconnected. Specifically, we have that:

$$\sigma_e^\varepsilon(t+1) = \gamma_1\sigma_e^\varepsilon(t) + \theta_1\sigma_a^\varepsilon(t)$$

$$\sigma_l(t+1) = \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\zeta_1\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t)-N(t)) + \zeta_1\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$\sigma_p(t+1) = \nu_1\sigma_p^\varepsilon(t) + \zeta_2 e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\zeta_2\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(v(t)-N(t)) + \zeta_2\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$\sigma_a^\varepsilon(t+1) = \nu_2\sigma_p^\varepsilon(t) + \theta_2\sigma_a^\varepsilon(t).$$

Recall that $\sigma_e^\varepsilon(0)$, $\sigma_l^\varepsilon(0)$, $\sigma_p^\varepsilon(0)$, $\sigma_a^\varepsilon(0) = 0$. Consider $t = 1$:

$$\sigma_e^\varepsilon(1) = \gamma_1\sigma_e^\varepsilon(0) + \theta_1\sigma_a^\varepsilon(0) = 0$$

$$\sigma_l^\varepsilon(1) = \gamma_2\sigma_e^\varepsilon(0) + \zeta_1 e^{-\alpha N^\varepsilon(0)}\sigma_l^\varepsilon(0) - 2\zeta_1\alpha e^{-\alpha N^\varepsilon(0)}\psi_l^\varepsilon(0)(\eta(0) - N(0))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))^2 = \zeta_1\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))^2$$

$$\sigma_p^\varepsilon(1) = \nu_1\sigma_p^\varepsilon(0) + \zeta_2 e^{-\alpha N^\varepsilon(0)}\sigma_l^\varepsilon(0) - 2\zeta_2\alpha e^{-\alpha N^\varepsilon(0)}\psi_l^\varepsilon(0)(\eta(0) - N(0))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))^2 = \zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0) - N(0))^2$$

$$\sigma_a^\varepsilon(1) = \nu_2\sigma_p^\varepsilon(0) + \theta_2\sigma_a^\varepsilon(0) = 0.$$

Next, $t = 2$

$$\sigma_e^\varepsilon(2) = \gamma_1\sigma_e^\varepsilon(1) + \theta_1\sigma_a^\varepsilon(1) = 0$$

$$\sigma_l^\varepsilon(2) = \gamma_2\sigma_e^\varepsilon(1) + \zeta_1 e^{-\alpha N^\varepsilon(1)}\sigma_l^\varepsilon(1) - 2\zeta_1\alpha e^{-\alpha N^\varepsilon(1)}\psi_l^\varepsilon(1)(\eta(1) - N(1))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))^2 =$$

$$= \zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))+$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))^2$$

$$= \zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))+$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(1)}(\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0))(\eta(1)-N(1))^2$$

$$= \zeta_1\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))^2$$

$$+2(\eta(0)-N(0))(\eta(1)-N(1))+(\eta(1)-N(1))^2]$$

$$= \zeta_1\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2$$

$$\sigma_p^\varepsilon(2) = \nu_1\sigma_p^\varepsilon(1)+\zeta_2 e^{-\alpha N^\varepsilon(1)}\sigma_l^\varepsilon(1)-2\zeta_2\alpha e^{-\alpha N^\varepsilon(1)}\psi_l^\varepsilon(1)(\eta(1)-N(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))^2 =$$

$$= \nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))^2$$

$$= \nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0)-N(0))^2 + \zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))} P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))} P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}(\gamma_2 P_e^\varepsilon(0) + \zeta_1 e^{-\alpha N(0)} P_l^\varepsilon(0))(\eta(1)-N(1))^2$$

$$= \nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0)-N(0))^2 + \zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))} P_l^\varepsilon(0)[(\eta(0)-N(0))^2$$

$$+2(\eta(0)-N(0))(\eta(1)-N(1)) + (\eta(1)-N(1))^2]$$

$$= \nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0)-N(0))^2 + \zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))} P_l^\varepsilon(0)[(\eta(0)-N(0)) + (\eta(1)-N(1))]^2$$

$$\sigma_a^\varepsilon(2) = \nu_2\sigma_p^\varepsilon(1) + \theta_2\sigma_a^\varepsilon(1) = \nu_2\left[\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0)-N(0))^2\right].$$

Then, $t = 3$

$$\sigma_e^\varepsilon(3) = \gamma_1\sigma_e^\varepsilon(2) + \theta_1\sigma_a^\varepsilon(2) = \theta_1\left[\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0)-N(0))^2\right]$$

$$\sigma_l^\varepsilon(3) = \gamma_2\sigma_e^\varepsilon(2) + \zeta_1 e^{-\alpha N^\varepsilon(2)}\sigma_l^\varepsilon(2)$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2)(\eta(2)-N(2)) + \zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)} P_l^\varepsilon(2)(\eta(2)-N(2))^2 =$$

$$= \zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))} P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))} P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))} P_l^\varepsilon(1)(\eta(1)-N(1))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(2)-N(2))$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(1)(\eta(1)-N(1))(\eta(2)-N(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))^2$$

$$=\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))^2+$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}[\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)](\eta(1)-N(1))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(2)-N(2))$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}[\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)](\eta(1)-N(1))(\eta(2)-N(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}[\gamma_2 P_e^\varepsilon(1)+\zeta_1 e^{-\alpha N(1)}P_l^\varepsilon(1)](\eta(2)-N(2))^2$$

$$=\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(2)-N(2))$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))(\eta(2)-N(2))$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(1)-N(1))(\eta(2)-N(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}[\gamma_2 P_e^\varepsilon(1)+\zeta_1 e^{-\alpha N(1)}[\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)]](\eta(2)-N(2))^2$$

$$=\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(1)-N(1))$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(1)-N(1))^2+$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(0)-N(0))(\eta(2)-N(2))$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))(\eta(2)-N(2))$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(1)-N(1))(\eta(2)-N(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)(\eta(2)-N(2))^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(2)-N(2))^2$$

$$=\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)[(\eta(0)-N(0))^2$$

$$+2(\eta(0)-N(0))(\eta(1)-N(1))+(\eta(1)-N(1))^2]$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)[2(\eta(0)-N(0))(\eta(2)-N(2))+2(\eta(1)-N(1))$$

$$(\eta(2)-N(2))]$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)(\eta(2)-N(2))^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))^2+(\eta(1)-N(1))(\eta(2)-N(2))+(\eta(2)-N(2))]^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2$$

$$=\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))]^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2$$

$$\sigma_p^\varepsilon(3)=\nu_1\sigma_p^\varepsilon(2)+\zeta_2 e^{-\alpha N^\varepsilon(2)}\sigma_l^\varepsilon(2)-2\zeta_2\alpha e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2)(\eta(2)-N(2))+$$

$$\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))^2=$$

$$=\nu_1[\nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2]$$

$$+\nu_1[\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2]$$

$$+\zeta_2[e^{-\alpha N^\varepsilon(2)}\sigma_l^\varepsilon(2)-2\alpha e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2)(\eta(2)-N(2))+\alpha^2 e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))^2]$$

using $\sigma_l^\varepsilon(3)=\zeta_1[e^{-\alpha N^\varepsilon(2)}\sigma_l^\varepsilon(2)-2\alpha e^{-\alpha N^\varepsilon(2)}\psi_l^\varepsilon(2)(\eta(2)-N(2))+\alpha^2 e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))^2]$

$$=\nu_1[\nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2]$$

$$+\nu_1[\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2]$$

$$+\frac{\zeta_2}{\zeta_1}[\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))]^2]$$

$$+\frac{\zeta_2}{\zeta_1}[\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))]^2]$$

$$+\frac{\zeta_2}{\zeta_1}[\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2]$$

$$\sigma_a^\varepsilon(3)=\nu_2\sigma_p^\varepsilon(2)+\theta_2\sigma_a^\varepsilon(2)=$$

$$=\nu_2[\nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2]$$

$$+\nu_2[\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2]$$

$$+\theta_2[\nu_2\left[\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2\right]]$$

$$=\nu_2\nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\nu_2\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2$$

$$+\theta_2\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

Next, $t=4$

$$\sigma_e^\varepsilon(4)=\gamma_1\sigma_e^\varepsilon(3)+\theta_1\sigma_a^\varepsilon(3)=$$

$$\gamma_1[\theta_1\left[\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2\right]]$$

$$+\theta_1\nu_2\nu_1\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2+\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(1)}\gamma_2 P_e^\varepsilon(0)(\eta(1)-N(1))^2$$

$$+\theta_1\nu_2\zeta_1\zeta_2\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))]^2$$

$$+\theta_1\theta_2\nu_2\zeta_2\alpha^2 e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$\sigma_l^\varepsilon(4)=\gamma_2\sigma_e^\varepsilon(3)+\zeta_1 e^{-\alpha N^\varepsilon(3)}\sigma_l^\varepsilon(3)$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3)(\eta(3)-N(3))+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))^2=$$

$$=\gamma_2\sigma_e^\varepsilon(3)$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))]^2]$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))]^2]$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2]$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1^3\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(1)(\eta(1)-N(1))]$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1\alpha e^{-\alpha N^\varepsilon(2)}P_l^\varepsilon(2)(\eta(2)-N(2))]$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))^2$$

$$=\gamma_2\sigma_e^\varepsilon(3)$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))]^2$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))]^2]$$

$$+\zeta_1 e^{-\alpha N^\varepsilon(3)}[\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(2)}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2]$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1^3\alpha e^{-\alpha N^\varepsilon(0)-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}P_l^\varepsilon(0)(\eta(0)-N(0)))$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1^2\alpha e^{-\alpha N^\varepsilon(1)-\alpha N^\varepsilon(2)}[\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)](\eta(1)-N(1))]$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}(\eta(3)-N(3))[-\zeta_1\alpha e^{-\alpha N^\varepsilon(2)}[\gamma_2 P_e^\varepsilon(1)+\zeta_1 e^{-\alpha N(1)}\{\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)\}](\eta(2)-N(2))]$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(3)}[\gamma_2 P_e^\varepsilon(2)+\zeta_1 e^{-\alpha N(2)}(\gamma_2 P_e^\varepsilon(1)+\zeta_1 e^{-\alpha N(1)}\{\gamma_2 P_e^\varepsilon(0)+\zeta_1 e^{-\alpha N(0)}P_l^\varepsilon(0)\})](\eta(3)-N(3))^2$$

$$=\gamma_2\sigma_e^\varepsilon(3)$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(1)(\eta(2)-N(2))^2$$

$$+2\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}P_l^\varepsilon(0)(\eta(3)-N(3))(\eta(0)-N(0))$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(0)(\eta(3)-N(3))(\eta(1)-N(1))$$

$$+2\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}P_l^\varepsilon(0)(\eta(3)-N(3))(\eta(1)-N(1))$$

$$+2\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}P_l^\varepsilon(0)(\eta(3)-N(3))(\eta(2)-N(2))$$

$$+2\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(0)(\eta(3)-N(3))(\eta(2)-N(2))$$

$$+2\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(1)(\eta(3)-N(3))(\eta(2)-N(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(3)}\gamma_2 P_e^\varepsilon(2)(\eta(3)-N(3))^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(1)(\eta(3)-N(3))^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(0)(\eta(3)-N(3))^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))} P_l^\varepsilon(0)(\eta(3)-N(3))^2$$

$$=\gamma_2\sigma_e^\varepsilon(3)$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))} P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(3)}\gamma_2 P_e^\varepsilon(2)(\eta(3)-N(3))^2$$

For $t=5$,

$$\sigma_l^\varepsilon(5)=\gamma_2\sigma_e^\varepsilon(4)+\zeta_1 e^{-\alpha N^\varepsilon(4)}\sigma_l^\varepsilon(4)$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}\psi_l^\varepsilon(4)(\eta(4)-N(4))+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)} P_l^\varepsilon(4)(\eta(4)-N(4))^2=$$

$$=\gamma_2\sigma_e^\varepsilon(4)+\zeta_1 e^{-\alpha N^\varepsilon(4)}[\gamma_2\sigma_e^\varepsilon(3)]$$

$$+\zeta_1^5\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))} P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(2)(\eta(3)-N(3))^2$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\gamma_2\psi_e^\varepsilon(3)+\zeta_1 e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3)-\zeta_1\alpha e^{-\alpha N^\varepsilon(3)} P_l^\varepsilon(3)(\eta(3)-N(3))]$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)} P_l^\varepsilon(4)(\eta(4)-N(4))^2$$

$$=\gamma_2\sigma_e^\varepsilon(4)+\zeta_1 e^{-\alpha N^\varepsilon(4)}[\gamma_2\sigma_e^\varepsilon(3)]$$

$$+\zeta_1^5\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(2)(\eta(3)-N(3))^2$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\gamma_2\psi_e^\varepsilon(3)]$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\zeta_1 e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3)-\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))]$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}P_l^\varepsilon(4)(\eta(4)-N(4))^2$$

$$=\gamma_2\sigma_e^\varepsilon(4)+\zeta_1 e^{-\alpha N^\varepsilon(4)}[\gamma_2\sigma_e^\varepsilon(3)]-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\gamma_2\psi_e^\varepsilon(3)]$$

$$+\zeta_1^5\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(2)(\eta(3)-N(3))^2$$

$$-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\zeta_1 e^{-\alpha N^\varepsilon(3)}\psi_l^\varepsilon(3)-\zeta_1\alpha e^{-\alpha N^\varepsilon(3)}P_l^\varepsilon(3)(\eta(3)-N(3))]$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}P_l^\varepsilon(4)(\eta(4)-N(4))^2$$

$$=\gamma_2\sigma_e^\varepsilon(4)+\zeta_1 e^{-\alpha N^\varepsilon(4)}[\gamma_2\sigma_e^\varepsilon(3)]-2\zeta_1\alpha e^{-\alpha N^\varepsilon(4)}(\eta(4)-N(4))[\gamma_2\psi_e^\varepsilon(3)]$$

$$+\zeta_1^5\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)N^\varepsilon(4))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))$$

$$+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(2)[(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}\gamma_2 P_e^\varepsilon(3)(\eta(4)-N(4))^2$$

$$= \gamma_2 \sigma_e^{\varepsilon}(4) +$$

$$+\zeta_1 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \left[\nu_2 \zeta_2 \alpha^2 e^{-\alpha N^{\varepsilon}(0)} P_l^{\varepsilon}(0)(\eta(0) - N(0))^2 \right]$$

$$+2\zeta_1 \zeta_2 \alpha^2 \nu_2 \theta_1 e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(4))} \gamma_2 P_l^{\varepsilon}(0)(\eta(4) - N(4))(\eta(0) - N(0)))$$

$$+\zeta_1^5 \alpha^2 e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(1)+N^{\varepsilon}(2)+N^{\varepsilon}(3)N^{\varepsilon}(4))} P_l^{\varepsilon}(0)[(\eta(0) - N(0)) + (\eta(1) - N(1)) + (\eta(2) - N(2))$$

$$+(\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1^4 \alpha^2 e^{-\alpha(N^{\varepsilon}(1)+N^{\varepsilon}(2)+N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^3 \alpha^2 e^{-\alpha(N^{\varepsilon}(2)+N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(1)[(\eta(2) - N(2)) + (\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1^2 \alpha^2 e^{-\alpha(N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(2)[(\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \gamma_1 P_e^{\varepsilon}(2)(\eta(4) - N(4))^2 +$$

$$\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \nu_2 \zeta_2 e^{-\alpha N^{\varepsilon}(0)} P_l(0)(\eta(4) - N(4))^2 +$$

$$\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \nu_2 \nu_1 P_p(0)))(\eta(4) - N(4))^2 +$$

$$\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \theta_2 P_a^{\varepsilon}(1)(\eta(4) - N(4))^2$$

$$= \gamma_2 \sigma_e^{\varepsilon}(4) +$$

$$+\zeta_1^5 \alpha^2 e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(1)+N^{\varepsilon}(2)+N^{\varepsilon}(3)N^{\varepsilon}(4))} P_l^{\varepsilon}(0)[(\eta(0) - N(0)) + (\eta(1) - N(1)) + (\eta(2) - N(2))$$

$$+(\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1^4 \alpha^2 e^{-\alpha(N^{\varepsilon}(1)+N^{\varepsilon}(2)+N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^3 \alpha^2 e^{-\alpha(N^{\varepsilon}(2)+N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(1)[(\eta(2) - N(2)) + (\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1^2 \alpha^2 e^{-\alpha(N^{\varepsilon}(3)+N^{\varepsilon}(4))} \gamma_2 P_e^{\varepsilon}(2)[(\eta(3) - N(3)) + (\eta(4) - N(4))]^2$$

$$+\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \gamma_1 P_e^{\varepsilon}(2)(\eta(4) - N(4))^2$$

$$+\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \nu_2 \nu_1 P_p(0)))(\eta(4) - N(4))^2$$

$$+\zeta_1 \alpha^2 e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \theta_2 P_a^{\varepsilon}(1)(\eta(4) - N(4))^2$$

$$+\zeta_1\zeta_2\alpha^2\nu_2\theta_1 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(4))}\gamma_2 P_l^\varepsilon(0)(\eta(0)-N(0))^2$$

$$+2\zeta_1\zeta_2\alpha^2\nu_2\theta_1 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(4))}\gamma_2 P_l^\varepsilon(0)(\eta(4)-N(4))(\eta(0)-N(0)))$$

$$\zeta_1\zeta_2\alpha^2\nu_2\theta_1 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(4))}\gamma_2 P_l(0)(\eta(4)-N(4))^2$$

$$=\gamma_2\sigma_e^\varepsilon(4)+$$

$$+\zeta_1^5\alpha^2 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)N^\varepsilon(4))}P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(1)-N(1))+(\eta(2)-N(2))$$

$$+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^4\alpha^2 e^{-\alpha(N^\varepsilon(1)+N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(0)[(\eta(1)-N(1))+(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^3\alpha^2 e^{-\alpha(N^\varepsilon(2)+N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(1)[(\eta(2)-N(2))+(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1^2\alpha^2 e^{-\alpha(N^\varepsilon(3)+N^\varepsilon(4))}\gamma_2 P_e^\varepsilon(2)[(\eta(3)-N(3))+(\eta(4)-N(4))]^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}\gamma_2\gamma_1 P_e^\varepsilon(2)(\eta(4)-N(4))^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}\gamma_2\theta_1\nu_2\nu_1 P_p(0)))(\eta(4)-N(4))^2$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(4)}\gamma_2\theta_1\theta_2 P_a^\varepsilon(1)(\eta(4)-N(4))^2$$

$$+\zeta_1\zeta_2\alpha^2\nu_2\theta_1 e^{-\alpha(N^\varepsilon(0)+N^\varepsilon(4))}\gamma_2 P_l^\varepsilon(0)[(\eta(0)-N(0))+(\eta(4)-N(4))]^2$$

Analysis Note the formula for $\sigma_e^\varepsilon(t)$ is a combination of positive parameters, exponential functions, and squares of the various $(\eta(i)-N(i))$. Therefore, we have in cases $t=1,2,3$ that $\sigma_e^\varepsilon(t)=0$ and when $t=3,4$ we have that $\sigma_e^\varepsilon(t)>0$. In cases $t=1,2,3,4,5$ we have that $\sigma_l^\varepsilon(t)>0$ since the formula for $\sigma_l^\varepsilon(t)$ is a combination of $\sigma_e^\varepsilon(t)$, positive parameters, exponential functions, and squares of the various $(\eta(i)-N(i))$. We achieved the formulation of $\sigma_l^\varepsilon(t)$ by grouping terms so that we had summations of $(\eta(i)-N(i))$ squared, ensuring a positive answer. In the above iterations we can see a pattern for the formulation of σ_l^ε. Consider the formulations of

σ_l^ε:

$$\sigma_l(t+1) = \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\zeta_1\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t) - N(t))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))^2$$

$$= \gamma_2\sigma_e^\varepsilon(t)+\zeta_1[e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t)-2\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t)-N(t))+\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2]$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1\Omega(t)$$

where

$$\Omega(t) = e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t) - N(t)) + \alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))^2.$$

Now we can restate the formulas for the various σ^ε as follows:

$$\sigma_e^\varepsilon(t+1) = \gamma_1\sigma_e^\varepsilon(t) + \theta_1\sigma_a^\varepsilon(t)$$

$$\sigma_l(t+1) = \gamma_2\sigma_e^\varepsilon(t) + \zeta_1\Omega(t)$$

$$\sigma_p(t+1) = \nu_1\sigma_p^\varepsilon(t) + \zeta_2\Omega(t)$$

$$\sigma_a^\varepsilon(t+1) = \nu_2\sigma_p^\varepsilon(t) + \theta_2\sigma_a^\varepsilon(t).$$

Using this formulation we can see how all the functions rely on $\Omega(t)$. For instance:

$$\sigma_e^\varepsilon(1) = \gamma_1\sigma_e^\varepsilon(0) + \theta_1\sigma_a^\varepsilon(0) = 0$$

$$\sigma_l(t+1) = \gamma_2\sigma_e^\varepsilon(0) + \zeta_1\Omega(0) = \zeta_1\Omega(0)$$

$$\sigma_p(t+1) = \nu_1\sigma_p^\varepsilon(0) + \zeta_2\Omega(0) = \zeta_2\Omega(0)$$

$$\sigma_a^\varepsilon(t+1) = \nu_2\sigma_p^\varepsilon(0) + \theta_2\sigma_a^\varepsilon(0) = 0$$

$$\sigma_e^\varepsilon(2) = \gamma_1\sigma_e^\varepsilon(1) + \theta_1\sigma_a^\varepsilon(1) = 0$$

$$\sigma_l(2) = \gamma_2\sigma_e^\varepsilon(1) + \zeta_1\Omega(1) = \zeta_1\Omega(1)$$

$$\sigma_p(2) = \nu_1\sigma_p^\varepsilon(1) + \zeta_2\Omega(1) = \nu_1\zeta_2\Omega(0) + \zeta_2\Omega(1) = \zeta_2\sum_{h=0}^{1}\nu_1^h\Omega(1-h)$$

$$\sigma_a^\varepsilon(2) = \nu_2\sigma_p^\varepsilon(1) + \theta_2\sigma_a^\varepsilon(1) = \nu_2\zeta_2\Omega(0)$$

$$\sigma_e^\varepsilon(3) = \gamma_1\sigma_e^\varepsilon(2) + \theta_1\sigma_a^\varepsilon(2) = \theta_1\nu_2\zeta_2\Omega(0)$$

$$\sigma_l(3) = \gamma_2\sigma_e^\varepsilon(2) + \zeta_1\Omega(2) = \zeta_1\Omega(2)$$

$$\sigma_p(3) = \nu_1\sigma_p^\varepsilon(2) + \zeta_2\Omega(2) = \nu_1^2\zeta_2\Omega(0) + \nu_1\zeta_2\Omega(1) + \zeta_2\Omega(2) = \zeta_2\sum_{h=0}^{2}\nu_1^h\Omega(2-h)$$

$$\sigma_a^\varepsilon(3) = \nu_2\sigma_p^\varepsilon(2) + \theta_2\sigma_a^\varepsilon(2) = \nu_2\nu_1\zeta_2\Omega(0) + \nu_2\zeta_2\Omega(1) + \theta_2\nu_2\zeta_2\Omega(0)$$

$$= \nu_2\zeta_2\sum_{h=0}^{1}\nu_1^h\Omega(1-h) + \theta_2\nu_2\zeta_2\sum_{h=0}^{0}\nu_1^h\Omega(0-h)$$

$$\sigma_e^\varepsilon(4) = \gamma_1\sigma_e^\varepsilon(3)+\theta_1\sigma_a^\varepsilon(3) = \gamma_1\theta_1\nu_2\zeta_2\Omega(0)+\theta_1[\nu_2\zeta_2\sum_{h=0}^{1}\nu_1^h\Omega(1-h)+\theta_2\nu_2\zeta_2\sum_{h=0}^{0}\nu_1^h\Omega(0-h)]$$

$$\sigma_l(4) = \gamma_2\sigma_e^\varepsilon(3) + \zeta_1\Omega(3) = \gamma_2\theta_1\nu_2\zeta_2\Omega(0) + \zeta_1\Omega(3)$$

$$\sigma_p(4) = \nu_1\sigma_p^\varepsilon(3) + \zeta_2\Omega(3) = \zeta_2\sum_{h=0}^{3}\nu_1^h\Omega(3-h)$$

$$\sigma_a^\varepsilon(4) = \nu_2\sigma_p^\varepsilon(3) + \theta_2\sigma_a^\varepsilon(3) =$$

$$= \left(\nu_2\zeta_2\sum_{h=0}^{2}\nu_1^h\Omega(2-h)\right) + \theta_2\left(\nu_2\zeta_2\sum_{h=0}^{1}\nu_1^h\Omega(1-h)\right) + \theta_2^2\left(\nu_2\zeta_2\sum_{h=0}^{0}\nu_1^h\Omega(0-h)\right)$$

$$= \nu_2\zeta_2 \sum_{g=0}^{2} \theta_2^{2-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)]$$

$$\sigma_e^\varepsilon(5) = \gamma_1\sigma_e^\varepsilon(4) + \theta_1\sigma_a^\varepsilon(4) = \gamma_1[\gamma_1\theta_1\nu_2\zeta_2\Omega(0) + \theta_1[\nu_2\zeta_2 \sum_{g=0}^{1} \theta_2^{1-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)]]$$

$$+\theta_1[\nu_2\zeta_2 \sum_{g=0}^{2} \theta_2^{2-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)]]$$

$$= \theta_1\nu_2\zeta_2 \left(\gamma_1^2\Omega(0) + \gamma_1 \sum_{g=0}^{1} \theta_2^{1-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] + \sum_{g=0}^{2} \theta_2^{2-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)] \right)$$

$$= \theta_1\nu_2\zeta_2 \left(\gamma_1^2 \sum_{g=0}^{0} \theta_2^{0-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] + \gamma_1 \sum_{g=0}^{1} \theta_2^{1-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] + \sum_{g=0}^{2} \theta_2^{2-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)] \right)$$

$$= \theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{2} \gamma_1^{2-d} \sum_{g=0}^{d} \theta_2^{d-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] \right)$$

$$\sigma_l(5) = \gamma_2\sigma_e^\varepsilon(4) + \zeta_1\Omega(4) = \gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{1} \gamma_1^{1-d} \sum_{g=0}^{d} \theta_2^{d-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] \right) + \zeta_1\Omega(4)$$

$$\sigma_p(5) = \nu_1\sigma_p^\varepsilon(4) + \zeta_2\Omega(4) = \zeta_2 \sum_{h=0}^{4} \nu_1^h \Omega(4-h)$$

$$\sigma_a^\varepsilon(5) = \nu_2\sigma_p^\varepsilon(4) + \theta_2\sigma_a^\varepsilon(4) = \nu_2\zeta_2 \sum_{g=0}^{3} \theta_2^{3-g} [\sum_{h=0}^{g} \nu_1^h \Omega(g-h)]$$

Thus, we can write for $t \geq 5$:

$$\sigma_e^\varepsilon(t+1) = \gamma_1\sigma_e^\varepsilon(t) + \theta_1\sigma_a^\varepsilon(t) = \theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{t-2} \gamma_1^{t-2-d} \sum_{g=0}^{d} \theta_2^{d-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] \right)$$

$$\sigma_l(t+1) = \gamma_2\sigma_e^\varepsilon(t) + \zeta_1\Omega(t) = \gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{t-3} \gamma_1^{t-3-d} \sum_{g=0}^{d} \theta_2^{d-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] \right) + \zeta_1\Omega(t)$$

$$\sigma_p(t+1) = \nu_1\sigma_p^\varepsilon(t) + \zeta_2\Omega(t) = \zeta_2 \sum_{h=0}^{t} \nu_1^h \Omega(t-h)$$

$$\sigma_a^\varepsilon(t+1) = \nu_2\sigma_p^\varepsilon(t) + \theta_2\sigma_a^\varepsilon(t) = \nu_2\zeta_2\sum_{g=0}^{t-1}\theta_2^{t-1-g}\left[\sum_{h=0}^{g}\nu_1^h\Omega(g-h)\right].$$

Hence, we have $\sigma_e^\varepsilon(t+1), \sigma_l^\varepsilon(t+1), \sigma_p^\varepsilon(t+1), \sigma_a^\varepsilon(t+1)$ defined as functions of model parameters and $\Omega(t)$. Recall,

$$\Omega(t) = e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t) - N(t)) + \alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))^2.$$

Our next step is to examine ψ_l^ε. We define

$$\omega(t) = e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t) - N(t))$$

then we have:

$$\psi_e^\varepsilon(t+1) = \gamma_1\psi_e^\varepsilon(t) + \theta_1\psi_a^\varepsilon(t)$$

$$\psi_l^\varepsilon(t+1) = \gamma_2\psi_e^\varepsilon(t) + \zeta_1\omega(t)$$

$$\psi_p^\varepsilon(t+1) = \nu_1\psi_p^\varepsilon(t) + \zeta_2\omega(t)$$

$$\psi_a^\varepsilon(t+1) = \nu_2\psi_p^\varepsilon(t) + \theta_2\psi_a^\varepsilon(t).$$

Using this formulation we can see how all ψ^ε function rely on $\omega(t)$. Note how the formulation of ψ^ε with ω looks similar to the formulation of σ^ε with Ω. Consider:

$$\psi_e^\varepsilon(1) = \gamma_1\psi_e^\varepsilon(0) + \theta_1\psi_a^\varepsilon(0) = 0$$

$$\psi_l^\varepsilon(1) = \gamma_2\psi_e^\varepsilon(0) + \zeta_1\omega(0) = \zeta_1\omega(0)$$

$$\psi_p^\varepsilon(1) = \nu_1\psi_p^\varepsilon(0) + \zeta_2\omega(0) = \zeta_2\omega(0)$$

$$\psi_a^\varepsilon(1) = \nu_2\psi_p^\varepsilon(0) + \theta_2\psi_a^\varepsilon(0) = 0$$

$$\psi_e^\varepsilon(2) = \gamma_1\psi_e^\varepsilon(1) + \theta_1\psi_a^\varepsilon(1) = 0$$

$$\psi_l^\varepsilon(2) = \gamma_2\psi_e^\varepsilon(1) + \zeta_1\omega(1) = \zeta_1\omega(1)$$

$$\psi_p^\varepsilon(2) = \nu_1\psi_p^\varepsilon(1) + \zeta_2\omega(1) = \nu_1\zeta_2\omega(0) + \zeta_2\omega(1) = \zeta_2\sum_{h=0}^{1}\nu_1^h\omega(1-h)$$

$$\psi_a^\varepsilon(2) = \nu_2\psi_p^\varepsilon(1) + \theta_2\psi_a^\varepsilon(1) = \nu_2\zeta_2\omega(0)$$

Using a similar method as we did with the σ^ε functions with Ω, we have for $t \geq 5$:

$$\psi_e^\varepsilon(t+1) = \gamma_1\psi_e^\varepsilon(t) + \theta_1\psi_a^\varepsilon(t) = \theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-2}\gamma_1^{t-2-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)$$

$$\psi_l(t+1) = \gamma_2\psi_e^\varepsilon(t)+\zeta_1\omega(t) = \gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-3}\gamma_1^{t-3-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)+\zeta_1\omega(t)$$

$$\psi_p(t+1) = \nu_1\psi_p^\varepsilon(t) + \zeta_2\omega(t) = \zeta_2\sum_{h=0}^{t}\nu_1^h\omega(t-h)$$

$$\psi_a^\varepsilon(t+1) = \nu_2\psi_p^\varepsilon(t) + \theta_2\psi_a^\varepsilon(t) = \nu_2\zeta_2\sum_{g=0}^{t-1}\theta_2^{t-1-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right].$$

We now consider values of $\omega(t)$, note we will only need to input ψ_l^ε.

$$\omega(0) = e^{-\alpha N^\varepsilon(0)}\psi_l^\varepsilon(0) - \alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0)) = -\alpha e^{-\alpha N^\varepsilon(0)}P_l^\varepsilon(0)(\eta(0)-N(0))$$

$$\omega(1) = e^{-\alpha N^\varepsilon(1)}\psi_l^\varepsilon(1) - \alpha e^{-\alpha N^\varepsilon(1)}P_l^\varepsilon(1)(\eta(1)-N(1))$$

$$= e^{-\alpha N^{\varepsilon}(1)}[\zeta_1 \omega(0)] - \alpha e^{-\alpha N^{\varepsilon}(1)} P_l^{\varepsilon}(1)(\eta(1) - N(1))$$

$$= -\zeta_1 \alpha e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(1))} P_l^{\varepsilon}(0)(\eta(0) - N(0)) - \alpha e^{-\alpha N^{\varepsilon}(1)} P_l^{\varepsilon}(1)(\eta(1) - N(1))$$

$$\omega(2) = e^{-\alpha N^{\varepsilon}(2)} \psi_l^{\varepsilon}(2) - \alpha e^{-\alpha N^{\varepsilon}(2)} P_l^{\varepsilon}(2)(\eta(2) - N(2))$$

$$= e^{-\alpha N^{\varepsilon}(2)}[\zeta_1 \omega(1)] - \alpha e^{-\alpha N^{\varepsilon}(2)} P_l^{\varepsilon}(2)(\eta(2) - N(2))$$

$$= -\zeta_1^2 \alpha e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(1)+N^{\varepsilon}(2))} P_l^{\varepsilon}(0)(\eta(0)-N(0)) - \zeta_1 \alpha e^{-\alpha(N^{\varepsilon}(1)+N^{\varepsilon}(2))} P_l^{\varepsilon}(1)(\eta(1)-N(1))$$

$$-\alpha e^{-\alpha N^{\varepsilon}(2)} P_l^{\varepsilon}(2)(\eta(2) - N(2))$$

$$= -\alpha \sum_{c=0}^{2} \zeta_1^{2-c} e^{-\alpha(\sum_{a=0}^{2-c} N^{\varepsilon}(a))} P_l^{\varepsilon}(c)(\eta(c) - N(c))$$

$$\omega(3) = e^{-\alpha N^{\varepsilon}(t)} \psi_l^{\varepsilon}(3) - \alpha e^{-\alpha N^{\varepsilon}(3)} P_l^{\varepsilon}(t)(\eta(3) - N(3))$$

$$= e^{-\alpha N^{\varepsilon}(t)}[\zeta_1 \omega(2)] - \alpha e^{-\alpha N^{\varepsilon}(3)} P_l^{\varepsilon}(t)(\eta(3) - N(3))$$

$$= -\alpha \sum_{c=0}^{3} \zeta_1^{3-c} e^{-\alpha(\sum_{a=0}^{3-c} N^{\varepsilon}(a))} P_l^{\varepsilon}(c)(\eta(c) - N(c))$$

$$\omega(4) = e^{-\alpha N^{\varepsilon}(4)} \psi_l^{\varepsilon}(4) - \alpha e^{-\alpha N^{\varepsilon}(4)} P_l^{\varepsilon}(4)(\eta(4) - N(4))$$

$$= e^{-\alpha N^{\varepsilon}(4)}[\gamma_2 \theta_1 \nu_2 \zeta_2 \omega(0) + \zeta_1 \omega(3)] - \alpha e^{-\alpha N^{\varepsilon}(4)} P_l^{\varepsilon}(4)(\eta(4) - N(4))$$

$$= e^{-\alpha N^{\varepsilon}(4)} \gamma_2 \theta_1 \nu_2 \zeta_2 \omega(0) - \alpha \sum_{c=0}^{4} \zeta_1^{4-c} e^{-\alpha(\sum_{a=0}^{4-c} N^{\varepsilon}(a))} P_l^{\varepsilon}(c)(\eta(c) - N(c))$$

$$= -\alpha \gamma_2 \theta_1 \nu_2 \zeta_2 \alpha e^{-\alpha(N^{\varepsilon}(0)+N^{\varepsilon}(4))} P_l^{\varepsilon}(0)(\eta(0) - N(0))]$$

$$-\alpha \sum_{c=0}^{4} \zeta_1^{4-c} e^{-\alpha(\sum_{a=0}^{4-c} N^{\varepsilon}(a))} P_l^{\varepsilon}(c)(\eta(c) - N(c))$$

$$\omega(5) = e^{-\alpha N^\varepsilon(5)}\psi_l^\varepsilon(5) - \alpha e^{-\alpha N^\varepsilon(5)} P_l^\varepsilon(5)(\eta(5) - N(5))$$

$$= e^{-\alpha N^\varepsilon(5)}[\gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{4-3}\gamma_1^{4-3-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right) + \zeta_1\omega(4)] - \alpha e^{-\alpha N^\varepsilon(5)} P_l^\varepsilon(5)(\eta(5)-N(5))$$

$$= e^{-\alpha N^\varepsilon(5)}\gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{1}\gamma_1^{1-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)$$

$$-\alpha\sum_{c=0}^{5}\zeta_1^{5-c}e^{-\alpha(\sum_{a=0}^{5-c}N^\varepsilon(a))}P_l^\varepsilon(c)(\eta(c) - N(c)$$

$$\omega(t) = e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t) - \alpha e^{-\alpha N^\varepsilon(t)} P_l^\varepsilon(t)(\eta(t) - N(t))$$

$$= e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{t-1-3}\gamma_1^{t-1-3-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)$$

$$-\alpha\sum_{c=0}^{t}\zeta_1^{t-c}e^{-\alpha(\sum_{a=0}^{t-c}N^\varepsilon(a))}P_l^\varepsilon(c)(\eta(c) - N(c)$$

$$= e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2 \left(\sum_{d=0}^{t-1-3}\gamma_1^{t-1-3-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)$$

$$-\alpha\sum_{c=0}^{t}\zeta_1^{t-c}e^{-\alpha(\sum_{a=0}^{t-c}N^\varepsilon(a))}P_l^\varepsilon(c)(\eta(c) - N(c)$$

We can similarly study $\Omega(t)$:

$$\Omega(0) = e^{-\alpha N^\varepsilon(0)}\sigma_l^\varepsilon(0) - 2\alpha e^{-\alpha N^\varepsilon(0)}\psi_l^\varepsilon(0)(\eta(0) - N(0)) + \alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))^2$$

$$= \alpha^2 e^{-\alpha N^\varepsilon(0)} P_l^\varepsilon(0)(\eta(0) - N(0))^2.$$

We have already calculated the values of $\Omega(t)$ for $t = 1, 2, 3, 4, 5$ since we have calculated $\sigma_l^\varepsilon(t)$ for $t = 1,2,3,4,5$, and recall $\sigma_l^\varepsilon(t) = \gamma_2\sigma_e^\varepsilon(t) + \zeta_1\Omega(t)$. In these equations we found we could group terms by common parameters and then simplify the associated $(\eta(i) - N(i))$ terms into a sum which is squared. Below we have a formula for $\Omega(t)$ in terms of parameters, $\omega(t)$, and $\Omega(t)$.

$$\Omega(t) = e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t) - 2\alpha e^{-\alpha N^\varepsilon(t)}\psi_l^\varepsilon(t)(\eta(t)-N(t)) + \alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$= e^{-\alpha N^\varepsilon(t)}\sigma_l^\varepsilon(t)$$

$$-2\alpha e^{-\alpha N^\varepsilon(t)}\left[\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)+\zeta_1\omega(t-1)\right](\eta(t)-N(t))$$

$$+\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$= e^{-\alpha N^\varepsilon(t)}\left[\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\Omega(g-h)\right]\right)+\zeta_1\Omega(t-1)\right]$$

$$-2\alpha e^{-\alpha N^\varepsilon(t)}\left[\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)+\zeta_1\omega(t-1)\right](\eta(t)-N(t))$$

$$+\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$= e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\Omega(g-h)\right]\right)+e^{-\alpha N^\varepsilon(t)}\zeta_1\Omega(t-1)$$

$$-2\alpha e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)(\eta(t)-N(t))$$

$$-2\alpha e^{-\alpha N^\varepsilon(t)}\zeta_1\omega(t-1)(\eta(t)-N(t)) + \alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$= e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\Omega(g-h)\right]\right)$$

$$-2\alpha e^{-\alpha N^\varepsilon(t)}\gamma_2\theta_1\nu_2\zeta_2\left(\sum_{d=0}^{t-4}\gamma_1^{t-4-d}\sum_{g=0}^{d}\theta_2^{d-g}\left[\sum_{h=0}^{g}\nu_1^h\omega(g-h)\right]\right)(\eta(t)-N(t))$$

$$+e^{-\alpha N^\varepsilon(t)}\zeta_1[\Omega(t-1) - 2\alpha\omega(t-1)(\eta(t)-N(t))]$$

$$+\alpha^2 e^{-\alpha N^\varepsilon(t)}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

We simplify this as we did in the $t = 5$ case. Note how both the summations will result in the same parameters associated with the values of Ω and ω. The last term, $\alpha^2 e^{-\alpha N^{\varepsilon}(t)} P_l^{\varepsilon}(t)(\eta(t) - N(t))^2$, will be expanded by using the formula for $P_l^{\varepsilon}(t)$. By expanding $P_l^{\varepsilon}(t)$ we will have various terms multiplied by $(\eta(t) - N(t))^2$ which will aid in forming summations of $(\eta(i) - N(i))$ which are squared. Meanwhile, the other terms will result in the other various $(\eta(i) - N(i))$, and since the summations parameters match we will be able to group appropriate terms. Note from above we do know that every $\omega(t)$ has a negative throughout the term, and this will allow us to switch the sign on the second summation and term $-2\alpha e^{-\alpha N^{\varepsilon}(t)} \zeta_1 \omega(t-1)(\eta(t) - N(t))$. The result will be the summations of $(\eta(i) - N(i))$ squared multiplied by associated parameters, making $\Omega(t) > 0$.

If we have that $\Omega(t) > 0$, then

$$\sigma_e^{\varepsilon}(t+1) = \gamma_1 \sigma_e^{\varepsilon}(t) + \theta_1 \sigma_a^{\varepsilon}(t) = \theta_1 \nu_2 \zeta_2 \left(\sum_{d=0}^{t-2} \gamma_1^{t-2-d} \sum_{g=0}^{d} \theta_2^{d-g} \left[\sum_{h=0}^{g} \nu_1^h \Omega(g-h) \right] \right) > 0.$$

Hence, we have that $\sigma_l^{\varepsilon}(t) = \gamma_2 \sigma_e^{\varepsilon}(t) + \zeta_1 \Omega(t) > 0$

Thus, we have that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^{\varepsilon}(t) P_l^{\varepsilon}(t) + \psi_l^{\varepsilon}(t)^2] > 0$. This establishes the convexity of J, which guarantees uniqueness of the minimum.

□

Chapter 3

Biological Control Persists

To make the model closer to reality, we decided to include in our model that control agents can persist for some time in the environment, and will not necessarily die off after one time step. Our goal is to minimize the objective functional which incorporates the cost functional that allows for the control agent to decay over several time steps. To account for this decay we let N_n be the new control agents being added to the field by the farmer and N_o be the decayed control from previous time steps.

3.1 Updated Model

We have the objective functional using the same cost function as previously but now purchasing new control agents, N_n.

$$J(N_n) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N_n(t)$$

Pest Dynamics with the control agents applied to the second, larva, stage and the possible survival of the control agents serves as a constraint to the optimization problem, including now the old control agents, N_o.

$$
\begin{aligned}
P_e(t+1) &= \gamma_1 P_e(t) + \theta_1 P_a(t) && P_e(0) = \phi_e \\
P_l(t+1) &= \gamma_2 P_e(t) + \zeta_1 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) && P_l(0) = \phi_l \\
P_p(t+1) &= \zeta_2 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) + \nu_1 P_p(t) && P_p(0) = \phi_p \\
P_a(t+1) &= \nu_2 P_p(t) + \theta_2 P_a(t) && P_a(0) = \phi_a \\
N_o(t+1) &= N_o(t)e^{-\mu} + N_n(t) && N_o(0) = 0
\end{aligned}
$$

Furthermore the control $N_n(t) \geq 0$, since we cannot add a negative quantity of nematodes, which also bounds $N_o(t)$.

We assume exponential decay models control agents survival, based on control agents life expectancy. Specifically, μ determines the rate of decay. When we use large values for μ the $N_o(t)e^{-\mu}$ term approaches zero and we have results that resemble the basic model, reflecting the control agents surviving for less time. For instance, if $\mu = \ln(2)$ we have,

$$
N_o(t+1) = N_o(t)e^{-\ln(2)} + N_n(t) = \frac{1}{2}N_o(t) + N_n(t)
$$

meaning half the old control agent are active from one time step to the next.

3.2 Optimal Control Problem

The goal for our Optimal Control Problem is to minimize the objective functional

$$
J(N_n) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N_n(t)
$$

subject to

$$
\begin{aligned}
P_e(t+1) &= \gamma_1 P_e(t) + \theta_1 P_a(t) & P_e(0) &= \phi_e \\
P_l(t+1) &= \gamma_2 P_e(t) + \zeta_1 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) & P_l(0) &= \phi_l \\
P_p(t+1) &= \zeta_2 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) + \nu_1 P_p(t) & P_p(0) &= \phi_p \\
P_a(t+1) &= \nu_2 P_p(t) + \theta_2 P_a(t) & P_a(0) &= \phi_a \\
N_o(t+1) &= N_o(t)e^{-\mu} + N_n(t) & N_o(0) &= 0
\end{aligned}
\tag{3.1}
$$

where $N_n(t) \geq 0$ for all t and $N_n \in \mathbf{N} = \{N : \{1, ..., T\} \to \{x \in \mathbb{R} | 0 \leq x(t) \leq N_{max}, t = 1, 2, ..., T\}\}$.

Again we will prove the existence and uniqueness of the optimal control, which we denote $\mathcal{N}_n$. Additionally, we will prove necessary conditions for the optimal control $\mathcal{N}_n$. The proofs roughly follow the proofs in Theorems 2.3.1, 2.3.2, 2.3.3.

Note in the following proofs each $\mathcal{P}_e, \mathcal{P}_l, \mathcal{P}_p, \mathcal{P}_a, \mathcal{N}_o$ is a function of $\mathcal{N}_n$. Similarly each $\mathcal{P}_e^\varepsilon, \mathcal{P}_l^\varepsilon, \mathcal{P}_p^\varepsilon, \mathcal{P}_a^\varepsilon, \mathcal{N}_o^\varepsilon$ is a function of $\mathcal{N}_n + \eta\varepsilon$.

3.2.1 Existence

Theorem 3.2.1. *There exists $\mathcal{N}_n \in \boldsymbol{N}$ which minimizes $J(N_n)$.*

Proof. Each P_e, P_l, P_p, P_a, N_o is continuous as a function of N_n for every time step by Equation 3.1. Define $B^+ = \{N(1), ...N(T) | N \in \mathbf{N}\}$. We note that there is a natural isomorphism between $\mathbf{N}$ and B^+. Consideing $J : \mathbf{N} \Leftrightarrow B^+ \to \mathbb{R}$, we see that J is continuous as a function of N_n. We have that B^+ is a compact subset of $\mathbb{R}^T$ in the standard

Euclidean topology. Therefore $\inf_{N_n \in \mathbf{N}} J(N_n)$ exists. Hence, we have a sequence $N_{n_k} \in \mathbf{N}$ such that $\lim_{k\to\infty} J(N_{n_k}) = \inf_{N_n\in\mathbf{N}} J(N_n)$, with corresponding $P_{e_k}, P_{l_k}, P_{p_k}, P_{a_k}, N_{o_k}$ sequences. Thus we can find subsequences $N_{n_{k_j}}, P_{e_{k_j}}, P_{l_{k_j}}, P_{p_{k_j}}, P_{a_{k_j}}, N_{o_{k_j}}$, such that $\lim_{j\to\infty} J(N_{n_{k_j}}) = \inf_{N_n\in\mathbf{N}} J(N_n)$, $N_{n_{k_j}} \to \mathcal{N}_n, P_{e_{k_j}} \to \mathcal{P}_e, P_{l_{k_j}} \to \mathcal{P}_l, P_{p_{k_j}} \to \mathcal{P}_p, P_{a_{k_j}} \to \mathcal{P}_a, N_{o_{k_j}} \to \mathcal{N}_o$. Therefore, there exists $\mathcal{N}_n \in \mathbf{N}$ which minimizes $J(N_n)$.

□

3.2.2 Necessary Conditions

Adjoint System: Define the following terminal value system:

$$\lambda_e(t) = \lambda_e(t+1)\gamma_1 + \lambda_l(t+1)\gamma_2$$

$$\lambda_l(t) = 2\beta_1\mathcal{P}_l(t) + \lambda_l(t+1)\zeta_1 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} + \lambda_p(t+1)\zeta_2 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}$$

$$\lambda_p(t) = \lambda_p(t+1)\nu_1 + \lambda_a(t+1)\nu_2$$

$$\lambda_a(t) = \lambda_e(t+1)\theta_1 + \lambda_a(t+1)\theta_2$$

$$\lambda_o(t) = -\alpha\zeta_1 e^{-\mu}\lambda_l(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k) - \alpha\zeta_2 e^{-\mu}\lambda_p(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k)$$
$$+\lambda_o(k+1)e^{-\mu}$$

$$\lambda_e(T) = 0,\ \lambda_l(T) = 0,\ \lambda_p(T) = 0,\ \lambda_a(T) = 0.$$

Theorem 3.2.2. *If there exists an optimal control $\mathcal{N}_n$, then there exists and adjoint system 3.2.2, and*

$$\mathcal{N}_n(t) = \begin{cases} 0 & if\ e^{\alpha \mathcal{N}_o(t) e^{-\mu}} > \xi_n(t) \\ \frac{1}{\alpha}\ln(\xi_n) - \mathcal{N}_o(t)e^{-\mu} & if\ e^{\alpha \mathcal{N}_o(t) e^{-\mu}} \le \xi_n(t) \end{cases}.$$

with $\xi_n(t) = \frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)}$

Proof. Since we have that $\mathcal{N}_n$ minimizes $J(N_n)$; for all sufficiently small $\varepsilon > 0$ and for all $\eta \in \{\eta = (\eta(1), ..., \eta(T)) | \eta(t) \le 1, t = 1, ..., T\}$ we have that $J(\mathcal{N}_n + \eta\varepsilon) \ge J(\mathcal{N}_n)$. Now we will take a directional derivative of functional J; so for the directional derivative in the direction of η with sufficiently small $\varepsilon > 0$ and $0 \le \mathcal{N}_n + \eta\varepsilon = \mathcal{N}_n^\varepsilon \in \mathbf{N}$ we have that:

$$0 \le \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N}_n + \eta\varepsilon) - J(\mathcal{N}_n)] =$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}\left[\sum_{t=0}^{T-1} \beta_1 \mathcal{P}_l^\varepsilon(t)^2 + \beta_2 \mathcal{N}_n^\varepsilon(t) - \sum_{t=0}^{T-1} \beta_1 \mathcal{P}_l(t)^2 + \beta_2 \mathcal{N}_n(t)\right]$$

$$= \sum_{t=0}^{T-1}\left[\beta_1 \lim_{\varepsilon \to 0^+} \frac{\mathcal{P}_l^\varepsilon(t)^2 - \mathcal{P}_l(t)^2}{\varepsilon} + \beta_2 \lim_{\varepsilon \to 0^+} \frac{\mathcal{N}_n^\varepsilon(t) - \mathcal{N}_n(t)}{\varepsilon}\right] = \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) + \beta_2\eta(t).$$

Where we will define the sensitivities $\psi_e(t), \psi_l(t), \psi_p(t), \psi_a(t), \psi_o(t)$ as:

$$\psi_e(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_e^\varepsilon(t) - \mathcal{P}_e(t)}{\varepsilon}, \quad \psi_l(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_l^\varepsilon(t) - \mathcal{P}_l(t)}{\varepsilon}, \quad \psi_p(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_p^\varepsilon(t) - \mathcal{P}_p(t)}{\varepsilon},$$

$$\psi_a(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{P}_a^\varepsilon(t) - \mathcal{P}_a(t)}{\varepsilon}, \quad \psi_o(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{N}_o^\varepsilon(t) - \mathcal{N}_o(t)}{\varepsilon}$$

where $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$, $\psi_o(0) = 0$. We have the limits exists from Chapter 23 in Optimal Control Applied to Biological Models [LW07]

Hence, we can write:

$$\psi_e(t+1) = \gamma_1\psi_e(t) + \theta_1\psi_a(t)$$

$$\psi_l(t+1) = \gamma_2\psi_e(t) + \zeta_1 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\psi_l(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\psi_o(t)\mathcal{P}_l(t)$$

$$-\zeta_1\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)$$

$$\psi_p(t+1) = \nu_1\psi_p(t) + \zeta_2 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\psi_l(t) - \zeta_2\alpha e^{-\mu}e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\psi_o(t)\mathcal{P}_l(t)$$

$$-\zeta_2\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)$$

$$\psi_a(t+1) = \nu_2\psi_p(t) + \theta_2\psi_a(t)$$

$$\psi_o(t+1) = \psi_o(t)e^{-\mu} + \eta(t).$$

Now, returning to $0 \le \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}[J(\mathcal{N}_n+\eta\varepsilon) - J(\mathcal{N}_n)] = \sum_{t=0}^{T-1}\beta_1 2\mathcal{P}_l(t)\psi_l(t) + \beta_2\eta(t)$. To remove the sensitivities $\psi_l(t)$ we will manipulate the sensitivities and adjoints equations.

We have that:

$$\begin{bmatrix}\psi_e(t+1)\\ \psi_l(t+1)\\ \psi_p(t+1)\\ \psi_a(t+1)\\ \psi_o(t+1)\end{bmatrix} - B\begin{bmatrix}\psi_e(t)\\ \psi_l(t)\\ \psi_p(t)\\ \psi_a(t)\\ \psi_o(t)\end{bmatrix} = \begin{bmatrix}0\\ -\zeta_1\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\\ -\zeta_2\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\\ 0\\ \eta(t)\end{bmatrix}$$

where $B = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 & 0 \\ \gamma_2 & \zeta_1 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} & 0 & 0 & -\zeta_1\alpha e^{-\mu}e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t) \\ 0 & \zeta_2 e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} & \nu_1 & 0 & -\zeta_2\alpha e^{-\mu}e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t) \\ 0 & 0 & \nu_2 & \theta_2 & 0 \\ 0 & 0 & 0 & 0 & e^{-\mu} \end{bmatrix}$.

Now we have that:

$$\sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) = \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) & \psi_o(t) \end{bmatrix} \begin{bmatrix} 0 \\ \beta_1 2\mathcal{P}_l(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) & \psi_o(t) \end{bmatrix} \left(\begin{bmatrix} \lambda_e(t) \\ \lambda_l(t) \\ \lambda_p(t) \\ \lambda_a(t) \\ \lambda_o(t) \end{bmatrix} - B^T \begin{bmatrix} \lambda_e(t+1) \\ \lambda_l(t+1) \\ \lambda_p(t+1) \\ \lambda_a(t+1) \\ \lambda_o(t+1) \end{bmatrix} \right)$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) & \psi_o(t) \end{bmatrix} \begin{bmatrix} \lambda_e(t) \\ \lambda_l(t) \\ \lambda_p(t) \\ \lambda_a(t) \\ \lambda_o(t) \end{bmatrix}$$

$$-\sum_{t=0}^{T-1}\begin{bmatrix}\psi_e(t) & \psi_l(t) & \psi_p(t) & \psi_a(t) & \psi_o(t)\end{bmatrix} B^T\begin{bmatrix}\lambda_e(t+1)\\ \lambda_l(t+1)\\ \lambda_p(t+1)\\ \lambda_a(t+1)\\ \lambda_o(t+1)\end{bmatrix}.$$

Recall that $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$, $\psi_o(0) = 0$ and $\lambda_e(T) = 0$, $\lambda_l(T) = 0$, $\lambda_p(T) = 0$, $\lambda_a(T) = 0$, $\lambda_o(T) = 0$. Therefore we can change the indices, so that:

$$\sum_{t=0}^{T-1}\beta_1 2\mathcal{P}_l(t)\psi_l(t) = \sum_{t=0}^{T-1}\begin{bmatrix}\psi_e(t+1) & \psi_l(t+1) & \psi_p(t+1) & \psi_a(t+1) & \psi_o(t+1)\end{bmatrix}\begin{bmatrix}\lambda_e(t+1)\\ \lambda_l(t+1)\\ \lambda_p(t+1)\\ \lambda_a(t+1)\\ \lambda_o(t+1)\end{bmatrix}$$

$$-\sum_{t=0}^{T-1}\begin{bmatrix}\lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) & \lambda_o(t+1)\end{bmatrix} B\begin{bmatrix}\psi_e(t)\\ \psi_l(t)\\ \psi_p(t)\\ \psi_a(t)\\ \psi_o(t)\end{bmatrix}$$

$$=\sum_{t=0}^{T-1}\begin{bmatrix}\lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) & \lambda_o(t+1)\end{bmatrix}\left(\begin{bmatrix}\psi_e(t+1)\\ \psi_l(t+1)\\ \psi_p(t+1)\\ \psi_a(t+1)\\ \psi_o(t+1)\end{bmatrix} - B\begin{bmatrix}\psi_e(t)\\ \psi_l(t)\\ \psi_p(t)\\ \psi_a(t)\\ \psi_o(t)\end{bmatrix}\right)$$

$$=$$

$$\sum_{t=0}^{T-1}\begin{bmatrix}\lambda_e(t+1) & \lambda_l(t+1) & \lambda_p(t+1) & \lambda_a(t+1) & \lambda_o(t+1)\end{bmatrix}\begin{bmatrix}0\\ -\zeta_1\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\\ -\zeta_2\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\\ 0\\ \eta(t)\end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \lambda_l(t+1)\left(-\zeta_1 \alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\right)$$
$$+\lambda_p(t+1)\left(-\zeta_2 \alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\right) + \lambda_o(t+1)\eta(t)$$
$$= \sum_{t=0}^{T-1} [\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]\left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)\eta(t)\right] + \lambda_o(t+1)\eta(t)$$
$$= \sum_{t=0}^{T-1} \left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1)\right]\eta(t).$$

Now combining everything we have that:

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N}_n + \eta\varepsilon) - J(\mathcal{N}_n)] = \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_l(t)\psi_l(t) + \beta_2\eta(t)$$

$$= \sum_{t=0}^{T-1} \left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1)\right]\eta(t) + \beta_2\eta(t)$$

$$= \sum_{t=0}^{T-1} \eta(t)\left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1) + \beta_2\right].$$

Considering the previous equation with equality

$$0 = \sum_{t=0}^{T-1} \eta(t)\left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1) + \beta_2\right]$$

for all $\eta(t) \in \{\eta = (\eta(1), ..., \eta(T)) | \eta(t) \leq 1, t = 1, ..., T\}$. The we have that for all t,

$$0 = -\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1) + \beta_2$$

Consider:

$$0 = -\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))}\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1) + \beta_2 \iff$$

$$\frac{\lambda_o(t+1) + \beta_2}{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2)} = e^{-\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t))} \iff$$

$$\ln\left[\frac{\lambda_o(t+1)+\beta_2}{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}\right] = -\alpha(\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t)) \Longleftrightarrow$$

$$\mathcal{N}_o(t)e^{-\mu}+\mathcal{N}_n(t) = \frac{1}{\alpha}\ln\left[\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2}\right] \Longleftrightarrow$$

$$\mathcal{N}_n(t) = \frac{1}{\alpha}\ln\left[\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2}\right] - \mathcal{N}_o(t)e^{-\mu}.$$

Note that $\alpha > 0$. We need that $\mathcal{N}_n(t) \geq 0$, so

$$\frac{1}{\alpha}\ln\left[\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2}\right] - \mathcal{N}_o(t)e^{-\mu} \geq 0 \Longleftrightarrow$$

$$\ln\left[\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2}\right] \geq \alpha\mathcal{N}_o(t)e^{-\mu} \Longleftrightarrow$$

$$\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2} \geq e^{\alpha\mathcal{N}_o(t)e^{-\mu}}.$$

Hence if

$$\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2} \geq e^{\alpha\mathcal{N}_o(t)e^{-\mu}},$$

then we have

$$\mathcal{N}_n(t) = \frac{1}{\alpha}\ln[\frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}] - \mathcal{N}_o(t)e^{-\mu}.$$

Now we will consider if

$$\frac{\alpha\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2)}{\lambda_o(t+1)+\beta_2} < e^{\alpha\mathcal{N}_o(t)e^{-\mu}},$$

then we have

$$\mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2) < \frac{1}{\alpha}e^{\alpha\mathcal{N}_o(t)e^{-\mu}}[\lambda_o(t+1)+\beta_2].$$

Returning to:

$$0 = \sum_{t=0}^{T-1} \eta(t) \left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu} + \mathcal{N}_n(t))} \mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2) + \lambda_o(t+1) + \beta_2 \right]$$

$$< \sum_{t=0}^{T-1} \eta(t) \left[-\alpha e^{-\alpha(\mathcal{N}_o(t)e^{-\mu} + \mathcal{N}_n(t))} [\frac{1}{\alpha} e^{\alpha \mathcal{N}_o(t)e^{-\mu}} (\lambda_o(t+1) + \beta_2)] + \lambda_o(t+1) + \beta_2 \right]$$

$$= \sum_{t=0}^{T-1} \eta(t) \left[-e^{-\alpha \mathcal{N}_n(t)} [(\lambda_o(t+1) + \beta_2)] + \lambda_o(t+1) + \beta_2 \right]$$

$$= \sum_{t=0}^{T-1} \eta(t) \left[\lambda_o(t+1) - e^{-\alpha \mathcal{N}_n(t)} \lambda_o(t+1) + \beta_2 - e^{-\alpha \mathcal{N}_n(t)} \beta_2 \right].$$

Hence we have if

$$\frac{\alpha \mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2)}{\lambda_o(t+1) + \beta_2} < e^{\alpha \mathcal{N}_o(t)e^{-\mu}}$$

then

$$0 < \sum_{t=0}^{T-1} \eta(t)[\lambda_o(t+1) - e^{-\alpha \mathcal{N}_n(t)} \lambda_o(t+1) + \beta_2 - e^{-\alpha \mathcal{N}_n(t)} \beta_2]$$

$$= \eta(t)[\lambda_o(t+1)(1 - e^{-\alpha \mathcal{N}_n(t)}) + \beta_2(1 - e^{-\alpha \mathcal{N}_n(t)})].$$

Recall we have that $\mathcal{N}_n(t) \geq 0$.

If $\mathcal{N}_n(t) > 0$ we have that

$$\lambda_o(t+1)(1 - e^{-\alpha \mathcal{N}_n(t)}) + \beta_2(1 - e^{-\alpha \mathcal{N}_n(t)}) < 0,$$

which is a contradiction. Thus, if

$$\frac{\alpha \mathcal{P}_l(t)(\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2)}{\lambda_o(t+1) + \beta_2} < e^{\alpha \mathcal{N}_o(t)e^{-\mu}}$$

we must have that $\mathcal{N}_n(t) = 0$. Set $\xi_n(t) = \frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)}$

$$\mathcal{N}_n(t) = \begin{cases} 0 & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} > \xi_n(t) \\ \frac{1}{\alpha}\ln(\xi_n) - \mathcal{N}_o(t)e^{-\mu} & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} \leq \xi_n(t) \end{cases}.$$

□

3.2.2.1 Comparing Basic and Persist Model Equations

Back in Theorem 2.3.2 we established:

$$\mathcal{N}(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]] & \text{if } \frac{\beta 2}{\alpha} \leq \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] \end{cases}.$$

From Theorem 3.2.2 we established:

$$\mathcal{N}_n(t) = \begin{cases} 0 & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} > \frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)} \\ \frac{1}{\alpha}\ln[\frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)}] - \mathcal{N}_o(t)e^{-\mu} & \text{if } e^{\alpha \mathcal{N}_o(t) e^{-\mu}} \leq \frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)} \end{cases}.$$

We want to see if the Persist model will reduce to the Basic model if we reduce the time that control persists, meaning $\mu \to \infty$. Note if we take $\mu \to \infty$ in $P_e, P_l, P_p, P_a, \lambda_e, \lambda_l, \lambda_p, \lambda_a$ from Theorem 3.2.2 we have the same equations from Theorem 2.3.2. First consider N_o and λ_o.

$$\lim_{\mu\to\infty} N_o(t+1) = \lim_{\mu\to\infty} [N_o(t)e^{-\mu} + N_n(t)] = N_n(t)$$

$$\lim_{\mu\to\infty} \lambda_o(t) = \lim_{\mu\to\infty} [-\alpha\zeta_1 e^{-\mu}\lambda_l(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k)]$$

$$+\lim_{\mu\to\infty}[-\alpha\zeta_2 e^{-\mu}\lambda_p(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k)+\lambda_o(k+1)e^{-\mu}]=0$$

Therefore, we have that $N_o(t+1) = N_n(t)$ and $\lambda_o(t) = 0$, which relates to the Basic model. Firstly, $N_o(t+1) = N_n(t)$ states that the only old control is the new control from the previous step, no old control survives. While $\lambda_o(t) = 0$ eliminates the old control from the process, since it doesn't exist as a factor in the basic model.

Now we will look at the equation for $\mathcal{N}_n(t)$ and take the limit of $\mu \to \infty$. First consider when $e^{\alpha\mathcal{N}_o(t)e^{-\mu}} > \frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}$ then $\mathcal{N}_n(t) = 0$. So taking $\mu \to \infty$:

$$e^0 > \frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+0} \iff$$

$$1 > \frac{\alpha}{\beta_2}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2] \iff$$

$$\frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2].$$

So as $\mu \to \infty$, if

$$\frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]$$

then $\mathcal{N}_n(t) = 0$. This is the same as with $\mathcal{N}(t)$.

Next consider when

$$e^{\alpha\mathcal{N}_o(t)e^{-\mu}} \leq \frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}$$

then

$$\mathcal{N}_n(t) = \frac{1}{\alpha}\ln[\frac{\alpha\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}] - \mathcal{N}_o(t)e^{-\mu}.$$

So taking $\mu \to \infty$ for the first part we have:

$$e^0 \le \frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + 0} \Longleftrightarrow$$

$$\frac{\beta_2}{\alpha} \le \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]$$

then we have that:

$$\begin{aligned}\mathcal{N}_n(t) &= \frac{1}{\alpha}\ln[\frac{\alpha \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + 0}] - \mathcal{N}_o(t)\cdot 0 \\ &= \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2].\end{aligned}$$

So as $\mu \to \infty$:

$$\mathcal{N}_n(t) = \begin{cases} 0 & \text{if } \frac{\beta_2}{\alpha} > \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]] & \text{if } \frac{\beta_2}{\alpha} \le \mathcal{P}_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] \end{cases}$$

which is the same as the Basic model.

3.2.3 Uniqueness

The following proof is similar to the proof of Theorem 2.3.3. Differences occur in the additional variables associated with considering old Nematodes, $\mathcal{N}_o$, which affects

Theorem 3.2.3. *Uniqueness: If the optimal control $\mathcal{N}_n$ exists, then it is unique.*

Proof. In order to show $\mathcal{N}_n$ is unique we will show that $J(N_n) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N_n(t)$ is strictly convex. Recall that if a function is strictly convex then there exists a unique minimum such that $J(\mathcal{N}_n) < J(N_n)$ for all $N_n \in \mathbf{N}_n \setminus \mathcal{N}_n$. To show tha J is strictly convex we will look at J along a line segment from N_n to η by defining

$z(\varepsilon) = J((1-\varepsilon)N_n + \varepsilon\eta) = J(N_n + \varepsilon(\eta - N_n))$ for $N_n, \eta \in \mathbf{N}_n$, and $0 < \varepsilon < 1$. Note that if z, a one dimensional function, is convex in every possible direction then J will be convex. To establish convexity we will show that $z''(\varepsilon) > 0$. First take the derivative of z:

$$z'(\varepsilon) = \lim_{\tau \to 0} \left(\frac{J(N_n + (\tau + \varepsilon)(\eta - N_n)) - J(N_n + \varepsilon(\eta - N_n))}{\tau} \right) =$$

$$= \lim_{\tau \to 0} \sum_{t=0}^{T-1} \frac{\beta_1}{\tau} \left[P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2 \right]$$

$$+ \frac{\beta_2}{\tau} \Big([N_n(t) + (\tau + \varepsilon)(\eta(t) - N_n(t))] - [N_n(t) + \varepsilon(\eta(t) - N_n(t))] \Big)$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2}{\tau} \right] + \beta_2 \left[\lim_{\tau \to 0} \frac{\tau(\eta(t) - N_n(t))}{\tau} \right]$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_l^{\tau+\varepsilon}(t)^2 - P_l^{\varepsilon}(t)^2}{\tau} \right] + \beta_2(\eta(t) - N_n(t)).$$

By The Chain Rule:

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2 P_l^{\varepsilon}(t) \psi_l^{\varepsilon}(t) + \beta_2(\eta(t) - N_n(t)).$$

Note we define sensitivities similar to in Theorem 3.2.2:

$$\psi_e^{\varepsilon}(t+1) = \gamma_1 \psi_e^{\varepsilon}(t) + \theta_1 \psi_a^{\varepsilon}(t)$$

$$\psi_l^{\varepsilon}(t+1) = \gamma_2 \psi_e^{\varepsilon}(t) + \zeta_1 e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu} + N_n^{\varepsilon}(t))} \psi_l^{\varepsilon}(t) - \zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu} + N_n^{\varepsilon}(t))} \psi_o^{\varepsilon}(t) P_l^{\varepsilon}(t)$$

$$-\zeta_1 \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon(t)(\eta(t)-N_n(t))$$

$$\psi_p^\varepsilon(t+1) = \nu_1 \psi_p^\varepsilon(t) + \zeta_2 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} \psi_l^\varepsilon(t) - \zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} \psi_o^\varepsilon(t) P_l^\varepsilon(t)$$

$$-\zeta_2 \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon(t)(\eta(t)-N_n(t))$$

$$\psi_a^\varepsilon(t+1) = \nu_2 \psi_p^\varepsilon(t) + \theta_2 \psi_a^\varepsilon(t)$$

$$\psi_o^\varepsilon(t+1) = \psi_o^\varepsilon(t) e^{-\mu} + (\eta(t)-N_n(t))$$

where $\psi_e(0) = 0$, $\psi_l(0) = 0$, $\psi_p(0) = 0$, $\psi_a(0) = 0$, $\psi_o(0) = 0$.

In order to continue we must define derivatives of sensitives, $\sigma_e(t), \sigma_l(t), \sigma_p(t), \sigma_a(t), \sigma_o(t)$ as:

$$\sigma_e^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_e^{\tau+\varepsilon}(t+1) - \psi_e^\varepsilon(t+1)}{\tau}, \quad \sigma_l^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_l^{\tau+\varepsilon}(t+1) - \psi_l^\varepsilon(t+1)}{\tau},$$

$$\sigma_p^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_p^{\tau+\varepsilon}(t+1) - \psi_p^\varepsilon(t+1)}{\tau}, \quad \sigma_a^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_a^{\tau+\varepsilon}(t+1) - \psi_a^\varepsilon(t+1)}{\tau},$$

$$\sigma_o^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_o^{\tau+\varepsilon}(t+1) - \psi_o^\varepsilon(t+1)}{\tau}.$$

Hence, we can write:

$$\sigma_e^\varepsilon(t+1) = \lim_{\tau\to 0} \frac{\psi_e^{\tau+\varepsilon}(t+1) - \psi_e^\varepsilon(t+1)}{\tau} = \gamma_1 \lim_{\tau\to 0} \frac{\psi_e^{\tau+\varepsilon}(t) - \psi_e^\varepsilon(t)}{\tau} + \theta_1 \lim_{\tau\to 0} \frac{\psi_a^{\tau+\varepsilon}(t) - \psi_a^\varepsilon(t)}{\tau}$$

$$= \gamma_1 \sigma_e^{\varepsilon}(t) + \theta_1 \sigma_a^{\varepsilon}(t)$$

$$\sigma_a^{\varepsilon}(t+1) = \lim_{\tau \to 0} \frac{\psi_a^{\tau+\varepsilon}(t+1) - \psi_a^{\varepsilon}(t+1)}{\tau} = \nu_2 \lim_{\tau \to 0} \frac{\psi_p^{\tau+\varepsilon}(t) - \psi_p^{\varepsilon}(t)}{\tau} + \theta_2 \lim_{\tau \to 0} \frac{\psi_a^{\tau+\varepsilon}(t) - \psi_a^{\varepsilon}(t)}{\tau}$$

$$= \nu_2 \sigma_p^{\varepsilon}(t) + \theta_2 \sigma_a^{\varepsilon}(t)$$

$$\sigma_o^{\varepsilon}(t+1) = \lim_{\tau \to 0} \frac{\psi_o^{\tau+\varepsilon}(t+1) - \psi_o^{\varepsilon}(t+1)}{\tau}$$

$$= e^{-\mu} \lim_{\tau \to 0} \frac{\psi_o^{\tau+\varepsilon}(t) - \psi_o^{\varepsilon}(t)}{\tau} + (\eta(t) - N_n(t)) - (\eta(t) - N_n(t)) = e^{-\mu} \sigma_o^{\varepsilon}(t).$$

Now, we will compute $\sigma_l^\varepsilon(t+1)$.

$$\psi_l^\varepsilon(t+1) = \gamma_2\psi_e^\varepsilon(t) + \zeta_1 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)+$$

$$-\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t)-N_n(t))$$

$$\sigma_l^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_l^{\tau+\varepsilon}(t+1)-\psi_l^\varepsilon(t+1)}{\tau} =$$

$$= \gamma_2 \lim_{\tau\to 0}\frac{\psi_e^{\tau+\varepsilon}(t)-\psi_e^\varepsilon(t)}{\tau} + \zeta_1 \lim_{\tau\to 0}\frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}\psi_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)}{\tau}$$

$$-\zeta_1\alpha e^{-\mu}\lim_{\tau\to 0}\frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}\psi_o^{\tau+\varepsilon}(t)P_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)}{\tau}$$

$$-\zeta_1\alpha(\eta(t)-N(t))\lim_{\tau\to 0}\frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}P_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)}{\tau}$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1\left[e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\sigma_l^\varepsilon(t) + \psi_l^\varepsilon(t)\left(-\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) - \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))\right)\right]$$

$$-\zeta_1\alpha e^{-\mu}\sigma_o^\varepsilon(t)e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)\psi_l^\varepsilon(t)$$

$$-\zeta_1\alpha e^{-\mu}\psi_o^\varepsilon P_l^\varepsilon(t)\left(-\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) - \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))\right)$$

$$-\zeta_1\alpha(\eta(t)-N(t))\left[\psi_l^\varepsilon(t)e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} + P_l^\varepsilon(t)\left(-\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) - \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))\right)\right]$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\sigma_l^\varepsilon(t) - \zeta_1\psi_l^\varepsilon(t)\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) - \zeta_1\psi_l^\varepsilon(t)\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))$$

$$-\zeta_1\alpha e^{-\mu}\sigma_o^\varepsilon(t)e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)\psi_l^\varepsilon(t)$$

$$-\zeta_1\alpha e^{-\mu}\psi_o^\varepsilon P_l^\varepsilon(t)\left(-\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) - \alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))\right)$$

$$-\zeta_1\psi_l^\varepsilon(t)\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))$$

$$+\zeta_1\alpha(\eta(t)-N(t))P_l^\varepsilon(t)\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t) + \zeta_1\alpha(\eta(t)-N(t))P_l^\varepsilon(t)\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}(\eta(t)-N(t))$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\sigma_l^\varepsilon(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)\psi_o^\varepsilon(t) - \zeta_1\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)(\eta(t)-N(t))$$

$$-\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\sigma_o^\varepsilon(t) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)\psi_o^\varepsilon(t)$$

$$+\zeta_1\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon\psi_o^\varepsilon(t)^2 + \zeta_1\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\psi_o^\varepsilon(t)(\eta(t)-N(t))$$

$$-\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)(\eta(t)-N(t))$$

$$+\zeta_1\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\psi_o^\varepsilon(t)(\eta(t)-N(t)) + \zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t)-N(t))^2$$

$$= \gamma_2\sigma_e^\varepsilon(t) + \zeta_1 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\sigma_l^\varepsilon(t) - 2\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)\psi_o^\varepsilon(t) - 2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)(\eta(t) - N(t))$$

$$-\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\sigma_o^\varepsilon(t)+\zeta_1\alpha^2e^{-2\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon\psi_o^\varepsilon(t)^2+2\zeta_1\alpha^2e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\psi_o^\varepsilon(t)(\eta(t)-N(t))$$

$$+\zeta_1\alpha^2e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t) - N(t))^2.$$

Now, we will compute $\sigma_p^\varepsilon(t+1)$.

$$\psi_p^\varepsilon(t+1) = \nu_1\psi_p^\varepsilon(t)+\zeta_2e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t)-\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)-\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t)-N_n(t))$$

$$\sigma_p^\varepsilon(t+1) = \lim_{\tau\to 0}\frac{\psi_p^{\tau+\varepsilon}(t+1) - \psi_p^\varepsilon(t+1)}{\tau} =$$

$$= \nu_1\lim_{\tau\to 0}\frac{\psi_p^{\tau+\varepsilon}(t) - \psi_p^\varepsilon(t)}{\tau} + \zeta_2\lim_{\tau\to 0}\frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}\psi_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t)}\psi_l^\varepsilon(t))}{\tau}$$

$$-\zeta_2\alpha e^{-\mu}\lim_{\tau\to 0}\frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}\psi_o^{\tau+\varepsilon}(t)P_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)}{\tau}$$

$$-\zeta_2\alpha(\eta(t) - N_n(t)) \lim_{\tau\to 0} \frac{e^{-\alpha(N_o^{\tau+\varepsilon}(t)e^{-\mu}+N_n^{\tau+\varepsilon}(t))}P_l^{\tau+\varepsilon}(t) - e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)}{\tau}$$

$$= \nu_1\sigma_p^{\varepsilon}(t) + \zeta_2 \left[e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\sigma_l^{\varepsilon}(t) - \alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))\psi_o^{\varepsilon}(t) - \alpha e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))((\eta(t) - N_n(t))\right]$$

$$-\zeta_2\alpha e^{-\mu}\left[e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_o^{\varepsilon}(t)\psi_l^{\varepsilon}(t) - \alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_o^{\varepsilon}(t)P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t) - \alpha e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_o^{\varepsilon}(t)P_l^{\varepsilon}(t)(\eta(t) - N_n(t))\right]$$

$$-\zeta_2\alpha e^{-\mu}\left[e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\sigma_o^{\varepsilon}(t)P_l^{\varepsilon}(t)\right]$$

$$-\zeta_2\alpha(\eta(t)-N_n(t))\left[e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t) - \alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t) - \alpha e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)(\eta(t) - N_n(t))\right]$$

$$= \nu_1\sigma_p^{\varepsilon}(t) + \zeta_2 e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\sigma_l^{\varepsilon}(t) - \zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))\psi_o^{\varepsilon}(t) - \zeta_2\alpha e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))((\eta(t) - N_n(t))$$

$$-\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t) + \zeta_2\alpha^2 e^{-2\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)^2$$

$$+\zeta_2\alpha^2 e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t) - N_n(t)) - \zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)\sigma_o^{\varepsilon}(t)$$

$$-\zeta_2\alpha e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t)(\eta(t) - N_n(t)) + \zeta_2\alpha^2 e^{-\mu}e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t) - N_n(t))$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t)-N_n(t))^2$$

$$=\nu_1\sigma_p^\varepsilon(t)+\zeta_2 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\sigma_l^\varepsilon(t)-2\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t))\psi_o^\varepsilon(t)-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_l^\varepsilon(t))((\eta(t)-N_n(t))$$

$$+\zeta_2\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\psi_o^\varepsilon(t)^2+2\zeta_2\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\psi_o^\varepsilon(t)(\eta(t)-N_n(t))$$

$$-\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)\sigma_o^\varepsilon(t)+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}P_l^\varepsilon(t)(\eta(t)-N_n(t))^2.$$

Next we compute

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2P_l^\varepsilon(t)\psi_l^\varepsilon(t) + \beta_2(\eta(t) - N_n(t))$$

$$z''(\varepsilon) = \lim_{\tau\to 0}\left(\frac{z'(\tau+\varepsilon) - z'(\varepsilon)}{\tau}\right) =$$

$$= \lim_{\tau\to 0}\sum_{t=0}^{T-1} \beta_1 2P_l^{\tau+\varepsilon}(t)\psi_l^{\tau+\varepsilon}(t) + \beta_2(\eta(t) - N_n(t)) - [\beta_1 2P_l^\varepsilon(t)\psi_l^\varepsilon(t) + \beta_2(\eta(t) - N_n(t))]$$

$$= \sum_{t=0}^{T-1} \beta_1 2 \lim_{\tau\to 0} \frac{P_l^{\tau+\varepsilon}(t)\psi_l^{\tau+\varepsilon}(t) - P_l^\varepsilon(t)\psi_l^\varepsilon(t)}{\tau} = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^\varepsilon(t)P_l^\varepsilon(t) + \psi_l^\varepsilon(t)^2].$$

We now need to show that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^\varepsilon(t)P_l^\varepsilon(t) + \psi_l^\varepsilon(t)^2] > 0$. To bound $z''(\varepsilon) > 0$ we will show that $\sigma_l^\varepsilon(t) > 0$.

We start by calculating some of the terms for both ψ^ε and σ^ε functions.

We have that $\psi_e^\varepsilon(0) = 0$, $\psi_l^\varepsilon(0) = 0$, $\psi_p^\varepsilon(0) = 0$, $\psi_a^\varepsilon(0) = 0$, $\psi_o^\varepsilon(0) = 0$, so for $t = 1$:

$$\psi_e^\varepsilon(1) = \gamma_1\psi_e^\varepsilon(0) + \theta_1\psi_a^\varepsilon(0) = 0$$

$$\psi_l^\varepsilon(1) = \gamma_2\psi_e^\varepsilon(0) + \zeta_1 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_o^\varepsilon(0)P_l^\varepsilon(0)$$

$$-\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0) - N_n(0))$$

$$= -\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0) - N_n(0))$$

$$\psi_p^\varepsilon(1) = \nu_1\psi_p^\varepsilon(0) + \zeta_2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0) - \zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_o^\varepsilon(0)P_l^\varepsilon(0)$$

$$-\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0)-N_n(0))$$

$$= -\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0)-N_n(0))$$

$$\psi_a^\varepsilon(1) = \nu_2\psi_p^\varepsilon(0) + \theta_2\psi_a^\varepsilon(0) = 0$$

$$\psi_o^\varepsilon(1) = \psi_o^\varepsilon(0)e^{-\mu} + (\eta(0)-N_n(0)) = (\eta(0)-N_n(0)).$$

Next, for $t = 2$

$$\psi_e^\varepsilon(2) = \gamma_1\psi_e^\varepsilon(1) + \theta_1\psi_a^\varepsilon(1) = 0$$

$$\psi_l^\varepsilon(2) = \gamma_2\psi_e^\varepsilon(1) + \zeta_1 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_o^\varepsilon(1)P_l^\varepsilon(1)$$

$$-\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}P_l^\varepsilon(1)(\eta(1)-N_n(1)) =$$

$$= \zeta_1 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}(\eta(0)-N_n(0))P_l^\varepsilon(1)$$

$$-\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}P_l^\varepsilon(1)(\eta(1)-N_n(1))$$

$$\psi_p^\varepsilon(2) = \nu_1\psi_p^\varepsilon(1) + \zeta_2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1) - \zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_o^\varepsilon(1)P_l^\varepsilon(1)$$

$$-\zeta_2 \alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1) - N_n(1))$$

$$= \nu_1 \psi_p^\varepsilon(1) + \zeta_2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} \psi_l^\varepsilon(1) - \zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}(\eta(0) - N_n(0)) P_l^\varepsilon(1)$$

$$-\zeta_2 \alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1) - N_n(1))$$

$$\psi_a^\varepsilon(2) = \nu_2 \psi_p^\varepsilon(1) + \theta_2 \psi_a^\varepsilon(1) = \nu_2 \psi_p^\varepsilon(1)$$

$$\psi_o^\varepsilon(2) = \psi_o^\varepsilon(1)e^{-\mu} + (\eta(1) - N_n(1)) = (\eta(0) - N_n(0))e^{-\mu} + (\eta(1) - N_n(1)).$$

Lastly, for $t = 3$

$$\psi_e^\varepsilon(3) = \gamma_1 \psi_e^\varepsilon(2) + \theta_1 \psi_a^\varepsilon(2) = \theta_1 \psi_a^\varepsilon(2)$$

$$\psi_l^\varepsilon(3) = \gamma_2 \psi_e^\varepsilon(2) + \zeta_1 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_l^\varepsilon(2) - \zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_o^\varepsilon(2) P_l^\varepsilon(2)$$

$$-\zeta_1 \alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} P_l^\varepsilon(2)(\eta(2) - N_n(2))$$

$$= \zeta_1 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_l^\varepsilon(2) - \zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_o^\varepsilon(2) P_l^\varepsilon(2)$$

$$-\zeta_1 \alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} P_l^\varepsilon(2)(\eta(2) - N_n(2))$$

$$\psi_p^\varepsilon(3) = \nu_1 \psi_p^\varepsilon(2) + \zeta_2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_l^\varepsilon(2) - \zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} \psi_o^\varepsilon(2) P_l^\varepsilon(2)$$

$$-\zeta_2 \alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))} P_l^\varepsilon(2)(\eta(2)-N_n(2))$$

$$\psi_a^\varepsilon(3)=\nu_2\psi_p^\varepsilon(2)+\theta_2\psi_a^\varepsilon(2)$$

$$\psi_o^\varepsilon(3)=\psi_o^\varepsilon(2)e^{-\mu}+(\eta(2)-N_n(2)).$$

Recall that $\sigma_e^\varepsilon(0)$, $\sigma_l^\varepsilon(0)$, $\sigma_p^\varepsilon(0)$, $\sigma_a^\varepsilon(0)=0$, $\sigma_o^\varepsilon(0)=0$. Consider $t=1$:

$$\sigma_e^\varepsilon(1)=\gamma_1\sigma_e^\varepsilon(0)+\theta_1\sigma_a^\varepsilon(0)=0$$

$$\sigma_l^\varepsilon(1)=\gamma_2\sigma_e^\varepsilon(0)+\zeta_1 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\sigma_l^\varepsilon(0)-2\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0)\psi_o^\varepsilon(0)$$

$$-2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0)(\eta(0)-N_n(0))-\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)\sigma_o^\varepsilon(0)$$

$$+\zeta_1\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon\psi_o^\varepsilon(0)^2$$

$$+2\zeta_1\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)\psi_o^\varepsilon(0)(\eta(0)-N_n(0))$$

$$+\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0)-N_n(0))^2$$

$$=\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)(\eta(0)-N_n(0))^2$$

$$\sigma_p^\varepsilon(1)=\nu_1\sigma_p^\varepsilon(0)+\zeta_2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\sigma_l^\varepsilon(0)-2\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0))\psi_o^\varepsilon(0)$$

$$-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}\psi_l^\varepsilon(0))((\eta(0)-N_n(0))+\zeta_2\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))}P_l^\varepsilon(0)\psi_o^\varepsilon(0)^2$$

$$+2\zeta_2\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))} P_l^\varepsilon(0)\psi_o^\varepsilon(0)(\eta(0)-N_n(0))$$

$$-\zeta_2\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))} P_l^\varepsilon(0)\sigma_o^\varepsilon(0)$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))} P_l^\varepsilon(0)(\eta(0)-N_n(0))^2$$

$$=\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(0)e^{-\mu}+N_n^\varepsilon(0))} P_l^\varepsilon(0)(\eta(0)-N_n(0))^2$$

$$\sigma_a^\varepsilon(1)=\nu_2\sigma_p^\varepsilon(0)+\theta_2\sigma_a^\varepsilon(0)=0$$

$$\sigma_o^\varepsilon(1)=e^{-\mu}\sigma_o^\varepsilon(0)=0.$$

Next, $t=2$

$$\sigma_e^\varepsilon(2)=\gamma_1\sigma_e^\varepsilon(1)+\theta_1\sigma_a^\varepsilon(1)=0$$

$$\sigma_l^\varepsilon(2)=\gamma_2\sigma_e^\varepsilon(1)+\zeta_1 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\sigma_l^\varepsilon(1)-2\zeta_1\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1)\psi_o^\varepsilon(1)$$

$$-2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1)(\eta(1)-N_n(1))-\zeta_1\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\sigma_o^\varepsilon(1)$$

$$+\zeta_1\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon\psi_o^\varepsilon(1)^2$$

$$+2\zeta_1\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)(\eta(1)-N_n(1))$$

$$+\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1)-N_n(1))^2$$

$$=\gamma_2\sigma_e^\varepsilon(1)+\zeta_1 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\sigma_l^\varepsilon(1)-2\zeta_1\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1)\psi_o^\varepsilon(1)$$

$$-2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1)(\eta(1)-N_n(1))+\zeta_1\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon\psi_o^\varepsilon(1)^2$$

$$+2\zeta_1\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)(\eta(1)-N_n(1))$$

$$+\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1)-N_n(1))^2$$

$$\sigma_p^\varepsilon(2) = \nu_1\sigma_p^\varepsilon(1) + \zeta_2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\sigma_l^\varepsilon(1) - 2\zeta_2\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1))\psi_o^\varepsilon(1)$$

$$-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1))((\eta(1)-N_n(1))+\zeta_2\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)^2$$

$$+2\zeta_2\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)(\eta(1)-N_n(1))$$

$$-\zeta_2\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\sigma_o^\varepsilon(1)$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1)-N_n(1))^2$$

$$= \nu_1\sigma_p^\varepsilon(1) + \zeta_2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\sigma_l^\varepsilon(1) - 2\zeta_2\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1))\psi_o^\varepsilon(1)$$

$$-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))}\psi_l^\varepsilon(1))((\eta(1)-N_n(1))+\zeta_2\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)^2$$

$$+2\zeta_2\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)\psi_o^\varepsilon(1)(\eta(1)-N_n(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(1)e^{-\mu}+N_n^\varepsilon(1))} P_l^\varepsilon(1)(\eta(1)-N_n(1))^2$$

$$\sigma_a^\varepsilon(2) = \nu_2\sigma_p^\varepsilon(1) + \theta_2\sigma_a^\varepsilon(1) = \nu_2\sigma_p^\varepsilon(1)$$

$$\sigma_o^\varepsilon(2) = e^{-\mu}\sigma_o^\varepsilon(1) = 0.$$

Next, $t = 3$

$$\sigma_e^\varepsilon(3) = \gamma_1\sigma_e^\varepsilon(2) + \theta_1\sigma_a^\varepsilon(2) = \theta_1\sigma_a^\varepsilon(2)$$

$$\sigma_l^\varepsilon(3) = \gamma_2\sigma_e^\varepsilon(2) + \zeta_1 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\sigma_l^\varepsilon(2) - 2\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2)\psi_o^\varepsilon(2)$$

$$-2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2)(\eta(2) - N_n(2)) - \zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\sigma_o^\varepsilon(2)$$

$$+\zeta_1\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon\psi_o^\varepsilon(2)^2$$

$$+2\zeta_1\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)(\eta(2) - N_n(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)(\eta(2) - N_n(2))^2$$

$$= \gamma_2\sigma_e^\varepsilon(2) + \zeta_1 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\sigma_l^\varepsilon(2) - 2\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2)\psi_o^\varepsilon(2)$$

$$-2\zeta_1\alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2)(\eta(2) - N_n(2)) + \zeta_1\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon\psi_o^\varepsilon(2)^2$$

$$+2\zeta_1\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)(\eta(2) - N_n(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)(\eta(2) - N_n(2))^2$$

$$\sigma_p^\varepsilon(3) = \nu_1\sigma_p^\varepsilon(2) + \zeta_2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\sigma_l^\varepsilon(2) - 2\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2))\psi_o^\varepsilon(2)$$

$$-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2))((\eta(2) - N_n(2)) + \zeta_2\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)^2$$

$$+2\zeta_2\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)(\eta(2) - N_n(2))$$

$$-\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\sigma_o^\varepsilon(2)$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)(\eta(2) - N_n(2))^2$$

$$= \nu_1\sigma_p^\varepsilon(2) + \zeta_2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\sigma_l^\varepsilon(2) - 2\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2))\psi_o^\varepsilon(2)$$

$$-2\zeta_2\alpha e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}\psi_l^\varepsilon(2))((\eta(2)-N_n(2))+\zeta_2\alpha^2 e^{-2\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)^2$$

$$+2\zeta_2\alpha^2 e^{-\mu}e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)\psi_o^\varepsilon(2)(\eta(2)-N_n(2))$$

$$+\zeta_2\alpha^2 e^{-\alpha(N_o^\varepsilon(2)e^{-\mu}+N_n^\varepsilon(2))}P_l^\varepsilon(2)(\eta(2)-N_n(2))^2$$

$$\sigma_a^\varepsilon(3)=\nu_2\sigma_p^\varepsilon(2)+\theta_2\sigma_a^\varepsilon(2)$$

$$\sigma_o^\varepsilon(3)=e^{-\mu}\sigma_o^\varepsilon(2)=0.$$

For these terms we can note some similarities and differences to the Basic Model proof for Theorem 2.3.3. We will use Ψ^ε and Σ^ε to denote the Basic model sensitivities and derivatives of sensitivities. For this comparison note that the difference of N^ε and $N_o^\varepsilon + N_n^\varepsilon$ in the exponents will not affect the pattern of our formulation we found in the Basic model for proving $\sigma_l^\varepsilon > 0$, hence we will use $\approx$ to associated similar terms in the model. We have then that $\Psi_e^\varepsilon \approx \psi_e^\varepsilon$, $\Psi_a^\varepsilon \approx \psi_a^\varepsilon$, $\Sigma_e^\varepsilon \approx \sigma_e^\varepsilon$, and $\Sigma_a^\varepsilon \approx \sigma_a^\varepsilon$. Note we have $\sigma_o^\varepsilon(t+1)=0$. Consider:

$$\psi_l^\varepsilon(t+1)\approx\Psi_l^\varepsilon(t+1)-\zeta_1\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)$$

$$\psi_p^\varepsilon(t+1)\approx\Psi_p^\varepsilon(t+1)-\zeta_2\alpha e^{-\mu}e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))}\psi_o^\varepsilon(t)P_l^\varepsilon(t)$$

$$\psi_o^\varepsilon(t+1)=\psi_o^\varepsilon(t)e^{-\mu}+(\eta(t)-N_n(t))=\sum_{c=0}^{t}e^{-\mu\cdot c}(\eta(t-c)-N_n(t-c))$$

$$\sigma_l^{\varepsilon}(t+1) \approx \Sigma_l^{\varepsilon}(t+1) - 2\zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)$$

$$-\zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\sigma_o^{\varepsilon}(t)$$

$$+\zeta_1 \alpha^2 e^{-2\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}\psi_o^{\varepsilon}(t)^2 + 2\zeta_1 \alpha^2 e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t)-N_n(t))$$

$$\approx \Sigma_l^{\varepsilon}(t+1) - 2\zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)$$

$$+\zeta_1 \alpha^2 e^{-2\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}\psi_o^{\varepsilon}(t)^2 + 2\zeta_1 \alpha^2 e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t)-N_n(t))$$

$$\sigma_p^{\varepsilon}(t+1) \approx \Sigma_p^{\varepsilon}(t+1) - 2\zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))\psi_o^{\varepsilon}(t)$$

$$+\zeta_2 \alpha^2 e^{-2\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)^2$$

$$+2\zeta_2 \alpha^2 e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t)-N_n(t)) - \zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\sigma_o^{\varepsilon}(t)$$

$$\approx \Sigma_p^{\varepsilon}(t+1) - 2\zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t))\psi_o^{\varepsilon}(t) + \zeta_2 \alpha^2 e^{-2\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)^2$$

$$+2\zeta_2 \alpha^2 e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} P_l^{\varepsilon}(t)\psi_o^{\varepsilon}(t)(\eta(t)-N_n(t)).$$

Since most of this follows the Basic model proof in Theorem 2.3.3, the only difference is in:

$$\psi_l^{\varepsilon}(t+1) \approx \Psi_l^{\varepsilon}(t+1) - \zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c) - N_n(t-1-c))\right] P_l^{\varepsilon}(t)$$

$$\psi_p^{\varepsilon}(t+1) \approx \Psi_p^{\varepsilon}(t+1) - \zeta_2 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))} \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c) - N_n(t-1-c))\right] P_l^{\varepsilon}(t)$$

$$\sigma_l^{\varepsilon}(t+1) \approx \Sigma_l^{\varepsilon}(t+1) - 2\zeta_1 \alpha e^{-\mu} e^{-\alpha(N_o^{\varepsilon}(t)e^{-\mu}+N_n^{\varepsilon}(t))}\psi_l^{\varepsilon}(t) \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c) - N_n(t-1-c))\right]$$

$$+\zeta_1\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c)-N_n(t-1-c))\right]^2$$

$$+2\zeta_1\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon(t) \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c)-N_n(t-1-c))\right](\eta(t)-N(t))$$

$$\sigma_p^\varepsilon(t+1) \approx \Sigma_p^\varepsilon(t+1) - 2\zeta_2\alpha e^{-\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} \psi_l^\varepsilon(t)) \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c)-N_n(t-1-c))\right]$$

$$+\zeta_2\alpha^2 e^{-2\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon(t) \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c)-N_n(t-1-c))\right]^2 +$$

$$+2\zeta_2\alpha^2 e^{-\mu} e^{-\alpha(N_o^\varepsilon(t)e^{-\mu}+N_n^\varepsilon(t))} P_l^\varepsilon(t) \left[\sum_{c=0}^{t-1} e^{-\mu\cdot c}(\eta(t-1-c)-N_n(t-1-c))\right](\eta(t)-N_n(t))$$

Considering the terms in $\psi_l^\varepsilon, \psi_p^\varepsilon, \sigma_l^\varepsilon, \sigma_p^\varepsilon$ other than the basic model terms, which we know will combine, we can note a construction similar to the Basic model. The terms assocaietd with summations are similar across multiple terms and we expect these to combine, resulting in the summations of $(\eta(i) - N(i))$ squared. These patterns and similar factors will result in a formulation for $\sigma_l^\varepsilon(t)$ which is the summations of $(\eta(i) - N(i))$ squared with associated parameters, and thus resulting in $\sigma_l^\varepsilon(t) > 0$ for all t.

Therefore, we have that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma_l^\varepsilon(t)P_l^\varepsilon(t) + \psi_l^\varepsilon(t)^2] > 0$, and we have uniqueness by convexity of z.

□

Chapter 4

Case Study: Diaprepes abbreviatus

4.1 Introduction

We will investigate the invasive species Diaprepes abbreviatus(DRW). DRW originated in the Caribbean and was transported to the central and southern regions of Florida around 1964 [EGC04]. The introduction of DRW was not intentional, and in the past 50 years DRW have proven to be a troublesome invasive species, spreading throughout Florida and eventually to California in 2005 [EGC04, JG09b]. DRW infests citrus groves along with other plants, causing the most damage during the larva stage to the roots [MSDN00]. For the DRW dynamics we have a matrix model from the paper *Contributions of demography and dispersal parameters to the spatial spread of a stage-structured insect invasion* by Miller and Tenhumberg [MT10]. In past studies of DRW it has been found that the larva stage feeds upon the roots causing severe problem for the citrus plants. Furthermore, most monitoring is done with traps at the adult stage, since there is no effective method to monitor larva[LDG16]. It has also been found that pesticides are not useful in management of DRW larva [CHEBD15]. DRW can be controlled using entomopathohenic nematodes [BPK99, Gau02]. We use the previous models to determine a management plan specifying timing and amount of entomopathohenic nematodes, while also considering the cost of applying nematodes and cost of DRW damage to the farmer.

4.2 Parameter Values

4.2.1 Values for Martrix - DRW Life Cycle

For the DRW life cycles and dynamics, I reduced the 6×6 matrix from a paper by Tom E. Miller and Brigitte Tenhumberg [MT10], to a 4×4 matrix using Hooley's algorithm [SP10]. We reduced to a 4 stage matrix to account for the 4 major stages in most insect life cycles: egg, larva, pupa, and adults.

Hooley's Algorithm To reduce a stage structure matrix $\mathcal{A}$ we find the corresponding eigenvalues and eigenvectors. Next, we identify the largest eigenvalues and the corresponding eigenvector. In our case the largest eigenvalue is 1.42271091 and the eigenvector is $\Upsilon = [0.812321361, 0.433821424, 0.388876952, 0.025331093, 0.006721562, 0.004566305]$. In order to do any reduction we need to alter Υ,

$$u_\Upsilon = \frac{\Upsilon}{sum(\Upsilon)}$$

$$= [0.485943142, 0.2595186540.232632179, 0.015153450, 0.004020942, 0.002731634]$$

.

$$\mathcal{A} = \begin{bmatrix} 0.305 & 0 & 0 & 0 & 25.692 & 161.045 \\ 0.530 & 0.43 & 0 & 0 & 0 & 0 \\ 0 & 0.43 & 0.943 & 0 & 0 & 0 \\ 0 & 0 & 0.0420 & 0.778 & 0 & 0 \\ 0 & 0 & 0 & 0.202 & 0.662 & 0 \\ 0 & 0 & 0 & 0 & 0.313 & 0.962 \end{bmatrix}$$

We want to go from a 6×6 matrix to a 4×4, specifically we combine the two larva stages into one larva stage and the two adults stages into one adult stage. Meaning we need to combine the second and third rows into one and the fifth and sixth rows into one, to do this we use a matrix P, and for reducing the same columns we use Q which is constructed using u_Υ. The formulations of P and Q come from [SP10].

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u_\Upsilon[2]}{u_\Upsilon[2]+u_\Upsilon[3]} & 0 & 0 \\ 0 & \frac{u_\Upsilon[3]}{u_\Upsilon[2]+u_\Upsilon[3]} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{u_\Upsilon[5]}{u_\Upsilon[5]+u_\Upsilon[6}] \\ 0 & 0 & 0 & \frac{u_\Upsilon[6]}{u_\Upsilon[5]+u_\Upsilon[6]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5273153 & 0 & 0 \\ 0 & 0.4726847 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5954679 \\ 0 & 0 & 0 & 0.4045321 \end{bmatrix}$$

Now, using P and Q we will reduce $\mathcal{A}$ to A, by $P \cdot \mathcal{A} \cdot Q = A$.

$$A = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 \\ \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & \nu_2 & \theta_2 \end{bmatrix} = \begin{bmatrix} 0.3048413 & 0 & 0 & 80.446501 \\ 0.5301587 & 0.89923928 & 0 & 0 \\ 0 & 0.01984631 & 0.7781462 & 0 \\ 0 & 0 & 0.2018538 & 0.969731 \end{bmatrix}$$

By using this method we have that the eigenvalue of A is 1.422711, the same as $\mathcal{A}$. The reduced 4×4 matrix takes into account the DRW eggs (P_e), larva (P_l), pupa (P_p), and adults (P_a). The values are scaled to consider a one week time step.

4.2.2 Values for Initial Conditions

There are infinitely many possible distributions; we choose the stable stage distribution (SSD) as a starting point. This way we minimize the effect of transients on our control. Note, the methods would work with any other initial distribution which could be used if the farmer has information. Since the DRW is at SSD, the initial conditions for the DRW are derived from the eigenvector associated with the largest eigenvector of A. Note that we can scale the initial values by any constant to reflect number of DRW in a hectare.

$$\begin{bmatrix} P_e(0) \\ P_l(0) \\ P_p(0) \\ P_a(0) \end{bmatrix} = \begin{bmatrix} \phi_e \\ \phi_l \\ \phi_p \\ \phi_a \end{bmatrix} = \begin{bmatrix} 0.485943142 \\ 0.492150833 \\ 0.015153450 \\ 0.006752576 \end{bmatrix}$$

4.2.3 Values for the search efficiency, α

We could not find an estimate of α in the literature. Therefore, we made the assumption that the recommended number of nematodes per hectare would result in a negative population growth rate, and iteratively searched for α values that produced a decreasing population size if the recommended nematode density was applied.

First we calculated the suggested number of nematodes per hectare, N_S [Gau02]. Then we fixed $N(t) = N_S$ in the $A(t)$ matrix, resulting in a linear system. We choose an α such that the DRW population decays at a slow rate, to be conservative. Hence, we varied α until we found a value of α which produces the eigenvalue of A more than 1. Recall this is the asymptotic population growth rate for linear PPMs. We choose α so that the eigenvalue was close to 1, meaning that an increase in nematode density would speed up population decline. Note the model never predicts extinction, and farmers rarely succeed in driving a pest extinct

The recommended nematode density is 22 nematodes per cm^2, and 1 cm^2=1×10^{-8} hecatre.

$$\frac{22\,nematodes}{cm^2} \times \frac{1cm^2}{10^{-8}\,hectare} = \frac{22\,nematodes}{10^{-8}\,hectare} = 22 \times 10^8\,per\,hectare$$

We iteratively found a value of α which with N(t)=22×10^8 per hectare produces a largest eigenvalue of A_l close to 1.

α	Largest eigenvalue
0.00000001	1.0622
0.000000015	1.009835
0.000000016	1.003221
0.0000000165	1.000286
0.00000001655	1.000005
0.000000016551	0.9999996
0.0000000166	0.9997267
0.000000017	0.9975792
0.00000002	0.9853918
0.0000001	0.97
0.0000005	0.97

When $\alpha = 0.00000001655$, A_l has an eigenvalue slightly larger than 1, meaning the DRW would just persist.

Note invasive behavior refers to population growth in the absence of control. Our choice of α means that the recommended dose according to the manufacture specifications is not sufficient to produce population decline. Hence, we would expect that our model predicts higher than recommended nematode applications. We expect a nematode manufacture to suggest applying too many nematodes as a safety net in case nematodes don't work as well as expected; or, nematodes may die because of unfavorable environmental conditions.

4.2.4 Values for the Cost Function

Now that we have the values for the parameters in the dynamic system we need to assign values to the cost function variables. To find the values of the cost function, $Cost\ at\ time\ t == \beta_1 P_l(t)^2 + \beta_2 N(t)$, from the objective functional we will investigate the two parameters separately.

Cost of DRW - β_1 For DRW cost, a literature review did not provide any estimate on how much damage a single DRW larvae causes in terms of loss in harvest. We did find in the literature an estimation for the number of weevils present at halfway to full infestation in a hectare [MSN03]. Additionally, we found how much farmers expect to make per hectare for citrus, and then divided this by the 52 weeks in a year, finding the cost of harvest [Gau02]. Therefore, we estimated a value for the DRW cost as the loss of harvest due to the feeding activity of the larvae.

Combining the above information we have that estimation that $\beta_1 = 7.9515 \times 10^{-12}$.

Cost of Nematodes - β_2 Nematodes can be purchased at 22 nematodes per cm^2 for \$62 per hectare [Gau02]. Hence we have that the cost of Nematodes, N - \$62/22/(1/10^8) per hectare per nematode $= \beta_2$.

Note, while we have configured the cost to be per one Nematodes, typically Nematodes are purchases in bulk. For instance you can purchase for your personal use 50 million nematodes for about ninety dollars [PR12].

So the total cost for any time is cost of diaprepes weevil damage, $\beta_1 P_l(t)^2$, plus cost of purchasing nematodes, $\beta_2 N(t)$.

$$Cost\ at\ time\ t = \beta_1 P_l(t)^2 + \beta_2 N(t)$$

where $\beta_1 = 7.9515 \times 10^{-12}$ and $\beta_2 = 2.8182 \times 10^{-8}$.

4.3 Basic Model with Parameter Values

Recall, our goal is to minimize the objective functional:

$$J(N) = \sum_{k=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N(t)$$

subject to:

$$P_e(t+1) = \gamma_1 P_e(t) + \theta_1 P_a(t) \qquad P_e(0) = \phi_e$$

$$P_l(t+1) = \gamma_2 P_e(t) + \zeta_1 e^{-\alpha N(t)} P_l(t) \quad P_l(0) = \phi_l$$

$$P_p(t+1) = \zeta_2 e^{-\alpha N(t)} P_l(t) + \nu_1 P_p(t) \quad P_p(0) = \phi_p$$

$$P_a(t+1) = \nu_2 P_p(t) + \theta_2 P_a(t) \qquad P_a(0) = \phi_a.$$

We additionally need $N(k) \geq 0$, because nematode densities cannot be negative. From the previous sections we have that

$$N(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > P_l(k)(\zeta_1 \lambda_l(t+1) + \zeta_2 \lambda_p(t+1)) \\ \frac{1}{\alpha} \ln[\frac{\alpha P_l(k)}{\beta_2}(\zeta_1 \lambda_l(t+1) + \zeta_2 \lambda_p(t+1)))] & \text{if } \frac{\beta 2}{\alpha} \leq P_l(t)(\zeta_1 \lambda_l(t+1) + \zeta_2 \lambda_p(t+1)) \end{cases}.$$

For the following simulations we will consider 52 weeks of application, so $T = 52$ with time steps of one week.

4.3.1 Forward-Backward Sweep (FBS)

An algorithm typically used to find an estimation for the solution to an Optimal Control problem is the forward-backward sweep[LW07]. The algorithm utilizes the pest dynamics and the adjoints to find a solution for how many nematodes to apply. The process is described in general by:

1. Let $N = 0$ and use this to calculate P_e, P_l, P_p, P_a using the initial conditions $\phi_e, \phi_l, \phi_p, \phi_a$.

2. Now calculate $\lambda_e, \lambda_l, \lambda_p, \lambda_a$ using the terminal condition $\lambda_e(T) = 0, \lambda_l(T) = 0, \lambda_p(T) = 0, \lambda_a(T) = 0$.

3. Using the calculations in 1 and 2 find N.

4. Check if the differences between the newly calculated P_e, P_l, P_p, P_a, $\lambda_e, \lambda_l, \lambda_p, \lambda_a$, and N are within an acceptable error, δ. If so, stop, since you have your value for the optimal control. If not use the N in 3 and repeat the process.

In the above description of the FBS we mention the acceptable error δ. Figures 4.1 and 4.2 varying δ, we decided to use $\delta = 0.1$ due to the speed of computation and the accuracy of answer.

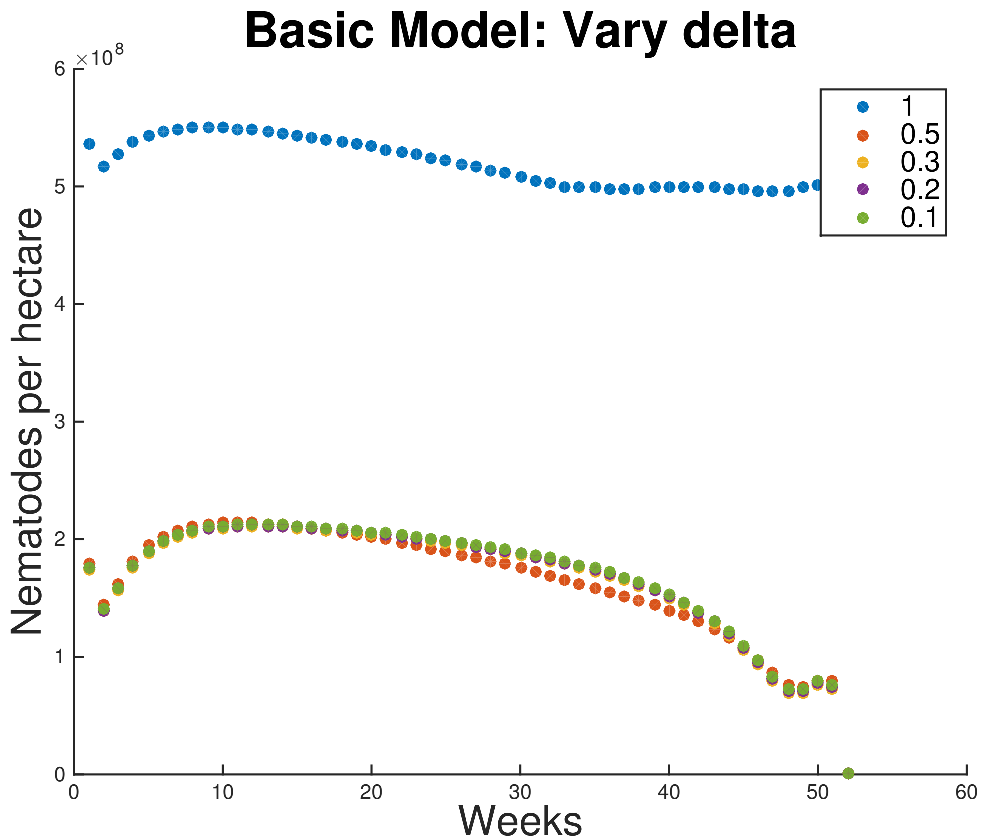

Figure 4.1: Using the Forward-Backward Sweep we calculate the number of nematodes to apply for initial populations 1100000 for various values of δ. Note how once we start using 0.5 we get a close estimate to 0.1.

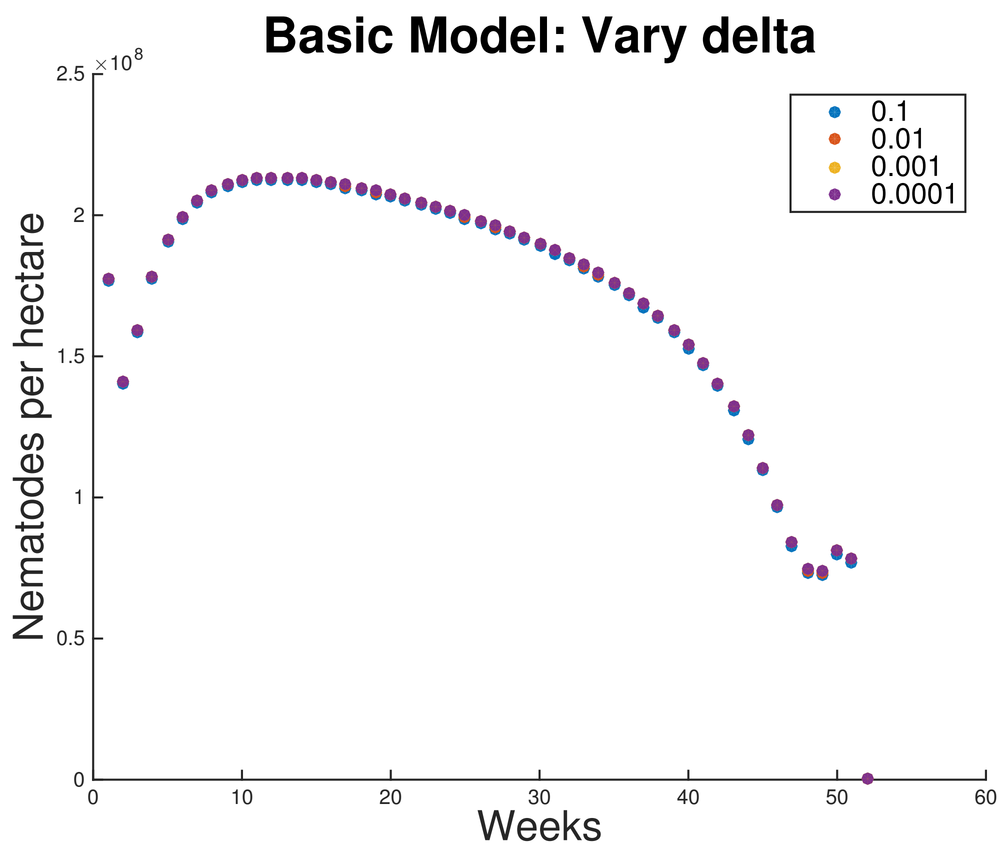

Figure 4.2: Using the Forward-Backward Sweep we calculate the number of nematodes to apply for initial populations 1100000 for various values of δ. Note how all the curves practically overlap.

4.3.1.1 Varying Initial Population

Using the FBS we varied the initial population to see the change in number of nematodes to apply and the corresponding DRW larvae population.

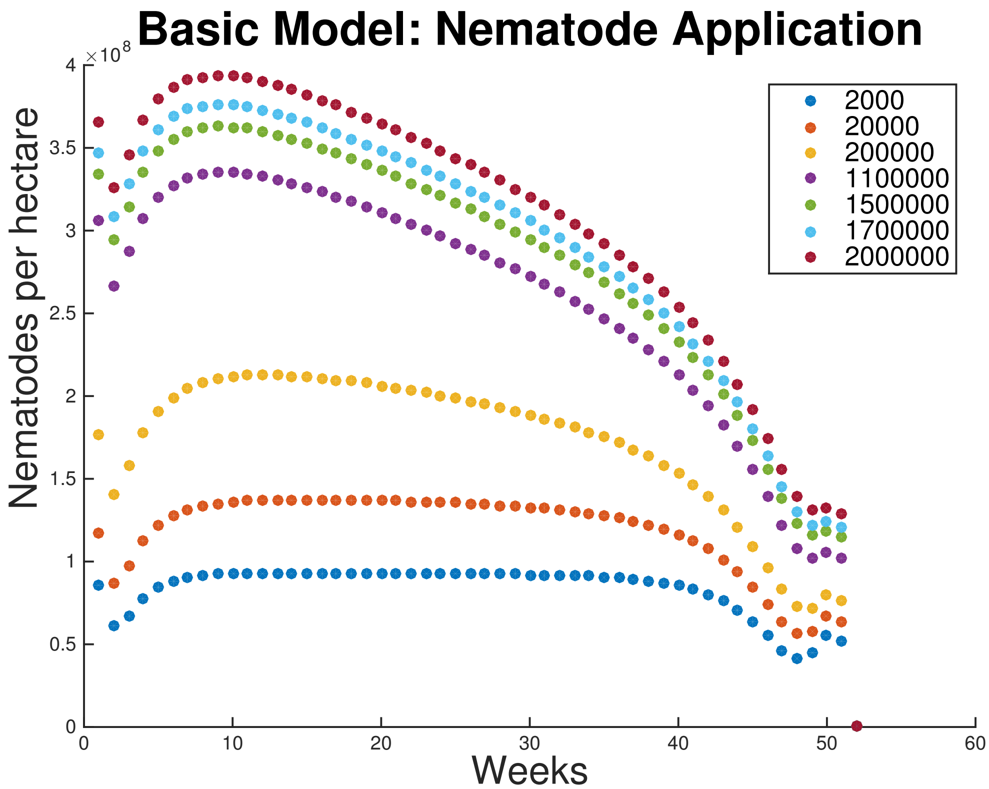

Figure 4.3: Using the Forward-Backward Sweep we calculate the number of nematodes to apply for various initial populations: 2000, 20000, 200000, 1100000, 1500000, 1700000, 2000000

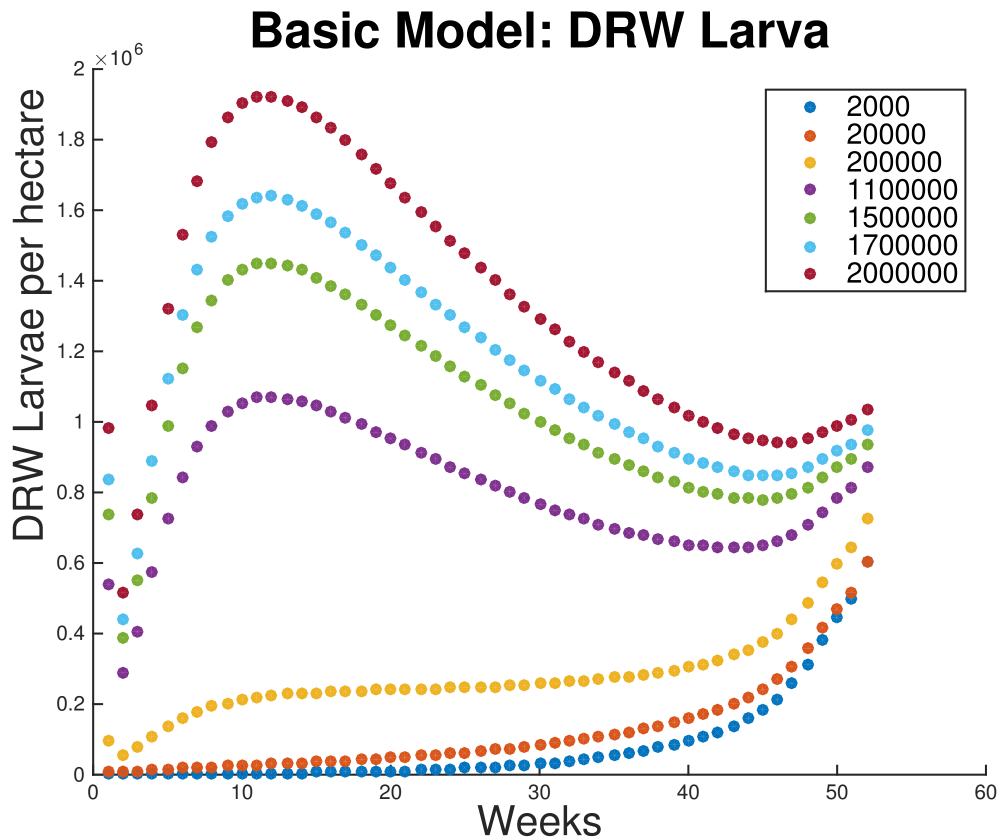

Figure 4.4: The corresponding DRW larva populations for the nematode application in Figure 4.3 for various initial populations: 2000, 20000, 200000, 1100000, 1500000, 1700000, 2000000

4.3.1.2 Varying Search Efficiency, α

In section 4.2.3 we calculated the value for α. Now we will vary the search efficiency by various percent changes for the initial population 1100000. The results for the initial population 1100000 in Figures 4.3 and 4.4 are similar to the other initial populations we considered.

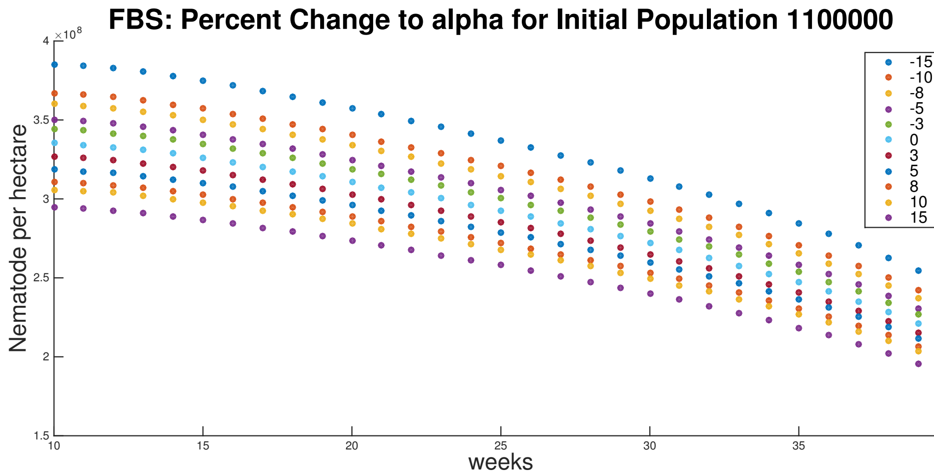

Figure 4.5: For initial population 1100000 we vary the value of α by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we focus on 10 to 40 weeks of the 52 week simulation to see the variance with the percentage change.

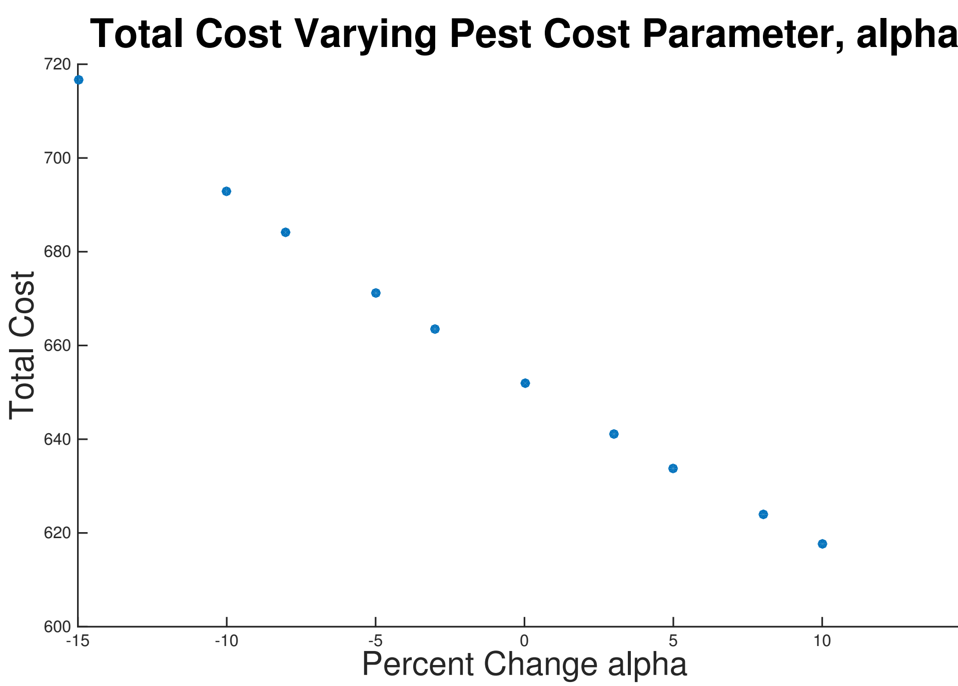

Figure 4.6: For initial population 1100000 we vary the value of α by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we display the Total Cost.

4.3.1.3 Varying Cost Associated with Loss of Harvest, β_1

In section 4.2.4 we calculated the value for β_1. Now we will vary the cost associated with loss of harvest by various percent changes for the initial population 1100000. The results for the initial population 1100000 in Figures 4.3 and 4.4 are similar to the other initial populations we considered.

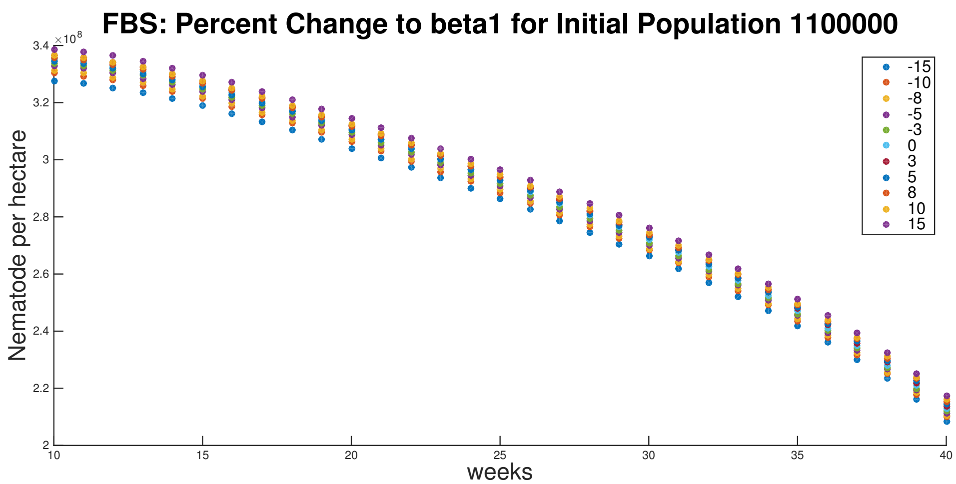

Figure 4.7: For initial population 1100000 we vary the value of β_1 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we focus on 10 to 40 weeks of the 52 week simulation to see the variance with the percentage change.

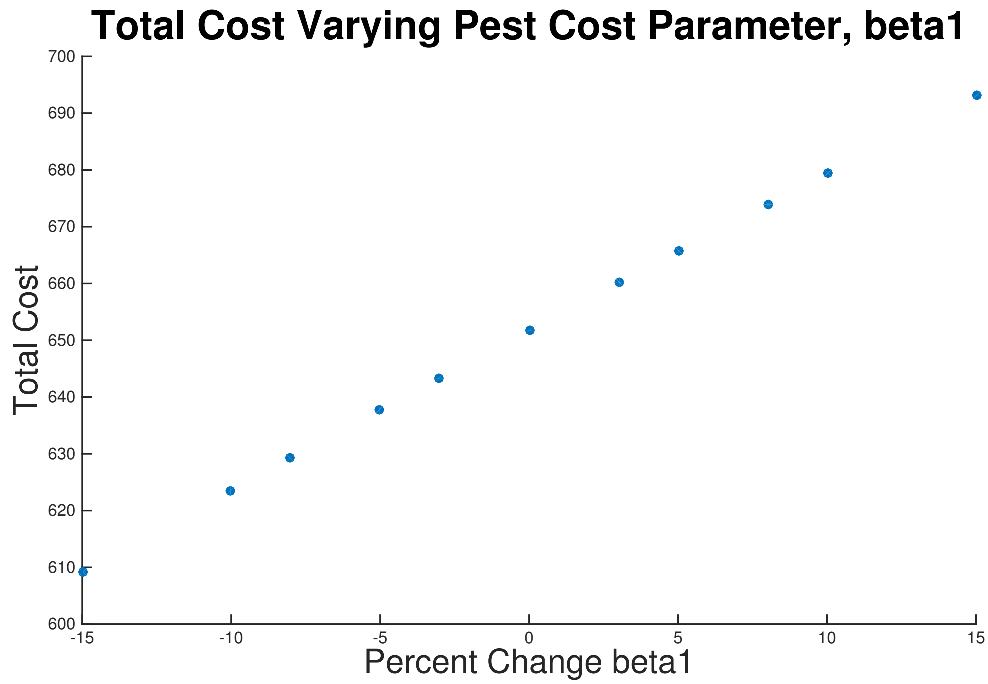

Figure 4.8: For initial population 1100000 we vary the value of β_1 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we display the Total Cost.

4.3.1.4 Varying Cost Associated with the Purchase of Nematodes, β_2

In section 4.2.4 we calculated the value for β_2. Now we will vary the cost associated with the purchase of nematodes by various percent changes for the initial population 1100000. The results for the initial population 1100000 in Figures 4.3 and 4.4 are similar to the other initial populations we considered.

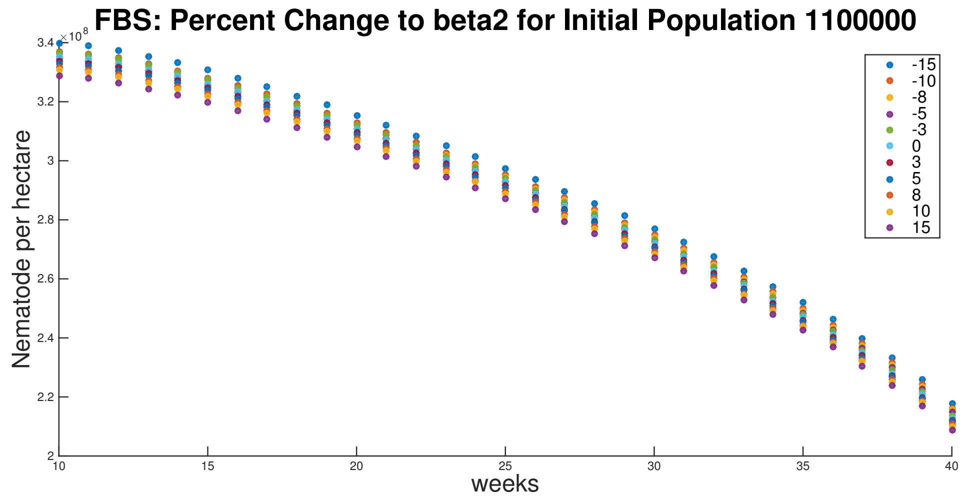

Figure 4.9: For initial population 1100000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we focus on 10 to 40 weeks of the 52 week simulation to see the variance with the percentage change.

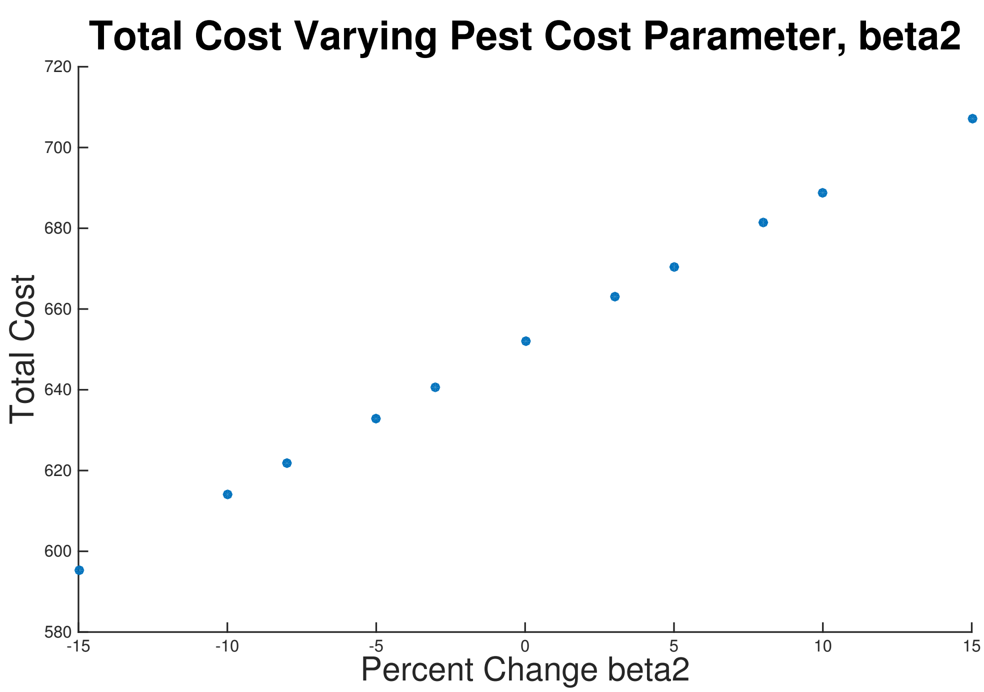

Figure 4.10: For initial population 1100000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we display the Total Cost.

4.3.1.5 Varying Search Efficiency, Control Cost, and Pest Cost Parameter, α, β_1, and β_2

In order to compare the difference in percent changes for α, β_1, and β_2 we combine the results of Figures 4.6, 4.8, and 4.10.

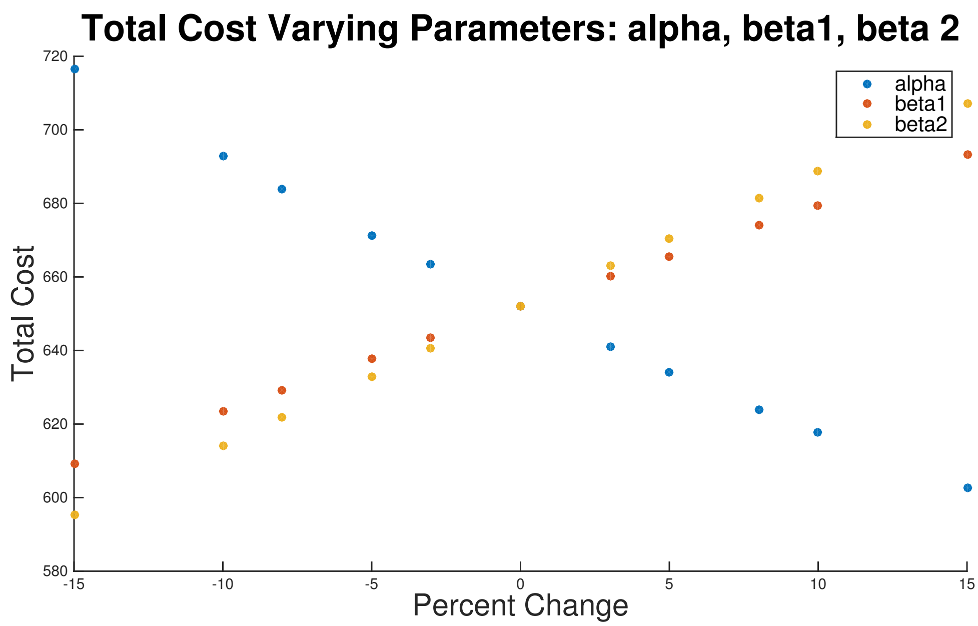

Figure 4.11: For initial population 1100000 we vary the value of α, β_1, and β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we display the Total Cost, combining Figures 4.6, 4.8, and 4.9.

4.3.2 MultiStart

Another method to find the number of Nematodes to apply is implementing the MultiStart Algorithm in MATLAB. The algorithm implements the fmincon function, which given a starting point will search for nearby local minimum. The issue is that fmincon cannot definitively say whether the point is a global minimum. To search for the global minimum we use MultiStart which allows you to input how many randomly generated points you would like MATLAB to run through fmincon. For instance, if you choose 100 then fmincon will be run on 100 different randomly selected points, and MultiStart will output the best possible option for the global minimum. As with fmincon, the output of MultiStart might not be the global minimum but it allows us to approximate and with more inputs for randomly generated points we can get a better approximation.

In this case, we can use MultiStart to compare with the results of Forward-Backward Sweep to approximate the global minimum by comparing the two algorithms outcomes. For MultiStart, we used the Holland Computing Center to run more inputs for MultiStart; specifically the results we display had 500,000 randomly chosen points.

4.3.2.1 Varying Initial Density

Using MultiStart we varied the initial population to see the change in number of nematodes to apply. We found that the results in MultiStart were seemingly converging to the same constant value of Nematodes to apply for approximately weeks from 6 to 46, see Figure 4.12. So we we graphed the average number of nematodes Multistart instructs to apply from weeks 6 to 46.

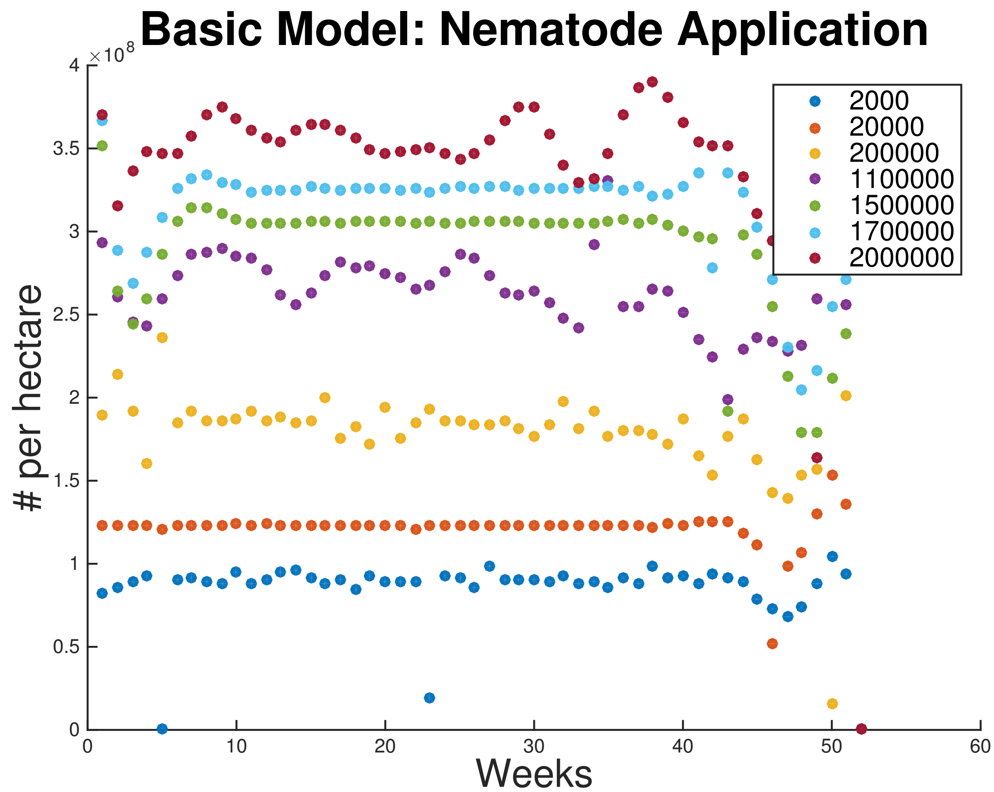

Figure 4.12: Using MultiStart we calculate the number of nematodes to apply for various initial populations: 2000, 20000, 200000, 1100000, 1500000, 1700000, 2000000

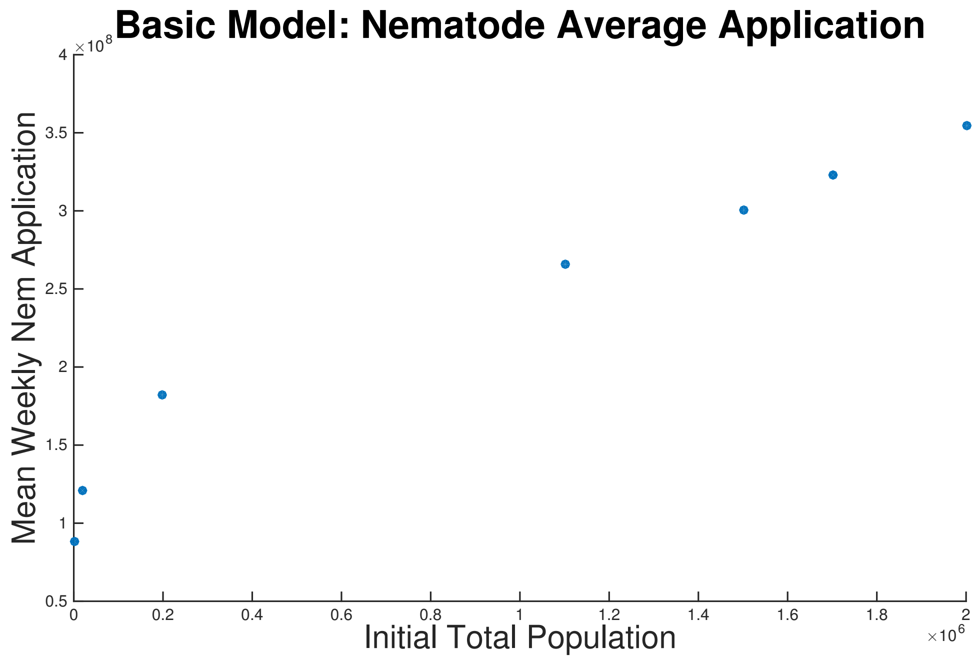

Figure 4.13: In Figure 4.12 we can note that the applications seem to stabilize somewhat after 6 weeks until 46 weeks. We took the average value for each initial population of nematodes application between 6 and 46 weeks and plotted them above.

4.3.2.2 Varying Cost Associated with the Purchase of Nematodes, β_2

Similar to Section 4.3.1.4 we will vary the cost associated with the purchase of nematodes. We plot the number of nematodes to apply for initial population 200000 for various percent changes of β_2. Also, we look again at the average value of nematodes to apply for weeks 6 through 46 and the total cost associated with the nematode application. Varying α and β_1 the results are similar to that in the FBS.

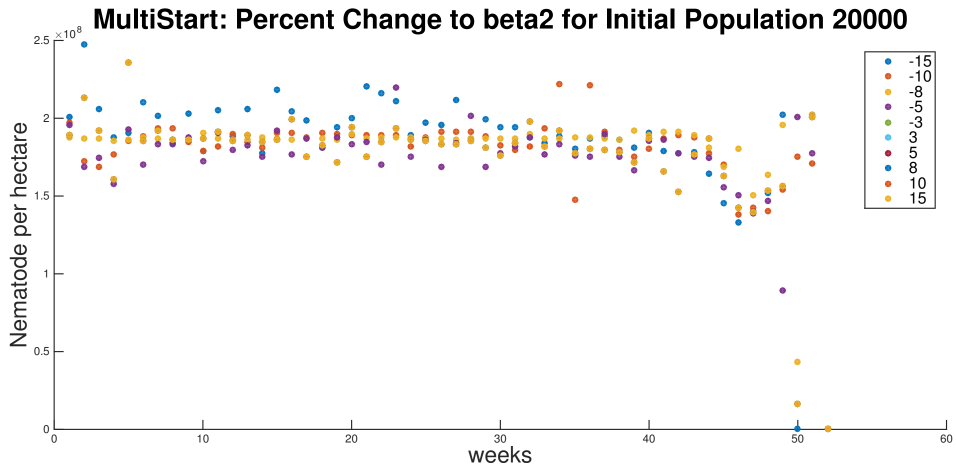

Figure 4.14: For initial population 200000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. Note that for certain percentage differences we do not have very linear curves from time 6 to 46 weeks.

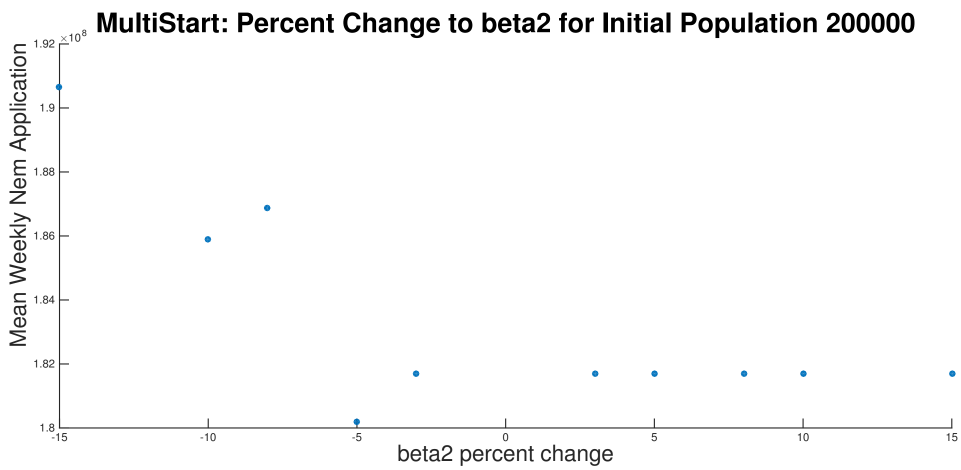

Figure 4.15: For initial population 200000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. As in Figure 4.13 took the average value for each initial population of nematodes application between 6 and 46 weeks. Note issues arise due to the curves erratic behavior for certain percents in Figure 4.14

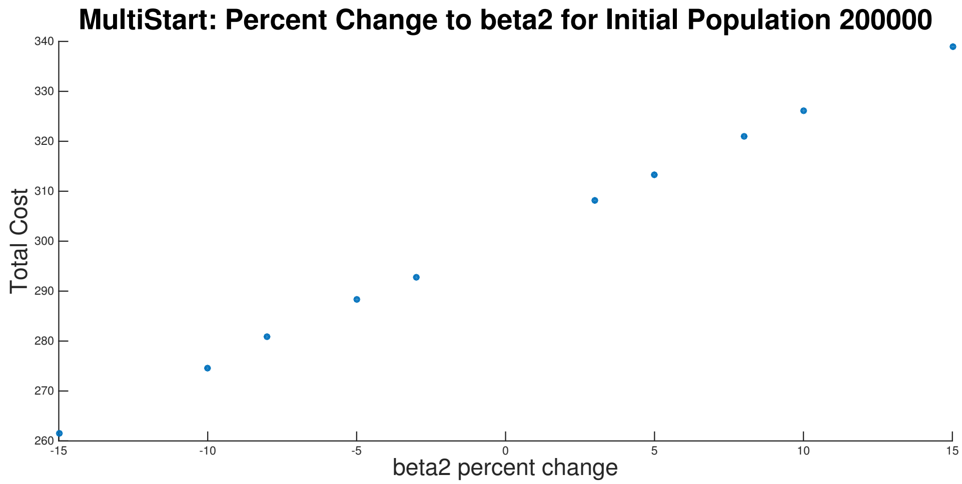

Figure 4.16: For initial population 200000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. The figure shows the different total costs for the nematodes applied in 4.14

4.3.3 Discussion/Summary

First we will consider the results from the FBS algorithm. Looking at Figure 4.3 we can see the FBS algorithm is outputting the expected biological response, the larger the initial population the more nematodes must be applied. The shape of the FBS curves of nematode application are all similar with bumps for the first few weeks at the start and end, which is natural with Optimal Control. Recall in our optimal control problem we are only worrying about the cost of the 52 weeks of application, and not afterwards, which explains the bumps at beginning and end. Now considering 4.4 we can see at low initial populations, there are not many nematodes applied and hence the DRW larvae are persisting and starting to grow. This is because the cost of DRW larvae damage does not yet outweigh the cost of purchasing nematodes. When we start with an initial population of 1100000 and more of DRW larvae, we can see in Figure 4.3 and 4.4 that there more nematodes applied to combat the DRW larvae.

Specifically, we can see the nematodes are starting to eliminate the DRW larvae at 1700000, with the curve reaching a peak in 4.4 and then decreasing before the last few weeks, which is again the nature of an Optimal Control problem.

When we vary the parameters α, β_1, and β_2 the results align with what we expect biologically. If we reduce the search efficiency, α, then we increase the number of nematodes we must apply and total cost, see Figures 4.5 and 4.6. Biologically this makes sense, since the worse nematodes are at finding the DRW larvae the more nematodes we must apply to the system and the larger the total cost. Meanwhile for cost associated with loss of harvest, β_1, and cost associated with the purchase of nematodes, β_2, if either increases the total cost increases as well, see Figures 4.7, 4.8, 4.9, and 4.10. Again, this makes biological sense, if we increase the cost of nematodes we still need to apply nematodes so the total cost increase. Similarly, if we increases the cost associated with loss of harvest, then we need to control the DRW larvae more and will result in a higher total cost. Looking at Figure 4.11, we can see the resulting total costs associated with changing α, β_1, and β_2. Note the greater change in the curve associated with α and the difference in the curves associated with β_1 and β_2. Recalling where these parameters are applied in the Optimal Control problem in Section 4.3 we see the curves in Figure 4.11 makes biological sense. Since search efficiency, α, is applied to the start variables through an exponential term, it makes sense changing α will more drastically effect the total cost. Comparing β_1 and β_2, since β_2 is associated with a linear term we see a more dramatic change of the total cost. Additionally, we can note that the fluctuation of total cost is at most 120 dollars, which is not too great considering we are varying parameters by possible 15% and our total cost without varying parameters is around 650 dollars per hectare.

We notice similar behavior in the MultiStart method with Figure 4.12 and 4.13, with a more approximate nature. We used MultiStart in this case to compare with

Forward-Backward Sweep. In Figure 4.17 we compare the Total Cost, J, of the two methods. We can note that the Forward-Backward Sweep always has a lower cost than the MultiStart, and the MultiStart seems to be converging, possibly to the Forward-Backward Sweep. Note there is a difference in the outputs of the FBS and MultiStart. Recall, the main part is Multistart is searching for the best choice by picking 500,000 random possibilities for the nematode vector and then running fmincon for these. Therefore, the comparison in Figure 4.18 of FBS and MultiStart makes more sense. For the smaller initial population, the curves seem closer for the two algorithms. For both initial populations we are still allowing MultiStart the same number of start points, so it makes sense that at smaller population would produce a closer approximation to the optimal FBS solution. Therefore, if we could run MultiStart for more start points we could find a less approximate solution in MultiStart.

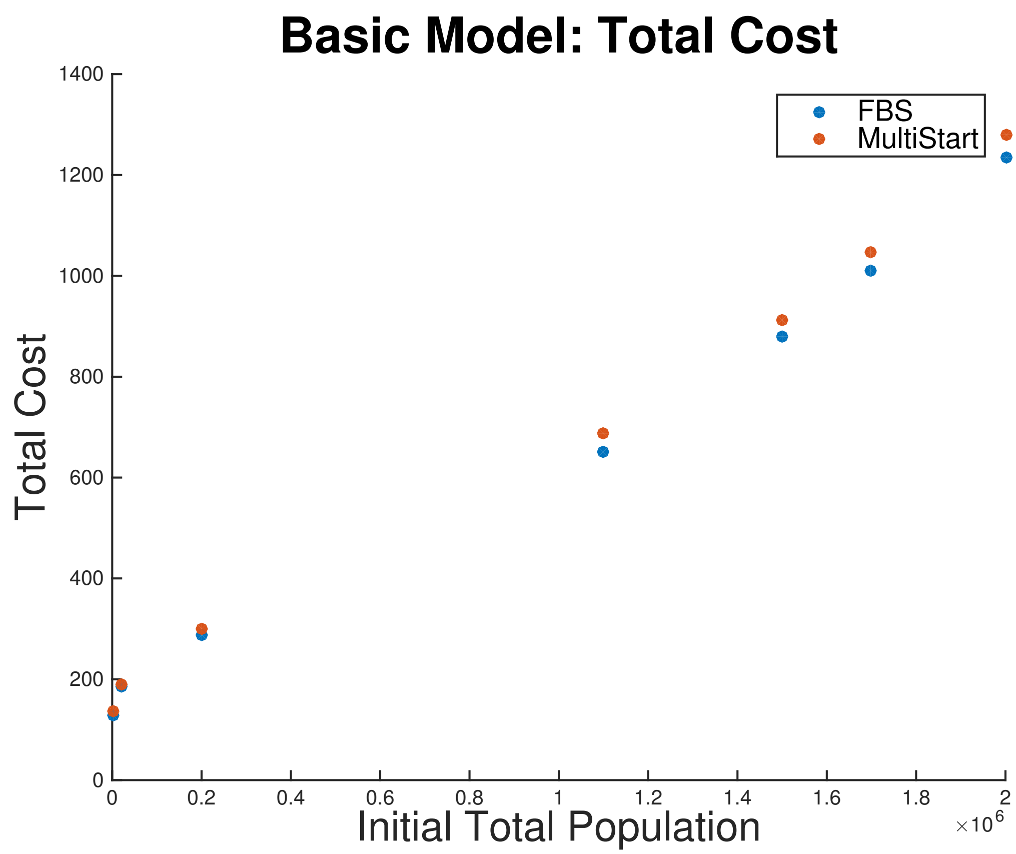

Figure 4.17: The Total Cost, or value of J, which corresponds to all the initial populations and nematodes applications from Figure 4.3 for Forward-Backward Sweep (FBS) and Figure 4.12 for MultiStart.

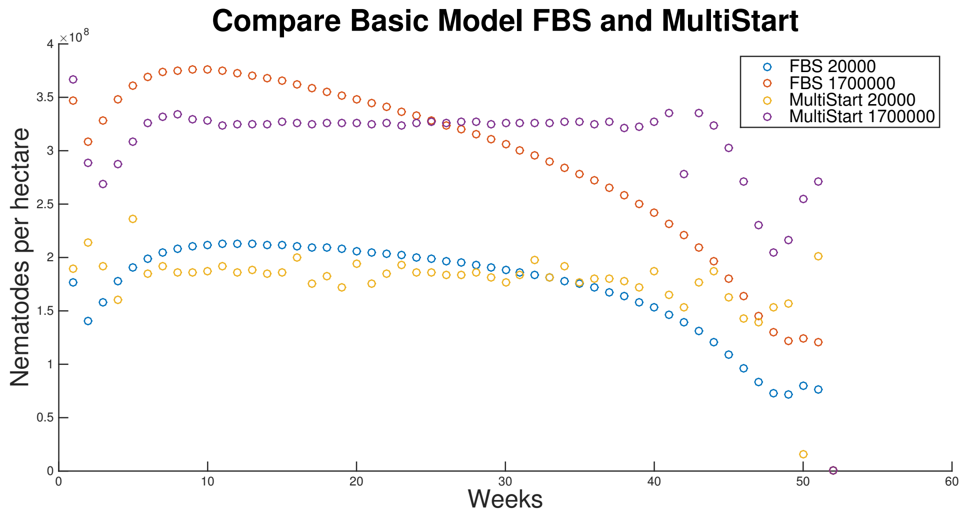

Figure 4.18: Displaying the number of nematodes to apply for initial populations 200000 and 1700000 for both FBS and MultiStart.

Additionally, we have simulations looking at varying the parameter β_2. Comparing Figures 4.19 and 4.14 we can see how issues can arise with MultiStart. In the FBS case of Figure 4.19 we can see the smooth transitions with varying the percentages of β_2. Meanwhile, in Figure 4.14, we have issues and there does not seem to be the similar smooth transition, which is further highlighted in Figure 4.15. However, note in Figure 4.16 the total cost is increasing in MultiStart as we would expect with the varying percentages of β_2. Hence, while MultiStart might be giving an approximate number of Nematodes to apply, the total cost of the system still make biological sense. To increase our understanding we need more starting points for MultiStart to see more exact solutions.

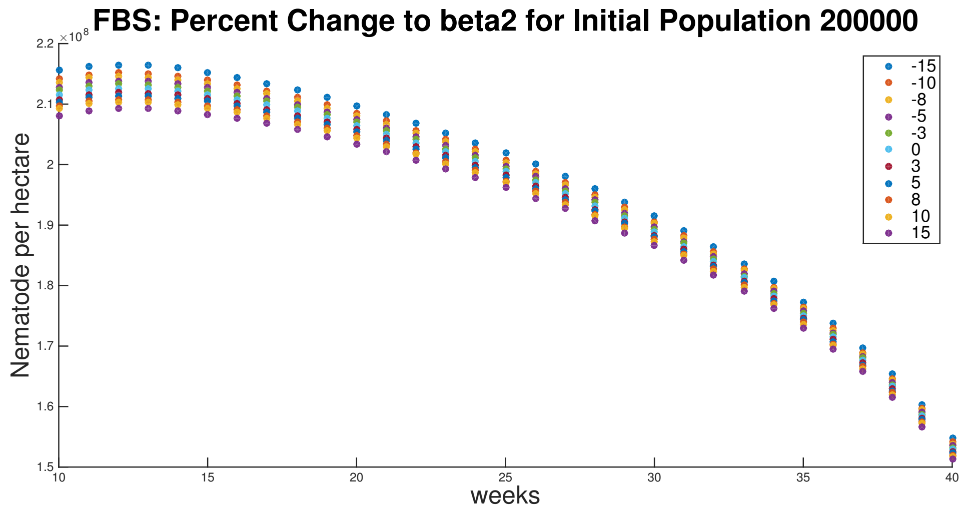

Figure 4.19: For initial population 200000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. In the figure we focus on 10 to 40 weeks of the 52 week simulation to see the variance with the percentage change.

For the next model we cannot use FBS, so by comparing the two methods for this Basic model we get insight into how the algorithms compare. While MultiStart might not find the optimal solution it does come close to the FBS for application purposes to eradicate DRW without incurring too much additional cost, Figure 4.17. Addtionally, MultiStart does find solutions whose total cost increase with initial populations, even with associated nematode applications that seem erratic.

4.4 Persist Model with Parameter Values

Our goal is to minimize the objective functional:

$$J(N_n) = \sum_{t=0}^{T-1} \beta_1 P_l(t)^2 + \beta_2 N_n(t)$$

subject to:

$$P_e(t+1) = \gamma_1 P_e(t) + \theta_1 P_a(t) \qquad P_e(0) = \phi_e$$

$$P_l(t+1) = \gamma_2 P_e(t) + \zeta_1 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) \qquad P_l(0) = \phi_l$$

$$P_p(t+1) = \zeta_2 e^{-\alpha(N_o(t)e^{-\mu}+N_n(t))} P_l(t) + \nu_1 P_p(t) \qquad P_p(0) = \phi_p$$

$$P_a(t+1) = \nu_2 P_p(t) + \theta_2 P_a(t) \qquad P_a(0) = \phi_a$$

$$N_o(t+1) = N_o(t)e^{-\mu} + N_n(t) \qquad N_o(0) = 0.$$

We additionally need $N_n(t) \geq 0$ because nematode densities cannot be negative. Note that this also bounds $N_o(t) \geq 0$.

From the previous proofs we have that

$$N_n(t) = \begin{cases} 0 & \text{if } e^{\alpha N_o(t)e^{-\mu}} > \frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)} \\ \frac{1}{\alpha}\ln[\frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}] - N_o(t)e^{-\mu} & \text{if } e^{\alpha N_o(t)e^{-\mu}} \leq \frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)} \end{cases}.$$

4.4.1 Forward-Backward Sweep

Unlike in the Basic Model we cannot use the Forward-Backward Sweep to find the number of Nematodes to apply for many options of μ. After running simulations we noted that the FBS was always outputting that the number of Nematodes to apply was zero. This can be explained as follows. Looking back at the equation for nematodes, N:

$$N_n(t) = \begin{cases} 0 & \text{if } e^{\alpha N_o(t)e^{-\mu}} > \frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)} \\ \frac{1}{\alpha}\ln[\frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}] - N_o(t)e^{-\mu} & \text{if } e^{\alpha N_o(t)e^{-\mu}} \leq \frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)} \end{cases}.$$

Note that for $N(t) > 0$ we need that

$$e^{\alpha N_o(t)e^{-\mu}} \leq \frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2]}{\beta_2 + \lambda_o(t+1)}.$$

Now we know that $1 \leq e^{\alpha N_o(t)e^{-\mu}}$ since $N_o(t)e^{-\mu} \geq 0$, so $\frac{\alpha P_l(t)[\lambda_l(t+1)\zeta_1+\lambda_p(t+1)\zeta_2]}{\beta_2+\lambda_o(t+1)}$ will need to be positive. We have that $\alpha P_l(t)[\lambda_l(t+1)\zeta_1 + \lambda_p(t+1)\zeta_2] > 0$. We also need to consider $\beta_2 + \lambda_o(t+1)$. Recall that

$$\lambda_o(t) = -\alpha\zeta_1 e^{-\mu}\lambda_l(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k)$$

$$-\alpha\zeta_2 e^{-\mu}\lambda_p(k+1)e^{-\alpha(\mathcal{N}_o(k)e^{-\mu}+\mathcal{N}_n(k))}\mathcal{P}_l(k) + \lambda_o(k+1)e^{-\mu}.$$

Since $\lambda_o(T) = 0$, we have that $\lambda_o(t) < 0$ for all t, which means for $\beta_2 + \lambda_o(t+1)$ to be positive we need that $\beta_2 > \lambda_o(t+1)$, as β_2 is a positive constant. In our case study of DRW, $\beta_2 = 2.8182 \times 10^{-8}$. After running simulations we found that for DRW, the only cases when the FBS is converging is if $\mu \to \infty$, meaning we choose large values for μ to mimic the Basic model. Meanwhile, for cases with μ as a fraction it is not possible to have $\beta_2 > \lambda_o(t+1)$, so the Forward-Backward Sweep was always assigning $N(t) = 0$ or having issues with convergence for the entire time frame.

Thus, for the Nematodes Persist Model, we must use the MultiStart algorithm to find approximations of the amount of nematodes to apply for various choices of μ considering nematodes survive for longer lengths of time. Figure 4.20 shows varying μ for initial population 1100000 and how issues occur for smaller values of μ but larger values converge to the Basic model results.

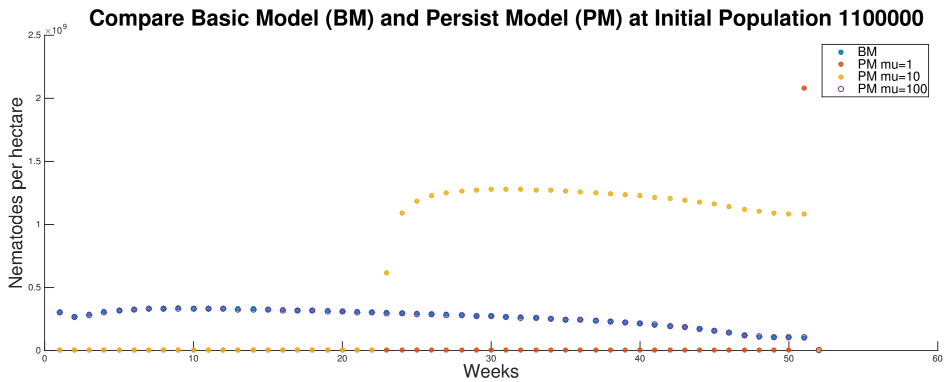

Figure 4.20: We have initial population of 1100000 and compare the Basic Model with the Persist Model with $\mu = 1, 10, 100$.

4.4.2 Multistep

4.4.2.1 Varying Initial Population and Survival per Time Step Parameter, μ

Using MultiStart we first varied the initial population with value $\mu = \ln(2)$ to see the change in number of nematodes to apply. Again, we found that the results in MultiStart were seemingly converging to the same constant value of Nematodes to apply for approximately weeks from 6 to 46, see Figure 4.12. So we we graphed the average number of nematodes Multistart instructs to apply from weeks 6 to 46 for varying both initial population and values of μ. Additionally we graphed the total cost associated with varying both initial population and values of μ.

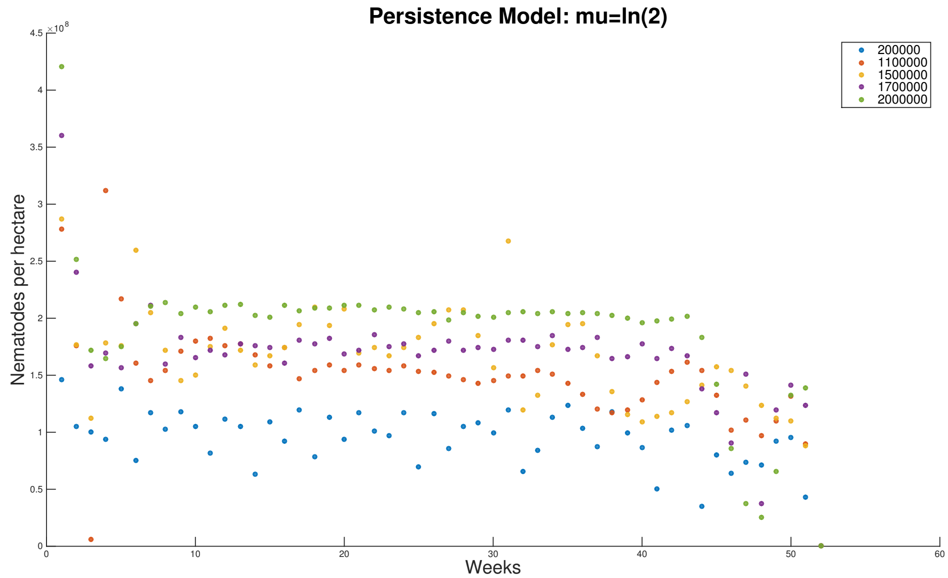

Figure 4.21: Vary initial populations 200000, 1100000, 1500000, 1700000, 2000000 with $\mu = \ln(2)$.

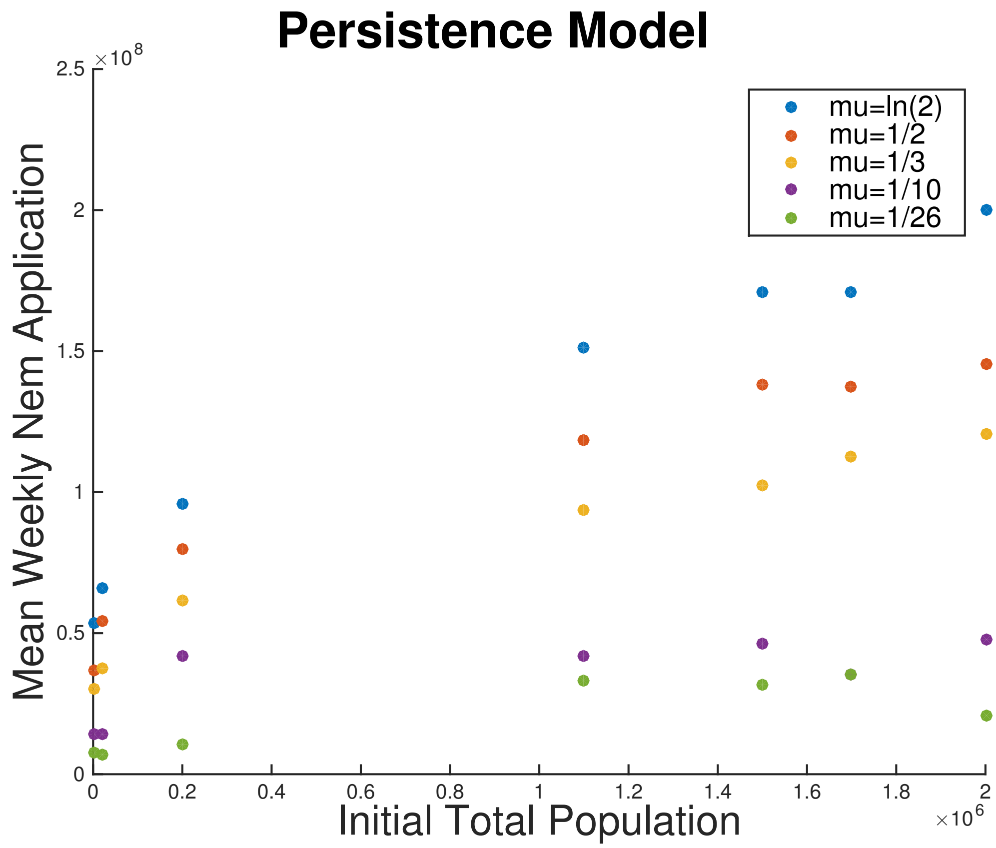

Figure 4.22: Similar to in Figure 4.13 we took the average value for each initial population of nematodes application between 6 and 46 weeks and plotted them above for the Persist case various μ (mu) values.

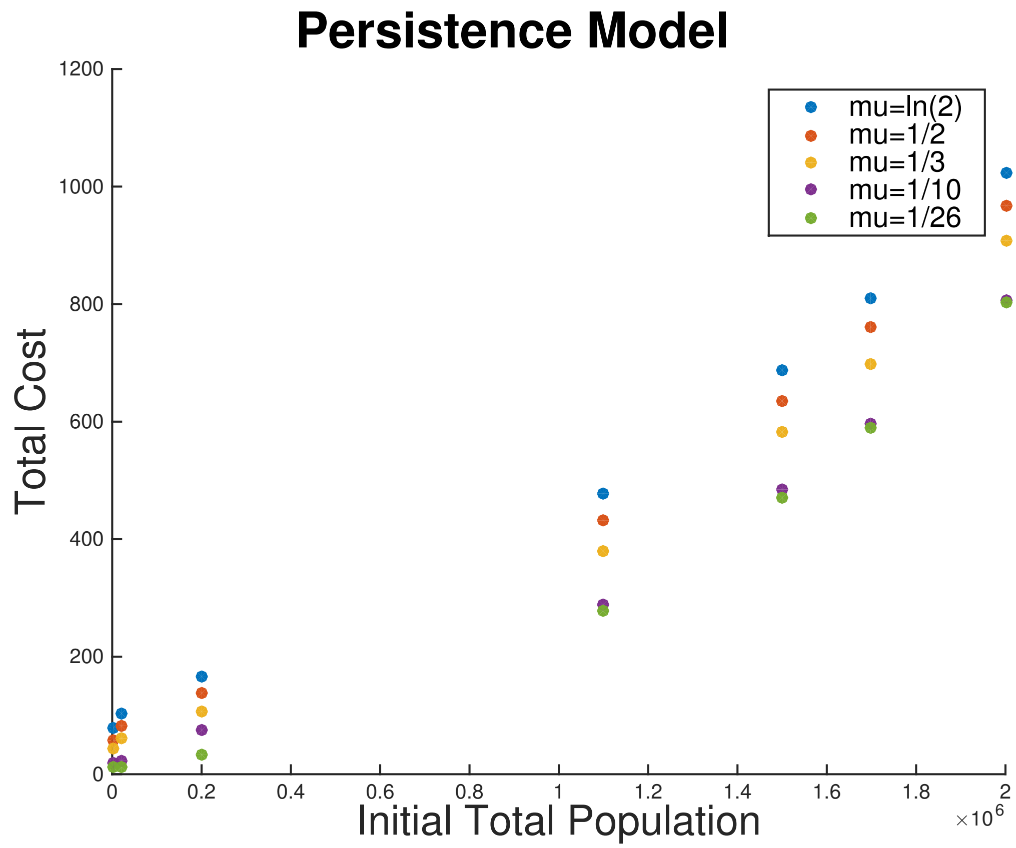

Figure 4.23: The Total Cost, or value of J, which corresponds to all the initial populations, μ values and nematodes applications from Figure 4.22.

4.4.2.2 Varying Nematode Cost Parameter, β_2

We varied the cost associated with the purchase of nematodes. We plot the number of nematodes to apply for initial population 1100000 for various percent changes of β_2. Also, we look again at the average value of nematodes to apply for weeks 6 through 46 and the total cost associated with the nematode application.

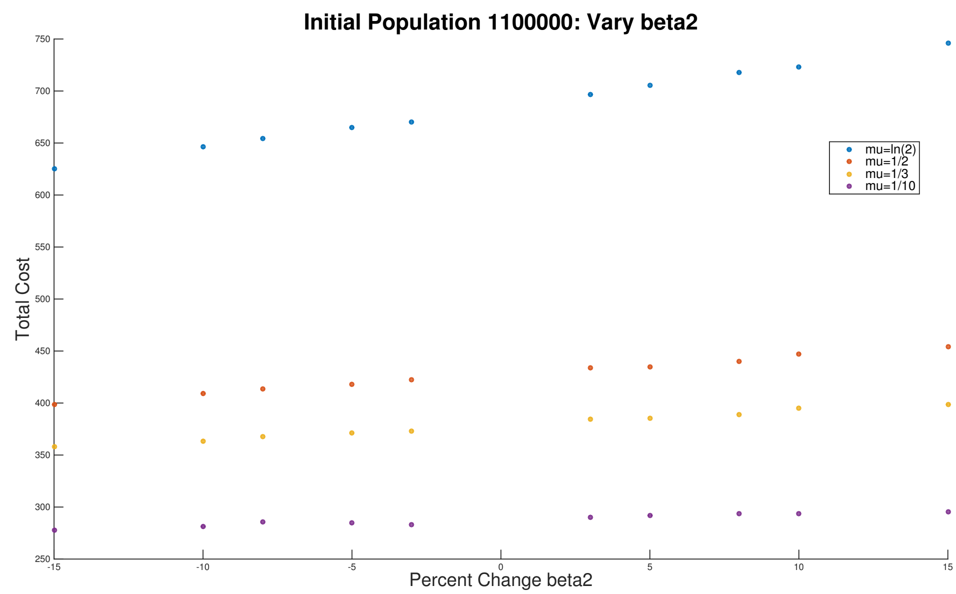

Figure 4.24: For initial population 1100000 we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15 and μ by $\ln(2), 1/2, 1/3, 1/10$. We plot the Total Cost.

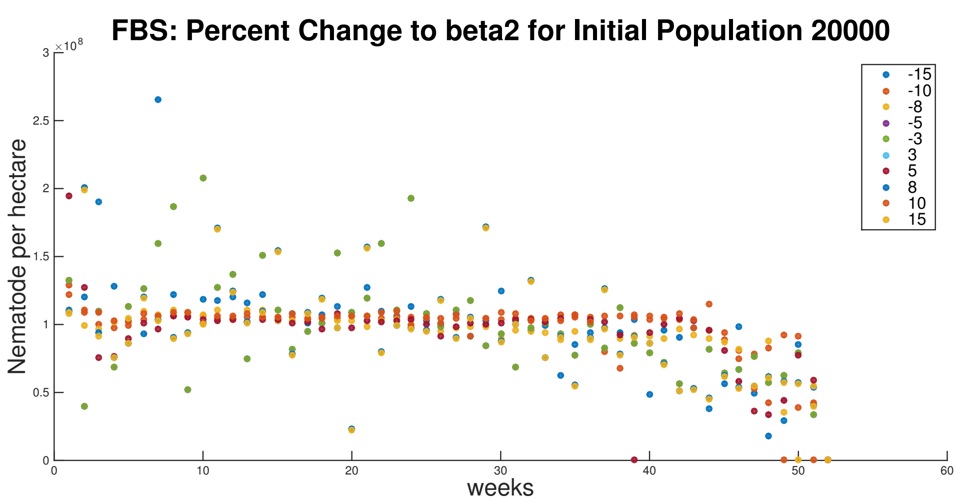

Figure 4.25: For initial population 200000 and $\ln(2)$ we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15.

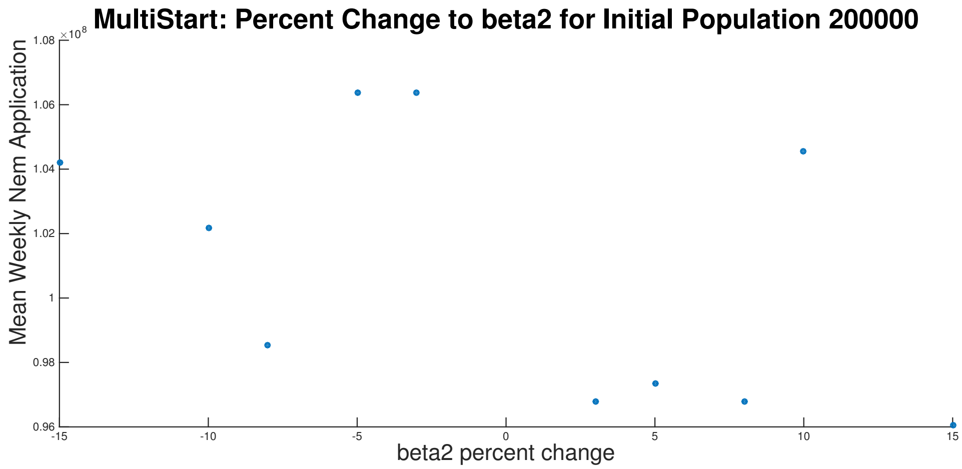

Figure 4.26: For initial population 200000 and ln(2) we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. As in Figure 4.13 took the average value for each initial population of nematodes application between 6 and 46 weeks. Note issues arise due to the curves erratic behavior for certain percents in Figure 4.25

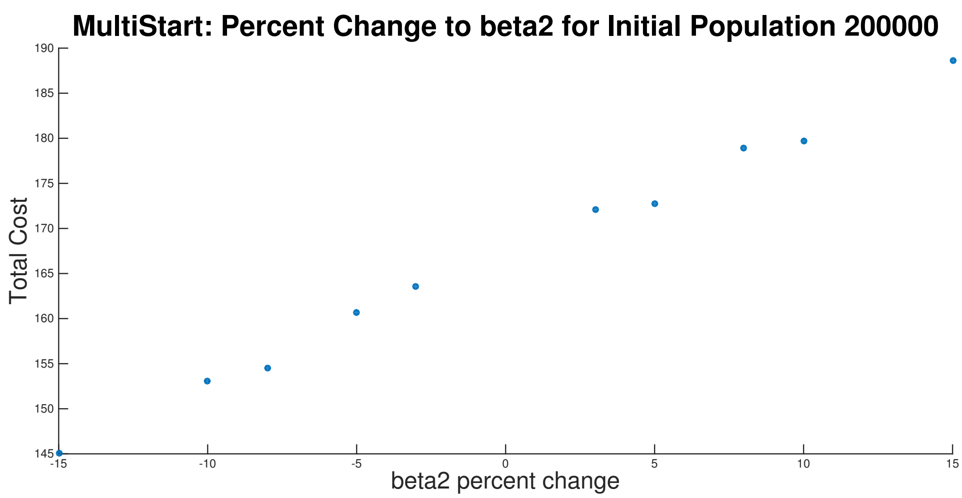

Figure 4.27: For initial population 200000 and ln(2) we vary the value of β_2 by the percents -15, -10, -8, -5, -3, 0, 3, 5, 8, 10, 15. The figure shows the different total costs for the nematodes applied in 4.25

4.4.2.3 Basic Model Comparison

We compare the total costs for various initial populations from the Basic Model FBS and MultiStart methods with the Persist model varying the value of μ.

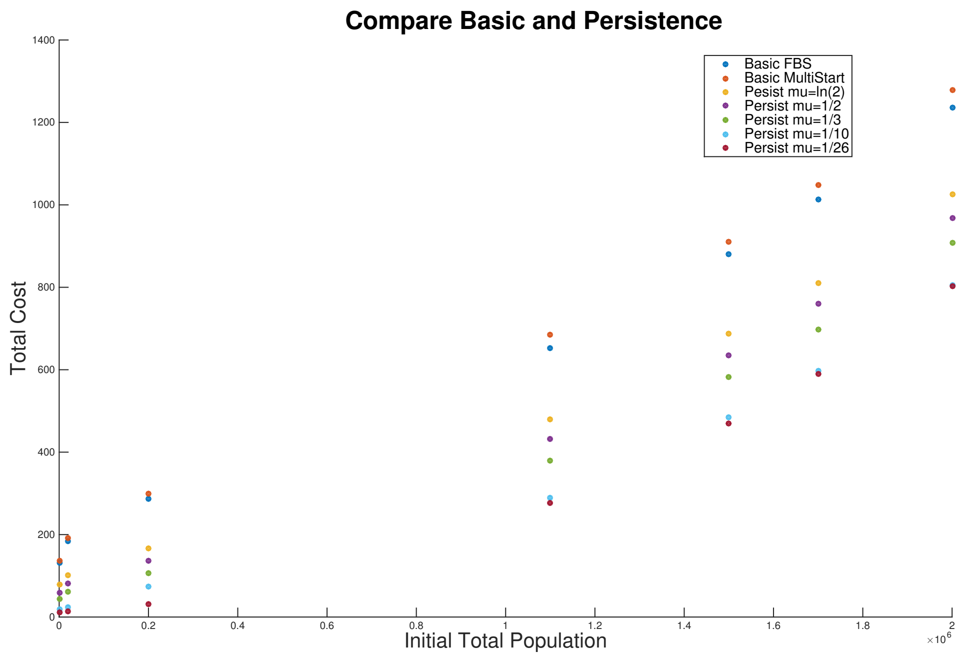

Figure 4.28: Vary initial populations 2000, 20000, 200000, 1100000, 1500000, 1700000, 2000000 for Basic FBS, Basic MultiStart, and the DRW Persistence Model for $\mu = \ln(2), 1/2, 1/3, 1/10, 1/26$. We map the Total Cost.

4.4.3 Nematodes Persist Discussion/Summary

We consider fractional values for μ since it is the survival per time step parameter, and for fractional values we have higher percents of nematodes survive per week. Specifically recall if $\mu = \ln(2)$, then half the nematodes survive one week. In Figure 4.21 we start as we did in the Basic model by varying the initial population for $\mu = \ln(2)$. As with the basic model lower initial density requires less nematodes, and we can note in general less nematodes are required since 50% survive.

Since it is likely that some nematodes survive from one week to the next, we vary μ to see how different survival percentages affect the number of nematodes we apply and the total cost. In Figures 4.22 and 4.23 we plot the average number of nematodes to apply each week for varying values of μ and initial populations. As expected if μ is decreased, so the percent of nematodes that survive the week increases, then we decrease the nematode application and total cost. The total cost curves in Figure 4.23 seem to be linear and have an even spread, while Figure 4.22 are less linear due to the nature of MultiStart, which is reflected in 4.21.

As with the Basic model we varied other parameters in the model, specifically shown in Figure 4.24 we vary β_2, the nematode cost parameter, for initial population 1100000 and various values of μ. As we would expect when we decrease β_2 the total cost decreases, and the various μ values also correspond as we saw in previous analysis without varying β_2. In Figure 4.25 we plot the number of nematodes to apply every week with initial population 2000 for various percent changes of β_2. Note again we are using MultiStart, so the erratic behavior is not surprising and reflected in Figure 4.26 for the average nematode application. If we look at Figure 4.27 we can see that even though the nematode application seems erratic, we still have a close to linear growth in the total cost with varying β_2 which does correspond to smaller β_2 means smaller cost. We have similar results when varying β_1 and α as we did in the Basic model, with expected decreases in nematode application and total cost when we incorporate nematode survival, μ.

In Figure 4.28 we compare the total cost for the Basic and Persist models at various initial total populations. Specifically we have the total cost of the FBS and MultiStart of the Basic model as in Figure 4.17, and the Persist model for $\mu = \ln(2), 1/2, 1/3, 1/10, 1/26$. We can note that the Basic model incurs a higher total cost, since no nematodes survive one week. Meanwhile, we can see the drop in the total cost when we utilize $\mu = \ln(2)$, 50% of nematodes survive, for some initial populations a reduction of over $100 dollars as initial populations increase. As the value of μ drops so does the total cost.

From the Persist model we have learned that if nematodes can survive for more than a week we can reduce the total cost, and ensure we do not over apply nematodes to the system. We do have similar behavior results when varying α, β_1, and β_2 as in the Basic model, with natural changes when we vary μ.

Part II

Optimal Control of Invasive Species with Spatial Dispersal

Chapter 5

Introduction

5.1 Spatial Spread

Since invasive species often spread spatially, it is often incorporated into models. In 1937 the first mathematical spatial spread ecology models were developed by Fisher. These early models used partial differential equations, with an aim to derive conclusions relating to asymptotic rate of spread [Fis37, HCD$^+$05].

An important area of studying spatial spread is Metapopulation Ecology, see the book and articles by Ilkka Hanski [Han94, Han98, Han99].

5.2 Overview

Our plan is to explore management of a spreading invasive species using optimal control theory. In Part II we will consider a multiple patch model.

In Chapter 6, we will introduce a two patch model which uses the basic model from Part 1 but allow movement between two patches. We will explore adult pests movement between the two patches. Furthermore we will prove existence, necessary conditions, and uniqueness for the optimal control.

In Chapter 7, we will introduce a four patch model which uses the basic model from Part 1 but allows movement between four patches. We will consider two ways that adult pests can dispersal between the four patches. As in part 1, we will consider the models for the case study investigating Diaprepes abbreviatus, DRW, and run simulations.

5.3 Reference Chart

	Notation	Description
	$P_{i,e}$	Number of eggs in patch i
Pest	$P_{i,l}$	Number of larvae in patch i
Vector	$P_{i,p}$	Number of pupae in patch i
	$P_{i,a}$	Number of adults in patch i
Pest	θ_1	Fecundity rate of female Pest adults
Matrix	$\theta_{i,i}$	Pest adult survival in specific patches
Changes	$\theta_{i,j}$	Pest adult survival in different patches

Chapter 6

Two Patches - Adults Spread

6.1 Model Formulation

In this section we consider a population that has as its habitat two patches. We be using the Basic model on two patches, so no control agent persistence. Additionally, we will consider that the adult pest can fly and travel between patches, so our new matrix for the pest dynamics will be as follows:

$$A_2 = \left[\begin{array}{cccc:cccc} \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} \\ \hdashline 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\ 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} \end{array}\right].$$

This model uses two copies of A for the two patches, but rather than θ_2 for adult survival we have $\theta_{1,1}$ and $\theta_{2,2}$ specifying how many adults survive and remain in their original patch. Meanwhile, $\theta_{1,2}$ and $\theta_{2,1}$ are how many adults survive and transition to the other patch, for instance $\theta_{1,2}$ are adults from patch 1 which travel to patch 2. Below is the formulation of the pest dynamics for the two patch model. Note this does not include the biological control in the larva stage.

$$\begin{bmatrix} P_{1,e}(k+1) \\ P_{1,l}(k+1) \\ P_{1,p}(k+1) \\ P_{1,a}(k+1) \\ P_{2,e}(k+1) \\ P_{2,l}(k+1) \\ P_{2,p}(k+1) \\ P_{2,a}(k+1) \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} \\ 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\ 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} \end{bmatrix} \begin{bmatrix} P_{1,e}(k) \\ P_{1,l}(k) \\ P_{1,p}(k) \\ P_{1,a}(k) \\ P_{2,e}(k) \\ P_{2,l}(k) \\ P_{2,p}(k) \\ P_{2,a}(k) \end{bmatrix}$$

Cost in Two Patches Since we are considering two independent patches, the cost in each patch would be the same formula as the cost in our Basic model. So, if we consider the total cost in two patches we combine the cost in each these two isolated patches,

$$\textit{Cost Two Patches} = \beta_1 P_{1,l}(t)^2 \beta_2 N_1(t) + \beta_1 P_{2,l}(t)^2 + \beta_2 N_2(t).$$

6.2 Optimal Control Problem

The set-up for our Optimal Control Problem is to minimize the objective functional

$$J(N_1, N_2) = \sum_{t=0}^{T-1} \beta_1 [P_{1,l}(t)^2 + P_{2,l}(t)^2] + \beta_2 [N_1(t) + N_2(t)]$$

subject to

$$
\begin{aligned}
P_{1,e}(t+1) &= \gamma_1 P_{1,e}(t) + \theta_1 P_{1,a}(t) && P_{1,e}(0) = \phi_{1,e} \\
P_{1,l}(t+1) &= \gamma_2 P_{1,e}(t) + \zeta_1 e^{-\alpha N_1(t)} P_{1,l}(t) && P_{1,l}(0) = \phi_{1,l} \\
P_{1,p}(t+1) &= \zeta_2 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_1 P_{1,p}(t) && P_{1,p}(0) = \phi_{1,p} \\
P_{1,a}(t+1) &= \nu_2 P_{1,p}(t) + \theta_{1,1} P_{1,a}(t) + \theta_{2,1} P_{2,a}(t) && P_{1,a}(0) = \phi_{1,a} \\
P_{2,e}(t+1) &= \gamma_1 P_{2,e}(t) + \theta_1 P_{2,a}(t) && P_{2,e}(0) = \phi_{2,e} \\
P_{2,l}(t+1) &= \gamma_2 P_{2,e}(t) + \zeta_1 e^{-\alpha N_2(t)} P_{2,l}(t) && P_{2,l}(0) = \phi_{2,l} \\
P_{2,p}(t+1) &= \zeta_2 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_1 P_{2,p}(t) && P_{2,p}(0) = \phi_{2,p} \\
P_{2,a}(t+1) &= \nu_2 P_{2,p}(t) + \theta_{2,2} P_{2,a}(t) + \theta_{1,2} P_{1,a}(t) && P_{2,a}(0) = \phi_{2,a}
\end{aligned}
\tag{6.1}
$$

where $N_1, N_2 \in \mathbf{N} = \{N : \{1, ..., T\} \to \{x \in \mathbb{R} | 0 \leq x(t) \leq N_{max}, t = 1, 2, ..., T\}\}$.

Now we will prove the existence and uniqueness of the optimal control, which we denote $\mathcal{N}_1$ and $\mathcal{N}_2$. Additionally, we will prove necessary conditions for the optimal control $\mathcal{N}_1$ and $\mathcal{N}_2$. The proofs roughly follow the proofs in Theorems 2.3.1, 2.3.2, 2.3.3.

Note in the following proofs each $\mathcal{P}_{1,e}, \mathcal{P}_{1,l}, \mathcal{P}_{1,p}, \mathcal{P}_{1,a}, \mathcal{P}_{2,e}, \mathcal{P}_{2,l}, \mathcal{P}_{2,p}, \mathcal{P}_{2,a}$ is a function of $\mathcal{N}_1$ and $\mathcal{N}_2$. Similarly each $\mathcal{P}^{\varepsilon}_{1,e}, \mathcal{P}^{\varepsilon}_{1,l}, \mathcal{P}^{\varepsilon}_{1,p}, \mathcal{P}^{\varepsilon}_{1,a}, \mathcal{P}^{\varepsilon}_{2,e}, \mathcal{P}^{\varepsilon}_{2,l}, \mathcal{P}^{\varepsilon}_{2,p}, \mathcal{P}^{\varepsilon}_{2,a}$ is a function of $\mathcal{N}_1 + \eta_1\varepsilon$ and $\mathcal{N}_2 + \eta_2\varepsilon$.

6.2.1 Existence

Theorem 6.2.1. *There exists* $\mathcal{N}_1, \mathcal{N}_2 \in \boldsymbol{N}$ *which minimizes* $J(N_1, N_2)$.

Proof. We have that each $P_{1,e}, P_{1,l}, P_{1,p}, P_{1,a}, P_{2,e}, P_{2,l}, P_{2,p}, P_{2,a}$ is continuous as a function of N_1, N_2 at every time step by Equation 6.1. Define $B^+ = \{(N(1), ...N(T))|N \in \mathbf{N}\}$. We note that there is a natural isomorphism between $\mathbf{N} \times \mathbf{N}$ and $B^+ \times B^+$. Considering $J : \mathbf{N} \times \mathbf{N} \leftrightarrow B^+ \times B^+ \to \mathbb{R}$, we see that J is continuous as a function of N_1 and N_2. We have that B^+ is a compact subset of $\mathbb{R}^T$ in the standard Euclidean topology. Thus, $\inf_{N_1, N_2 \in \mathbf{N}} J(N_1, N_2)$ exists. Hence, we have sequences $N_{1_k}, N_{2_k} \in \mathbf{N}$ such that $\lim_{k\to\infty} J(N_{1_k}, N_{2_k}) = \inf_{N_1, N_2 \in \mathbf{N}} J(N_1, N_2)$, with corresponding $P_{1,e_k}, P_{1,l_k}, P_{1,p_k}, P_{1,a_k}, P_{2,e_k}, P_{2,l_k}, P_{2,p_k}, P_{2,a_k}$ sequences. Thus we can find subsequences $N_{1_{k_j}}, N_{2_{k_j}}, P_{1,e_{k_j}}, P_{1,l_{k_j}}, P_{1,p_{k_j}}, P_{1,a_{k_j}}, P_{2,e_{k_j}}, P_{2,l_{k_j}}, P_{2,p_{k_j}}, P_{2,a_{k_j}}$, such that $\lim_{j\to\infty} J(N_{1_{k_j}}, N_{2_{k_j}}) = \inf_{N_1, N_2 \in \mathbf{N}} J(N_1, N_2)$ and converge to $N_{1_{k_j}} \to \mathcal{N}_1$, $N_{2_{k_j}} \to \mathcal{N}_2$, $P_{1,e_{k_j}} \to \mathcal{P}_{1,e}$, $P_{1,l_{k_j}} \to \mathcal{P}_{1,l}$, $P_{1,p_{k_j}} \to \mathcal{P}_{1,p}$, $P_{1,a_{k_j}} \to \mathcal{P}_{1,a}$, $P_{2,e_{k_j}} \to \mathcal{P}_{2,e}$, $P_{2,l_{k_j}} \to \mathcal{P}_{2,l}$, $P_{2,p_{k_j}} \to \mathcal{P}_{2,p}$, $P_{2,a_{k_j}} \to \mathcal{P}_{2,a}$. Therefore, there exists $\mathcal{N}_1, \mathcal{N}_2 \in \mathbf{N}$ which minimizes $J(N_1, N_2)$.

□

6.2.2 Necessary Conditions

Adjoint System: Define the following terminal value system:

$$\lambda_{1,e}(t) = \lambda_{1,e}(t+1)\gamma_1 + \lambda_{1,l}(t+1)\gamma_2$$

$$\lambda_{1,l}(t) = 2\beta_1 \mathcal{P}_{1,l}(t) + \lambda_{1,l}(t+1)\zeta_1 e^{-\alpha \mathcal{N}_1(t)} + \lambda_{1,p}(t+1)\zeta_2 e^{-\alpha \mathcal{N}_1(t)}$$

$$\lambda_{1,p}(t) = \lambda_{1,p}(t+1)\nu_1 + \lambda_{1,a}(t+1)\nu_2$$

$$\lambda_{1,a}(t) = \lambda_{1,e}(t+1)\theta_1 + \lambda_{1,a}(t+1)\theta_{1,1} + \lambda_{2,a}(t+1)\theta_{1,2}$$

$$\lambda_{2,e}(t) = \lambda_{2,e}(t+1)\gamma_1 + \lambda_{2,l}(t+1)\gamma_2$$

$$\lambda_{2,l}(t) = 2\beta_1 \mathcal{P}_{2,l}(t) + \lambda_{2,l}(t+1)\zeta_1 e^{-\alpha \mathcal{N}_2(t)} + \lambda_{2,p}(t+1)\zeta_2 e^{-\alpha \mathcal{N}_2(t)}$$

$$\lambda_{2,p}(t) = \lambda_{2,p}(t+1)\nu_1 + \lambda_{2,a}(t+1)\nu_2$$
$$\lambda_{2,a}(t) = \lambda_{2,e}(t+1)\theta_1 + \lambda_{2,a}(t+1)\theta_{2,2} + \lambda_{1,a}(t+1)\theta_{2,1}$$

$\lambda_{1e}(T) = 0$, $\lambda_{1,l}(T) = 0$, $\lambda_{1,p}(T) = 0$, $\lambda_{1,a}(T) = 0$, $\lambda_{2,e}(T) = 0$, $\lambda_{2,l}(T) = 0$, $\lambda_{2,p}(T) = 0$, $\lambda_{2,a}(T) = 0$.

Theorem 6.2.2. *If there exist optimal controls $\mathcal{N}_1$ and $\mathcal{N}_2$, then there exists adjoint system 6.2.2, and*

$$\mathcal{N}_1(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi_1(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi_1(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi_1(t) \end{cases}$$

$$\mathcal{N}_2(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi_2(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi_2(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi_2(t) \end{cases}.$$

with

$$\xi_1(t) = \zeta_1\lambda_{1,l}(t+1)\mathcal{P}_{1,l}(t) + \zeta_2\lambda_{1,p}(t+1)\mathcal{P}_{1,l}(t)$$

and

$$\xi_2(t) = \zeta_1\lambda_{2,l}(t+1)\mathcal{P}_{2,l}(t) + \zeta_2\lambda_{2,p}(t+1)\mathcal{P}_{2,l}(t)$$

Proof. Since we have that $\mathcal{N}_1$ and $\mathcal{N}_2$ minimize $J(N_1, N_2)$; for all $\eta_1, \eta_2 \in \{\eta = (\eta(1), ..., \eta(T)) | \eta(t) \leq 1, t = 1, ..., T\}$ we have that $J(\mathcal{N}_1 + \eta_1\varepsilon, \mathcal{N}_2 + \eta_2\varepsilon) \geq J(\mathcal{N}_1, \mathcal{N}_2)$ for all sufficiently small $\varepsilon > 0$. Now we will take a directional derivative of functional J; so for the directional derivative in direction of $\eta = [\eta_1, \eta_2]^T$ with sufficiently small $\varepsilon > 0$ and $0 \leq \mathcal{N}_1 + \eta_1\varepsilon = \mathcal{N}_1^\varepsilon, \mathcal{N}_2 + \eta_2\varepsilon = \mathcal{N}_2^\varepsilon \in \mathbf{N}_n$ we have that:

$$0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N}_1 + \eta_1\varepsilon, \mathcal{N}_2 + \eta_2\varepsilon) - J(\mathcal{N}_1, \mathcal{N}_2)]$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \sum_{t=0}^{T-1} \beta_1(\mathcal{P}_{1,l}^{\varepsilon}(t)^2 + \mathcal{P}_{2,l}^{\varepsilon}(t)^2) + \beta_2(\mathcal{N}_1^{\varepsilon}(t) + \mathcal{N}_2^{\varepsilon}(t))$$

$$- \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \sum_{t=0}^{T-1} \beta_1(\mathcal{P}_{1,l}(t)^2 + \mathcal{P}_{2,l}(t)^2) + \beta_2(\mathcal{N}_1(t) + \mathcal{N}_2(t))$$

$$= \sum_{t=0}^{T-1} \beta_1 \lim_{\varepsilon\to 0^+} \left(\frac{\mathcal{P}_{1,l}^{\varepsilon}(t)^2 - \mathcal{P}_{1,l}(t)^2}{\varepsilon} + \frac{\mathcal{P}_{2,l}^{\varepsilon}(t)^2 - \mathcal{P}_{2,l}(t)^2}{\varepsilon} \right)$$

$$+ \sum_{t=0}^{T-1} \beta_2 \lim_{\varepsilon\to 0^+} \left(\frac{\mathcal{N}_1^{\varepsilon}(t) - \mathcal{N}_1(t)}{\varepsilon} + \frac{\mathcal{N}_2^{\varepsilon}(t) - \mathcal{N}_2(t)}{\varepsilon} \right)$$

$$= \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \beta_1 2\mathcal{P}_{2,l}(t)\psi_{2,l}(t) + \beta_2\eta_1(t) + \beta_2\eta_2(t).$$

We define the sensitivities, $\psi_{1,e}(t), \psi_{1,l}(t), \psi_{1,p}(t), \psi_{1,a}(t), \psi_{2,e}(t), \psi_{2,l}(t), \psi_{2,p}(t), \psi_{2,a}(t)$ as:

$$\psi_{1,e}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{1,e}^{\varepsilon}(t) - \mathcal{P}_{1,e}(t)}{\varepsilon}, \quad \psi_{1,l}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{1,l}^{\varepsilon}(t) - \mathcal{P}_{1,l}(t)}{\varepsilon},$$

$$\psi_{1,p}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{1,p}^{\varepsilon}(t) - \mathcal{P}_{1,p}(t)}{\varepsilon}, \quad \psi_{1,a}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{1,a}^{\varepsilon}(t) - \mathcal{P}_{1,a}(t)}{\varepsilon},$$

$$\psi_{2,e}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{2,e}^{\varepsilon}(t) - \mathcal{P}_{2,e}(t)}{\varepsilon}, \quad \psi_{2,l}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{2,l}^{\varepsilon}(t) - \mathcal{P}_{2,l}(t)}{\varepsilon},$$

$$\psi_{2,p}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{2,p}^{\varepsilon}(t) - \mathcal{P}_{2,p}(t)}{\varepsilon}, \quad \psi_{2,a}(t) = \lim_{\varepsilon\to 0} \frac{\mathcal{P}_{2,a}^{\varepsilon}(t) - \mathcal{P}_{2,a}(t)}{\varepsilon}$$

where

$$\psi_{1,e}(0) = 0,\ \psi_{1,l}(0) = 0,\ \psi_{1,p}(0) = 0,\ \psi_{1,a}(0) = 0, \psi_{2,e}(0) = 0,\ \psi_{2,l}(0) = 0,$$

$$\psi_{2,p}(0) = 0,\ \psi_{2,a}(0) = 0.$$

We have that the limits exists from Miller and Lenhart [LW07].

Hence, we can write:

$$\psi_{1,e}(t+1) = \gamma_1\psi_{1,e}(t) + \theta_1\psi_{1,a}(t)$$

$$\psi_{1,l}(t+1) = \gamma_2\psi_{1,e}(t) + \zeta_1 e^{-\alpha\mathcal{N}_1(t)}\psi_{1,l}(t) - \zeta_1\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t)$$

$$\psi_{1,p}(t+1) = \nu_1\psi_{1,p}(t) + \zeta_2 e^{-\alpha\mathcal{N}_1(t)}\psi_{1,l}(t) - \zeta_2\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t)$$

$$\psi_{1,a}(t+1) = \nu_2\psi_{1,p}(t) + \theta_{1,1}\psi_{1,a}(t) + \theta_{2,1}\psi_{2,a}(t)$$

$$\psi_{2,e}(t+1) = \gamma_1\psi_{2,e}(t) + \theta_1\psi_{2,a}(t)$$

$$\psi_{2,l}(t+1) = \gamma_2\psi_{2,e}(t) + \zeta_1 e^{-\alpha\mathcal{N}_2(t)}\psi_{2,l}(t) - \zeta_1\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t)$$

$$\psi_{2,p}(t+1) = \nu_1\psi_{2,p}(t) + \zeta_2 e^{-\alpha\mathcal{N}_2(t)}\psi_{2,l}(t) - \zeta_2\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t)$$

$$\psi_{2,a}(t+1) = \nu_2\psi_{2,p}(t) + \theta_{2,2}\psi_{2,a}(t) + \theta_{1,2}\psi_{1,a}(t).$$

Now, returning to

$$0 \leq \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}[J(\mathcal{N}_1+\eta\varepsilon, \mathcal{N}_2+\eta\varepsilon) - J(\mathcal{N}_1, \mathcal{N}_2)]$$

$$= \sum_{t=0}^{T-1}\beta_1 2\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \beta_1 2\mathcal{P}_{2,l}(t)\psi_{2,l}(t) + \beta_2\eta_1(t) + \beta_2\eta_2(t),$$

to remove the sensitivities $\psi_{1,l}(t), \psi_{2,l}(t)$ we will manipulate the sensitivities and adjoints equations.

We have that:

$$\begin{bmatrix} \psi_{1,e}(t+1) \\ \psi_{1,l}(t+1) \\ \psi_{1,p}(t+1) \\ \psi_{1,a}(t+1) \\ \psi_{2,e}(t+1) \\ \psi_{2,l}(t+1) \\ \psi_{2,p}(t+1) \\ \psi_{2,a}(t+1) \end{bmatrix} - B \begin{bmatrix} \psi_{1,e}(t) \\ \psi_{1,l}(t) \\ \psi_{1,p}(t) \\ \psi_{1,a}(t) \\ \psi_{2,e}(t) \\ \psi_{2,l}(t) \\ \psi_{2,p}(t) \\ \psi_{2,a}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -\zeta_1 \alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t) \eta_1(t) \\ -\zeta_2 \alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t) \eta_1(t) \\ 0 \\ 0 \\ -\zeta_1 \alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t) \eta_2(t) \\ -\zeta_2 \alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t) \eta_2(t) \\ 0 \end{bmatrix}$$

$$\text{where } B = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \zeta_1 e^{-\alpha \mathcal{N}_1(k)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 e^{-\alpha \mathcal{N}_1(k)} & \nu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} \\ 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\ 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 e^{-\alpha \mathcal{N}_2(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_2 e^{-\alpha \mathcal{N}_2(k)} & \nu_1 & 0 \\ 0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} \end{bmatrix}.$$

Now we have that:

$$\sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \beta_1 2\mathcal{P}_{2,l}(t)\psi_{2,l}(t) =$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_{1,e}(t) & \psi_{1,l}(t) & \psi_{1,p}(t) & \psi_{1,a}(t) & \psi_{2,e}(t) & \psi_{2,l}(t) & \psi_{2,p}(t) & \psi_{2,a}(t) \end{bmatrix} \begin{bmatrix} 0 \\ \beta_1 2\mathcal{P}_{1,l}(t) \\ 0 \\ 0 \\ 0 \\ \beta_1 2\mathcal{P}_{2,l}(t) \\ 0 \\ 0 \end{bmatrix}$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_{1,e}(t) & \psi_{1,l}(t) & \psi_{1,p}(t) & \psi_{1,a}(t) & \psi_{2,e}(t) & \psi_{2,l}(t) & \psi_{2,p}(t) & \psi_{2,a}(t) \end{bmatrix} \left(\begin{bmatrix} \lambda_{1,e}(t) \\ \lambda_{1,l}(t) \\ \lambda_{1,p}(t) \\ \lambda_{1,a}(t) \\ \lambda_{2,e}(t) \\ \lambda_{2,l}(t) \\ \lambda_{2,p}(t) \\ \lambda_{2,a}(t) \end{bmatrix} - B^T \begin{bmatrix} \lambda_{1,e}(t+1) \\ \lambda_{1,l}(t+1) \\ \lambda_{1,p}(t+1) \\ \lambda_{1,a}(t+1) \\ \lambda_{2,e}(t+1) \\ \lambda_{2,l}(t+1) \\ \lambda_{2,p}(t+1) \\ \lambda_{2,a}(t+1) \end{bmatrix} \right)$$

$$= \sum_{t=0}^{T-1} \left[\begin{array}{cccccccc} \psi_{1,e}(t) & \psi_{1,l}(t) & \psi_{1,p}(t) & \psi_{1,a}(t) & \psi_{2,e}(t) & \psi_{2,l}(t) & \psi_{2,p}(t) & \psi_{2,a}(t) \end{array} \right] \left[\begin{array}{c} \lambda_{1,e}(t) \\ \lambda_{1,l}(t) \\ \lambda_{1,p}(t) \\ \lambda_{1,a}(t) \\ \lambda_{2,e}(t) \\ \lambda_{2,l}(t) \\ \lambda_{2,p}(t) \\ \lambda_{2,a}(t) \end{array} \right]$$

$$- \sum_{t=0}^{T-1} \left[\begin{array}{cccccccc} \psi_{1,e}(t) & \psi_{1,l}(t) & \psi_{1,p}(t) & \psi_{1,a}(t) & \psi_{2,e}(t) & \psi_{2,l}(t) & \psi_{2,p}(t) & \psi_{2,a}(t) \end{array} \right] B^T \left[\begin{array}{c} \lambda_{1,e}(t+1) \\ \lambda_{1,l}(t+1) \\ \lambda_{1,p}(t+1) \\ \lambda_{1,a}(t+1) \\ \lambda_{2,e}(t+1) \\ \lambda_{2,l}(t+1) \\ \lambda_{2,p}(t+1) \\ \lambda_{2,a}(t+1) \end{array} \right].$$

Recall that

$$\psi_{1,e}(0) = 0,\ \psi_{1,l}(0) = 0,\ \psi_{1,p}(0) = 0,\ \psi_{1,a}(0) = 0, \psi_{2,e}(0) = 0,\ \psi_{2,l}(0) = 0,\ \psi_{2,p}(0) = 0,\ \psi_{2,a}(0) = 0$$

and

$$\lambda_{1,e}(T) = 0,\ \lambda_{1,l}(T) = 0,\ \lambda_{1,p}(T) = 0,\ \lambda_{1,a}(T) = 0, \lambda_{2,e}(T) = 0,\ \lambda_{2,l}(T) = 0,\ \lambda_{2,p}(T) = 0,\ \lambda_{2,a}(T) = 0.$$

Therefore we can change the indices, so that:

$$\sum_{t=0}^{T-1} \beta_1 2 P_l(t) \psi_l(t) =$$

$$= \sum_{t=0}^{T-1} \begin{bmatrix} \psi_{1,e}(t+1) & \psi_{1,l}(t+1) & \psi_{1,p}(t+1) & \psi_{1,a}(t+1) & \psi_{2,e}(t+1) & \psi_{2,l}(t+1) & \psi_{2,p}(t+1) & \psi_{2,a}(t+1) \end{bmatrix} \begin{bmatrix} \lambda_{1,e}(t+1) \\ \lambda_{1,l}(t+1) \\ \lambda_{1,p}(t+1) \\ \lambda_{1,a}(t+1) \\ \lambda_{2,e}(t+1) \\ \lambda_{2,l}(t+1) \\ \lambda_{2,p}(t+1) \\ \lambda_{2,a}(t+1) \end{bmatrix}$$

$$- \sum_{t=0}^{T-1} \begin{bmatrix} \lambda_{1,e}(t+1) & \lambda_{1,l}(t+1) & \lambda_{1,p}(t+1) & \lambda_{1,a}(t+1) & \lambda_{2,e}(t+1) & \lambda_{2,l}(t+1) & \lambda_{2,p}(t+1) & \lambda_{2,a}(t+1) \end{bmatrix} B \begin{bmatrix} \psi_{1,e}(t) \\ \psi_{1,l}(t) \\ \psi_{1,p}(t) \\ \psi_{1,a}(t) \\ \psi_{2,e}(t) \\ \psi_{2,l}(t) \\ \psi_{2,p}(t) \\ \psi_{2,a}(t) \end{bmatrix}$$

$$=$$

$$\sum_{t=0}^{T-1}\begin{bmatrix}\lambda_{1,e}(t+1) & \lambda_{1,l}(t+1) & \lambda_{1,p}(t+1) & \lambda_{1,a}(t+1) & \lambda_{2,e}(t+1) & \lambda_{2,l}(t+1) & \lambda_{2,p}(t+1) & \lambda_{2,a}(t+1)\end{bmatrix}\left(\begin{bmatrix}\psi_{1,e}(t+1)\\ \psi_{1,l}(t+1)\\ \psi_{1,p}(t+1)\\ \psi_{1,a}(t+1)\\ \psi_{2,e}(t+1)\\ \psi_{2,l}(t+1)\\ \psi_{2,p}(t+1)\\ \psi_{2,a}(t+1)\end{bmatrix} - B\begin{bmatrix}\psi_{1,e}(t)\\ \psi_{1,l}(t)\\ \psi_{1,p}(t)\\ \psi_{1,a}(t)\\ \psi_{2,e}(t)\\ \psi_{2,l}(t)\\ \psi_{2,p}(t)\\ \psi_{2,a}(t)\end{bmatrix}\right)$$

$$= \sum_{t=0}^{T-1}\begin{bmatrix}\lambda_{1,e}(t+1) & \lambda_{1,l}(t+1) & \lambda_{1,p}(t+1) & \lambda_{1,a}(t+1) & \lambda_{2,e}(t+1) & \lambda_{2,l}(t+1) & \lambda_{2,p}(t+1) & \lambda_{2,a}(t+1)\end{bmatrix}\begin{bmatrix}0\\ -\zeta_1\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t)\\ -\zeta_2\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t)\\ 0\\ 0\\ -\zeta_1\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t)\\ -\zeta_2\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t)\\ 0\end{bmatrix}$$

$$= \textstyle\sum_{t=0}^{T-1}\lambda_{1,l}(t+1)(-\zeta_1\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t)) + \lambda_{1,p}(t+1)(-\zeta_2\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t))$$
$$+\lambda_{2,l}(t+1)(-\zeta_1\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t)) + \lambda_{2,p}(t+1)(-\zeta_2\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t))$$
$$= \textstyle\sum_{t=0}^{T-1}[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2][-\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t))] + [\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2][-\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t))].$$

Now combining everything we have that:

$$0 \le \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[J(\mathcal{N}_1 + \eta\varepsilon, \mathcal{N}_2 + \eta\varepsilon) - J(\mathcal{N}_1, \mathcal{N}_2)]$$

$$= \sum_{t=0}^{T-1} \beta_1 2\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \beta_1 2\mathcal{P}_{2,l}(t)\psi_{2,l}(t) + \beta_2\eta_1(t) + \beta_2\eta_2(t)$$

$$= \sum_{t=0}^{T-1}[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2][-\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)\eta_1(t))]$$

$$+[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2][-\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)\eta_2(t))]$$

$$= \sum_{t=0}^{T-1} \eta_1(t)\left[-\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_2(t)\left[-\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right].$$

Considering the previous equation with equality

$$0 = \sum_{t=0}^{T-1} \eta_1(t)\left[-\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_2(t)\left[-\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right].$$

Since this must hold for all η_1 and η_2, we have that for all t,

$$0 = -\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2$$

and

$$0 = -\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2.$$

Solution for $\mathcal{N}_1$: We will consider

$$0 = -\alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2,$$

then:

$$e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] = \frac{\beta_2}{\alpha} \iff$$

$$e^{-\alpha \mathcal{N}_1(t)} = \frac{\beta_2}{\alpha \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]} \iff$$

$$-\alpha \mathcal{N}_1(t) = \ln\left[\frac{\beta_2}{\alpha \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]}\right] \iff$$

$$\alpha \mathcal{N}_1(t) = \ln\left[\frac{\alpha \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]}{\beta_2}\right].$$

Note $\alpha > 0$. We need that $\mathcal{N}_1(t) \geq 0$, so

$$\ln\left[\frac{\alpha \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]}{\beta_2}\right] \geq 0$$

meaning

$$\frac{\alpha \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]}{\beta_2} \geq 1.$$

Hence when

$$\frac{\beta 2}{\alpha} \leq \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]$$

and we have

$$\mathcal{N}_1(t) = \frac{1}{\alpha} \ln(\frac{\alpha}{\beta_2} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]).$$

Now we will consider if

$$\frac{\beta 2}{\alpha} > \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2],$$

then we have:

$$0 = \sum_{t=0}^{T-1} \eta_1(t) \left[-\alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_2(t) \left[-\alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right]$$

$$= \sum_{t=0}^{T-1} \eta_1(t) \left[-\alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right] + \eta_2(t) \cdot 0$$

$$< \sum_{t=0}^{T-1} \eta_1(t) \left[-\alpha e^{-\alpha \mathcal{N}_1(t)} \left(\frac{\beta_2}{\alpha}\right) + \beta_2\right]$$

$$= \sum_{t=0}^{T-1} \eta_1(t) \left[-\beta_2 e^{-\alpha \mathcal{N}_1(t)} + \beta_2\right] = \sum_{t=0}^{T-1} \eta_1(t)\beta_2 \left[-e^{-\alpha \mathcal{N}_1(t)} + 1\right].$$

If $\mathcal{N}_1(t) > 0$ we have that $\beta_2(-e^{-\alpha \mathcal{N}_1(t)} + 1) < 0$, which is a contradiction. Thus, if

$$\frac{\beta 2}{\alpha} > \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2]$$

we must have that $\mathcal{N}_1(t) = 0$. Set

$$\xi_1(t) = \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2],$$

so

$$\mathcal{N}_1(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi_1(t) \\ \frac{1}{\alpha} \ln[\frac{\alpha}{\beta_2}\xi_1(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi_1(t) \end{cases}$$

Solution for $\mathcal{N}_2$: We consider

$$0 = -\alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2,$$

then:

$$e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] = \frac{\beta_2}{\alpha} \iff$$

$$e^{-\alpha \mathcal{N}_2(t)} = \frac{\beta_2}{\alpha \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]} \iff$$

$$-\alpha \mathcal{N}_2(t) = \ln\left[\frac{\beta_2}{\alpha \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]}\right] \iff$$

$$\alpha \mathcal{N}_2(t) = \ln\left[\frac{\alpha \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]}{\beta_2}\right].$$

Note $\alpha > 0$. We need that $\mathcal{N}_2(t) \geq 0$, so

$$\ln\left[\frac{\alpha \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]}{\beta_2}\right] \geq 0$$

meaning

$$\frac{\alpha \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]}{\beta_2} \geq 1.$$

Hence when

$$\frac{\beta 2}{\alpha} \leq \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]$$

and we have

$$\mathcal{N}_2(t) = \frac{1}{\alpha} \ln\left(\frac{\alpha}{\beta_2} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]\right).$$

If

$$\frac{\beta 2}{\alpha} > \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2],$$

then we have:

$$0 = \sum_{t=0}^{T-1} \eta_1(t) \left[-\alpha e^{-\alpha \mathcal{N}_1(t)} \mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_2(t) \left[-\alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right]$$

$$= \sum_{t=0}^{T-1} \eta_1(t) \cdot 0 + \eta_2(t) \left[-\alpha e^{-\alpha \mathcal{N}_2(t)} \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right]$$

$$< \sum_{t=0}^{T-1} \eta_2(t) \left[-\alpha e^{-\alpha \mathcal{N}_2(t)} \left(\frac{\beta_2}{\alpha}\right) + \beta_2\right]$$

$$= \sum_{t=0}^{T-1} \eta_2(t) \left[-\beta_2 e^{-\alpha \mathcal{N}_2(t)} + \beta_2\right] = \sum_{t=0}^{T-1} \eta_2(t) \beta_2 \left[-e^{-\alpha \mathcal{N}_2(t)} + 1\right].$$

If $\mathcal{N}_2(t) > 0$ we have that $\beta_2(-e^{-\alpha \mathcal{N}_2(t)} + 1) < 0$ contradiction. Thus, if

$$\frac{\beta 2}{\alpha} > \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2]$$

we must have that $\mathcal{N}_2(t) = 0$. Set

$$\xi_2(t) = \mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2].$$

$$\mathcal{N}_2(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi_2(t) \\ \frac{1}{\alpha} \ln[\frac{\alpha}{\beta_2} \xi_2(t)] & \text{if } \frac{\beta 2}{\alpha} \leq \xi_2(t) \end{cases}$$

□

6.2.3 Uniqueness

Theorem 6.2.3. *If the optimal controls $\mathcal{N}_1$ and $\mathcal{N}_2$ exist, then they are unique.*

Proof. In order to show $\mathcal{N}_1$ and $\mathcal{N}_2$ are unique we will show that $J(N_1, N_2) = \sum_{t=0}^{T-1} \beta_1[P_{1,l}(t)^2 + P_{2,l}(t)^2] + \beta_2[N_1(t) + N_2(t)]$ is strictly convex. Recall if a function is strictly convex then there exists a unique minimum such that $J(\mathcal{N}_1, \mathcal{N}_2) < J(N_1, N_2)$ for all $N_1, N_2 \in \mathbf{N} \setminus \{\mathcal{N}_1, \mathcal{N}_2\}$. To show J is strictly convex we will look at J along a line from $N = [N_1, N_2]^T$ to $\eta = [\eta_1, \eta_2]^T$ by defining $z(\varepsilon) = J((1-\varepsilon)N_1 + \varepsilon\eta_1, (1-\varepsilon)N_2 + \varepsilon\eta_2) = J(N_1 + \varepsilon(\eta_1 - N_1), N_2 + \varepsilon(\eta_2 - N_2))$ for $N_1, N_2, \eta_1, \eta_2 \in \mathbf{N}$, and $0 < \varepsilon < 1$. Note that if z, a one dimensional function, is convex in every possible direction then J will be convex. To establish convexity we will show that $z''(\varepsilon) > 0$. First take the derivative of z:

$$z'(\varepsilon) =$$

$$= \lim_{\tau \to 0} \left(\frac{J[N_1 + (\tau+\varepsilon)(\eta_1 - N_1), N_2 + (\tau+\varepsilon)(\eta_2 - N_2)] - J[N_1 + \varepsilon(\eta_1 - N_1), N_2 + \varepsilon(\eta_2 - N_2)]}{\tau} \right)$$

$$= \lim_{\tau \to 0} \sum_{t=0}^{T-1} \frac{\beta_1}{\tau} \left[P_{1,l}^{\tau+\varepsilon}(t)^2 - P_{1,l}^{\varepsilon}(t)^2 \right] + \frac{\beta_2}{\tau} \Big([N_1(t) + (\tau+\varepsilon)(\eta_1(t) - N_1(t))] - [N_1(t) + \varepsilon(\eta_1(t) - N_1(t))] \Big)$$

$$+ \lim_{\tau \to 0} \sum_{t=0}^{T-1} \frac{\beta_1}{\tau} \left[P_{2,l}^{\tau+\varepsilon}(t)^2 - P_{2,l}^{\varepsilon}(t)^2 \right] + \frac{\beta_2}{\tau} \Big([N_2(t) + (\tau+\varepsilon)(\eta_2(t) - N_2(t))] - [N_2(t) + \varepsilon(\eta_2(t) - N_2(t))] \Big)$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_{1,l}^{\tau+\varepsilon}(t)^2 - P_{1,l}^{\varepsilon}(t)^2}{\tau} + \lim_{\tau \to 0} \frac{P_{2,l}^{\tau+\varepsilon}(t)^2 - P_{2,l}^{\varepsilon}(t)^2}{\tau} \right]$$

$$+ \beta_2 \left[\lim_{\tau \to 0} \frac{\tau(\eta_1(t) - N_1(t))}{\tau} + \lim_{\tau \to 0} \frac{\tau(\eta_2(t) - N_2(t))}{\tau} \right]$$

$$= \sum_{t=0}^{T-1} \beta_1 \left[\lim_{\tau \to 0} \frac{P_{1,l}^{\tau+\varepsilon}(t)^2 - P_{1,l}^{\varepsilon}(t)^2}{\tau} + \lim_{\tau \to 0} \frac{P_{2,l}^{\tau+\varepsilon}(t)^2 - P_{2,l}^{\varepsilon}(t)^2}{\tau} \right] + \beta_2(\eta_1(t) - N_1(t)) + \beta_2(\eta_2(t) - N_2(t)).$$

By The Chain Rule:

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2 P_{1,l}^{\varepsilon}(t) \psi_{1,l}^{\varepsilon}(t) + \beta_1 2 P_{2,l}^{\varepsilon}(t) \psi_{2,l}^{\varepsilon}(t) + \beta_2(\eta_1(t) - N_1(t)) + \beta_2(\eta_2(t) - N_2(t)).$$

Note we define sensitivities similar to in Theorem 6.2.2:

$$\psi^{\varepsilon}_{1,e}(t+1) = \gamma_1 \psi^{\varepsilon}_{1,e}(t) + \theta_1 \psi^{\varepsilon}_{1,a}(t)$$

$$\psi^{\varepsilon}_{1,l}(t+1) = \gamma_2 \psi^{\varepsilon}_{1,e}(t) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(t)} \psi^{\varepsilon}_{1,l}(t) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_1(t)} P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))$$

$$\psi^{\varepsilon}_{1,p}(t+1) = \nu_1 \psi^{\varepsilon}_{1,p}(t) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(t)} \psi^{\varepsilon}_{1,l}(t) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}_1(t)} P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))$$

$$\psi^{\varepsilon}_{1,a}(t+1) = \nu_2 \psi^{\varepsilon}_{1,p}(t) + \theta_{1,1} \psi^{\varepsilon}_{1,a}(t) + \theta_{2,1} \psi^{\varepsilon}_{2,a}(t)$$

$$\psi^{\varepsilon}_{2,e}(t+1) = \gamma_1 \psi^{\varepsilon}_{2,e}(t) + \theta_1 \psi^{\varepsilon}_{2,a}(t)$$

$$\psi^{\varepsilon}_{2,l}(t+1) = \gamma_2 \psi^{\varepsilon}_{2,e}(t) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(t)} \psi^{\varepsilon}_{2,l}(t) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_2(t)} P^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))$$

$$\psi^{\varepsilon}_{2,p}(t+1) = \nu_1 \psi^{\varepsilon}_{2,p}(t) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(t)} \psi^{\varepsilon}_{2,l}(t) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}_2(t)} P^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))$$

$$\psi^{\varepsilon}_{2,a}(t+1) = \nu_2 \psi^{\varepsilon}_{2,p}(t) + \theta_{2,2} \psi^{\varepsilon}_{2,a}(t) + \theta_{1,2} \psi^{\varepsilon}_{1,a}(t)$$

where $\psi_{1,e}(0) = 0$, $\psi_{1,l}(0) = 0$, $\psi_{1,p}(0) = 0$, $\psi_{1,a}(0) = 0, \psi_{2,e}(0) = 0$, $\psi_{2,l}(0) = 0$, $\psi_{2,p}(0) = 0$, $\psi_{2,a}(0) = 0$.

In order to continue we must define derivatives for the sensitivities, $\sigma_{1,e}(t)$, $\sigma_{1,l}(t)$, $\sigma_{1,p}(t)$, $\sigma_{1,a}(t)$, $\sigma_{2,e}(t)$, $\sigma_{2,l}(t)$, $\sigma_{2,p}(t)$, $\sigma_{2,a}(t)$ as:

$$\sigma^{\varepsilon}_{1,e}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,e}(t+1) - \psi^{\varepsilon}_{1,e}(t+1)}{\tau}, \quad \sigma^{\varepsilon}_{1,l}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,l}(t+1) - \psi^{\varepsilon}_{1,l}(t+1)}{\tau},$$

$$\sigma^{\varepsilon}_{1,p}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,p}(t+1) - \psi^{\varepsilon}_{1,p}(t+1)}{\tau}, \quad \sigma^{\varepsilon}_{1,a}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,a}(t+1) - \psi^{\varepsilon}_{1,a}(t+1)}{\tau}$$

$$\sigma^{\varepsilon}_{2,e}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{2,e}(t+1) - \psi^{\varepsilon}_{2,e}(t+1)}{\tau}, \quad \sigma^{\varepsilon}_{2,l}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{2,l}(t+1) - \psi^{\varepsilon}_{2,l}(t+1)}{\tau},$$

$$\sigma^{\varepsilon}_{2,p}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{2,p}(t+1) - \psi^{\varepsilon}_{2,p}(t+1)}{\tau}, \quad \sigma^{\varepsilon}_{2,a}(t{+}1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{2,a}(t+1) - \psi^{\varepsilon}_{2,a}(t+1)}{\tau}.$$

Hence, we can write:

$$\begin{aligned}\sigma^{\varepsilon}_{1,e}(t+1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,e}(t+1) - \psi^{\varepsilon}_{1,e}(t+1)}{\tau} &= \gamma_1 \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,e}(t) - \psi^{\varepsilon}_{1,e}(t)}{\tau} \\ &+\theta_1 \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,a}(t) - \psi^{\varepsilon}_{1,a}(t)}{\tau} \\ &= \gamma_1 \sigma^{\varepsilon}_{1,e}(t) + \theta_1 \sigma^{\varepsilon}_{1,a}(t)\end{aligned}$$

$$\sigma^{\varepsilon}_{1,a}(t+1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,a}(t+1) - \psi^{\varepsilon}_{1,a}(t+1)}{\tau}$$

$$= \nu_2 \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,p}(t) - \psi^{\varepsilon}_{1,p}(t)}{\tau} + \theta_{1,1} \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,a}(t) - \psi^{\varepsilon}_{1,a}(t)}{\tau} + \theta_{2,1} \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{2,a}(t) - \psi^{\varepsilon}_{2,a}(t)}{\tau}$$

$$= \nu_2 \sigma^{\varepsilon}_{1,p}(t) + \theta_{1,1} \sigma^{\varepsilon}_{1,a}(t) + \theta_{2,1} \sigma^{\varepsilon}_{2,a}(t)$$

$$\sigma_{2,e}^{\varepsilon}(t+1) = \lim_{\tau \to 0} \frac{\psi_{2,e}^{\tau+\varepsilon}(t+1) - \psi_{2,e}^{\varepsilon}(t+1)}{\tau} = \gamma_1 \lim_{\tau \to 0} \frac{\psi_{2,e}^{\tau+\varepsilon}(t) - \psi_{2,e}^{\varepsilon}(t)}{\tau}$$

$$+\theta_1 \lim_{\tau \to 0} \frac{\psi_{2,a}^{\tau+\varepsilon}(t) - \psi_{2,a}^{\varepsilon}(t)}{\tau}$$

$$= \gamma_1 \sigma_{2,e}^{\varepsilon}(t) + \theta_1 \sigma_{2,a}^{\varepsilon}(t)$$

$$\sigma_{2,a}^{\varepsilon}(t+1) = \lim_{\tau \to 0} \frac{\psi_{2,a}^{\tau+\varepsilon}(t+1) - \psi_{2,a}^{\varepsilon}(t+1)}{\tau}$$

$$= \nu_2 \lim_{\tau \to 0} \frac{\psi_{2,p}^{\tau+\varepsilon}(t) - \psi_{2,p}^{\varepsilon}(t)}{\tau} + \theta_{2,2} \lim_{\tau \to 0} \frac{\psi_{2,a}^{\tau+\varepsilon}(t) - \psi_{2,a}^{\varepsilon}(t)}{\tau} + \theta_{1,2} \lim_{\tau \to 0} \frac{\psi_{1,a}^{\tau+\varepsilon}(t) - \psi_{1,a}^{\varepsilon}(t)}{\tau}$$

$$= \nu_2 \sigma_{2,p}^{\varepsilon}(t) + \theta_{2,2} \sigma_{2,a}^{\varepsilon}(t) + \theta_{1,2} \sigma_{1,a}^{\varepsilon}(t).$$

Now, we will compute $\sigma^{\varepsilon}_{1,l}(t+1)$.

$$\sigma_{1,l}(t+1) = \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,l}(t+1) - \psi^{\varepsilon}_{1,l}(t+1)}{\tau} =$$

$$= \lim_{\tau\to 0} \frac{\gamma_2\psi^{\tau+\varepsilon}_{1,e}(t) + \zeta_1 e^{-\alpha N_1^{\tau+\varepsilon}(t)}\psi^{\tau+\varepsilon}_{1,l}(t) - \zeta_1\alpha e^{-\alpha N_1^{\tau+\varepsilon}(t)}P^{\tau+\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t)) - \gamma_2\psi^{\varepsilon}_{1,e}(t) - \zeta_1 e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t) + \zeta_1\alpha e^{-\alpha N_1^{\varepsilon}(t)}P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))}{\tau}$$

$$= \gamma_2 \lim_{\tau\to 0} \frac{\psi^{\tau+\varepsilon}_{1,e}(t) - \psi^{\varepsilon}_{1,e}(t)}{\tau} + \zeta_1 \lim_{\tau\to 0} \frac{e^{-\alpha N_1^{\tau+\varepsilon}(t)}\psi^{\tau+\varepsilon}_{1,l}(t) - e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t)}{\tau} - \zeta_1\alpha(\eta_1(t) - N_1(t)) \lim_{\tau\to 0} \frac{e^{-\alpha N_1^{\tau+\varepsilon}(t)}P^{\tau+\varepsilon}_{1,l}(t) - e^{-\alpha N_1^{\varepsilon}(t)}P^{\varepsilon}_{1,l}(t)}{\tau}$$

$$= \gamma_2\sigma^{\varepsilon}_{1,e}(t) + \zeta_1[e^{-\alpha N_1^{\varepsilon}(t)}\sigma^{\varepsilon}_{1,l}(t) - \alpha e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))] - \zeta_1\alpha(\eta_1(t) - N_1(t))[e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t) - \alpha e^{-\alpha N_1^{\varepsilon}(t)}P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))]$$

$$= \gamma_2\sigma^{\varepsilon}_{1,e}(t) + \zeta_1 e^{-\alpha N_1^{\varepsilon}(t)}\sigma^{\varepsilon}_{1,l}(t) - \zeta_1\alpha e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t)) - \zeta_1\alpha(\eta_1(t) - N_1(t))e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t) + \zeta_1(\eta_1(t) - N_1(t))^2\alpha^2 e^{-\alpha N_1^{\varepsilon}(t)}P^{\varepsilon}_{1,l}(t)$$

$$= \gamma_2\sigma^{\varepsilon}_{e}(t) + \zeta_1 e^{-\alpha N_1^{\varepsilon}(t)}\sigma^{\varepsilon}_{1,l}(t) - 2\zeta_1\alpha e^{-\alpha N_1^{\varepsilon}(t)}\psi^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t)) + \zeta_1\alpha^2 e^{-\alpha N_1^{\varepsilon}(t)}P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))^2.$$

Now, we will compute $\sigma^{\varepsilon}_{1,p}(t+1)$.

$$\sigma_{1,p}(t+1) = \lim_{\tau\to 0} \frac{\psi_{1,p}^{\tau+\varepsilon}(t+1) - \psi_{1,p}^{\varepsilon}(t+1)}{\tau} =$$

$$= \nu_1\sigma_{1,p}^{\varepsilon}(t) + \zeta_2[e^{-\alpha N_1^{\varepsilon}(t)}\sigma_{1,l}^{\varepsilon}(t) - \alpha e^{-\alpha N_1^{\varepsilon}(t)}\psi_{1,l}^{\varepsilon}(t)(\eta_1(t) - N_1(t))] - \zeta_2\alpha(\eta_1(t) - N_1(t))[e^{-\alpha N_1^{\varepsilon}(t)}\psi_{1,l}^{\varepsilon}(t) - \alpha e^{-\alpha N_1^{\varepsilon}(t)}P_{1,l}^{\varepsilon}(t)(\eta_1(t) - N_1(t))]$$

$$= \nu_1\sigma_{1,p}^{\varepsilon}(t) + \zeta_2 e^{-\alpha N_1^{\varepsilon}(t)}\sigma_{1,l}^{\varepsilon}(t) - 2\zeta_2\alpha e^{-\alpha N_1^{\varepsilon}(t)}\psi_{1,l}^{\varepsilon}(t)(\eta_1(t) - N_1(t)) + \zeta_2\alpha^2 e^{-\alpha N_1^{\varepsilon}(t)}P_{1,l}^{\varepsilon}(t)(\eta_1(t) - N_1(t))^2.$$

Similarly we have that:

$$\sigma_{2,l}(t+1) = \lim_{\tau\to 0} \frac{\psi_{2,l}^{\tau+\varepsilon}(t+1) - \psi_{2,l}^{\varepsilon}(t+1)}{\tau} =$$

$$= \gamma_2\sigma_e^{\varepsilon}(t) + \zeta_1 e^{-\alpha N_2^{\varepsilon}(t)}\sigma_{2,l}^{\varepsilon}(t) - 2\zeta_1\alpha e^{-\alpha N_2^{\varepsilon}(t)}\psi_{2,l}^{\varepsilon}(t)(\eta_2(t) - N_2(t)) + \zeta_1\alpha^2 e^{-\alpha N_2^{\varepsilon}(t)}P_{2,l}^{\varepsilon}(t)(\eta_2(t) - N_2(t))^2$$

$$\sigma_{2,p}(t+1) = \lim_{\tau\to 0} \frac{\psi_{2,p}^{\tau+\varepsilon}(t+1) - \psi_{2,p}^{\varepsilon}(t+1)}{\tau} =$$

$$= \nu_1\sigma_{2,p}^{\varepsilon}(t) + \zeta_2 e^{-\alpha N_2^{\varepsilon}(t)}\sigma_{2,l}^{\varepsilon}(t) - 2\zeta_2\alpha e^{-\alpha N_2^{\varepsilon}(t)}\psi_{2,l}^{\varepsilon}(t)(\eta_2(t) - N_2(t)) + \zeta_2\alpha^2 e^{-\alpha N_2^{\varepsilon}(t)}P_{2,l}^{\varepsilon}(t)(\eta_2(t) - N_2(t))^2.$$

Thus,

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2P_{1,l}^{\varepsilon}(t)\psi_{1,l}^{\varepsilon}(t) + \beta_1 2P_{2,l}^{\varepsilon}(t)\psi_{2,l}^{\varepsilon}(t) + \beta_2(\eta_1(t) - N_1(t)) + \beta_2(\eta_2(t) - N_2(t))$$

$$z''(\varepsilon) = \lim_{\tau \to 0} \left(\frac{z'(\tau + \varepsilon) - z'(\varepsilon)}{\tau} \right)$$

$$= \lim_{\tau \to 0} \sum_{t=0}^{T-1} \beta_1 2 P_{1,l}^{\tau+\varepsilon}(t) \psi_{1,l}^{\tau+\varepsilon}(t) + \beta_1 2 P_{2,l}^{\tau+\varepsilon}(t) \psi_{2,l}^{\tau+\varepsilon}(t) + \beta_2(\eta_1(t) - N_1(t)) + \beta_2(\eta_2(t) - N_2(t)) +$$

$$-[\beta_1 2 P_{1,l}^{\varepsilon}(t) \psi_{1,l}^{\varepsilon}(t) + \beta_1 2 P_{2,l}^{\varepsilon}(t) \psi_{2,l}^{\varepsilon}(t) + \beta_2(\eta_1(t) - N_1(t)) + \beta_2(\eta_2(t) - N_2(t))]$$

$$= \sum_{t=0}^{T-1} \beta_1 2 \left[\lim_{\tau \to 0} \frac{P_{1,l}^{\tau+\varepsilon}(t) \psi_{1,l}^{\tau+\varepsilon}(t) - P_{1,l}^{\varepsilon}(t) \psi_{1,l}^{\varepsilon}(t)}{\tau} + \lim_{\tau \to 0} \frac{P_{2,l}^{\tau+\varepsilon}(t) \psi_{2,l}^{\tau+\varepsilon}(t) - P_{2,l}^{\varepsilon}(t) \psi_{2,l}^{\varepsilon}(t)}{\tau} \right] =$$

$$= \sum_{t=0}^{T-1} \beta_1 2 [\sigma_{1,l}^{\varepsilon}(t) P_{1,l}^{\varepsilon}(t) + \psi_{1,l}^{\varepsilon}(t)^2] + \beta_1 2 [\sigma_{2,l}^{\varepsilon}(t) P_{2,l}^{\varepsilon}(t) + \psi_{2,l}^{\varepsilon}(t)^2].$$

We now need to show that

$$z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t) P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2] + \beta_1 2[\sigma^{\varepsilon}_{2,l}(t) P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2] > 0.$$

Specifically we will show that $\sigma^{\varepsilon}_{1,l}(t) > 0$ and $\sigma^{\varepsilon}_{2,l}(t) > 0$

We start by calculating the terms for $t = 1, 2, 3$.

We have that $\psi^{\varepsilon}_{1,e}(0) = 0$, $\psi^{\varepsilon}_{1,l}(0) = 0$, $\psi^{\varepsilon}_{1,p}(0) = 0$, $\psi^{\varepsilon}_{1,a}(0) = 0, \psi^{\varepsilon}_{2,e}(0) = 0$, $\psi^{\varepsilon}_{2,l}(0) = 0$, $\psi^{\varepsilon}_{2,p}(0) = 0$, $\psi^{\varepsilon}_{2,a}(0) = 0$, so for $t = 1$:

$$\psi^{\varepsilon}_{1,e}(1) = \gamma_1 \psi^{\varepsilon}_{1,e}(0) + \theta_1 \psi^{\varepsilon}_{1,a}(0) = 0$$

$$\psi^{\varepsilon}_{1,l}(1) = \gamma_2 \psi^{\varepsilon}_{1,e}(0) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(0)} \psi^{\varepsilon}_{1,l}(0) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_1(0)} P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$

$$= -\zeta_1 \alpha e^{-\alpha N^{\varepsilon}_1(0)} P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$

$$\psi^{\varepsilon}_{1,p}(1) = \nu_1 \psi^{\varepsilon}_{1,p}(0) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(0)} \psi^{\varepsilon}_{1,l}(0) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}_1(0)} P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$

$$= -\zeta_2 \alpha e^{-\alpha N^{\varepsilon}_1(0)} P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$

$$\psi^{\varepsilon}_{1,a}(1) = \nu_2 \psi^{\varepsilon}_{1,p}(0) + \theta_{1,1} \psi^{\varepsilon}_{1,a}(0) + \theta_{2,1} \psi^{\varepsilon}_{2,a}(0) = 0$$

$$\psi^{\varepsilon}_{2,e}(1) = \gamma_1 \psi^{\varepsilon}_{2,e}(0) + \theta_1 \psi^{\varepsilon}_{2,a}(0) = 0$$

$$\psi^{\varepsilon}_{2,l}(1) = \gamma_2 \psi^{\varepsilon}_{2,e}(0) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(0)} \psi^{\varepsilon}_{2,l}(0) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_2(0)} P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$= -\zeta_1 \alpha e^{-\alpha N^{\varepsilon}_2(0)} P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$\psi^{\varepsilon}_{2,p}(1) = \nu_1 \psi^{\varepsilon}_{2,p}(0) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(0)} \psi^{\varepsilon}_{2,l}(0) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}_2(0)} P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$= -\zeta_2 \alpha e^{-\alpha N^{\varepsilon}_2(0)} P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$\psi^{\varepsilon}_{2,a}(1) = \nu_2 \psi^{\varepsilon}_{2,p}(0) + \theta_{2,2} \psi^{\varepsilon}_{2,a}(0) + \theta_{1,2} \psi^{\varepsilon}_{1,a}(0) = 0.$$

Next, for $t = 2$

$$\psi^{\varepsilon}_{1,e}(2) = \gamma_1 \psi^{\varepsilon}_{1,e}(1) + \theta_1 \psi^{\varepsilon}_{1,a}(1) = 0$$

$$\psi^{\varepsilon}_{1,l}(2) = \gamma_2 \psi^{\varepsilon}_{1,e}(1) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(1)} \psi^{\varepsilon}_{1,l}(1) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_1(1)} P^{\varepsilon}_{1,l}(1)(\eta_1(1) - N_1(1))$$

$$= \zeta_1 e^{-\alpha N^{\varepsilon}_1(1)} \psi^{\varepsilon}_{1,l}(1) - \zeta_1 \alpha e^{-\alpha N^{\varepsilon}_1(1)} P^{\varepsilon}_{1,l}(1)(\eta_1(1) - N_1(1))$$

$$\psi^{\varepsilon}_{1,p}(2) = \nu_1 \psi^{\varepsilon}_{1,p}(1) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(1)} \psi^{\varepsilon}_{1,l}(1) - \zeta_2 \alpha e^{-\alpha N^{\varepsilon}_1(1)} P^{\varepsilon}_{1,l}(1)(\eta_1(1) - N_1(1))$$

$$\psi^{\varepsilon}_{1,a}(2) = \nu_2\psi^{\varepsilon}_{1,p}(1) + \theta_{1,1}\psi^{\varepsilon}_{1,a}(1) + \theta_{2,1}\psi^{\varepsilon}_{2,a}(1) = \nu_2\psi^{\varepsilon}_{1,p}(1)$$

$$\psi^{\varepsilon}_{2,e}(2) = \gamma_1\psi^{\varepsilon}_{2,e}(1) + \theta_1\psi^{\varepsilon}_{2,a}(1) = 0$$

$$\psi^{\varepsilon}_{2,l}(2) = \gamma_2\psi^{\varepsilon}_{2,e}(1) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(1)}\psi^{\varepsilon}_{2,l}(1) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(1)}P^{\varepsilon}_{2,l}(1)(\eta_2(1) - N_2(1))$$

$$= \zeta_1 e^{-\alpha N^{\varepsilon}_2(1)}\psi^{\varepsilon}_{2,l}(1) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(1)}P^{\varepsilon}_{2,l}(1)(\eta_2(1) - N_2(1))$$

$$\psi^{\varepsilon}_{2,p}(2) = \nu_1\psi^{\varepsilon}_{2,p}(1) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(1)}\psi^{\varepsilon}_{2,l}(1) - \zeta_2\alpha e^{-\alpha N^{\varepsilon}_2(1)}P^{\varepsilon}_{2,l}(1)(\eta_2(1) - N_2(1))$$

$$\psi^{\varepsilon}_{2,a}(2) = \nu_2\psi^{\varepsilon}_{2,p}(1) + \theta_{2,2}\psi^{\varepsilon}_{2,a}(1) + \theta_{1,2}\psi^{\varepsilon}_{1,a}(1) = \nu_2\psi^{\varepsilon}_{2,p}(1).$$

Lastly, for $t = 3$

$$\psi^{\varepsilon}_{1,e}(3) = \gamma_1\psi^{\varepsilon}_{1,e}(2) + \theta_1\psi^{\varepsilon}_{1,a}(2) = \theta_1\nu_2\psi^{\varepsilon}_{1,p}(1) < 0$$

$$\psi^{\varepsilon}_{1,l}(3) = \gamma_2\psi^{\varepsilon}_{1,e}(2) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(2)}\psi^{\varepsilon}_{1,l}(2) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_1(2)}P^{\varepsilon}_{1,l}(2)(\eta_1(2) - N_1(2))$$

$$= \zeta_1 e^{-\alpha N^{\varepsilon}_1(2)}\psi^{\varepsilon}_{1,l}(2) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_1(2)}P^{\varepsilon}_{1,l}(2)(\eta_1(2) - N_1(2))$$

$$\psi^{\varepsilon}_{1,p}(3) = \nu_1\psi^{\varepsilon}_{1,p}(2) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(2)}\psi^{\varepsilon}_{1,l}(2) - \zeta_2\alpha e^{-\alpha N^{\varepsilon}_1(2)}P^{\varepsilon}_{1,l}(2)(\eta_1(2) - N_1(2))$$

$$\psi^{\varepsilon}_{1,a}(3) = \nu_2\psi^{\varepsilon}_{1,p}(2) + \theta_{1,1}\psi^{\varepsilon}_{1,a}(2) + \theta_{2,1}\psi^{\varepsilon}_{2,a}(2)$$

$$\psi^{\varepsilon}_{2,e}(3) = \gamma_1\psi^{\varepsilon}_{2,e}(2) + \theta_1\psi^{\varepsilon}_{2,a}(2) = \theta_1\nu_2\psi^{\varepsilon}_{2,p}(1)$$

$$\psi^{\varepsilon}_{2,l}(3) = \gamma_2\psi^{\varepsilon}_{2,e}(2) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(2)}\psi^{\varepsilon}_{2,l}(2) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(2)}P^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))$$

$$= \zeta_1 e^{-\alpha N^{\varepsilon}_2(2)}\psi^{\varepsilon}_{2,l}(2) - \zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(2)}P^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))$$

$$\psi^{\varepsilon}_{2,p}(3) = \nu_1\psi^{\varepsilon}_{2,p}(2) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(2)}\psi^{\varepsilon}_{2,l}(2) - \zeta_2\alpha e^{-\alpha N^{\varepsilon}_2(2)}P^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))$$

$$\psi^{\varepsilon}_{2,a}(3) = \nu_2\psi^{\varepsilon}_{2,p}(2) + \theta_{2,2}\psi^{\varepsilon}_{2,a}(2) + \theta_{1,2}\psi^{\varepsilon}_{1,a}(2).$$

We have that:

$$\sigma^{\varepsilon}_{1,e}(t+1) = \gamma_1\sigma^{\varepsilon}_{1,e}(t) + \theta_1\sigma^{\varepsilon}_{1,a}(t)$$

$$\sigma^{\varepsilon}_{1,l}(t+1) = \gamma_2\sigma^{\varepsilon}_{e}(t) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(t)}\sigma^{\varepsilon}_{1,l}(t) - 2\zeta_1\alpha e^{-\alpha N^{\varepsilon}_1(t)}\psi^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_1(t)}P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))^2$$

$$\sigma^{\varepsilon}_{1,p}(t+1) = \nu_1\sigma^{\varepsilon}_{1,p}(t) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(t)}\sigma^{\varepsilon}_{1,l}(t) - 2\zeta_2\alpha e^{-\alpha N^{\varepsilon}_1(t)}\psi^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))$$
$$+\zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_1(t)}P^{\varepsilon}_{1,l}(t)(\eta_1(t) - N_1(t))^2$$

$$\sigma^{\varepsilon}_{1,a}(t+1) = \nu_2\sigma^{\varepsilon}_{1,p}(t) + \theta_{1,1}\sigma^{\varepsilon}_{1,a}(t) + \theta_{2,1}\sigma^{\varepsilon}_{2,a}(t)$$

$$\sigma^{\varepsilon}_{2,e}(t+1) = \gamma_1\sigma^{\varepsilon}_{2,e}(t) + \theta_1\sigma^{\varepsilon}_{2,a}(t)$$

$$\sigma^{\varepsilon}_{2,l}(t+1) = \gamma_2\sigma^{\varepsilon}_{e}(t) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(t)}\sigma^{\varepsilon}_{2,l}(t) - 2\zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(t)}\psi^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))$$
$$+\zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_2(t)}P^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))^2$$

$$\sigma^{\varepsilon}_{2,p}(t+1) = \nu_1\sigma^{\varepsilon}_{2,p}(t) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(t)}\sigma^{\varepsilon}_{2,l}(t) - 2\zeta_2\alpha e^{-\alpha N^{\varepsilon}_2(t)}\psi^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))$$
$$+\zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_2(t)}P^{\varepsilon}_{2,l}(t)(\eta_2(t) - N_2(t))^2$$

$$\sigma^{\varepsilon}_{2,a}(t+1) = \nu_2\sigma^{\varepsilon}_{2,p}(t) + \theta_{2,2}\sigma^{\varepsilon}_{2,a}(t) + \theta_{1,2}\sigma^{\varepsilon}_{1,a}(t).$$

Recall that $\sigma^{\varepsilon}_{1,e}(0), \sigma^{\varepsilon}_{1,l}(0), \sigma^{\varepsilon}_{1,p}(0), \sigma^{\varepsilon}_{1,a}(0), \sigma^{\varepsilon}_{2,e}(0), \sigma^{\varepsilon}_{2,l}(0), \sigma^{\varepsilon}_{2,p}(0), \sigma^{\varepsilon}_{2,a}(0) = 0$. Consider $t = 1$:

$$\sigma^{\varepsilon}_{1,e}(1) = \gamma_1\sigma^{\varepsilon}_{1,e}(0) + \theta_1\sigma^{\varepsilon}_{1,a}(0) = 0$$

$$\sigma^{\varepsilon}_{1,l}(1) = \gamma_2\sigma^{\varepsilon}_{e}(0) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(0)}\sigma^{\varepsilon}_{1,l}(0) - 2\zeta_1\alpha e^{-\alpha N^{\varepsilon}_1(0)}\psi^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$
$$+\zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_1(0)}P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))^2$$
$$= \zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_1(0)}P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))^2$$

$$\sigma^{\varepsilon}_{1,p}(1) = \nu_1\sigma^{\varepsilon}_{1,p}(0) + \zeta_2 e^{-\alpha N^{\varepsilon}_1(0)}\sigma^{\varepsilon}_{1,l}(0) - 2\zeta_2\alpha e^{-\alpha N^{\varepsilon}_1(0)}\psi^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_1(0)}P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))^2$$

$$= \zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_1(0)}P^{\varepsilon}_{1,l}(0)(\eta_1(0) - N_1(0))^2$$

$$\sigma^{\varepsilon}_{1,a}(1) = \nu_2\sigma^{\varepsilon}_{1,p}(0) + \theta_{1,1}\sigma^{\varepsilon}_{1,a}(0) + \theta_{2,1}\sigma^{\varepsilon}_{2,a}(0) = 0$$

$$\sigma^{\varepsilon}_{2,e}(1) = \gamma_1\sigma^{\varepsilon}_{2,e}(0) + \theta_1\sigma^{\varepsilon}_{2,a}(0) = 0$$

$$\sigma^{\varepsilon}_{2,l}(1) = \gamma_2\sigma^{\varepsilon}_{e}(0) + \zeta_1 e^{-\alpha N^{\varepsilon}_2(0)}\sigma^{\varepsilon}_{2,l}(0) - 2\zeta_1\alpha e^{-\alpha N^{\varepsilon}_2(0)}\psi^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$+\zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_2(0)}P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))^2$$

$$= \zeta_1\alpha^2 e^{-\alpha N^{\varepsilon}_2(0)}P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))^2$$

$$\sigma^{\varepsilon}_{2,p}(1) = \nu_1\sigma^{\varepsilon}_{2,p}(0) + \zeta_2 e^{-\alpha N^{\varepsilon}_2(0)}\sigma^{\varepsilon}_{2,l}(0) - 2\zeta_2\alpha e^{-\alpha N^{\varepsilon}_2(0)}\psi^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))$$

$$+\zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_2(0)}P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))^2$$

$$= \zeta_2\alpha^2 e^{-\alpha N^{\varepsilon}_2(0)}P^{\varepsilon}_{2,l}(0)(\eta_2(0) - N_2(0))^2$$

$$\sigma^{\varepsilon}_{2,a}(1) = \nu_2\sigma^{\varepsilon}_{2,p}(0) + \theta_{2,2}\sigma^{\varepsilon}_{2,a}(0) + \theta_{1,2}\sigma^{\varepsilon}_{1,a}(0) = 0.$$

Next, $t = 2$

$$\sigma^{\varepsilon}_{1,e}(2) = \gamma_1\sigma^{\varepsilon}_{1,e}(1) + \theta_1\sigma^{\varepsilon}_{1,a}(1) = 0$$

$$\sigma^{\varepsilon}_{1,l}(2) = \gamma_2\sigma^{\varepsilon}_{e}(1) + \zeta_1 e^{-\alpha N^{\varepsilon}_1(1)}\sigma^{\varepsilon}_{1,l}(1) - 2\zeta_1\alpha e^{-\alpha N^{\varepsilon}_1(1)}\psi^{\varepsilon}_{1,l}(1)(\eta_1(1) - N_1(1))$$

$$+\zeta_1\alpha^2 e^{-\alpha N_1^\varepsilon(1)} P_{1,l}^\varepsilon(1)(\eta_1(1) - N_1(1))^2$$

$$\sigma_{1,p}^\varepsilon(2) = \nu_1\sigma_{1,p}^\varepsilon(1) + \zeta_2 e^{-\alpha N_1^\varepsilon(1)}\sigma_{1,l}^\varepsilon(1) - 2\zeta_2\alpha e^{-\alpha N_1^\varepsilon(1)}\psi_{1,l}^\varepsilon(1)(\eta_1(1) - N_1(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha N_1^\varepsilon(1)} P_{1,l}^\varepsilon(1)(\eta_1(1) - N_1(1))^2$$

$$\sigma_{1,a}^\varepsilon(2) = \nu_2\sigma_{1,p}^\varepsilon(1) + \theta_{1,1}\sigma_{1,a}^\varepsilon(1) + \theta_{2,1}\sigma_{2,a}^\varepsilon(1)$$

$$\sigma_{2,e}^\varepsilon(2) = \gamma_1\sigma_{2,e}^\varepsilon(1) + \theta_1\sigma_{2,a}^\varepsilon(1) = 0$$

$$\sigma_{2,l}^\varepsilon(2) = \gamma_2\sigma_e^\varepsilon(1) + \zeta_1 e^{-\alpha N_2^\varepsilon(1)}\sigma_{2,l}^\varepsilon(1) - 2\zeta_1\alpha e^{-\alpha N_2^\varepsilon(1)}\psi_{2,l}^\varepsilon(1)(\eta_2(1) - N_2(1))$$

$$+\zeta_1\alpha^2 e^{-\alpha N_2^\varepsilon(1)} P_{2,l}^\varepsilon(1)(\eta_2(1) - N_2(1))^2$$

$$\sigma_{2,p}^\varepsilon(2) = \nu_1\sigma_{2,p}^\varepsilon(1) + \zeta_2 e^{-\alpha N_2^\varepsilon(1)}\sigma_{2,l}^\varepsilon(1) - 2\zeta_2\alpha e^{-\alpha N_2^\varepsilon(1)}\psi_{2,l}^\varepsilon(1)(\eta_2(1) - N_2(1))$$

$$+\zeta_2\alpha^2 e^{-\alpha N_2^\varepsilon(1)} P_{2,l}^\varepsilon(1)(\eta_2(1) - N_2(1))^2$$

$$\sigma_{2,a}^\varepsilon(2) = \nu_2\sigma_{2,p}^\varepsilon(1) + \theta_{2,2}\sigma_{2,a}^\varepsilon(1) + \theta_{1,2}\sigma_{1,a}^\varepsilon(1).$$

Next, $t = 3$

$$\sigma_{1,e}^\varepsilon(3) = \gamma_1\sigma_{1,e}^\varepsilon(2) + \theta_1\sigma_{1,a}^\varepsilon(2)$$

$$\sigma_{1,l}^\varepsilon(3) = \gamma_2\sigma_e^\varepsilon(2) + \zeta_1 e^{-\alpha N_1^\varepsilon(2)}\sigma_{1,l}^\varepsilon(2) - 2\zeta_1\alpha e^{-\alpha N_1^\varepsilon(2)}\psi_{1,l}^\varepsilon(2)(\eta_1(2) - N_1(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N_1^\varepsilon(2)} P_{1,l}^\varepsilon(2)(\eta_1(2) - N_1(2))^2$$

$$\sigma^{\varepsilon}_{1,p}(3) = \nu_1\sigma^{\varepsilon}_{1,p}(2) + \zeta_2 e^{-\alpha N_1^{\varepsilon}(2)}\sigma^{\varepsilon}_{1,l}(2) - 2\zeta_2\alpha e^{-\alpha N_1^{\varepsilon}(2)}\psi^{\varepsilon}_{1,l}(2)(\eta_1(2) - N_1(2))$$

$$+\zeta_2\alpha^2 e^{-\alpha N_1^{\varepsilon}(2)}P^{\varepsilon}_{1,l}(2)(\eta_1(2) - N_1(2))^2$$

$$\sigma^{\varepsilon}_{1,a}(3) = \nu_2\sigma^{\varepsilon}_{1,p}(2) + \theta_{1,1}\sigma^{\varepsilon}_{1,a}(2) + \theta_{2,1}\sigma^{\varepsilon}_{2,a}(2)$$

$$\sigma^{\varepsilon}_{2,e}(3) = \gamma_1\sigma^{\varepsilon}_{2,e}(2) + \theta_1\sigma^{\varepsilon}_{2,a}(2)$$

$$\sigma^{\varepsilon}_{2,l}(3) = \gamma_2\sigma^{\varepsilon}_{e}(2) + \zeta_1 e^{-\alpha N_2^{\varepsilon}(2)}\sigma^{\varepsilon}_{2,l}(2) - 2\zeta_1\alpha e^{-\alpha N_2^{\varepsilon}(2)}\psi^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))$$

$$+\zeta_1\alpha^2 e^{-\alpha N_2^{\varepsilon}(2)}P^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))^2$$

$$\sigma^{\varepsilon}_{2,p}(3) = \nu_1\sigma^{\varepsilon}_{2,p}(2) + \zeta_2 e^{-\alpha N_2^{\varepsilon}(2)}\sigma^{\varepsilon}_{2,l}(2) - 2\zeta_2\alpha e^{-\alpha N_2^{\varepsilon}(2)}\psi^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))$$

$$+\zeta_2\alpha^2 e^{-\alpha N_2^{\varepsilon}(2)}P^{\varepsilon}_{2,l}(2)(\eta_2(2) - N_2(2))^2$$

$$\sigma^{\varepsilon}_{2,a}(3) = \nu_2\sigma^{\varepsilon}_{2,p}(1) + \theta_{2,2}\sigma^{\varepsilon}_{2,a}(1) + \theta_{1,2}\sigma^{\varepsilon}_{1,a}(1).$$

The proof is similar to that in the Basic Model proof of Theorem 2.3.3. We can see similarities and differences between the previously calculated terms and the terms in Theorem 2.3.3. If we consider the patches individually we have the Basic Model. Therefore, we will consider the differences that arise when the adults can travel between patches. The terms which will differ from Theorem 2.3.3 are:

$$\psi^{\varepsilon}_{1,a}(t+1) = \nu_2\psi^{\varepsilon}_{1,p}(t) + \theta_{1,1}\psi^{\varepsilon}_{1,a}(t) + \theta_{2,1}\psi^{\varepsilon}_{2,a}(t)$$

$$\sigma^{\varepsilon}_{1,a}(t+1) = \nu_2 \sigma^{\varepsilon}_{1,p}(t) + \theta_{1,1}\sigma^{\varepsilon}_{1,a}(t) + \theta_{2,1}\sigma^{\varepsilon}_{2,a}(t)$$

$$\psi^{\varepsilon}_{2,a}(t+1) = \nu_2 \psi^{\varepsilon}_{2,p}(t) + \theta_{2,2}\psi^{\varepsilon}_{2,a}(t) + \theta_{1,2}\psi^{\varepsilon}_{1,a}(t)$$

$$\sigma^{\varepsilon}_{2,a}(t+1) = \nu_2 \sigma^{\varepsilon}_{2,p}(t) + \theta_{2,2}\sigma^{\varepsilon}_{2,a}(t) + \theta_{1,2}\sigma^{\varepsilon}_{1,a}(t).$$

These equations are similar to the Basic case, but each patch now has some adults from the other patch traveling inward and adults leaving. This won't cause an issue since in each patch we have the same parameter on the term and have both the $\psi^{\varepsilon}_{i,a}(t)$ and $\sigma^{\varepsilon}_{i,a}(t)$. As we have seen in the basic case we can manipulate these two terms to create a summation of $(\eta(i) - N_n(i))$ which are squared and multiplied by parameters. Due to the model design, and relation to the Basic model, we can expand our previous finding and will have that both $\sigma^{\varepsilon}_{1,l}(t) > 0$ and $\sigma^{\varepsilon}_{2,l}(t) > 0$ for all t.

Thus, we have that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2] + \beta_1 2[\sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2] > 0$, and we have uniqueness by convexity of z.

□

Chapter 7

Four Patches

7.1 Four Connected Patches

Now, we will consider using four neighboring connected patches, aka connected patches. As before adults can travel between patches, and we will assume patches are arranged as in Figure 7.1.

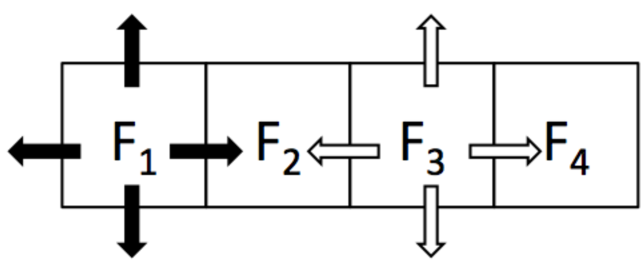

Figure 7.1: Four Connected Patches: Note any patch is connected to the patches next to it. Specifically consider patch F_1 which is connected to patch F_2. Note the black arrows from F_1 demonstrate how the adult pest can disperse from the patch, specifically the pest adult can only spread to F_2 and to none of the other patches. Meanwhile, F_3 is attached to F_2 and F_4 and the pest adults in F_3 can spread to F_2 and F_4 along the white arrows but not to F_1. Note that F_2 spreads to F_1 and F_3, while F_4 only spreads to F_3, these patches arrows are not show in the figure.

Then the resulting matrix for our pest dynamics will be as follows:

$$A_{4c} = \left[\begin{array}{cccc:cccc:cccc:cccc}
\gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2,3} & 0 & 0 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{3,4} & 0 & 0 & \nu_2 & \theta_{4,4}
\end{array}\right]$$

Note this matrix, A_{4c}, is very similar to A_2 for the two patch model since we have 4 copies of the basic matrix A and pest adults travelling between neighboring patches. The parameters $\theta_{1,1}, \theta_{2,2}, \theta_{3,3}, \theta_{4,4}$ relate to pest adults which survive and remain in their original patch. Meanwhile, $\theta_{1,2}, \theta_{2,1}, \theta_{2,3}, \theta_{3,2}, \theta_{3,4}, \theta_{4,3}$ relate to pest adults which survive and move to a neighboring patch. Below is the formulation of the pest dynamics for the four patch model. Note this does not include the biological control in the larva stage.

$$
\begin{bmatrix} P_{1,e}(k+1) \\ P_{1,l}(k+1) \\ P_{1,p}(k+1) \\ P_{1,a}(k+1) \\ P_{2,e}(k+1) \\ P_{2,l}(k+1) \\ P_{2,p}(k+1) \\ P_{2,a}(k+1) \\ P_{3,e}(k+1) \\ P_{3,l}(k+1) \\ P_{3,p}(k+1) \\ P_{3,a}(k+1) \\ P_{4,e}(k+1) \\ P_{4,l}(k+1) \\ P_{4,p}(k+1) \\ P_{4,a}(k+1) \end{bmatrix}
=
\begin{bmatrix}
\gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2,3} & 0 & 0 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{3,4} & 0 & 0 & \nu_2 & \theta_{4,4}
\end{bmatrix}
\begin{bmatrix} P_{1,e}(k) \\ P_{1,l}(k) \\ P_{1,p}(k) \\ P_{1,a}(k) \\ P_{2,e}(k) \\ P_{2,l}(k) \\ P_{2,p}(k) \\ P_{2,a}(k) \\ P_{3,e}(k) \\ P_{3,l}(k) \\ P_{3,p}(k) \\ P_{3,a}(k) \\ P_{4,e}(k) \\ P_{4,l}(k) \\ P_{4,p}(k) \\ P_{4,a}(k) \end{bmatrix}
$$

Cost in Four Patches Since we are considering four independent patches, the cost in each patch would be the same formula as the cost in our Basic model. So, if we consider the total cost of four patches we combine the cost in each these four isolated patches, Cost at time t is

$$\beta_1 P_{1,l}(t)^2 + \beta_2 N_1(t) + \beta_1 P_{2,l}(t)^2 + \beta_2 N_2(t) + \beta_1 P_{3,l}(t)^2 + \beta_2 N_3(t) + \beta_1 P_{4,l}(t)^2 + \beta_2 N_4(t).$$

7.1.1 Optimal Control Problem Formulation

The goal for our Optimal Control Problem is to minimize the objective functional

$$J(N_1, N_2, N_3, N_4) = \sum_{t=0}^{T-1} \beta_1 [P_{1,l}(t)^2 + P_{2,l}(t)^2 + P_{3,l}(t)^2 + P_{4,l}(t)^2]$$
$$+\beta_2 [N_1(t) + N_2(t) + N_3(t) + N_4(t)]$$

subject to

$$
\begin{aligned}
P_{1,e}(t+1) &= \gamma_1 P_{1,e}(t) + \theta_1 P_{1,a}(t) && P_{1,e}(0) = \phi_{1,a} \\
P_{1,l}(t+1) &= \gamma_2 P_{1,e}(t) + \zeta_1 e^{-\alpha N_1(t)} P_{1,l}(t) && P_{1,l}(0) = \phi_{1,l} \\
P_{1,p}(t+1) &= \zeta_2 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_1 P_{1,p}(t) && P_{1,p}(0) = \phi_{1,p} \\
P_{1,a}(t+1) &= \nu_2 P_{1,p}(t) + \theta_{1,1} P_{1,a}(t) + \theta_{2,1} P_{2,a}(t) && P_{1,a}(0) = \phi_{1,a} \\
P_{2,e}(t+1) &= \gamma_1 P_{2,e}(t) + \theta_1 P_{2,a}(t) && P_{2,e}(0) = \phi_{2,e} \\
P_{2,l}(t+1) &= \gamma_2 P_{2,e}(t) + \zeta_1 e^{-\alpha N_2(t)} P_{2,l}(t) && P_{2,l}(0) = \phi_{2,l} \\
P_{2,p}(t+1) &= \zeta_2 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_1 P_{2,p}(t) && P_{2,p}(0) = \phi_{2,p} \\
P_{2,a}(t+1) &= \nu_2 P_{2,p}(t) + \theta_{2,2} P_{2,a}(t) + \theta_{1,2} P_{1,a}(t) + \theta_{3,2} P_{3,a}(t) && P_{2,a}(0) = \phi_{2,a} \\
P_{3,e}(t+1) &= \gamma_1 P_{3,e}(t) + \theta_1 P_{3,a}(t) && P_{3,e}(0) = \phi_{3,e} \\
P_{3,l}(t+1) &= \gamma_2 P_{3,e}(t) + \zeta_1 e^{-\alpha N_3(t)} P_{3,l}(t) && P_{1,l}(0) = \phi_{3,l} \\
P_{3,p}(t+1) &= \zeta_2 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_1 P_{3,p}(t) && P_{3,p}(0) = \phi_{3,p} \\
P_{3,a}(t+1) &= \nu_2 P_{3,p}(t) + \theta_{3,3} P_{3,a}(t) + \theta_{2,3} P_{2,a}(t) + \theta_{4,3} P_{4,a}(t) && P_{3,a}(0) = \phi_{3,a} \\
P_{4,e}(t+1) &= \gamma_1 P_{4,e}(t) + \theta_1 P_{4,a}(t) && P_{4,e}(0) = \phi_{4,e} \\
P_{4,l}(t+1) &= \gamma_2 P_{4,e}(t) + \zeta_1 e^{-\alpha N_4(t)} P_{4,l}(t) && P_{4,l}(0) = \phi_{4,l} \\
P_{4,p}(t+1) &= \zeta_2 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_1 P_{4,p}(t) && P_{4,p}(0) = \phi_{4,p} \\
P_{4,a}(t+1) &= \nu_2 P_{4,p}(t) + \theta_{4,4} P_{4,a}(t) + \theta_{3,4} P_{3,a}(t) && P_{4,a}(0) = \phi_{4,a}
\end{aligned}
\tag{7.1}
$$

where $N_1, N_2, N_3, N_4 \in \mathbf{N} = \{N : \{1, ..., T\} \to \{x \in \mathbb{R} | 0 \leq x(t) \leq N_{max}, t = 1, 2, ..., T\}\}$.

7.1.2 Optimal Control Problem

Now we will prove the existence and uniqueness of the optimal control, which we denote $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$. Additionally, we will prove necessary conditions for the optimal control $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$. The proofs roughly follow the proofs of Theorems 6.2.1, 6.2.2, 6.2.3.

7.1.2.1 Existence

Theorem 7.1.1. *There exists* $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4 \in \boldsymbol{N}$ *which minimizes* $J(N_1, N_2, N_3, N_4)$.

Proof. We have that each $P_{1,e}, P_{1,l}, P_{1,p}, P_{1,a},\ P_{2,e}, P_{2,l}, P_{2,p}, P_{2,a},\ P_{3,e}, P_{3,l}, P_{3,p}, P_{3,a}$, $P_{4,e}, P_{4,l}, P_{4,p}, P_{4,a}$ is continuous as a function of N_1, N_2, N_3, N_4 at every time step by Equation 7.1. Define $B^+ = \{(N(1), ...N(T)) | N \in \mathbf{N}\}$. We note that there is a natural isomorphism between $\mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ and $B^+ \times B^+ \times B^+ \times B^+$. Considering $J : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \times \mathbf{N} \leftrightarrow B^+ \times B^+ \times B^+ \times B^+ \to \mathbb{R}$, we see that J is continuous as a function of N_1, N_2, N_3 and N_4. We have that B^+ is a compact subset of $\mathbb{R}^T$ in the standard Euclidean topology. Therefore, $\inf\limits_{N_1,N_2,N_3,N_4 \in \mathbf{N}} J(N_1, N_2, N_3, N_4)$ exists. Hence, we have sequences $N_{1_k}, N_{2_k}, N_{3_k}, N_{4_k} \in \mathbf{N}$ such that $\lim\limits_{k\to\infty} J(N_{1_k}, N_{2_k}, N_{3_k}, N_{4_k}) = \inf\limits_{N_1,N_2,N_3,N_4 \in \mathbf{N}} J(N_1, N_2, N_3, N_4)$, with corresponding $P_{1,e_k}, P_{1,l_k}, P_{1,p_k}, P_{1,a_k}, P_{2,e_k}, P_{2,l_k}$, $P_{2,p_k}, P_{2,a_k}, P_{3,e_k}, P_{3,l_k}, P_{3,p_k}, P_{3,a_k},\ P_{4,e_k}, P_{4,l_k}, P_{4,p_k}, P_{4,a_k}$ sequences. Thus we can find subsequences $N_{1_{k_j}}, N_{2_{k_j}}, N_{3_{k_j}}, N_{4_{k_j}}, P_{1,e_{k_j}}, P_{1,l_{k_j}}, P_{1,p_{k_j}}, P_{1,a_{k_j}}, P_{2,e_{k_j}}, P_{2,l_{k_j}}, P_{2,p_{k_j}}, P_{2,a_{k_j}}$, $P_{3,e_{k_j}}, P_{3,l_{k_j}}, P_{3,p_{k_j}}, P_{3,a_{k_j}}, P_{4,e_{k_j}}, P_{4,l_{k_j}}, P_{4,p_{k_j}}, P_{4,a_{k_j}}$, such that $\lim\limits_{j\to\infty} J(N_{1_{k_j}}, N_{2_{k_j}}, N_{3_{k_j}}, N_{4_{k_j}})$ $= \inf_{N_1,N_2,N_3,N_4 \in \mathbf{N}} J(N_1, N_2, N_3, N_4)$ and converge to $N_{1_{k_j}} \to \mathcal{N}_1, N_{2_{k_j}} \to \mathcal{N}_2, N_{3_{k_j}} \to$

$\mathcal{N}_3, N_{4_{k_j}} \to \mathcal{N}_4, P_{1,e_{k_j}} \to \mathcal{P}_{1,e}, P_{1,l_{k_j}} \to \mathcal{P}_{1,l}, P_{1,p_{k_j}} \to \mathcal{P}_{1,p}, P_{1,a_{k_j}} \to \mathcal{P}_{1,a}, P_{2,e_{k_j}} \to \mathcal{P}_{2,e}, P_{2,l_{k_j}} \to \mathcal{P}_{2,l}, P_{2,p_{k_j}} \to \mathcal{P}_{2,p}, P_{2,a_{k_j}} \to \mathcal{P}_{2,a}, P_{3,e_{k_j}} \to \mathcal{P}_{3,e}, P_{3,l_{k_j}} \to \mathcal{P}_{3,l}, P_{3,p_{k_j}} \to \mathcal{P}_{3,p}, P_{3,a_{k_j}} \to \mathcal{P}_{3,a}, P_{4,e_{k_j}} \to \mathcal{P}_{4,e}, P_{4,l_{k_j}} \to \mathcal{P}_{4,l}, P_{4,p_{k_j}} \to \mathcal{P}_{4,p}, P_{4,a_{k_j}} \to \mathcal{P}_{4,a}$. Therefore, there exists $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4 \in \mathbf{N}$ which minimizes $J(N_1, N_2, N_3, N_4)$.

□

7.1.2.2 Necessary Conditions

Adjoint System: Define the following terminal value system:

$$\lambda_{1,e}(t) = \lambda_{1,e}(t+1)\gamma_1 + \lambda_{1,l}(t+1)\gamma_2$$

$$\lambda_{1,l}(t) = 2\beta_1\mathcal{P}_{1,l}(t) + \lambda_{1,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_1(t)} + \lambda_{1,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_1(t)}$$

$$\lambda_{1,p}(t) = \lambda_{1,p}(t+1)\nu_1 + \lambda_{1,a}(t+1)\nu_2$$

$$\lambda_{1,a}(t) = \lambda_{1,e}(t+1)\theta_1 + \lambda_{1,a}(t+1)\theta_{1,1} + \lambda_{2,a}(t+1)\theta_{1,2}$$

$$\lambda_{2,e}(t) = \lambda_{2,e}(t+1)\gamma_1 + \lambda_{2,l}(t+1)\gamma_2$$

$$\lambda_{2,l}(t) = 2\beta_1\mathcal{P}_{2,l}(t) + \lambda_{2,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_2(t)} + \lambda_{2,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_2(t)}$$

$$\lambda_{2,p}(t) = \lambda_{2,p}(t+1)\nu_1 + \lambda_{2,a}(t+1)\nu_2$$

$$\lambda_{2,a}(t) = \lambda_{2,e}(t+1)\theta_1 + \lambda_{2,a}(t+1)\theta_{2,2} + \lambda_{1,a}(t+1)\theta_{2,1} + \lambda_{3,a}(t+1)\theta_{2,3}$$

$$\lambda_{3,e}(t) = \lambda_{3,e}(t+1)\gamma_1 + \lambda_{3,l}(t+1)\gamma_2$$

$$\lambda_{3,l}(t) = 2\beta_1\mathcal{P}_{3,l}(t) + \lambda_{3,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_3(t)} + \lambda_{3,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_3(t)}$$

$$\lambda_{3,p}(t) = \lambda_{3,p}(t+1)\nu_1 + \lambda_{3,a}(t+1)\nu_2$$

$$\lambda_{3,a}(t) = \lambda_{3,e}(t+1)\theta_1 + \lambda_{3,a}(t+1)\theta_{3,3} + \lambda_{2,a}(t+1)\theta_{3,2} + \lambda_{4,a}(t+1)\theta_{3,4}$$

$$\lambda_{4,e}(t) = \lambda_{4,e}(t+1)\gamma_1 + \lambda_{4,l}(t+1)\gamma_2$$

$$\lambda_{4,l}(t) = 2\beta_1\mathcal{P}_{4,l}(t) + \lambda_{4,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_4(t)} + \lambda_{4,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_4(t)}$$

$$\lambda_{4,p}(t) = \lambda_{4,p}(t+1)\nu_1 + \lambda_{4,a}(t+1)\nu_2$$

$$\lambda_{4,a}(t) = \lambda_{4,e}(t+1)\theta_1 + \lambda_{4,a}(t+1)\theta_{4,4} + \lambda_{3,a}(t+1)\theta_{4,3}$$

$\lambda_{1e}(T) = 0,\ \lambda_{1,l}(T) = 0,\ \lambda_{1,p}(T) = 0,\ \lambda_{1,a}(T) = 0,\ \lambda_{2,e}(T) = 0,\ \lambda_{2,l}(T) = 0,\ \lambda_{2,p}(T) = 0,\ \lambda_{2,a}(T) = 0,\ \lambda_{3,e}(T) = 0,\ \lambda_{3,l}(T) = 0,\ \lambda_{3,p}(T) = 0,\ \lambda_{3,a}(T) = 0,\ \lambda_{4,e}(T) = 0,\ \lambda_{4,l}(T) = 0,\ \lambda_{4,p}(T) = 0,\ \lambda_{4,a}(T) = 0.$

Theorem 7.1.2. *If there exists optimal controls $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$, then there exists adjoint system 7.1.2.2, and*

$$\mathcal{N}_j(t) = \begin{cases} 0 & if\ \frac{\beta 2}{\alpha} > \xi_j(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi_j(t)] & if\ \frac{\beta 2}{\alpha} \leq \xi_j(t) \end{cases}.$$

for $j = 1, 2, 3, 4$ we have that $\xi_j(t) = \zeta_1\lambda_{j,l}(t+1)\mathcal{P}_{j,l}(t) + \zeta_2\lambda_{j,p}(t+1)\mathcal{P}_{j,l}(t)$

Proof. The proof is similar to that of Theorem 6.2.2.

Since we have that $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$ minimize $J(N_1, N_2, N_3, N_4)$; for all sufficiently small $\varepsilon > 0$ and for all

$$\eta_1, \eta_2, \eta_3, \eta_4 \in \{\eta = (\eta(1), ..., \eta(T))|\eta(t) \leq 1, t = 1, ..., T\}$$

we have that

$$J(\mathcal{N}_1 + \eta_1\varepsilon, \mathcal{N}_2 + \eta_2\varepsilon, \mathcal{N}_3 + \eta_3\varepsilon, \mathcal{N}_4 + \eta_4\varepsilon) \geq J(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4).$$

Similar to Theorem 6.2.2, we will take the directional derivative with $\mathcal{N}_j + \eta_j\varepsilon = \mathcal{N}_j^\varepsilon \in \mathbf{N}$. Then we have that:

$$0 \leq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[(\mathcal{N}_1 + \eta_1\varepsilon, \mathcal{N}_2 + \eta_2\varepsilon, \mathcal{N}_3 + \eta_3\varepsilon, \mathcal{N}_4 + \eta_4\varepsilon) - J(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4)]$$

$$= \sum_{t=0}^{T-1} \beta_1 2[\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \mathcal{P}_{2,l}(t)\psi_{2,l}(t) + \mathcal{P}_{3,l}(t)\psi_{3,l}(t) + \mathcal{P}_{4,l}(t)\psi_{4,l}(t)]$$

$$+\beta_2[\eta_1(t)+\eta_2(t)+\eta_3(t)+\eta_4(t)].$$

Additionally we define the sensitivities, $\psi_{j,e}(t), \psi_{j,l}(t), \psi_{j,p}(t), \psi_{j,a}(t)$ for $j = 1, 2, 3, 4$ similar to as in Theorem 6.2.2.

Now, returning to

$$0 \leq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[(\mathcal{N}_1+\eta_1\varepsilon, \mathcal{N}_2+\eta_2\varepsilon, \mathcal{N}_3+\eta_3\varepsilon, \mathcal{N}_4+\eta_4\varepsilon) - J(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3,\mathcal{N}_4)]$$

$$= \sum_{t=0}^{T-1} \beta_1 2[\mathcal{P}_{1,l}(t)\psi_{1,l}(t) + \mathcal{P}_{2,l}(t)\psi_{2,l}(t) + \mathcal{P}_{3,l}(t)\psi_{3,l}(t) + \mathcal{P}_{4,l}(t)\psi_{4,l}(t)]$$

$$+\beta_2[\eta_1(t)+\eta_2(t)+\eta_3(t)+\eta_4(t)].$$

To remove the sensitivities $\psi_{1,l}(t), \psi_{2,l}(t), \psi_{3,l}(t), \psi_{4,l}(t)$ we will manipulate the sensitivities and adjoints equations as in Theorem 6.2.2. The process of switching limits of summation and using properties of matrices and vectors results in:

$$0 \leq \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}[(\mathcal{N}_1+\eta_1\varepsilon, \mathcal{N}_2+\eta_2\varepsilon, \mathcal{N}_3+\eta_3\varepsilon, \mathcal{N}_4+\eta_4\varepsilon) - J(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3,\mathcal{N}_4)]$$

$$= \sum_{t=0}^{T-1} \eta_1(t)\left[-\alpha e^{-\alpha\mathcal{N}_1(t)}\mathcal{P}_{1,l}(t)[\lambda_{1,l}(t+1)\zeta_1 + \lambda_{1,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_2(t)\left[-\alpha e^{-\alpha\mathcal{N}_2(t)}\mathcal{P}_{2,l}(t)[\lambda_{2,l}(t+1)\zeta_1 + \lambda_{2,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_3(t)\left[-\alpha e^{-\alpha\mathcal{N}_3(t)}\mathcal{P}_{3,l}(t)[\lambda_{3,l}(t+1)\zeta_1 + \lambda_{3,p}(t+1)\zeta_2] + \beta_2\right]$$

$$+\eta_4(t)\left[-\alpha e^{-\alpha\mathcal{N}_4(t)}\mathcal{P}_{4,l}(t)[\lambda_{4,l}(t+1)\zeta_1 + \lambda_{4,p}(t+1)\zeta_2] + \beta_2\right] = \chi_4.$$

Consider the previous equation with equality, $0 = \chi_4$. Since this must hold for all η_1, η_2, η_3 and η_4, we can find the solutions to $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ similar to the process in 6.2.2

□

7.1.2.3 Uniqueness

Theorem 7.1.3. *If the optimal controls $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$ exist, then they are unique.*

Proof. In order to show $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$ are unique we will show that

$$J(N_1, N_2, N_3, N_4) = \sum_{t=0}^{T-1} \beta_1[P_{1,l}(t)^2 + P_{2,l}(t)^2 + P_{3,l}(t)^2 + P_{4,l}(t)^2]$$
$$+\beta_2[N_1(t) + N_2(t) + N_3(t) + N_4(t)]$$

is strictly convex. To show that J is strictly convex we use a method similar to Theorem 6.2.3 by defining $z(\varepsilon) = J((1-\varepsilon)N_1+\varepsilon\eta_1, (1-\varepsilon)N_2+\varepsilon\eta_2, (1-\varepsilon)N_3+\varepsilon\eta_3, (1-\varepsilon)N_4+\varepsilon\eta_4)$ for $N_1, N_2, N_3, N_4, \eta_1, \eta_2, \eta_3, \eta_4 \in \mathbf{N}$, and $0 < \varepsilon < 1$. Note that if z, a one dimensional function, is convex for every choice of η then J will be convex. To establish convexity of z we will show that $z''(\varepsilon) > 0$. First take the derivative of z, as in Theorem 6.2.3:

$$z'(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[P^{\varepsilon}_{1,l}(t)\psi^{\varepsilon}_{1,l}(t) + P^{\varepsilon}_{2,l}(t)\psi^{\varepsilon}_{2,l}(t) + P^{\varepsilon}_{3,l}(t)\psi^{\varepsilon}_{3,l}(t) + P^{\varepsilon}_{4,l}(t)\psi^{\varepsilon}_{4,l}(t)]$$

$$+\beta_2[(\eta_1(t) - N_1(t)) + (\eta_2(t) - N_2(t)) + (\eta_3(t) - N_3(t)) + (\eta_4(t) - N_4(t))].$$

We define derivatives of sensitivities, $\sigma_{j,e}(t), \sigma_{j,l}(t), \sigma_{j,p}(t), \sigma_{j,a}(t)$, for $j = 1, 2, 3, 4$ as in Theorem 6.2.3 with $\psi^{\varepsilon}_{j,e}(t+1), \psi^{\varepsilon}_{j,l}(t+1), \psi^{\varepsilon}_{j,p}(t+1), \psi^{\varepsilon}_{4,a}(t+1)$.

Thus,

$$z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2 + \sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2 + \sigma^{\varepsilon}_{3,l}(t)P^{\varepsilon}_{3,l}(t) + \psi^{\varepsilon}_{3,l}(t)^2$$

$$+\sigma^{\varepsilon}_{4,l}(t)P^{\varepsilon}_{4,l}(t) + \psi^{\varepsilon}_{4,l}(t)^2].$$

We need $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t)+\psi^{\varepsilon}_{1,l}(t)^2+\sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t)+\psi^{\varepsilon}_{2,l}(t)^2+\sigma^{\varepsilon}_{3,l}(t)P^{\varepsilon}_{3,l}(t)+ \psi^{\varepsilon}_{3,l}(t)^2 + \sigma^{\varepsilon}_{4,l}(t)P^{\varepsilon}_{4,l}(t) + \psi^{\varepsilon}_{4,l}(t)^2] > 0$, meaning we need to bound $\sigma^{\varepsilon}_{j,l}(t) > 0$. The argument $\sigma^{\varepsilon}_{l}(t) > 0$ for all t is similar to that in Theorem 6.2.3.

Therefore, we have that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2 + \sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2 + \sigma^{\varepsilon}_{3,l}(t)P^{\varepsilon}_{3,l}(t) + \psi^{\varepsilon}_{3,l}(t)^2 + \sigma^{\varepsilon}_{4,l}(t)P^{\varepsilon}_{4,l}(t) + \psi^{\varepsilon}_{4,l}(t)^2] > 0$, and we have uniqueness by convexity of z.

□

7.1.3 Parameters

While most of the parameters are the same as in Part 1, we do need to consider the new θ parameters which characterize adult dispersal. To start define $\theta_{i,i} = p \cdot \theta_2$, where p is the percent of pest adults which do not travel.

Suppose there is equal probability that the pests will travel east, west, north, and south as seem in Figure 7.1. So we have $\theta_{i,j} = \frac{1-p}{4} \cdot \theta_2$, where $i \neq j$. Then we have:

$$\theta_{1,1} = \theta_{2,2} = \theta_{3,3} = \theta_{4,4} = p \cdot \theta_2$$

$$\theta_{1,2} = \theta_{2,1} = \theta_{2,3} = \theta_{3,2} = \theta_{3,4} = \theta_{4,3} = \frac{1-p}{4} \cdot \theta_2.$$

Later we will vary the value p for a specific case study.

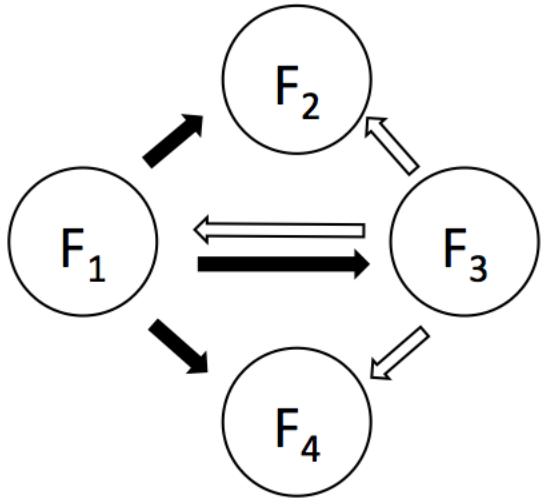

Figure 7.2: Four Isolated Patches: We have any patch is connected to any other patch. Specifically consider patch F_1 which is connected to patch F_2, F_3, and F_4 by the black arrows. Note the black arrows from F_1 demonstrate how the adult pest can disperse from the patch, specifically the pest adult can spread to any of the other patches. Similarly, F_3 the white arrows from F_3 demonstrate how the adult pest can spread from the patch, specifically the pest adult can spread to any of the other patches. Note that F_2 and F_4 also spread to all other patches, these patches arrows are not show in the figure.

7.2 Four Isolated Patches

Now, we will consider using four isolated patches. Again adults can travel between patches, and we will assume patches are arranged as the Figure 7.2 suggests.

The resulting matrix for out pest dynamics will be as follows:

$$A_{4s} = \left[\begin{array}{cccc:cccc:cccc:cccc}
\gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & \theta_{3,1} & 0 & 0 & 0 & \theta_{4,1} \\
\hdashline
0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & \theta_{4,2} \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,3} & 0 & 0 & 0 & \theta_{2,3} & 0 & 0 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\
0 & 0 & 0 & \theta_{1,4} & 0 & 0 & 0 & \theta_{2,4} & 0 & 0 & 0 & \theta_{3,4} & 0 & 0 & \nu_2 & \theta_{4,4}
\end{array}\right].$$

Note that the matrix, A_{4s}, is very similar to A_{4c}. Again we have the parameters $\theta_{1,1}, \theta_{2,2}, \theta_{3,3}, \theta_{4,4}$ relate to pest adults which survive and remain in their original patch. Meanwhile, $\theta_{1,2}, \theta_{2,1}, \theta_{2,3}, \theta_{3,2}, \theta_{3,4}, \theta_{4,3}$ relate to pest adults which survive and move to a neighboring patch. However in A_{4s} we also have $\theta_{3,1}, \theta_{1,3}, \theta_{4,1}, \theta_{1,4}, \theta_{4,2}, \theta_{2,4}$ for pest adult movement between the other patches. Below is the formulation of the pest dynamics for the four patch model, note this does not include the biological control in the larva stage.

$$\begin{bmatrix} P_{1,e}(k+1) \\ P_{1,l}(k+1) \\ P_{1,p}(k+1) \\ P_{1,a}(k+1) \\ P_{2,e}(k+1) \\ P_{2,l}(k+1) \\ P_{2,p}(k+1) \\ P_{2,a}(k+1) \\ P_{3,e}(k+1) \\ P_{3,l}(k+1) \\ P_{3,p}(k+1) \\ P_{3,a}(k+1) \\ P_{4,e}(k+1) \\ P_{4,l}(k+1) \\ P_{4,p}(k+1) \\ P_{4,a}(k+1) \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & \theta_{3,1} & 0 & 0 & 0 & \theta_{4,1} \\ 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{1,2} & 0 & 0 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & \theta_{4,2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{1,3} & 0 & 0 & 0 & \theta_{2,3} & 0 & 0 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \theta_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 & \nu_1 & 0 \\ 0 & 0 & 0 & \theta_{1,4} & 0 & 0 & 0 & \theta_{2,4} & 0 & 0 & 0 & \theta_{3,4} & 0 & 0 & \nu_2 & \theta_{4,4} \end{bmatrix} \begin{bmatrix} P_{1,e}(k) \\ P_{1,l}(k) \\ P_{1,p}(k) \\ P_{1,a}(k) \\ P_{2,e}(k) \\ P_{2,l}(k) \\ P_{2,p}(k) \\ P_{2,a}(k) \\ P_{3,e}(k) \\ P_{3,l}(k) \\ P_{3,p}(k) \\ P_{3,a}(k) \\ P_{4,e}(k) \\ P_{4,l}(k) \\ P_{4,p}(k) \\ P_{4,a}(k) \end{bmatrix}$$

Cost in Four Patches As in the Four Connected Patches we are considering four independent patches, so Cost at time t is

$$\beta_1 P_{1,l}(t)^2 + \beta_2 N_1(t) + \beta_1 P_{2,l}(t)^2 + \beta_2 N_2(t) + \beta_1 P_{3,l}(t)^2 + \beta_2 N_3(t) + \beta_1 P_{4,l}(t)^2 + \beta_2 N_4(t).$$

7.2.1 Optimal Control Problem Formulation

The set-up for our Optimal Control Problem is to minimize the objective functional

$$J(N_1, N_2, N_3, N_4) = \sum_{t=0}^{T-1} \beta_1 [P_{1,l}(t)^2 + P_{2,l}(t)^2 + P_{3,l}(t)^2 + P_{4,l}(t)^2]$$
$$+\beta_2 [N_1(t) + N_2(t) + N_3(t) + N_4(t)]$$

subject to

$$
\begin{aligned}
P_{1,e}(t+1) &= \gamma_1 P_{1,e}(t) + \theta_1 P_{1,a}(t) & P_{1,e}(0) &= \phi_{1,e} \\
P_{1,l}(t+1) &= \gamma_2 P_{1,e}(t) + \zeta_1 e^{-\alpha N_1(t)} P_{1,l}(t) & P_{1,l}(0) &= \phi_{1,l} \\
P_{1,p}(t+1) &= \zeta_2 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_1 P_{1,p}(t) & P_{1,p}(0) &= \phi_{1,p} \\
P_{1,a}(t+1) &= \nu_2 P_{1,p}(t) + \theta_{1,1} P_{1,a}(t) + \theta_{2,1} P_{2,a}(t) + \theta_{3,1} P_{3,a}(t) + \theta_{4,1} P_{4,a}(t) & P_{1,a}(0) &= \phi_{1,a} \\
P_{2,e}(t+1) &= \gamma_1 P_{2,e}(t) + \theta_1 P_{2,a}(t) & P_{2,e}(0) &= \phi_{2,e} \\
P_{2,l}(t+1) &= \gamma_2 P_{2,e}(t) + \zeta_1 e^{-\alpha N_2(t)} P_{2,l}(t) & P_{2,l}(0) &= \phi_{2,l} \\
P_{2,p}(t+1) &= \zeta_2 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_1 P_{2,p}(t) & P_{2,p}(0) &= \phi_{2,p} \\
P_{2,a}(t+1) &= \nu_2 P_{2,p}(t) + \theta_{2,2} P_{2,a}(t) + \theta_{1,2} P_{1,a}(t) + \theta_{3,2} P_{3,a}(t) + \theta_{4,2} P_{4,a}(t) & P_{2,a}(0) &= \phi_{2,a} \\
P_{3,e}(t+1) &= \gamma_1 P_{3,e}(t) + \theta_1 P_{3,a}(t) & P_{3,e}(0) &= \phi_{3,e} \\
P_{3,l}(t+1) &= \gamma_2 P_{3,e}(t) + \zeta_1 e^{-\alpha N_3(t)} P_{3,l}(t) & P_{1,l}(0) &= \phi_{3,l} \\
P_{3,p}(t+1) &= \zeta_2 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_1 P_{3,p}(t) & P_{3,p}(0) &= \phi_{3,p} \\
P_{3,a}(t+1) &= \nu_2 P_{3,p}(t) + \theta_{3,3} P_{3,a}(t) + \theta_{2,3} P_{2,a}(t) + \theta_{4,3} P_{4,a}(t) + \theta_{1,3} P_{1,a}(t) & P_{3,a}(0) &= \phi_{3,a} \\
P_{4,e}(t+1) &= \gamma_1 P_{4,e}(t) + \theta_1 P_{4,a}(t) & P_{4,e}(0) &= \phi_{4,e} \\
P_{4,l}(t+1) &= \gamma_2 P_{4,e}(t) + \zeta_1 e^{-\alpha N_4(t)} P_{4,l}(t) & P_{4,l}(0) &= \phi_{4,l} \\
P_{4,p}(t+1) &= \zeta_2 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_1 P_{4,p}(t) & P_{4,p}(0) &= \phi_{4,p} \\
P_{4,a}(t+1) &= \nu_2 P_{4,p}(t) + \theta_{4,4} P_{4,a}(t) + \theta_{3,4} P_{3,a}(t) + \theta_{1,4} P_{1,a}(t) + \theta_{2,4} P_{2,a}(t) & P_{4,a}(0) &= \phi_{4,a}
\end{aligned}
\tag{7.2}
$$

where $N_1(t), N_2(t), N_3(t), N_4(t) \geq 0$ for all t and $N_1, N_2, N_3, N_4 \in \mathbf{N} = \{N : \{1, ..., T\} \rightarrow \{x \in \mathbb{R} | 0 \leq x(t) \leq N_{max}, t = 1, 2, ..., T\}\}$.

7.2.2 Optimal Control Problem

Now we will prove the existence and uniqueness of the optimal control $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$. Additionally, we will prove necessary conditions for the optimal control $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_4$. The proofs roughly follow the proofs in Theorems 7.1.1, 7.1.2, and 7.1.3.

7.2.2.1 Existence

Theorem 7.2.1. *There exists* $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4 \in \boldsymbol{N}$ *which minimizes* $J(N_1, N_2, N_3, N_4)$.

Proof. This theorem is analogous to Theorem 7.1.1, since $P_{1,e}, P_{1,l}, P_{1,p}, P_{1,a}, P_{2,e}, P_{2,l}, P_{2,p}, P_{2,a}, P_{3,e}, P_{3,l}, P_{3,p}, P_{3,a}, P_{4,e}, P_{4,l}, P_{4,p}, P_{4,a}$ are all continuous with respect to N_1, N_2, N_3, N_4 by Equations 7.2. Additionally, we have J is continuous as a function of N_1, N_2, N_3, N_4 and B^+ is a compact subset of $\mathbb{R}^T$, so $\inf_{N_1, N_2, N_3, N_4 \in \mathbf{N}} J(N_1, N_2, N_3, N_4)$ exists.

□

7.2.2.2 Necessary Conditions

Adjoint System: Consider the following terminal value system:

$$\lambda_{1,e}(t) = \lambda_{1,e}(t+1)\gamma_1 + \lambda_{1,l}(t+1)\gamma_2$$

$$\lambda_{1,l}(t) = 2\beta_1 \mathcal{P}_{1,l}(t) + \lambda_{1,l}(t+1)\zeta_1 e^{-\alpha \mathcal{N}_1(t)} + \lambda_{1,p}(t+1)\zeta_2 e^{-\alpha \mathcal{N}_1(t)}$$

$$\lambda_{1,p}(t) = \lambda_{1,p}(t+1)\nu_1 + \lambda_{1,a}(t+1)\nu_2$$

$$\lambda_{1,a}(t) = \lambda_{1,e}(t+1)\theta_1 + \lambda_{1,a}(t+1)\theta_{1,1} + \lambda_{2,a}(t+1)\theta_{1,2} + \lambda_{3,a}(t+1)\theta_{1,3} + \lambda_{4,a}(t+1)\theta_{1,4}$$

$$\lambda_{2,e}(t) = \lambda_{2,e}(t+1)\gamma_1 + \lambda_{2,l}(t+1)\gamma_2$$
$$\lambda_{2,l}(t) = 2\beta_1\mathcal{P}_{2,l}(t) + \lambda_{2,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_2(t)} + \lambda_{2,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_2(t)}$$
$$\lambda_{2,p}(t) = \lambda_{2,p}(t+1)\nu_1 + \lambda_{2,a}(t+1)\nu_2$$
$$\lambda_{2,a}(t) = \lambda_{2,e}(t+1)\theta_1 + \lambda_{2,a}(t+1)\theta_{2,2} + \lambda_{1,a}(t+1)\theta_{2,1} + \lambda_{3,a}(t+1)\theta_{2,3} + \lambda_{4,a}(t+1)\theta_{2,4}$$
$$\lambda_{3,e}(t) = \lambda_{3,e}(t+1)\gamma_1 + \lambda_{3,l}(t+1)\gamma_2$$
$$\lambda_{3,l}(t) = 2\beta_1\mathcal{P}_{3,l}(t) + \lambda_{3,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_3(t)} + \lambda_{3,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_3(t)}$$
$$\lambda_{3,p}(t) = \lambda_{3,p}(t+1)\nu_1 + \lambda_{3,a}(t+1)\nu_2$$
$$\lambda_{3,a}(t) = \lambda_{3,e}(t+1)\theta_1 + \lambda_{3,a}(t+1)\theta_{3,3} + \lambda_{2,a}(t+1)\theta_{3,2} + \lambda_{4,a}(t+1)\theta_{3,4} + \lambda_{1,a}(t+1)\theta_{3,1}$$
$$\lambda_{4,e}(t) = \lambda_{4,e}(t+1)\gamma_1 + \lambda_{4,l}(t+1)\gamma_2$$
$$\lambda_{4,l}(t) = 2\beta_1\mathcal{P}_{4,l}(t) + \lambda_{4,l}(t+1)\zeta_1 e^{-\alpha\mathcal{N}_4(t)} + \lambda_{4,p}(t+1)\zeta_2 e^{-\alpha\mathcal{N}_4(t)}$$
$$\lambda_{4,p}(t) = \lambda_{4,p}(t+1)\nu_1 + \lambda_{4,a}(t+1)\nu_2$$
$$\lambda_{4,a}(t) = \lambda_{4,e}(t+1)\theta_1 + \lambda_{4,a}(t+1)\theta_{4,4} + \lambda_{3,a}(t+1)\theta_{4,3} + \lambda_{1,a}(t+1)\theta_{4,1} + \lambda_{2,a}(t+1)\theta_{4,2}$$

$\lambda_{1e}(T) = 0$, $\lambda_{1,l}(T) = 0$, $\lambda_{1,p}(T) = 0$, $\lambda_{1,a}(T) = 0$, $\lambda_{2,e}(T) = 0$, $\lambda_{2,l}(T) = 0$, $\lambda_{2,p}(T) = 0$, $\lambda_{2,a}(T) = 0$, $\lambda_{3,e}(T) = 0$, $\lambda_{3,l}(T) = 0$, $\lambda_{3,p}(T) = 0$, $\lambda_{3,a}(T) = 0$, $\lambda_{4,e}(T) = 0$, $\lambda_{4,l}(T) = 0$, $\lambda_{4,p}(T) = 0$, $\lambda_{4,a}(T) = 0$.

Theorem 7.2.2. *If there exists optimal controls $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$, then there exists adjoint system 7.2.2.2, and*

$$\mathcal{N}_j(t) = \begin{cases} 0 & \text{if } \frac{\beta 2}{\alpha} > \xi_j(t) \\ \frac{1}{\alpha}\ln[\frac{\alpha}{\beta_2}\xi_j(t)] & \text{if } \frac{\beta 2}{\alpha} \le \xi_j(t) \end{cases}.$$

for $j = 1,2,3,4$ we have that $\xi_j(t) = \zeta_1\lambda_{j,l}(t+1)\mathcal{P}_{j,l}(t) + \zeta_2\lambda_{j,p}(t+1)\mathcal{P}_{j,l}(t)$

Proof. The proof is similar to that of Theorem 7.1.2. The difference comes in the additional terms in the adjoint and sensitivity equations for the adults. The change does not alter the proof process, since the directional derivative will be the same, and the difference arises in the manipulation of sensitivities to adjoints. Hence we find the same equation for the directional derivative and thus the formulas of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$ are as in Theorem 7.1.2.

□

7.2.2.3 Uniqueness

Theorem 7.2.3. *If the optimal controls $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ and $\mathcal{N}_4$ exist, then they are unique.*

Proof. The proof is similar to that of Theorem 7.1.3. Again the difference comes in the additional terms in the adjoint, sensitivity, and σ equations for the adults. The change does not alter the proof process, the terms are incorporated with the same method as in the proof of Theorem 6.2.3 and 7.1.3. Then we have that

$$z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2 + \sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2 + \sigma^{\varepsilon}_{3,l}(t)P^{\varepsilon}_{3,l}(t) + \psi^{\varepsilon}_{3,l}(t)^2 + \sigma^{\varepsilon}_{4,l}(t)P^{\varepsilon}_{4,l}(t) + \psi^{\varepsilon}_{4,l}(t)^2]$$

and $\sigma^{\varepsilon}_{j,l}(t) > 0$.

Therefore, we have that $z''(\varepsilon) = \sum_{t=0}^{T-1} \beta_1 2[\sigma^{\varepsilon}_{1,l}(t)P^{\varepsilon}_{1,l}(t) + \psi^{\varepsilon}_{1,l}(t)^2 + \sigma^{\varepsilon}_{2,l}(t)P^{\varepsilon}_{2,l}(t) + \psi^{\varepsilon}_{2,l}(t)^2 + \sigma^{\varepsilon}_{3,l}(t)P^{\varepsilon}_{3,l}(t) + \psi^{\varepsilon}_{3,l}(t)^2 + \sigma^{\varepsilon}_{4,l}(t)P^{\varepsilon}_{4,l}(t) + \psi^{\varepsilon}_{4,l}(t)^2] > 0$, and we have uniqueness by convexity of z.

□

7.2.3 Parameters

Most of the parameters are the same as in Part 1, and we do need to consider the new θ parameters which characterize adult spread. As in the four connected patches case define $\theta_{i,i} = p \cdot \theta_2$, where p is the percent of pest adults which do not travel.

Unlike the four connected patches case there is equal probability that the weevils will travel between the three other patches as seen in Figure 7.2. So we have $\theta_{i,j} = \frac{1-p}{3} \cdot \theta_2$ when $i \neq j$. Then we have:

$$\theta_{1,1} = \theta_{2,2} = \theta_{3,3} = \theta_{4,4} = p \cdot \theta_2$$

$$\theta_{1,2} = \theta_{1,3} = \theta_{1,4} = \theta_{2,1} = \theta_{2,3} = \theta_{2,4} = \theta_{3,1} = \theta_{3,2} = \theta_{3,4} = \theta_{4,1} = \theta_{4,2} = \theta_{4,3} = \frac{1-p}{3} \cdot \theta_2.$$

Later we will vary the value p for a specific case study.

Since these patches are not adjacent in space it is possible that some of the pest adults will die along the trip, so later we will incorporate a mortality factor.

7.3 Case Study: DRW

Once again we will use the Diaprepes abbreviatus as a case study, making most of the parameters the same as in Part 1. The only new parameter is p, the percent of DRW adults which do not travel. We have that DRW adult can fly an average dispersal distance is less than 0.03 hectares[TJWJK16]. For an estimation of p we must also include the possibility that wind and human interaction allow the DRW adults to spread further [JG+09a].

7.3.1 Four Connected Patches Simulations

We will use the Forward-Backward Sweep to estimate for the four patches how many nematodes to use and when and where to use them. Since we have spreading to neighboring patches, if the infestation starts everywhere, when the DRW spread patch 1 and 4 look alike and patch 2 and 3 look alike.

We will explore the behavior of this model more by varying p and where the infestation starts.

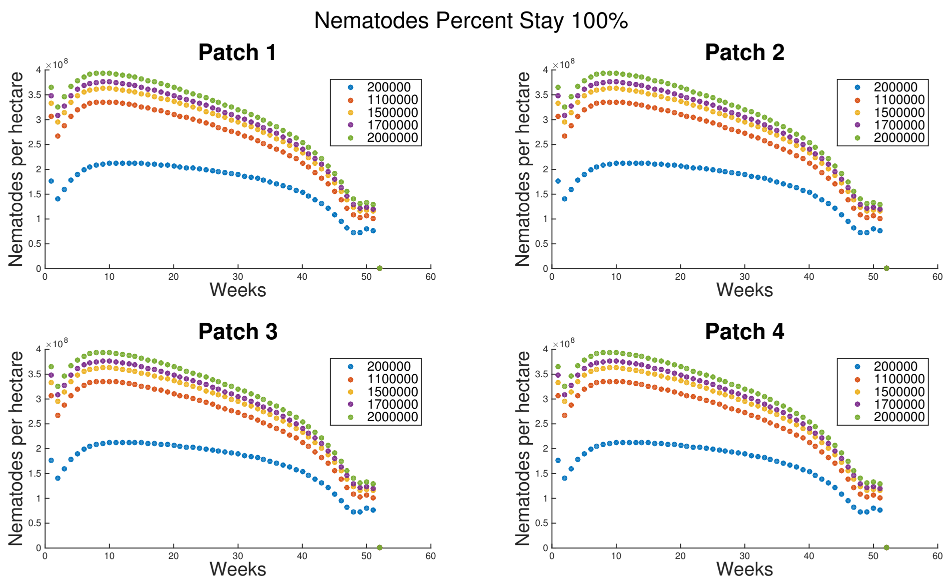

Figure 7.3: Using the Forward-Backward Sweep we calculate the number of nematodes to apply for various initial populations: 200000, 1100000, 1500000, 1700000, 2000000. Here we have all DRW stay in their original patch.

In Figure 7.3 we show the FBS for various initial populations without any spread. Each individual patch looks the same as that in the Basic model, Figure 4.3. When we start to run simulations varying p and where the infestation starts we run into issues with the number of runs the simulation needs to perform due to the choice of δ. Due to computational restraints we reexamine Figures 4.1 and 4.2. We will now shift our choice of δ to be from 0.2 up to 1, allowing for computational ease and answer accuracy.

7.3.1.1 Varying Initial Population and Percent of DRW Adults which Remain, p

We will vary the initial population and consider various percentages for how many adult DRW will remain in their patches.

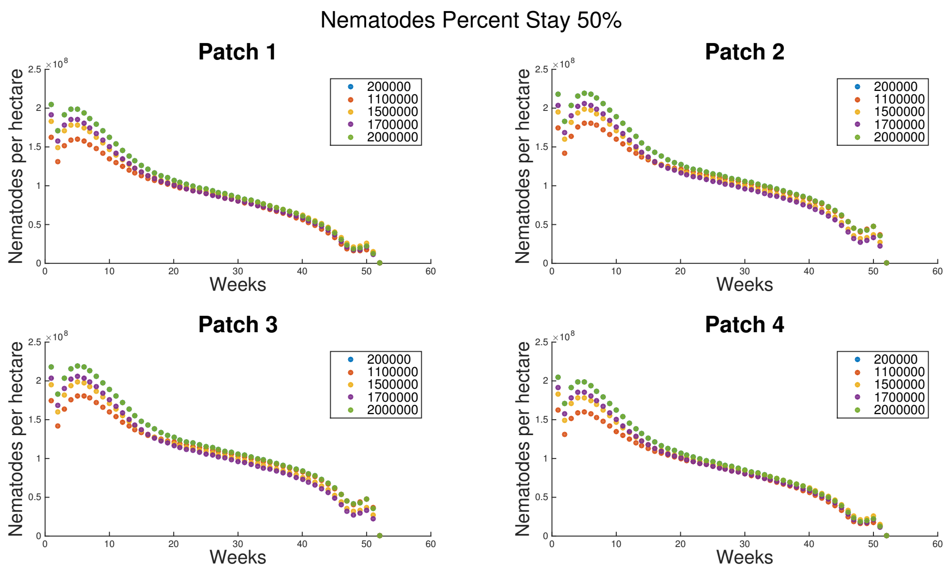

Figure 7.4: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $p = .5$, so 50% leave, for various initial populations: 200000, 1100000, 1500000, 1700000, 2000000. Notice the that all but 200000 makes sense for the spreading of 50% since patches 1 and 4 would be the same and patches 3 and 4 would be the same. Note $\delta = 0.3$.

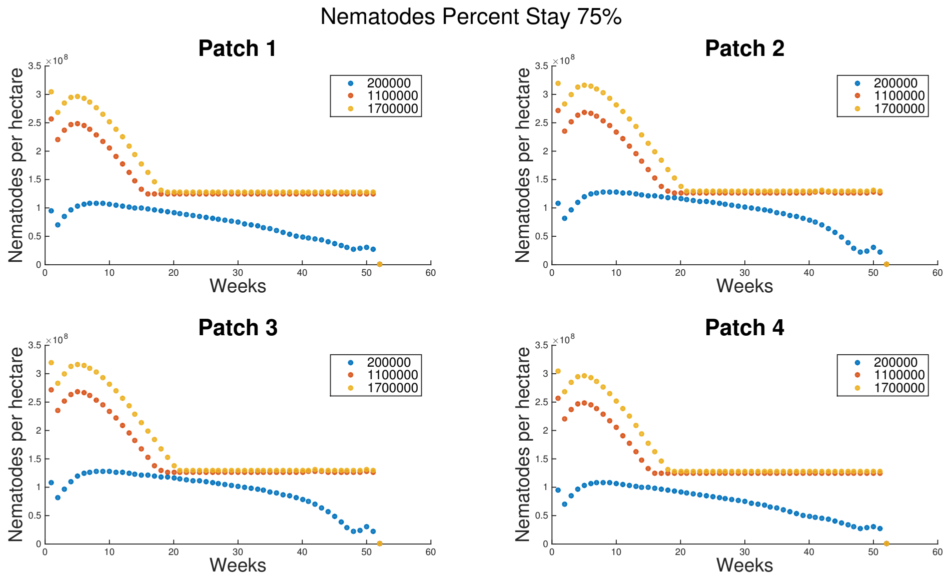

Figure 7.5: sing the Forward-Backward Sweep we calculate the number of nematodes to apply when $p = .75$, so 75% leave, for various initial populations: 200000, 1100000, 1700000. Notice the that all but 200000 makes sense for the spreading of 75% since patches 1 and 4 would be the same and patches 3 and 4 would be the same. Note $\delta = 0.75$.

7.3.1.2 Starts in Patch 1

We start the infestation in patch one with 50% of adults remaining in their patches, and we vary the initial population for the infestation.

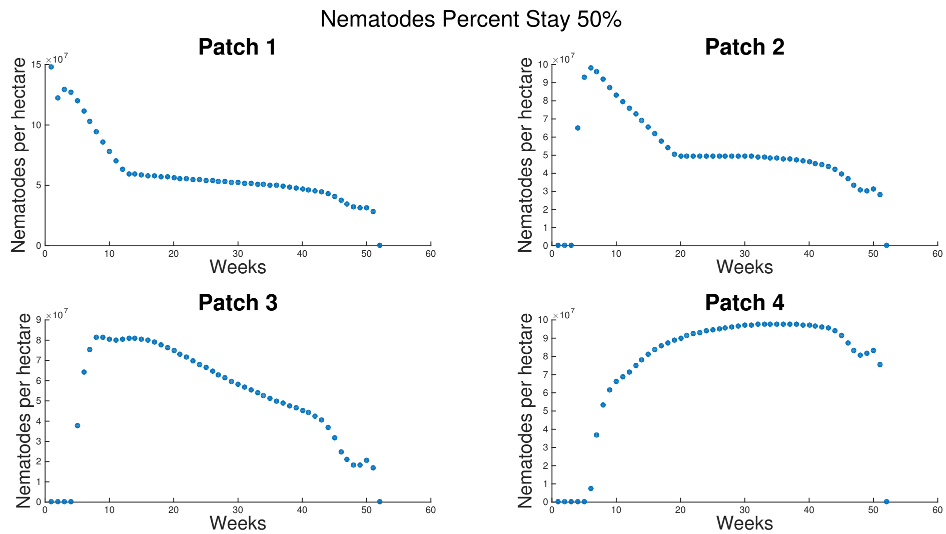

Figure 7.6: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .99$ and initial populations 200000 for $p = .5$.

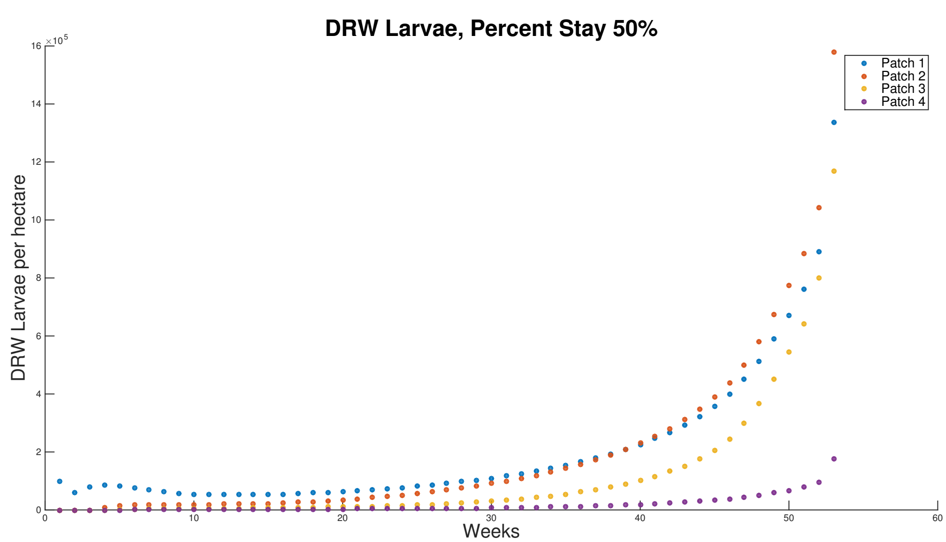

Figure 7.7: Number of DRW larvae to apply when $\delta = .99$ and initial populations 200000 for $p = .5$, associated with figure 7.6

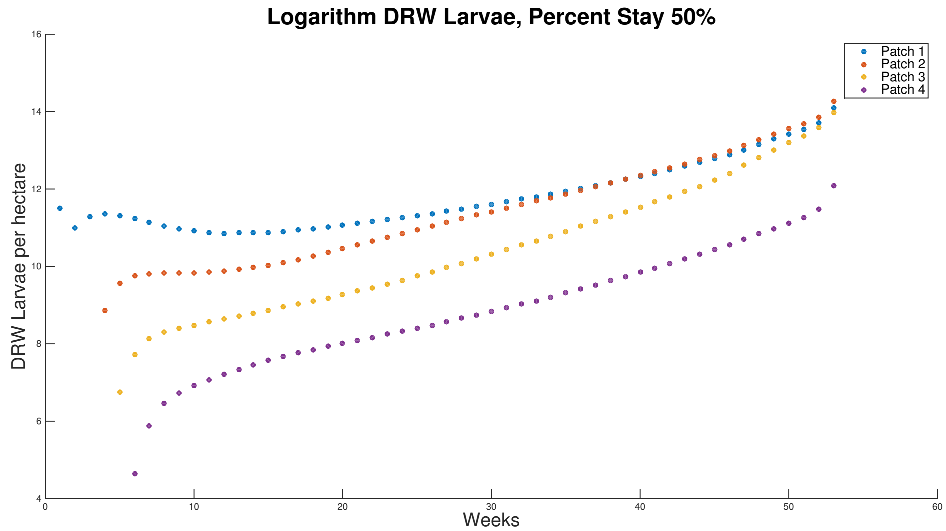

Figure 7.8: Logarithm of number of DRW larvae to apply when $\delta = .99$ and initial populations 200000 for $p = .5$, associated with figure 7.6

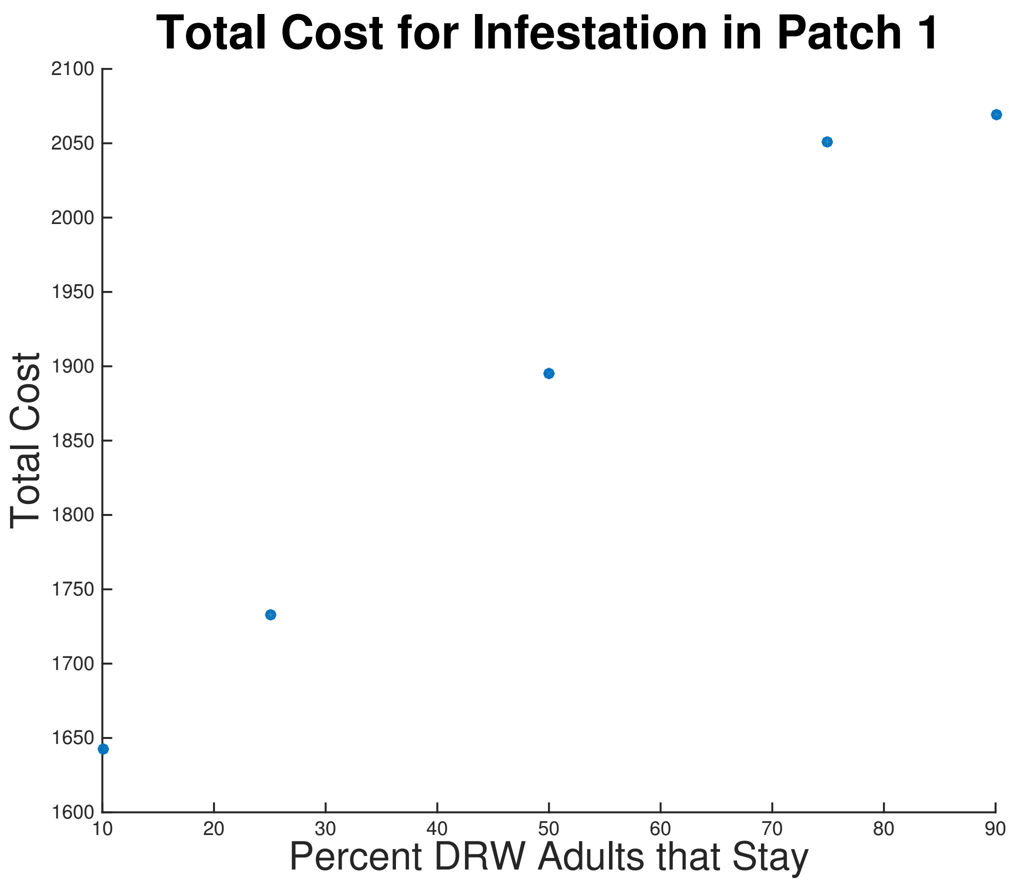

Figure 7.9: Using the Forward-Backward Sweep we calculate the Total Cost when $\delta = 1$ and initial populations 200000 for $p = .0.1, 0.25, 0.5, 0.75, 0.9$.

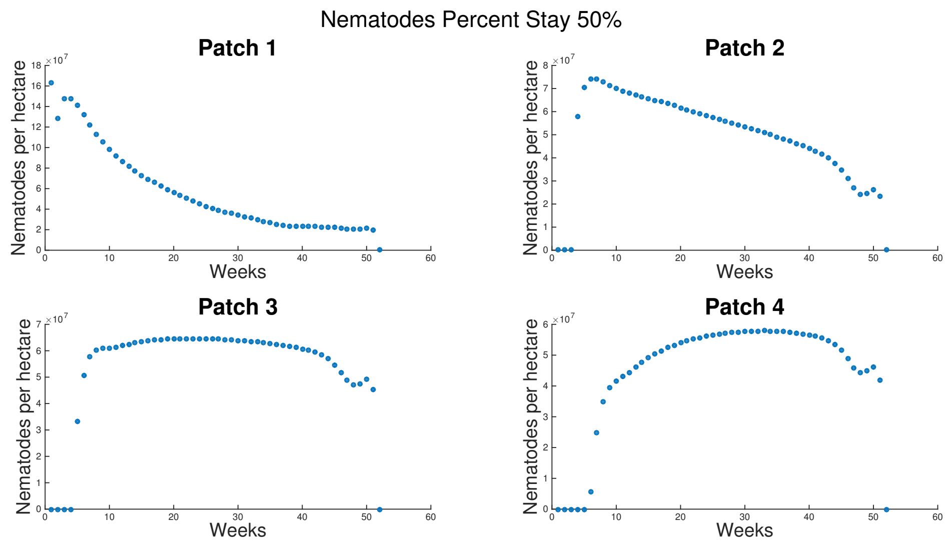

Figure 7.10: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .9999$ and initial populations 1100000 for $p = .5$.

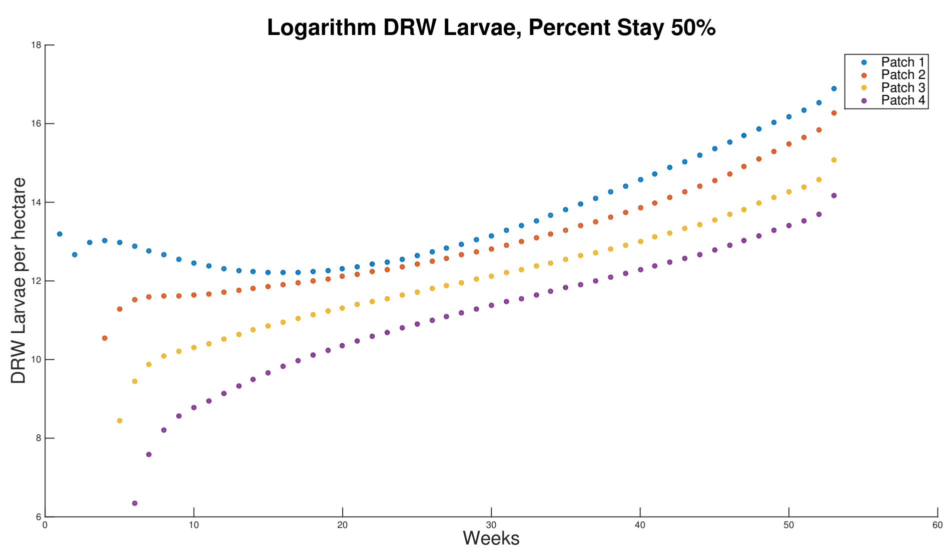

Figure 7.11: Logarithm of number of DRW larvae to apply when $\delta = .9999$ and initial populations 1100000 for $p = .5$, associated with figure 7.10

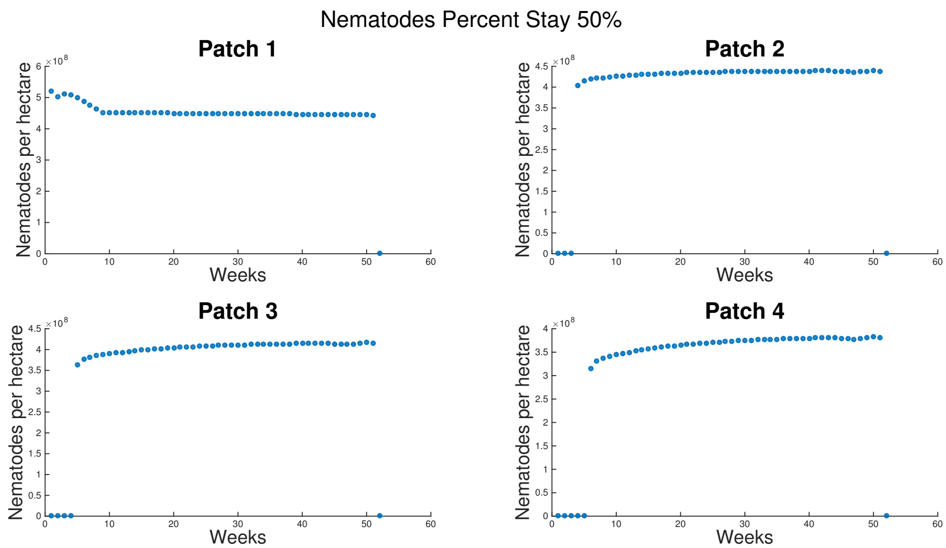

Figure 7.12: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .9999999999999$ and initial populations 1700000 for $p = .5$.

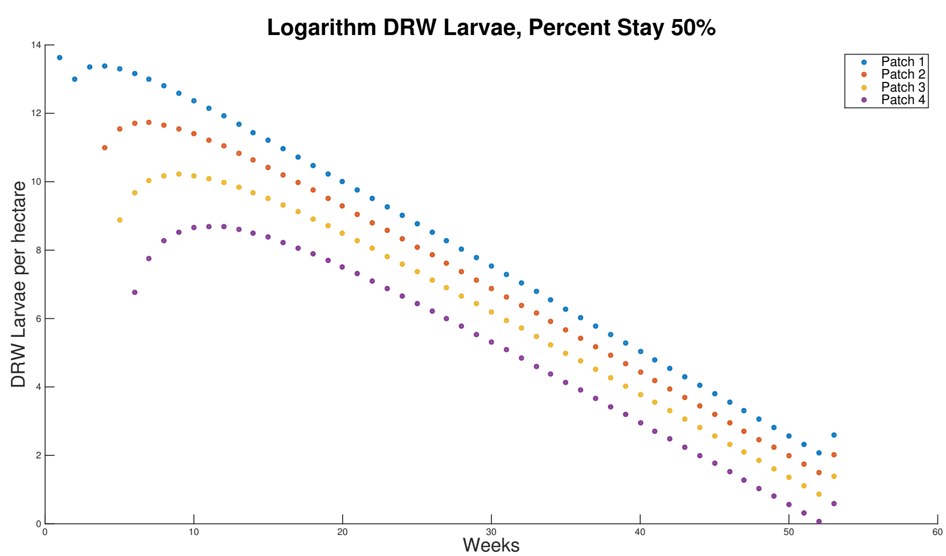

Figure 7.13: Logarithm of number of DRW larvae to apply when $\delta = .9999999999999$ and initial populations 1700000 for $p = .5$, associated with figure 7.12

7.3.1.3 Starts in Patch 2

We start the infestation in patch two with 50% of adults remaining in their patches, and we have initial population of 200000.

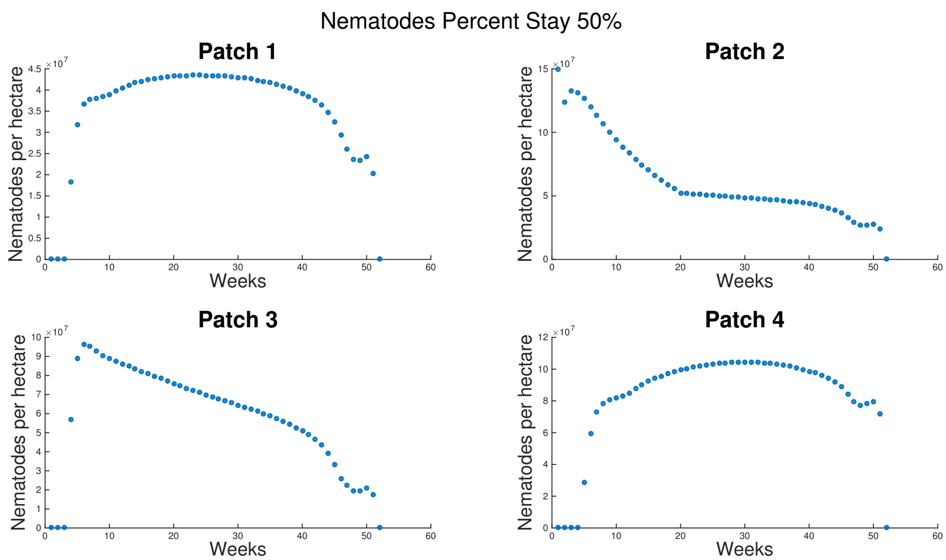

Figure 7.14: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .99$ and initial populations 200000 for $p = .5$.

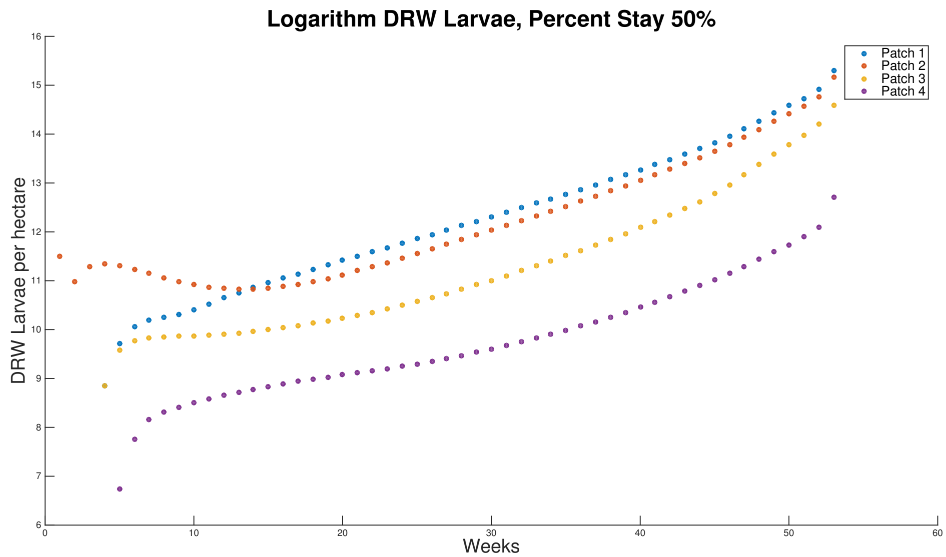

Figure 7.15: Logarithm of number of DRW larvae to apply when $\delta = .99$ and initial populations 200000 for $p = .5$, associated with figure 7.14

7.3.1.4 Starts in Patch 1 and 3

We start the infestation in patches one and three with 50% of adults remaining in their patches, and we have initial population of 200000.

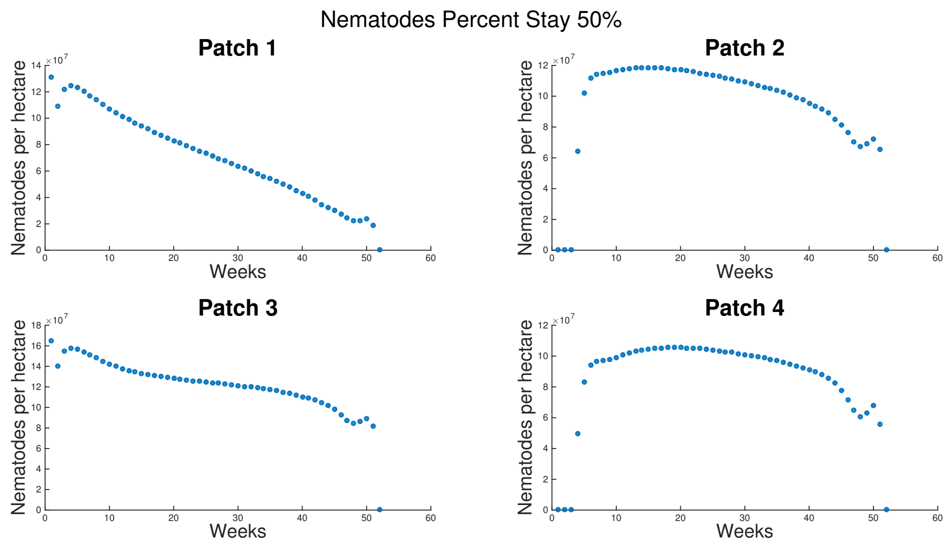

Figure 7.16: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .85$ and initial populations 200000 for $p = .5$.

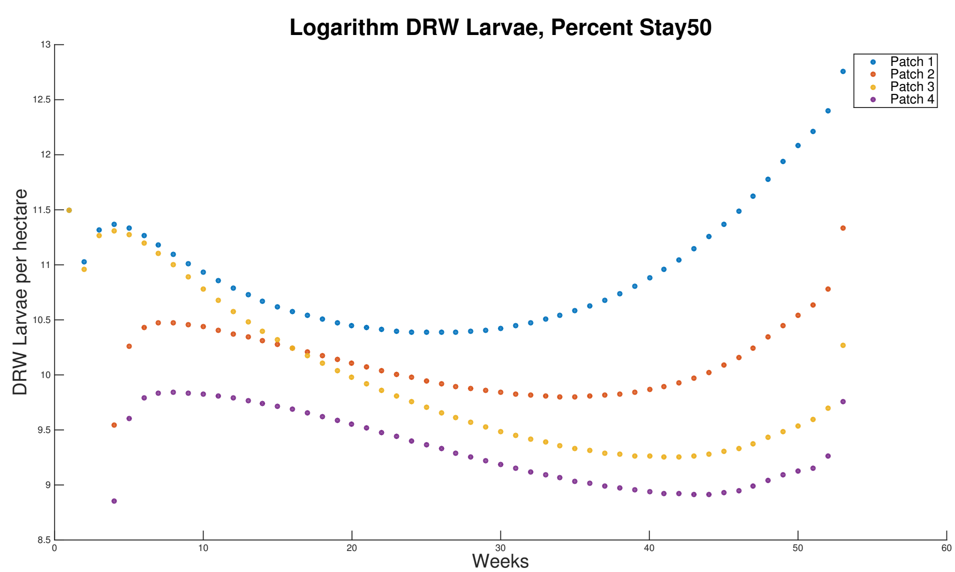

Figure 7.17: Logarithm of number of DRW larvae to apply when $\delta = .85$ and initial populations 200000 for $p = .5$, associated with figure 7.16

7.3.1.5 Starts in Patch 1 and 4

We start the infestation in patches one and four with 50% of adults remaining in their patches, and we have initial population of 200000.

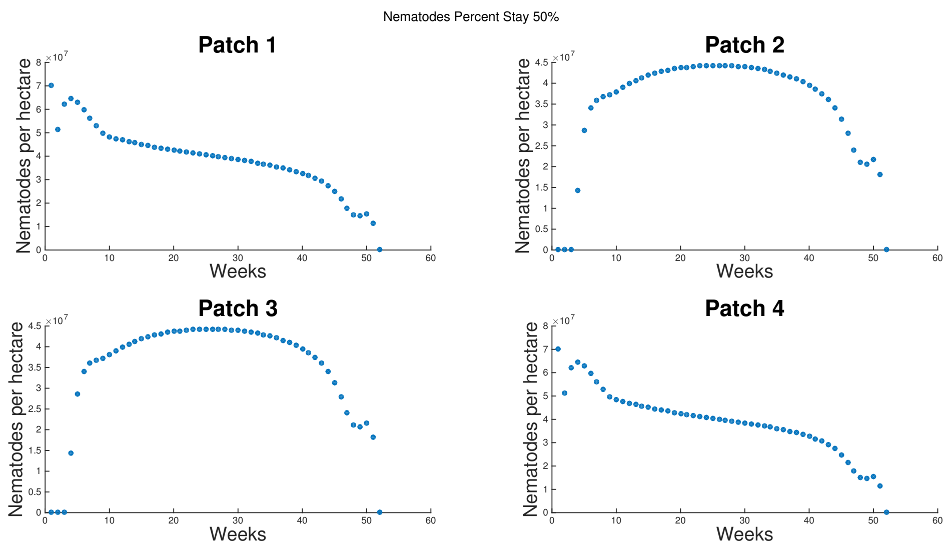

Figure 7.18: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .99$ and initial populations 200000 for $p = .5$.

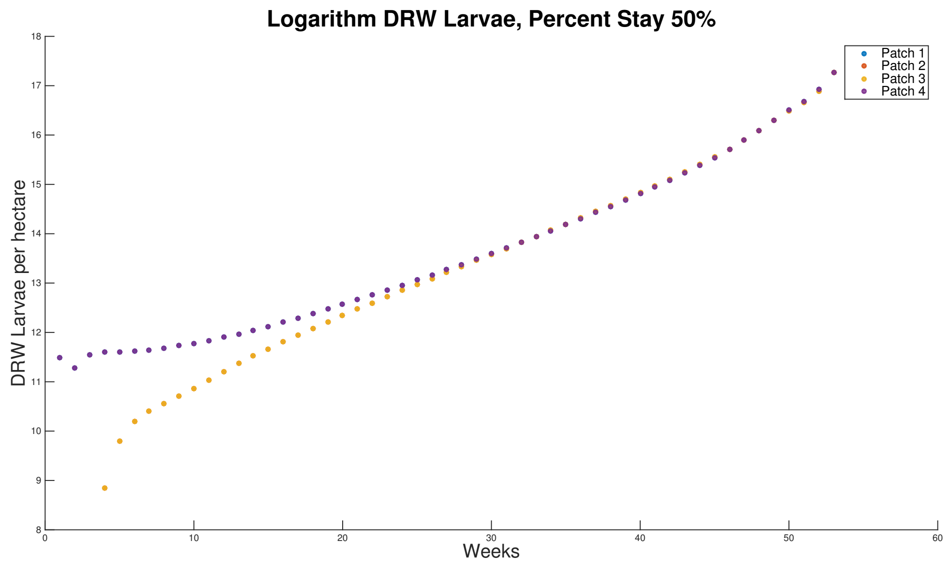

Figure 7.19: Logarithm of number of DRW larvae to apply when $\delta = .99$ and initial populations 200000 for $p = .5$, associated with figure 7.18

7.3.1.6 Starts in Patch 2 and 3

We start the infestation in patches two and three with 50% of adults remaining in their patches, and we have initial population of 200000.

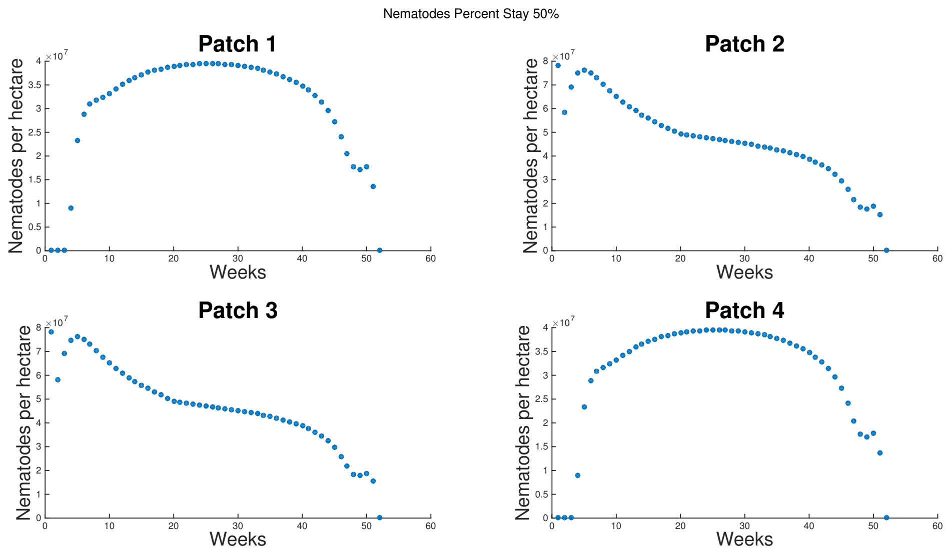

Figure 7.20: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .9$ and initial populations 200000 for $p = .5$.

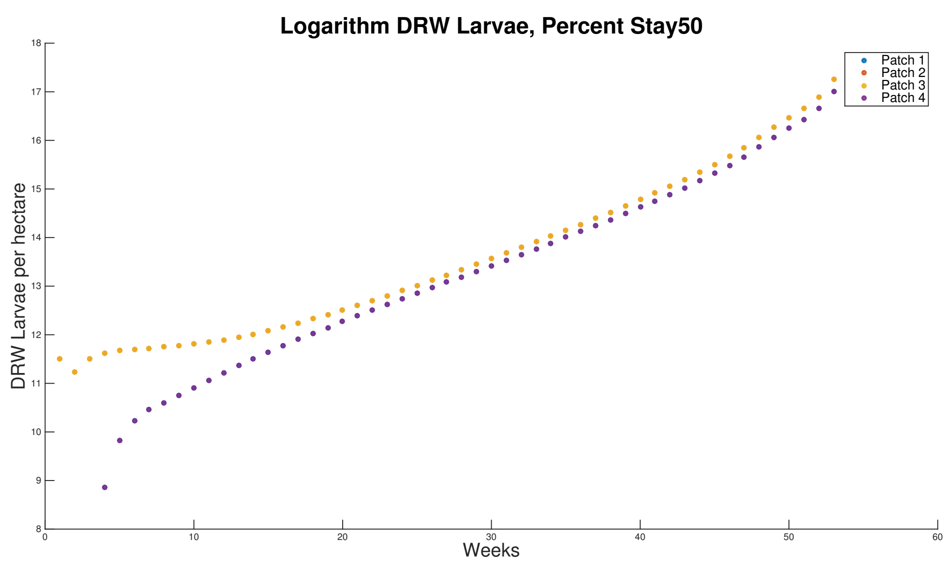

Figure 7.21: Logarithm of number of DRW larvae to apply when $\delta = .9$ and initial populations 200000 for $p = .5$, associated with figure 7.20

7.3.1.7 Starts in Patch 1, 2, and 3

We start the infestation in patches one, two, and three with 50% of adults remaining in their patches, and we have initial population of 200000.

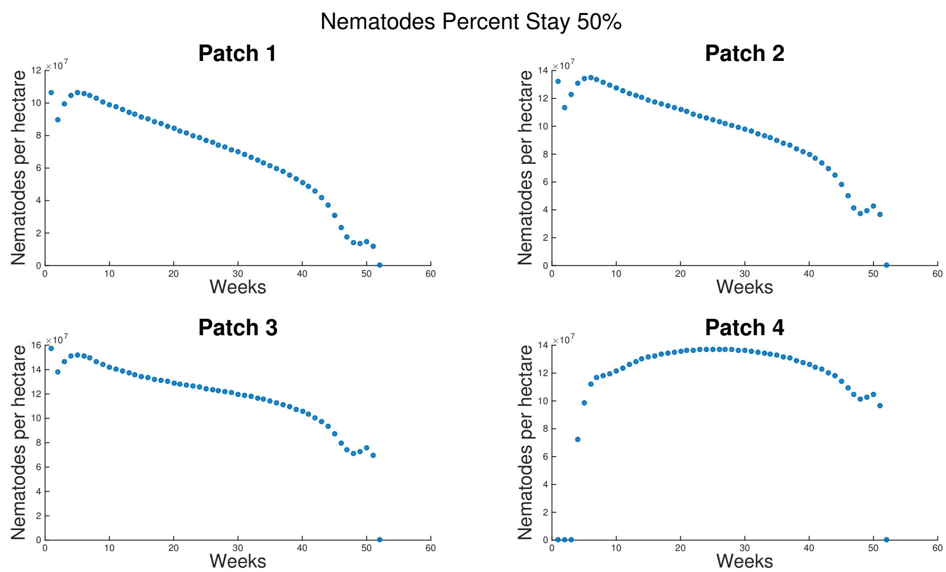

Figure 7.22: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .8$ and initial populations 200000 for $p = .5$.

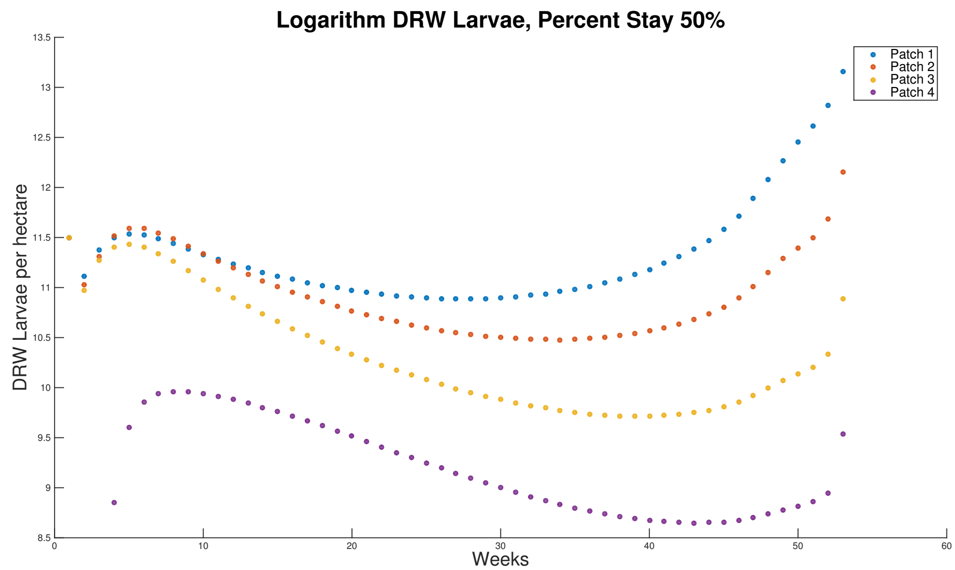

Figure 7.23: Logarithm of number of DRW larvae to apply when $\delta = .8$ and initial populations 200000 for $p = .5$, associated with figure 7.22

7.3.1.8 Starts in Patch 1, 2, and 4

We start the infestation in patches one, two, and four with 50% of adults remaining in their patches, and we have initial population of 200000.

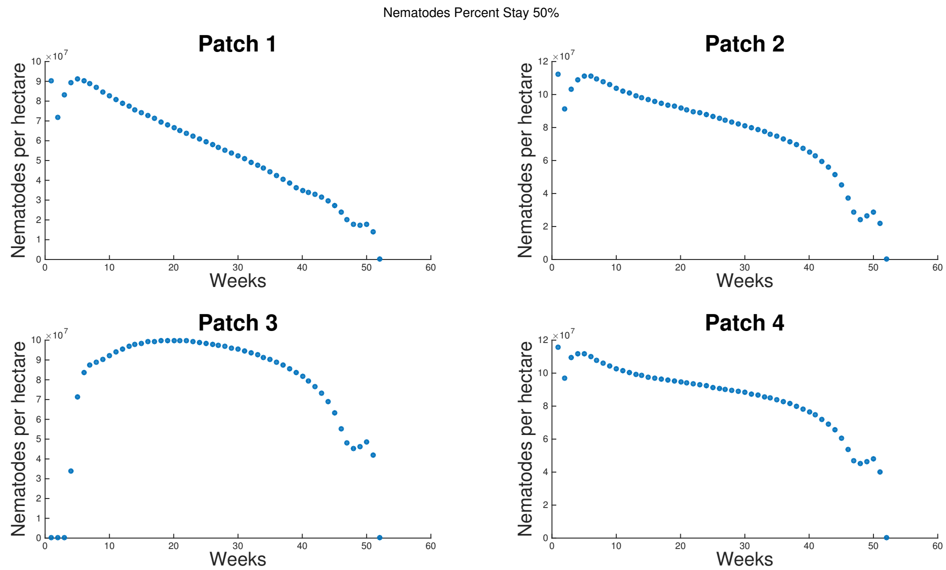

Figure 7.24: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .9$ and initial populations 200000 for $p = .5$.

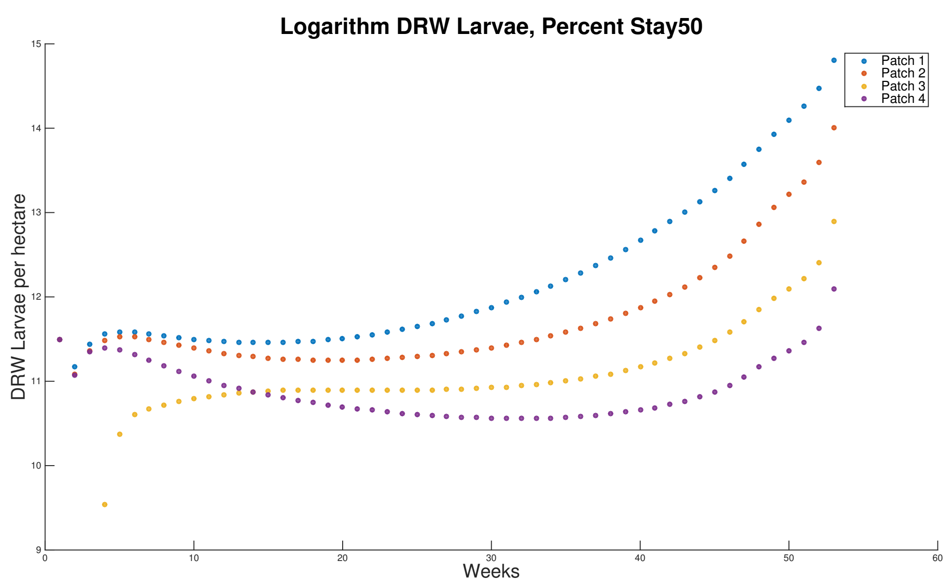

Figure 7.25: Logarithm of number of DRW larvae to apply when $\delta = .9$ and initial populations 200000 for $p = .5$, associated with figure 7.24

7.3.2 Discussion and Results

To start with we vary the values of DRW adults that remain in the original patch with infestation starting in all patches. In Figures 7.4 and 7.5 we vary initial populations for 50% remain and 75% remain respectively. In both cases the first and fourth patch have the same nematode application and the second and third patch have the same nematode application. This makes sense since the DRW dynamics are the same in the first and fourth patches, and similarly the same for second and third. So if we start the infestation equally in every patch the first and fourth will be the same and require less nematodes than the second and fourth, since the first patch only receives DRW adults from patch 2 but still loses DRW adults in all four directions. We can also note that the amount of nematodes required is less with a smaller percent that remain, p, since if more DRW adults remain we don't lose as many to the surrounding area through at least north and south travel.

Next we varied where the infestation would start in the four patches. If we start in patch one, it is the same as starting in patch 4 by the DRW dynamics, similarly for starting in patch 2 or 3. By using this knowledge we were able to run simulations for all possibilities without redundancy.

To start with we look at the infestation starting in patch one with 50% of the adults remaining so 50% leave their original patches. If we have initial population 200000, the Figure 7.6 demonstrates how many nematodes to apply in each patch. We note how the DRW spread between the patches in Figure 7.7. Meanwhile, in Figure 7.8 we took the logarithm of Figure 7.7, and can see that changes in the DRW in the patches. Specifically, how the DRW start in patch one and spread to the other three patches in order. Then since patch 2 and 3 are similar, both in receiving more adults and losing less to the enviroment, the DRW grow quickly and the two patches seem to over take patch 1 at the end. Meanwhile, patch 4 mimics patch one will a slower rise since it is furthest from the initial infestation and only receives from patch 3. In Figure 7.9 we consider varying the values of adults that remain during a patch 1 infestation and note that the larger the percent of adults that remain the larger the cost since less adults are lost to the surrounding system. For Figures 7.10 and 7.12 we plot how many nematodes to apply for initial populations 1100000 and 1700000 for 50% remain. We can note how there is a similar pattern for the application of nematodes, and in Figures 7.11 and 7.13 we have the logarithm of how many DRW larvae correspond with the nematode application. As in the Basic model with higher initial populations we apply more nematodes, and at 1700000 there is a larger initial population and we apply enough nematodes to reduce the DRW larvae.

Next, we start the infestation in patch two, again in 50% of adults remaining and initial population 200000. Figure 7.14 plots nematode application and we can note the differences to the patch one infestation case in Figure 7.6, for instance in patch 1 where the number of nematodes required is a different shape and less in the second patch infestation. Meanwhile in Figure 7.15 is the logarithm of how many DRW larvae are associated with the nematodes applied. Note how again patch four is below the other and seems similar to patch one, but patch one, two, and three are all increasing rapidly. The reverse would be true for patch one and four if the infestation started in patch 3. Now that we have explored the possibilities of starting the infestation in one patch we will consider the possibilities for the infestation starting in two patches.

We considered the possible combinations for the infestation starting in two patches. In each case we plot nematode application for the four patches and the logarithm of how many DRW larvae are associated with the nematodes applied. First in patches one and three, with Figures 7.16 and 7.17. Then in patches one and four, with Figures 7.18 and 7.19. Lastly, in patches two and three, with Figures 7.20 and 7.21. The case of patch one and three is different than the other two, as expected from the DRW dynamics. As we see in Figure 7.17, the three patches seem to move almost in unison, but are all separate in the amount of DRW larvae. Meanwhile in the cases of patches one and four and patches two and three, the DRW larvae results are very close, with slight deviation in case on patches two and three.

Lastly we explore the possible combinations for the infestation starting in three patches. In each case we plot nematode application for the four patches and the logarithm of how many DRW larvae are associated with the nematodes applied. First in patches one, two, and three, with Figures 7.22 and 7.23. Then in patches one, two, and four, with Figures 7.24 and 7.25. The difference in the DRW graphs is interesting, with a steeper increase when the infestation doesn't start with two and three utilized.

From these simulations we can see how drastically the origins of the infestation can affect the amount of nematodes required. We also have application methods for the various infestation starting points. Overall, the fact that we lose adult DRW to the surrounding environment affects the simulations, especially when more adults travel. Therefore, how our patches are situated in space has an affect on our nematode application and total cost. We will explore this more by looking at the Isolated model, which has a different organization of the patches.

7.3.3 Four Isolated Patches Simulations

Note when none of the DRW spread we have the same Figure 7.3.

7.3.4 Vary Percent of DRW Adults which Remain, p

We explore the behavior of the model by varying p. Note, if all patches are infested then no matter the value of p we will have the same cost; since all that leave also return, see Figure 7.26. Specifically, if $p = 40\%$, then 60% of adults leave a patch one but 20% of adults from the other three patches travel to patch one, resulting in 60% traveling to patch one. Therefore, we maintain the same amount of DRW adults in every patch. Hence, we need to look at varying where the infestation start and then vary p.

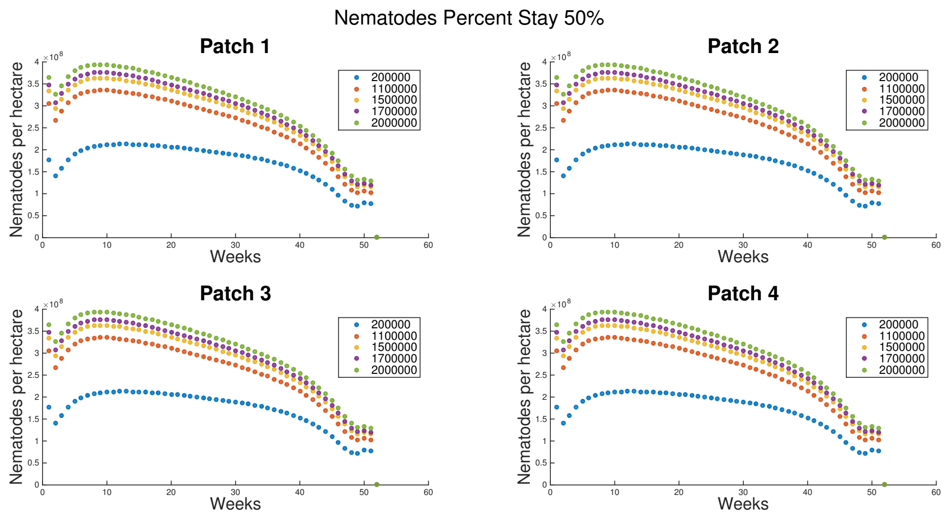

Figure 7.26: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $p = .5$, so 50% leave, for various initial populations:200000, 1100000, 1500000, 1700000, 2000000. Notice how all the patches are still the same since we have equal spread between patches. Note $\delta = 0.1$

7.3.4.1 Starts in Patch 1

We start the infestation in patch one with 50% of adults remaining in their patches, and we have initial population of 200000.

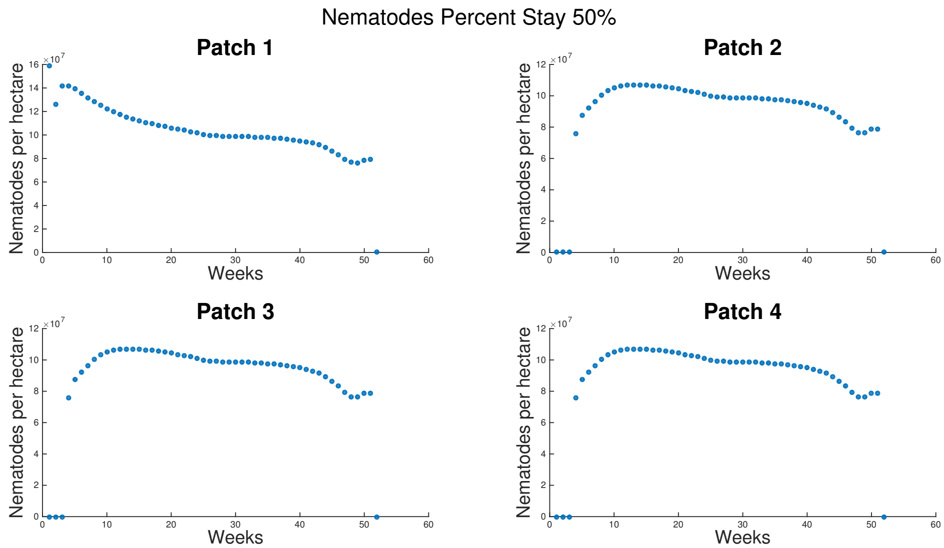

Figure 7.27: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .9999$ and initial populations 200000 for $p = .5$.

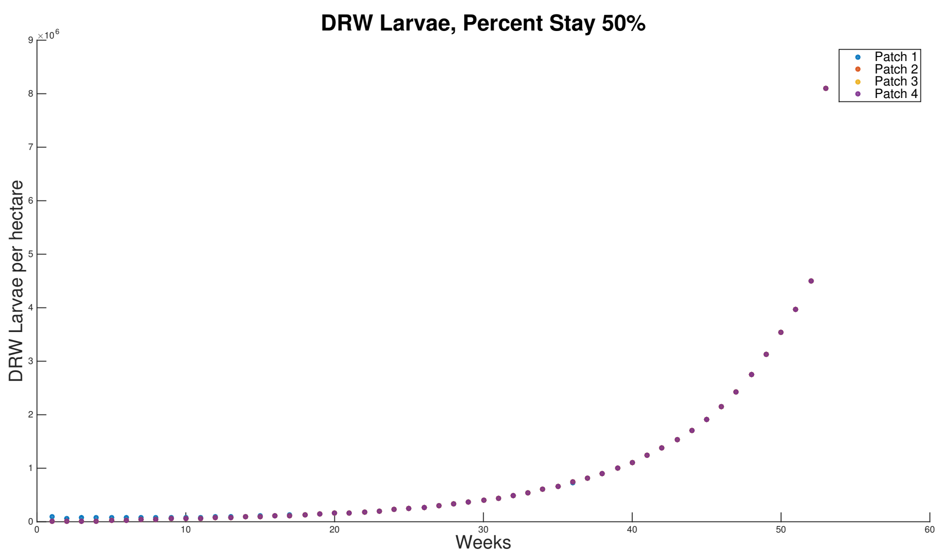

Figure 7.28: Number of DRW larvae to apply when $\delta = .9999$ and initial populations 200000 for $p = .5$, associated with figure 7.27

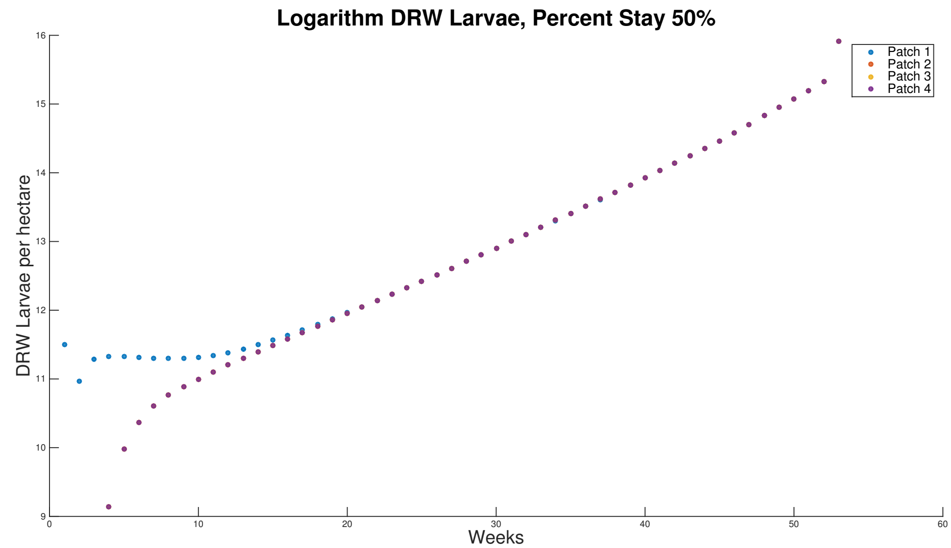

Figure 7.29: Logarithm of number of DRW larvae to apply when $\delta = .9999$ and initial populations 200000 for $p = .5$, associated with figure 7.27

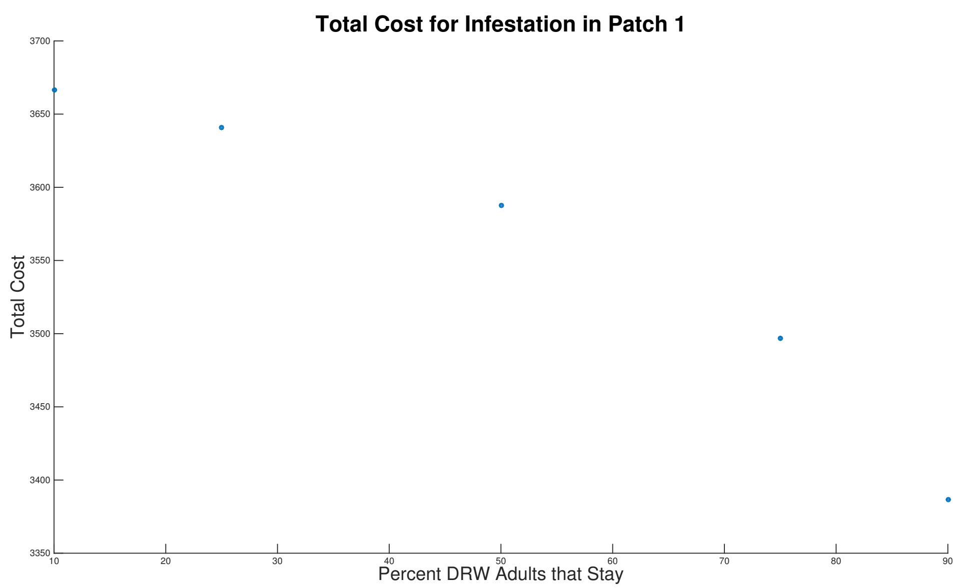

Figure 7.30: Using the Forward-Backward Sweep we calculate the Total Cost when $\delta = .9999$ and initial populations 200000 for $p = 0.1, 0.25, 0.5, 0.75, 0.9$.

As more stay less other patches to treat so lower cost.

7.3.4.2 Starts in Patch 1 and 2

We start the infestation in patches one and two with 50% of adults remaining in their patches, and we have initial population of 200000.

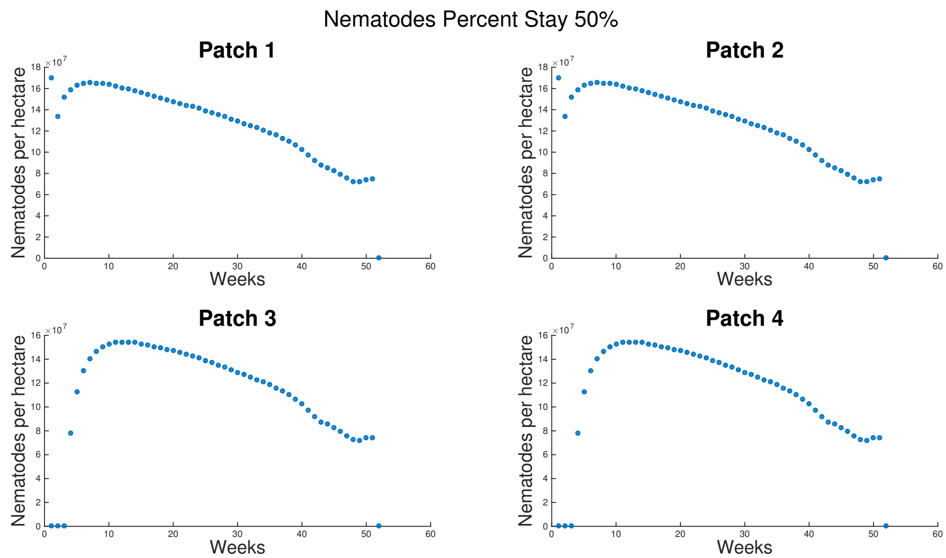

Figure 7.31: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .999$ and initial populations 200000 for $p = .5$.

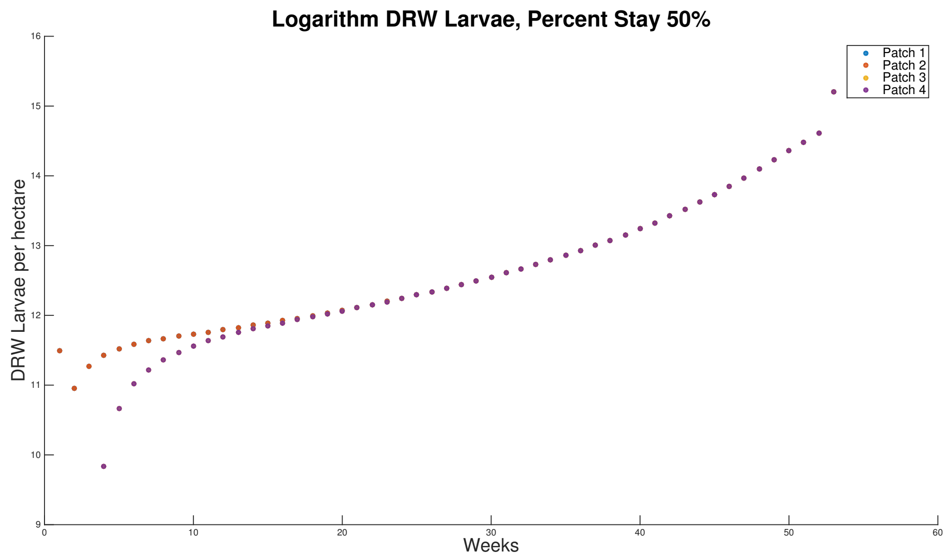

Figure 7.32: Logarithm of number of DRW larvae to apply when $\delta = .999$ and initial populations 200000 for $p = .5$, associated with figure 7.31

7.3.4.3 Starts in Patch 1, 2, and 3

We start the infestation in patches one, two, and three with 50% of adults remaining in their patches, and we have initial population of 200000.

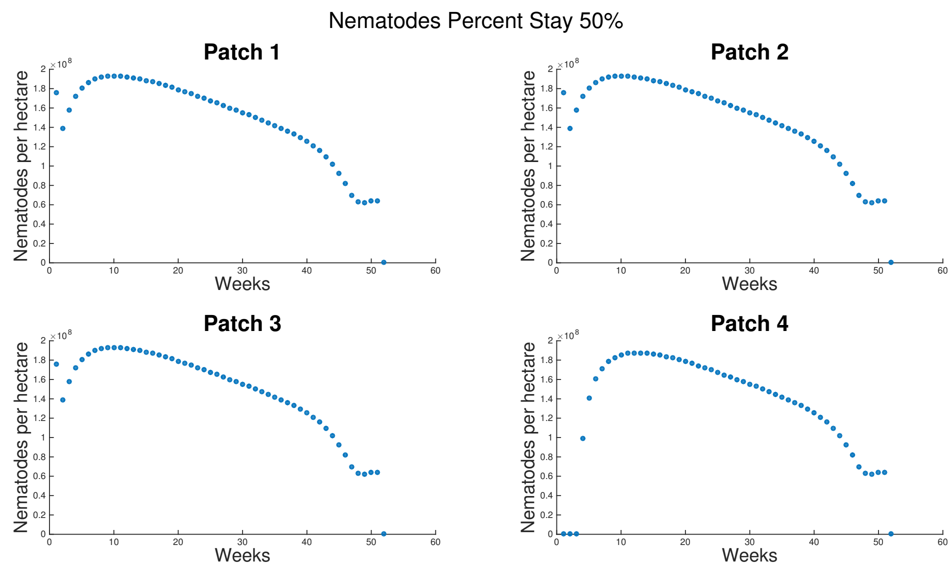

Figure 7.33: Using the Forward-Backward Sweep we calculate the number of nematodes to apply when $\delta = .999$ and initial populations 200000 for $p = .5$.

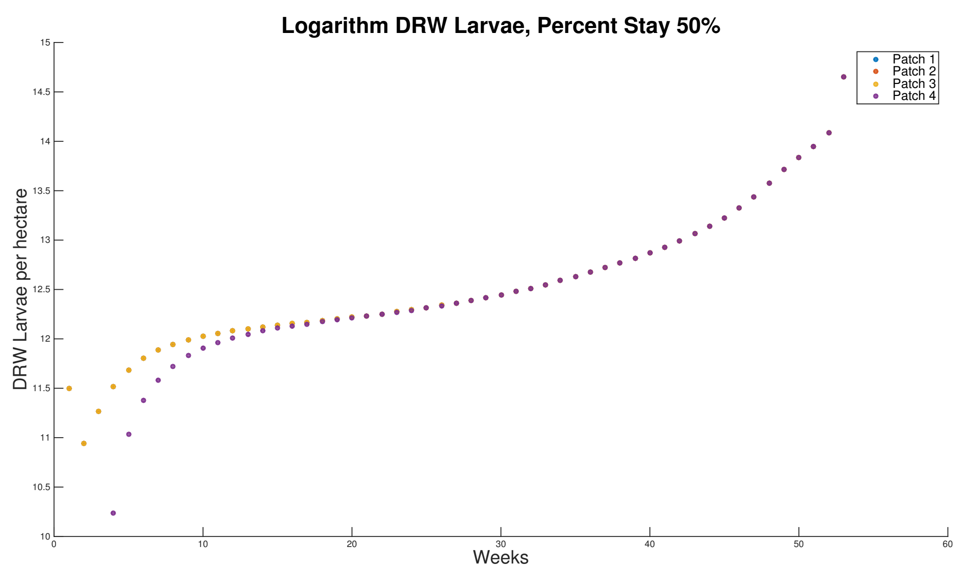

Figure 7.34: Logarithm of number of DRW larvae to apply when $\delta = .999$ and initial populations 200000 for $p = .5$, associated with figure 7.33

7.3.5 Discussion and Summary

As we stated earlier with Figure 7.26, if the infestation starts in all patches, then no matter the value of the DRW adults that remain, p, we will have the same results. This is because all adults only travel equally between the three patches, so no matter the percent that leave that same percent will return from the other three patches.

Next, we consider if the infestation starts in a single patch, and due to the DRW dynamics we can consider starting in any patch, so we chose patch one. We will use initial population 200000. Unlike the Connected model, there is equal spread and the other three patches look identical in Figures 7.27, 7.28, and 7.29. For instance in Figure 7.28, the number of DRW larvae looks almost identical, which is also visible in the logarithm of DRW graph, with all four the same after about time step 20 weeks. In Figure 7.30 we plot various percentages for how many adult DRW remain. At higher percentages the DRW do not distribute between the patches, so there is a higher cost to treat a larger infestation in one patch rather than a smaller cost to treat smaller infestations in four patches.

Now, we consider the infestation starting in two patches, we picked patches one and two. We plot nematode application for the four patches, 7.31, and the logarithm of how many DRW larvae are associated with the nematodes applied, 7.32. As with the single patch the DRW spread evenly and the result happens in a similar amount of time.

Next, we consider the infestation starting in three patches, we picked patches one, two, and three. We plot nematode application for the four patches, 7.33, and the logarithm of how many DRW larvae are associated with the nematodes applied, 7.34. As with the two patch the DRW spread evenly and the result happens in a similar amount of time.

We note the general similarities between the infestation starting one, two, and three patch, which is a result of the four patches having the same DRW dynamics and spread. This means all patches are likely equidistant and so far we have assumed all adults that leave a patch reach another patch. The next step is to consider a mortality rate.

7.3.5.1 Mortality Rate

Our next step with the Four Patch Isolated Model is to include a mortality rate, taking into consideration the possibility some adult DRW will not reach another patch. If we apply the mortality rate, m, to the adults that leave patches we would have:

$$\theta_{i,j} = \frac{1-p}{3} \cdot \theta_2 \cdot m$$

While this would reduce the amount of DRW in the system, results with this model would be similar to those above. Suppose the infestation starts in all patches. If we have 60% of DRW adults travel, then each patch loses 60% of DRW adults and received $60\% \cdot m$ adults from the other patches combined. Hence, we will have less DRW and require less nematodes, but the distribution of the four patches will be the same. To allow for varying patch distances and possible mortality rates during travel, we could need to introduce different mortality rates for the patches.

Part III

Future Work

Chapter 8

Discrete Time Step 4 weeks

Parts 1 and 2 explored discrete models that implemented a one week time step. In some cases it is not practical for farmers to apply control every week. Therefore we will consider a discrete model which has a 4 week time step.

8.1 Basic Model

Recall that in the Basic model we had a matrix A which characterized the pest dynamics for a one week time step. The resulting matrix for our pest dynamics with a four week time step will be as follows:

$$A^4 = \begin{bmatrix} \gamma_1 & \zeta_4 & \nu_3 & \theta_1 \\ \gamma_2 & \zeta_1 & \nu_4 & \theta_3 \\ \gamma_3 & \zeta_2 & \nu_1 & \theta_4 \\ \gamma_4 & \zeta_3 & \nu_2 & \theta_2 \end{bmatrix}.$$

Below is the formulation of the pest dynamics for the basic model with a four week time step, note this does not include the biological control in the larva stage.

$$\begin{bmatrix} P_e(k+1) \\ P_l(k+1) \\ P_p(k+1) \\ P_a(k+1) \end{bmatrix} = \begin{bmatrix} \gamma_1 & \zeta_4 & \nu_3 & \theta_1 \\ \gamma_2 & \zeta_1 & \nu_4 & \theta_3 \\ \gamma_3 & \zeta_2 & \nu_1 & \theta_4 \\ \gamma_4 & \zeta_3 & \nu_2 & \theta_2 \end{bmatrix} \begin{bmatrix} P_e(k) \\ P_l(k) \\ P_p(k) \\ P_a(k) \end{bmatrix}$$

Cost of Basic Model We will need to update the cost of pests, so

$$Cost = \beta_3 P_l(t)^2 + \beta_2 N_1(t).$$

8.1.1 Optimal Control Problem

The set-up for our Optimal Control Problem is to minimize the objective functional

$$J(N) = \sum_{t=0}^{T-1} \beta_3 P_l(t)^2 + \beta_2 N_1(t)$$

subject to

$$\begin{aligned} P_e(t+1) &= \gamma_1 P_e(t) + \zeta_4 e^{-\alpha N(t)} P_l(t) + \nu_3 P_p(t) + \theta_1 P_a(t) \quad P_e(0) = \Phi_e \\ P_l(t+1) &= \gamma_2 P_e(t) + \zeta_1 e^{-\alpha N(t)} P_l(t) + \nu_4 P_p(t) + \theta_3 P_a(t) \quad P_l(0) = \Phi_l \\ P_p(t+1) &= \gamma_3 P_e(t) + \zeta_2 e^{-\alpha N(t)} P_l(t) + \nu_1 P_p(t) + \theta_4 P_a(t) \quad P_p(0) = \Phi_p \\ P_a(t+1) &= \gamma_4 P_e(t) + \zeta_3 e^{-\alpha N(t)} P_l(t) + \nu_2 P_p(t) + \theta_2 P_a(t) \quad P_a(0) = \Phi_a \end{aligned} \tag{8.1}$$

where $N \in \mathbf{N} = \{N : \{1, ..., T\} \to \{x \in \mathbb{R} | 0 \le x(t) \le N_{max}, t = 1, 2, ..., T\}\}$.

Again we will prove the existence of the optimal control $\mathcal{N}$. In the future, we will prove necessary conditions and uniqueness for the optimal control $\mathcal{N}$.

8.1.2 Existence

Theorem 8.1.1. *There exists $\mathcal{N} \in \boldsymbol{N}$ which minimizes $J(N)$.*

Proof. This theorem is analogous to Theorem 2.3.1, since P_e, P_l, P_p, P_a are all continuous with respect to N by Equation 8.1. Additionally, we have J is continuous as a function of N and B^+ is a compact subset of $\mathbb{R}^T$, so $\inf_{N \in \mathbf{N}} J(N)$ exists.

□

8.2 Four Connected Patch Model

Recall in the Basic model we had a matrix A_{4c} which characterized the pest dynamics for a one week time step. The resulting matrix for our pest dynamics with a four week time step will be as follows:

$$A_{4c}^{4} = \left[\begin{array}{cccc:cccc:cccc:cccc}
\gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_4 & \zeta_3 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,2} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2,3} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{3,4} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{4,4}
\end{array}\right]$$

Below is the formulation of the pest dynamics for the four patch model with four week time step, note this does not include the biological control in the larva stage.

$$
\begin{bmatrix} P_{1,e}(k+1) \\ P_{1,l}(k+1) \\ P_{1,p}(k+1) \\ P_{1,a}(k+1) \\ P_{2,e}(k+1) \\ P_{2,l}(k+1) \\ P_{2,p}(k+1) \\ P_{2,a}(k+1) \\ P_{3,e}(k+1) \\ P_{3,l}(k+1) \\ P_{3,p}(k+1) \\ P_{3,a}(k+1) \\ P_{4,e}(k+1) \\ P_{4,l}(k+1) \\ P_{4,p}(k+1) \\ P_{4,a}(k+1) \end{bmatrix}
=
\begin{bmatrix}
\gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_4 & \zeta_3 & \nu_2 & \theta_{1,1} & 0 & 0 & 0 & \theta_{2,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta_{1,2} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{2,2} & 0 & 0 & 0 & \theta_{3,2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{2,3} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{3,3} & 0 & 0 & 0 & \theta_{4,3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_1 & \zeta_4 & \nu_3 & \theta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_2 & \zeta_1 & \nu_4 & \theta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_3 & \zeta_2 & \nu_1 & \theta_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_{3,4} & \gamma_4 & \zeta_3 & \nu_2 & \theta_{4,4}
\end{bmatrix}
\begin{bmatrix} P_{1,e}(k) \\ P_{1,l}(k) \\ P_{1,p}(k) \\ P_{1,a}(k) \\ P_{2,e}(k) \\ P_{2,l}(k) \\ P_{2,p}(k) \\ P_{2,a}(k) \\ P_{3,e}(k) \\ P_{3,l}(k) \\ P_{3,p}(k) \\ P_{3,a}(k) \\ P_{4,e}(k) \\ P_{4,l}(k) \\ P_{4,p}(k) \\ P_{4,a}(k) \end{bmatrix}
$$

Note, as in Part 2, we have $\theta_{1,1} = \theta_{2,2} = \theta_{3,3} = \theta_{4,4} = p * \theta_2$ and $\theta_{1,2} = \theta_{2,1} = \theta_{2,3} = \theta_{3,2} = \theta_{3,4} = \theta_{4,3} = \frac{1-p}{4} * \theta_2$, where p is the percent of adult pests which remain in the original patch.

In the future we will vary the value p for a specific case study.

Cost of Four Patches Same as in the Four Connected Patches case we are considering four independent patches, and using the basic model with a four week time step we have

$$Cost = \beta_3 P_{1,l}(t)^2 + \beta_2 N_1(t) + \beta_3 P_{2,l}(t)^2 + \beta_2 N_2(t) + \beta_3 P_{3,l}(t)^2 + \beta_2 N_3(t) + \beta_3 P_{4,l}(t)^2 + \beta_2 N_4(t).$$

8.2.1 Optimal Control Problem Formulation

The set-up for our Optimal Control Problem is to minimize the objective functional

$$J(N_1, N_2, N_3, N_4) = \sum_{t=0}^{T-1} \beta_3 [P_{1,l}(t)^2 + P_{2,l}(t)^2 + P_{3,l}(t)^2 + P_{4,l}(t)^2]$$
$$+\beta_2 [N_1(t) + N_2(t) + N_3(t) + N_4(t)]$$

subject to

$$P_{1,e}(t+1) = \gamma_1 P_{1,e}(t) + \zeta_4 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_3 P_{1,p}(t) + \theta_1 P_{1,a}(t) \qquad P_{1,e}(0) = \Phi_{1,e}$$

$$P_{1,l}(t+1) = \gamma_2 P_{1,e}(t) + \zeta_1 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_4 P_{1,p}(t) + \theta_3 P_{1,a}(t) \qquad P_{1,l}(0) = \Phi_{1,l}$$

$$P_{1,p}(t+1) = \gamma_3 P_{1,e}(t) + \zeta_2 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_1 P_{1,p}(t) + \theta_4 P_{1,a}(t) \qquad P_{1,p}(0) = \Phi_{1,p}$$

$$P_{1,a}(t+1) = \gamma_4 P_{1,e}(t) + \zeta_3 e^{-\alpha N_1(t)} P_{1,l}(t) + \nu_2 P_{1,p}(t) + \theta_{1,1} P_{1,a}(t) + \theta_{2,1} P_{2,a}(t) \qquad P_{1,a}(0) = \Phi_{1,a}$$

$$P_{2,e}(t+1) = \gamma_1 P_{2,e}(t) + \zeta_4 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_3 P_{2,p}(t) + \theta_1 P_{2,a}(t) \qquad P_{2,e}(0) = \Phi_{2,e}$$

$$P_{2,l}(t+1) = \gamma_2 P_{2,e}(t) + \zeta_1 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_4 P_{2,p}(t) + \theta_3 P_{2,a}(t) \qquad P_{2,l}(0) = \Phi_{2,l}$$

$$P_{2,p}(t+1) = \gamma_3 P_{2,e}(t) + \zeta_2 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_1 P_{2,p}(t) + \theta_4 P_{2,a}(t) \qquad P_{2,p}(0) = \Phi_{2,p}$$

$$P_{2,a}(t+1) = \gamma_4 P_{2,e}(t) + \zeta_3 e^{-\alpha N_2(t)} P_{2,l}(t) + \nu_2 P_{2,p}(t) + \theta_{2,2} P_{2,a}(t) + \theta_{1,2} P_{1,a}(t) + \theta_{3,2} P_{3,a}(t) \qquad P_{2,a}(0) = \Phi_{2,a}$$

$$P_{3,e}(t+1) = \gamma_1 P_{3,e}(t) + \zeta_4 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_3 P_{3,p}(t) + \theta_1 P_{3,a}(t) \qquad P_{3,e}(0) = \Phi_{3,e}$$

$$P_{3,l}(t+1) = \gamma_2 P_{3,e}(t) + \zeta_1 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_4 P_{3,p}(t) + \theta_3 P_{3,a}(t) \qquad P_{1,l}(0) = \Phi_{3,l}$$

$$P_{3,p}(t+1) = \gamma_3 P_{3,e}(t) + \zeta_2 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_1 P_{3,p}(t) + \theta_4 P_{3,a}(t) \qquad P_{3,p}(0) = \Phi_{3,p}$$

$$P_{3,a}(t+1) = \gamma_4 P_{3,e}(t) + \zeta_3 e^{-\alpha N_3(t)} P_{3,l}(t) + \nu_2 P_{3,p}(t) + \theta_{3,3} P_{3,a}(t) + \theta_{2,3} P_{2,a}(t) + \theta_{4,3} P_{4,a}(t) \qquad P_{3,a}(0) = \Phi_{3,a}$$

$$P_{4,e}(t+1) = \gamma_1 P_{4,e}(t) + \zeta_4 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_3 P_{4,p}(t) + \theta_1 P_{4,a}(t) \qquad P_{4,e}(0) = \Phi_{4,e}$$

$$P_{4,l}(t+1) = \gamma_2 P_{4,e}(t) + \zeta_1 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_4 P_{4,p}(t) + \theta_3 P_{4,a}(t) \qquad P_{4,l}(0) = \Phi_{4,l}$$

$$P_{4,p}(t+1) = \gamma_3 P_{4,e}(t) + \zeta_2 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_1 P_{4,p}(t) + \theta_4 P_{4,a}(t) \qquad P_{4,p}(0) = \Phi_{4,p}$$

$$P_{4,a}(t+1) = \gamma_4 P_{4,e}(t) + \zeta_3 e^{-\alpha N_4(t)} P_{4,l}(t) + \nu_2 P_{4,p}(t) + \theta_{4,4} P_{4,a}(t) + \theta_{3,4} P_{3,a}(t) \qquad P_{4,a}(0) = \Phi_{4,a}$$

where $N_1, N_2, N_3, N_4 \in \mathbf{N} = \{N : \{1, ..., T\} \to \{x \in \mathbb{R} | 0 \le x(t) \le N_{max}, t = 1, 2, ..., T\}\}$.

8.2.2 Proofs

Again we will prove the existence of the optimal control $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$. In the future, we will prove necessary conditions and uniqueness for the optimal control.

8.2.2.1 Existence

Theorem 8.2.1. *There exists $\mathcal{N} \in \boldsymbol{N}$ which minimizes $J(N)$.*

Proof. This theorem is analogous to Theorem 7.1.1, since P_e, P_l, P_p, P_a are all continuous with respect to N_1, N_2, N_3, N_4 by the equations in Section 8.2.1. Additionally, we have J is continuous as a function of N_1, N_2, N_3, N_4 and B^+ is a compact subset of $\mathbb{R}^T$, so $\inf_{N_1, N_2, N_3, N_4 \in \mathbf{N}} J(N_1, N_2, N_3, N_4)$ exists.

□

8.3 Case Study: DRW

We will establish parameter values using the basic model for a four week time step and then expand these to the four connected patch model. Note most of the parameters will be the same in both models, the notable difference will be in the four connected patch model having the additional p parameter.

8.3.1 Values for DRW Dynamics

Using the original matrix A from above, we transition to time steps of 4 weeks rather than a week.

$$A^4 = \begin{bmatrix} \gamma_1 & \zeta_4 & \nu_3 & \theta_1 \\ \gamma_2 & \zeta_1 & \nu_4 & \theta_3 \\ \gamma_3 & \zeta_2 & \nu_1 & \theta_4 \\ \gamma_4 & \zeta_3 & \nu_2 & \theta_2 \end{bmatrix} = \begin{bmatrix} 0.1795 & 0.9513 & 47.5175 & 105.9500 \\ 0.5755 & 0.8247 & 25.4132 & 140.0478 \\ 0.0286 & 0.0471 & 0.5375 & 2.4986 \\ 0.0063 & 0.0188 & 0.5454 & 1.0552 \end{bmatrix}$$

8.3.2 Initial Conditions

Similar to Part 1, we assume for any field the DRW are at SSD. Meaning we have the same initial conditions as in Part 1.

$$\begin{bmatrix} P_e(0) \\ P_l(0) \\ P_p(0) \\ P_a(0) \end{bmatrix} = \begin{bmatrix} \Phi_e \\ \Phi_l \\ \Phi_p \\ \Phi_a \end{bmatrix} = \begin{bmatrix} 0.485943142 \\ 0.492150833 \\ 0.015153450 \\ 0.006752576 \end{bmatrix}$$

8.3.3 Cost Function

Cost of Nematodes The cost of nematodes does not depend on the length of time step so we still have that, Cost of Nematodes N - \$62/22/(1/10^8) per hectare per nematode = β_2.

Cost of DRW Recall that β_1 was dependent on one week as a time step, so we know have Cost of DRW P_l - $\beta_3 = \beta_1 * 4 = 3.1806 \times 10^{-11}$ per hectare per 4 weeks.

So we have the cost for any time is cost of diaprepes weevil damage, $\beta_3 P_l(t)^2$, plus cost of using nematodes, $\beta_2 N(t)$.

$$Cost = \beta_3 P_l(t)^2 + \beta_2 N(t)$$

where $\beta_3 = 3.1806 \times 10^{-11}$ and $\beta_2 = 2.8182 \times 10^{-8}$.

Chapter 9

Future Work

9.1 Different Biological Approaches

There are various ways we can explore how changing more biological components changes the model and the dynamics. For instance, we could consider applying the control to a different stage in the matrix. Alternatively, we could change the model to consider a predator prey component.

Additionally with our current or these new models we can consider an integrodifference model for continuous time, using a dispersal kennel and model longer spreading for the population.

9.2 Robustness

I intend to study the robustness of my Optimal Control management solutions. Optimal Control is not designed to be robust to uncertainties, parameter drift, or unmodeled dynamics, since it doesn't respond to new information. Specifically, I will be testing how well the optimal control management solutions fare when uncertainties and parameter drift are incorporated. It is not only important to find a solution that minimizes the cost to the farmers but also accounts for the possibility that changes might occur and a slight perturbation should not result in great loss to the farmer.

9.3 Stochasticity

Currently we are using a deterministic matrix model for the pest dynamics and in the future I would like formulate a stochastic model. This will allow for the natural changes in the environment to be reflected in the model.

9.4 Collaboration

My work on using Optimal Control theory to aid population management for DRW is part of a collaboration that started in May 2014 with Richard Rebarber, Brigitte Tenhumberg, Yu Jin (University of Nebraska-Lincoln),Chris Guiver, Stuart Townley (University of Exeter - Cornwall), and Jim Powell (Utah State University), and has since grown to include and Stephanie Lloyd (Exeter). We consider different control theory approaches resulting in management methods which we will compare. Since the initial meeting I have been working on an Optimal Control theory approach, while other members have been working on feedback control methods such as adaptive control. A paper by Chris Guiver is published in the SIAM Journal on Applied Mathematics (SIAP), "Simple adaptive control for positive linear systems with applications to pest management." [GEJ+16].

Our plan is to compare the various control theory methods by cost, reduction of DRW, and robustness. Feedback controls are known to be more robust, but require monitoring of the system. Meanwhile Optimal Control is known to minimize the cost, but requires initial data. Hence, we will be looking closely at the robustness of Optimal Control and the cost efficiency of feedback control. Once we have done the initial comparison, we can extend the research to other systems deducing which method of control theory outputs the best result for different purposes.

Bibliography

[BPK99] R. C. Bullock, R. R. Pelosi, and E. E. Killer. Management of citrus root weevils (coleoptera: Curculionidae) on florida citrus with soil-applied entomopathogenic nematodes (nematoda: Rhabditida). *The Florida Entomologist*, 82(1):pp. 1–7, 1999. 4.1

[BS78] Dimitri P Bertsekas and Steven E Shreve. *Stochastic optimal control: The discrete time case*, volume 23. Academic Press New York, 1978.

[CAP16] Paula Castro, Anabela Marisa Azul, and Jorge Paiva. Conservation of biological resources: Why does it matter? In *Biodiversity and Education for Sustainable Development*, pages 13–28. Springer, 2016. 1.2

[CHEBD15] Raquel Campos-Herrera, Fahiem E El-Borai, and Larry W Duncan. It takes a village: entomopathogenic nematode community structure and conservation biological control in florida (us) orchards. In *Nematode Pathogenesis of Insects and Other Pests*, pages 329–351. Springer, 2015. 4.1

[Dab10] Kathryn Dabbs. Optimal control in discrete pest control models. 2010. 1.1, 2.3

[DL09] Sebastian Dewhirst and Frithjof Lutscher. Dispersal in heterogeneous habitats: thresholds, spatial scales, and approximate rates of spread. *Ecology*, 90(5):1338–1345, 2009. 1.2

[EGC04] K.G.J.P.C.M.R.L. E. Grafton-Cardwell. *Diaprepes Root Weevil*. UCANR Publications, 2004. 4.1

[Fah02] Lenore Fahrig. Effect of habitat fragmentation on the extinction threshold: a synthesis. *Ecological applications*, 12(2):346–353, 2002. 1.2

[FCZI03] Enrique Fernández Cara and Enrique Zuazua Iriondo. Control theory: History, mathematical achievements and perspectives. *Boletín de la Sociedad Española de Matemática Aplicada, 26, 79-140.*, 2003. 1.1

[Fil62] AoF Filippov. On certain questions in the theory of optimal control. *Journal of the Society for Industrial and Applied Mathematics, Series A: Control*, 1(1):76–84, 1962. 1.1

[Fis37] Ronald Aylmer Fisher. The wave of advance of advantageous genes. *Annals of eugenics*, 7(4):355–369, 1937. 5.1

[Gau02] R. Gaugler. *Entomopathogenic Nematology*. CABI, 2002. 4.1, 4.2.3, 4.2.4, 4.2.4

[GEJ+16] Chris Guiver, Christina Edholm, Yu Jin, Markus Mueller, Jim Powell, Richard Rebarber, Brigitte Tenhumberg, and Stuart Townley. Simple adaptive control for positive linear systems with applications to pest management. *SIAM Journal on Applied Mathematics (SIAP)*, 76(1):238–275, 2016. 9.4

[Gra10] Rachel Natalie Graves. A method to accomplish the optimal control of continuous dynamical systems with impulse controls via discrete optimal control and utilizing optimal control theory to explore the emergence of synchrony. 2010. 1.1

[Han94] Ilkka Hanski. A practical model of metapopulation dynamics. *Journal of animal ecology*, pages 151–162, 1994. 5.1

[Han98] Ilkka Hanski. Metapopulation dynamics. *Nature*, 396(6706):41–49, 1998. 5.1

[Han99] Ilkka Hanski. *Metapopulation ecology*. Oxford University Press, 1999. 5.1

[HCD+05] Alan Hastings, Kim Cuddington, Kendi F Davies, Christopher J Dugaw, Sarah Elmendorf, Amy Freestone, Susan Harrison, Matthew Holland, John Lambrinos, Urmila Malvadkar, et al. The spatial spread of invasions: new developments in theory and evidence. *Ecology Letters*, 8(1):91–101, 2005. 5.1

[Hob00] Richard J Hobbs. Land-use changes and invasions. *Invasive species in a changing world*, pages 55–64, 2000. 1.2

[JG+09a] K Jetter, Kris Godfrey, et al. Diaprepes root weevil, a new california pest, will raise costs for pest control and trigger quarantines. *California agriculture*, 63(3):121–126, 2009. 7.3

[JG09b] Karen M. Jetter and Kris Godfrey. Diaprepes root weevil, a new california pest, will raise costs for pest control and trigger quarantines. *California Agriculture*, 63(3), 2009. 4.1

[JLPB05] Eunok Jung, Suzanne Lenhart, Vladimir Protopopescu, and Charles F Babbs. Optimal control theory applied to a difference equation model for cardiopulmonary resuscitation. *Mathematical Models and Methods in Applied Sciences*, 15(10):1519–1531, 2005. 1.1

[Kot01] Mark Kot. *Elements of mathematical ecology*. Cambridge University Press, 2001.

[LDG16] S.H. Futch L.W. Duncan, M.E. Rogers and J.H. Graham. 2016 florida citrus pest management guide: Ch. 13 citrus root weevils. 2016. 4.1

[Leu93] Siu Fai Leung. Optimality for controlled jump processes: a simple approach. *Economic Theory*, 3(4):765–774, 1993. 1.1

[LW07] S. Lenhart and J.T. Workman. *Optimal Control Applied to Biological Models*. Taylor & Francis, 2007. 1.1, 2.3, 2.3.2, 2.3.2, 3.2.2, 4.3.1, 6.2.2

[MLW15] Marco V. Martinez, Suzanne Lenhart, and K. A. Jane White. Optimal control of integrodifference equations in a pest-pathogen system. *Discrete and Continuous Dynamical Systems - Series B*, 20(6):1759–1783, 2015. 1.1

[MS12] Jack Macki and Aaron Strauss. *Introduction to optimal control theory*. Springer Science & Business Media, 2012. 1.1

[MSDN00] Clayton W McCoy, David I Shapiro, Larry W Duncan, and Khuong Nguyen. Entomopathogenic nematodes and other natural enemies as mortality factors for larvae of diaprepes abbreviatus (coleoptera: Curculionidae). *Biological Control*, 19(2):182 – 190, 2000. 4.1

[MSN03] C. W. McCoy, R. J. Stuart, and H. N. Nigg. Seasonal life stage abundance of diaprepes abbreviatus in irrigated and non-irrigated citrus plantings in central florida. *The Florida Entomologist*, 86(1):pp. 34–42, 2003. 4.2.4

[MT10] Tom EX Miller and Brigitte Tenhumberg. Contributions of demography and dispersal parameters to the spatial spread of a stage-structured insect invasion. *Ecological Applications*, 20(3):620–633, 2010. 4.1, 4.2.1

[Oer06] E-C Oerke. Crop losses to pests. *The Journal of Agricultural Science*, 144(01):31–43, 2006. 1.2

[PG97] D Pimentel and A Greiner. Environmental and socio-economic impacts of pesticides use.(51-78 pp). *Techniques for reducing pesticide use: environmental an economics benefits. Chichester (UK): Jhon Wiley Sons*, 1997. 1.2

[Pim05] David Pimentel. Environmental and economic costs of the application of pesticides primarily in the united states. *Environment, development and sustainability*, 7(2):229–252, 2005. 1.2

[PMZ+91] David Pimentel, Lori McLaughlin, Andrew Zepp, Benyamin Lakitan, Tamara Kraus, Peter Kleinman, Fabius Vancini, W John Roach, Ellen Graap, William S Keeton, et al. Environmental and economic effects of reducing pesticide use. *BioScience*, 41(6):402–409, 1991. 1.2

[Pon87] Lev Semenovich Pontryagin. *Mathematical theory of optimal processes.* CRC Press, 1987. 1.1

[PR12] BioLogic Biocontrol Products and Research. Biologic products, 2012. 4.2.4

[PU03] Pan-UK. Current pesticide spectrum, global use and major concerns. 2003. 1.2

[Ris77] R. Rishel. Optimality for completely observed controlled jump processes. *IEEE Transactions on Automatic Control*, 22(6):906–908, Dec 1977. 1.1

[SP10] Roberto Salguero_Gòmez and Joshua B. Plotkin. Matrix dimensions bias demographic inferences: Implications for comparative plant demography. *The American Naturalist*, 176(6):pp. 710–722, 2010. 4.2.1, 4.2.1

[TJWJK16] R.M. Giblin-Davis Thomas J. Weissling, J.E. Pea and Jr. J.L. Knapp. Diaprepes abbreviatus. *Entomology & Nematology*, 2016. 7.3

[Wit02] Kimberly A With. The landscape ecology of invasive spread. *Conservation Biology*, 16(5):1192–1203, 2002. 1.2

[WLW08] Andrew Whittle, Suzanne Lenhart, and K.A.J. White. Optimal control of gypsy moth populations. *Bulletin of Mathematical Biology*, 70(2):398–411, 2008. 1.1, 2.3