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A Caputo Boundary Value Problem in Nabla Fractional Calculus

Julia St. Goar
University of Nebraska-Lincoln, jstgoar@gmail.com

A CAPUTO BOUNDARY VALUE PROBLEM IN NABLA FRACTIONAL CALCULUS

by

Julia St. Goar

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A CAPUTO BOUNDARY VALUE PROBLEM IN NABLA FRACTIONAL CALCULUS

Julia St. Goar, Ph.D.

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Adviser: Allan C. Peterson

Boundary value problems have long been of interest in the continuous differential equations context. However, with the advent of new areas like Nabla Fractional Calculus, we may consider such problems in new contexts. In this work, we will consider several right focal boundary value problems, involving a Caputo fractional difference operator, in the Nabla Fractional Calculus context. Properties of the Green's functions for each of these boundary value problems will be investigated and, in the case of a particular boundary value problem, used to establish the existence of positive solutions to a nonlinear version of the boundary value problem.

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DEDICATION

To Edward and Rebecca St. Goar

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Chapter 1

Introduction

Two ways of approaching Discrete Fractional Calculus are extensively introduced in Goodrich and Peterson [31] and differ from the start in the way the difference operator is defined. Delta Fractional Calculus makes use of the forward difference operator, for $f : \mathbb{N}_a \to \mathbb{R}$, defined by

$$\Delta f(t) := f(t+1) - f(t), \quad \text{for } t \in \mathbb{N}_a$$

where $\mathbb{N}_a := \{a, a+1, a+2, ...\}$, whereas Nabla Fractional Calculus makes use of the nabla (backward) difference operator, for $f : \mathbb{N}_a \to \mathbb{R}$, defined by

$$\nabla f(t) := f(t) - f(t-1), \quad \text{where } t \in \mathbb{N}_{a+1}.$$

These initial differences give rise to differing definitions of the integral and fractional derivative. This thesis will focus on the Nabla Fractional Calculus case, which, of the two areas, is the relatively newer focus among researchers and thus has been less extensively elaborated. However, a few features of the most prominent operators in Nabla Fractional Calculus indicate not only distinctions in behavior but also some properties which appear particularly advantageous.

In Nabla Fractional Calculus, the fractional nabla sum, which is related to the continuous integral, is defined as follows for $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\mu \in \mathbb{R}^+$:

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t, \rho(s)) f(s) \nabla s$$

for $t \in \mathbb{N}_a$, where by convention $\nabla_a^{\mu} f(a) = 0$. In the above, $H_\nu(t,s)$ is notation for Taylor monomials in this context. Notice that the domain of the function f and of $\nabla_a^{-\mu} f(t)$ differ only by a single value; that is, the domain is shifted to the left by 1. Hence, the domains before and after the application of the fractional sum are quite similar. This state of affairs differs markedly from the fractional sum operator in the Delta Fractional Calculus context. In the Delta Fractional Calculus context, the application of the fractional sum operator may result in the domain of the result being shifted by a non-integer value, resulting in a domain that differs entirely from that of the original function. Such an operator in the Delta Fractional Calculus still holds much mathematical meaning and usefulness. However, the relative consistency in domains before and after the application of such fractional operators in Nabla Fractional Calculus is certainly pleasant. Some papers in the field of discrete nabla fractional calculus include [13], [15], [2], and the references therein. Nabla fractional calculus is considered in the context of the more general area of time scales by Anastassiou in [4] and [7] and by Anderson in [8]. Some papers in the field of delta fractional calculus include [14], [16], [29], [30]. In [17], Atici and Şengül use fractional operators in the context of delta fractional calculus to model tumor growth. In [39], Baoguo, Erbe and Peterson discuss some relationships between asymptotic behavior of nabla and delta fractional difference equations.

Before discussing the particular focus of this work, it is important to refer to the prominent field of fractional calculus in the context of standard calculus. This field has a long history and extensive research that can only be touched on here. A few of the many instances of research in fractional differential equations include [11], [20] which considers a boundary value problem involving a Caputo operator, [47], and [44]. Finally Oldham and Spanier survey and discuss the many applications of fractional calculus in [45].

The particular focus will be on various boundary value problems in the context of nabla fractional calculus. In these problems one looks for functions which satisfy a difference equation on a given domain as well as several conditions on the boundaries of the domain.

Additionally all boundary value problems in this thesis involve a fractional Caputo difference operator, denoted by ∇_{a*}^{ν} for $\nu > 0$. The Caputo operator has a few useful properties that distinguish it from the Riemann-Liouville operator, including the fact that $\nabla_{a*}^{\nu} C = 0$ for constant C and $\nu \geq 1$, which does not necessarily hold for the Riemann-Liouville operator. The papers [5] and [6] by Anastassiou introduce the Caputo operator in the nabla fractional calculus context and shows some other properties in nabla fractional calculus. There are also monotonicity results related to the Caputo operator, as discussed by B. Jia, etal. [40], as well as in [31] and [28]. The Riemann-Liouville operator, briefly introduced in Chapter 2, is also frequently used in research and its properties are discussed at length in [28].

Similar boundary value problems to the one considered in this paper have been investigated by Holm [37], whose problem involved a Riemann-Liouville fractional operator in the context of the Delta Fractional Calculus, and by Erbe and Peterson [27], whose problem involved whole order differences in the context of Time Scales. A linear boundary value problem similar to the one in Chapter 5 was investigated by Ahrendt et al. [3] in the Nabla Fractional Calculus context and involved a Caputo operator.

Boundary value problems are also considered in Discrete Fractional Calculus contexts by Atici and Eloe [16], Goodrich [32], Awasthi [19], and Brackins [23]. A boundary value problem involving a whole order delta and nabla operator on a discrete domain was investigated by [24].

Within the consideration of each boundary value problem, I make use of and investigate Green's functions. Green's functions are specific functions typically unique to each boundary value problem. They lead directly to solutions in the case of linear boundary value problems and can be useful tools in showing the existence of solutions to nonlinear boundary value problems. In the papers by Holm [37] and Erbe and Peterson [27], positive solutions to the nonlinear case were sought using Green's Functions and a fixed point theorem from Krasnosel′skiĭ [42] and Deimling [26]. The fixed point theorem stated as norm type was shown by Guo in [34] and [33]. In the present work, a boundary value problem we consider is

$$\begin{cases} -\nabla^{\nu}_{a^*}x(t) = h(t, x(t-1)), \quad t \in \mathbb{N}^b_{a+1} \\ x(a-1) = 0, \\ \nabla x(b) = 0, \end{cases} \tag{1.1}$$

where $1 < \nu \leq 2$, $h : \mathbb{N}^b_{a+1} \times \mathbb{R}^+ \to \mathbb{R}^+$, $b - a \in \mathbb{Z}$, $b - a \geq 1$, and where the solutions x are defined on $\mathbb{N}^b_{a-1}$.

The Guo-Krasnosel′skiĭ fixed point theorem is a frequently used theorem within the larger context of cone theory. Kwong discusses Krasnosel′skiĭ's theorem as well as its connection to other fixed-point theorems in [43]. The larger context of cone theory and its applications to nonlinear problems is more fully elaborated in the text by Guo and Lakshmikantham [35]. Also, fixed-point theory is surveyed by Agarwal, Meehan, and O'Regan in [1] and by Zeidler in [48].

The area of nabla fractional calculus and discrete fractional calculus, especially as presented in this work, may be seen within the context of time scales. The area of time scales works with functions whose domains are any nonempty closed subsets of the real numbers and was inagurated by Hilger [36]. Hence, both standard calculus and discrete calculus may be viewed as contained within the larger context of time scales. The area of time scales has been extensively surveyed by Bohner and Peterson in [21] and [22]. Boundary value problems with whole order operators have been considered in the context of time scales in [27], [10], [12], [25] to name a few.

After an overview of background notation and theorems in Chapter 2, we will focus on boundary value problems. In Chapter 3, we will consider the BVP (1.1). The Green's function for this BVP is found, bounds are established for this Green's function, and the Guo-Krasnosel′skiĭ fixed point theorem is used to show existence of positive solutions in some cases and existence of multiple positive solutions in others. In Chapter 4, we consider the BVP

$$\begin{cases} \nabla^{\nu}_{a^*}x(t) = h(t), & t \in \mathbb{N}^b_{a+1} \\ \nabla^k x(a-1) = 0, & 0 \leq k \leq N-2 \\ \nabla^{N-1}x(b) = 0, \end{cases} \tag{1.2}$$

where $1 < \nu$, $h : \mathbb{N}^b_{a+1} \to \mathbb{R}^+$, $b - a \in \mathbb{Z}$, $b - a \geq N - 1$, and where the solutions x are defined on $\mathbb{N}^b_{a-N+1}$. We first expand our knowledge of the BVP (1.2) for the case where $1 < \nu \leq 2$. We then continue on to consider the Green's function of the BVP (1.2) for $2 < \nu \leq 3$ and establish bounds on the Green's function. In so doing, we show that the Green's function for the $2 < \nu \leq 3$ case differs markedly in its behavior from that of $1 < \nu \leq 2$. Lastly, we consider a generalization of the BVP (1.2). In

Chapter 5, for $0 < \nu \leq 1$, we consider the BVP

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2} \\ x(a) = \nabla x(b) = 0, \end{cases} . \tag{1.3}$$

where a, b are positive integers such that $b - a \geq 2$ and $h : \mathbb{N}^{b}_{a+2} \to \mathbb{R}$. In particular, we establish several bounds on the Green's function for the BVP (1.3) and compare the revealed behavior to the behavior of the Green's function for an analogous BVP in the continuous differential equations context.

Chapter 2

Background

In this chapter we will introduce relevant definitions and theorems from the area of Nabla Fractional Calculus.

2.1 Discrete Nabla Operators

In this section we will address the notation for the domains of functions in the area of Nabla Fractional Calculus. Additionally we will introduce some basic operators in this context along with some of their properties.

All definitions and theorems found in this chapter may be found in [31], along with a more general introduction to the area of Nabla Fractional Calculus.

Definition 2.1. Let $a \in \mathbb{R}$ and $b - a \in \mathbb{Z}^+$. Then

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$$

and

$$\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b\}.$$

The notation in Definition 2.1 shows the typical notation for domains of functions in the area of Nabla Fractional Calculus. Definition 2.2 shows additional notation commonly used in this area.

Definition 2.2 (Backward Jump Operator)**.** We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \to \mathbb{N}_a$, by

$$\rho(t) = t - 1.$$

If we consider the derivative, we notice that the standard definition of the derivative will not work when considering functions with discrete domains because in order to take a derivative at a point, the function must be defined on some open interval around the point. However, we wish to define an analogue of this operator for discrete functions. While the definition must be different from the definition given in standard calculus, we still want the operator we use to relate to the idea of slope. Since there are no points in the domain arbitrarily close to any other points, the next best option is to use the slope of the line between two adjacent points as our derivative. Hence, we define the nabla operator, or backwards difference operator, for this context.

Definition 2.3 (Nabla Operator)**.** For an arbitrary $f : \mathbb{N}_a \to \mathbb{R}$, we define the nabla operator, ∇, by

$$(\nabla f)(t) := f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Note additionally that ∇^0 is the identity, that is it is defined by $\nabla^0 f(t) = f(t)$. For $N \in \mathbb{N}$, we define ∇^N by

$$\nabla^N f(t) := \nabla(\nabla^{N-1} f(t)),$$

for $t \in \mathbb{N}_{a+N}$.

The theorem below gives several basic properties of the nabla operator, including two properties analogous to the product rule and quotient rule from standard calculus.

Theorem 2.4. *Assume $f, g : \mathbb{N}_a \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Then for $t \in \mathbb{N}_{a+1}$,*

(i.) $\nabla \alpha = 0$;

(ii.) $\nabla \alpha f(t) = \alpha \nabla f(t)$;

(iii.) $\nabla(f(t) + g(t)) = \nabla f(t) + \nabla g(t)$;

(iv.) *if* $\alpha \neq 0$, *then* $\nabla \alpha^{t+\beta} = \frac{\alpha-1}{\alpha}\alpha^{t+\beta}$;

(v.) $\nabla(f(t)g(t)) = f(\rho(t))\nabla g(t) + \nabla f(t) g(t)$;

(vi.) $\nabla\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\nabla f(t) - f(t)\nabla g(t)}{g(t)g(\rho(t))}$, *if* $g(t) \neq 0$, $t \in \mathbb{N}_{a+1}$.

In addition to an operator similar to the derivative, we also need an operator similar to the integral from standard calculus. Of course, as before, the definition must differ from the standard definition in this discrete context.

Definition 2.5 (Discrete Nabla Integral)**.** Assume $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$. Then

$$\int_c^d f(t)\nabla t := \begin{cases} \sum_{t=c+1}^{d} f(t), & \text{if} \quad d > c \\ 0, & \text{if} \quad d \leq c. \end{cases}$$

Notice that Definition 2.5 is essentially a right Riemann sum, involving rectangles of width one. The theorem below shows some basic properties of the discrete nabla integral.

Theorem 2.6. *Assume $f, g : \mathbb{N}_a \to \mathbb{R}$, $b, c, d \in \mathbb{N}_a$, $b < c < d$, and $\alpha \in \mathbb{R}$. Then*

(i.) $\int_b^c \alpha f(t) \nabla t = \alpha \int_b^c f(t) \nabla t$;

(ii.) $\int_b^c (f(t) + g(t)) \nabla t = \int_b^c f(t) \nabla t + \int_b^c g(t) \nabla t$;

(iii.) $\int_b^b f(t) \nabla t = 0$;

(iv.) $\int_b^d f(t) \nabla t = \int_b^c f(t) \nabla t + \int_c^d f(t) \nabla t$;

(v.) $|\int_b^c f(t) \nabla t| \leq \int_b^c |f(t)| \nabla t$;

(vi.) if $F(t) := \int_b^t f(s) \nabla s$, *for* $t \in \mathbb{N}_b^c$, *then* $\nabla F(t) = f(t)$, $t \in \mathbb{N}_{b+1}^c$;

(vii.) if $f(t) \geq g(t)$ *for* $t \in \{b+1, b+2, \cdots, c\}$, *then* $\int_b^c f(t) \nabla t \geq \int_b^c g(t) \nabla t$.

2.2 The Rising Function and Taylor Monomials

Among the properties of the nabla operator listed in Theorem 2.4, notice that there is no property similar to the power rule listed. In order to have a property analogous to the power rule, as stated in Theorem 2.8, we must define the rising function.

Definition 2.7 (Rising Function)**.** Assume n is a positive integer and $t \in \mathbb{R}$. Then we define the rising function, $t^{\overline{n}}$, by

$$t^{\overline{n}} := t(t+1) \cdots (t+n-1).$$

Theorem 2.8 (Nabla Power Rules)**.** *For $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$,*

$$\nabla(t+\alpha)^{\overline{n}} = n(t+\alpha)^{\overline{n-1}},$$

for $t \in \mathbb{R}$.

Taylor monomials will be used in Section 2.3 to define fractional sums and differences and are often used throughout this work.

Definition 2.9 (Nabla Taylor Monomials)**.** We define the nabla Taylor monomials, $H_n(t,a)$, $n \in \mathbb{N}_0$ by $H_0(t,a) = 1$, $t \in \mathbb{N}_a$ and

$$H_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!},$$

for $t \in \mathbb{N}_{a-n+1}$ and $n \in \mathbb{N}$.

2.3 Fractional Sums and Differences

In this section we introduce more definitions and theorems in order to generalize the nabla operator and the discrete nabla integral. In order to make these generalizations, we use the Gamma function, which is defined to be

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt,$$

where z is such that the real part of z is positive. By integration by parts, one can show that

$$\Gamma(z+1) = z\Gamma(z). \tag{2.1}$$

It follows that $\Gamma(N+1) = N \cdot \Gamma(N)$ for N a positive integer. Hence $\Gamma(N+1) = N!$ for $N \in \mathbb{N}$. The definition of $\Gamma(z)$ is extended, using (2.1), to all complex z such that $z \neq 0, -1, -2, -3, \ldots$. Additionally, $\lim_{z \to -n} |\Gamma(z)| = \infty$ for $n = 0, -1, -2, -3, \ldots$. Below, we use the Gamma function to generalize the rising function. Note that for a positive integer N, $t^{\overline{N}} = \frac{(t+N+1)!}{(t+1)!}$. With this in mind, we define the generalization of the rising function as follows.

Definition 2.10 ((Generalized) Rising Function)**.** The (generalized) rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)} \tag{2.2}$$

for those values of t and r such that the right hand side of equation (2.2) makes sense. We also use the convention that if t is a nonpositive integer, but $t+r$ is not a nonpositive integer then $t^{\overline{r}} := 0$.

Note that $t^{\overline{0}} = 1$ for $t \neq 0, -1, -2, \ldots$

Theorem 2.11 (Generalized Nabla Power Rules)**.** *The formulas*

$$\nabla(t+\alpha)^{\overline{r}} = r(t+\alpha)^{\overline{r-1}}, \tag{2.3}$$

and

$$\nabla(\alpha - t)^{\overline{r}} = -r(\alpha - \rho(t))^{\overline{r-1}}, \tag{2.4}$$

hold for those values of t, r and α so that the expressions (2.3) and (2.4) makes sense.

Additionally we define the analogue of the Taylor monomial, $\frac{(t-a)^n}{n!}$, from standard calculus and describe some of its properties.

Definition 2.12. Let $\mu \neq -1, -2-3, \cdots$, then we define the μ-th order nabla fractional Taylor monomial, $H_\mu(t,a)$, by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} \tag{2.5}$$

whenever the right hand side of the equation (2.5) makes sense.

Theorem 2.13. *The following hold whenever the expressions below are well-defined:*

(i.) $H_\mu(a,a) = 0$;

(ii.) $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$;

(iii.) $\int_a^t H_\mu(s,a)\nabla s = H_{\mu+1}(t,a)$;

(iv.) $\int_a^t H_\mu(t,\rho(s))\nabla s = H_{\mu+1}(t,a)$;

(v.) for $k \in \mathbb{N}$, $H_{-k}(t,a) = 0$, $t \in \mathbb{N}_a$.

Next we define the nabla fractional sum, denoted $\nabla_a^{-\nu}$ for $a \in \mathbb{R}$ and $\nu \geq 0$, which is an extension of the discrete nabla integral. The operator is defined so that $\nabla_a^{-1} f(t) = \int_a^t f(t)\nabla t$. The definition is additionally motivated by the fact that

$$\int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\nabla\tau_n \cdots \nabla\tau_2\nabla\tau_1 = \int_a^t H_{n-1}(t,\rho(s))f(s)\nabla s.$$

Definition 2.14 (Nabla Fractional Sum)**.** Let $f \in \mathbb{N}_{a+1} \to \mathbb{R}$ be given and $\mu \in \mathbb{R}^+$, then

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t,\rho(s))f(s)\nabla s$$

for $t \in \mathbb{N}_a$, where by convention $\nabla_a^{\mu}(a) = 0$.

The following theorem gives us an important property of the nabla fractional sum.

Theorem 2.15. *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ and $\nu + \mu$ are not negative integers. Then we have that*

$$\nabla_a^{-\nu} H_\mu(t, a) = H_{\mu+\nu}(t, a)$$

for $t \in \mathbb{N}_a$.

This property is analogous to the fact that $\int_a^t \frac{(\tau-a)^n}{n!} d\tau = \frac{(t-a)^{n+1}}{(n+1)!}$ in standard calculus. Next we use the nabla fractional sum in the definition of the nabla fractional difference.

Definition 2.16 (Nabla Fractional Difference)**.** Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu \in \mathbb{R}^+$, and choose N so that $N - 1 < \nu \leq N$. Then we define the ν-th order nabla fractional difference, $\nabla_a^\nu f(t)$ by

$$\nabla_a^\nu f(t) = \nabla^N \nabla_a^{-(N-\nu)} f(t) \quad \text{for} \quad t \in \mathbb{N}_{a+N}.$$

Note that for $N \in \mathbb{N}$,

$$\nabla_a^N f(t) = \nabla^N \nabla_a^{-(N-N)} f(t) = \nabla^N f(t) \quad \text{for} \quad t \in \mathbb{N}_{a+N}.$$

2.4 Caputo Fractional Difference

Definition 2.16 represents the Riemann-Liouville definition of the fractional difference. However, one may also define the Caputo fractional difference by changing the order of the operators in Definition 2.16.

Definition 2.17 (Caputo Fractional Difference)**.** Assume $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$ and $\mu > 0$. Then the μ-th Caputo nabla fractional difference of f is defined by

$$\nabla_{a^*}^{\mu} f(t) = \nabla_a^{-(N-\mu)} \nabla^N f(t)$$

for $t \in \mathbb{N}_{a+1}$, where $N = \lceil \mu \rceil$.

Note that for constant C, $\nabla_{a^*}^{\mu} C = \nabla_a^{-(N-\mu)} \nabla^N C = \nabla_a^{-(N-\mu)} 0 = 0$. Such a property is reasonable for an operator extending the difference operator. Yet this property does not hold for the Riemann-Liouville operator defined in Definition 2.16. This work is primarily concerned with the Caputo Fractional Difference as the extension of the nabla difference operator. We describe a basic property of this operator below from [31, Theorem 3.118 on p. 229].

Theorem 2.18. *Assume $\mu > 0$ and $N = \lceil \mu \rceil$. Then the nabla Taylor monomials, $H_k(t, a)$, $0 \leq k \leq N-1$, are N linearly independent solutions of $\nabla_{a^*}^{\mu} x = 0$ on $\mathbb{N}_{a-N+1}$.*

Chapter 3

A Nonlinear Right Focal Boundary Value Problem Involving a Caputo Operator with $1 < \nu \leq 2$

3.1 Introduction

In this chapter we will consider the nonlinear right focal boundary value problem (3.1) in the context of Nabla Fractional Calculus, as shown below

$$\begin{cases} -\nabla_{a^*}^{\nu} x(t) = h(t, x(t-1)), \quad t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = 0, \\ \nabla x(b) = 0, \end{cases} \tag{3.1}$$

where $1 < \nu \leq 2$, $h : \mathbb{N}_{a+1}^{b} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$, $b - a \in \mathbb{N}_1$, and where the solutions x are defined on $\mathbb{N}_{a-1}^{b}$. Recall that the Caputo difference operator is denoted by $\nabla_{a^*}^{\nu}$ for $\nu > 0$.

The eventual goal of this chapter is to establish the existence of positive solutions and, in some cases, even of multiple positive solutions to the boundary value problem (3.1). However, this chapter will begin by considering a linear version of the BVP (3.1) for any $\nu > 1$. This more general linear BVP is

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k x(a-1) = A_k, & 0 \leq k \leq N-2 \\ \nabla^{N-1} x(b) = B, \end{cases} \tag{3.2}$$

where $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $\nu > 1$, $N = \lceil \nu \rceil$, $A_k \in \mathbb{R}$ for $0 \leq k \leq N-2$, $B \in \mathbb{R}$, $b - a \in \mathbb{Z}$, $b - a \geq N - 1$, and where the solutions x are defined on $\mathbb{N}_{a-N+1}^{b}$.

In Section 3.2, the uniqueness of solutions will be established under certain conditions for the BVP (3.2). In Section 3.3, we will consider the Green's function for the BVP

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k x(a-1) = 0, & 0 \leq k \leq N-2 \\ \nabla^{N-1} x(b) = 0, \end{cases} \tag{3.3}$$

where $\nu > 1$, $N = \lceil \nu \rceil$, $b - a \in \mathbb{Z}$, $b - a \geq N - 1$, and where the solutions x are defined on $\mathbb{N}_{a-N+1}^{b}$. The Green's function is useful in calculating solutions to the BVP (3.2). In particular, the Green's function is defined so that $x(t) = \int_a^b G(t,s)h(s)\nabla s$ for $t \in \mathbb{N}_{a-N+1}^{b}$ solves the BVP (3.2). In Subsection 3.3.2, the Green's function will be considered only for the case where $1 < \nu \leq 2$. Bounds will be established on the Green's function in this context in Subsection 3.3.2 so that these bounds may then be used to solve the nonlinear BVP (3.1). In Section 3.4, existence of positive solutions to the BVP (3.1) will be established by finding a fixed point for the operator

$$Ax(t) = \int_a^b G(t,s)h(s,x(s-1))\nabla s$$

for x in a cone in an appropriate Banach space and $t \in \mathbb{N}_{a-1}^b$. The general definition of a cone is given in Definition 3.14. This fixed point will be found by using the Guo-Krasnosel′skiĭ theorem. In Subsection 3.4.2, conditions are established under which multiple solutions to the BVP (3.1) may be found. Finally, directions for future work will be discussed in Section 3.5. Note that many of the results in this chapter are included in [46].

3.2 Uniqueness of Solutions

First consider the following nth order initial value problem:

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^k x(a) = c_k, & 0 \le k \le N-1. \end{cases}$$

where $a, \nu \in \mathbb{R}$, $\nu > 0$, $N := \lceil \nu \rceil$, $c_k \in \mathbb{R}$ for $0 \le k \le N-1$, and $h : \mathbb{N}_{a+1} \to \mathbb{R}$. By the theorem below, the solution to the above IVP is defined on $\mathbb{N}_{a-N+1}$.

By [31, Theorem 3.120 on p. 230], we have the following theorem:

Theorem 3.1. *The unique solution to the IVP is given by*

$$x(t) = \sum_{k=0}^{N-1} H_k(t,a)c_k + \nabla_a^{-\nu} h(t)$$

for $t \in \mathbb{N}_{a-N+1}$, where by convention $\nabla_a^{-\nu} h(t) = 0$ for $a - N + 1 \le t \le a$.

Noting that $\nabla_a^{-\nu}h(t) = \int_a^t H_{\nu-1}(t,\rho(s))h(s)\nabla s$ for $t \in \mathbb{N}_{a+1}$, we define the Cauchy function for the IVP.

Definition 3.2. The **Cauchy function** for $\nabla_{a^*}^{\nu} x(t) = 0$ is defined to be

$$x(t,s) := H_{\nu-1}(t,\rho(s))$$

for those values of t, s, and ν such that the right side makes sense.

Then by Theorem 3.1, we have the following variation of constants formula.

Theorem 3.3 (Variation of Constants)**.** *For $h : \mathbb{N}_{a+1} \to \mathbb{R}$ and for $t \in \mathbb{N}_{a-N+1}$, the solution to the initial value problem*

$$\begin{cases} \nabla_{a^*}^{\nu} y(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^k y(a) = 0, & 0 \leq k \leq N-1. \end{cases}$$

is given by

$$y(t) = \int_a^t x(t,s)h(s)\nabla s.$$

In Section 3.3 the Cauchy function will be used in the definition of the Green's function.

The proofs of the two theorems that follow are similar to proofs in [31] and [3].

Theorem 3.4 (General Solution of the Homogeneous Equation)**.** *Let $a, \nu \in \mathbb{R}$, $\nu > 0$, and $N = \lceil \nu \rceil$. Assume $x_1, x_2, ..., x_N$ are N linearly independent solutions of $\nabla_{a^*}^{\nu} x = 0$ on $\mathbb{N}_{a-N+1}$. Then the general solution to $\nabla_{a^*}^{\nu} x = 0$ is given by*

$$x(t) = c_1 x_1(t) + \cdots + c_N x_N(t), \qquad t \in \mathbb{N}_{a-N+1}$$

where $c_1, c_2, ..., c_N \in \mathbb{R}$ are arbitrary constants.

Proof. Let $x_1, x_2, ..., x_N$ be linearly independent solutions of $\nabla^\nu_{a^*} x = 0$ on $\mathbb{N}_{a-N+1}$. Let $\nabla^k x_1(a) = A_{1,k}$ for $0 \le k \le N-1$, $\nabla^k x_2(a) = A_{2,k}$ for $0 \le k \le N-1$, and in general $\nabla^k x_j(a) = A_{j,k}$ for $0 \le k \le N-1$ and $1 \le j \le N$. Then for each $1 \le j \le N$, it holds that x_j is the unique solution to the IVP

$$\begin{cases} \nabla^\nu_{a^*} x_j(t) = 0, & t \in \mathbb{N}_{a+1} \\ \nabla^k x_j(a) = A_{j,k}, & 0 \le k \le N-1. \end{cases}$$

By the linearity of $\nabla^\nu_{a^*}$,

$$\nabla^\nu_{a^*}\left[c_1 x_1(t) + \cdots + c_N x_N(t)\right] = c_1 \nabla^\nu_{a^*} x_1(t) + ... + c_N \nabla^\nu_{a^*} x_N(t) = 0,$$

so $x(t) := c_1 x_1(t) + \cdots + c_N x_N(t)$ solves $\nabla^\nu_{a^*} x(t) = 0$.

Now suppose $x : \mathbb{N}_{a-N+1} \to \mathbb{R}$ solves $\nabla^\nu_{a^*} x(t) = 0$. Let $\nabla^k x(a) = M_k$ for $0 \le k \le N-1$. Then $x(t)$ is the unique solution of

$$\begin{cases} \nabla^\nu_{a^*} x(t) = 0, & t \in \mathbb{N}_{a+1} \\ \nabla^k x(a) = M_k, & 0 \le k \le N-1. \end{cases}$$

Now it will be shown that

$$\begin{bmatrix} x_1(a) & x_2(a) & \dots & x_N(a) \\ \nabla x_1(a) & \nabla x_2(a) & \dots & \nabla x_N(a) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{N-1} x_1(a) & \nabla^{N-1} x_2(a) & \dots & \nabla^{N-1} x_N(a) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \end{bmatrix}$$

has a unique solution for $c_1, c_2, ..., c_N \in \mathbb{R}$. Express the above equation as follows:

$$\begin{bmatrix} A_{1,0} & A_{2,0} & \dots & A_{N,0} \\ A_{1,1} & A_{2,1} & \dots & A_{N,1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,N-1} & A_{2,N-1} & \dots & A_{N,N-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ M_{N-1} \end{bmatrix}.$$

Suppose towards a contradiction that

$$\begin{vmatrix} A_{1,0} & A_{2,0} & \dots & A_{N,0} \\ A_{1,1} & A_{2,1} & \dots & A_{N,1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,N-1} & A_{2,N-1} & \dots & A_{N,N-1} \end{vmatrix} = 0$$

Then the columns of the above matrix are not linearly independent, so at least one column is a linear combination of the other columns. Without loss of generality, say that the Nth column is a linear combination of the other $N-1$ columns. Then

$$\alpha_1 \begin{bmatrix} A_{1,0} \\ A_{1,1} \\ \vdots \\ A_{1,N-1} \end{bmatrix} + \alpha_2 \begin{bmatrix} A_{2,0} \\ A_{2,1} \\ \vdots \\ A_{2,N-1} \end{bmatrix} + \cdots + \alpha_{N-1} \begin{bmatrix} A_{N-1,0} \\ A_{N-1,1} \\ \vdots \\ A_{N-1,N-1} \end{bmatrix} = \begin{bmatrix} A_{N,0} \\ A_{N,1} \\ \vdots \\ A_{N,N-1} \end{bmatrix}$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_{N-1} \in \mathbb{R}$. Note that $X(t) := \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_{N-1} x_{N-1}(t)$ is a solution of $\nabla^\nu_{a^*} x(t) = 0$. Now both $X(t)$ and $x_N(t)$ solve

$$\begin{cases} \nabla^\nu_{a^*} y(t) = 0, & t \in \mathbb{N}_{a+1} \\ \nabla^k y(a) = \alpha_1 A_{1,k} + \alpha_2 A_{2,k} + \cdots + \alpha_{N-1} A_{N-1,k}, & 0 \le k \le N-1, \end{cases}$$

so by uniqueness $X(t) = x_N(t)$, so $x_N(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) + \cdots + \alpha_{N-1} x_{N-1}(t)$. But then $x_1, x_2, \ldots, x_N$ are linearly dependent. Thus we have a contradiction. $\square$

Theorem 3.5 (General Solution of the Nonhomogeneous Equation)**.** *Let $\nu > 0$ and $N = \lceil \nu \rceil$. Assume $x_1, x_2, \ldots, x_n$ are linearly independent solutions of $\nabla_{a^*}^{\nu} x(t) = 0$ on $\mathbb{N}_{a-N+1}$ and y_0 is a particular solution to $\nabla_{a^*}^{\nu} x(t) = h(t)$ on $\mathbb{N}_{a-N+1}$ for some $h : \mathbb{N}_{a+1} \to \mathbb{R}$. Then the general solution of $\nabla_{a^*}^{\nu} x(t) = h(t)$ is given by*

$$x(t) = c_1 x_1(t) + \cdots + c_N x_N(t) + y_0(t),$$

where $c_1, c_2, \ldots, c_N \in \mathbb{R}$ are arbitrary constants and $t \in \mathbb{N}_{a-N+1}$.

Proof. By linearity of the operator $\nabla_{a^*}^{\nu}$,

$$\nabla_{a^*}^{\nu} x(t) = c_1 \nabla_{a^*}^{\nu} x_1(t) + \cdots + c_N \nabla_{a^*}^{\nu} x_N(t) + \nabla_{a^*}^{\nu} y_0(t) = h(t),$$

so $x(t)$ solves $\nabla_{a^*}^{\nu} x(t) = h(t)$.

Conversely, assume $x : \mathbb{N}_{a-N+1} \to \mathbb{R}$ solves $\nabla_{a^*}^{\nu} x(t) = h(t)$. Let $\nabla^k x(a) = b_k$ for $0 \leq k \leq N-1$. Then $x(t)$ uniquely solves the IVP

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1} \\ \nabla^k x(a) = b_k, & 0 \leq k \leq N-1. \end{cases}$$

Now there exist constants $c_1, c_2, \ldots, c_N \in \mathbb{R}$ such that $c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t)$ solves

$$\begin{cases} \nabla_{a^*}^{\nu} x_h(t) = 0, & t \in \mathbb{N}_{a+1} \\ \nabla^k x_h(a) = b_k - \nabla^k y_0(a), & 0 \leq k \leq N-1. \end{cases}$$

Note that $x_h(t) + y_0(t)$ solves $\nabla_{a^*}^{\nu} x(t) = h(t)$. Also

$$\nabla^k x_h(a) + \nabla^k y_0(a) = b_k - \nabla^k y_0(a) + \nabla^k y_0(a) = b_k$$

for $0 \leq k \leq N-1$. Hence $x(t)$ and $x_h(t) + y_0(t)$ both solve the same IVP. Hence by uniqueness,

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) + y_0(t).$$

Therefore any solution to $\nabla_{a^*}^{\nu} x(t) = 0$ can be written in this form. □

Now consider the non-homogeneous boundary value problem (3.2) and the corresponding homogeneous boundary value problem (3.3). Note that some arguments in the proof below are analogous to those given in [3].

Theorem 3.6. *The nonhomogeneous boundary value problem (3.2) has a unique solution.*

Proof. By Theorem 3.4, a general solution to $\nabla_{a^*}^{\nu} x(t) = 0$ is given by $x(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t)$, where $c_0, \ldots, c_{N-1} \in \mathbb{R}$ and $x_0, x_1, \ldots, x_{N-1}$ are N linearly independent solutions to $\nabla_{a^*}^{\nu} x(t) = 0$ on $\mathbb{N}_{a-N+1}$. Thus, by Theorem 3.1, the general solution to $\nabla_{a^*}^{\nu} x(t) = 0$ is given by

$$x(t) = c_0 + c_1 H_1(t,a) + c_2 H_2(t,a) + \cdots + c_{N-1} H_{N-1}(t,a),$$

for $t \in \mathbb{N}_{a-N+1}^{b}$, where $c_0, c_1, \ldots, c_{N-1} \in \mathbb{R}$. Without loss of generality, say $x_0(t) = 1, x_1(t) = H_1(t,a), \ldots, x_{N-1}(t) = H_{N-1}(t,a)$ for $t \in \mathbb{N}_{a-N+1}$.

Say $x(t)$ solves (3.3). It will be shown that $x(t)$ is the trivial solution. It holds that

$x(t)$ is the trivial solution if and only if $c_0 = c_1 = \ldots = c_{N-1} = 0$. This holds if and only if the system of equations

$$\begin{cases} c_0 \nabla^k x_0(a-1) + \cdots + c_{N-1} \nabla^k x_{N-1}(a-1) = 0, \quad 0 \le k \le N-2 \\ c_0 \nabla^{N-1} x_0(b) + \cdots + c_{N-1} \nabla^{N-1} x_{N-1}(b) = 0 \end{cases}$$

has only the trivial solution. Define

$$M := \begin{bmatrix} x_0(a-1) & x_1(a-1) & \ldots & x_{N-1}(a-1) \\ \nabla x_0(a-1) & \nabla x_1(a-1) & \ldots & \nabla x_{N-1}(a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{N-2} x_0(a-1) & \nabla^{N-2} x_1(a-1) & \ldots & \nabla^{N-2} x_{N-1}(a-1) \\ \nabla^{N-1} x_0(b) & \nabla^{N-1} x_1(b) & \ldots & \nabla^{N-1} x_{N-1}(b) \end{bmatrix}.$$

If $det(M) \neq 0$, then the system of equations has only the trivial solution.

For $0 \le i \le N-1$ and $0 \le j \le N-2$, it holds that

$$\nabla^j x_i(a-1) = \nabla^j H_i(t,a)|_{t=a-1} = H_{i-j}(a-1,a).$$

Now, for $i = j$, $H_{i-j}(a-1,a) = H_0(a-1,a) = 1$. Also $\nabla^{N-1} x_{N-1}(b) = H_0(b,a) = 1$. Thus M has 1's on the diagonal. For $i = j+1$,

$$H_{i-j}(a-1,a) = H_{(j+1)-j}(a-1,a) = \frac{(-1)^{\overline{1}}}{\Gamma(2)} = -1.$$

This implies that M has -1's on the superdiagonal.

Next, if $0 \le i < j$ and $0 \le j \le N-2$, then $H_{i-j}(a-1,a) = 0$, since $i - j < 0$. If $j+1 \le i \le N-1$ for $0 \le j \le N-2$, then $H_{i-j}(a-1,a) = \frac{(-1)^{\overline{i-j}}}{\Gamma(i-j+1)} = 0$. Finally, if

$0 \leq i \leq N-2$, then $\nabla^{N-1}x_i(b) = H_{i-N+1}(b,a) = H_{i-N+1}(b,a) = 0$. Thus

$$M = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

so $det(M) = 1$. Then the above system has a unique solution, so homogeneous BVP (3.3) has only the trivial solution.

Next, by Theorem 3.5, the general solution to $\nabla^{\nu}_{a^*}x(t) = h(t)$ is given by

$$\begin{aligned} y(t) &= c_0x_0(t) + \cdots + c_{N-1}x_{N-1}(t) + y_0(t) \\ &= c_0 + c_1H_1(t,a) + \cdots + c_{N-1}H_{N-1}(t,a) + y_0(t), \end{aligned}$$

where $c_0, \dots, c_{N-1} \in \mathbb{R}$ and $y_0 : \mathbb{N}_{a-N+1} \to \mathbb{R}$ is a particular solution to $\nabla^{\nu}_{a^*}y_0(t) = h(t)$. Then $y(t)$ solves (3.2) if and only if

$$\begin{cases} c_0\nabla^k x_0(a-1) + \cdots + c_{N-1}\nabla^k x_{N-1}(a-1) + \nabla^k y_0(a-1) = A_k, \\ \hfill 0 \leq k \leq N-2 \\ c_0\nabla^{N-1}x_0(b) + \cdots + c_{N-1}\nabla^{N-1}x_{N-1}(b) + \nabla^{N-1}y_0(b) = B \end{cases}$$

has a unique solution if and only if

$$\begin{cases} c_0\nabla^k x_0(a-1)+\cdots+c_{N-1}\nabla^k x_{N-1}(a-1)=A_k-\nabla^k y_0(a-1), \\ \hfill 0\le k\le N-2 \\ c_0\nabla^{N-1}x_0(b)+\cdots+c_{N-1}\nabla^{N-1}x_{N-1}(b)=B-\nabla^{N-1}y_0(b) \end{cases}$$

has a unique solution. Thus $y(t)$ uniquely satisfies the boundary conditions in (3.2) if and only if $\det(M)\neq 0$. Therefore, since $\det(M)=1$, the BVP (3.2) has a unique solution. □

3.3 The Green's Function

In this section, the Green's function for the BVP (3.3) will be defined. Then in Theorem 3.11, three bounds will be established for the Green's function in the case where $1<\nu\le 2$ that will be key in considering the nonlinear case in Section 3.4.

3.3.1 Definition and Basic Properties of the Green's Function

Note here that the proofs through Theorem 3.9 use arguments analogous to the derivation of a different Green's function in [3].

Definition 3.7 (Green's Function)**.** Let $x(t,s)$ be the Cauchy function for $\nabla^\nu_{a^*}x(t)=0$. The Green's function, $G(t,s):\mathbb{N}^b_{a-N+1}\times\mathbb{N}^b_{a+1}\to\mathbb{R}$, for the BVP (3.3) is given by

$$G(t,s):=\begin{cases} u(t,s), & t\le\rho(s) \\ v(t,s), & \rho(s)\le t \end{cases} \tag{3.4}$$

where $u(t,s)$ is defined to be the unique solution of the BVP

$$\begin{cases} \nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k u(a-1,s) = 0, & 0 \le k \le N-2 \\ \nabla^{N-1} u(b,s) = -\nabla^{N-1} x(b,s) \end{cases}$$

for each fixed $s \in \mathbb{N}_{a+1}^{b}$, and

$$v(t,s) := u(t,s) + x(t,s).$$

Theorem 3.8 below shows that the Green's function, among other things, is useful in calculating solutions to the BVP (3.3).

Theorem 3.8. *Let $G(t,s)$ be the Green's function for the BVP (3.3), and assume $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$. Then the unique solution to*

$$\begin{cases} \nabla_{a^*}^{\nu} y(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k y(a-1) = 0, & 0 \le k \le N-2 \\ \nabla^{N-1} y(b) = 0, \end{cases} \tag{3.5}$$

is given by

$$y(t) = \int_a^b G(t,s)h(s)\nabla s, \quad t \in \mathbb{N}_{a-N+1}^{b}.$$

Proof. Let $y(t) := \int_a^b G(t,s)h(s)\nabla s$ for $t \in \mathbb{N}_{a-N+1}^{b}$. Then, for $t \in \mathbb{N}_{a-N+1}^{b}$,

$$\begin{aligned} y(t) &= \int_a^b G(t,s)h(s)\nabla s \\ &= \int_a^{t+1} v(t,s)h(s)\nabla s + \int_{t+1}^{b} u(t,s)h(s)\nabla s \end{aligned}$$

$$
\begin{aligned}
&= \int_a^{t+1} [u(t,s) + x(t,s)]h(s)\nabla s + \int_{t+1}^b u(t,s)h(s)\nabla s \\
&= \int_a^b u(t,s)h(s)\nabla s + \int_a^{t+1} x(t,s)h(s)\nabla s \\
&= \int_a^b u(t,s)h(s)\nabla s + \int_a^t x(t,s)h(s)\nabla s + x(t,t+1)h(t+1) \\
&= \int_a^b u(t,s)h(s)\nabla s + \int_a^t x(t,s)h(s)\nabla s + H_{\nu-1}(t,\rho(t+1))h(t+1) \\
&= \int_a^b u(t,s)h(s)\nabla s + \int_a^t x(t,s)h(s)\nabla s.
\end{aligned}
$$

Define $z(t) := \int_a^t x(t,s)h(s)\nabla s$. By the variation of constants formula in Theorem 3.3, $z(t)$ solves the IVP

$$
\begin{cases}
\nabla_{a^*}^{\nu} z(t) = h(t), & t \in \mathbb{N}_{a+1}^b \\
\nabla^k z(a) = 0, & 0 \le k \le N-1
\end{cases}
$$

for $t \in \mathbb{N}_{a-N+1}^b$. Thus

$$
\begin{aligned}
\nabla_{a^*}^{\nu} y(t) &= \nabla_{a^*}^{\nu} \left[\int_a^b u(t,s)h(s)\nabla s + z(t) \right] \\
&= \int_a^b \nabla_{a^*}^{\nu} u(t,s)h(s)\nabla s + \nabla_{a^*}^{\nu} z(t) \\
&= h(t)
\end{aligned}
$$

for $t \in \mathbb{N}_{a-N+1}^b$. It will now be shown that the boundary conditions hold. First, say $0 \le k \le N-2$. Then

$$
\begin{aligned}
\nabla^k y(a-1) &= \int_a^b \nabla^k u(a-1,s)h(s)\nabla s + \nabla^k z(a-1) \\
&= 0 + \int_a^{a-1} \nabla^k x(a-1,s)h(s)\nabla s
\end{aligned}
$$

$$= 0.$$

According to the Leibniz Formula in [31, Theorem 3.41 on p. 175], if $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$, then $\nabla \left(\int_a^t f(t,\tau)\nabla\tau \right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t)$. Hence,

$$\begin{aligned}
\nabla^{N-1} y(b) &= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \nabla^{N-1} z(b) \\
&= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \left[\nabla^{N-1} \int_a^t x(t,s)h(s)\nabla s \right]_{t=b} \\
&= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \left[\nabla^{N-2} \int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} \\
&\qquad + x(\rho(b),b)h(b) \\
&= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \left[\nabla^{N-2} \int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} \\
&\qquad + H_{\nu-1}(\rho(b),\rho(b))h(b)
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \left[\nabla^{N-2} \int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} \\
&\;\;\vdots \\
&= \int_a^b \nabla^{N-1} u(b,s)h(s)\nabla s + \int_a^b \nabla^{N-1} x(b,s)h(s)\nabla s \\
&= -\int_a^b \nabla^{N-1} x(b,s)h(s)\nabla s + \int_a^b \nabla^{N-1} x(b,s)h(s)\nabla s \\
&= 0.
\end{aligned}$$

Hence the boundary conditions hold. □

Theorem 3.9. *The Green's function* $G(t,s) : \mathbb{N}^b_{a-N+1} \times \mathbb{N}^b_{a+1} \to \mathbb{R}$ *for the homoge-*

neous BVP (3.3) is given by (3.4) where

$$u(t,s) := -H_{N-1}(t,a-1)H_{\nu-N}(b,\rho(s))$$

and

$$v(t,s) := -H_{N-1}(t,a-1)H_{\nu-N}(b,\rho(s)) + H_{\nu-1}(t,\rho(s)).$$

Proof. By Theorem 3.8, it suffices to show that, for each fixed $s \in \mathbb{N}_{a+1}^{b}$ when $t \leq \rho(s)$, $u(t,s)$ is the solution to the BVP

$$\begin{cases} \nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k u(a-1,s) = 0, & 0 \leq k \leq N-2 \\ \nabla^{N-1} u(b,s) = -\nabla^{N-1} x(b,s). & \end{cases}$$

Let $t \leq \rho(s)$. First,

$$\begin{aligned} \nabla_{a^*}^{\nu} u(t,s) &= \nabla_a^{-(N-\nu)} \nabla^N u(t,s) \\ &= \nabla_a^{-(N-\nu)} H_{\nu-N}(b,\rho(s)) \nabla - H_{N-1-(N-1)}(t,a-1) \\ &= \nabla_a^{-(N-\nu)} \nabla H_{\nu-N}(b,\rho(s)) \nabla 1 \\ &= \nabla_a^{-(N-\nu)} 0 \\ &= 0. \end{aligned}$$

Next, it will be shown that the boundary conditions hold. Let $0 \leq k \leq N-3$. Then

$$\begin{aligned} \nabla^k u(a-1,s) &= -H_{\nu-N}(b,\rho(s)) \cdot H_{N-k-1}(t,a-1)|_{t=a-1} \\ &= -H_{\nu-N}(b,\rho(s)) \cdot \frac{\Gamma(t-a+N-k)}{\Gamma(t-a+1)\Gamma(N-k)}|_{t=a-1} \end{aligned}$$

$$= -H_{\nu-N}(b,\rho(s)) \cdot \frac{\Gamma(N-k-1)}{\Gamma(0)\Gamma(N-k)}$$

$$= 0.$$

Also

$$\begin{aligned}\nabla^{N-2}u(a-1,s) &= -H_{\nu-N}(b,\rho(s)) \cdot H_{N-1-N+2}(t,a-1)|_{t=a-1} \\ &= -H_{\nu-N}(b,\rho(s)) \cdot \frac{t-a+1}{\Gamma(N-k)}|_{t=a-1} \\ &= 0.\end{aligned}$$

Next, note that $\nabla^{N-1}x(t,s) = \nabla^{N-1}H_{\nu-1}(t,\rho(s)) = H_{\nu-N}(t,\rho(s))$. Then,

$$\begin{aligned}\nabla^{N-1}u(b,s) &= -H_{\nu-N}(b,\rho(s)) \cdot H_{N-1-N+1}(t,a-1)|_{t=b} \\ &= -H_{\nu-N}(b,\rho(s)) \\ &= -\nabla^{N-1}x(b,s).\end{aligned}$$

□

Consider the case where $\nu = N \in \mathbb{N}$. Then

$$u(t,s) = H_{N-1}(t,a-1) = -\frac{(t-a+1)^{\overline{N-1}}}{\Gamma(N)},$$

and

$$\begin{aligned}v(t,s) &= H_{N-1}(t,a-1) + H_{N-1}(t,\rho(s)) \\ &= -\frac{(t-a+1)^{\overline{N-1}}}{\Gamma(N)} + \frac{(t-s+1)^{\overline{N-1}}}{\Gamma(N)}.\end{aligned}$$

Note that this aligns well with the Green's function for the continuous case with non-fractional derivatives. In the continuous case, according to [41, Example 6.30 on p. 296], the Green's function for the BVP

$$\begin{cases} x^{(n)} = 0, \\ x^{(i)}(a) = 0, \qquad 0 \le i \le n-2 \\ x^{(n-1)}(b) = 0. \end{cases}$$

is given by

$$G(t,s) := \begin{cases} -\frac{(t-a)^{n-1}}{(n-1)!}, & a \le t \le s \le b \\ -\frac{(t-a)^{n-1}}{(n-1)!} + \frac{(t-s)^{n-1}}{(n-1)!}, & a \le s \le t \le b. \end{cases}$$

In Example 3.10, Theorems 3.8 and 3.9 will be used to calculate the solution to a special case of the BVP (3.3).

Example 3.10. Say $\nu = \frac{15}{8}$, $a = 0$, $b = 6$, and $h(t) \equiv 1$. That is, consider the following BVP

$$\begin{cases} \nabla_{0^*}^{\frac{15}{8}} x(t) = 1, \quad t \in \mathbb{N}_1^6 \\ x(-1) = 0, \\ \nabla x(6) = 0. \end{cases} \tag{3.6}$$

By Theorem 3.9, the Green's function for the BVP (3.6) is given by

$$G(t,s) = \begin{cases} u(t,s), & t \le \rho(s) \\ v(t,s), & \rho(s) \le t \end{cases}$$

where

$$u(t,s) = -(t+1)H_{-\frac{1}{8}}(6,\rho(s))$$

and

$$v(t,s) = u(t,s) + x(t,s),$$

where $x(t,s) = H_{\frac{7}{8}}(t,\rho(s))$. By Theorem 3.8 and by its proof, the solution for $t \in \mathbb{N}_{-1}^{6}$ to the BVP (3.6) is given by

$$\begin{aligned}
x(t) &= \int_0^6 G(t,s)h(s)\nabla s \\
&= \int_0^6 u(t,s)h(s)\nabla s + \int_0^t x(t,s)h(s)\nabla s \\
&= -(t+1)\int_0^6 H_{-\frac{1}{8}}(6,\rho(s))h(s)\nabla s + \int_0^t H_{\frac{7}{8}}(t,\rho(s))h(s)\nabla s \\
&= (t+1)\left[H_{\frac{7}{8}}(6,s)\right]_{s=0}^{s=6} - \left[H_{\frac{15}{8}}(t,s)\right]_{s=0}^{s=6} \\
&= -(t+1)H_{\frac{7}{8}}(6,0) - H_{\frac{15}{8}}(t,6) + H_{\frac{15}{8}}(t,0).
\end{aligned}$$

Note that by rewriting the above solution as

$$x(t) = \frac{\Gamma\left(6+\frac{7}{8}\right)}{\Gamma(6)\Gamma\left(\frac{15}{8}\right)} - \frac{\Gamma\left(t-6+\frac{15}{8}\right)}{\Gamma(t-6)\Gamma\left(\frac{23}{8}\right)} + \frac{\Gamma\left(t+\frac{15}{8}\right)}{\Gamma(t)\Gamma\left(\frac{23}{8}\right)},$$

for $t \in \mathbb{N}_{-1}^{6}$, we may provide a graphical solution to the BVP (3.6), as shown in Figure 3.1.

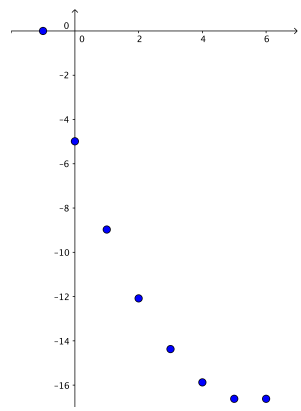

Figure 3.1: A solution to the homogenous BVP (3.6).

3.3.2 Bounds on the Green's Function

We will now establish bounds on $G(t,s)$ with the goal of eventually approaching the nonlinear case using this Green's function.

Consider $G(t,s)$ for the BVP (3.3). If $t = \rho(s)$, it holds that the Cauchy function $H_{\nu-1}(t,\rho(s)) = 0$ so $u(t,s) = v(t,s)$.
Additionally, for $t \in \mathbb{N}_a$, define $\lceil \cdot \rceil_a : \mathbb{R} \to \mathbb{N}_a$ such that

$$\lceil t \rceil_a = \lceil t - a \rceil + a, \quad \text{for } t \geq a$$

and such that $\lceil t \rceil_a = a$ for $t < a$. Then, to be used in Theorem 3.11, define the constant

$$r := \left\lceil b - \frac{b-(a+1)}{4} \right\rceil_a .$$

Note that, in order to establish the bound in part (iii) of Theorem 3.11, the proof requires a restriction on the domain of solutions to the BVP (3.3). In particular, it requires that $1 \leq b-a \leq \frac{1}{2-\nu}-1$. Note that $1 \leq \frac{1}{2-\nu}-1$, which implies that $\nu \geq \frac{3}{2}$. Hence we get the restriction mentioned in part (iii) of Theorem 3.11 and in Section 3.4.

Theorem 3.11. *Let $1<\nu<2$ and $1 \leq b-a \leq \frac{1}{2-\nu}$. Let $G(t,s) : \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ be the Green's function for the homogeneous BVP (3.3). Then the following hold:*

(i) $G(t,s) \leq 0$ for $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$, and

(ii) $\min_{t \in \mathbb{N}_{a-1}^{b}} G(t,s) = G(\rho(s),s)$ for each fixed $s \in \mathbb{N}_{a+1}^{b}$.

If we further assume $1 \leq b-a \leq \frac{1}{2-\nu}-1$ and thus that $\frac{3}{2} \leq \nu \leq 2$, then

(iii) $G(t,s) \leq k \cdot G(\rho(s),s)$, where k is a constant satisfying $0<k \leq 1$ for $t \in \mathbb{N}_{r}^{b}$ and $s \in \mathbb{N}_{a+1}^{b}$.

Proof. (i) Note that $N=2$. Let $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$ and $t \leq \rho(s)$. Then

$$\begin{aligned}
G(t,s) &= u(t,s) \\
&= -H_1(t,a-1)H_{\nu-2}(b,\rho(s)) \\
&= -\frac{(t-a+1)(b-s+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\
&= -\frac{(t-a+1)\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu-1)} \\
&\leq 0
\end{aligned}$$

Next let $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$ and $\rho(s) \leq t$. Then

$$\begin{aligned}
G(t,s) &= v(t,s) \\
&= -H_1(t,a-1)H_{\nu-2}(b,\rho(s)) + H_{\nu-1}(t,\rho(s)).
\end{aligned}$$

Now it will be shown that $v(t,s)$ is increasing with respect to t where $t \in \mathbb{N}_{a-1}^{b}$ and $\rho(s) \leq t$ for each fixed $s \in \mathbb{N}_{a+1}^{b}$. Note that for $\nu = 2$,

$$\nabla_t H_{\nu-2}(t,\rho(s)) = \nabla_t H_0(t,\rho(s)) = \nabla_t 1 = 0.$$

For $1 < \nu < 2$,

$$\begin{aligned}\nabla_t H_{\nu-2}(t,\rho(s)) &= H_{\nu-3}(t,\rho(s)) \\ &= \frac{(t-s+1)^{\overline{\nu-3}}}{\Gamma(\nu-2)} \\ &= \frac{\Gamma(t-s+\nu-2)}{\Gamma(t-s+1)\Gamma(\nu-2)} \\ &\leq 0,\end{aligned}$$

provided $t \neq s$, since $\Gamma(\nu-2) \leq 0$. Hence

$$\nabla_t v(t,s) = -H_{\nu-2}(b,\rho(s)) + H_{\nu-2}(t,\rho(s)) \geq 0$$

for $t \neq s$. If $t = s = b$, then

$$\begin{aligned}\nabla_t v(b,b) &= -H_{\nu-2}(b,\rho(b)) + H_{\nu-2}(b,\rho(b)) \\ &= 0.\end{aligned}$$

Also, in general if $t = s$,

$$\begin{aligned}\nabla_t v(t,t) &= -H_{\nu-2}(b,\rho(t)) + H_{\nu-2}(t,\rho(t)) \\ &= -\frac{(b-t+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} + \frac{1^{\overline{\nu-2}}}{\Gamma(\nu-1)}\end{aligned}$$

$$= -\frac{\Gamma(b-t+\nu-1)}{\Gamma(b-t+1)\Gamma(\nu-1)} + 1$$
$$\geq 0$$

since $0 < \nu - 1 \leq 1$. Hence $\nabla_t v(t,s) \geq 0$ for $t \geq s$ for every fixed $s \in \mathbb{N}_{a-1}^b$. Thus $v(t,s)$ is increasing with respect to t where $t \in \mathbb{N}_{a-1}^b$ and $\rho(s) \leq t$ for each fixed $s \in \mathbb{N}_{a-1}^b$.

Next let $(t,s) \in \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b$ and $t \geq \rho(s)$. Then

$$\begin{aligned}
v(b,s) &= -H_1(b,a-1)H_{\nu-2}(b,\rho(s)) + H_{\nu-1}(b,\rho(s)) \\
&= -\frac{(b-a+1)(b-s+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} + \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= -\frac{(b-a+1)\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu-1)} + \frac{\Gamma(b-s+\nu)}{\Gamma(b-s+1)\Gamma(\nu)} \\
&= -\frac{(b-a+1)\Gamma(b-s+\nu-1)(\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)} + \frac{(b-s+\nu-1)\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)} \\
&= \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)}\left[-(b-a+1)(\nu-1) + (b-s) + \nu - 1\right] \\
&\leq \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)}\left[-(b-a+1)(\nu-1) + (b-a) + \nu - 2\right] \\
&= \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)}\left[(b-a)(1-\nu) - (\nu-1) + (b-a) + \nu - 2\right] \\
&= \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)}\left[(b-a)(2-\nu) - 1\right] \\
&\leq \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu)}\left[\frac{1}{2-\nu}(2-\nu) - 1\right] \\
&= 0.
\end{aligned}$$

Therefore $v(t,s) \leq 0$.

(ii) Let $t \leq \rho(s)$ and $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$. Then

$$\begin{aligned}\nabla_t u(t,s) &= -\frac{(b-s+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\ &= -\frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu-1)} \\ &\leq 0.\end{aligned}$$

Note that by the proof of (i) it also holds that $\nabla_t v(t,s) \geq 0$ for $\rho(s) \leq t$ where $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$. Hence $\min_{t \in \mathbb{N}_{a-1}^{b}} G(t,s) = G(\rho(s),s)$ for each fixed $s \in \mathbb{N}_{a+1}^{b}$.

(iii.) Let for $t \in \mathbb{N}_{r}^{b}$, $s \in \mathbb{N}_{a+1}^{b}$, $1 \leq b-a \leq \frac{1}{2-\nu} - 1$. Define

$$k := \min\left\{\frac{3(b-a)+5}{4(b-a)}, \frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right]\right\}.$$

It will first be shown that $\frac{G(t,s)}{G(\rho(s),s)} \geq k$.

Let $t \geq \rho(s)$. Then

$$\begin{aligned}\frac{G(t,s)}{G(\rho(s),s)} &= \frac{-H_1(t,a-1)H_{\nu-2}(b,\rho(s)) + H_{\nu-1}(t,\rho(s))}{-H_1(\rho(s),a-1)H_{\nu-2}(b,\rho(s)) + H_{\nu-1}(\rho(s),\rho(s))} \\ &= \frac{(t-a+1)H_{\nu-2}(b,\rho(s)) - H_{\nu-1}(t,\rho(s))}{(s-a)H_{\nu-2}(b,\rho(s))} \\ &= \frac{1}{s-a}\left[t-a+1-\frac{H_{\nu-1}(t,\rho(s))}{H_{\nu-2}(b,\rho(s))}\right].\end{aligned}$$

Note

$$\nabla_s\left[\frac{H_{\nu-1}(t,\rho(s))}{H_{\nu-2}(b,\rho(s))}\right] = \frac{H_{\nu-2}(b,\rho(s))[-H_{\nu-2}(t,s-2)] - H_{\nu-1}(t,\rho(s))[-H_{\nu-3}(b,s-2)]}{H_{\nu-2}(b,\rho(s))H_{\nu-2}(b,s-2)}.$$

Then, considering the numerator of the above expression,

$$\begin{aligned}
&H_{\nu-2}(b,\rho(s))[-H_{\nu-2}(t,s-2)] - H_{\nu-1}(t,\rho(s))[-H_{\nu-3}(b,s-2)] \qquad (3.7)\\
&= -\frac{(b-s+1)^{\overline{\nu-2}}(t-s+2)^{\overline{\nu-2}}}{\Gamma(\nu-1)\Gamma(\nu-1)} + \frac{(t-s+1)^{\overline{\nu-1}}(b-s+2)^{\overline{\nu-3}}}{\Gamma(\nu)\Gamma(\nu-2)}\\
&= -\frac{\Gamma(b-s+\nu-1)\Gamma(t-s+\nu)}{(\Gamma(\nu-1))^2\Gamma(b-s+1)\Gamma(t-s+2)} + \frac{\Gamma(t-s+\nu)\Gamma(b-s+\nu-1)}{\Gamma(\nu)\Gamma(\nu-2)\Gamma(t-s+1)\Gamma(b-s+2)}\\
&= \frac{\Gamma(b-s+\nu-1)\Gamma(t-s+\nu)}{\Gamma(b-s+1)\Gamma(t-s+1)\Gamma(\nu-1)\Gamma(\nu-2)}\left[\frac{-1}{(t-s+1)(\nu-2)} + \frac{1}{(b-s+1)(\nu-1)}\right]\\
&\leq 0,
\end{aligned}$$

since, provided $\nu \neq 2$, $-1 < \nu - 2 < 0$ and $\Gamma(\nu-2) \leq 0$. It can be shown that if $\nu = 2$, the expression in (3.7) is nonpositive. Additionally

$$\begin{aligned}
\nabla_t\left[(t-a+1)H_{\nu-2}(b,\rho(s)) - H_{\nu-1}(t,\rho(s))\right] &= H_{\nu-2}(b,\rho(s)) - H_{\nu-2}(t,\rho(s))\\
&\leq 0
\end{aligned}$$

by the proof of part (i). Hence,

$$\begin{aligned}
\frac{1}{s-a}\left[t-a+1-\frac{H_{\nu-1}(t,\rho(s))}{H_{\nu-2}(b,\rho(s))}\right] &\geq \frac{1}{b-a}\left[t-a+1-\frac{H_{\nu-1}(t,\rho(a+1))}{H_{\nu-2}(b,\rho(a+1))}\right]\\
&= \frac{1}{b-a}\left[t-a+1-\frac{(t-a)^{\overline{\nu-1}}}{(\nu-1)(b-a)^{\overline{\nu-2}}}\right]\\
&\geq \frac{1}{b-a}\left[b-a+1-\frac{(b-a)^{\overline{\nu-1}}}{(\nu-1)(b-a)^{\overline{\nu-2}}}\right]\\
&= \frac{1}{b-a}\Bigg[b-a+1\\
&\quad -\frac{\Gamma(b-a+\nu-1)\Gamma(b-a)}{(\nu-1)\Gamma(b-a+\nu-2)\Gamma(b-a)}\Bigg]\\
&= \frac{1}{b-a}\left[b-a+1-\frac{b-a+\nu-2}{\nu-1}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b-a}\left[\frac{(b-a+1)(\nu-1)-(b-a+\nu-2)}{\nu-1}\right] \\
&= \frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right] \\
&\geq k.
\end{aligned}$$

Let $t \leq \rho(s)$. Because $t \in \mathbb{N}_r^b$,

$$\begin{aligned}
t &\geq \left\lceil b - \frac{b-(a+1)}{4} \right\rceil_a \\
&= \left\lceil b - \frac{b-(a+1)}{4} - a \right\rceil + a \\
&\geq b - \frac{b-(a+1)}{4} - a + a \\
&= b - \frac{b-(a+1)}{4}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{G(t,s)}{G(\rho(s),s)} &= \frac{-H_1(t,a-1)H_{\nu-2}(b,\rho(s))}{-H_1(\rho(s),a-1)H_{\nu-2}(b,\rho(s))} \\
&= \frac{t-a+1}{s-a} \\
&\geq \frac{b-\frac{b-(a+1)}{4}-a+1}{b-a} \\
&= \frac{3(b-a)+5}{4(b-a)} \\
&\geq k.
\end{aligned}$$

Next it will be shown that $0 < k \leq 1$. First, for $\frac{3}{2} \leq \nu \lneq 2$,

$$\frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right] \geq \frac{1}{b-a}\left[\frac{\left(\frac{1}{2-\nu}-1\right)(\nu-2)+1}{\nu-1}\right]$$

$$= \frac{1}{b-a}\left[\frac{\frac{\nu-1}{2-\nu}(\nu-2)+1}{\nu-1}\right]$$
$$= \frac{1}{b-a}\left[\frac{2-\nu}{\nu-1}\right]$$
$$\geq 0.$$

For $\nu = 2$, note

$$\frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right] = \frac{1}{b-a} > 0.$$

Also,

$$\begin{aligned}\frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right] &\leq \frac{1}{b-a}\left[\frac{1}{\nu-1}\right]\\ &\leq \frac{1}{b-a}\left[\frac{1}{\left(\frac{1}{2}\right)}\right]\\ &\leq 1.\end{aligned}$$

Note that if $1 \leq b-a \leq 4$, then $\mathbb{N}_r^b$ contains only the element b. Thus the cases where $1 \leq b-a \leq 4$ are contained within the case where $\rho(s) \leq t$. If $b-a \geq 5$, then, for the $t \leq \rho(s)$ case, $\frac{3(b-a)+5}{4(b-a)} \leq 1$, and certainly $\frac{3(b-a)+5}{4(b-a)} \geq 0$. $\square$

Note that Theorem 3.11 also holds for $\nu = 2$. However, the bound $1 \leq b-a \leq \frac{1}{2-\nu}$ is not necessary for parts (i) and (ii) to hold in this case. Additionally part (iii) holds without the bound $1 \leq b-a \leq \frac{1}{2-\nu} - 1$.

Next we give some examples to illustrate our results.

Example 3.12. Consider the boundary value problem

$$\begin{cases} \nabla_{0^*}^{\frac{15}{8}} x(t) = h(t), \quad t \in \mathbb{N}_1^8 \\ x(-1) = 0 \\ \nabla x(8) = 0 \end{cases}$$

That is, $\nu = \frac{15}{8}$, $b = 8$, and $a = 0$. Note that

$$\frac{1}{2-\nu} = \frac{1}{2-\frac{15}{8}} = 8.$$

Hence, Theorem 3.11, part (i) and (ii) applies to this example. Set $s = 5$. Then, for $-1 \leq t \leq 4$,

$$\begin{aligned} G(t,5) &= u(t,5) \\ &= -H_1(t, a-1) \cdot H_{-\frac{1}{8}}(b, \rho(s)) \\ &= -\frac{(t-a+1)\Gamma\left(b-s+\frac{7}{8}\right)}{\Gamma(b-s+1)\Gamma\left(\frac{7}{8}\right)} \\ &= -\frac{(t+1)\Gamma\left(3+\frac{7}{8}\right)}{\Gamma(4)\Gamma\left(\frac{7}{8}\right)} \\ &= -\frac{(t+1)\left(2+\frac{7}{8}\right)\left(1+\frac{7}{8}\right)\left(\frac{7}{8}\right)}{6} \\ &= -\frac{805}{1024}(t+1). \end{aligned}$$

Also, for $4 \leq t \leq 8$,

$$\begin{aligned} G(t,5) &= v(t,5) \\ &= -\frac{805}{1024}(t+1) + H_{\frac{7}{8}}(t, \rho(s)) \end{aligned}$$

$$= -\frac{805}{1024}(t+1) + \frac{(t-4)^{\overline{\frac{7}{8}}}}{\Gamma\left(\frac{15}{8}\right)}$$
$$= -\frac{805}{1024}(t+1) + \frac{\Gamma\left(t-4+\frac{7}{8}\right)}{\Gamma(t-4)\Gamma\left(\frac{15}{8}\right)}.$$

Note that in Figure 3.2, $G(t,5) \leq 0$ for $1 \leq t \leq 8$ and $\min_{t \in \{1,2,\cdots,8\}} G(t,5) = G(4,5)$.

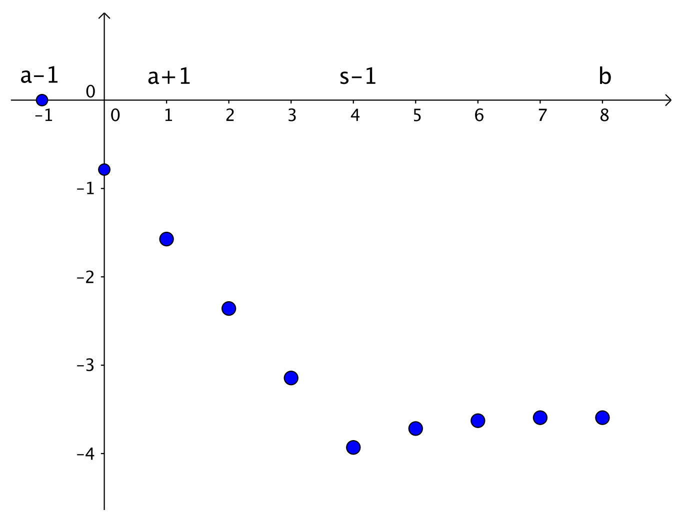

Figure 3.2: A graph of $G(t,5)$ for $1 \leq t \leq 8$.

In Example 3.13, we increase the domain in Example 3.12 by 1 so that the bound in Theorem 3.11 no longer holds. That is, we find in this instance that the Green's function already no longer satisfies $G(t,s) \leq 0$ for all $(t,s) \in \mathbb{N}_{a-1} \times \mathbb{N}_{a+1}$.

Example 3.13. Now consider the boundary value problem

$$\begin{cases} \nabla_{0^*}^{\frac{15}{8}} x(t) = h(t), \quad t \in \mathbb{N}_1^9 \\ x(-1) = 0 \\ \nabla x(9) = 0 \end{cases}$$

That is $\nu = \frac{15}{8}$, $b = 9$, and $a = 0$. Set $s = 1$. Then if $t \in \mathbb{N}_1^9$,

$$G(t,1) = v(t,1) = -\frac{(t-a+1)\Gamma\left(b-s+\frac{7}{8}\right)}{\Gamma(b-s+1)\Gamma\left(\frac{7}{8}\right)} + \frac{\Gamma\left(t-\rho(s)+\frac{7}{8}\right)}{\Gamma(t-\rho(s))\Gamma\left(\frac{15}{8}\right)}$$

Then,

$$\begin{aligned} v(8,1) &= -\frac{(9)\Gamma\left(8+\frac{7}{8}\right)}{\Gamma(9)\Gamma\left(\frac{7}{8}\right)} + \frac{\Gamma\left(8+\frac{7}{8}\right)}{\Gamma(8)\Gamma\left(\frac{15}{8}\right)} \\ &= \frac{215643285}{2147483648} \\ &\approx 0.01 \geq 0. \end{aligned}$$

Note that in this example, increasing the value of $b-a$ by 1 past the region to which Theorem 3.11, part (i) applies causes $G(t,s) \leq 0$ to no longer be true.

3.4 The Nonlinear Case

Consider the following nonlinear boundary value problem for $\frac{3}{2} \leq \nu \leq 2$

$$\begin{cases} -\nabla_{a^*}^{\nu} x(t) = h(t, x(t-1)), \quad t \in \mathbb{N}_{a+1}^b \\ x(a-1) = 0, \\ \nabla x(b) = 0, \end{cases} \tag{3.8}$$

where $h : \mathbb{N}_{a+1}^b \times \mathbb{R}^+ \to \mathbb{R}^+$, $b-a \in \mathbb{Z}$, $\dfrac{1}{2-\nu} - 1 \geq b-a \geq 1$, and where the solutions x are defined on $\mathbb{N}_{a-1}^b$. Write $r := \left\lceil b - \frac{b-(a+1)}{4} \right\rceil_a$.

In this section, we will use Theorem 3.11 and Theorem 3.15 (Guo-Krasnosel′skiĭ Theorem) to find solutions to the BVP (3.8). In the approach to this problem, arguments analogous to those used in [27, Section 3], where Erbe and Peterson found positive solutions to a boundary value problem involving whole-order derivatives in a time scales context, are used below. Define the Banach space

$$\mathbb{E} := \{x : \mathbb{N}_{a-1}^{b} \to \mathbb{R} : x(a-1) = 0 \text{ and } \nabla x(b) = 0\}$$

with norm

$$||x|| := \max\{|x(t)|, \quad t \in \mathbb{N}_{a-1}^{b}\}.$$

That is, all elements of $\mathbb{E}$ satisfy the boundary conditions in BVP (3.8). Define the cone

$$K := \{x \in \mathbb{E} : \text{both } x(t) \geq 0 \text{ for } t \in \mathbb{N}_{a-1}^{b} \text{ and } x(t) \geq k||x|| \text{ for } t \in \mathbb{N}_{r}^{b}\},$$

where $0 < k \leq 1$ is constant as defined in Theorem 3.11 part (iii). The general definition of a cone is given in Definition 3.14. Define the operator A by

$$Ax(t) = \int_{a}^{b} G(t,s)h(s,x(s-1))\nabla s = \sum_{s=a+1}^{b} G(t,s)h(s,x(s-1))\nabla s$$

for $t \in \mathbb{N}_{a-1}^{b}$ and $x \in K$. By Theorem 3.11 part (i), $G(t,s) \geq 0$ for $(t,s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$, and $h : \mathbb{N}_{a+1}^{b} \times \mathbb{R}^{+} \to \mathbb{R}^{+}$. Hence $Ax(t) \geq 0$ for $t \in \mathbb{N}_{a-1}^{b}$. Also by Theorem 3.11 part (ii),

$$\begin{aligned}
\min_{t \in \mathbb{N}_{r}^{b}} Ax(t) &\geq \int_{a}^{b} \min_{t \in \mathbb{N}_{r}^{b}} G(t,s)h(s,x(s-1))\nabla s \\
&\geq k \int_{a}^{b} G(\rho(s),s)h(s,x(s-1))\nabla s \\
&\geq k \int_{a}^{b} \max_{t \in \mathbb{N}_{a-1}^{b}} G(t,s)h(s,x(s-1))\nabla s \\
&\geq k \max_{t \in \mathbb{N}_{a-1}^{b}} \int_{a}^{b} G(t,s)h(s,x(s-1))\nabla s
\end{aligned}$$

$$= k||Ax||.$$

Thus it holds that $Ax \in K$, so $A : K \to K$. In the theorems that follow, we will be finding fixed points of the operator A, since fixed points give solutions to the BVP (3.8). The general definition of a cone from [38] is given below.

Definition 3.14. Let $\mathbb{E}$ be a Banach space. A nonempty closed convex set $C \subset \mathbb{E}$ is called a cone if it satisfies the following two conditions:

(i.) $x \in C$, $\lambda \geq 0$ implies $\lambda x \in C$;

(ii.) $x \in C$, $-x \in C$ implies $x = 0$.

where 0 denotes the identity element of $\mathbb{E}$.

Next we will be making use of a theorem from Krasnosel′skiĭ [42] and Deimling [26] stated below. The larger context of the theorem in cone theory and its applications to nonlinear problems is more fully elaborated in the text by Guo and Lakshmikantham [35].

Theorem 3.15 (Guo-Krasnosel′skiĭ)**.** *Let $\mathbb{E}$ be a Banach space and let $P \subset \mathbb{E}$ be a cone. Assume Ω_1, Ω_2 are open subsets of $\mathbb{E}$ with $\overline{\Omega_1} \subset \Omega_2$ and $0 \in \Omega_1$, and assume that*

(i.) $||Ax|| \leq ||x||$, $x \in P \cap \partial\Omega_1$, *and* $||Ax|| \geq ||x||$, $x \in P \cap \partial\Omega_2$, *or*

(ii.) $||Ax|| \geq ||x||$, $x \in P \cap \partial\Omega_1$, *and* $||Ax|| \leq ||x||$, $x \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3.4.1 Existence of Positive Solutions

In Theorems 3.16 and Theorem 3.17, we will be making use of Theorem 3.15 to find fixed points of A and thus to show the existence of positive solutions. Define

$$m := \frac{1}{\int_a^b G(\rho(s), s)\nabla s} \quad \text{, and}$$
$$w := \frac{1}{k\int_r^b G(t_0, s)\nabla s},$$

where $t_0 \in \mathbb{N}_r^b$ is fixed. Note that $w \geq m$, since

$$k\int_r^b G(t_0, s)\nabla s \leq \int_r^b G(t_0, s)\nabla s \leq \int_r^b G(\rho(s), s)\nabla s \leq \int_a^b G(\rho(s), s)\nabla s.$$

In the following, let $B_r(0) \subset \mathbb{E}$ denote an open ball of radius r about the origin in $\mathbb{E}$.

For all theorems in this section assume that $h : \mathbb{N}_{a-1}^b \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Theorem 3.16. *Consider the following hypotheses:*

(i.) $h(t, x) \leq mp_1$ *for all* $t \in \mathbb{N}_{a-1}^b$ *and* $0 \leq x \leq p_1$*;*

(ii.) *if* $r < b$, $h(t, x) \geq wx$ *for all* $t \in \mathbb{N}_{r+1}^b$ *and* $kp_2 \leq x \leq p_2$*, or if* $r = b$, $h(t, x) \geq wx$ *for* $t = b$ *and* $kp_2 \leq x \leq p_2$*;*

(iii.) $kp_2 \geq p_1$ *whenever* $p_1 < p_2$ *and* $w \neq m$*; and*

(iv.) $wp_2 < mp_1$ *whenever* $p_2 < p_1$.

If there exist $p_1, p_2 \in (0, \infty)$ *such that either (i), (ii), and (iii) hold and* $p_1 < p_2$ *OR (i), (ii), and (iv) hold and* $p_2 < p_1$*, then the boundary value problem (3.8) has a positive solution.*

Proof. Suppose first that $p_1 < p_2$. Then $p_1 \leq kp_2$, so $h(t, x(t-1))$ makes sense. Define $\Omega_1 := B_{p_1}(0) \subset \mathbb{E}$ and $\Omega_2 := B_{p_2}(0) \subset \mathbb{E}$. Say x is such that $||x|| = p_1$, $x \geq 0$, and $x \geq kp_1$ on $t \in \mathbb{N}_r^b$. Then $x \in K \cap \partial\Omega_1$. Hence,

$$\begin{aligned}
Ax(t) &= \int_a^b G(t,s)h(s, x(s-1))\nabla s \\
&\leq \int_a^b G(\rho(s), s)h(s, x(s-1))\nabla s \\
&\leq mp_1 \int_a^b G(\rho(s), s)\nabla s \\
&= p_1.
\end{aligned}$$

Then $||Ax|| \leq p_1 = ||x||$. Next say x is such that $||x|| = p_2$, $x \geq 0$, and $x \geq kp_2$ on $t \in \mathbb{N}_r^b$. Then $x \in K \cap \partial\Omega_2$. Hence,

$$\begin{aligned}
Ax(t_0) &= \int_a^b G(t_0,s)h(s, x(s-1))\nabla s \\
&\geq \int_r^b G(t_0,s)h(s, x(s-1))\nabla s \\
&\geq w \int_r^b G(t_0,s)x(s-1)\nabla s \\
&\geq wk||x|| \int_r^b G(t_0,s)\nabla s \\
&= ||x||
\end{aligned}$$

so since $Ax(t_0) \leq ||Ax||$, it holds that $||Ax|| \geq ||x||$. Therefore the boundary value problem (3.8) has a solution in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$. That is, the solution $x(t)$ is such that $p_1 \leq ||x|| \leq p_2$, $x \geq 0$ on $t \in \mathbb{N}_{a-1}^b$, and $x \geq kp_1$.

Now say $p_2 < p_1$. Then $wp_2 < mp_1$, so $wx \leq wp_2 < mp_1$ for $x \leq p_2$, so $h(t, x(t-1))$ makes sense. Define $\Omega_1 := B_{p_2}(0) \subset \mathbb{E}$ and $\Omega_2 := B_{p_1}(0) \subset \mathbb{E}$. If x is such that $||x|| = p_2$, $x \geq 0$, and $x(t) \geq kp_2$ on $t \in \mathbb{N}_r^b$, then $x \in K \cap \partial\Omega_1$, and by the above,

$||Ax|| \geq ||x||$. If x is such that $||x|| = p_1$, $x \geq 0$, and $x(t) \geq kp_1$ on $t \in \mathbb{N}_r^b$, then $x \in K \cap \partial\Omega_2$, and by the above, $||Ax|| \leq ||x||$. Then by the second part of the Guo-Krasnosel′skiĭ Theorem (Theorem 3.15), the boundary value problem has a solution $x(t)$ such that $p_2 \leq ||x|| \leq p_1$, $x \geq 0$ on $t \in \mathbb{N}_{a-1}^b$, and $x \geq kp_2$. □

Next, define

$$\lambda = \lambda(t) := \lim_{x \to 0^+} \frac{h(t,x)}{x}$$

and

$$\gamma = \gamma(t) := \lim_{x \to \infty} \frac{h(t,x)}{x}.$$

for $t \in \mathbb{N}_{a-1}^b$.

Theorem 3.17. *Assume λ and γ exist pointwise on the extended real numbers and furthermore that either*

(i) $\lambda = 0$ *and* $\gamma = \infty$

or

(ii) $\lambda = \infty$ *and* $\gamma = 0$.

Then the nonlinear boundary value problem (3.8) has a solution.

Proof. (i) Assume $\lambda = 0$ and $\gamma = \infty$.
Because $\lambda = 0$, we define $\delta > 0$ such that $0 < x \leq \delta$ implies, for all $t \in \mathbb{N}_{a-1}^b$,

$$\frac{h(t,x)}{x} \leq m.$$

Thus for $0 < x \leq \delta$, $h(t,x) \leq mx$, for all $t \in \mathbb{N}_{a-1}^{b}$. Define $\Omega_1 := B_\delta(0) \subset \mathbb{E}$ and say $x \in \Omega_1 \cap K$. Then for $t \in \mathbb{N}_{a-1}^{b}$,

$$\begin{aligned} Ax(t) &= \int_a^b G(t,s)h(s,x(s-1))\nabla s \\ &\leq m\int_a^b G(t,s)x(s-1)\nabla s \\ &\leq m\delta\int_a^b G(t,s)\nabla s \\ &\leq m\delta\int_a^b G(\rho(s),s)\nabla s \\ &= \delta. \end{aligned}$$

Since $||x|| = \delta$, $||Ax|| \leq ||x||$. Next, because $\gamma = \infty$, we define $\delta' > 0$ such that $x \geq \delta'$ implies, for all $t \in \mathbb{N}_{a-1}^{b}$,

$$\frac{h(t,x)}{x} \geq w.$$

Thus for $x \geq \delta'$, $h(t,x) \geq wx$, for all $t \in \mathbb{N}_{a-1}^{b}$. Then define

$$r := \max\{\delta', 2\delta\}$$

Define $\Omega_2 := B_r(0) \subset \mathbb{E}$ and say $x \in \Omega_2 \cap K$. Then

$$\begin{aligned} Ax(t_0) &= \int_a^b G(t_0,s)h(s,x(s-1))\nabla s \\ &\geq \int_r^b G(t_0,s)h(s,x(s-1))\nabla s \\ &\geq w\int_r^b G(t_0,s)x(s-1)\nabla s \\ &\geq wk||x||\int_r^b G(t_0,s)\nabla s \end{aligned}$$

$$= ||x||.$$

Since $Ax(t_0) \leq ||Ax||$, $||Ax|| \geq ||x||$. Therefore the nonlinear boundary value problem (3.8) has a solution in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

(ii) Assume $\lambda = \infty$ and $\gamma = 0$.

Because $\lambda = \infty$, define $\zeta > 0$ such that $0 < x \leq \zeta$ implies, for all $t \in \mathbb{N}_{a-1}^{b}$,

$$\frac{h(t,x)}{x} \geq w.$$

Thus for $0 < x \leq \zeta$, $h(t,x) \geq wx$. Define $\Omega_1 := B_\zeta(0) \subset \mathbb{E}$ and say $x \in \Omega_1 \cap K$. Then

$$\begin{aligned} Ax(t_0) &= \int_a^b G(t_0,s)h(s,x(s-1))\nabla s \\ &\geq \int_r^b G(t_0,s)h(s,x(s-1))\nabla s \\ &\geq w\int_r^b G(t_0,s)x(s-1)\nabla s \\ &\geq kw||x|| \int_r^b G(t_0,s)\nabla s \\ &= ||x||, \end{aligned}$$

so $||Ax|| \geq ||x||$.

Because $\gamma = 0$, define $\xi > 0$ such that $x \geq \xi$ implies, for all $t \in \mathbb{N}_{a-1}^{b}$,

$$\frac{h(t,x)}{x} \leq m.$$

Then for $x \geq \xi$, $h(t,x) \leq mx$ for all $t \in \mathbb{N}_{a-1}^{b}$. Define $\alpha := \max\{\xi, 2\zeta\}$, and define

$\Omega_2 := B_\alpha(0) \subset \mathbb{E}$. Then if $x \in \Omega_2 \cap K$,

$$\begin{aligned}
Ax(t) &= \int_a^b G(t,s)h(s,x(s-1))\nabla s \\
&\le m \int_a^b G(t,s)x(s-1)\nabla s \\
&\le m||x|| \int_a^b G(t,s)\nabla s \\
&\le m||x|| \int_a^b G(\rho(s),s)\nabla s \\
&= ||x||.
\end{aligned}$$

Thus $||Ax|| \le ||x||$. Therefore the nonlinear boundary value problem (3.8) has a solution in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$. □

3.4.2 Conditions for the Existence of Multiple Positive Solutions

Next note that Theorems 3.16 and Theorem 3.17, while showing the existence of positive solutions, did not guarantee uniqueness of solutions.

Theorem 3.18 below shows the possibility of not only two solutions but even countably many solutions when particular conditions on the function h are met.

Recall that, as in the previous two theorems, the constant k is as defined in Theorem 3.11 and is $0 < k \le 1$. Additionally recall that

$$\begin{aligned}
m :=& \frac{1}{\int_a^b G(\rho(s),s)\nabla s} \quad \text{, and} \\
w :=& \frac{1}{k\int_r^b G(t_0,s)\nabla s}
\end{aligned}$$

where $t_0 \in \mathbb{N}_r^b$ is fixed, and recall that $w \ge m$.

Theorem 3.18. *Suppose we have a sequence of positive real values such that*

$$0 < p_1 < kp_2 \leq p_2 < p_3 < kp_4 \leq p_4 < p_5 < kp_6 \leq p_6 < \cdots .$$

Consider the following hypotheses:

(i.) $h(t,x) \leq mp_1$ *for all* $t \in \mathbb{N}_{a-1}^b$ *and* $0 \leq x \leq p_1$*,*

(ii.) $h(t,x) \leq mp_i$ *for* i *odd for all* $t \in \mathbb{N}_{a-1}^b$ *and* $p_{i-1} + \varepsilon \leq x \leq p_i$*, where* $0 < \varepsilon < p_i - p_{i-1}$*,*

(iii.) $h(t,x) \geq wx$ *for* i *even for all* $t \in \mathbb{N}_{r+1}^b$ *or* $t = b$ *if* $r = b$ *and* $kp_i \leq x \leq p_i$*.*

Then there are countably many positive solutions to the BVP (3.8).

Proof. For each $n \in \mathbb{N}$, define $\Omega_n := B_{p_n}(0) \subset \mathbb{E}$. Then by Theorem 3.16, there exists a solution in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Assume $||x|| = p_i$ when i is odd. Then $h(t,x) \leq mp_i$. Also assume $x \geq 0$ and $x \geq kp_i$ on $t \in \mathbb{N}_r^b$. Then $x \in K \cap \partial\Omega_i$. Hence

$$\begin{aligned} Ax(t) &= \int_a^b G(t,s)h(s,x(s-1))\nabla s \\ &\leq \int_a^b G(\rho(s),s)h(s,x(s-1))\nabla s \\ &\leq mp_i \int_a^b G(\rho(s),s)\nabla s \\ &= p_i = ||x||. \end{aligned}$$

Hence $||Ax|| \leq ||x||$. Next assume $||x|| = p_{i+1}$. Then $h(t,x) \geq wx$. Also assume that $x \geq 0$ and $x \geq kp_{i+1}$ on $t \in \mathbb{N}_r^b$. Then $x \in K \cap \partial\Omega_{i+1}$.

$$Ax(t) = \int_a^b G(t,s)h(s,x(s-1))\nabla s$$

$$\geq \int_r^b G(t_0, s)h(s, x(s-1))\nabla s$$
$$\geq w \int_r^b G(t_0, s)x(s-1)\nabla s$$
$$\geq k||x||w \int_r^b G(t_0, s)\nabla s$$
$$= ||x||.$$

Hence $||Ax|| \geq ||x||$.

From the above one can conclude that there exists a solution in $K \cap (\overline{\Omega_{i+1}} \setminus \Omega_i)$ for all $i \in \mathbb{N}$. Hence, there exist countably many positive solutions to the BVP (3.8). $\square$

Note that while the theorem above specifies conditions on $h(t, x)$ under which countably many solutions may be guaranteed, the proof of the above theorem may be used to guarantee other finite numbers of solutions. The example below will illustrate how to use the above proof to guarantee two solutions to the boundary value problem (3.8).

Example 3.19. Consider the boundary value problem

$$\begin{cases} -\nabla_{0^*}^{\frac{15}{8}} x(t) = h(t, x(t-1)), \quad t \in \mathbb{N}_1^6 \\ x(-1) = 0, \\ \nabla x(6) = 0, \end{cases} \tag{3.9}$$

where solutions x are defined on $\mathbb{N}_{-1}^6$ and $h : \mathbb{N}_1^6 \times \mathbb{R}^+ \to \mathbb{R}^+$. In this example, we will find conditions on h that guarantee two solutions to the BVP (3.9). First, we must find the value of constants k, m, and w. By the definition of k from Theorem

3.11 and from the fact that $b-a=6$,

$$\begin{aligned}
k &= \min\left\{\frac{3(b-a)+5}{4(b-a)}, \frac{1}{b-a}\left[\frac{(b-a)(\nu-2)+1}{\nu-1}\right]\right\} \\
&= \min\left\{\frac{23}{24}, \frac{1}{21}\right\} \\
&= \frac{1}{21}.
\end{aligned}$$

Next, by the definition of m, and using the fact that $u(\rho(s), s) = v(\rho(s), s)$,

$$\begin{aligned}
m &= \frac{1}{\int_0^6 G(\rho(s), s)\nabla s} \\
&= \frac{1}{\int_0^6 u(\rho(s), s)\nabla s} \\
&= \frac{1}{\int_0^6 \frac{(\rho(s)-0+1)(6-s+1)^{\overline{\frac{15}{8}-2}}}{\Gamma\left(\frac{15}{8}-1\right)}\nabla s} \\
&= \frac{1}{\int_0^6 \frac{(s)(7-s)^{\overline{-\frac{1}{8}}}}{\Gamma\left(\frac{7}{8}\right)}\nabla s} \\
&= \frac{1}{\sum_{s=1}^{6} \frac{s\Gamma\left(7-s-\frac{1}{8}\right)}{\Gamma\left(\frac{7}{8}\right)\Gamma(7-s)}} \\
&\approx 0.0547.
\end{aligned}$$

Note that

$$\begin{aligned}
r &= \left\lceil b - \frac{b-(a+1)}{4}\right\rceil_0 - 0 \\
&= \left\lceil 6 - \frac{5}{4}\right\rceil \\
&= 5.
\end{aligned}$$

Since $t_0 \in \mathbb{N}_r^b$, we have that $t_0 \in \mathbb{N}_5^6$. Say $t_0 = 6$. Then

$$\begin{aligned}
w &= \frac{1}{k\int_r^b G(t_0,s)\nabla s} \\
&= \frac{1}{\frac{1}{21}\int_5^6 G(6,s)\nabla s} \\
&= \frac{21}{G(6,6)} \\
&= \frac{21}{v(6,6)} \\
&= \frac{21}{-(6+1)H_{-\frac{1}{8}}(6,\rho(6)) + H_{\frac{7}{8}}(6,\rho(6))} \\
&= \frac{21}{\frac{7\cdot\Gamma\left(\frac{7}{8}\right)}{\Gamma\left(\frac{7}{8}\right)} - \frac{\Gamma\left(\frac{15}{8}\right)}{\Gamma\left(\frac{15}{8}\right)}} \\
&= \frac{7}{2}.
\end{aligned}$$

We now look for p_1, p_2, and p_3.

Say, referring to Theorem 3.18, part (i), that we wish $h(t,x)$ to be such that $h(t,x) \leq 2$ for all $t \in \mathbb{N}_{-1}^6$ for $0 \leq x \leq p_1$. Because we need $h(t,x) \leq mp_1$, we can find p_1 such that the above holds. That is we need $mp_1 = 2$, so $p_1 = \frac{2}{m} \approx 36.5606$.

Next, we want, by Theorem 3.18 part (iii), for $h(t,x) \geq wx = \frac{7}{2}\cdot x$ for $t = 6$ and $p_1 < kp_2 \leq x \leq p_2$. Because we need $p_1 < kp_2$, say $kp_2 = 38$. Then $p_2 = \frac{38}{k} = 798$.

Lastly, we want, by Theorem 3.18 part (ii), $h(t,x) \leq mp_3$ for $t \in \mathbb{N}_{-1}^6$ and $p_2 + \epsilon \leq x \leq p_3$ where $\epsilon > 0$. So say $\epsilon = 2$. Since, $p_2 < p_3$, say $p_3 = 805$. Then $h(t,x) \leq mp_3 \approx 44.0365$ for $t \in \mathbb{N}_{-1}^6$ and $800 \leq x \leq 805$.

Define $c := \frac{7}{2}\cdot 38 + \frac{1}{38^2+1} + 1 = \frac{193631}{1445}$ and $d := \frac{7}{2}\cdot 798 + \frac{1}{798^2+1} + 1 = \frac{1779233171}{636805}$. An example of $h(t,x)$ that guarantees two solutions is as follows:

$$h(t,x) = \begin{cases} \sin(t) + 1, & 0 \le x \le 37 \\ \left(c - \sin(37) - 1\right)(x - 37) + \sin(37) + 1, & 37 \le x \le 38 \\ \frac{7}{2} \cdot x + \frac{1}{t^2+1} + 1, & 38 < x \le 798 \\ \left(\frac{\cos(800)+40-d}{2}\right)(x - 798) + d, & 798 \le x \le 800 \\ \cos(t) + 40, & 800 < x, \end{cases}$$

for $t \in \mathbb{N}_{-1}^{6}$. Since we also assume that $h(t,x)$ is continuous, the above includes lines connecting the separate portions of the graph of $h(t,x)$ for $37 \le x \le 38$ and for $798 \le x \le 800$.

Note that by the proof of Theorem 3.18, the two solutions exist in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$ and $K \cap (\overline{\Omega_3} \setminus \Omega_2)$, where $\Omega_1 = B_{p_1}(0)$, $\Omega_2 = B_{p_2}(0)$, and $\Omega_3 = B_{p_3}(0)$. Hence both solutions satisfy $x(-1) = 0$, $\nabla x(6) = 0$, $x(t) \ge 0$ for $t \in \mathbb{N}_{-1}^{6}$, $x(t) \ge k||x|| = \frac{1}{21}||x||$ for $t \in \mathbb{N}_{5}^{6}$. Additionally, one solution satisfies $36.5606 \approx p_1 < ||x|| \le p_2 = 798$, while the other solution satisfies $800 < ||x|| \le 805$.

3.5 Future Work

In the current chapter, existence of solutions to the BVP (3.8) were established for $\frac{3}{2} \le \nu \le 2$. Hence, a natural next step would be to look for possibilities for extending this result to cover more ν values between 1 and 2. That is, one could investigate the behavior of $G(t,s)$ outside of $\frac{3}{2} \le \nu \le 2$ to see whether there are other behaviors one could take advantage of in order to establish existence of solutions to the BVP (3.8) for $1 < \nu < \frac{3}{2}$. Theorem 4.1 in Section 4.2 is a first attempt at doing so, but there is much left to be done.

Additionally, in a recent paper by Anderson and Avery [9], they use the omitted ray fixed point theorem [18] to find solutions to a nonlinear right-focal boundary value problem in continuous time. What is particularly worth noting here is that using the omitted ray fixed point theorem, they are able to relax the conditions under which fixed points, and thus solutions, can be found. Hence, a possible future direction could involve seeing whether the omitted ray fixed point theorem could help extend the results from this chapter.

Finally, in Subsection 3.3.1, the Green's function for the linear BVP (3.2) where $\nu > 1$ was found. The Green's function as demonstrated in Example 3.10 may be used to calculate solutions to the BVP (3.2). Thus a future direction may involve investigating the behavior of solutions to the BVP (3.2).

Chapter 4

A Right Focal Boundary Value Problem Involving a Caputo Operator for Larger Values of ν

4.1 Introduction

This chapter expands upon results obtained in Chapter 3 with regard to the linear BVP (3.2), which is

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{k} x(a-1) = A_k, & 0 \le k \le N-2 \\ \nabla^{N-1} x(b) = B, \end{cases}$$

where $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $\nu > 1$, $N = \lceil \nu \rceil$, $A_k \in \mathbb{R}$ for $0 \le k \le N-2$, $b-a \in \mathbb{Z}$, $b-a \ge N-1$, and where the solutions x are defined on $\mathbb{N}_{a-N+1}^{b}$.

In Section 4.2, we approach the behavior of the Green's function for the BVP (3.2).

We know from Theorem 3.11 in Chapter 3 that the Green's function is negative provided $1 < \nu \leq 2$ and $1 \leq b - a \leq \frac{1}{2-\nu}$, so Theorem 4.1 provides some conditions outside of this region that would guarantee $G(t,s)|_{\mathbb{N}_{a+1}^b \times \mathbb{N}_{a+1}^b} \leq 0$.

In Section 4.3, we consider the the BVP (3.2) when $2 < \nu \leq 3$, and we provide bounds on the Green's function similar to those given Theorem 3.11 parts (i) and (ii).

In Section 4.4, we consider a generalized form of the BVP (3.2). In particular the second boundary condition in the BVP (3.2) is replaced with $\nabla^i x(b) = 0$ where $i \in \mathbb{N}_0^{N-1}$ is a constant. We show that this general BVP has a unique solution, define its Green's function, and prove some basic properties of the Green's function.

Finally in Section 4.5, we discuss possibilities for future work.

4.2 Further Results for $1 < \nu \leq 2$

In this section, we continue to consider the BVP (3.2) for the case where $1 < \nu \leq 2$. In Theorem 4.1, we give a condition that, if satisfied, guarantees $G(t,s)|_{\mathbb{N}_{a+1}^b \times \mathbb{N}_{a+1}^b} \leq 0$ for all $1 < \nu \leq 2$.

Theorem 4.1. *Let $1 \leq \nu \lneq 2$, $G(t,s) : \mathbb{N}_{a-1}^b \times \mathbb{N}_{a+1}^b \to \mathbb{R}$ be the Green's function for the homogeneous BVP (3.3). Then, for $\rho(s) \leq t$, if*

$$t - a + 1 \leq \frac{s - a - \nu + 2}{1 - H_{\nu-2}(b, \rho(s))},$$

then $G(t,s)|_{\mathbb{N}_{a+1}^b \times \mathbb{N}_{a+1}^b} \leq 0$.

Proof. First, by the proof of Theorem 3.11, $u(t,s) \leq 0$ for $t \leq \rho(s)$.

Let $\rho(s) \leq t$. Then

$$v(t,s) = -H_1(t, a-1)H_{\nu-2}(b, \rho(s)) + H_{\nu-1}(t, \rho(s))$$

$$
\begin{aligned}
&= -(t-a+1)H_{\nu-2}(b,\rho(s)) + \frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \\
&= -(t-a+1)H_{\nu-2}(b,\rho(s)) + \frac{\Gamma(t-s+\nu)}{\Gamma(t-s+1)\Gamma(\nu)} \\
&= -(t-a+1)H_{\nu-2}(b,\rho(s)) + \\
&\qquad (t-s-1+\nu)\cdot\frac{t-s-2+\nu}{t-s}\cdot\frac{t-s-3+\nu}{t-s-1}\cdot\cdots\cdot\frac{\nu}{2} \\
&\le -(t-a+1)H_{\nu-2}(b,\rho(s)) + t-s-1+\nu.
\end{aligned}
$$

Next note that, because $0 \le \nu - 1 \lneq 1$,

$$
\begin{aligned}
1 - H_{\nu-2}(b,\rho(s)) &= 1 - \frac{(b-s+1)^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\
&= 1 - \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu-1)} \\
&\ge 1 - \frac{1}{\Gamma(\nu-1)} \\
&\ge 0.
\end{aligned}
$$

Now

$$
\begin{aligned}
-(t-a+1)H_{\nu-2}(b,\rho(s)) + t-s-1+\nu \le 0 &\quad \text{iff} \\
t-s-1+\nu \le (t-a+1)H_{\nu-2}(b,\rho(s)) &\quad \text{iff} \\
(t-a+1)-(s-a-\nu+2) \le (t-a+1)H_{\nu-2}(b,\rho(s)) &\quad \text{iff} \\
(t-a+1)(1-H_{\nu-2}(b,\rho(s))) \le s-a-\nu+2 &\quad \text{iff} \\
t-a+1 \le \frac{s-a-\nu+2}{1-H_{\nu-2}(b,\rho(s))}. &
\end{aligned}
$$

Thus if $t-a+1 \le \frac{s-a-\nu+2}{1-H_{\nu-2}(b,\rho(s))}$, then $v(t,s) \le 0$. $\square$

4.3 Bounds on the Green's Function for $2 < \nu \leq 3$

In this section, we continue to consider the BVP (3.3) for the case where $2 < \nu \leq 3$. In this case the BVP may be viewed as follows

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), \quad t \in \mathbb{N}_{a+1}^{b} \\ x(a-1) = 0, \\ \nabla x(a-1) = 0, \\ \nabla^2 x(b) = 0, \end{cases} \tag{4.1}$$

where $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $\nu > 1$, $N = \lceil \nu \rceil = 3$, $b - a \in \mathbb{N}_2$, and where the solutions x are defined on $\mathbb{N}_{a-2}^{b}$.

Theorem 4.2 below has two results similar to results in Theorem 3.11. However, the Green's function for the BVP (4.1) exhibits different behavior from the Green's function for the $1 < \nu \leq 2$ case. For instance notice below that the minimum value of $G(t, s)$ occurs at $t = b$ instead of $t = \rho(s)$ for fixed $s \in \mathbb{N}_{a+1}^{b}$.

Note that, in order to establish the bounds in Theorem 4.2, the proof requires a restriction on the domain of solutions to the BVP (4.1). In particular, it requires that $2 \leq b - a \leq \frac{1}{3-\nu}$. Note that $2 \leq \frac{1}{3-\nu}$, which implies that $\nu \geq \frac{5}{2}$. Hence we get the restriction mentioned in Theorem 4.2.

Theorem 4.2. *Let $\frac{5}{2} \leq \nu \leq 3$ and $2 \leq b - a \leq \frac{1}{3-\nu}$. Let $G(t, s) : \mathbb{N}_{a-2}^{b} \times \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ be the Green's function for the homogeneous BVP (4.1). Then the following hold:*

(i) $G(t, s)|_{\mathbb{N}_{a-2}^{b} \times \mathbb{N}_{a+1}^{b}} \leq 0$ *and*

(ii) $\min_{t \in \{a-1, a, \cdots, b\}} G(t, s) = G(b, s)$ *for each fixed* $s \in \mathbb{N}_{a+1}^{b}$.

Proof. (i) Note that $N = 3$. Let $t \leq \rho(s)$ and $(t, s) \in \mathbb{N}_{a-1}^{b} \times \mathbb{N}_{a+1}^{b}$. Then

$$\begin{aligned}
G(t,s) &= u(t,s) \\
&= -H_2(t, a-1)H_{\nu-3}(b, \rho(s)) \\
&= -\frac{(t-a+1)^{\overline{2}}(b-s+1)^{\overline{\nu-3}}}{\Gamma(3)\Gamma(\nu-2)} \\
&= -\frac{\Gamma(t-a+3)\Gamma(b-s+\nu-2)}{2\Gamma(t-a+1)\Gamma(b-s+1)\Gamma(\nu-2)} \\
&\leq 0.
\end{aligned}$$

Note that if $t = a - 2$ and $t \leq \rho(s)$, then

$$\begin{aligned}
G(t,s) &= -H_2(a-2, a-1)H_{\nu-3}(b, \rho(s)) \\
&= -\frac{(-1)^{\overline{2}}(b-s+1)^{\overline{\nu-3}}}{\Gamma(3)\Gamma(\nu-2)} \\
&= 0
\end{aligned}$$

by convention. Next let $\rho(s) \leq t$. Note then that, since $s \geq a + 1$, we must have $a \leq \rho(a+1) \leq t$. Then

$$v(t,s) = -H_2(t, a-1)H_{\nu-3}(b, \rho(s)) + H_{\nu-1}(t, \rho(s)).$$

so

$$\begin{aligned}
\nabla_s v(t,s) &= \nabla_s \left[-\frac{H_2(t,a-1)(b-s+1)^{\overline{\nu-3}}}{\Gamma(3)\Gamma(\nu-2)} + \frac{(t-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)} \right] \\
&= \frac{H_2(t,a-1)(b-s+2)^{\overline{\nu-4}}}{2\Gamma(\nu-3)} - \frac{(t-s+2)^{\overline{\nu-2}}}{\Gamma(\nu-1)} \\
&= \frac{\Gamma(t-a+3)\Gamma(b-s+\nu-2)}{2\Gamma(t-a+1)\Gamma(b-s+2)\Gamma(\nu-3)} - \frac{\Gamma(t-s+\nu)}{\Gamma(t-s+2)\Gamma(\nu-1)}
\end{aligned}$$

$$\leq 0,$$

since $-1 < \nu - 3 \leq 0$ and hence $\Gamma(\nu - 3) \leq 0$. Because $v(t,s)$ is decreasing with respect to s, it remains to show that $v(t, a+1) \leq 0$.

Note that

$$\nabla_t v(t,s) = -H_1(t, a-1)H_{\nu-3}(b, \rho(s)) + H_{\nu-2}(t, \rho(s))$$

and

$$\nabla_{tt} v(t,s) = -H_{\nu-3}(b, \rho(s)) + H_{\nu-3}(t, \rho(s)).$$

Also,

$$\begin{aligned}
\nabla_t H_{\nu-3}(t, \rho(s)) &= H_{\nu-4}(t, \rho(s)) \\
&= \frac{\Gamma(t - s + \nu - 3)}{\Gamma(t - s + 1)\Gamma(\nu - 3)} \\
&\leq 0
\end{aligned}$$

provided $t \neq s$. Since

$$\begin{aligned}
\nabla_{tt} v(t,s)|_{t=s} &= -\frac{(b - s + 1)^{\overline{\nu-3}}}{\Gamma(\nu - 2)} + \frac{1^{\overline{\nu-3}}}{\Gamma(\nu - 2)} \\
&= -\frac{\Gamma(b - s + \nu - 2)}{\Gamma(b - s + 1)\Gamma(\nu - 2)} + 1 \\
&\geq -\frac{1}{\Gamma(\nu - 2)} + 1 \\
&\geq 0,
\end{aligned}$$

it holds that $\nabla_{tt} v(t,s) \geq 0$. Next, because $b - a \leq \frac{1}{3-\nu}$ and $s \geq a+1$,

$$\nabla_t v(t,s)|_{t=b} = -H_1(b, a-1)H_{\nu-3}(b, \rho(s)) + H_{\nu-2}(b, \rho(s))$$

$$
\begin{aligned}
&= -\frac{(b-a+1)\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-2)} + \frac{\Gamma(b-s+\nu-1)}{\Gamma(b-s+1)\Gamma(\nu-1)}\\
&= -\frac{(b-a+1)\Gamma(b-s+\nu-2)(\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)} + \frac{(b-s+\nu-2)\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}\\
&= \frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}[-(b-a+1)(\nu-2)+b-s+\nu-2]\\
&= \frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}[(b-a)(2-\nu)+(b-s)]\\
&\le \frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}[(b-a)(2-\nu)+(b-a-1)]\\
&= \frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}[(b-a)(3-\nu)-1]\\
&\le \frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-1)}\left[\left(\frac{1}{3-\nu}\right)(3-\nu)-1\right]\\
&= 0.
\end{aligned}
$$

Hence, since $\nabla_{tt}v(t,s) \ge 0$, it holds that $\nabla_t v(t,s) \le 0$, so in particular $\nabla_t v(t,a+1) \le 0$. Thus if $v(a,a+1) \le 0$, then $v(t,a+1) \le 0$. Now

$$
\begin{aligned}
v(a,a+1) &= -H_2(t,a-1)H_{\nu-3}(b,\rho(s)) + H_{\nu-1}(t,\rho(s))\\
&= -H_2(a,a-1)H_{\nu-3}(b,a) + H_{\nu-1}(a,a)\\
&= -\frac{(1)^{\overline{2}}}{\Gamma(3)} \cdot \frac{(b-a)^{\overline{\nu-3}}}{\Gamma(\nu-2)} + \frac{0^{\overline{\nu-1}}}{\Gamma(\nu)}\\
&= -\frac{\Gamma(3)}{\Gamma(1)\Gamma(3)} \cdot \frac{\Gamma(b-a+\nu-3)}{\Gamma(b-a)\Gamma(\nu-2)} + \frac{\Gamma(\nu-1)}{\Gamma(0)\Gamma(\nu)}\\
&= -\frac{\Gamma(3)}{\Gamma(1)\Gamma(3)} \cdot \frac{\Gamma(b-a+\nu-3)}{\Gamma(b-a)\Gamma(\nu-2)}\\
&\le 0.
\end{aligned}
$$

Hence, we are done.

(ii.) Note that for $a-1 \leq t \leq \rho(s) \leq b$,

$$\begin{aligned}
\nabla_t G(t,s) &= \nabla_t u(t,s) \\
&= \nabla_t \left[-H_2(t,a-1)H_{\nu-3}(b,\rho(s))\right] \\
&= -(t-a+1)H_{\nu-3}(b,\rho(s)) \\
&= -\frac{(t-a+1)\Gamma(b-s+\nu-2)}{\Gamma(b-s+1)\Gamma(\nu-2)} \\
&\leq 0.
\end{aligned}$$

Since by the proof of part (i), it also holds that $\nabla_t v(t,s) \leq 0$, it holds that

$$\min_{t \in \{a-1,a,\cdots,b\}} G(t,s) = G(b,s)$$

for each fixed $s \in \mathbb{N}_{a+1}^{b}$. □

4.4 Generalization of the Boundary Value Problem

In this section we generalize BVP (3.2) by replacing $\nabla^{N-1}x(b) = B$ with $\nabla^i x(b) = B$ for constant $i \in \mathbb{N}_0^{N-1}$ in the final boundary condition. First we show that BVP (4.2) has a unique solution in Theorem 4.3 and then move on to consider its Green's function.

Now consider the following non homogeneous boundary value problem

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k x(a-1) = A_k, & 0 \le k \le N-2 \\ \nabla^i x(b) = B, \end{cases} \tag{4.2}$$

and the corresponding homogeneous boundary value problem

$$\begin{cases} \nabla_{a^*}^{\nu} x(t) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^k x(a-1) = 0, & 0 \le k \le N-2 \\ \nabla^i x(b) = 0, \end{cases} \tag{4.3}$$

where $i \in \mathbb{N}_0^{N-1}$, $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$, $\nu > 1$, $N = \lceil \nu \rceil$, $A_k \in \mathbb{R}$ for $0 \le k \le N-2$, $b-a \in \mathbb{Z}$, $b - a \ge N - 1$, and where the solutions x are defined on $\mathbb{N}_{a-N+1}^{b}$.

Theorem 4.3. *The nonhomogeneous boundary value problem (4.2) has a unique solution.*

Proof. By Theorem 3.4, a general solution to $\nabla_{a^*}^{\nu} x(t) = 0$ is given by

$$x(t) = c_0 x_0(t) + \cdots + c_{N-1} x_{N-1}(t),$$

where $c_0, \ldots, c_{N-1} \in \mathbb{R}$ and $x_0, x_1, \ldots, x_{N-1}$ are N linearly independent solutions to $\nabla_{a^*}^{\nu} x(t) = 0$ on $\mathbb{N}_{a-N+1}$. Thus, by Theorem 2.18, the general solution to $\nabla_{a^*}^{\nu} x(t) = 0$ is given by

$$x(t) = c_0 + c_1 H_1(t, a) + c_2 H_2(t, a) + \cdots + c_{N-1} H_{N-1}(t, a),$$

for $t \in \mathbb{N}_{a-N+1}^{b}$, where $c_0, c_1, \ldots, c_{N-1} \in \mathbb{R}$. Without loss of generality, say $x_0(t) = 1, x_1(t) = H_1(t,a), \ldots, x_{N-1}(t) = H_{N-1}(t,a)$ for $t \in \mathbb{N}_{a-N+1}$.

Suppose $x(t)$ solves (4.3). Let $i \in \mathbb{N}_0^{N-1}$. Then $x(t)$ is the trivial solution if and only if $c_0 = c_1 = \ldots = c_{N-1} = 0$. This holds if and only if the system of equations

$$\begin{cases} c_0 \nabla^k x_0(a-1) + c_1 \nabla^k x_1(a-1) + \cdots + c_{N-1} \nabla^k x_{N-1}(a-1) = 0, \quad 0 \leq k \leq N-2 \\ c_0 \nabla^i x_0(b) + c_1 \nabla^i x_1(b) + \cdots + c_{N-1} \nabla^i x_{N-1}(b) = 0 \end{cases}$$

has only the trivial solution. Define

$$M := \begin{bmatrix} x_0(a-1) & x_1(a-1) & \ldots & x_{N-1}(a-1) \\ \nabla x_0(a-1) & \nabla x_1(a-1) & \ldots & \nabla x_{N-1}(a-1) \\ \vdots & \vdots & \ddots & \vdots \\ \nabla^{N-2} x_0(a-1) & \nabla^{N-2} x_1(a-1) & \ldots & \nabla^{N-2} x_{N-1}(a-1) \\ \nabla^i x_0(b) & \nabla^i x_1(b) & \ldots & \nabla^i x_{N-1}(b) \end{bmatrix}.$$

If $det(M) \neq 0$, then the system of equations has only the trivial solution. By the proof of Theorem 3.6,

$$M = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ \nabla^i x_0(b) & \nabla^i x_1(b) & \nabla^i x_2(b) & \ldots & \nabla^i x_{N-2}(b) & \nabla^i x_{N-1}(b) \end{bmatrix}.$$

Then, assuming that N is odd,

$$det(M) = \nabla^i x_0(b) \begin{vmatrix} -1 & 0 & \dots & 0 & 0 \\ 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{vmatrix} - \nabla^i x_1(b) \begin{vmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{vmatrix} +$$

$$\nabla^i x_3(b) \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix} + \cdots - \nabla^i x_{N-1}(b) \begin{vmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

Note that the above matrices are block matrices, where each block is a triangular matrix. Hence,

$$\begin{aligned} det(M) &= \nabla^i x_0(b)(-1)^{N-1} - \nabla^i x_1(b)(-1)^{N-2} + \nabla^i x_2(b)(-1)^{N-3} - \cdots + \nabla^i x_{N-1}(b) \\ &= \nabla^i x_0(b) + \nabla^i x_1(b) + \nabla^i x_2(b) + \cdots + \nabla^i x_{N-1}(b) \\ &= H_{-i}(b,a) + H_{1-i}(b,a) + H_{2-i}(b,a) + \cdots + H_{N-1-i}(b,a) \\ &\neq 0. \end{aligned}$$

If N is even,

$$\begin{aligned} det(M) &= -\nabla^i x_0(b)(-1)^{N-1} + \nabla^i x_1(b)(-1)^{N-2} - \nabla^i x_2(b)(-1)^{N-3} + \cdots + \nabla^i x_{N-1}(b) \\ &= \nabla^i x_0(b) + \nabla^i x_1(b) + \nabla^i x_2(b) + \cdots + \nabla^i x_{N-1}(b) \end{aligned}$$

$\neq 0.$

Thus, (4.3) has only the trivial solution. Next, by Theorem 3.5, the general solution to $\nabla^{\nu}_{a^*}x(t) = h(t)$ is given by

$$y(t) = c_1x_1(t) + \cdots + c_Nx_N(t) + y_0(t)$$

where $c_1, \dots, c_N \in \mathbb{R}$ and $y_0 : \mathbb{N}_{a-N+1} \to \mathbb{R}$ is a particular solution to $\nabla^{\nu}_{a^*}y_0(t) = h(t)$. Then $y(t)$ satisfies (3.2) if and only if

$$\begin{cases} \nabla^k\left[c_1x_1(a-1) + c_2x_2(a-1) + \cdots + c_Nx_N(a-1) + y_0(a-1)\right] = A_k, \quad 0 \leq k \leq N-2 \\ \nabla^i\left[c_1x_1(b) + c_2x_2(b) + \cdots + c_Nx_N(b) + y_0(b)\right] = B \end{cases}$$

if and only if

$$\begin{cases} \nabla^k\left[c_1x_1(a-1) + c_2x_2(a-1) + \cdots + c_Nx_N(a-1)\right] = A_k - \nabla^k y_0(a-1), \quad 0 \leq k \leq N-2 \\ \nabla^i\left[c_1x_1(b) + c_2x_2(b) + \cdots + c_Nx_N(b)\right] = B - \nabla^i y_0(b). \end{cases}$$

Thus $y(t)$ uniquely satisfies the boundary conditions in (4.2) if and only if $det(M) \neq 0$. Therefore the BVP (4.2) has a unique solution. □

Next we define the Green's function for the BVP (4.3) and prove some of its basic properties.

Definition 4.4 (Green's Function)**.** The Green's function, $G(t,s) : \mathbb{N}^b_{a-N+1} \times \mathbb{N}^b_{a+1} \to$

$\mathbb{R}$, for the BVP (4.3) is given by

$$G(t,s) := \begin{cases} u(t,s), & t \le \rho(s) \\ v(t,s), & \rho(s) \le t \end{cases} \tag{4.4}$$

where for each fixed $s \in \mathbb{N}_{a+1}^{b}$, $u(t,s)$ is defined to be the unique solution of

$$\begin{cases} \nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{k} u(a-1,s) = 0, & 0 \le k \le N-2 \\ \nabla^{i} u(b,s) = -\nabla^{i} x(b,s) & \end{cases}$$

and

$$v(t,s) := u(t,s) + x(t,s),$$

where $x(t,s)$ is the Cauchy function for $\nabla_{a^*}^{\nu} x(t) = 0$ and where i is some constant in $\mathbb{N}_0^{N-1}$.

Theorem 4.5. *Let $h : \mathbb{N}_{a+1}^{b} \to \mathbb{R}$ and $i \in \mathbb{N}_0^{N-1}$ be a constant. Then the unique solution to*

$$\begin{cases} \nabla_{a^*}^{\nu} y(t) = h(t), & t \in \mathbb{N}_{a+1}^{b} \\ \nabla^{k} y(a-1) = 0, & 0 \le k \le N-2 \\ \nabla^{i} y(b) = 0, & \end{cases} \tag{4.5}$$

is given by

$$y(t) = \int_a^b G(t,s) h(s) \nabla s, \quad t \in \mathbb{N}_{a-N+1}^{b},$$

where $G(t,s)$ is the Green's function for the BVP (4.3).

Proof. Let $i \in \mathbb{N}_0^{N-1}$. By the proof of Theorem 3.8, $\nabla_{a^*}^{\nu} y(t) = h(t)$ and $\nabla^k y(a-1) = 0$ for $0 \leq k \leq N-2$. It remains to show that $\nabla^i y(b) = 0$. Recall that according to the Leibniz Formula in [31, Theorem 3.41 on p. 175], if $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$, then $\nabla \left(\int_a^t f(t,\tau)\nabla\tau \right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t)$. Hence, it holds that

$$\begin{aligned}
\nabla^i y(b) &= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \nabla^i z(b) \\
&= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \nabla^i \left[\int_a^t x(t,s)h(s)\nabla s \right]_{t=b} \\
&= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \nabla^{i-1} \left[\int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} + x(\rho(b),b)h(b) \\
&= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \nabla^{i-1} \left[\int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} + H_{\nu-1}(\rho(b),\rho(b)) \\
&= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \nabla^{i-1} \left[\int_a^t \nabla x(t,s)h(s)\nabla s \right]_{t=b} \\
&\ \vdots \\
&= \int_a^b \nabla^i u(b,s)h(s)\nabla s + \int_a^b \nabla^i x(b,s)h(s)\nabla s \\
&= -\int_a^b \nabla^i x(b,s)h(s)\nabla s + \int_a^b \nabla^i x(b,s)h(s)\nabla s \\
&= 0.
\end{aligned}$$

□

Theorem 4.6. *The Green's function $G(t,s) : \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b \to \mathbb{R}$ for the homogeneous BVP (4.3) is given by (4.4) where*

$$u(t,s) := -\frac{H_{N-1}(t,a-1)H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)}$$

and

$$v(t,s) := -\frac{H_{N-1}(t,a-1)H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)} + H_{\nu-1}(t,\rho(s)).$$

Proof. Let $t \leq \rho(s)$ and $i \in \mathbb{N}_0^{N-1}$. Then it will be shown that, for each fixed $s \in \mathbb{N}_{a+1}^b$, $u(t,s)$ is the solution to the BVP

$$\begin{cases} \nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+1}^b \\ \nabla^k u(a-1,s) = 0, & 0 \leq k \leq N-2 \\ \nabla^i u(b,s) = -\nabla^i x(b,s). \end{cases}$$

By the proof of Theorem 3.9, $\nabla_{a^*}^{\nu} u(t,s) = 0$. Let $0 \leq k \leq N-3$. Then

$$\begin{aligned} \nabla^k u(a-1,s) &= -\frac{H_{N-k-1}(t,a-1)H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)}|_{t=a-1} \\ &= -\frac{\Gamma(t-a+N-k)}{\Gamma(t-a+1)\Gamma(N-k)} \cdot \frac{H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)}|_{t=a-1} \\ &= -\frac{\Gamma(N-k-1)}{\Gamma(0)\Gamma(N-k)} \cdot H_{\nu-N}(b,\rho(s)) \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} \nabla^{N-2} u(a-1,s) &= \frac{H_{N-1-N+2}(t,a-1)H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)}|_{t=a-1} \\ &= -\frac{t-a+1}{\Gamma(N-k)} \cdot \frac{H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)}|_{t=a-1} \\ &= 0. \end{aligned}$$

Next, note that $\nabla^i x(t,s) = \nabla^i H_{\nu-1}(t,\rho(s)) = H_{\nu-i-1}(t,\rho(s))$. Then,

$$\begin{aligned} \nabla^i u(b,s) &= -\frac{H_{N-i-1}(b,a-1)H_{\nu-i-1}(b,\rho(s))}{H_{N-i-1}(b,a-1)} \\ &= -H_{\nu-i-1}(b,\rho(s)) \end{aligned}$$

$$= -\nabla^i x(b, s).$$

□

4.5 Future Work

As stated at the end of Chapter 3, more work remains to be done to expand on Theorem 3.11 and Theorem 4.3 to eventually entirely specify when the Green's function is negative for $1 < \nu \leq 2$. Furthermore, there is more to be done to classify the behavior of the Green's function in general for $1 < \nu \leq 2$. Doing so would be a first step in expanding on the nonlinear case and instances of existence of solutions to BVPs.

In addition to the results in Theorem 4.2, it still remains to find constants $0 < k \leq 1$ and $r_1, r_2 \in \mathbb{N}_{a+1}^b$ such that $G(t, s) \leq k \cdot G(b, s)$ for $t \in \mathbb{N}_{r_1}^{r_2}$ and $s \in \mathbb{N}_{a+1}^b$. Proving a result along these lines would allow one to approach the nonlinear case for $2 < \nu \leq 3$ and to prove existence of solutions.

Also, as the proofs to Theorem 4.2 indicates, the behavior of the Green's function for $2 < \nu \leq 3$ differs from that of $1 \leq \nu \leq 2$, so investigating the behavior for $2 < \nu \leq 3$ is an interesting possibility.

More broadly it remains to prove theorems similar to Theorem 3.11 for all $\nu > 1$, to investigate the behavior of the Green's function in these cases, and to prove the existence of solutions to the BVPs in the nonlinear case.

One could also investigate the BVP (4.2) in general but also for other specific values of $i \in \mathbb{N}_0^{N-1}$. Given some of the difficulties that have arisen for the $i = N - 1$ case, perhaps some issues could be resolved by varying i. Additionally, given that a formula for the Green's function was provided in Theorem 4.6, one could use it to calculate solutions to the BVP (4.2) for varying values of $i \in \mathbb{N}_0^{N-1}$ and investigate their behavior.

Chapter 5

A Right Focal Boundary Value Problem Involving the Operator $\nabla\nabla_{a^*}^{\nu}$

5.1 Introduction

In this section we will consider a right focal boundary value problem involving the operator $\nabla\nabla_{a^*}^{\nu}$ for $0 < \nu \leq 1$ as follows:

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} x(t) = h(t), \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = \nabla x(b) = 0, \end{cases} \tag{5.1}$$

where $b-a \in \mathbb{N}_2$ and $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$. In the rest of the introduction to this chapter, we give some theorems similar to theorems proved in [31] and [3] with some corrections incorporated. Because corrections have been made, the proofs have been included for completeness.

First, we show uniqueness of the solution to the IVP related to the BVP (5.1).

Theorem 5.1. *Let* $A, B \in \mathbb{R}$, $h : \mathbb{N}_{a+2} \to \mathbb{R}$, *and* $0 < \nu \leq 1$. *Then the IVP*

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = h(t), \quad t \in \mathbb{N}_{a+2} \\ x(a) = A \\ \nabla x(a+1) = B, \end{cases} \tag{5.2}$$

has a unique solution.

Proof. Let $x \in \mathbb{N}_a \to \mathbb{R}$ satisfy $x(a) = A$ and $x(a+1) = A + B$. Say first that $\nu = 1$. Then say that $x : \mathbb{N}_a \to \mathbb{R}$ satisfies the recursive equation

$$x(t) = h(t) + 2x(t-1) - x(t-2)$$

for $t \in \mathbb{N}_{a+2}$. Hence $x(t)$ is uniquely defined by $x(t-1)$ and $x(t-2)$ for $t \in \mathbb{N}_{a+2}$. Note that the above equation is equivalent to $\nabla^2 x(t) = h(t)$. Thus $x : \mathbb{N}_a \to \mathbb{R}$ satisfies the IVP (5.2), so a solution to the IVP (5.2) exists and the solution is unique.

Next say $0 < \nu < 1$ and furthermore that $x : \mathbb{N}_a \to \mathbb{R}$ satisfies the recursive equation

$$\nabla x(t) = h(t) - \sum_{s=a+1}^{t-1} H_{-\nu}(t,s)\left(\frac{t-s-\nu}{t-s} - 1\right)\nabla x(s). \tag{5.3}$$

Note $H_{-\nu}(t,s)\left(\frac{t-s-\nu}{t-s} - 1\right)$ is never 0 or undefined for $t \in \mathbb{N}_{a+1}$ and where $s \in \mathbb{N}_{a+1}^{t-1}$. Hence $x(t)$ is uniquely defined by $x(a)$, $x(a+1)$,..., $x(t-2)$, and $x(t-1)$ for $t \in \mathbb{N}_{a+2}$. Additionally note

$$H_{-\nu}(t,\rho(s)) - H_{-\nu}(\rho(t),\rho(s)) = \frac{(t-s+1)^{\overline{-\nu}}}{\Gamma(1-\nu)} - \frac{(t-s)^{\overline{-\nu}}}{\Gamma(1-\nu)}$$

$$
\begin{aligned}
&= \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)\Gamma(1-\nu)} - \frac{\Gamma(t-s-\nu)}{\Gamma(t-s)\Gamma(1-\nu)} \\
&= \frac{(t-s-\nu)\Gamma(t-s-\nu)}{(t-s)\Gamma(t-s)\Gamma(1-\nu)} - \frac{\Gamma(t-s-\nu)}{\Gamma(t-s)\Gamma(1-\nu)} \\
&= \frac{\Gamma(t-s-\nu)}{\Gamma(t-s)\Gamma(1-\nu)} \left[\frac{t-s-\nu}{t-s} - 1\right] \\
&= H_{-\nu}(t,s) \left[\frac{t-s-\nu}{t-s} - 1\right],
\end{aligned}
$$

for $t \in \mathbb{N}_{a+2}$ and where $s \in \mathbb{N}_{a+1}^{t-1}$. Combining the above with (5.3), we get

$$
\nabla x(t) = h(t) - \sum_{s=a+1}^{t-1} (H_{-\nu}(t,\rho(s)) - H_{-\nu}(\rho(t),\rho(s)))\nabla x(s).
$$

Then because $\frac{1^{\overline{-\nu}}}{\Gamma(1-\nu)} = 1$, it holds that

$$
\begin{aligned}
h(t) &= \frac{1^{\overline{-\nu}}}{\Gamma(1-\nu)}\nabla x(t) + \sum_{s=a+1}^{t-1} (H_{-\nu}(t,\rho(s)) - H_{-\nu}(\rho(t),\rho(s)))\nabla x(s) \\
&= H_{-\nu}(t,\rho(t))\nabla x(t) + \sum_{s=a+1}^{t-1} H_{-\nu}(t,\rho(s))\nabla x(s) \\
&\qquad\qquad - \sum_{s=a+1}^{t-1} H_{-\nu}(\rho(t),\rho(s))\nabla x(s) \\
&= \sum_{s=a+1}^{t} H_{-\nu}(t,\rho(s))\nabla x(s) - \sum_{s=a+1}^{t-1} H_{-\nu}(t-1,\rho(s))\nabla x(s) \\
&= \nabla_t \int_a^t H_{-\nu}(t,\rho(s))\nabla x(s) \\
&= \nabla\nabla_a^{-(1-\nu)}\nabla x(t) \\
&= \nabla\nabla_{a^*}^{\nu} x(t).
\end{aligned}
$$

Thus, $x : \mathbb{N}_a \to \mathbb{R}$ satisfies the IVP (5.2). Hence a solution to the IVP (5.2) exists. Next reverse the above algebraic steps and notice this shows that if x satisfies the IVP (5.2), then it satisfies the recursive equation (5.3). Therefore, there exists a unique solution to IVP (5.2).

$\square$

Note that the solution to the IVP (5.2) is defined on $\mathbb{N}_a$. Next, we introduce the Cauchy function for the $\nabla\nabla^{\nu}_{a^*}$ operator and use the Cauchy function in the Variation of Constants formula.

Definition 5.2. The Cauchy function for

$$\nabla\nabla^{\nu}_{a^*}y(t) = 0, \quad t \in \mathbb{N}_{a+2}$$

is given by the formula

$$x(t,s) = H_{\nu}(t, \rho(s)),$$

where $s \in \mathbb{N}_{a+2}$.

Theorem 5.3 (Variation of Constants)**.** *Let $h : \mathbb{N}_{a+2} \to \mathbb{R}$ and $0 < \nu \leq 1$. Then the solution to the IVP*

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}y(t) = h(t), \quad t \in \mathbb{N}_{a+2} \\ y(a) = 0 \\ \nabla y(a+1) = 0 \end{cases} \tag{5.4}$$

is given by

$$y(t) = \int_{a+1}^{t} x(t,s)h(s)\nabla s.$$

Proof. First, note that

$$y(a) = \int_{a+1}^{a} x(a,s)h(s)\nabla s = 0$$

and

$$y(a+1) = \int_{a+1}^{a+1} x(a+1,s)\nabla s = 0.$$

Hence, $y : \mathbb{N}_a \to \mathbb{R}$ satisfies the initial conditions. By [31, Theorem 3.107 on p. 218], if $f : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu, \mu > 0$, then $\nabla_a^{-\nu}\nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t)$. Also, according to the Leibniz Formula in [31, Theorem 3.41 on p. 175], if $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$, then $\nabla \left(\int_a^t f(t,\tau)\nabla\tau \right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t)$. Hence

$$\begin{aligned}
\nabla\nabla_{a^*}^{\nu} y(t) &= \nabla\nabla_{a^*}^{\nu} \int_{a+1}^{t} H_\nu(t,\rho(s))h(s)\nabla s \\
&= \nabla\nabla_a^{-(1-\nu)}\nabla \int_{a+1}^{t} H_\nu(t,\rho(s))h(s)\nabla s \\
&= \nabla\nabla_a^{-(1-\nu)} \left[\int_{a+1}^{t} \nabla_t H_\nu(t,\rho(s))h(s)\nabla s + H_\nu(\rho(t),\rho(t))h(t) \right] \\
&= \nabla\nabla_a^{-(1-\nu)} \int_{a+1}^{t} H_{\nu-1}(t,\rho(s))h(s)\nabla s.
\end{aligned}$$

Now if $\nu = 1$, then

$$\begin{aligned}
\nabla\nabla_a^{-(1-\nu)} \int_{a+1}^{t} H_{\nu-1}(t,\rho(s))h(s)\nabla s &= \nabla \int_{a+1}^{t} H_0(t,\rho(s))h(s)\nabla s \\
&= \nabla \int_{a+1}^{t} h(s) \\
&= h(t).
\end{aligned}$$

If $0<\nu<1$, then

$$\begin{aligned}
\nabla\nabla_a^{-(1-\nu)} & \int_{a+1}^t H_{\nu-1}(t,\rho(s))h(s)\nabla s \\
&= \nabla \int_a^t H_{-\nu}(t,\rho(s)) \int_{a+1}^s H_{\nu-1}(s,\rho(\tau))h(\tau)\nabla\tau\nabla s \\
&= \nabla[\int_{a+1}^t H_{-\nu}(t,\rho(s)) \int_{a+1}^s H_{\nu-1}(s,\rho(\tau))h(\tau)\nabla\tau\nabla s \\
&\qquad + H_{-\nu}(t,\rho(a+1)) \int_{a+1}^{a+1} H_{\nu-1}(a+1,\rho(\tau))h(\tau)\nabla\tau] \\
&= \nabla \int_{a+1}^t H_{-\nu}(t,\rho(s)) \int_{a+1}^s H_{\nu-1}(s,\rho(\tau))h(\tau)\nabla\tau\nabla s \\
&= \nabla\nabla_{a+1}^{-(1-\nu)}\nabla_{a+1}^{-\nu}h(t) \\
&= \nabla\nabla_{a+1}^{-(1-\nu)-\nu}f(t) \\
&= \nabla\nabla_{a+1}^{-1}f(t) \\
&= \nabla \sum_{s=a+2}^{t} H_0(t,\rho(s))h(s) \\
&= \nabla \sum_{s=a+2}^{t} h(s) \\
&= h(t).
\end{aligned}$$

Therefore $y:\mathbb{N}_a\to\mathbb{R}$ satisfies the IVP (5.4). $\square$

Lemma 5.4. *The function* $y(t):=(\nabla_a^{-\nu}1)(t)$ *solves* $\nabla[\nabla_{a^*}^{\nu}y(t)]=0$ *for* $0<\nu\leq 1$.

Proof. By [31, Theorem 3.57, part (iv), on p. 186], it holds that $\int_a^t H_\mu(t,\rho(s))\nabla s = H_{\mu+1}(t,a)$ whenever the expressions in this equation are defined. Additionally, by [31, Theorem 3.93, part (i), on p. 207], when $\nu\in\mathbb{R}^+$, $\mu\in\mathbb{R}$ such that μ and $\nu+\mu$ are not negative integers, then $\nabla_a^{-\nu}H_\mu(t,a)=H_{\mu+\nu}(t,a)$ for $t\in\mathbb{N}_a$. By [31], for $k\in\mathbb{N}_0$

and $\mu > 0$, it holds that $\nabla^k \nabla_a^{\mu} f(t) = \nabla_a^{k-\mu} f(t)$ for $t \in \mathbb{N}_{a+k}$. Then if $0 < \nu < 1$

$$\begin{aligned}
\nabla \nabla_{a^*}^{\nu} (\nabla_a^{-\nu} 1)(t) &= \nabla \nabla_a^{-(1-\nu)} \nabla (\nabla_a^{-\nu} 1)(t) \\
&= \nabla \nabla_a^{-(1-\nu)} (\nabla_a^{1-\nu} 1)(t) \\
&= \nabla \nabla_a^{-(1-\nu)} \int_a^t H_{\nu-2}(t, \rho(s)) \nabla s \\
&= \nabla \nabla_a^{-(1-\nu)} H_{\nu-1}(t, a) \\
&= \nabla^2 \nabla_a^{-(2-\nu)} H_{\nu-1}(t, a) \\
&= \nabla^2 H_{(\nu-1)+(2-\nu)}(t, a) \\
&= \nabla^2 H_1(t, a) \\
&= 0.
\end{aligned}$$

Note additionally that if, $\nu = 1$

$$\begin{aligned}
\nabla \nabla_{a^*}^{\nu} (\nabla_a^{-\nu} 1)(t) &= \nabla^2 (\nabla_a^{-1} 1)(t) \\
&= \nabla^2 \int_a^t H_0(t, \rho(s)) \nabla s \\
&= \nabla^2 \int_a^t 1 \nabla s \\
&= 0.
\end{aligned}$$

Hence we are done. $\square$

Note that one can prove Theorem 5.5 and Theorem 5.6. Theorems similar to these, together with proofs, may be found in [3] and [31].

Theorem 5.5. *Suppose $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ are linearly independent solutions to* $\nabla \nabla_{a^*}^{\nu} x(t) =$

0 for $0 < \nu \leq 1$. Then the general solution to $\nabla\nabla^{\nu}_{a^*}x(t) = 0$ is given by

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

for $t \in \mathbb{N}_a$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Theorem 5.6. *Suppose $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ are linearly independent solutions of $\nabla\nabla^{\nu}_{a^*}x(t) = 0$ for $0 < \nu \leq 1$ and $y_0 : \mathbb{N}_a \to \mathbb{R}$ is a particular solution to $\nabla\nabla^{\nu}_{a^*}x(t) = h(t)$ for some $h : \mathbb{N}_{a+1} \to \mathbb{R}$. Then the general solution of $\nabla\nabla^{\nu}_{a^*}x(t) = h(t)$ is given by*

$$x(t) = c_1x_1(t) + c_2x_2(t) + y_0(t),$$

for $t \in \mathbb{N}_a$ and where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Next consider the boundary value problems

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^b_{a+2} \\ x(a) = \nabla x(b) = 0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = h(t), \quad t \in \mathbb{N}^b_{a+2} \\ x(a) = A, \\ \nabla x(b) = B, \end{cases} \tag{5.6}$$

where $0 < \nu \leq 1$, $a - b \in \mathbb{N}_2$, and $h : \mathbb{N}^b_{a+2} \to \mathbb{R}$.

Theorem 5.7. *The BVP (5.6) has a unique solution.*

Proof. First we show that the BVP (5.5) has only the trivial solution. Certainly, $x_1(t) = 1$ is a solution of $\nabla\nabla^{\nu}_{a^*}x(t) = 0$ for $t \in \mathbb{N}^b_a$. Also, by Lemma 5.4, $x_2(t) := (\nabla_a^{-\nu}1)(t)$ is also a solution of $\nabla\nabla^{\nu}_{a^*}x(t) = 0$. Note that

$$\begin{aligned}(\nabla_a^{-\nu}1)(t) &= \int_a^t H_{\nu-1}(t, \rho(s))\nabla s \\ &= H_\nu(t, a) \\ &= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.\end{aligned}$$

As $x_1(t)$ and $x_2(t)$ are linearly independent on $\mathbb{N}^b_a$, the general solution is given by

$$y(t) = c_1 + c_2(\nabla_a^{-\nu}1)(t) = c_1 + c_2\frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

By $x(a) = 0$, we get $c_1 = 0$. Note

$$\nabla x(b) = c_2\left[\frac{(t-a)^{\overline{\nu-1}}}{\Gamma(\nu)}\right]_{t=b} = c_2\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)},$$

so by $\nabla x(b) = 0$, we get that $c_2 = 0$. Hence the BVP (5.5) has only the trivial solution.

Next let $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ be linearly independent solutions to $\nabla\nabla^{\nu}_{a^*}x(t) = 0$. By Theorem 5.5, a general solution to $\nabla\nabla^{\nu}_{a^*}x(t) = 0$ is given by

$$x(t) = c_1x_1(t) + c_2x_2(t),$$

where c_1, c_2 are arbitrary real constants. As shown above, if $x(t)$ solves the BVP (5.5), then $x(t)$ is the trivial solution, which holds if and only if $c_1 = c_2 = 0$, which holds if and only if the system of equations

$$\begin{cases} c_1 x_1(a) + c_2 x_2(a) = 0 \\ c_1 \nabla x_1(b) + c_2 \nabla x_2(b) = 0 \end{cases}$$

has only the trivial solution. That is, $x(t)$ solves the BVP (5.5) if and only if

$$D := \begin{vmatrix} x_1(a) & x_2(a) \\ \nabla x_1(b) & \nabla x_2(b) \end{vmatrix} \neq 0$$

Therefore $D \neq 0$.

Now consider the BVP (5.6). By Theorem 5.6, a general solution to $\nabla\nabla^{\nu}_{a^*}x(t) = h(t)$ is given by

$$x(t) = a_1 x_1(t) + a_2 x_2(t) + y_0(t),$$

where $a_1, a_2 \in \mathbb{R}$ are arbitrary constants and $y_0 : \mathbb{N}_a \to \mathbb{R}$ is a particular solution of $\nabla\nabla^{\nu}_{a^*}x(t) = h(t)$. Consider the system of equations

$$\begin{cases} a_1 x_1(a) + a_2 x_2(a) + y_0(a) = A \\ \nabla[a_1 x_1(b) + a_2 x_2(b) + y_0(b)] = B, \end{cases}$$

where $A, B \in \mathbb{R}$ are as in the BVP (5.6), which is equivalent to

$$\begin{cases} a_1 x_1(a) + a_2 x_2(a) = A - y_0(a) \\ a_1 \nabla x_1(b) + a_2 \nabla x_2(b) = B - \nabla y_0(b), \end{cases}$$

which has a unique solution since $D \neq 0$. Therefore the BVP (5.6) has a unique solution. $\square$

Next we define the Green's function for the BVP (5.5) and its basic property in Theorem 5.9. The formula for this Green's function can be found in Section 5.2.

Definition 5.8. Define the Green's function, $G(t,s) : \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$, for the homogeneous BVP (5.5), by

$$G(t,s) := \begin{cases} u(t,s), & a \le t \le s \le b, \\ v(t,s), & a \le s \le t \le b, \end{cases}$$

where $u(t,s)$ solves the BVP

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+2}^b \\ u(a,s) = 0, \\ \nabla u(b) = -\nabla x(b,s), \end{cases}$$

for each fixed $s \in \mathbb{N}_{a+2}^b$ and where $x(t,s)$ is the Cauchy function for $\nabla\nabla_{a^*} x(t)$. Then we define

$$v(t,s) := u(t,s) + x(t,s).$$

Theorem 5.9. *The solution to the BVP (5.6) where $A = B = 0$ is given by*

$$y(t) = \int_{a+1}^{b} G(t,s)h(s)\nabla s$$

where $G(t,s) : \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$ is the Green's function for the homogeneous BVP (5.5).

Proof. Let $y(t) := \int_{a+1}^{b} G(t,s)h(s)\nabla s$ for $t \in \mathbb{N}_a^b$. Then for $t \in \mathbb{N}_a^b$

$$\begin{aligned} y(t) &= \int_{a+1}^{t} v(t,s)h(s)\nabla s + \int_{t}^{b} u(t,s)h(s)\nabla s \\ &= \int_{a+1}^{t} (u(t,s) + x(t,s))h(s)\nabla s + \int_{t}^{b} u(t,s)h(s)\nabla s \\ &= \int_{a+1}^{b} u(t,s)h(s)\nabla s + \int_{a+1}^{t} x(t,s)h(s)\nabla s. \end{aligned}$$

Define $z(t) := \int_{a+1}^{t} x(t,s)h(s)\nabla s$ for $t \in \mathbb{N}_a^b$. By the Variation of Constants formula in Theorem 5.3, $z(t)$ solves IVP

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} z(t) = h(t), \quad t \in \mathbb{N}_{a+2} \\ z(a) = \nabla z(a+1) = 0, \end{cases} \tag{5.7}$$

for $t \in \mathbb{N}_a^b$. Note that $\nabla\nabla_{a^*}^{\nu}$ is a linear operator. Thus for $t \in \mathbb{N}_{a+2}^b$

$$\begin{aligned} \nabla\nabla_{a^*}^{\nu} y(t) &= \nabla\nabla_{a^*}^{\nu} \left[\int_{a+1}^{b} u(t,s)h(s)\nabla s + z(t)\right] \\ &= \int_{a+1}^{b} \nabla\nabla_{a^*}^{\nu} u(t,s)h(s)\nabla s + \nabla\nabla_{a^*}^{\nu} z(t) \\ &= 0. \end{aligned}$$

It will now be shown that the boundary conditions hold. First,

$$y(a) = \int_{a+1}^{b} u(a,s)h(s)\nabla s + z(a) = 0 + 0 = 0.$$

Recall that according to the Leibniz Formula in [31, Theorem 3.41 on p. 175], if

$f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$, then $\nabla \left(\int_a^t f(t,\tau)\nabla\tau \right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t)$. So

$$\begin{aligned}
\nabla y(b) &= \int_{a+1}^{b} \nabla u(b,s)h(s)\nabla s + \nabla z(b) \\
&= \int_{a+1}^{b} \nabla u(b,s)h(s)\nabla s + \left[\nabla \int_{a+1}^{t} H_\nu(t,\rho(s))h(s)\nabla s \right]_{t=b} \\
&= \int_{a+1}^{b} \nabla u(b,s)h(s)\nabla s + \left[\int_{a+1}^{t} \nabla H_\nu(t,\rho(s))h(s)\nabla s + H_\nu(\rho(t),\rho(t)) \right]_{t=b} \\
&= \int_{a+1}^{b} \nabla u(b,s)h(s)\nabla s + \int_{a+1}^{b} \nabla H_\nu(b,\rho(s))h(s)\nabla s \\
&= -\int_{a+1}^{b} \nabla x(b,s)h(s)\nabla s + \int_{a+1}^{b} \nabla x(b,s)h(s)\nabla s \\
&= 0.
\end{aligned}$$

Hence the boundary conditions hold. $\square$

5.2 The Green's Function

In this section, we find the formula for the Green's function for the BVP (5.5). The proof of Theorem (5.10) is analogous to a proof from [3], in which the formula for a Green's function for a different BVP was found.

Theorem 5.10. *Assume $a, b \in \mathbb{R}$, that $b-a$ is a positive integer, $b-a \geq 2$, and $0 < \nu \leq 1$. The Green's function, $G(t,s) : \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$, for the BVP (5.5) is given by*

$$G(t,s) = \begin{cases} u(t,s), & a \leq t \leq s \leq b, \\ v(t,s), & a \leq s \leq t \leq b. \end{cases}$$

where

$$u(t,s) = -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}$$

and

$$v(t,s) = u(t,s) + \frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

Proof. By the definition of the Green's function it is true that

$$G(t,s) = \begin{cases} u(t,s), & a \leq t \leq s \leq b, \\ v(t,s), & a \leq s \leq t \leq b. \end{cases}$$

where $u(t,s)$ for each fixed $s \in \mathbb{N}_{a+2}^{b}$ solves the BVP

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} u(t,s) = 0, & t \in \mathbb{N}_{a+2}^{b} \\ u(a,s) = 0 \\ \nabla u(b,s) = -\nabla x(b,s). \end{cases}$$

for $t \in \mathbb{N}_{a}^{b}$, and $v(t,s) = u(t,s) + x(t,s)$. As shown in the proof of Theorem 5.7, the general solution to $\nabla\nabla_{a^*}^{\nu} y(t) = 0$ is given by

$$y(t) = c_1 + c_2(\nabla_a^{-\nu} 1)(t) = c_1 + c_2 \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)}.$$

Hence, it follows that

$$u(t,s) = c_1(s) + c_2(s) \frac{(t-a)^{\overline{\nu}}}{\Gamma(1+\nu)}.$$

The boundary condition $u(a,s) = 0$ implies that $c_1(s) = 0$. Now consider the boundary condition $\nabla u(b,s) = -\nabla x(b,s)$. It holds that

$$x(b,s) = H_\nu(t,\rho(s))|_{t=b} = \frac{(b-s+1)^{\overline{\nu}}}{\Gamma(1+\nu)}.$$

Hence

$$\nabla x(b,s) = H_{\nu-1}(t,\rho(s))|_{t=b} = \frac{(b-s+1)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Also,

$$-\nabla x(b,s) = \nabla u(b,s) = c_2(s)\frac{(b-a)^{\overline{\nu-1}}}{\Gamma(\nu)}.$$

Therefore,

$$c_2(s) = -\frac{(b-s+1)^{\overline{\nu-1}}}{(b-a)^{\overline{\nu-1}}}.$$

This completes the proof.

□

Note that if $\nu = 1$, then the BVP (5.5) appears as follows

$$\begin{cases} \nabla^2 x(t) = 0, \quad t \in \mathbb{N}_{a+2}^b \\ x(a) = \nabla x(b) = 0, \end{cases} . \tag{5.8}$$

where $b - a \in \mathbb{N}_2$ and $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$.

Also, the Green's function for the BVP (5.8)

$$G(t,s) = \begin{cases} -(t-a), \quad a \leq t \leq s \leq b \\ -(s-a-1), \quad a \leq s \leq t \leq b. \end{cases}$$

Next consider the following boundary value problem from standard ordinary differential equations

$$\begin{cases} x'' = 0, \\ x(a) = x'(b) = 0. \end{cases} \tag{5.9}$$

By [41], the Green's function for BVP (5.9) is

$$G(t,s) = \begin{cases} -(t-a), & a \le t \le s \le b \\ -(s-a), & a \le s \le t \le b. \end{cases}$$

Hence, the Green's function for the BVP (5.5) is very similar to the Green's function for a comparable BVP from standard differential equations.

5.3 Bounds on the Green's Function

In this section, we establish three bounds on the Green's function for the BVP (5.5). Then we will discuss how these bounds compare to bounds on the Green's function in the continuous case.

Theorem 5.11. *Assume $b-a \in \mathbb{N}_2$, and $0 < \nu \le 1$. The Green's function, $G(t,s) : \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$, for the BVP (5.5) satisfies the inequalities*

(i) $G(t,s) \le 0$,

(ii) $G(t,s) \ge -\frac{(b-a+\nu-1)}{\nu}$.

(iii) $\int_{a+1}^b |G(t,s)|\nabla s \le \frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)}$,

for $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^b$.

Proof. (i.) First, write

$$\begin{aligned} u(t,s) &= -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} \\ &= -\frac{\Gamma(b-s+\nu)\Gamma(t-a+\nu)\Gamma(b-a)}{\Gamma(b-a+\nu-1)\Gamma(b-s+1)\Gamma(t-a)\Gamma(1+\nu)}. \end{aligned}$$

Then, if we assume that $a \leq t \leq s \leq b$, from the above it can be seen that $u(t,s) \leq 0$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.

Next, let $a \leq s \leq t \leq b$. Note that $v(t,a) = 0$. It will now be shown that, for each fixed t, $v(t,s)$ is decreasing with respect to s for $s \in \mathbb{N}_a^b$:

$$\begin{aligned}
\nabla_s v(t,s) &= \nabla_s \left(-\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} + \frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)} \right) \\
&= -\frac{-(\nu-1)(b-\rho(s)+1)^{\overline{\nu-2}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} - \frac{\nu(t-\rho(s)+1)^{\overline{\nu-1}}}{\Gamma(1+\nu)} \\
&= -\frac{(1-\nu)(b-s+2)^{\overline{\nu-2}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} - \frac{\nu(t-s+2)^{\overline{\nu-1}}}{\Gamma(1+\nu)} \\
&= -\frac{(1-\nu)\Gamma(b-s+\nu)\Gamma(b-a)\Gamma(t-a+\nu)}{\Gamma(b-s+2)\Gamma(b-a+\nu-1)\Gamma(t-a)\Gamma(1+\nu)} - \frac{\nu\Gamma(t-s+\nu+1)}{\Gamma(1+\nu)\Gamma(t-s+2)},
\end{aligned}$$

where one can see that the last line is nonpositive when $a \leq s < t \leq b$. Thus $v(t,s) \leq 0$ when $a \leq s \leq t \leq b$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_a^b$, so certainly $v(t,s) \leq 0$ when $a \leq s \leq t \leq b$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. Therefore $G(t,s) \leq 0$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.

(ii.) First let $a \leq s \leq t \leq b$. Then

$$v(t,s) = u(t,s) + \frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)} \geq u(t,s).$$

Hence it suffices to show that

$$u(t,s) = -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} \geq -\frac{(b-a+\nu-1)}{\nu}.$$

for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. First an upper bound of $(b-s+1)^{\overline{\nu-1}}$ will be found. Note that

$$\begin{aligned}(b-b+1))^{\overline{\nu-1}} &= 1^{\overline{\nu-1}}\\ &= \frac{\Gamma(1+\nu-1)}{\Gamma(1)}\\ &= \Gamma(\nu).\end{aligned}$$

Next, say $s \leq b$. The following shows that $(b-s+1)^{\overline{\nu-1}}$ is increasing with respect to s. That is

$$\begin{aligned}\nabla_s(b-s+1)^{\overline{\nu-1}} &= -(\nu-1)(b-\rho(s)+1)^{\overline{\nu-2}}\\ &= (1-\nu)(b-s+2)^{\overline{\nu-2}}\\ &= \frac{(1-\nu)\Gamma(b-s+\nu)}{\Gamma(b-s+2)},\end{aligned}$$

noting that the final expression is nonnegative. Hence $(b-s)^{\overline{\nu-1}}$ is increasing for $s \leq b$. Thus

$$(b-s+1)^{\overline{\nu-1}} \leq \Gamma(\nu),$$

for $s \leq b$. Also,

$$\begin{aligned}\nabla_t(t-a)^{\overline{\nu}} &= \nu(t-a)^{\overline{\nu-1}}\\ &= \frac{\nu\Gamma(t-a+\nu-1)}{\Gamma(t-a)}\\ &\geq 0,\end{aligned}$$

so $(t-a)^{\overline{\nu}}$ is increasing with respect to t for $t \in \mathbb{N}_a^b$. Hence for $t \in \mathbb{N}_a^b$,

$$(t-a)^{\overline{\nu}} \leq (b-a)^{\overline{\nu}}.$$

Therefore,

$$\begin{aligned} u(t,s) &= -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} \\ &\geq -\frac{\Gamma(\nu)(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} \\ &= -\frac{\Gamma(b-a)(t-a)^{\overline{\nu}}}{\nu\Gamma(b-a+\nu-1)} \\ &\geq -\frac{\Gamma(b-a)(b-a)^{\overline{\nu}}}{\nu\Gamma(b-a+\nu-1)} \\ &= -\frac{\Gamma(b-a)\Gamma(b-a+\nu)}{\nu\Gamma(b-a+\nu-1)\Gamma(b-a)} \\ &= -\frac{(b-a+\nu-1)\Gamma(b-a+\nu-1)}{\nu\Gamma(b-a+\nu-1)} \\ &= -\frac{b-a+\nu-1}{\nu}. \end{aligned}$$

Note that the above steps hold for both $a \leq s \leq t \leq b$ and $a \leq t \leq s \leq b$, so it is also true that

$$v(t,s) \geq -\frac{b-a+\nu-1}{\nu}.$$

Therefore $G(t,s) \geq -\frac{(b-a+\nu-1)}{\nu}$ for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$.

(iii.) For $t \in \mathbb{N}_a^b$, the following calculations hold:

$$\begin{aligned} \int_{a+1}^{b} |G(t,s)|\nabla s &= \int_{a+1}^{t} |v(t,s)|\nabla s + \int_{t}^{b} |u(t,s)|\nabla s \\ &= \int_{a+1}^{t} \left| -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} + \frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)} \right| \nabla s \\ &\quad + \int_{t}^{b} \left| -\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} \right| \nabla s \\ &= \int_{a+1}^{t} \frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)} - \frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)} \nabla s \end{aligned}$$

$$
\begin{aligned}
&+\int_{t}^{b}\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}\nabla s\\
&=\int_{a+1}^{b}\frac{(b-s+1)^{\overline{\nu-1}}(t-a)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}\nabla s-\int_{a+1}^{t}\frac{(t-s+1)^{\overline{\nu}}}{\Gamma(\nu+1)}\nabla s\\
&=\left[\frac{-(b-s)^{\overline{\nu}}(t-a)^{\overline{\nu}}}{\nu(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}\right]_{s=a+1}^{s=b}-\left[\frac{-(t-s)^{\overline{\nu+1}}}{(\nu+1)\Gamma(\nu+1)}\right]_{s=a+1}^{s=t}\\
&=\frac{(b-a-1)^{\overline{\nu}}(t-a)^{\overline{\nu}}}{\nu(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}-\frac{(t-a-1)^{\overline{\nu+1}}}{\Gamma(\nu+2)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{(b-a-1)^{\overline{\nu}}}{(b-a)^{\overline{\nu-1}}}&=\frac{\Gamma(b-a-1+\nu)\Gamma(b-a)}{\Gamma(b-a-1+\nu)\Gamma(b-a-1)}\\
&=\frac{(b-a-1)\Gamma(b-a-1)}{\Gamma(b-a-1)}\\
&=b-a-1.
\end{aligned}
$$

Hence, using the fact that $(t-a-1)^{\overline{\nu+1}}=(t-a-1)(t-a)^{\overline{\nu}}$ from [3], for $t\in\mathbb{N}_a^b$,

$$
\begin{aligned}
\frac{(b-a-1)^{\overline{\nu}}(t-a)^{\overline{\nu}}}{\nu(b-a)^{\overline{\nu-1}}\Gamma(1+\nu)}-\frac{(t-a-1)^{\overline{\nu+1}}}{\Gamma(\nu+2)}&=\frac{(b-a-1)(t-a)^{\overline{\nu}}}{\nu\Gamma(1+\nu)}-\frac{(t-a-1)^{\overline{\nu+1}}}{\Gamma(\nu+2)}\\
&=\frac{(b-a-1)(t-a)^{\overline{\nu}}}{\nu\Gamma(1+\nu)}-\frac{(t-a-1)(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}\\
&=\frac{(t-a)^{\overline{\nu}}}{\Gamma(1+\nu)}\left[\frac{b-a-1}{\nu}-\frac{t-a-1}{1+\nu}\right]\\
&=\frac{(t-a)^{\overline{\nu}}}{\Gamma(1+\nu)}\left[\frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu(1+\nu)}\right]\\
&\le\frac{t-a}{\Gamma(1+\nu)}\left[\frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu(1+\nu)}\right].
\end{aligned}
\tag{5.9}
$$

Next, it will be shown that (5.9) is increasing with respect to t for $t\in\mathbb{N}_a^b$. Using the product rule, it holds that

$$
\begin{aligned}
\nabla_t \frac{t-a}{\Gamma(1+\nu)} & \left[\frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu(1+\nu)}\right] \\
&= \frac{t-1-a}{\Gamma(1+\nu)} \cdot \frac{-\nu}{\nu(1+\nu)} + \frac{1}{\Gamma(1+\nu)} \cdot \frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu(1+\nu)} \\
&= \frac{-(t-a-1)}{\Gamma(2+\nu)} + \frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu\Gamma(2+\nu)} \\
&= \frac{-\nu(t-a-1)+(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu\Gamma(2+\nu)} \\
&= \frac{-2\nu(t-a-1)+(1+\nu)(b-a-1)}{\nu\Gamma(2+\nu)} \\
&\geq \frac{-2\nu(b-a-1)+(1+\nu)(b-a-1)}{\nu\Gamma(2+\nu)} \\
&= \frac{(1-\nu)(b-a-1)}{\nu\Gamma(2+\nu)} \\
&\geq 0.
\end{aligned}
$$

Hence (5.9) is increasing for $t \in \mathbb{N}_a^b$. Thus

$$
\begin{aligned}
\frac{t-a}{\Gamma(1+\nu)} & \left[\frac{(1+\nu)(b-a-1)-\nu(t-a-1)}{\nu(1+\nu)}\right] \\
&\leq \frac{b-a}{\Gamma(1+\nu)} \left[\frac{(1+\nu)(b-a-1)-\nu(b-a-1)}{\nu(1+\nu)}\right] \\
&= \frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)}.
\end{aligned}
$$

This completes the proof. □

Recall that if $\nu = 1$, then the BVP (5.5) appears as BVP (5.8).

Then the bounds from Theorem 5.11 lead to the following for $\nu = 1$:

(i.) $G(t,s) \leq 0$,

(ii.) $G(t,s) \geq -(b-a)$, and

(iii.) $\int_{a+1}^{b} |G(t,s)|\nabla s \leq \frac{(b-a)(b-a-1)}{\Gamma(3)} \leq \frac{(b-a)^2}{2}$,

for $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. Next recall the boundary value problem (5.9) from standard ordinary differential equations. By [41], we have

(i.) $G(t,s) \leq 0$ for $s,t \in [a,b]$,

(ii.) $G(t,s) \geq -(b-a)$ for $a \leq t \leq b$, and

(iii.) $\int_a^b |G(t,s)|ds \leq \frac{(b-a)^2}{2}$ for $a \leq t \leq b$.

Between the analogous BVP (5.5) and the BVP (5.9), note the strong similarities between the bounds on the Green's functions for each.

5.4 Future Work

In addition to the bounds on Theorem 5.11, I conjecture that

$$\int_{a+1}^{b} |\nabla_t G(t,s)|\nabla s \leq \frac{b-a}{\nu}$$

for $t \in \mathbb{N}_a^b$. However, this conjecture remains to be completely proven.

More generally, it remains to build on Theorem 5.11 to eventually approach a non-linear case for the BVP (5.1). This case could be approached through a fixed-point or through a contraction mapping theorem.

Since the Green's function in Theorem 5.2 may be used to calculate solutions to the BVP (5.1), one can investigate the behavior of these solutions.

Lastly, one could consider other variations on the BVP (5.1). That is, one could consider the BVP

$$\begin{cases} \nabla\left[p(t)\nabla_{a^*}^{\nu}x(t)\right] + q(t)x(t-1) = h(t), \quad t \in \mathbb{N}_{a+2}^{b} \\ x(a) = \nabla x(b) = 0, \end{cases} \tag{5.10}$$

for $0 < \nu < 1$, $b - a \in \mathbb{N}_2$, $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, $q : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, and $p : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$, either in full generality or the various cases resulting from the BVP (5.1). Additionally one could consider other variations in the boundary conditions of the BVP (5.1). One other such combination has been investigated in [3], but many other combinations remain to be explored.

Bibliography

[1] Ravi P. Agarwal, Maria Meehan, and Donal O'Regan. *Fixed point theory and applications*, volume 141 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[2] K. Ahrendt, L. Castle, M. Holm, and K. Yochman. Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula. *Commun. Appl. Anal.*, 16(3):317–347, 2012.

[3] K. Ahrendt, T. Rolling, L. Dewolf, L. Mazurowski, K. Mitchell, and D. Veconi. Initial and boundary value problems for the caputo fractional self-adjoint difference equations. *EPAM*, 2(1):105–141, 2016.

[4] George A. Anastassiou. Foundations of nabla fractional calculus on time scales and inequalities. *Comput. Math. Appl.*, 59(12):3750–3762, 2010.

[5] George A. Anastassiou. Nabla discrete fractional calculus and nabla inequalities. *Math. Comput. Modelling*, 51(5-6):562–571, 2010.

[6] George A. Anastassiou. Right nabla discrete fractional calculus. *Int. J. Difference Equ.*, 6(2):91–104, 2011.

[7] George A. Anastassiou. Nabla fractional calculus on time scales and inequalities. *J. Concr. Appl. Math.*, 11(1):96–111, 2013.

[8] Douglas R. Anderson. Taylor polynomials for nabla dynamic equations on time scales. *Panamer. Math. J.*, 12(4):17–27, 2002.

[9] Douglas R. Anderson and Richard I. Avery. Application of the omitted ray fixed point theorem. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 17, 9, 2014.

[10] Douglas R. Anderson and Joan Hoffacker. Existence of solutions for a cantilever beam problem. *J. Math. Anal. Appl.*, 323(2):958–973, 2006.

[11] Sadia Arshad and Vasile Lupulescu. On the fractional differential equations with uncertainty. *Nonlinear Anal.*, 74(11):3685–3693, 2011.

[12] F. Merdivenci Atici and G. Sh. Guseinov. On Green's functions and positive solutions for boundary value problems on time scales. *J. Comput. Appl. Math.*, 141(1-2):75–99, 2002. Dynamic equations on time scales.

[13] Ferhan M. Atıcı and Paul W. Eloe. Discrete fractional calculus with the nabla operator. *Electron. J. Qual. Theory Differ. Equ.*, (Special Edition I):No. 3, 12, 2009.

[14] Ferhan M. Atici and Paul W. Eloe. Initial value problems in discrete fractional calculus. *Proc. Amer. Math. Soc.*, 137(3):981–989, 2009.

[15] Ferhan M. Atici and Paul W. Eloe. Linear systems of fractional nabla difference equations. *Rocky Mountain J. Math.*, 41(2):353–370, 2011.

[16] Ferhan M. Atıcı and Paul W. Eloe. Two-point boundary value problems for finite fractional difference equations. *J. Difference Equ. Appl.*, 17(4):445–456, 2011.

[17] Ferhan M. Atıcı and Sevgi Şengül. Modeling with fractional difference equations. *J. Math. Anal. Appl.*, 369(1):1–9, 2010.

[18] Richard Avery, Johnny Henderson, and Xueyan Liu. Omitted ray fixed point theorem. *J. Fixed Point Theory Appl.*, 17(2):313–330, 2015.

[19] Pushp Awasthi. *Boundary value problems for discrete fractional equations*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Nebraska - Lincoln.

[20] M. Benchohra, S. Hamani, and S. K. Ntouyas. Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Anal.*, 71(7-8):2391–2396, 2009.

[21] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser Boston, 2012.

[22] Martin Bohner and Allan Peterson, editors. *Advances in dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[23] Abigail Brackins. *Boundary value problems of nabla fractional difference equations*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–The University of Nebraska - Lincoln.

[24] Wing-Sum Cheung, Jingli Ren, Patricia J. Y. Wong, and Dandan Zhao. Multiple positive solutions for discrete nonlocal boundary value problems. *J. Math. Anal. Appl.*, 330(2):900–915, 2007.

[25] Jeffrey J. DaCunha, John M. Davis, and Parmjeet K. Singh. Existence results for singular three point boundary value problems on time scales. *J. Math. Anal. Appl.*, 295(2):378–391, 2004.

[26] Klaus Deimling. *Nonlinear functional analysis*. Springer-Verlag, Berlin, 1985.

[27] L. Erbe and A. Peterson. Positive solutions for a nonlinear differential equation on a measure chain. *Math. Comput. Modelling*, 32(5-6):571–585, 2000. Boundary value problems and related topics.

[28] Lynn Erbe, Christopher S Goodrich, Baoguo Jia, and Allan Peterson. Survey of the qualitative properties of fractional difference operators: monotonicity, convexity, and asymptotic behavior of solutions. *Adv. Difference Equ.*, page 2016:43, 2016.

[29] Rui A. C. Ferreira. Existence and uniqueness of solution to some discrete fractional boundary value problems of order less than one. *J. Difference Equ. Appl.*, 19(5):712–718, 2013.

[30] Rui A. C. Ferreira and Christopher S. Goodrich. Positive solution for a discrete fractional periodic boundary value problem. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 19(5):545–557, 2012.

[31] C. Goodrich and A. Peterson. *Discrete Fractional Calculus*. Springer, 2015.

[32] Christopher S. Goodrich. *An analysis of nonlocal boundary value problems of fractional and integer order*. ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–The University of Nebraska - Lincoln.

[33] Da Jun Guo. A new fixed-point theorem. *Acta Math. Sinica*, 24(3):444–450, 1981.

[34] Da Jun Guo. Some fixed point theorems on cone maps. *Kexue Tongbao (English Ed.)*, 29(5):575–578, 1984.

[35] Da Jun Guo and V. Lakshmikantham. *Nonlinear problems in abstract cones*, volume 5 of *Notes and Reports in Mathematics in Science and Engineering*. Academic Press, Inc., Boston, MA, 1988.

[36] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[37] Michael Holm. *The theory of discrete fractional calculus: Development and application*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–The University of Nebraska - Lincoln.

[38] Rochdi Jebari and Abderrahman Boukricha. Solvability and positive solution of a system of second-order boundary value problem with integral boundary conditions. *Bound. Value Probl.*, pages 2014:262, 13, 2014.

[39] Baoguo Jia, Lynn Erbe, and Allan Peterson. Comparison theorems and asymptotic behavior of solutions of discrete fractional equations. *Electron. J. Qual. Theory Differ. Equ.*, pages Paper No. 89, 18, 2015.

[40] Baoguo Jia, Lynn Erbe, and Allan Peterson. Some relations between the Caputo fractional difference operators and integer-order differences. *Electron. J. Differential Equations*, pages No. 163, 7, 2015.

[41] Walter G. Kelley and Allan C. Peterson. *The theory of differential equations*. Universitext. Springer, New York, second edition, 2010. Classical and qualitative.

[42] M. A. Krasnosel′skiĭ. *Positive solutions of operator equations*. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron. P. Noordhoff Ltd. Groningen, 1964.

[43] Man Kam Kwong. On Krasnoselskii's cone fixed point theorem. *Fixed Point Theory Appl.*, pages Art. ID 164537, 18, 2008.

[44] V. Lakshmikantham and A. S. Vatsala. Basic theory of fractional differential equations. *Nonlinear Anal.*, 69(8):2677–2682, 2008.

[45] Keith B. Oldham and Jerome Spanier. *The fractional calculus*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974. Theory and applications of differentiation and integration to arbitrary order, With an annotated chronological bibliography by Bertram Ross, Mathematics in Science and Engineering, Vol. 111.

[46] J. St. Goar. A nonlinear right focal boundary value problem in nabla fractional calculus. *EPAM*, 2(1):25–52, 2016.

[47] Xinwei Su. Boundary value problem for a coupled system of nonlinear fractional differential equations. *Appl. Math. Lett.*, 22(1):64–69, 2009.

[48] Eberhard Zeidler. *Nonlinear functional analysis and its applications. I*. Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.