Saturated fusion systems and finite groups

by

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Abstract

This thesis is primarily concerned with saturated fusion systems over groups of shape $q^r : q$ where $q = p^n$ for some odd prime p and some natural number n. We shall present two results related to these fusion systems.

Our first result is a complete classification of saturated fusion systems over a Sylow p-subgroup of $\mathrm{SL}_3(q)$ (which has shape $q^3 : q$). This extends a result of Albert Ruiz and Antonio Viruel, who studied the case when $q = p$ in [36]. As an immediate consequence of this result we shall have a complete classification of p-local finite groups over Sylow p-subgroups of $\mathrm{SL}_3(q)$.

In the second half of this thesis we shall construct an infinite family of exotic fusion systems over some groups of shape $p^r : p$. This extends some work of Broto, Levi and Oliver, who studied the case when $r = 3$ in [12].

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Contents

Introduction

In a finite group G, conjugate elements x and y have the same order. If $\pi(|x|) = \{p_1, \dots p_r\}$ (where for any $n \in \mathbb{N}$, $\pi(n)$ is the set of primes dividing n) then $x = x_1 \dots x_r$ and $y = y_1 \dots y_r$ where x_i, y_i are uniquely determined p_i-elements of $\langle x \rangle$ and $\langle y \rangle$ respectively. This means that if x and y are conjugate in G, then so are x_i and y_i. This property, together with Sylow's Theorem, shows that many problems involving conjugacy in finite groups can be reduced to problems about conjugacy of elements in Sylow p-subgroups of finite groups. This is the concept of *fusion*; more formally, we say that elements x and y of a Sylow p-subgroup S of a finite group G are *fused* if they are conjugate in G but not necessarily in S.

A *fusion system* over a finite group S is a category whose objects are the subgroups of S and whose morphisms consist of monomorphisms of groups, and which include all the maps induced by conjugation in S. A natural example of a fusion system is obtained by taking a subgroup S of a group G, and setting $\mathcal{F}_S(G)$ to be the fusion system whose morphisms are simply the maps induced by conjugation in G. The idea of encoding information about fusion in a category is due to Puig [33], who used them to study p-blocks of finite groups. This idea was subsequently picked up by Broto, Levi and Oliver in [10] and used to study classifying spaces of finite groups. In [11] they studied a class of topological spaces which behave similarly to classifying spaces of finite groups. This lead them to define *p-local finite groups*, algebraic objects which admit a notion of a 'classifying space' which gives rise to such topological spaces. Part of this definition formalizes the notion of a fusion system of a finite p-group, where p is a prime.

One of the most important properties of fusion in finite groups is explained by *Alperin's Fusion Theorem* (see [1]). This shows that all fusion in a finite group is *local*, *i.e.* that any fusion in a finite group G takes place inside the normalizers of certain non-identity p-subgroups of G. The concept of a *saturated* fusion system was first introduced by Puig in [33] (who called them *full Frobenius systems*). A saturated fusion system has the property that some form of Alperin's Fusion Theorem holds, *i.e.* that the fusion is local in some sense. In particular, a saturated fusion system is controlled by a subclass of subgroups which we call *Alperin subgroups*. The definitions were later reformulated by Broto, Levi and Oliver in [11], and more recently refined by Stancu in [38].

For a prime p, a p-local finite group consists of a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p-group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a category associated to $\mathcal{F}$ called a *centric linking system*. The most natural examples of p-local finite groups are derived from finite groups. However, there is considerable interest in finding p-local finite groups that *do not* come from finite groups. Such objects are called *exotic* p-local finite groups. Several examples have already been found, including the so-called *Solomon* 2-local finite groups discovered by Levi and Oliver in [25]. These were based on some considerations of Solomon in [37] on the Sylow 2-subgroups of Co_3, and some later work of Benson in [4]. It was later shown by Aschbacher and Chermak in [3] that using amalgams an infinite group can be constructed which gives rise to the saturated fusion system in some of the Solomon 2-local finite groups.

Further examples of exotic p-local finite groups were discovered by Ruiz and Viruel in [36]. In that paper, they classified the saturated fusion systems over the group p_+^{1+2}, the extraspecial group of order p^3 and of exponent p, and found several which did not come from finite groups for $p = 3$, 5, 7 and 13. They then used a theorem of Broto, Levi and

Oliver [11, Theorem E] to show that these fusion systems give rise to exotic p-local finite groups.

In some sense, the goal of this thesis is to find some new exotic fusion systems. In chapters 2-4, we investigate saturated fusion systems over Sylow p-subgroups of $\mathrm{SL}_3(p^n)$ for any n (note that p_+^{1+2} is a Sylow p-subgroup of $\mathrm{SL}_3(p)$). In fact this does not lead to any new exotic fusion systems, but it does lead to and extension of the classification of p-local finite groups over p_+^{1+2} given by Ruiz and Viruel to Sylow p-subgroups of $\mathrm{SL}_3(p^n)$. In chapters 5 and 6 we shall construct an infinite family of saturated fusion systems which we shall be able to show are exotic.

In our investigation of saturated fusion systems over Sylow p-subgroups of $\mathrm{SL}_3(p^n)$, we are able to show that all of these fusion systems, apart from the exceptions found by Ruiz and Viruel, come from finite groups. Let $q = p^n$ and $S \in \mathrm{Syl}_p(\mathrm{SL}_3(q))$. More specifically, we show that a saturated fusion system $\mathcal{F}$ over S comes from one of:

- the semidirect product $S : A$ where A is isomorphic to a p'-subgroup of $\Gamma\mathrm{L}_2(q) := \mathrm{Aut}(\mathrm{GL}_2(q))$;
- a group of the form $q^2 : B$ where $\mathrm{SL}_2(q) \leq B \leq \Gamma\mathrm{L}_2(q)$ and $B/\mathrm{SL}_2(q)$ has p'-order; or
- a group G where $\mathrm{PSL}_3(q) \leq G \leq \mathrm{Aut}(\mathrm{PSL}_3(q))$,

or it is one of the exceptional cases of Ruiz and Viruel, which includes fusion systems coming from the Tits group, sporadic simple groups and extensions of sporadic simple groups, as well as three exotic fusion systems.

The remaining chapters are devoted to the construction of an infinite family of exotic fusion systems, similar to those constructed by Broto *et al.* in [12]. These arise by considering the so-called *basic* modules for $\mathrm{GL}_2(p)$ over finite fields.

In the first chapter we introduce the basic theory of fusion systems, and we shall prove a form of the Frattini Lemma for saturated fusion systems. This will help us determine the structure of the fusion systems we shall be studying.

In chapter 2 we investigate the properties of the Sylow p-subgroups S of $\mathrm{SL}_3(p^n)$. As a result we shall be able to identify an important class of subgroups which contains the Alperin subgroups for any saturated fusion system $\mathcal{F}$ over our group S.

The third chapter uses results of Timmesfeld to identify a subgroup isomorphic to $\mathrm{SL}_2(p^n)$ in the so-called $\mathcal{F}$-normalizers of proper Alperin subgroups. This will be used in later chapters to apply the Frattini Lemma for saturated fusion systems.

In the fourth chapter we complete the picture by analysing the $\mathcal{F}$-normalizer of the whole group S, and investigating how the $\mathcal{F}$-normalizers of the proper Alperin subgroups interact. We do this by proving a result about certain subgroups of $\mathrm{PGL}_2(p^n)$, which also sheds some light on how the exceptional fusion systems discovered by Ruiz and Viruel come about.

In chapter 5 we shall construct a family of fusion systems (which we call $\mathcal{E}(n,p)$, where $2 \leq n \leq p-1$ and p is an odd prime), and show that they are saturated using a theorem of Broto *et al.*. These will be fusion systems over Sylow p-subgroups of certain semidirect products which arise naturally when considering the basic irreducible $\mathbb{F}_p\mathrm{GL}_2(p)$-modules. We call these subgroups $S(n,p)$. In fact, when $n = 2$ this is exactly the case considered by Broto, Levi and Oliver in [12].

In the final chapter, we prove that the fusion systems studied in chapter 5 are in fact exotic, whenever $n \geq 5$ and $p \geq 13$. We introduce these mild conditions on n and p in order to minimize technical difficulties. It may in fact be possible to show that the fusion systems $\mathcal{E}(n,p)$ are exotic for smaller values of n and p; indeed, Broto *et al.* proved that they are exotic for $n = 2$ and $p \geq 3$. To show that the systems $\mathcal{E}(n,p)$ are exotic we use another theorem of Broto *et al.* to show that it suffices to check that no almost simple group gives rise to one of our fusion systems. We then go on to check all the almost simple groups. For most almost simple groups G we are able to show that G does not contain a Sylow p-subgroup isomorphic to any group of the form $S(n, p)$. However, things are not so easy when G is a classical group. In this case we directly show that there are subgroups P of $S(n, p)$ such that the $\mathcal{E}$-normalizer of P is not the same as the $\mathcal{F}_S(G)$-normalizer of P.

Chapter 1

Preliminaries

In this chapter we shall introduce the basic theory of fusion systems.

1.1 Fusion Systems

We start by introducing some notation. Let G be a group and let $g \in G$. Denote by c_g the automorphism of G given by $c_g : x \mapsto gxg^{-1}$ for any $x \in G$. If P and Q are subgroups of G, then we define the *transporter* from P to Q to be the set $N_G(P,Q)$ of elements which conjugate P to Q. Let $\mathrm{Inj}(P,Q)$ be the set of injective homomorphisms from P to Q. Define $\mathrm{Hom}_G(P,Q) = \{c_g|_P | P^g \leq Q\}$, *i.e.* $\mathrm{Hom}_G(P,Q)$ is just the set of elements of $\mathrm{Inj}(P,Q)$ which are given by conjugation in G.

Note that if P is finite, then $N_G(P,P) = \{g \in G \mid gPg^{-1} = P\} = N_G(P)$ and so $\mathrm{Hom}_G(P,P) = \{c_g \in \mathrm{Aut}(P) \mid g \in N_G(P)\} \cong N_G(P)/C_G(P)$.

Remark We can identify $\mathrm{Hom}_G(P,Q)$ in a natural way with the collection of right coset in G of the form $C_G(P)g$ where $g \in N_G(P,Q)$.

Example A sensible way to encode all the fusion data about a subgroup S of a group G is to put all the information into a category. We can do this by defining a category $\mathcal{F}_S(G)$, whose objects are the subgroups of S, and where the morphisms are defined simply as

$$\mathrm{Hom}_{\mathcal{F}}(P,Q) := \mathrm{Hom}_G(P,Q)$$

for any pair P, Q of subgroups of S. Note that, as we saw above, $\mathrm{Hom}_{\mathcal{F}}(P,P)$ is a group isomorphic to $N_G(P)/C_G(P)$.

Before defining fusion systems, we shall introduce some terminology from category theory. Let A and B be objects in the category $\mathcal{C}$. Then an element of $\mathrm{Mor}_{\mathcal{C}}(A,B)$ is called a *$\mathcal{C}$-morphism*. A *$\mathcal{C}$-isomorphism* is an isomorphism in the category theoretic sense; that is, a $\mathcal{C}$-morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ such that there exists a morphism $\theta \in \mathrm{Mor}_{\mathcal{C}}(B,A)$ such that $\phi\theta = 1_A$ and $\theta\phi = 1_B$.

Definition 1.1.1 *Let S be a group. A* fusion system *over S is a category $\mathcal{F}$ whose objects are the subgroups of S and whose morphisms satisfy the following two properties:*

(i) for any two subgroups P, Q of S,

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q),$$

(ii) any $\mathcal{F}$-morphism can be factored as the composition of an $\mathcal{F}$-isomorphism followed by an inclusion.

(iii) The laws of composition in $\mathcal{F}$ are the same as the composition of injective homomorphisms.

Example The category $\mathcal{F}_S(G)$ as defined above is a fusion system.

All the morphisms in $\mathcal{F}$ are injective group homomorphisms, so in the case that S is finite (in the sequel we shall always assume that S is finite), every morphism in

$\mathrm{Hom}_{\mathcal{F}}(P,P)$ is in fact an automorphism of P. For these reason we shall write $\mathrm{Hom}_{\mathcal{F}}(P,P)$ as $\mathrm{Aut}_{\mathcal{F}}(P)$. The second condition ensures that if $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ then the isomorphism $P \to P^{\alpha}$ is an element of $\mathrm{Hom}_{\mathcal{F}}(P,P^{\alpha})$.

Proposition 1.1.2 *Let $\mathcal{F}$ be a fusion system over a group S. For any subgroup $P \leq S$, the set $\mathrm{Aut}_{\mathcal{F}}(P)$ is a group.*

Proof: Note that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a subset of $\mathrm{Aut}(P)$ with the same multiplication defined on it, so closure implies associativity. It remains to show closure and the existence of inverses. Since $\mathcal{F}$ is a category, there is a law of composition

$$\mathrm{Aut}_{\mathcal{F}}(P) \times \mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}}(P)$$

which ensures that $\mathrm{Aut}_{\mathcal{F}}(P)$ is closed. The existence of inverses is guaranteed by part (ii) of the definition of a fusion system, which states that any morphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ is an isomorphism. This means that $\alpha^{-1} \in \mathrm{Aut}_{\mathcal{F}}(P)$. Hence $\mathrm{Aut}_{\mathcal{F}}(P)$ is a subgroup of $\mathrm{Aut}(P)$. ∎

We write $\mathrm{Out}_{\mathcal{F}}(P)$ for the group $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(S)$.

Two subgroups P, Q of S are called *$\mathcal{F}$-conjugate* if there exists an $\mathcal{F}$-isomorphism between P and Q.

1.2 Equivalence of fusion systems

We now consider the problem of defining what it means for two fusion systems to be the same. It would seem reasonable that we would require some kind of equivalence of categories that respects the fact that the objects of our categories are groups. Recall that an *equivalence* of categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\Phi : \mathcal{C} \to \mathcal{D}$ with the property that there exists a functor $\Psi : \mathcal{D} \to \mathcal{C}$ such that $\Phi\Psi : \mathcal{C} \to \mathcal{C}$ is naturally equivalent to the identity functor of $\mathcal{C}$ and $\Psi\Phi : \mathcal{D} \to \mathcal{D}$ is naturally equivalent to the identity functor on $\mathcal{D}$ (see, for example, Hilton & Stammbach [23, Section II.4]). If Φ is an equivalence of categories between the fusion systems $\mathcal{F}$ and $\mathcal{G}$ (over p-groups S and T respectively) we should also require that for every subgroup $P \leq S$ there exist group isomorphisms $t_P : P \to P^\Phi$ such that for any $Q \leq S$ and any $f \in \mathrm{Hom}_\mathcal{F}(P, Q)$, the following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "t_P"] \arrow[d, "f"'] & P^\Phi \arrow[d, "f^\Phi"] \\ Q \arrow[r, "t_Q"'] & Q^\Phi. \end{tikzcd}$$

This is what is called an *isotypical equivalence* by Martino and Priddy in [27], in the case of fusion systems from finite groups.

Definition 1.2.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be fusion systems over p-groups S and T respectively. A group isomorphism $\theta : S \to T$ is said to* preserve fusion *from $\mathcal{F}$ to $\mathcal{G}$ if given subgroups $P, Q \leq S$ and an isomorphism $\alpha : P \to Q$, then $\alpha \in \mathrm{Hom}_\mathcal{F}(P, Q)$ if and only if $\theta^{-1}\alpha\theta \in \mathrm{Hom}_\mathcal{G}(P^\theta, Q^\theta)$.*

Note that an isomorphism θ preserves fusion from $\mathcal{F}$ to $\mathcal{G}$ if and only if the map

$$\begin{aligned} \mathrm{Hom}_\mathcal{F}(P, Q) &\to \mathrm{Hom}_\mathcal{G}(P^\theta, Q^\theta) \\ f &\mapsto \theta^{-1} f \theta \end{aligned}$$

is a bijection for all subgroups $P, Q \leq S$, and an isomorphism of groups if $P = Q$.

Example Let G be a finite group with $S, T \in \mathrm{Syl}_p(G)$. By Sylow's Theorem there exists a $g \in G$ such that $S^g = T$. It is easy to see that $c_g : S \to T$ is a fusion preserving isomorphism from $\mathcal{F}_S(G)$ to $\mathcal{F}_T(G)$. This shows that the fusion system $\mathcal{F}_S(G)$ does not depend on the choice of S, up to fusion preserving isomorphism.

The following proposition is a generalisation to arbitrary fusion systems of a result of Martino and Priddy [27, Corollary 1.2].

Proposition 1.2.2 *Let $\mathcal{F}$ and $\mathcal{G}$ be fusion systems over p-groups S and T respectively. Then $\mathcal{F}$ and $\mathcal{G}$ are isotypically equivalent if and only if there exists an isomorphism $S \to T$ which preserves fusion from $\mathcal{F}$ to $\mathcal{G}$. In fact, there is a natural one-to-one correspondence between isotypical equivalences of fusion systems and fusion preserving isomorphisms.*

Proof: Given a fusion preserving isomorphism $\theta : S \to T$, define a functor $\Theta : \mathcal{F} \to \mathcal{G}$ as follows: for any subgroup $P \leq S$, set $P^\Theta = P^\theta$ and for any morphism $f \in \mathrm{Hom}_\mathcal{F}(P, Q)$ set $f^\Theta = \theta^{-1} f \theta$. It is easy to see that Θ is isotypical, and the functor Θ^{-1} induced by the isomorphism θ^{-1} is clearly an inverse to Θ which is also isotypical. Hence Θ is an isotypical equivalence of the fusion systems $\mathcal{F}$ and $\mathcal{G}$.

Now suppose that $\Phi : \mathcal{F} \to \mathcal{G}$ is an isotypical equivalence of fusion systems. For each subgroup $P \leq S$ denote the inclusion map $P \hookrightarrow S$ by i_P, and let $(i_P)^\Phi = f_P \in \mathrm{Hom}_\mathcal{G}(P^\Phi, T)$. We shall define a functor $\Theta : \mathcal{F} \to \mathcal{G}$ which acts on objects by $P^\Theta = (P^\Phi)^{f_P}$. Now let $\alpha_P : P^\Phi \to P^\Theta$ be the group isomorphism given by $\alpha_P = f_P$. The action of Θ on morphisms shall be:

$$(g : P \to Q) \mapsto (\alpha_P^{-1} g^\Phi \alpha_Q : P^\Theta \to Q^\Theta).$$

Note that the following diagram commutes, for any $P, Q \leq S$ and any $g \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$:

$$\begin{tikzcd}
P \arrow[r, "\beta_P"] \arrow[d, "g"'] & P^{\Phi} \arrow[r, "\alpha_P"] \arrow[d, "g^{\Phi}"] & P^{\Theta} \arrow[d, "g^{\Theta} = \alpha_P^{-1} g^{\Phi} \alpha_Q"] \\
Q \arrow[r, "\beta_Q"'] & Q^{\Phi} \arrow[r, "\alpha_Q"'] & Q^{\Theta},
\end{tikzcd}$$

where β_P and β_Q are group isomorphisms. This shows that Θ is naturally isomorphic to Φ and that Θ is an isotypical equivalence of fusion systems which maps i_P to $i_{P^{\Theta}}$. For any subgroup P, let $\theta_P = \beta_P \alpha_P$. By setting $Q = S$ and $g = i_P$ in the diagram above, we see that $\theta_P = \theta_S|_P$. So the functor Θ is just the functor induced by the group isomorphism θ_S. Thus for any subgroups $P, Q \leq S$, the map

$$\begin{array}{rcl}
\mathrm{Hom}_{\mathcal{F}}(P, Q) & \rightarrow & \mathrm{Hom}_{\mathcal{G}}(P^{\Theta}, Q^{\Theta}) \\
f & \mapsto & f^{\Theta} = \theta_S^{-1} f \theta_S
\end{array}$$

is bijective. Thus by the remark above, the isomorphism θ_S is fusion preserving $\mathcal{F} \rightarrow \mathcal{G}$.

∎

Given $\alpha \in \mathrm{Aut}(S)$ we can define a functor Θ by $P^{\Theta} = P^{\alpha}$ and for $f \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, define $f^{\Theta} = \alpha^{-1} f \alpha \in \mathrm{Hom}_{\mathcal{F}^{\Theta}}(P^{\alpha}, Q^{\alpha})$. By an abuse of notation, we denote $\mathcal{F}^{\Theta}$ by $\mathcal{F}^{\alpha}$. The previous result shows that the fusion systems $\mathcal{F}$ and $\mathcal{F}^{\alpha}$ are isotypically equivalent, and that α is a fusion preserving isomorphism between $\mathcal{F}$ and $\mathcal{F}^{\alpha}$. Note that if $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ then $\mathcal{F}^{\alpha} = \mathcal{F}$.

Corollary 1.2.3 *Isotypically equivalent fusion systems are isomorphic as categories.*

If $\mathcal{F}$ and $\mathcal{G}$ are isotypically equivalent fusion systems then we write $\mathcal{F} \cong \mathcal{G}$.

1.3 Generating fusion systems

The following ideas were first published by Aschbacher and Chermak in [3].

Definition 1.3.1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be fusion systems over subgroups S_1, S_2 of a group S respectively. We say that $\mathcal{F}_1$ is a* fusion subsystem *of $\mathcal{F}_2$, and write $\mathcal{F}_1 \leq \mathcal{F}_2$ if $\mathcal{F}_1$ is a subcategory of $\mathcal{F}_2$,* i.e. *if $S_1 \leq S_2$ as groups, and for every pair $P, Q \leq S_1$, we have*

$$\mathrm{Hom}_{\mathcal{F}_1}(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}_2}(P, Q).$$

Example Let S_1, S_2, G_1 and G_2 be groups with $S_1 \leq S_2 \cap G_1$ and $G_1 \leq G_2$. Then $\mathcal{F}_{S_1}(G_1) \leq \mathcal{F}_{S_2}(G_2)$.

Note that there is always a unique largest fusion system $\mathcal{U}(S)$ and a unique smallest fusion system $\mathcal{F}_S(S)$ over a group S. These are defined as follows: for any $P, Q \leq S$ let $\mathrm{Hom}_{\mathcal{U}(S)}(P, Q) = \mathrm{Inj}(P, Q)$ and let $\mathrm{Hom}_{\mathcal{F}_S(S)}(P, Q) = \mathrm{Hom}_S(P, Q)$. Note that if $\mathcal{F}$ is any fusion system over a group S then $\mathcal{F}_S(S) \leq \mathcal{F} \leq \mathcal{U}(S)$.

Definition 1.3.2 *Let F be a set of fusion systems over a group S. Then define $\mathcal{F}_F = \bigcap_{\mathcal{F} \in F} \mathcal{F}$ to be the category with objects as the subgroups of S and where morphisms are defined by*

$$\mathrm{Hom}_{\mathcal{F}_F}(P, Q) = \bigcap_{\mathcal{F} \in F} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

We observe the following:

Proposition 1.3.3 [3, p10] *Given a set F of fusion systems over a group S, the category $\mathcal{F}_F$ is a fusion system over S.*

Given a set of fusion systems E, we define the fusion system *generated* by E to be the fusion system $\mathcal{F}_F$ where F is the set of all fusion systems containing every member of E; we write $\langle E \rangle$ for the fusion system generated by E. We note the following:

Proposition 1.3.4 **[3, Lemma 1.9]** *Let S be a group and let $\{S_i\}_{i\in I}$ be a collection of subgroups of S. For each $i \in I$, let $\mathcal{F}_i$ be a fusion system over S_i and let $F = \{\mathcal{F}_i\}_{i\in I}$. Suppose that there exists a $j \in I$ such that $S_j = S$, and define a fusion system $\mathcal{G}$ whose objects are the subgroups of S and where $\mathrm{Hom}_{\mathcal{G}}(P, Q)$ consists of maps of the form $\alpha_0 \ldots \alpha_r$ where for each $0 \le k \le r$ there exists an $i \in I$ such that $\alpha_k \in \mathrm{Hom}_{\mathcal{F}_i}(P_k, P_{k+1})$, and where $P_0 = P$ and $P_{r+1} = Q$. Then $\langle F\rangle = \mathcal{G}$.*

Remark We may restate Proposition 1.3.4 as follows: if F is a collection of fusion systems over subgroups of a finite group S, and at least one element of F is a fusion system over S, then the $\langle F\rangle$-morphisms are exactly the compositions of morphisms in fusion systems in F.

The following are refinements of some ideas of Broto, Levi and Oliver [12, p20], where we shall broaden the notion of generating fusion systems to the case when the data we are given are just a collection of maps, and not a collection of fusion systems.

Let $\mathcal{F}_0$ be a fusion system over a finite p-group S and let $\Delta_1, \ldots, \Delta_n$ be groups such that for each $1 \le i \le n$, there exists a subgroup $Q_i \le S$ with $\mathrm{Aut}_{\mathcal{F}_0}(Q_i) \le \Delta_i \le \mathrm{Aut}(Q_i)$. Define a category $\mathcal{F}$ where for any two subgroups $P, Q \le S$, $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the set of all composites

$$P = P_0 \xrightarrow{\phi_1} P_1 \xrightarrow{\phi_2} P_2 \longrightarrow \cdots \longrightarrow P_{k-1} \xrightarrow{\phi_k} P_k = Q$$

where for each $1 \le i \le k$, either $\phi_i \in \mathrm{Hom}_{\mathcal{F}_0}(P_{i-1}, P_i)$ or $\phi_i : P_{i-1} \to P_i$ is a group isomorphism and there exists a j such that $P_{i-1}, P_i \le Q_j$ with ϕ_i the restriction of an element of Δ_j. It is routine to check that $\mathcal{F}$ is indeed a category.

Proposition 1.3.5 *The category $\mathcal{F}$ as defined above is a fusion system. Furthermore, it is the intersection of all fusion systems that contain $\mathcal{F}_0$ as a subsystem and all of the maps in $\Delta_1 \cup \ldots \cup \Delta_n$.*

Proof: We show that every $\mathcal{F}$-morphism can be written as the composition of an $\mathcal{F}$-isomorphism and an inclusion. The other fusion system axioms are trivial to check.

We shall say an $\mathcal{F}$-morphism $\phi : P \to Q$ satisfies assumption $(*)$ if the restriction of ϕ to any subgroup of P is an $\mathcal{F}$-morphism that can be written as an $\mathcal{F}$-isomorphism composed with an inclusion.

Suppose we have two $\mathcal{F}$-morphisms $\phi_1 : A \to B$ and $\phi_2 : B \to C$, and both satisfy assumption $(*)$. Since $\mathcal{F}$ is a category, the map $\phi_1\phi_2 : A \to C$ is an $\mathcal{F}$-morphism. We show that the map $\phi_1\phi_2 : A \to C$ can be written as an $\mathcal{F}$-isomorphism composed with an inclusion.

By assumption $(*)$ for ϕ_1, ϕ_1 induces an $\mathcal{F}$-isomorphism $\phi_1' : A \to A^{\phi_1}$. But $A^{\phi_1} \leq B$ and so by assumption $(*)$ for ϕ_2, $\phi_2 \mid_{A^{\phi_1}}: A^{\phi_1} \to C$ is an $\mathcal{F}$-morphism that can be written as an $\mathcal{F}$-isomorphism composed with an inclusion. So $\phi_2 \mid_{A^{\phi_1}}$ induces an $\mathcal{F}$-isomorphism $\phi_2' : A^{\phi_1} \to A^{\phi_1\phi_2}$. The composition of two $\mathcal{F}$-isomorphisms is an $\mathcal{F}$-isomorphism and hence $\phi_1'\phi_2' : A \to A^{\phi_1\phi_2}$ is an $\mathcal{F}$-isomorphism. It is now clear that $\phi_1\phi_2$ can be written as the composition of $\phi_1'\phi_2'$ and the inclusion of $A^{\phi_1\phi_2}$ into C. Therefore by induction, any finite composition of $\mathcal{F}$-morphisms satisfying assumption $(*)$ can be written as an $\mathcal{F}$-isomorphism composed with an inclusion.

Now fix j. We show that every element of Δ_j satisfies assumption $(*)$. If the isomorphism $\phi : P \to P^{\phi}$ is the restriction of an element $\widehat{\phi}$ of Δ_j for some j, then ϕ is an $\mathcal{F}$-morphism. Note that $\phi^{-1} : P^{\phi} \to P$ is the restriction of $\widehat{\phi}^{-1}$, which is also an element of Δ_j since Δ_j is a group. Therefore for every element $\widehat{\phi} \in \Delta_j$ and every subgroup Q of Q_j, there exists an $\mathcal{F}$-isomorphism $\phi_Q \in \mathrm{Hom}_{\mathcal{F}}(Q, Q^{\widehat{\phi}})$, and so every element of Δ_j can be written as an $\mathcal{F}$-isomorphism followed by an inclusion.

Now, since $\mathcal{F}_0$ is a fusion system, any $\mathcal{F}_0$-morphism satisfies $(*)$, and we have shown above that for every j, every element of Δ_j also satisfies assumption $(*)$. Since every $\mathcal{F}$-morphism can, by definition, be written as a finite composition of restrictions of these maps, we have that every $\mathcal{F}$-morphism can be written as an $\mathcal{F}$-isomorphism composed with an inclusion. Hence $\mathcal{F}$ is a fusion system.

It is clear that $\mathcal{F}$ must be a fusion subsystem of any fusion system which contains $\mathcal{F}_0$ and the maps in $\Delta_1 \cup \ldots \cup \Delta_n$, and therefore $\mathcal{F}$ is the intersection of all of these fusion systems. ∎

We say that $\mathcal{F}$ is the fusion system *generated* by $\mathcal{F}_0$ and $\Delta_1, \ldots, \Delta_n$, and write $\mathcal{F} = \langle \mathcal{F}_0, \Delta_1, \ldots, \Delta_n \rangle$.

1.4 Further properties of fusion systems

In this section, we prove some general properties of fusion systems.

Definition 1.4.1 *Let $\mathcal{F}$ be a fusion system over a p-group S, and let $P \leq S$.*

- *P is* fully centralised *in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P^\alpha)|$ for all $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$;*
- *P is* fully normalised *in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P^\alpha)|$ for all $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$;*
- *P is* $\mathcal{F}$-centric *if $C_S(P^\alpha) \leq P^\alpha$ for all $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$;*
- *P is* $\mathcal{F}$-radical *if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$ where $\mathrm{Out}_{\mathcal{F}}(P) := \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.*

Remark Note that if $P \leq S$ is $\mathcal{F}$-centric then every subgroup of S which is $\mathcal{F}$-conjugate to P is $\mathcal{F}$-centric. Furthermore, if P and Q are $\mathcal{F}$-conjugate then $\mathrm{Aut}_{\mathcal{F}}(P) \cong \mathrm{Aut}_{\mathcal{F}}(Q)$ as groups. Hence if P is $\mathcal{F}$-radical then so is Q.

We now state some general facts about fusion systems from finite groups, and give their proofs for the convenience of the reader.

Lemma 1.4.2 [11, Proposition 1.3] *Let G be a finite group, $S \in \mathrm{Syl}_p(G)$, and $\mathcal{F}$ the fusion system $\mathcal{F}_S(G)$. Let $P \leq S$.*

(i) P is fully normalised if and only if $N_S(P) \in \mathrm{Syl}_p(N_G(P))$;

(ii) P is fully centralised if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$; and

(iii) if P is fully normalised then P is fully centralised.

Proof: If $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ then $|N_S(P)| = |N_G(P^g)|_p \geq |N_S(P^g)|$ for any $g \in G$, and therefore P is fully normalised. Similarly, if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$ then P is fully centralised.

Suppose that P is fully normalised. Let $R \in \mathrm{Syl}_p(N_G(P))$ such that $N_S(P) \leq R$. By Sylow's Theorem there exists a $g \in G$ such that $R^g \leq S$, since $S \in \mathrm{Syl}_p(G)$. Thus $R^g \in \mathrm{Syl}_p(N_G(P^g))$ and $R^g \leq S \cap N_G(P^g) = N_S(P^g)$. Since P is fully normalised we have

$$|R| = |R^g| = |N_S(P^g)| \leq |N_S(P)| \leq |R|.$$

Therefore $R = N_S(P) \in \mathrm{Syl}_p(N_G(P))$, proving (i).

Now suppose that P is fully centralised. Again, let $R \in \mathrm{Syl}_p(N_G(P))$ such that $N_S(P) \leq R$, and let $g \in G$ such that $R^g \leq S$. Since $C_G(P) \trianglelefteq N_G(P)$ we have that $R \cap C_G(P) \in \mathrm{Syl}_p(C_G(P))$. Then $R^g \cap C_G(P^g) \in \mathrm{Syl}_p(C_G(P^g))$, and so $C_S(P^g) \in \mathrm{Syl}_p(C_G(P^g))$, since $R^g \cap C_G(P^g) \leq C_S(P^g)$. But $C_S(P) = R \cap C_G(P)$ therefore $|C_S(P)| \leq |C_S(P^g)|$. As P is fully centralised, this means that $|C_S(P)| = |C_S(P^g)|$ and so $C_S(P) \in \mathrm{Syl}_p(C_G(P))$.

To prove *(iii)*, suppose $P \leq S$ is fully normalised in $\mathcal{F}$. By *(i)* and *(ii)*, $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ and $C_G(P) \trianglelefteq N_G(P)$ and so $C_S(P) = N_S(P) \cap C_G(P) \in \mathrm{Syl}_p(C_G(P))$, hence P is fully centralised. ∎

Proposition 1.4.3 *Let S be a finite p-group with $|S| > p$ and let $\mathcal{F}$ be a fusion system over S. Let P be an $\mathcal{F}$-centric subgroup of S. Then $|P| > p$.*

Proof: Suppose $|P| = p$. Then by comparing orders, it is clear that $P \lneqq S$, and so $N_S(P) \gneqq P$ since S is a finite p-group. As $|P| = p$, we have that $N_S(P) = C_S(P)$ since $N_S(P)/C_S(P)$ is a p-subgroup of $\mathrm{Aut}(P) \cong p-1$, and therefore trivial. Hence $C_S(P) > P$, so P cannot be $\mathcal{F}$-centric. ∎

1.5 Saturated fusion systems

The notion of *saturation* of fusion systems over finite p-groups arises naturally when considering fusion systems of the form $\mathcal{F}_S(G)$ where G is a finite group and $S \in \mathrm{Syl}_p(G)$.

For any subgroup $P \leq S$ and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, set

$$N_\phi = \{g \in N_S(P) | \phi^{-1} c_g \phi \in \mathrm{Aut}_S(P^\phi)\}.$$

Note that N_ϕ is a subgroup of $N_S(P)$ which contains P. It is also useful to observe that N_ϕ is the largest subgroup of S to which ϕ could possibly extend. This is because if $g \in N_\phi$ and $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ is an extension of ϕ, then $\overline{\phi}^{-1} c_g \overline{\phi} = c_{g^{\overline{\phi}}}$.

The definition of saturation presented here is the one used by Broto, Levi and Oliver in [11], and it is equivalent to the Puig's definition of full Frobenius systems. Stancu has produced a similar set of conditions which have been shown to be equivalent to the ones presented here [38]. Throughout this thesis we shall use the definition of Broto, Levi and Oliver.

Definition 1.5.1 *Let S be a finite p-group and let $\mathcal{F}$ be a fusion system over S. Then $\mathcal{F}$ is* saturated *if:*

(I) *Every fully normalized subgroup P in $\mathcal{F}$ is fully centralized in $\mathcal{F}$, and $\mathrm{Aut}_S(P) \in Syl_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

(II) *If $P \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ is a homomorphism such that P^ϕ is fully centralized then ϕ can be extended to a homomorphism $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ with $\bar{\phi}|_P = \phi$.*

Throughout this thesis we shall denote the above conditions by **(I)** and **(II)** respectively.

Lemma 1.5.2 *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then S is fully normalised, fully centralised, $\mathcal{F}$-centric and $\mathcal{F}$-radical. In particular,* $\mathrm{Out}_{\mathcal{F}}(S)$ *is a p'-group.*

Proof: Since S is $\mathcal{F}$-conjugate only to itself, S is fully normalised and fully centralised, and it is easy to see that S is $\mathcal{F}$-centric.

To show that S is $\mathcal{F}$-radical, we need to show that $O_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$. But $\mathcal{F}$ is saturated, and so $\mathrm{Aut}_S(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$. Thus $\mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_S(S)$ is a p'-group and hence $O_p(\mathrm{Out}_{\mathcal{F}}(S)) = 1$. ∎

The following proposition can be found in [11, Proposition 1.3]; we provide a slightly different proof, based on some ideas of Chermak [14].

Proposition 1.5.3 *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. Then $\mathcal{F} = \mathcal{F}_S(G)$ is a saturated fusion system.*

Proof: First we prove **(I)**. Lemma 1.4.2 shows that if P is fully normalised then P is fully centralised. Lemma 1.4.2 also implies that

$$\mathrm{Aut}_S(P) \cong \frac{N_S(P)}{C_S(P)} \cong \frac{N_S(P)C_G(P)}{C_G(P)} \in \mathrm{Syl}_p\left(\frac{N_G(P)}{C_G(P)}\right) = \mathrm{Syl}_p(\mathrm{Aut}_G(P)),$$

proving (I).

To prove (II), let $g \in N_G(P, S)$ such that P^g is fully centralised in $\mathcal{F}$. Set $M = C_G(P)N_{S^{g^{-1}}}(P)$.

Claim 1: $N_{c_g} \leq M$.

Let $x \in N_{c_g}$. Then $c_{g^{-1}xg} = c_h$ for some $h \in N_S(P^g)$, so $g^{-1}xg \in C_G(P^g)N_S(P^g)$. Hence $x \in M$ as claimed.

Claim 2: $N_{S^{g^{-1}}}(P) \in \mathrm{Syl}_p(M)$.

The group $C_S(P^g)$ is a Sylow p-subgroup of $C_G(P^g)$ because P^g is fully centralised. Hence

$$\frac{|C_G(P^g)N_S(P^g)|}{|C_G(P^g)|} = \frac{|N_S(P^g)|}{|C_S(P^g)|}$$

and therefore

$$\frac{|M|}{|N_{S^{g^{-1}}}(P)|} = \frac{|C_G(P^g)N_S(P^g)|}{|N_S(P^g)|} = \frac{|C_G(P^g)|}{|C_S(P^g)|}$$

is coprime to p. Since $N_{S^{g^{-1}}}(P)$ is a p-group, it must be the case that $N_{S^{g^{-1}}}(P) \in \mathrm{Syl}_p(M)$, proving claim 2.

Now, N_{c_g} is a p-subgroup of M so there exists an $h \in C_G(P)$ such that $N_{c_g}^h \leq N_{S^{g^{-1}}}(P)$, so we have $N_{c_g}^{hg} \leq N_S(P^g) \leq S$. Hence $hg \in N_G(N_{c_g}, S)$. Since $h \in C_G(P)$, we have that $c_{hg}|_P = c_g|_P$, and so $c_{hg}|_P \in \mathrm{Hom}_{\mathcal{F}}(N_{c_g}, S)$ which extends $c_g|_P$. ▮

We now construct some useful fusion subsystems of a fusion system $\mathcal{F}$.

Definition 1.5.4 *Let $\mathcal{F}$ be a fusion system over a finite group S, let $Q \leq S$ and let $K \leq \mathrm{Aut}(Q)$. Define the K-*normalizer fusion system *of Q in $\mathcal{F}$ to be the fusion system $N_{\mathcal{F}}^K(Q)$ over $N_S^K(Q) = \{g \in N_S(Q) | c_g \in K\}$ with*

$$\mathrm{Hom}_{N_{\mathcal{F}}^K(Q)}(P, P') = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(P, P') | \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, P'Q), \psi|_P = \phi, \psi|_Q \in K\}.$$

In particular, we write $N_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\mathrm{Aut}(Q)}(Q)$ and $C_{\mathcal{F}}(Q) = N_{\mathcal{F}}^{\{\mathrm{Id_Q}\}}(Q)$. These are called the *normalizer* and *centralizer* fusion systems respectively.

Note in particular that $N_{\mathcal{F}}(S)$ is the fusion system generated by the subcategory consisting of just S and $\mathrm{Aut}_{\mathcal{F}}(S)$, *i.e.* the fusion system all of whose morphisms are just compositions of restrictions of morphisms in $\mathrm{Aut}_{\mathcal{F}}(S)$.

A proof of the following proposition may be found in [11, Proposition A.6].

Proposition 1.5.5 *Let $\mathcal{F}$ be a saturated fusion system over a group S and let $Q \leq S$.*

(i) If Q is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a saturated fusion system over $N_S(Q)$.

(ii) If Q is fully $\mathcal{F}$-centralized then $C_{\mathcal{F}}(Q)$ is a saturated fusion system over $C_S(Q)$.

Lemma 1.5.6 *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $P \trianglelefteq S$ be a fully $\mathcal{F}$-centralized normal subgroup of S. Then there is a canonical isomorphism*

$$\frac{N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)}{C_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)} \to N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P)).$$

Proof: Firstly, note that, as P is normal, $N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ is the set of $\mathcal{F}$-automorphisms ϕ of P such that $N_\phi = N_S(P) = S$.

Also note that if $\alpha \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)$ then $\alpha|_P \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$. This is because, for any $x \in P$ and any $g \in S$,

$$x^{\alpha^{-1}c_g\alpha} = (g^{-1}x^{\alpha^{-1}}g)^\alpha = (g^{-1})^\alpha x g^\alpha = (g^\alpha)^{-1}xg^\alpha = x^{c_{g^\alpha}}.$$

Thus we may define a map $\Theta : N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P) \to N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ by $\alpha \mapsto \alpha|_P$.

Now, since P is fully normalized saturation implies that Θ is surjective because any $\phi \in N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$ extends to a map $\overline{\phi} \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\overline{\phi}|_P = \phi$. Furthermore Θ is a homomorphism of groups since if $\alpha, \beta \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)$ then $\alpha\beta|_P = \alpha|_P\beta|_P$. It is also easy to see that the kernel of Θ is $C_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)$. Therefore there is an isomorphism

$$\frac{N_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)}{C_{\mathrm{Aut}_{\mathcal{F}}(S)}(P)} \to N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P)).$$

∎

1.6 p-local finite groups

The notion of a p-local finite group was first developed by Broto, Levi and Oliver in [11], as a way of trying to understand the p-completed classifying spaces of finite groups.

A p-local finite group is an object which admits a notion of a classifying space. These classifying spaces have many of the same properties as the classifying spaces of finite groups.

Let G be a finite group and regard G as a category with one object, whose morphism set is simply the set of elements of G, and where composition of morphisms is just given by multiplication in the group G. Then the *nerve* NG of G is a simplicial set, where an n-simplex consists of an ordered sequence of elements of G, together with a face map

$$s_i : (x_0, \dots, x_n) \longmapsto (x_0, \dots, x_i, 1_G, x_{i+1}, \dots, x_n),$$

and a degeneracy map

$$d_i : (x_0, \dots, x_n) \longmapsto (x_0, \dots, x_{i-1}, x_i x_{i+1}, x_{i+2}, \dots, x_n).$$

The *classifying space* BG of G is defined to be the topological realization $|NG|$ of the nerve of G. The space BG has the property that its fundamental group $\pi_1(BG) \cong G$, and that all higher homotopy groups are trivial. For more details, see [5, Chapters 1,2].

The *p-completion* of BG is a space $BG_p^\wedge$ which encodes many properties of the group G which depend on the prime p, such as the cohomology modulo p. For example, the Martino-Priddy-Oliver Theorem (see [28] and [29]) states that given two finite groups G and H, with Sylow p-subgroups S and T respectively, then $BG_p^\wedge$ is homotopy equivalent to $BH_p^\wedge$ if and only if the fusion systems $\mathcal{F}_S(G)$ and $\mathcal{F}_T(H)$ are isotypically equivalent.

For a subgroup P of a finite group G, let $\theta(P) = O^p(C_G(P))$. That is, $\theta(P)$ is the smallest normal subgroup of $C_G(P)$ with p-power index in $C_G(P)$.

Definition 1.6.1 *Let G be a finite group with $S \in \mathrm{Syl}_p(G)$. Define a category $\mathcal{L} = \mathcal{L}_S(G)$, whose objects consist of the $\mathcal{F}$-centric subgroups of S and with $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ equal to the set of right cosets in $N_G(P, Q)$ of the form $\theta(P)g$ where $g \in N_G(P, Q)$. Define composition* *in $\mathcal{L}$ as follows: let $g \in N_G(P,Q)$ and let $h \in N_G(Q,R)$. Then define*

$$(\theta(P)g)(\theta(Q)h) = \theta(P)gh.$$

The category $\mathcal{L}$ is called the centric linking category *associated to G with respect to S.*

Lemma 1.6.2 *Composition in $\mathcal{L}$ is well-defined.*

Proof: We prove that composition is well-defined by showing that $(\theta(P)g)(\theta(Q)h) = \theta(P)gh$ as subsets of G. Let $x \in \theta(P)$, $y \in \theta(Q)$, $g \in N_G(P,Q)$ and $h \in N_G(Q,R)$, so that $xgyh$ is a typical element of $(\theta(P)g)(\theta(Q)h)$. Then $y \in C_G(Q) \leq C_G(P^g) = C_G(P)^g$, and so $y = z^g$ for some $z \in C_G(P)$. We now have

$$xgyh = xgg^{-1}zgh = xzgh \in \theta(P)gh,$$

since both y and z have order prime to p. Now let $xgh \in \theta(P)gh$, for some $x \in \theta(P)$. Then clearly $xgh = xg.1h \in (\theta(P)g)(\theta(Q)h)$. Hence $(\theta(P)g)(\theta(Q)h) = \theta(P)gh$. ∎

Now define a functor $\pi : \mathcal{L} \to \mathcal{F}$ which acts as the identity on objects, as natural projection on morphisms i.e. $\pi : \mathrm{Mor}_{\mathcal{L}}(P,Q) \to \mathrm{Mor}_{\mathcal{F}}(P,Q)$ is given by $\theta(P)g \mapsto C_G(P)g$ for any $g \in N_G(P,Q)$. Also define a homomorphism of groups $\delta_P : P \to \mathrm{Aut}_{\mathcal{L}}(P)$ given by $x \mapsto \theta(P)x$. Note that δ_P is injective since if $\theta(P)x = \theta(P)$ then $x \in \theta(P)$. But $x \in P$ and so has p power order. Since P is $\mathcal{F}_S(G)$-centric, we have that $Z(P) \in \mathrm{Syl}_p(C_G(P))$ (for a proof of this, see [10, Lemma A5]). This implies that $\theta(P)$ is a Hall p'-subgroup of $C_G(P)$ (that is, a p'-subgroup with p-power index in $C_G(P)$). Hence $x = 1$, hence $\ker(\delta_P)$ is trivial.

Proposition 1.6.3 ***[11, p786]*** *Let G be a finite group, let $S \in Syl_p(G)$ and write $\mathcal{F} = \mathcal{F}_S(G)$. Let $\mathcal{L} = \mathcal{L}_S(G)$ be the centric linking system associated to $\mathcal{F}$. Let π and δ_P be as above. Then the following hold:*

(A) *For each P, Q in $\mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ (via δ_P, each element of $Z(P)$ induces an element of $\mathrm{Aut}_{\mathcal{L}}(P)$, and so $Z(P)^{\delta_P}$ acts on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition) and π induces a bijection from the collection of distinct orbits in the action of $Z(P)^{\delta_P}$ on $Mor_{\mathcal{L}}(P,Q)$ to $\mathrm{Hom}_{\mathcal{F}}(P,Q)$.*

(B) *For each $\mathcal{F}$-centric subgroup P of S and each $g \in P$, π sends $g^{\delta_P} \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.*

(C) *For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$ the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g^{\delta_P}"'] & Q \arrow[d, "(g^{f^\pi})^{\delta_Q}"] \\ P \arrow[r, "f"] & Q \end{tikzcd}$$

Definition 1.6.4 *Let S be any finite p-group, and let $\mathcal{F}$ be a saturated fusion system over S. A* centric linking system *associated to the fusion system $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S together with a functor $\pi : \mathcal{L} \to \mathcal{F}$ which acts as the identity on objects and is surjective on morphisms and a distinguished set of monomorphisms $\delta_P : P \to \mathrm{Aut}_{\mathcal{L}}(P)$ for all $P \in \mathcal{L}$ satisfying the properties (A), (B) and (C) above.*

Definition 1.6.5 *A* p-local finite group *is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group, $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The* classifying space *of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|_p^\wedge$.*

There is a notion of an isomorphism between p-local finite groups, which consists of a pair (α, β) where α is an isotypical equivalence of fusion systems and β is an isomorphism of categories between two centric linking system. The maps α and β must also satisfy various compatability conditions which we shall not discuss here. For more details, see [3, Definition 2.10].

Definition 1.6.6 *A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is called* exotic *if it is not isomorphic to a p-local finite group of the form $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$, where G is a finite group with S isomorphic to a Sylow p-subgroup of G. Similarly, a saturated fusion system $\mathcal{F}$ over a p-group S is called* exotic *if there does not exist a finite group G such that $\mathcal{F}$ is isotypically equivalent to the fusion system $\mathcal{F}_S(G)$.*

The following theorem is an immediate consequence of [28, Theorem 4.5] and [11, Proposition 3.1].

Theorem 1.6.7 *Let p be an odd prime and let S be a finite p-group. Let $\mathcal{F}$ be a saturated fusion system over S, and suppose that $\mathcal{F}$ is not exotic,* i.e. *there exists a finite group G with $S \in \mathrm{Syl}_p(G)$ such that $\mathcal{F} \cong \mathcal{F}_S(G)$. Then there exists a unique centric linking system associated to $\mathcal{F}$, namely $\mathcal{L}_S(G)$.*

It has been conjectured by Oliver [28, Conjecture 2.2] that this result holds for all saturated fusion systems, and thus that a p-local finite group is uniquely determined (up to an isomorphism of p-local finite groups, a notion which we shall not discuss here) by its associated saturated fusion system.

There is a great deal of interest in finding examples of exotic p-local finite groups, as the study of these objects may shed some light on what it really means for a space to be the p-completed classifying space of a finite group. Examples may be found in [11], [12], [18], [25], and [36]. Indeed, one of the aims of this thesis is to construct some exotic p-local finite groups, which will be described in later chapters.

1.7 Subgroup families controlling fusion

In general it is very difficult to say much about fusion systems over a p-group S because it is necessary to know all the subgroups of S and all the morphisms between them. This makes the task of classifying fusion systems over a given group extremely hard in general. Fortunately, saturated systems have a more rigid structure, and the following version of a famous theorem of Alperin shows that a saturated fusion system is generated by its centric and radical subgroups (in a way which we define later). A stronger version was first proved by Puig [33, Corollory 3.9], but the version given here and the proof, which we reproduce here for the convenience of the reader, are from [11, Theorem A10].

Theorem 1.7.1 (Alperin's Fusion Theorem for Saturated Fusion Systems) *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Let P be a subgroup of S. Then for each morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P^{\phi} \text{ and } Q_1, Q_2, \ldots, Q_k,$$

and elements $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$ such that

(i) Q_i is fully normalized, $\mathcal{F}$-centric and $\mathcal{F}$-radical for $1 \leq i \leq k$.

(ii) $P_{i-1}, P_i \leq Q_i$ and $P_{i-1}^{\phi_i} = P_i$.

(iii) $\phi = \phi_1|_P \phi_2|_{P_1} \cdots \phi_k|_{P_{k-1}}$.

Proof: We proceed by backwards induction on the order of P. The theorem is true for $P = S$ since S is fully normalised, $\mathcal{F}$-centric and $\mathcal{F}$-radical by Lemma 1.5.

Assume now that $P \lneqq S$, and that the theorem holds for all subgroups of greater order. From the definition of fully normalized subgroups, it is clear that there exists a fully normalized subgroup $P^* < S$ which is $\mathcal{F}$-conjugate to P. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, P^*)$.

Now if the theorem holds for $\phi^{-1}\psi \in \mathrm{Hom}_{\mathcal{F}}(P^{\phi}, P^{*})$ and ψ then it holds for ϕ. This is a consequence of that fact that if $\psi = \psi_1\psi_2\dots\psi_k$ and $\phi^{-1}\psi = \phi_1\phi_2\dots\phi_l$ then $\phi = \psi(\phi^{-1}\psi)^{-1} = \psi_1\dots\psi_k\phi_l^{-1}\dots\phi_1^{-1}$. Therefore, it suffices to prove the theorem in the case when the image P^{ϕ} is fully normalized.

Assume now that P^{ϕ} is fully normalized. Note that $\phi^{-1}\mathrm{Aut}_S(P)\phi$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P^{\phi})$ (since $\phi^{-1}c_g\phi = c_{g^{\phi}}$). From **(II)** we have that $\mathrm{Aut}_S(P^{\phi}) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P^{\phi}))$ and so by Sylow's Theorem, there exists an automorphism $\psi \in \mathrm{Aut}_{\mathcal{F}}(P^{\phi})$ such that

$$\psi^{-1}\phi^{-1}\mathrm{Aut}_S(P)\phi\psi \leq \mathrm{Aut}_S(P^{\phi}).$$

So if we set $\chi = \phi\psi\phi^{-1} \in \mathrm{Aut}_{\mathcal{F}}(P)$ then we have

$$\phi^{-1}\chi^{-1}\mathrm{Aut}_S(P)\chi\phi \leq \mathrm{Aut}_S(P^{\phi}). \tag{1.1}$$

Now, $\mathcal{F}$ is saturated and P^{ϕ} is fully normalized, and so P^{ϕ} is fully centralized. From Definition 1.5.1 **(II)**, $\chi\phi$ extends to a homomorphism $\overline{\chi\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_{\chi\phi}, S)$. But if $g \in N_S(P)$ then by (1.1) we have $\phi^{-1}\chi^{-1}c_g\chi\phi \in \mathrm{Aut}_S(P^{\phi})$, and so $N_{\chi\phi} = N_S(P)$. Since P is a proper subgroup of S, $N_S(P) \gneqq P$, and so by our inductive hypothesis the result holds for $\overline{\chi\phi}$, and hence for $\chi\phi$. Therefore the result holds for ϕ if and only if it holds for χ. It now suffices to prove the result for $P = P^{\phi}$ with P fully normalized and $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$.

Assume now that this is the case. If P is $\mathcal{F}$-centric and $\mathcal{F}$-radical then the result holds trivially. So assume that P is not $\mathcal{F}$-centric. P is fully normalized and $\mathcal{F}$ is saturated, so P is fully centralized. It is easy to see that $C_S(P)P \leq N_{\phi}$, and so by saturation, we have that ϕ extends to a homomorphism $\overline{\phi} \in \mathrm{Aut}_{\mathcal{F}}(C_S(P)P)$. Since P is not $\mathcal{F}$-centric, $C_S(P)P \gneqq P$ and so the result holds by induction.

Finally, assume that P is not $\mathcal{F}$-radical. Let $K = O_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Note that since P is not $\mathcal{F}$-radical, $K \gneqq \mathrm{Inn}(P)$. Set $N_S^K(P) = \{g \in N_S(P) |\ c_g \in K\}$. Since P is fully normalized, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and so $K \leq \mathrm{Aut}_S(P)$. Now, if $g \in N_S^K(P)$, then $\phi^{-1}c_g\phi \in K$, since K is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. But $K \leq \mathrm{Aut}_S(P)$ so $\phi^{-1}c_g\phi \in \mathrm{Aut}_S(P)$. Thus $N_S^K(P) \leq N_\phi(P)$. By saturation, ϕ extends to a homomorphism $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_S^K(P), S)$. But $N_S^K(P) \gneqq P$ since $K \gneqq \mathrm{Inn}(P)$, and so, by induction, the result holds. ∎

Definition 1.7.2 *Let $\mathcal{F}$ be a saturated fusion system over a finite group S. We call a subgroup $P \leq S$ an $\mathcal{F}$-*Alperin subgroup* if P is fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric and $\mathcal{F}$-radical.*

Note that S is always an $\mathcal{F}$-Alperin subgroup. We shall denote the set of all proper Alperin subgroups in a fusion system $\mathcal{F}$ over a p-group S by $\mathrm{Alp}(\mathcal{F})$.

A form of converse to Alperin's Fusion Theorem is proved in the joint work of Broto, Castellana, Grodal, Levi and Oliver [8]. We give an outline of their main result.

Definition 1.7.3 *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{H}$ be a collection of subgroups of S that is closed under $\mathcal{F}$-conjugation.*

*(i) $\mathcal{F}$ is $\mathcal{H}$-*generated* if every morphism in $\mathcal{F}$ is a composite of restrictions of morphisms in $\mathcal{F}$ between subgroups in $\mathcal{H}$;*

*(ii) $\mathcal{F}$ is $\mathcal{H}$-*saturated* if the conditions **(I)** and **(II)** hold in $\mathcal{F}$ for all subgroups in $\mathcal{H}$.*

Thus we can restate Alperin's Fusion Theorem as follows: if $\mathcal{F}$ is a saturated fusion system over a finite p-group S, and $\mathcal{H}$ is the collection of $\mathcal{F}$-Alperin subgroups and their $\mathcal{F}$-conjugates, then $\mathcal{F}$ is $\mathcal{H}$-generated.

We note that the use of the term *$\mathcal{H}$-generated* does not conflict with our previous use of the term *generated*. This is because of the following observation:

Proposition 1.7.4 *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $\mathcal{H}$ be a collection of subgroups of S that is closed under $\mathcal{F}$-conjugation. If $\mathcal{F}$ is $\mathcal{H}$-generated* *then* $\mathcal{F} = \langle \mathcal{F}_S(S), \mathrm{Aut}_{\mathcal{F}}(H) | H \in \mathcal{H} \rangle$*, and if* $\mathcal{F} = \langle \mathcal{F}_S(S), \mathrm{Aut}_{\mathcal{F}}(H) | H \in \mathcal{H} \rangle$ *then* $\mathcal{F}$ *is* $\mathcal{H} \cup \{S\}$*-generated.*

Proof: Follows immediately from the definitions. ∎

We shall now state a theorem which simplifies the task of proving saturation in a given fusion system. First we shall state a Lemma which makes statement 1.7.6 equivalent to [8, Theorem 2.2].

Lemma 1.7.5 *Suppose that* $\mathcal{F}$ *is a fusion system over a finite p-group* S *and that* $\mathcal{H}$ *is a collection of subgroups of* S *which is closed under* $\mathcal{F}$*-conjugation. Suppose that* $\mathcal{H}$ *has the property that for each* $\mathcal{F}$*-conjugacy class* $\mathcal{P}$ *of* $\mathcal{F}$*-centric subgroups not contained in* $\mathcal{H}$ *there exists a* $P \in \mathcal{P}$ *such that*

$$\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}_{\mathcal{F}}(P)) \neq 1. \tag{1.2}$$

Then $\mathcal{H}$ *contains all the* $\mathcal{F}$*-Alperin subgroups of* S*.*

Proof: Let P be an $\mathcal{F}$-Alperin subgroup of S, and suppose that $P \notin \mathcal{H}$. In particular, P is $\mathcal{F}$-centric, and since $\mathcal{H}$ is closed under $\mathcal{F}$-conjugation, the $\mathcal{F}$-conjugacy class $\mathcal{P}$ of P is not contained in $\mathcal{H}$. Therefore there exists a subgroup $Q \in \mathcal{P}$ satisfying 1.2. In particular, $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) \neq 1$ and so Q is not $\mathcal{F}$-radical. But P is $\mathcal{F}$-conjugate to Q and so P is not $\mathcal{F}$-radical either, contrary to our assumption. ∎

Theorem 1.7.6 [8, Theorem 2.2] *Let* $\mathcal{F}$ *be a fusion system over a finite p-group* S*. Let* $\mathcal{H}$ *be a collection of subgroups of* S *as in Lemma 1.7.5. If* $\mathcal{F}$ *is* $\mathcal{H}$*-generated and* $\mathcal{H}$*-saturated then* $\mathcal{F}$ *is saturated.*

The next result shows that if $\mathcal{F}$ is $\mathcal{H}$-generated then $\mathcal{F}$ is determined up to isotypical equivalence by $\mathcal{H}$.

Proposition 1.7.7 *Let $\mathcal{F}$ and $\mathcal{G}$ be fusion systems over p-groups S and T respectively, and suppose that $\theta : S \to T$ is a group isomorphism. Suppose that $\mathcal{H}$ is a collection of subgroups of S which is closed under $\mathcal{F}$-conjugacy such that $\mathcal{F}$ is $\mathcal{H}$-generated. Then the map θ preserves fusion from $\mathcal{F}$ to $\mathcal{G}$ if and only if the following hold:*

(i) $\mathcal{G}$ is $\mathcal{H}^\theta$-generated; and

(ii) for all subgroups $H, H' \in \mathcal{H}$ we have $\alpha \in \mathrm{Hom}_{\mathcal{F}}(H, H')$ if and only if $(\theta^{-1}\alpha\theta)|_{H^\theta} \in \mathrm{Hom}_{\mathcal{G}}(H^\theta, H'^\theta)$.

Proof: Suppose that θ preserves fusion. Then (ii) holds by definition, and we need to show that $\mathcal{G}$ is $\mathcal{H}^\theta$-generated. To see this, let $P, Q \leq S$ and let $\alpha \in \mathrm{Hom}_{\mathcal{G}}(P^\theta, Q^\theta)$. Since θ is fusion preserving, we have $\theta\alpha\theta^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. But $\mathcal{F}$ is $\mathcal{H}$-generated, so there exist subgroups $H_1, \ldots, H_n \in \mathcal{H}$ and, for each $1 \leq i \leq n-1$ there exist morphisms $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(H_i, H_{i+1})$ such that $\theta\alpha\theta^{-1} = (\phi_1 \cdots \phi_n)|_P$. Therefore $\alpha = ((\theta^{-1}\phi_1\theta)\cdots(\theta^{-1}\phi_n\theta))|_{P^\theta}$. Since θ is fusion preserving, for each $1 \leq i \leq n-1$ we have $\theta^{-1}\phi_i\theta \in \mathrm{Hom}_{\mathcal{G}}(H_i^\theta, H_{i+1}^\theta)$. Thus we have shown that α is a composition of restrictions of $\mathcal{G}$-morphisms between subgroups of $\mathcal{H}^\theta$. Hence $\mathcal{G}$ is $\mathcal{H}^\theta$-generated.

Now suppose conversely that conditions (i) and (ii) hold. Let $P, Q \leq S$ and $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Since $\mathcal{F}$ is $\mathcal{H}$-generated, there exist subgroups $H_1, \ldots, H_n \in \mathcal{H}$ and, for each $1 \leq i \leq n-1$ there exist morphisms $\phi_i \in \mathrm{Hom}_{\mathcal{F}}(H_i, H_{i+1})$ such that $\alpha = (\phi_1 \cdots \phi_n)|_P$. By (ii), we have, for each $1 \leq i \leq n-1$, $\theta^{-1}\phi_i\theta \in \mathrm{Hom}_{\mathcal{G}}(H_i^\theta, H_{i+1}^\theta)$. Hence $\theta^{-1}\alpha\theta = ((\theta^{-1}\phi_1\theta)\cdots(\theta^{-1}\phi_n\theta))|_{P^\theta} \in \mathrm{Hom}_{\mathcal{G}}(P^\theta, Q^\theta)$. A similar argument shows that if $\beta \in \mathrm{Hom}_{\mathcal{G}}(P^\theta, Q^\theta)$ then $\theta\beta\theta^{-1} \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. Hence θ is fusion preserving. ∎

We can now say that an isomorphism $\theta : S \to T$ is fusion preserving (with respect to $\mathcal{F}$ and $\mathcal{G}$) if and only if the following two conditions hold:

(i) a subgroup P is $\mathcal{F}$-Alperin if and only if P^θ is $\mathcal{G}$-Alperin;

(ii) for every $\mathcal{F}$-Alperin subgroup P, the map

$$\begin{array}{rcl} \mathrm{Aut}_{\mathcal{F}}(P) & \to & \mathrm{Aut}_{\mathcal{G}}(P^{\theta}) \\ f & \mapsto & \theta^{-1} f \theta \end{array}$$

is an isomorphism of groups.

The proof of this fact is the same as the proof of Proposition 1.7.7, but we observe that Alperin's Theorem shows that every $\mathcal{F}$-morphism is a composite of restrictions of $\mathcal{F}$-automorphisms of Alperin subgroups.

1.8 A Frattini lemma

Recall that if G is a finite group then $O^{p'}(G) := \langle \mathrm{Syl}_p(G) \rangle$. Furthermore, if $S \in \mathrm{Syl}_p(G)$ then the Frattini Lemma says that

$$G = O^{p'}(G) N_G(S). \tag{1.3}$$

We can prove a similar result for saturated fusion systems that will further reduce the amount of data we must collect to classify a saturated fusion system. An alternative, and independently discovered formulation of these results can be found in [9, Lemma 3.4].

First we need to define analogous notions of the subgroups $O^{p'}(G)$ and $N_G(S)$.

Given a fusion system $\mathcal{F}$ over a finite p-group S, the *normalizer* fusion system $N_{\mathcal{F}}(S)$ of S is simply the fusion subsystem of $\mathcal{F}$ which consists of all the $\mathcal{F}$-morphisms which are restrictions of elements of $\mathrm{Aut}_{\mathcal{F}}(S)$. There is a more general definition of the normalizer fusion system of an arbitrary subgroup of S, for details of which we refer the reader to Definition 1.5.4. An important property of $N_{\mathcal{F}}(S)$ is that if $\mathcal{F}$ is saturated then so is $N_{\mathcal{F}}(S)$. For a proof of this fact see [11, Proposition A.6].

Given a fusion system $\mathcal{F}$ over a finite group S, let $O^{p'}(\mathcal{F})$ be the fusion system $\langle \mathcal{F}_S(S), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q)) | Q \in \mathrm{Alp}(\mathcal{F}) \rangle$.

Note that $O^{p'}(\mathcal{F})$ is a fusion subsystem of $\mathcal{F}$. In fact it is a very important fusion subsystem, as we shall see in the next proposition.

Recall that if $\mathcal{F}$ is a fusion system over S and $\alpha \in \mathrm{Aut}(S)$ then α induces a new category $\mathcal{F}^\alpha$. For details of this category, see Proposition 1.2.2.

Proposition 1.8.1 *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. Then*

$$O^{p'}(\mathcal{F})^\alpha = O^{p'}(\mathcal{F}).$$

Proof: For each $V \in \mathrm{Alp}(\mathcal{F})$, the map $\alpha|_V$ is a group isomorphism $V \to V^\alpha$ and we have $\mathrm{Aut}_{\mathcal{F}}(V)^\alpha = \mathrm{Aut}_{\mathcal{F}}(V^\alpha)$. Now, $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ is a characteristic subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$ so $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))^\alpha = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)^\alpha) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V^\alpha))$. Hence

$$\begin{aligned} O^{p'}(\mathcal{F})^\alpha &= \langle \mathcal{F}_S(S), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V^\alpha)) | V \in \mathrm{Alp}(\mathcal{F})\rangle \\ &= \langle \mathcal{F}_S(S), O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)) | V \in \mathrm{Alp}(\mathcal{F})\rangle \\ &= O^{p'}(\mathcal{F}), \end{aligned}$$

and so the proposition is proved. ∎

Theorem 1.8.2 (Frattini Lemma for saturated fusion systems) *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S, then every $\mathcal{F}$-morphism ϕ can be written as a composite of $O^{p'}(\mathcal{F})$-morphisms and $N_{\mathcal{F}}(S)$-morphisms. In particular, $\mathcal{F} = \langle O^{p'}(\mathcal{F}), N_{\mathcal{F}}(S)\rangle = \langle O^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$.*

Proof: First we prove that every $\mathcal{F}$-automorphism of an $\mathcal{F}$-Alperin subgroup can be written as the composition of an $O^{p'}(\mathcal{F})$-morphism and an $N_{\mathcal{F}}(S)$-morphism. Let $V \in \mathrm{Alp}(\mathcal{F})$. We proceed by reverse induction on $|V|$. The result is clearly true for $V = S$. So assume that V is a proper subgroup of S, and that the result holds for all Alperin subgroups W with $|W| > |V|$.

By Proposition 1.1.2, $\mathrm{Aut}_{\mathcal{F}}(V)$ is a finite group, and since $\mathcal{F}$ is saturated and V is an $\mathcal{F}$-Alperin subgroup, $\mathrm{Aut}_S(V)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$. Therefore by the Frattini Lemma for finite groups, we have

$$\mathrm{Aut}_{\mathcal{F}}(V) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))N_{\mathrm{Aut}_{\mathcal{F}}(V)}(\mathrm{Aut}_S(V))$$

since $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(V)$. The group $N_{\mathrm{Aut}_{\mathcal{F}}(V)}(\mathrm{Aut}_S(V))$ is the set of $\phi \in \mathrm{Aut}_{\mathcal{F}}(V)$ for which $N_\phi = N_S(V) \gneqq V$. Thus any element $\phi \in N_{\mathrm{Aut}_{\mathcal{F}}(V)}(\mathrm{Aut}_S(V))$ extends by saturation to a map $\overline{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_S(V), S)$.

Now let $\phi \in \mathrm{Aut}_{\mathcal{F}}(V)$. Then $\phi = \mu\theta$ where $\mu \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ and $\theta \in N_{\mathrm{Aut}_{\mathcal{F}}(V)}(\mathrm{Aut}_S(V))$. As explained above, θ extends to a map $\overline{\theta} \in \mathrm{Hom}_{\mathcal{F}}(N_S(V), S)$. Since S is a p-group and $V \lneqq S$, we have that $N_S(V) \gneqq V$. By Alperin's Fusion Theorem, $\overline{\theta}$ is a composite of restrictions of $\mathcal{F}$-automorphisms of Alperin subgroups with order at least as large as $|N_S(V)|$. By induction, all of these $\mathcal{F}$-automorphisms are composites of $N_{\mathcal{F}}(S)$-morphisms and $O^{p'}(\mathcal{F})$-morphisms, and therefore $\overline{\theta}$ and θ are composites of $N_{\mathcal{F}}(S)$-morphisms and $O^{p'}(\mathcal{F})$-morphisms. Hence so is ϕ. The theorem now holds by induction. ▮

Proposition 1.8.3 *Let G be a finite group with S a Sylow p-subgroup of G. Then*

$$O^{p'}(\mathcal{F}_S(G)) = \mathcal{F}_S(O^{p'}(G)).$$

Proof: Let $V \leq S$ be fully $\mathcal{F}_S(G)$-normalized. Let $x \in N_S(V)$ and $g \in N_G(V)$ such that $c_x \in \mathrm{Aut}_S(V)$ and $c_g, c_{g^{-1}} \in \mathrm{Aut}_G(V)$. Then $c_{g^{-1}}c_xc_g = c_{g^{-1}xg} \in \mathrm{Aut}_{S^g}(V)$. Since $\mathcal{F}_S(G)$ is saturated, $\mathrm{Aut}_S(V)$ is a Sylow p-subgroup of $\mathrm{Aut}_G(V)$, and therefore $O^{p'}(\mathrm{Aut}_G(V)) = \langle \mathrm{Aut}_S(V)^{c_g} | g \in N_G(V)\rangle = \langle \mathrm{Aut}_{S^g}(V) | g \in N_G(V)\rangle = \mathrm{Aut}_{O^{p'}(G)}(V)$. In particular, this shows that $O^{p'}(\mathcal{F}_S(G)) \leq \mathcal{F}_S(O^{p'}(G))$.

Now suppose that V is $\mathcal{F}_S(O^{p'}(G))$-Alperin. First we show that V is fully $\mathcal{F}_S(G)$-normalized. Let $g \in G$ such that $c_g \in \mathrm{Hom}_{\mathcal{F}}(V, S)$. By the Frattini Lemma for finite groups, $G = O^{p'}(G)N_G(S)$ and so there exists an $x \in O^{p'}(G)$ and an $n \in N_G(S)$ such that $g = xn$. We have $|N_S(V)| \geq |N_S(V^x)|$ since V is fully $\mathcal{F}_S(O^{p'}(G))$-normalized, and $|N_S(V^x)| = |N_S(V^{xn})| = |N_S(V^g)|$ since $n \in N_G(S)$. Hence $|N_S(V)| \geq |N_S(V^g)|$ for all $g \in N_G(V, S)$ and so V is fully $\mathcal{F}_S(G)$-normalized. Also note that $C_S(V^g) = C_S(V^{xn}) = (C_S(V^x))^n = Z(V^x)^n = Z(V^g)$ since V is $\mathcal{F}_S(O^{p'}(G))$-centric. Hence V is $\mathcal{F}_S(G)$-centric.

We have shown that V is fully $\mathcal{F}_S(G)$-normalized. This means that $O_p(\mathrm{Aut}_G(V)) = O_p(O^{p'}(\mathrm{Aut}_G(V))) = O_p(\mathrm{Aut}_{O^{p'}(G)}(V))$. But V is $\mathcal{F}_S(O^{p'}(G))$-radical, so $O_p(\mathrm{Aut}_{O^{p'}(G)}(V)) \leq \mathrm{Inn}(V)$. Therefore $O_p(\mathrm{Aut}_G(V)) \leq \mathrm{Inn}(V)$ and so V is $\mathcal{F}_S(G)$-radical. Therefore $\mathrm{Alp}(\mathcal{F}_S(O^{p'}(G))) \subseteq \mathrm{Alp}(\mathcal{F}_S(G))$.

Now, by definition $O^{p'}(\mathcal{F}_S(G)) = \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)) | V \in \mathrm{Alp}(\mathcal{F}) \cup \{S\}\rangle$ and by Alperin's Theorem $\mathcal{F}_S(O^{p'}(G)) = \langle \mathrm{Aut}_{O^{p'}(G)}(W) | W \in \mathrm{Alp}(\mathcal{F}_S(O^{p'}(G))) \cup \{S\}\rangle$. Since we have shown that $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)) = \mathrm{Aut}_{O^{p'}(G)}(V)$ whenever V is an $\mathcal{F}_S(G)$-Alperin subgroup, and that $\mathrm{Alp}(\mathcal{F}_S(O^{p'}(G))) \subseteq \mathrm{Alp}(\mathcal{F}_S(G))$, we have that $\mathcal{F}_S(O^{p'}(G)) \leq O^{p'}(\mathcal{F}_S(G))$. Hence $\mathcal{F}_S(O^{p'}(G)) = O^{p'}(\mathcal{F}_S(G))$ and the theorem is proved. ∎

Chapter 2

Sylow p-Subgroups of $\mathrm{SL}_3(p^n)$

Let p be an odd prime, and let $q = p^n$ for some n. Consider groups of the form $S = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{smallmatrix} \right) \mid a, b, c \in \mathbb{F}_q \right\}$. These are Sylow p-subgroups of $\mathrm{SL}_3(q)$ and in the case $n = 1$ they are exactly the groups considered by Ruiz and Viruel [36]. In the next few chapters, S will always denote a group of this form. If the value of q is clear from the context, we may omit this part of the notation.

There are the $p^n + 1$ elementary abelian subgroups of S of order p^{2n}; these subgroups will prove to be very important in the study of saturated fusion systems over the groups S. These are all of the form $C_S(x)$ for some $x \in S \setminus Z(S)$. It turns out that for any saturated fusion system over S, the proper Alperin subgroups of S are of this form, and so we only need worry about these elementary abelian subgroups and S itself.

2.1 Basic facts

First we prove some well-known facts about the group S.

Lemma 2.1.1 *The following hold for S:*

(i) $|S| = q^3$ *and* S *has exponent* p.

(ii) $\Phi(S) = [S, S] = Z(S) = [S, s]$ *for any* $s \in S \setminus Z(S)$*, and* $|Z(S)| = q$.

(iii) *For every* $s \in S \setminus Z(S)$, $C_S(s)$ *is elementary abelian of order* q^2.

(iv) *An elementary abelian* p*-subgroup of* S *of order* q^2 *is one of*

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{pmatrix} | a, c \in \mathbb{F}_q \right\}, \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{pmatrix} | b, c \in \mathbb{F}_q \right\} \text{ or } \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & a\lambda & 1 \end{pmatrix} | a, c \in \mathbb{F}_q \right\}$$

where $\lambda \in \mathbb{F}_q \setminus \{0\}$. *In particular, there are exactly* $q+1$ *elementary abelian* p*-subgroups of order* q^2 *in* S.

(v) *If* V_1 *and* V_2 *are distinct elementary abelian* p*-subgroups of order* q^2 *then* $V_1 \cap V_2 = Z(S)$.

Proof: (i) It is easy to see that $|S| = q^3$ and so there exists an element of order p in S. This shows that Exponent$(S) \geq p$. Now, given any element $s \in S$ with

$$s = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix},$$

we have, for $1 \leq i \leq p$,

$$s^i = \begin{pmatrix} 1 & 0 & 0 \\ ia & 1 & 0 \\ ic + \frac{abi(i-1)}{2} & ib & 1 \end{pmatrix}.$$

This means that $s^p = 1$ for all $s \in S$, and thus S has exponent p.

(ii) Suppose that $r = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{smallmatrix}\right) \in Z(S)$. We have

$$rs = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ x+a & 1 & 0 \\ z+c+ay & y+b & 1 \end{pmatrix}$$

and

$$sr = \begin{pmatrix} 1 & 0 & 0 \\ x+a & 1 & 0 \\ z+c+bx & y+b & 1 \end{pmatrix}.$$

So r commutes with every $s \in S$ if and only if $ay = bx$ for all $a, b \in \mathbb{F}_q$. But this is the case if and only if $x = y = 0$, and so

$$Z(S) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix} | * \in \mathbb{F}_q \right\},$$

which is easily seen to have order q.

Now consider $[S, S]$. It is easy to see that $s^{-1} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab-c & -b & 1 \end{smallmatrix}\right)$ and so commutators have the form $r^{-1}s^{-1}rs = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ay-bx & 0 & 1 \end{smallmatrix}\right)$. Thus $[S, S] = Z(S) = [S, s]$ for any $s \in S$.

Since S has exponent p, $S/Z(S)$ is elementary abelian. Hence $\Phi(S) \leq Z(S) = [S, S]$, and so $\Phi(S) = Z(S) = [S, S]$.

(iii) Let r, s be as above. As in (i), r commutes with s if and only if $ay = bx$. This equation shows that we can freely choose the entries x and z, and then y is uniquely determined. Thus $C_S(s)$ has order q^2. If either a or b is equal to 0 then it is easy to check that $C_S(s)$ is abelian. If $a, b \neq 0$ then suppose that $r = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{smallmatrix}\right)$ and $r' = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x' & 1 & 0 \\ z' & y' & 1 \end{smallmatrix}\right) \in C_S(s)$. Then $ay = bx$ and $ay' = bx'$ and therefore $abx'y = ay(ay') = bxay' = abxy'$, hence $x'y = xy'$. This shows that r and r' commute. Hence $C_S(s)$ is abelian. By (i), $C_S(s)$ has exponent p and therefore $C_S(s)$ must be elementary abelian.

(iv) Let V be an elementary abelian p-subgroup of order q^2, and let $v \in V \setminus Z(S)$. Since V is abelian we have $V \leq C_S(v)$, but by (iii), $|C_S(v)| = q^2$ and so $V = C_S(v)$. A simple matrix calculation shows that the centralizer of an element of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \lambda & 1 \end{smallmatrix}\right)$, where $\lambda \in \mathbb{F}_q$, is $\left\{\left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & a\lambda & 1 \end{smallmatrix}\right) | a, c \in \mathbb{F}_q\right\}$. Similarly, the centralizer of the element $\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&1&1\end{smallmatrix}\right)$ is $\left\{\left(\begin{smallmatrix}1&0&0\\0&1&0\\c&b&1\end{smallmatrix}\right)|b,c\in\mathbb{F}_q\right\}$. It is now easy to see that every element of $S\setminus Z(S)$ is contained in one of the $q+1$ centralizers mentioned above, hence there are no other subgroups of the form $C_S(s)$ where $s\in S\setminus Z(S)$, *i.e.* there are no other elementary abelian p-subgroups of order q^2.

(v) Let $V_1=C_S(v_1)$ and $V_2=C_S(v_2)$ be distinct elementary abelian subgroups of order q^2. Clearly $V_1\cap V_2\geq Z(S)$, and if $w\in V_1\cap V_2\setminus Z(S)$ then $C_S(w)=V_1=V_2$, a contradiction. Hence $V_1\cap V_2=Z(S)$. ∎

2.2 The automorphism group

Now let us calculate the automorphism group of S. For this purpose, we state a general result from Parker and Rowley [32, Lemma 5.2].

Lemma 2.2.1 *Let G be a finite group. Then*

$$C_{\mathrm{Aut}(G)}(G/(Z(G)\cap G'))\cong \mathrm{Hom}(G,Z(G)\cap G'),$$

where G' denotes the derived subgroup of G.

Remark In the statement of this result, note that by $C_{\mathrm{Aut}(G)}(G/(Z(G)\cap G'))$ we mean the set $\{\alpha\in\mathrm{Aut}(G)|\ \forall x\in G,$ the coset $x(Z(G)\cap G')$ is α-invariant $\}$.

The automorphisms in $C_{\mathrm{Aut}(G)}(G/Z(G)\cap G')$ are called the *central automorphisms* of G. In the group S, we have that every inner automorphism of S is a central automorphism. To see this, let $x,y\in S$. Then $x^{-1}y^{-1}xy\in S'=Z(S)$ and so the cosets $xZ(S)$ and $(xZ(S))^y$ are equal.

Lemma 2.2.2 *The set of central automorphisms of S is a normal subgroup of* $\mathrm{Aut}(S)$ *which is isomorphic to an elementary abelian p-group of order p^{2n}. In particular, the set of central automorphisms consists of automorphisms of the form:*

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c+\psi(a,b) & b & 1 \end{pmatrix}$$

where $\psi \in \mathrm{Hom}((\mathbb{F}_q,+)^2,(\mathbb{F}_q,+))$.

Proof: It is easy to see that maps of the given form are indeed central automorphisms. But the group S is generated by the set of matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} | a \in \mathbb{F}_q \right\} \cup \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix} | b \in \mathbb{F}_q \right\}.$$

Hence every central automorphism $\mathfrak{z}$ is uniquely determined by its action on the elements of this set. Thus for each central automorphism $\mathfrak{z}$ we can define a map $\psi_{\mathfrak{z}} \in \mathrm{Hom}((\mathbb{F}_q,+)^2,(\mathbb{F}_q,+))$ by the following equation:

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ \psi_{\mathfrak{z}}(a,b) & b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & b & 1 \end{pmatrix}^{\mathfrak{z}}.$$

It is now easy to see that the map $\mathfrak{z}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c+\psi_{\mathfrak{z}}(a,b) & b & 1 \end{pmatrix}.$$

It is clear from the definition that the set of all central automorphisms is a normal subgroup of $\mathrm{Aut}(S)$. ∎

Recall that for any prime power $q = p^n$, the group $\Gamma\mathrm{L}_2(q)$ is the semidirect product of $\mathrm{GL}_2(q)$ with the cyclic group of order n generated by the automorphism $\sigma : (a_{ij}) \mapsto (a_{ij}^p)$. It is isomorphic to the semidirect product of $\mathrm{GL}_2(q)$ with the group $\langle\sigma\rangle$.

Lemma 2.2.3 [32, p29] $\mathrm{Aut}(S)$ *contains a subgroup isomorphic to* $\Gamma\mathrm{L}_2(q)$.

Proof: For each $x = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(q)$, define an automorphism of S by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ a\alpha + b\gamma & 1 & 0 \\ \frac{1}{2}a^2\alpha\beta + \frac{1}{2}b^2\gamma\delta + ab\beta\gamma + \det(x)c & a\beta + b\delta & 1 \end{pmatrix}.$$

It is easy to see that this defines an embedding of $\mathrm{GL}_2(q)$ into $\mathrm{Aut}(S)$. Now, for each $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ define an automorphism of S by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ a^\sigma & 1 & 0 \\ c^\sigma & b^\sigma & 1 \end{pmatrix}.$$

This gives a cyclic subgroup of $\mathrm{Aut}(S)$ of order n (where $q = p^n$).

It is clear that these two subgroups generate a subgroup of $\mathrm{Aut}(S)$ isomorphic to $\Gamma\mathrm{L}_2(q)$ as claimed. ∎

The following well-known result gives us the automorphism group of S. Proofs may be found in [32, 5.3] or [19, 20.8].

Proposition 2.2.4 [32, 5.3] $\mathrm{Aut}(S) = CH$ *where C is the normal elementary abelian p-subgroup of central automorphisms of order p^{2n} and $H \cong \Gamma\mathrm{L}_2(q)$.*

We require the following well-known facts.

Lemma 2.2.5 *Let V be a 2-dimensional $\mathbb{F}_q$-space and let $G = \mathrm{GL}(V)$. Let $\mathcal{L}$ denote the collection of 1-dimensional subspaces of V. Then G acts by conjugation on $\mathrm{Syl}_p(G)$ with kernel $Z(G)$, and the faithful action of $G/Z(G)$ on $\mathrm{Syl}_p(G)$ is equivalent to the natural action of $\mathrm{PGL}(V)$ on $\mathcal{L}$.*

Proof: Note that G acts naturally on $\mathcal{L}$ with kernel $Z(G)$. This is, by definition, equivalent to the natural action of $\mathrm{PGL}(V)$ on $\mathcal{L}$.

Fix an ordered basis of V and let $R = \{\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right) \in \mathcal{G} | x \in \mathbb{F}_q\}$. Then R is a Sylow p-subgroup of G, with $N_G(R) = \{\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in G | a, b, c \in \mathbb{F}_q,\ a, c \neq 0\} = \mathrm{Stab}_G(l)$, where $l \in \mathcal{L}$ is the subspace of V generated by the vector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

By Sylow's Theorem, every Sylow p-subgroup T of G can be written as $T = R^g$ for some $g \in G$. Therefore $N_G(T) = N_G(R^g) = N_G(R)^g = \mathrm{Stab}_G(l)^g = \mathrm{Stab}_G(l^g)$. Define a map $\mu : \mathrm{Syl}_p(G) \to \mathcal{L}$ by $T \mapsto l'$ where $N_G(T) = \mathrm{Stab}_G(l')$. We show that μ is a bijective map.

First we show that μ is well-defined. By transitivity of G on $\mathcal{L}$, every element of $\mathcal{L}$ may be written as l^g for some $g \in G$. Suppose that $\mathrm{Stab}_G(l^g) = \mathrm{Stab}_G(l^h)$ for some $g, h \in G$. Then $\mathrm{Stab}_G(l)^{gh^{-1}} = \mathrm{Stab}_G(l)$, *i.e.* $N_G(R)^{gh^{-1}} = N_G(R)$. But $N_G(R)$ is self-normalizing in G, and therefore $gh^{-1} \in N_G(R) = \mathrm{Stab}_G(l)$. Hence $l^{gh^{-1}} = l$, *i.e.* $l^g = l^h$. Thus μ is well-defined.

To see that μ is injective, note that if $l^g = l^h$ for some $g, h \in G$ then $gh^{-1} \in N_G(R)$ and therefore $R^g = R^h$. Also μ is surjective by the transitivity of G on $\mathcal{L}$.

The group G acts on $\mathrm{Syl}_p(G)$ by conjugation by Sylow's Theorem. The kernel of this action is equal to the set $\bigcap_{g \in G} N_G(R^g)$. Note that $\{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathcal{G} | x \in \mathbb{F}_q\}$ is also a Sylow p-subgroup of G, and the normalizer of this subgroup is given by $\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in G | a, b, c \in \mathbb{F}_q,\ a, c \neq 0\}$. Hence $\bigcap_{g \in G} N_G(R^g) \leq \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in G | a, c \in \mathbb{F}_q \setminus \{0\}\}$. Consider the stabilizer of the subspace of V spanned by the vector $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ (where $a, b \in \mathbb{F}_q \setminus \{0\}$) is contained in this stabilizer if and only if $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} c \\ c \end{smallmatrix}\right)$ for some $c \in \mathbb{F}_q$. But $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ and therefore $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ is in the stabilizer if and only if $a = b$. In particular, $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in \bigcap_{g \in G} N_G(R^g)$ if and only if $a = b$. Hence $\bigcap_{g \in G} N_G(R^g) = Z(G)$. This means that G acts on $\mathrm{Syl}_p(G)$ with kernel $Z(G)$, in particular, $G/Z(G)$ acts faithfully on $\mathrm{Syl}_p(G)$.

Thus we have a bijection $\mu : \mathrm{Syl}_p(G) \to \mathcal{L}$ and an isomorphism $G/Z(G) \to \mathrm{PGL}(V)$ which together show that the action of $G/Z(G)$ on $\mathrm{Syl}_p(G)$ is equivalent to the action of $\mathrm{PGL}(V)$ on $\mathcal{L}$. ∎

Lemma 2.2.6 *The action by conjugation of* $\mathrm{GL}_2(q)$ *on* $\mathrm{Syl}_p(\mathrm{GL}_2(q))$ *is* 2*-transitive.*

Proof: Let L be the set of lower unitriangular matrices in $\mathrm{GL}_2(q)$ and let U be the set of upper unitriangular matrices. These are both Sylow p-subgroups of $\mathrm{GL}_2(q)$, and $L \cap U = 1$.

But U is the unique Sylow p-subgroup of $N_{\mathrm{GL}_2(q)}(U)$, and so $N_L(U) = \{g \in L | U^g = U\}$, which is a p-group, must be contained in U. Hence $N_L(U) = 1$.

The group L acts by conjugation on the set $\mathrm{Syl}_p(\mathrm{GL}_2(q)) \setminus \{L\}$ of q Sylow p-subgroups of $\mathrm{GL}_2(q)$ distinct from L. We have that $|\mathrm{Orb}_L(U)| = |L : N_L(U)| = |L| = q$, and so L acts transitively on $\mathrm{Syl}_p(\mathrm{GL}_2(q)) \setminus \{L\}$. Sylow's Theorem shows that a group acts transitively on the set of all its Sylow subgroups, and so $\mathrm{GL}_2(q)$ acts 2-transitively on its Sylow p-subgroups. ∎

Corollary 2.2.7 *The group* $\mathrm{PGL}_2(q)$ *acts 2-transitively on the* $q+1$ *points of the projective line* $\mathcal{P}$.

Proof: The projective line $\mathcal{P}$ can be identified with the collection $\mathcal{L}$ of 1-dimensional subspaces of $\mathbb{F}_q^2$.

By Lemma 2.2.5, the action of $\mathrm{PGL}_2(q)$ on $\mathcal{L}$ is equivalent to the action of $\mathrm{GL}_2(q)$ on its Sylow p-subgroups. Thus the result is equivalent to Lemma 2.2.6. ∎

We can now deduce the following:

Corollary 2.2.8 *The automorphism group of* S *acts 2-transitively on the collection of elementary abelian subgroups of* S *of order* q^2.

Proof: Consider $S/Z(S)$ as a vector space of dimension 2 over $\mathbb{F}_q$. By Proposition 2.2.4, $\mathrm{Aut}(S)$ contains a subgroup G isomorphic to $\mathrm{GL}_2(q)$, and the action of G on $S/Z(S)$ is equivalent to the natural action of $\mathrm{GL}_2(q)$ on $\mathbb{F}_q^2$. By considering $S/Z(S)$ as an $\mathbb{F}_q$-space, we see that the elementary abelian subgroups of order q^2 are just the 1-dimensional subspaces spanned by $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & b & 1 \end{smallmatrix}\right) Z(S)$ (for $b \in \mathbb{F}_q$) and $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right) Z(S)$. Thus we can identify the collection of elementary abelian subgroups of S of order q^2 with the collection $\mathcal{L}$ of 1-dimensional subspaces of $\mathbb{F}_q^2$. Lemma 2.2.7 now shows that G, and hence $\mathrm{Aut}(S)$, acts 2-transitively on the collection of elementary abelian subgroups of order q^2. ∎

This result will allow us to map two of these subgroups to any other two of these subgroups whilst preserving (up to isotypical equivalence) the properties of the fusion system we are studying.

We need the next result in the following chapter.

Lemma 2.2.9 *Let V be the subgroup of S consisting of matrices of the form* $\left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right)$ *where $a, c \in \mathbb{F}_q$, and let $Z = Z(S)$. We regard V as a $2n$-dimensional vector space over $\mathbb{F}_p$, and regard Z as a vector subspace of V. Then the group of central automorphisms of S acts transitively on the collection of complementary subspaces to Z in V.*

Proof: Let Y be the subspace of matrices of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Clearly this is a complementary subspace to Z in V. Now define projections

$$\begin{aligned} \pi_1 : V &\rightarrow Y \\ z + y &\mapsto y \end{aligned}$$

and

$$\begin{aligned} \pi_2 : V &\rightarrow Z \\ z + y &\mapsto z. \end{aligned}$$

Now let K be any complementary subspace to Z. Observe that $\pi_1|_K$ is an isomorphism, so $\theta_K := (\pi_1|_K)^{-1}\pi_2$ is a well-defined map $Y \rightarrow Z$.

Suppose $\theta_K = \theta_{K'}$ for some complementary subspace K'. Then for every $y \in Y$,

$$y^{(\pi_1|_K)^{-1}\pi_2} = y^{(\pi_1|_{K'})^{-1}\pi_2}. \tag{2.1}$$

But if $y^{(\pi_1|_K)^{-1}} = z + y$ and $y(\pi_1|_{K'})^{-1} = z' + y$ then the equation above gives $z = z'$. Hence $(\pi_1|_K)^{-1} = (\pi_1|_{K'})^{-1}$, and so $K = K'$.

We now show that $K = \{y + y^{\theta_K} \in Y \oplus Z | y \in Y\} =: A$. To see this, first note that if $y, y' \in Y$ and $\lambda \in \mathbb{F}_p$ then $(y + y^{\theta_K}) + \lambda(y' + y'^{\theta_K}) = (y + \lambda y') + (y^{\theta_K} + \lambda y'^{\theta_K}) \in A$. So A

is a vector subspace of V. But $(y + y^{\theta_K}) \in Z$ if and only if $y = 0$, and therefore $y^{\theta_K} = 0$. Hence $A \cap Z = 0$. Finally, we show that $A \oplus Z = V$. To do this we show that $Y \subseteq A \oplus Z$. If $y \in Y$ then $y^{\theta_K} \in Z$ and so $y = (y + y^{\theta_K}) - y^{\theta_K} \in A \oplus Z$. Thus A is a complementary subspace to Z in V.

Now, given any $y \in Y$, $y^{(\pi_1|_A)^{-1}\pi_2} = y^{\theta_K}$ and so $\theta_A = \theta_K$. Hence $K = A$. To complete the proof, define a homomorphism $\psi_K \in \mathrm{Hom}((\mathbb{F}_q, +)^2, (\mathbb{F}_q, +))$ by $\psi_K(a, b) = a^{\theta_K}$. We now have that the map

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c + \psi_K(a,b) & b & 1 \end{pmatrix}$$

is a central automorphism of S mapping Y to K. ∎

2.3 Potentially radical subgroups

By Alperin's Fusion Theorem, a saturated fusion system is completely determined by the $\mathcal{F}$-Alperin subgroups and their $\mathcal{F}$-automorphisms. So to classify all the saturated fusion systems over S we need to consider which subgroups can be $\mathcal{F}$-centric and $\mathcal{F}$-radical. To this end, we make the following definition:

Definition 2.3.1 *A subgroup Q of any finite p-group P is called* potentially radical *if there exists a subgroup $H \leq \mathrm{Aut}(Q)$ such that*

(i) $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(H)$*; and*

(ii) $O_p(H) = Q/Z(Q) = \mathrm{Inn}(Q)$.

Remark Note that if $\mathcal{F}$ is a saturated fusion system over P and $Q \leq P$ is fully normalized and $\mathcal{F}$-radical then Q is potentially radical since $H = \mathrm{Aut}_{\mathcal{F}}(Q) \leq \mathrm{Aut}(Q)$ satisfies the definition above.

Definition 2.3.2 *We shall say that a subgroup V of S is* self-centralizing *if $C_S(V) = Z(V)$. Denote by $\mathcal{C}$ the collection of all subgroups V of S which are both self-centralizing and potentially radical.*

Lemma 2.3.3 *If $Q \in \mathcal{C}$ and $Q \neq S$ then $Z(S) < Q$ and $Q \trianglelefteq S$.*

Proof: Firstly, note that $Q \neq Z(S)$ since $C_S(Z(S)) = S \neq Z(S)$. It is also clear that $Z(S) \leq C_S(Q)$. But Q is self-centralizing and so $Z(S) \leq C_S(Q) \leq Q$. Since $[S, S] = Z(S) \leq Q$, we have that Q is normal in S. ∎

We now prove a sharper version of a result from Gorenstein [20, 5.3.2].

Proposition 2.3.4 *Let P be a finite p-group and let*

$$P = P_0 \geq P_1 \geq \cdots \geq P_{k-1} \geq P_k = 1$$

be a series of characteristic subgroups of P. Then

$$\bigcap_{i=0}^{k} C_{\mathrm{Aut}(P)}(P_i/P_{i+1}) \leq \mathrm{O}_p(\mathrm{Aut}(P)).$$

Proof: Let $C_i := C_{\mathrm{Aut}(P)}(P_i/P_{i+1})$. Applying Gorenstein's result [20, 5.3.3] gives that $\bigcap_{i=1}^{k} C_i$ is a p-subgroup of $\mathrm{Aut}(P)$, so it remains to show that $\bigcap_{i=1}^{k} C_i \trianglelefteq \mathrm{Aut}(P)$. Let $\alpha \in C_i$, $\beta \in \mathrm{Aut}(P)$ and let $xP_{i+1} \in P_i/P_{i+1}$. Then $\beta^{-1}\alpha\beta(xP_{i+1}) = \beta^{-1}\alpha(\beta(x)P_{i+1}) = \beta^{-1}(\beta(x)P_{i+1}) = xP_{i+1}$ since P_i and P_{i+1} are characteristic in P. Hence for every i, $C_i \trianglelefteq \mathrm{Aut}(P)$, and therefore $\bigcap_{i=1}^{k} C_i \trianglelefteq \mathrm{Aut}(P)$. Thus $\bigcap_{i=1}^{k} C_i \leq \mathrm{O}_p(\mathrm{Aut}(P))$. ∎

We shall introduce some convenient terminology from Gorenstein [20, p178].

Definition 2.3.5 *Let G be a finite group and let A be a subgroup of* $\mathrm{Aut}(G)$. *We say that A* stabilizes *the series*

$$G = G_0 \geq G_1 \geq \cdots \geq G_{k-1} \geq G_k = 1$$

if each G_i is A-invariant and A acts trivially on each factor G_i/G_{i+1} for $1 \leq i \leq k$.

Corollary 2.3.6 *Let P be a finite p-group. Suppose that $Q \lneqq P$ and that $\mathrm{Aut}_P(Q)$ stabilizes some series of characteristic subgroups of Q. Then Q is not potentially radical.*

Proof: Suppose that $Q \in \mathcal{C}$ and that $\mathrm{Aut}_P(Q)$ stabilizes the series

$$Q = Q_0 \geq Q_1 \geq \cdots \geq Q_{k-1} \geq Q_k = 1$$

of characteristic subgroups of Q. Then $\mathrm{Aut}_P(Q) \leq \bigcap_{i=0}^{k} C_{\mathrm{Aut}(Q)}(Q_i/Q_{i+1})$ and so by Proposition 2.3.4 $\mathrm{Aut}_P(Q) \leq O_p(\mathrm{Aut}(Q))$. Now suppose that $H \leq \mathrm{Aut}(Q)$ satisfies Definition 2.3.1 for Q. We have $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(H)$, and so $\mathrm{Aut}_P(Q) = H \cap O_p(\mathrm{Aut}(Q)) \trianglelefteq H$. Therefore $\mathrm{Aut}_P(Q) = P/Z(Q) \leq O_p(H) = Q/Z(Q)$, which contradicts the fact that Q is a proper subgroup of P. ∎

Proposition 2.3.7 *No self-centralizing non-abelian proper subgroup of S is potentially radical.*

Proof: Let Q be a self-centralizing non-abelian proper subgroup of S. Note that since Q is self-centralizing, $[S, S] = Z(S) < Q$. Thus Q is normal in S. We consider two cases.

Case 1: $Z(Q) = Z(S)$. In this case $[Q, S] \leq [S, S] = Z(S) = Z(Q)$ and it is easy to see that $\mathrm{Aut}_S(Q)$ centralizes $Z(Q)$ and $Q/Z(Q)$. This means that $\mathrm{Aut}_S(Q)$ stabilizes the characteristic series $Q \geq Z(Q) \geq 1$, and so by Corollary 2.3.6, Q is not potentially radical.

Case 2: $Z(Q) \gneqq Z(S)$. If $x \in Z(Q) \setminus Z(S)$ then $Q \leq C_S(x)$. But $C_S(x)$ is abelian by Lemma 2.1.1(iii), which contradicts the assumption that Q is non-abelian.

This completes the proof. ∎

Proposition 2.3.8 *Let $Q \in \mathcal{C}$ be a proper abelian subgroup of S. Then there exists a $y \in Q \setminus Z(S)$ such that $Q = C_S(y)$.*

Proof: By Lemma 2.3.3, we have $Z(S) < Q$. Let $y \in Q \setminus Z(S)$. Since Q is abelian, $Q \leq C_S(y)$. But Lemma 2.1.1(*iii*) shows that $C_S(y)$ is abelian, and so $C_S(y)$ centralizes Q. Since Q is self-centralizing, we therefore have $Q = C_S(y)$. ∎

Putting together Propositions 2.1.1(iii), 2.3.7 and 2.3.8, we see that the set $\mathcal{C}$ of potentially radical subgroups contains just S and elementary abelian groups of order q^2. This means that the set $\mathrm{Alp}(\mathcal{F})$ is contained in the set of all elementary abelian subgroups of S of order q^2. Let us collect some facts about the elementary abelian elements of $\mathcal{C}$. Recall from Lemma 2.2.2 that if $\mathfrak{z}$ is a central automorphism then there exists a homomorphism $\psi_{\mathfrak{z}} : \mathbb{F}_q^2 \to \mathbb{F}_q$ associated to $\mathfrak{z}$.

Lemma 2.3.9 *Let V be an elementary abelian subgroup in $\mathcal{C}$. Then the following hold:*

(i) *V is a normal subgroup of S;*

(ii) $\mathrm{Aut}_S(V) \cong S/V$ *is elementary abelian of order q;*

(iii) $\mathrm{Aut}(V) \cong \mathrm{GL}_{2n}(p)$;

(iv) $C_{\mathrm{Aut}(S)}(V) = \{\mathfrak{z} \in C_{\mathrm{Aut}(S)}(S/Z(S)) | \psi_{\mathfrak{z}}(x, 0) = 0$ *for all* $x \in \mathbb{F}_q\}$.

(v) $\mathrm{Aut}(V)$ *contains a subgroup isomorphic to* $\mathrm{GL}_2(q)$.

Proof: We have $S' = Z(S) \leq V$ and so (*i*) and (*ii*) now follow immediately. To see (*iii*), we note that V may be considered as a $2n$-dimensional vector space over $\mathrm{GF}(p)$, from which it is clear that $\mathrm{Aut}(V) \cong \mathrm{GL}_{2n}(p)$.

We now prove (*iv*). By Corollary 2.2.8 we may assume that V is the subgroup $\left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{smallmatrix} \right) | x, y \in \mathbb{F}_q \right\}$. Let $\mathfrak{z} \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \sigma^t \in C_{\mathrm{Aut}(S)}(V)$, where $\mathfrak{z}$ is a central automorphism and σ is the automorphism induced by the Frobenius automorphism of $\mathbb{F}_q$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & 0 & 1 \end{pmatrix}^{\mathfrak{z} \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \sigma^t} = \begin{pmatrix} 1 & 0 & 0 \\ (x\alpha)^{p^t} & 1 & 0 \\ (\frac{1}{2}x^2\alpha\beta + (\alpha\delta - \beta\gamma)(y + \psi_{\mathfrak{z}}))^{p^t} & (x\beta)^{p^t} & 1 \end{pmatrix}$$

for all $x, y \in \mathbb{F}_q$. First, note that we have $(x\beta)^{p^t} = 0$ and so $\beta = 0$. By setting $x = 1$ we see that $\alpha^{p^t} = 1$ and if ω is a primitive element of $\mathbb{F}_q$ then setting $x = \omega$ gives $\alpha^{p^t}\omega^{p^t-1} = 1$. Thus $\omega^{p^t-1} = 1$ and therefore $p^t = q$ or 1. By setting $x = 0$ we see that $(\alpha\delta)y = y$, and so $\alpha\delta = 1$. Therefore for any x we have $y + \psi_{\mathfrak{z}}(x, 0) = y$, which means that $\psi_{\mathfrak{z}}(x, 0) = 0$ for all x.

To see (v), we show that the elementary abelian groups of order q^2 may be regarded as a vector space over $\mathbb{F}_q$. Suppose that $V = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & a\lambda & 1 \end{smallmatrix} \right) | a, c\mathbb{F}_q \right\}$ where $\lambda \in \mathbb{F}_q$. Let $\omega \in \mathbb{F}_q$ be a primitive element of $\mathbb{F}_q$. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix} \right)$, $e_2 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega & 0 & 1 \end{smallmatrix} \right)$, $\ldots$, $e_n = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega^{n-1} & 0 & 1 \end{smallmatrix} \right)$, $f_1 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & \lambda & 1 \end{smallmatrix} \right)$, $\ldots$, $f_n = \left(\begin{smallmatrix} 1 & 0 & 0 \\ \omega^{n-1} & 1 & 0 \\ 0 & \omega^{n-1}\lambda & 1 \end{smallmatrix} \right)$. Since V is elementary abelian, it is clear that V can be regarded as a $2n$-dimensional $\mathbb{F}_p$-space, with ordered $\mathbb{F}_p$-basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. We can define an action of $\mathbb{F}_q$ on V as follows: for every $\mu \in \mathbb{F}_q$, with $\mu = \mu_1 + \mu_2\omega + \cdots + \mu_n\omega^{n-1}$ where $\mu_i \in \mathbb{F}_p$ define a matrix

$$M = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_n & \mu_1 & \cdots & \mu_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_2 & \mu_3 & \cdots & \mu_1 \end{pmatrix}.$$

Then for every element $v \in V$ (considered as a $2n$-dimensional $\mathbb{F}_p$-space), define $\mu.v = \left(\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix} \right) v$. This multiplication makes V into a 2-dimensional $\mathbb{F}_q$-space with basis $\{e_1, f_1\}$. A similar argument works for the case when $V = \left\{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{smallmatrix} \right) | b, c \in \mathbb{F}_q \right\}$. This shows that $\mathrm{Aut}(V)$ has a subgroup isomorphic to $\mathrm{GL}_2(q)$, namely the collection of automorphisms of V which preserve this $\mathbb{F}_q$-structure. ∎

Corollary 2.3.10 *Let $x \in S \setminus Z(S)$ and let $V = C_S(x)$. Then V is potentially radical in S.*

Proof: The group V is elementary abelian of order q^2 by 2.1.1. In the proof of 2.3.9(v), we constructed a subgroup of $\mathrm{Aut}(V)$ isomorphic to $\mathrm{GL}_2(q)$; denote this subgroup by H. If we can show that $\mathrm{Aut}_S(V) \in \mathrm{Syl}_p(H)$ then we will have shown that V is potentially radical since $O_p(H) = 1$.

The Sylow p-subgroups of H have order q, and so it remains to show that $\mathrm{Aut}_S(V) \leq H$ since $\mathrm{Aut}_S(V)$ also has order q. Recall that if $s = \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{smallmatrix}\right)$ then $s^{-1}vs = \left(\begin{smallmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z+ay-bx & y & 1 \end{smallmatrix}\right)$. To see that $\mathrm{Aut}_S(V)$ is contained in H, we shall write the elements of $\mathrm{Aut}_S(V)$ as $2n$-dimensional matrices over $\mathbb{F}_p$ and show that they commute with the matrices representing scalar multiplication of elements of V by elements of $\mathbb{F}_q$.

We observe that with respect to the ordered $\mathbb{F}_p$-basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ described in 2.3.9, elements of $\mathrm{Aut}_S(V)$ have the form $\left(\begin{smallmatrix} I & A \\ 0 & I \end{smallmatrix}\right)$ where A is an $n \times n$ matrix over $\mathbb{F}_p$ and I is the $n \times n$ identity matrix. But matrices of this form commute with matrices of the form $\left(\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}\right)$. In particular, elements of $\mathrm{Aut}_S(V)$ commute with scalar multiplication of elements of V by elements of $\mathbb{F}_q$, as required. This completes the proof that $\mathrm{Aut}_S(V) \leq H$, thereby showing that V is potentially radical. ∎

Corollary 2.3.11 *If $\mathcal{F}$ is a saturated fusion system over S then* $\mathrm{Alp}(\mathcal{F})$ *is closed under $\mathcal{F}$-conjugation.*

Proof: Lemma 2.3.9 shows that $\mathcal{C} = \{S, C_S(x) | x \in S \setminus Z(S)\}$, and therefore $\mathrm{Alp}(\mathcal{F}) \subseteq \{C_S(x) | x \in S \setminus Z(S)\}$. So let V be an elementary abelian Alperin subgroup of S; then $|V| = q^2$. Since every elementary abelian subgroup of S which has order q^2 is in $\mathcal{C}$, we have that every $\mathcal{F}$-conjugate of V is an elementary abelian member of $\mathcal{C}$. So by 2.3.9, V is normal in S, and is therefore fully $\mathcal{F}$-normalized.

Now, every $\mathcal{F}$-conjugate of an $\mathcal{F}$-radical subgroup of S is $\mathcal{F}$-radical and every $\mathcal{F}$-conjugate of an $\mathcal{F}$-centric subgroup is $\mathcal{F}$-centric. Therefore every $\mathcal{F}$-conjugate of V is $\mathcal{F}$-radical, $\mathcal{F}$-centric, and fully $\mathcal{F}$-normalized. Hence every $\mathcal{F}$-conjugate of V is an Alperin subgroup, and so $\mathrm{Alp}(\mathcal{F})$ is closed under $\mathcal{F}$-conjugation. ∎

Corollary 2.3.12 *If $\mathcal{F}$ is a saturated fusion system over S then $\mathcal{F}$ is* $\mathrm{Alp}(\mathcal{F}) \cup \{S\}$*-generated.*

Proof: This follows immediately from 2.3.11 and Alperin's Fusion Theorem. ∎

Chapter 3

Determining $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$

In this chapter, as before, S continues to be a Sylow p-subgroup of $\mathrm{SL}_3(q)$, where $q = p^n$ for some n and some prime p. It turns out that for any saturated fusion system $\mathcal{F}$ over our group S, there is a lot we can say about $O^{p'}(\mathcal{F})$. Specifically, we shall be able to prove that whenever V is an elementary abelian $\mathcal{F}$-Alperin subgroup of S of order q^2, the group $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ satisfies the hypothesis of a theorem of Timmesfeld [40, Theorem 3.2], thereby proving that it is isomorphic to $\mathrm{SL}_2(q)$. Using Theorem 1.8.2, the fusion system shall then be determined by $\mathrm{Aut}_{\mathcal{F}}(S)$.

The paper of Ruiz and Viruel [36] deals with the case $n = 1$ (*i.e.* $q = p$). They proved that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains a normal subgroup isomorphic to $\mathrm{SL}_2(p)$. Of course, in their case $\mathrm{Aut}(V) \cong \mathrm{GL}_2(p)$, whereas our situation is rather more complicated as $\mathrm{Aut}(V) \cong \mathrm{GL}_{2n}(p)$. The result proved in this chapter includes that of Ruiz and Viruel as a special case.

3.1 A sufficient condition for $\mathcal{F}$-radical subgroups

First we observe that V is $\mathcal{F}$-radical if $\mathrm{Aut}_{\mathcal{F}}(V)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$.

Proposition 3.1.1 *Let $\mathcal{F}$ be a saturated fusion system over S, and let V be an elementary abelian subgroup of S of order q^2. If $\mathrm{Aut}_{\mathcal{F}}(V)$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$*

then V is $\mathcal{F}$-radical.

Proof: Suppose $L \le \mathrm{Aut}_{\mathcal{F}}(V)$ is isomorphic to $\mathrm{SL}_2(q)$. The group V is abelian so $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{Aut}_{\mathcal{F}}(V)/\mathrm{Inn}(V) \cong \mathrm{Aut}_{\mathcal{F}}(V)$. Since $\mathcal{F}$ is saturated and V is fully normalised, we know from Lemma 2.3.9 that $\mathrm{Aut}_{\mathcal{F}}(V)$ has elementary abelian Sylow p-subgroups of order q. But $\mathrm{SL}_2(q)$ also has elementary abelian Sylow p-subgroups of order q and so $\mathrm{Syl}_p(L) \subseteq \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(V))$. Therefore $O_p(\mathrm{Aut}_{\mathcal{F}}(V)) \le O_p(L) = 1$ and so V is $\mathcal{F}$-radical.
∎

3.2 Rank one groups

In this section we shall introduce the notions of a *rank one group* and *quadratic action*. We shall show that these notions apply to $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$.

Definition 3.2.1 *A group X is called a* rank one group *if it is generated by two distinct nilpotent subgroups A and B with the property that for each $a \in A \setminus \{1\}$ there exists $b \in B \setminus \{1\}$ such that $A^b = B^a$, and vice versa.*

The conjugates of A and B are called the *unipotent* subgroups of X. To say that X is a *rank* 1 group is simply a shorthand way of saying that X is a group with a split (B, N)-pair of rank 1, as noted in [40, p.3]. We shall not discuss groups with a (B, N)-pair explicitly here; for more details about groups with a (B, N)-pair see [6, Chapter IV, §2].

Definition 3.2.2 *Let X be a rank one group with unipotent subgroup A. The abelian group V is called a* quadratic X-module *if X acts faithfully on V and $[V, A, A] = 0$.*

Let $\mathcal{F}$ be a saturated fusion system over S and V an elementary abelian p-subgroup in $\mathcal{C}$. Let $G = \mathrm{Aut}_{\mathcal{F}}(V)$. Then we may regard V as a $2n$-dimensional $\mathbb{F}_pG$-module.

We have the following lemma:

Lemma 3.2.3 *Let V be an elementary abelian p-subgroup of S in $\mathcal{C}$. By regarding V as a $\mathbb{F}_pG$-module, we have that for all $s \in S \setminus V$,*

$$C_V(\mathrm{Aut}_S(V)) = [V, \mathrm{Aut}_S(V)] = [V, c_s] = C_V(c_s) = Z(S).$$

Proof: Firstly, note that

$$C_V(\mathrm{Aut}_S(V)) = \{x \in V | s^{-1}xs = x \,\forall\, s \in S\} = V \cap Z(S) = Z(S).$$

Now

$$C_V(c_s) = \{x \in V | s^{-1}xs = x\} = C_S(s) \cap V \geq Z(S).$$

If $s \in V$ then c_s acts trivially on V because V is abelian. So by Lemma 2.1.1(iii) $C_S(s)$ is an elementary abelian p-group of order q^2 distinct from V. Hence by part (v) of that lemma, $C_V(s) = Z(S)$.

It is easy to see that $[V, \mathrm{Aut}_S(V)] = [V, S]$ and $[V, c_s] = [V, s]$. Let $s = \left(\begin{smallmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ w & v & 1 \end{smallmatrix}\right)$ and let $x = \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{smallmatrix}\right) \in V$. Then $[x, s] = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ub-va & 0 & 1 \end{smallmatrix}\right)$ and so $[V, s] = [V, S] = Z(S)$ since a and b range over all of $\mathrm{GF}(q)$. This completes the proof. ∎

The lemma above implies that with $V \in \mathcal{C} \setminus \{S\}$, $G = \mathrm{Aut}_{\mathcal{F}}(V)$ and $R = \mathrm{Aut}_S(V)$ the hypotheses of the following result are satisfied.

Lemma 3.2.4 [32, Lemma 4.16] *Suppose that p is a prime, G is a finite group, $R \in \mathrm{Syl}_p(G)$ has order p^n and V is a faithful $\mathbb{F}_pG$-module of dimension $2n$. If, for all non-identity $r \in R$,*

$$C_V(R) = [V, R] = [V, r] = C_V(r)$$

has $\mathbb{F}_p$-dimension n, then either $|\mathrm{Syl}_p(G)| = 1$ or $p^n + 1$. Furthermore, if $|\mathrm{Syl}_p(G)| = p^n + 1$ then the following hold:

(i) for any pair $P \neq P' \in \mathrm{Syl}_p(G)$ we have $P \cap P' = 1$ and $C_V(P) \cap C_V(P') = 0$, and

(ii) G acts 2-transitively on $\mathrm{Syl}_p(G)$.

Proof: Suppose that $|\mathrm{Syl}_p(G)| > 1$, and let $R, T \in \mathrm{Syl}_p(G)$. Note that if $v + [V, R] \in V/[V, R]$ and $r \in R$ then

$$(v + [V, R])^r = v^r + [V, R] = v^r + (-v)^r - (-v) + [V, R] = v + [V, R].$$

This means that if $C_V(T) = C_V(R) = [V, R] = [V, T]$ then $\langle R, T\rangle$ centralizes the series $V \rhd [V, R] \rhd 0$. Thus by Proposition 2.3.4, we have that $\langle R, T\rangle$ is a p-group. But $R \in \mathrm{Syl}_p(G)$ and hence $R = T$. We conclude that for distinct $R, T \in \mathrm{Syl}_p(G)$, $C_V(R) \neq C_V(T)$. This in turn means that $R \cap T = 1$, since if x is a non-trivial element of $R \cap T$ then $C_V(R) = C_V(x) = C_V(T)$, which is a contradiction. This proves the first part of (i).

From among all the distinct pairs of members of $\mathrm{Syl}_p(G)$, choose R, T such that the dimension of $C_V(R) \cap C_V(T)$ is maximal. Set $U = C_V(R) \cap C_V(T)$ and $W = C_V(R) + C_V(T)$. Since R is conjugate to T we have $\dim(C_V(R)) = \dim(C_V(T))$. Hence $\dim(C_V(R) + C_V(T)) = \dim(C_V(R)) + \dim(C_V(T)) - \dim(C_V(R) \cap C_V(T))$. Therefore $\dim(W/U) = \dim(C_V(R)) + \dim(C_V(T)) - 2\dim(C_V(R) \cap C_V(T))$. This means that $\dim(W/U)$ is even, *i.e.* W/U has dimension $2b$ for some $0 < b \leq n$. Let $L = \langle R, T\rangle$ and let $P, Q \in \mathrm{Syl}_p(L)$ with $P \neq Q$. Then the subspaces $C_V(P)/U$ and $C_V(Q)/U$ of W/U both have dimension b, and by the maximality of $\dim(U)$, they intersect trivially. Now, R acts on $\mathrm{Syl}_p(L)$ by conjugation, and so $N_S(T)$ is a p-subgroup of $N_L(T)$. But T is the unique Sylow p-subgroup of $N_L(T)$ and so $N_R(T) \leq T$. But $R \cap T = 1$ so $N_R(T) = 1$ and thus $|\mathrm{Orb}_R(T)| = |R : N_R(T)| = |R| = p^n$ by the Orbit-Stabilizer Theorem. Hence $|\mathrm{Syl}_p(L)| \geq p^n + 1$ and so W/U has at least $(p^n + 1)(p^b - 1)$ non-trivial elements. Thus we have $p^{2b} - 1 \geq (p^n + 1)(p^b - 1)$, which means that $b = n$. This allows us to conclude that $C_V(R) \cap C_V(T) = 0$ for all pairs of distinct members R and T of $\mathrm{Syl}_p(G)$, proving (i).

We now have that $|\mathrm{Syl}_p(G)| \geq p^n + 1$ and so

$$p^{2n} = |V| \geq (p^n + 1)(p^n - 1) + 1 = p^{2n}.$$

From this we can conclude that $|\mathrm{Syl}_p(G)| = p^n + 1$. Finally, we note that any $R \in \mathrm{Syl}_p(G)$ acts on $\mathrm{Syl}_p(G) \setminus \{R\}$. If $T \in \mathrm{Syl}_p(G) \setminus \{R\}$ then $|\mathrm{Orb}_R(T)| = p^n$, and thus R acts regularly on $\mathrm{Syl}_p(G) \setminus \{R\}$. In addition, Sylow's Theorem shows that G acts transitively on $\mathrm{Syl}_p(G)$ and so it follows that G is 2-transitive on $\mathrm{Syl}_p(G)$. This proves (ii). ∎

Proposition 3.2.5 *Let G be a finite group, and suppose that $R \in \mathrm{Syl}_p(G)$ has order p^n. Let V be a faithful $\mathbb{F}_pG$-module of dimension $2n$. Suppose that for all non-identity $r \in R$,*

$$C_V(R) = [V, R] = [V, r] = C_V(r)$$

has $\mathbb{F}_p$-dimension n, and that $|\mathrm{Syl}_p(G)| = q + 1$. Let $X = O^{p'}(G)$. Then

(i) X is a rank 1 group with unipotent subgroup R,

(ii) V is a quadratic X-module.

Proof: Let $R, T \in \mathrm{Syl}_p(G)$. As we saw in the proof of Lemma 3.2.4, R acts regularly on $\mathrm{Syl}_p(G) \setminus \{R\}$ and T acts regularly on $\mathrm{Syl}_p(G) \setminus \{T\}$. Hence $\langle R, T\rangle$ contains every Sylow p-subgroup of G, *i.e.* $O^{p'}(G) = \langle R, T\rangle$.

Now, if $r \in R$ then (by regularity of R) $T^r \in \mathrm{Syl}_p(G) \setminus \{R, T\}$. By the transitivity of T on $\mathrm{Syl}_p(G) \setminus \{T\}$, there exists a $t \in T$ such that $R^t = T^r$. This completes the proof that $X = O^{p'}(G)$ is a rank one group.

To prove (ii), note that by hypothesis, $[V, R, R] = [C_V(R), R] = 0$. ∎

Corollary 3.2.6 *Let $\mathcal{F}$ be a saturated fusion system over S and let V be an elementary abelian $\mathcal{F}$-Alperin subgroup of S. Let $X = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ and let $A = \mathrm{Aut}_S(V)$. Then X is a rank one group with unipotent subgroup A, and V is a quadratic X-module.*

Proof: Let $G = \mathrm{Aut}_{\mathcal{F}}(V)$. Then V is a faithful $2n$-dimensional $\mathbb{F}_pG$-module, and by 3.2.3 we have that $C_V(S) = [V,S] = [V,s] = C_V(s)$ for all non-identity $s \in S$. Hence by 3.2.4, $|\mathrm{Syl}_p(G)| = 1$ or $|\mathrm{Syl}_p(G)| = q+1$. But if $|\mathrm{Syl}_p(G)| = 1$ then $\mathrm{Aut}_S(V)$ would be a non-trivial normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(V) \cong \mathrm{Out}_{\mathcal{F}}(V)$, contradicting the fact that V is $\mathcal{F}$-radical. Hence $|\mathrm{Syl}_p(G)| = q+1$. The result now follows from 3.2.5. ∎

Lemma 3.2.7 *Under the conditions of Lemma 3.2.4, we have*

$$V = \bigcup_{T \in \mathrm{Syl}_p(G)} C_V(T).$$

Proof: By hypothesis, $\dim(C_V(T)) = n$, and therefore $|C_V(T)| = p^n$ for all $T \in \mathrm{Syl}_p(G)$. By Theorem 3.2.4 (i), for any two distinct Sylow subgroups P, Q we have $C_V(P) \cap C_V(Q) = 0$, thus

$$\left| \bigcup_{T \in \mathrm{Syl}_p(G)} C_V(T) \right| = (p^n+1)(p^n-1)+1 = p^{2n}.$$

This means that

$$V = \bigcup_{T \in \mathrm{Syl}_p(G)} C_V(T)$$

as required. ∎

Definition 3.2.8 *Let D be a set with two operations; an addition $(a,b) \mapsto a+b$ and a multiplication $(a,b) \mapsto ab$. We say that D is an* alternative division ring *if the following hold:*

(i) D is an abelian group with respect to the operation $+$;

(ii) there exists a multiplicative unit element $1 \in D$;

(iii) every non-zero element $x \in D$ has a two-sided inverse x^{-1} such that $x^{-1}(xy) = y = (yx)x^{-1}$ for all $y \in D \setminus \{0\}$;

(iv) the operations on D satisfy the distributive laws.

Remark Note that we do not assume that the multiplication in D is associative.

Lemma 3.2.9 *A finite alternative division ring is a field.*

Proof: Let D be a finite alternative division ring. A basic property of alternative division rings is that any two elements generate an (associative) subring (see [41, Theorem ADR1, p439]). By the uniqueness of inverses in D, we have that any such subring R is an (associative) division ring. But D, and therefore R, is finite, and so by Wedderburn's Theorem (see [15, p178]) R is a field. In particular, R is commutative. Since this holds for any two elements of D, we have that every two elements of D commute. Hence D is commutative. Now by [41, Lemma 1, p439], a commutative alternative division ring is associative, and therefore D is associative. But then D is a commutative associative division ring; *i.e.* D is a field. ∎

The following theorem from Timmesfeld will us give the generalization of the Ruiz-Viruel result we want.

Theorem 3.2.10 [40, Theorem 3.2] *Let X be a rank 1 group with unipotent subgroup A. Suppose that V is a quadratic X-module with the following properties:*

(i) $V = [V, X]$ and $C_V(X) = 0$,

(ii) $[V, A] = [v, A]$ for every $v \in V - [V, A]$.

Then there exists an alternative division ring K whose additive group is isomorphic to A such that $X \cong \mathrm{SL}_2(K)$, and V is the natural $\mathbb{Z}X$-module.

Proposition 3.2.11 *Let V be an elementary abelian p-subgroup of S in $\mathcal{C}$, let $X = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$. Then $X \cong \mathrm{SL}_2(q)$ and acts on V as the natural $\mathbb{F}_q\mathrm{SL}_2(q)$-module.*

Proof: Let $A = \mathrm{Aut}_S(V)$. We showed in 3.2.6 that X is a rank 1 group with unipotent subgroup A and that V is a quadratic X-module. It remains to prove conditions (i) and (ii) from Theorem 3.2.10.

Now, since $[V, A] = C_V(A)$, we have that for any $x \in X$, $[V, A^x] = [V, A]^x = C_V(A)^x = C_V(A^x)$. Let $v \in V$. Then by Lemma 3.2.7, there exists a $T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(V))$ such that $v \in C_V(T) = [V, T] \subseteq [V, X]$. Hence $V = [V, X]$. Also $C_V(X) \subseteq \bigcap_{T \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(V))} C_V(T) = 0$ by Lemma 3.2.4, proving condition (i).

To prove (ii), let $v \in V \setminus Z(S)$. By Lemma 3.2.3, it suffices to show that $[v, S] = Z(S)$. Using Lemma 2.1.1, this is easily seen to be true. Hence by Theorem 3.2.10, $X \cong \mathrm{SL}_2(K)$ where K is an alternative division ring whose additive group is isomorphic to $\mathrm{Aut}_S(V)$. In particular, K is finite and has order q. Hence by 3.2.9, and the fact that finite fields are determined by order, $K = \mathbb{F}_q$. Therefore $X \cong \mathrm{SL}_2(q)$ as claimed, and we also have that X acts on V as a natural $\mathrm{SL}_2(q)$-module. ∎

We now show that $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ is the only subgroup of $\mathrm{Aut}_{\mathcal{F}}(V)$ containing $\mathrm{Aut}_S(V)$ which is isomorphic to $\mathrm{SL}_2(q)$.

Lemma 3.2.12 *Suppose that V is a $2n$-dimensional vector space over $\mathbb{F}_p$. Let Z be an n-dimensional subspace of V and Y a complementary subspace to Z in V. Assume that $X = \langle A, B\rangle$ and $X' = \langle A, B'\rangle$ (where $A, B \in \mathrm{Syl}_p(X)$ and $B' \in \mathrm{Syl}_p(X')$) are groups isomorphic to $\mathrm{SL}_2(q)$ whose actions on V are equivalent to the action of $\mathrm{SL}_2(q)$ on its natural module. Suppose additionally that $Z = C_V(A) = [V, A]$, $Y = C_V(B) = C_V(B')$. Then $B = B'$ and $X = X'$.*

Proof: The group X permutes the set $\mathcal{P} = \{C_V(R) | R \in \mathrm{Syl}_p(X)\}$. But X and A satisfy the hypotheses of Lemma 3.2.4, and so A acts transitively on $\mathrm{Syl}_p(X)\setminus\{A\}$. Thus we have that $\mathcal{P} = Y^A \cup \{Z\}$. Similarly we set $\mathcal{P}' = \{C_V(R') | R' \in \mathrm{Syl}_p(X')\}$ and get $\mathcal{P}' = Y^A \cup \{Z\}$. Hence $\mathcal{P} = \mathcal{P}'$.

The actions of X and X' on $\mathcal{P}$ are equivalent to the natural action of $\mathrm{PSL}_2(q)$ on $q+1$ points. Note that $[V, B] = [V, B'] = Y = C_V(B) = C_V(B')$ and so $[V, B, B'] = [V, B', B] = 0$. By considering V, B and B' as subgroups of the semidirect product $V \ltimes \langle B, B' \rangle$ the Three Subgroup Lemma (see, for example, [20, Theorem 2.2.3]) implies that $[B, B'] = 1$, since the actions of B and B' on V are faithful.

Now let N be the kernel of the action of $H := \langle X, X' \rangle$ on $\mathcal{P}$ (note that $Z(X) \leq N$). Then H/N can be embedded in $\mathrm{Sym}(q+1)$. Now, since $[B, B'] = 1$ we have that BN/N and $B'N/N$ commute. Furthermore, Lemma 3.2.4 shows that both these groups act regularly on the set $\mathcal{P} \setminus \{Y\}$. But B and B' are isomorphic to Sylow p-subgroups of $\mathrm{SL}_2(q)$, and so are abelian. This means that BN/N and $B'N/N$ are abelian, and since a regular abelian permutation group is its own centralizer (see, for example, Wielandt [43, Proposition 4.4]), we deduce that $BN/N = B'N/N$. Therefore $XN = X'N$.

Let $B_1 = \langle B, B' \rangle$. Then we have that $B_1N = BN = B'N$. Thus $|B_1N| = |B_1||N|/|B_1 \cap N| = |B||N|/|B \cap N|$ and so $|B_1|/|B| = |B_1 \cap N|/|B \cap N|$. Hence if $B_1 > B$ then $|B_1 \cap N| > 1$. Let $B_0 := B_1 \cap N$. Note that since $[V, B, B'] = [V, B', B] = 0$, we have $[V, B_1] = Y$. This means that $[V, B_0] \leq [V, B_1] = Y$. But $B_0 \leq N$ normalizes Z, and so

$$[V, B_0] = [Z + Y, B_0] = [Z, B_0] + [Y, B_0] = [Z, B_0] \leq Z.$$

Thus $[V, B_0] \leq Y \cap Z = 0$. Hence $B_0 = 1$ and so $B = B'$, therefore $X = X'$. ∎

Corollary 3.2.13 *A saturated fusion system $\mathcal{F}$ over S is uniquely determined by* $\mathrm{Aut}_{\mathcal{F}}(S)$ *and the set* $\mathrm{Alp}(\mathcal{F})$.

Proof: Theorem 1.8.2 tells us that a saturated fusion system is determined by the groups $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ for $V \in \mathrm{Alp}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$, and Proposition 3.2.11 shows that $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V)) \cong \mathrm{SL}_2(q)$ with the natural action on V whenever $V \in \mathrm{Alp}(\mathcal{F})$. ∎

Chapter 4

The Classification

Throughout this section, let $\mathcal{F}$ be a saturated fusion system over S. The results proved in the previous chapter put us well on the way to being able to apply the Frattini Lemma for saturated fusion systems, by determining the structure of $O^{p'}(V)$ for Alperin subgroups V. It remains to determine the possibilities for $\mathrm{Aut}_{\mathcal{F}}(S)$. In this chapter we shall classify the possibilities for $\mathrm{Aut}_{\mathcal{F}}(S)$. Mostly we shall be able to do this by using saturation to lift $\mathcal{F}$-automorphisms of the proper Alperin subgroups to S. However, this lifting process depends on the number of proper Alperin subgroups, and we shall prove a result that shows that there are (usually) no more than 2.

4.1 The structure of $\mathrm{Aut}_{\mathcal{F}}(S)$

Let V be a 2-dimensional $\mathbb{F}_q$-vector space. We may consider V as a $2n$-dimensional vector space over the prime subfield $\mathbb{F}_p$. It is clear that any $\mathbb{F}_q$-linear map of V is also $\mathbb{F}_p$-linear; denote this set of maps by A. Since a field automorphism of $\mathbb{F}_q$ fixes the prime subfield, a map of V induced by a field automorphism of $\mathbb{F}_q$ is $\mathbb{F}_p$-linear. This means that the natural action of $\Gamma\mathrm{L}_2(q)$ on V is $\mathbb{F}_p$-linear. Since this action is faithful, it defines an embedding of $\Gamma\mathrm{L}_2(q)$ into $\mathrm{GL}_{2n}(p)$ as a group of endomorphisms of V. We denote the image of $\Gamma\mathrm{L}_2(q)$ in $\mathrm{GL}_{2n}(p)$ by H.

Now let V be an elementary abelian $\mathcal{F}$-Alperin subgroup of S of order q^2. We may consider V as a $2n$-dimensional vector space over $\mathbb{F}_p$, and so $\mathrm{Aut}_{\mathcal{F}}(V) \leq \mathrm{Aut}(V) \cong \mathrm{GL}_{2n}(p)$. Let $L = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$. By Proposition 3.2.11, L is isomorphic to $\mathrm{SL}_2(q)$ and V is a natural $\mathbb{F}_q L$-module. We can extend L to subgroups A and H of $\mathrm{GL}_{2n}(p)$ with $L \leq A \leq H$, $A \cong \mathrm{GL}_2(q)$ and $H \cong \Gamma\mathrm{L}_2(q)$.

Lemma 4.1.1 *Let $V \leq S$ be an element of* $\mathrm{Alp}(\mathcal{F})$*, and let H be the image of* $\Gamma\mathrm{L}_2(q)$ *in $G := \mathrm{GL}_{2n}(p)$ as described above. Then* $\mathrm{Aut}_{\mathcal{F}}(V) \leq H \cong \Gamma\mathrm{L}_2(q)$.

Proof: By Proposition 3.2.4, we have that V is an irreducible $\mathbb{F}_p L$-module of dimension $2n$. By Schur's Lemma, $\mathrm{End}_L(V)$ is a field containing $\mathbb{F}_p$. Now if $M := M(V)$ denotes the set of $\mathbb{F}_p$-linear maps $V \to V$, then by definition $\mathrm{End}_L(V) = C_M(L)$. Since $\mathbb{F}_p \subseteq C_M(L)$, we have that V is a $C_M(L)$-vector space. Now, $C_M(L)$ is a field of characteristic p so $C_M(L)$ is a $\mathbb{F}_p$ space of dimension b for some positive integer b. Thus $|V| = p^{2n} = p^{mb}$ where $m = \dim_{C_M(L)}(V)$.

We have $G \geq H \geq A \geq L$ and therefore $C_M(L) \supseteq C_G(L) \supseteq C_A(L) \cong Z(\mathrm{GL}_2(q)) \cong \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. This means that $\dim_{\mathbb{F}_p}(C_M(L)) \geq n$, and

$$\dim_{\mathbb{F}_p}(V) = \dim_{\mathbb{F}_p}(C_M(L)).\dim_{C_M(L)}(V)$$

hence $2n \geq nm$, giving us that $m = 1$ or 2. Note that there exists an embedding $L \to \mathrm{GL}_m(C_M(L))$ (since V is an m-dimensional $C_M(L)L$-module) and so if $m = 1$ then we have an embedding $L \to \mathrm{GL}_1(C_M(L)) \cong C_M(L)^* = C_M(L) \setminus \{0\}$. This is a contradiction since the multiplicative group of $C_M(L)$ is abelian. This means that $m = 2$, therefore $\dim_{\mathbb{F}_p}(C_M(L)) = n$ and $|C_M(L)| = p^n$, thus $|C_G(L)| = p^n - 1$. But $C_H(L) \cong \mathbb{F}_q^*$ and so $|C_H(L)| = p^n - 1$, hence $C_G(L) = C_H(L) \leq H$.

Now consider $N_G(L)$. It is clear that $H \leq N_G(L)$, and we claim that $H = N_G(L)$.

Note that

$$\frac{HC_G(L)}{C_G(L)} \cong \frac{H}{C_H(L)} \cong \mathrm{Aut}(L).$$

But $HC_G(L)/C_G(L) \leq N_G(L)/C_G(L)$, and $N_G(L)/C_G(L)$ is isomorphic to a subgroup of $\mathrm{Aut}(L)$, therefore $N_G(L) = HC_G(L)$. But $C_G(L) \leq H$ so $N_G(L) = H$ as claimed.

Now, since $L \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(V)$, we must have that $\mathrm{Aut}_{\mathcal{F}}(V) \leq N_G(L) = H$. ▮

Lemma 4.1.2 *Let $V \in \mathrm{Alp}(\mathcal{F})$ and let $\phi \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(V))$ be the element which acts as $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right)$ on V, where ω is a primitive element in $\mathbb{F}_q$. Then any extension $\overline{\phi}$ of ϕ in $\mathrm{Aut}_{\mathcal{F}}(S)$ is of the form $\mathfrak{z}A$, where $\mathfrak{z}$ is a central automorphism and $A \in \mathrm{GL}_2(q)$ is a matrix conjugate to $\left(\begin{smallmatrix} \omega & 0 \\ \gamma & \omega^{-2} \end{smallmatrix}\right)$ for some $\gamma \in \mathbb{F}_q$.*

Proof: By Corollary 2.2.8, we may assume that V is the subgroup of matrices of the form $\left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right)$ where $a, c \in \mathbb{F}_q$. An easy matrix calculation shows that $N_\phi = S$: if $x = \left(\begin{smallmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \nu & \mu & 1 \end{smallmatrix}\right) \in S$ then the map $\phi^{-1}c_x\phi$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}^{\phi^{-1}c_x\phi} = \begin{pmatrix} 1 & 0 & 0 \\ a\omega^{-1} & 1 & 0 \\ c\omega & 0 & 1 \end{pmatrix}^{c_x\phi} = \begin{pmatrix} 1 & 0 & 0 \\ a\omega^{-1} & 1 & 0 \\ c\omega - \mu a\omega^{-1} & 0 & 1 \end{pmatrix}^{\phi} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c - \mu a\omega^{-2} & 0 & 1 \end{pmatrix},$$

which is clearly a central automorphism, and hence in $\mathrm{Aut}_S(V)$. Thus $N_\phi = S$.

Now, by saturation ϕ extends to a map in $\mathrm{Aut}_{\mathcal{F}}(S)$. Fix some extension $\overline{\phi}$ in $\mathrm{Aut}_{\mathcal{F}}(S)$. From Proposition 2.2.4 we see that $\overline{\phi}$ is of the form $\mathfrak{z}.\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right).\sigma^t$, where $\mathfrak{z}$ is a central automorphism, $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ acts as in Lemma 2.2.3, and σ is the map induced by the Frobenius automorphism of the field $\mathbb{F}_q$. Since we already know the action of $\overline{\phi}$ on V, we can perform some simple calculations to determine $\overline{\phi}$. Suppose that $\mathfrak{z}$ is given by

$$\mathfrak{z} : \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z + \psi(x, y) & y & 1 \end{pmatrix}$$

where $\psi \in \mathrm{Hom}(q^2, q)$. Then we have

$$\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\1&1&0\\\psi(1,0)&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\\alpha&1&0\\\frac{1}{2}\alpha\beta+(\alpha\delta-\beta\gamma)\psi(1,0)&\beta&1\end{pmatrix}$$

$$\to \begin{pmatrix}1&0&0\\\alpha^{p^t}&1&0\\(\frac{1}{2}\alpha\beta+(\alpha\delta-\beta\gamma)\psi(1,0))^{p^t}&\beta^{p^t}&1\end{pmatrix}.$$

But

$$\overline{\phi}: \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&0\\\omega&1&0\\0&0&1\end{pmatrix}$$

so we have that $\alpha^{p^t} = \omega$ and $\beta^{p^t} = 0$ from which we can deduce that $\beta = 0$ and that $\psi(1,0) = 0$.

Now we consider:

$$\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix} \longmapsto \begin{pmatrix}1&0&0\\0&1&0\\\alpha\delta&0&1\end{pmatrix} \longmapsto$$

$$\begin{pmatrix}1&0&0\\0&1&0\\(\alpha\delta)^{p^t}&0&1\end{pmatrix},$$

from which we deduce that $(\alpha\delta)^{p^t} = \omega^{-1}$, and hence that $\delta = \omega^{-2}$. But also

$$\begin{pmatrix}1&0&0\\0&1&0\\\omega&0&1\end{pmatrix} \longmapsto \begin{pmatrix}1&0&0\\0&1&0\\\omega&0&1\end{pmatrix} \longmapsto \begin{pmatrix}1&0&0\\0&1&0\\\omega\alpha\delta&0&1\end{pmatrix}$$

$$\longmapsto \begin{pmatrix}1&0&0\\0&1&0\\(\omega\alpha\delta)^{p^t}&0&1\end{pmatrix}$$

and so $\omega^{p^t} = \omega$. But ω is a primitive element of $\mathrm{GF}(p^n)$ and so we must have $\omega^{p^n} = \omega$. Therefore $p^t = p^n$ gives the trivial map. Finally,

$$\begin{pmatrix} 1 & 0 & 0 \\ \omega & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ \omega & 1 & 0 \\ \psi(\omega, 0) & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ \alpha\omega & 1 & 0 \\ \alpha\delta\psi(\omega, 0) & 0 & 1 \end{pmatrix},$$

which shows that $\alpha = \omega$ and $\psi(\omega, 0) = 0$.

Putting this all together we can see that $\overline{\phi}$ is an automorphism of the form $\mathfrak{z}.\left(\begin{smallmatrix} \omega & 0 \\ \gamma & \omega^{-2} \end{smallmatrix}\right)$ where $\mathfrak{z}$ is a central automorphism with $\psi_{\mathfrak{z}}(a, 0) = 0$ for all $a \in \mathbb{F}_q$. ∎

Lemma 4.1.3 *Let $\omega \in \mathbb{F}_q$ be a primitive element and let $\overline{\phi}$ denote the automorphism of S which acts as the matrix $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-2} \end{smallmatrix}\right)$. Then $C_{\mathrm{Aut}(S)}(\overline{\phi}) = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in \mathrm{Aut}(S) | \lambda, \mu \in \mathbb{F}_q\}$.*

Proof: A routine calculation in $\mathrm{Aut}(S)$ shows that

$$C_{\mathrm{Aut}(S)}(\overline{\phi}) = \{\mathfrak{z}\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) \in \mathrm{Aut}(S) | \psi_{\mathfrak{z}}(a, b) = \omega\psi_{\mathfrak{z}}(a\omega, b\omega^{-2}) \text{ for all } a, b \in \mathbb{F}_q\}.$$

Let ψ be a homomorphism $(\mathbb{F}_q, +)^2 \to (\mathbb{F}_q, +)$ such that $\psi(a, b) = \omega\psi(a\omega, b\omega^{-2})$. Since ψ is an additive homomorphism, we have that $\psi(a, b) = \psi(a, 0) + \psi(0, b)$ for all $a, b \in \mathbb{F}_q$. Let $\psi_1(a) = \psi(a, 0)$ and $\psi_2(b) = \psi(0, b)$. Note that these are also additive group homomorphisms.

Suppose that ψ_1 is not the trivial map. We show that ψ_1 is an isomorphism of additive groups by showing that $\ker(\psi_1) = 0$. To do this, let $a \in \ker(\psi_1)$ with $a \neq 0$. Then $0 = \psi_1(a) = \omega\psi_1(a\omega)$, so $\psi_1(a\omega) = 0$ since $\omega \neq 0$. Now, if $\psi_1(a\omega^k) = 0$ then $\psi_1(a\omega^k) = \omega\psi_1(a\omega^{k+1}) = 0$. Hence by induction, $\psi_1(a\omega^k) = 0$ for all $1 \leq k \leq q - 1$. Since we assumed that ψ_1 is non-trivial, this means that $a = 0$. Hence $\ker(\psi_1) = 0$, and therefore ψ_1 is an automorphism of the additive group of $\mathbb{F}_q$.

In particular, ψ_1 is surjective, and so there exists an integer k such that $\psi_1(\omega^k) = 1$.

Note that for any l we have

$$\begin{aligned}
\psi_1(\omega^l) &= \psi_1(\omega^k \omega^{l-k}) \\
&= \omega^{k-l}\psi_1(\omega^k) \\
&= \omega^{k-l}.
\end{aligned}$$

By additivity, we also have $\psi_1(2\omega^k) = 2$. Now, since ω is primitive in $\mathbb{F}_q$, there exists an integer f such that $\omega^f = 2$. Therefore $\psi_1(\omega^{f+k}) = \omega^f$. But $\psi_1(\omega^{f+k}) = \omega^{(k-k-f)} = \omega^{-f}$, and so $\omega^f = \psi_1(\omega^{f+k}) = \omega^{-f}$. This means that $2 \equiv -1 \bmod p$, and therefore that $p = 3$.

So assume that $p = 3$. Then ω^f is the unique involution in the multiplicative group $\mathbb{F}_q$. Hence $f = (q-1)/2$. As $\omega^k = 1$, we also have that $\psi_1(\omega^f) = \omega^{k-f} = \omega^k\omega^{-f} = \omega^{-f} = \omega^f$. Therefore $k - f \equiv f \bmod q-1$, hence $k \equiv 0 \bmod q-1$. Thus $k = 0$ and so $\psi_1(\omega^l) = \omega^{-l}$ for all l, *i.e.* $\psi_1(a) = a^{-1}$ for all non-zero $a \in \mathbb{F}_q$. In particular, $\psi_1(1+\omega) = (1+\omega)^{-1}$. But by the linearity of ψ_1, $\psi_1(1+\omega) = \psi_1(1) + \psi_1(\omega) = 1 + \omega^{-1}$. This means that

$$\begin{aligned}
1 &= (1+\omega^{-1})(1+\omega) \\
&= 1 + \omega + \omega^{-1} + 1
\end{aligned}$$

and hence that $\omega^2 + \omega + 1 \equiv 0 \bmod 3$. But the polynomial $x^2 + x + 1 \equiv (x-1)^2 \bmod 3$, and so over fields of characteristic 3, 1 is the only root of the polynomial x^2+x+1. But 1 is not a primitive element of $\mathbb{F}_q$. Hence $\omega^2 + \omega + 1 \neq 0$ in any finite field of characteristic 3. Therefore ψ_1 is trivial.

Now suppose that ψ_2 is non-trivial. Let $b \in \ker(\psi_2)$ with $b \neq 0$ so that $\psi_2(b) = 0$. Then $\omega\psi_2(\omega^{-2}b) = 0$ and hence $\psi_2(\omega^{-2}b) = 0$. If $\psi_2(\omega^{-2k}b) = 0$ then $0 = \psi_2(\omega^{-2k}b) = \omega\psi_2(\omega^{-2(k+1)}b)$ and so $\psi_2(\omega^{-2(k+1)}b) = 0$. Hence $\psi_2(\omega^{-2k}b) = 0$ for all $1 \leq k \leq q-1$. This shows that $\ker(\psi_2)$ has index 2 in $(\mathbb{F}_q, +)$. Since p is odd, this means that $\ker(\psi_2) = 0$.

Hence ψ_2 is an automorphism of the additive group $(\mathbb{F}_q, +)$. So there exists an integer k with $\psi_2(\omega^k) = 1$. Now, $\psi_2(\omega^k) = \omega^{-1}\psi_2(\omega^{k+2})$, and an easy induction argument (similar to the one above) shows that $1 = \psi_2(\omega^k) = \omega^{-l}\psi_2(\omega^{k+2l})$ for all integers l. But this means that $\omega^l = \psi_2(\omega^{k+2l})$ for all l. Since ω is primitive, we have that every element of $\mathbb{F}_q$ can be written as $\psi_2(\omega^{k+2l})$. But ψ_2 is injective and $\{\omega^{k+2l} | 0 \leq l \leq q-1\}$ is a coset of the subgroup of all square elements in $\mathbb{F}_q$, which has index 2. This is a contradiction. Thus ψ_2 is trivial. This shows that ψ is trivial. ∎

We now proceed to classify all the possibilities for $\mathrm{Out}_{\mathcal{F}}(S)$ and $\mathrm{Alp}(\mathcal{F})$.

Lemma 4.1.4 *The group* $\mathrm{Out}_{\mathcal{F}}(S)$ *is isomorphic to a subgroup of* $\Gamma\mathrm{L}_2(q)$.

Proof: Note that by Lemma 2.2.2, every element of $\mathrm{Inn}(S) \cong S/Z(S)$ is a central automorphism. However, every central automorphism has p-power order and $\mathrm{Out}_{\mathcal{F}}(S)$ is a p'-group by saturation. Hence by Lemma 2.2.4, $\mathrm{Out}_{\mathcal{F}}(S)$ is contained in a subgroup of $\mathrm{Aut}(S)$ which is isomorphic to a subgroup of $\Gamma\mathrm{L}_2(q)$. ∎

4.2 The case $|\mathrm{Alp}(\mathcal{F})| < 2$

We are now ready to use the full power of the Frattini Lemma for Saturated Fusion Systems (Theorem 1.8.2). In this section we consider which saturated fusion systems can occur when there are less than two Alperin subgroups.

Lemma 4.2.1 *Let* $\mathcal{G}$ *be a saturated fusion system over* S. *Then* S *contains no proper* $\mathcal{G}$*-Alperin subgroups if and only if* $\mathcal{G} \cong \mathcal{F}_S(S \rtimes W)$ *where* W *is a* p'*-subgroup of* $\Gamma\mathrm{L}_2(q)$

Proof: Suppose that there are no $\mathcal{G}$-Alperin subgroups. Then by 1.8.2, $\mathcal{G} = \langle \mathrm{Aut}_{\mathcal{G}}(S) \rangle$. We know from 4.1.4 and condition **(I)** of saturation that $\mathrm{Out}_{\mathcal{G}}(S)$ must be isomorphic to a p'-subgroup W of $\Gamma\mathrm{L}_2(q)$. In this case we have that $\mathrm{Aut}_{\mathcal{G}}(S) \cong \mathrm{Inn}(S) \rtimes W \cong \mathrm{Aut}_{S \rtimes W}(S)$. Hence $\mathcal{F}_S(S \rtimes W)$ contains $\mathcal{G}$ as a fusion subsystem.

Now fix a p'-subgroup $W \leq \Gamma L_2(q)$ and consider the fusion system $\mathcal{F}_S(S \rtimes W)$. We claim that $\mathrm{Alp}(\mathcal{F}_S(S \rtimes W)) = \emptyset$. Suppose that $V \in \mathrm{Alp}(\mathcal{F}_S(S \rtimes W))$. Then $O^{p'}(\mathrm{Aut}_{S\rtimes W}(V))$ is isomorphic to $\mathrm{SL}_2(q)$ by 3.2.11, and acts on V as the natural module. So there exist elements of $O^{p'}(\mathrm{Aut}_{S\rtimes W}(V))$ which do not normalize $Z(S)$. But $Z(S)$ is invariant under conjugation in the group $S \rtimes W$, so every $\mathcal{F}_S(S \rtimes W)$-morphism normalizes $Z(S)$. This is a contradiction. Hence $\mathrm{Alp}(\mathcal{F}_S(S \rtimes W)) = \emptyset$. Thus by 1.8.2, $\mathcal{F}_S(S \rtimes W) = \langle \mathrm{Aut}_{S\rtimes W}(S)\rangle = \langle \mathrm{Aut}_{\mathcal{G}}(S)\rangle = \mathcal{G}$. ∎

Proposition 4.2.2 *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be saturated fusion systems over S with $\mathrm{Alp}(\mathcal{F}_1) = \mathrm{Alp}(\mathcal{F}_2) = \{V\}$ for some $V \leq S$. Suppose that $\mathrm{Aut}_{\mathcal{F}_1}(V) = \mathrm{Aut}_{\mathcal{F}_2}(V)$. Then $\mathcal{F}_1 \cong \mathcal{F}_2$.*

Remark Firstly, note that the assumption that $\mathrm{Aut}_{\mathcal{F}_1}(V) = \mathrm{Aut}_{\mathcal{F}_2}(V)$ is stronger than the assumption that the groups $\mathrm{Aut}_{\mathcal{F}_1}(V)$ and $\mathrm{Aut}_{\mathcal{F}_2}(V)$ are isomorphic; the assumption in the proposition is that the $\mathcal{F}_1$- and $\mathcal{F}_2$-morphisms of V are exactly the same maps.

Proof: Let $A_1 = \mathrm{Aut}_{\mathcal{F}_1}(S)$ and $A_2 = \mathrm{Aut}_{\mathcal{F}_2}(S)$. We know that V is a normal subgroup of S by 2.3.9. Since V is the only $\mathcal{F}$-Alperin subgroup, we also have that every element of A_1 and A_2 normalizes V.

Let $J = C_{\mathrm{Aut}(S)}(V)\mathrm{Inn}(S)$. Now, A_1, A_2 and J are all contained in $N_{\mathrm{Aut}(S)}(V)$ and J is a normal subgroup of $N_{\mathrm{Aut}(S)}(V)$, so we may consider the groups A_1J and A_2J.

Let $\phi \in A_1$. Then $\phi|_V \in \mathrm{Aut}_{\mathcal{F}_1}(V) = \mathrm{Aut}_{\mathcal{F}_2}(V)$. Since $\phi|V$ lifts to ϕ_1 in $\mathcal{F}_1$, we have that $\phi|V \in N_{\mathrm{Aut}_{\mathcal{F}_1}(V)}(\mathrm{Aut}_S(V)) = N_{\mathrm{Aut}_{\mathcal{F}_2}(V)}(\mathrm{Aut}_S(V))$. Hence $\phi|V$ lifts to some $\phi' \in \mathrm{Aut}_{\mathcal{F}_2}(S)$. This means that $\phi' = \phi\theta$ where $\theta \in C_{\mathrm{Aut}(S)}(V)$ and so $\phi J = \phi' J$. From this we deduce that $A_1J \leq A_2J$. By symmetry we also have that $A_2J \leq A_1J$, thus $A_1J = A_2J$.

Now, J is a p-group by Lemma 2.3.9(iv), therefore so is $J/\mathrm{Inn}(S)$. By saturation, $A_1/\mathrm{Inn}(S)$ is a p'-group. Hence $(|A_1J/\mathrm{Inn}(S)|, |A_1/\mathrm{Inn}(S)|) = 1$, and therefore by the Schur-Zassenhaus Theorem (see, for example, [2, 18.1]), there exists a $g \in A_1J/\mathrm{Inn}(S)$

such that $(A_1/\mathrm{Inn}(S))^g = A_2/\mathrm{Inn}(S)$. Thus we can find an $\alpha \in C_{\mathrm{Aut}(S)}(V)$ such that $\alpha\mathrm{Inn}(S) = g$. Hence $A_1^\alpha = A_2$. Since α centralizes V, we have $\mathrm{Aut}_{\mathcal{F}_1}(V)^\alpha = \mathrm{Aut}_{\mathcal{F}_1}(V)$. Now Proposition 1.7.7 shows that α is an automorphism of S which preserves fusion from $\mathcal{F}_1$ to $\mathcal{F}_2$. Hence $\mathcal{F}_1$ and $\mathcal{F}_2$ are isotypically equivalent. ∎

Proposition 4.2.3 *Let $\mathcal{F}$ be a saturated fusion system over S. Then $|\mathrm{Alp}(\mathcal{F})| = 1$ if and only if $\mathcal{F} \cong \mathcal{F}_S(q^2 : W)$ where $\mathrm{SL}_2(q) \leq W \leq \Gamma\mathrm{L}_2(q)$ and $W/\mathrm{SL}_2(q)$ has p' order.*

Proof: By 3.2.11, $\mathrm{Aut}_{\mathcal{F}}(V)$ contains a normal subgroup isomorphic to $\mathrm{SL}_2(q)$. Therefore $\mathrm{Aut}_{\mathcal{F}}(V)$ is isomorphic to a group W where $\mathrm{SL}_2(q) \leq W \leq \Gamma\mathrm{L}_2(q)$ by Lemma 4.1.1. But $\mathrm{Aut}_S(V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(V))$ has order q, and so W must have the same Sylow p-subgroups as $\mathrm{SL}_2(q)$. This is equivalent to the condition that $W/\mathrm{SL}_2(q)$ is a p'-group.

Now let $W' \cong W$ be a group of automorphisms of V. Then the group $V \rtimes W$ contains a Sylow p-subgroup isomorphic to S and the fusion system $\mathcal{F}_S(V \rtimes W)$ has exactly one Alperin subgroup. Hence by 4.2.2, $\mathcal{F} \cong \mathcal{F}_S(V \rtimes W)$.

To see the converse, we simply note that given any group W with $\mathrm{SL}_2(q) \leq W \leq \Gamma\mathrm{L}_2(q)$ and $W/\mathrm{SL}_2(q)$ a p'-group, we have that the group $q^2 \rtimes W$ has a Sylow p-subgroup isomorphic to S, and that the saturated fusion system $\mathcal{F}_S(q^2 \rtimes W)$ has exactly one Alperin subgroup. ∎

4.3 Subgroups of $\mathrm{PGL}_2(q)$

In this section we prove an important lemma concerning certain subgroups of $\mathrm{PGL}_2(q)$, which we use in the proof of Theorem 4.4.1. Denote by $\mathcal{P}$ the projective line over $\mathbb{F}_q$. As a set, this space has $q+1$ points.

The following is a well-known fact about projective linear groups:

Lemma 4.3.1 *Let $q = p^n$ for some prime p. Then $\mathrm{PGL}_2(q)$ acts 2-transitively on $\mathcal{P}$ and an element of $\mathrm{PGL}_2(q)$ which fixes three points in the natural action on $\mathcal{P}$ is trivial.*

Proof: By assigning coordinates to the elements of $\mathcal{P}$ in the usual way, so that each point in $\mathcal{P}$ corresponds to an element of $\mathbb{F}_q$ or the symbol ∞, we can identify $\mathrm{PGL}_2(q)$ with the set of projective transformations

$$z \mapsto \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{F}_q$ and $ad-bc \neq 0$. To see that $\mathrm{PGL}_2(q)$ is 2-transitive, we note that if $x, y \in \mathcal{P}$ with $x \neq y$ then the transformation $z \mapsto \frac{yz+x}{z+1}$ maps 0 to x and ∞ to y.

We also observe that the stabilizer of the points 0 and ∞ consists of transformations of the form $z \mapsto az$, with $a \in \mathbb{F}_q$. But such a transformation does not fix another point unless $a = 1$, *i.e.* unless it is the identity transformation. Therefore, by 2-transitivity, the only element which fixes three points in $\mathcal{P}$ is the identity. $\blacksquare$

Lemma 4.3.2 *The order of a point stabilizer in* $\mathrm{PGL}_2(q)$ *with respect to the action on* $\mathcal{P}$ *is* $q(q-1)$ *and every* p'*-subgroup of a point stabilizer in* $\mathrm{PGL}_2(q)$ *is cyclic.*

Proof: Let V be a 2-dimensional $\mathbb{F}_q$-space. Note that the stabilizer in $\mathrm{GL}_2(q)$ of the 1-dimensional subspace of V spanned by the vector $(1,0)$ is $A := \{\left(\begin{smallmatrix} a & 0 \\ c & b \end{smallmatrix}\right) | a,b,c \in \mathbb{F}_q,\ a,b \neq 0\}$. For any vector $v \in V$, there exists an element $x \in \mathrm{GL}_2(q)$ such that $vx = (1,0)$. Hence $\mathrm{Stab}_{\mathrm{GL}_2(q)}(\langle v \rangle) = A^x$.

Note that the stabilizer of a projective point in $\mathrm{PGL}_2(q)$ is the projection of a line stabilizer in $\mathrm{GL}_2(q)$. Since every line stabilizer in $\mathrm{GL}_2(q)$ is conjugate to A, the stabilizer of every projective point in $\mathrm{PGL}_2(q)$ is conjugate to the projection $\overline{A}$ of the group A in $\mathrm{PGL}_2(q)$. The order of A is $q(q-1)^2$, thus each projective point stabilizer has order $q(q-1)$.

It is easy to see that in $\mathrm{GL}_2(q)$, the p-subgroup $P := \{\left(\begin{smallmatrix} 1 & 0 \\ d & 1 \end{smallmatrix}\right) | d \in \mathbb{F}_q\}$ is normal in A, and so is the unique Sylow p-subgroup of A. Note that if $\overline{A}$ and $\overline{P}$ are the projections of A and P respectively in $\mathrm{PGL}_2(q)$, then $\overline{P}$ is the unique Sylow p-subgroup of $\overline{A}$. Also note that $\overline{A}/\overline{P}$ is cyclic, and generated by the image of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right)$, where ω is a primitive element of $\mathbb{F}_q$.

Now, if $\overline{H}$ is a p'-subgroup of $\overline{A}$ then $\overline{H} \cap \overline{P} = 1$ and so by the Parallelogram Law $\overline{H} \cong \overline{H}\,\overline{P}/\overline{P} \leq \overline{A}/\overline{P}$, and so $\overline{H}$ is cyclic. ∎

In the next lemma, the primes at which Ruiz and Viruel's exceptional fusion systems occur make their first appearance.

Lemma 4.3.3 *Suppose $q = p^n$ for some odd prime p. Let X be a p'-subgroup of $\mathrm{PGL}_2(q)$. Suppose that there exist $\alpha, \beta \in \mathcal{P}$ such that $|\mathrm{Stab}_X(\alpha)|$ and $|\mathrm{Stab}_X(\beta)|$ are divisible by $(q-1)/(3, q-1)$, and that $X = \langle \mathrm{Stab}_X(\alpha), \mathrm{Stab}_X(\beta)\rangle$. Then either $\mathrm{Stab}_X(\alpha) = \mathrm{Stab}_X(\beta)$ or $q = 3$, 5, 7 or 13.*

Remark It is possible to prove this theorem using Dickson's Theorem (see Suzuki [39, Theorem 3.6.25]) which lists the subgroups of $\mathrm{PSL}_2(q)$. However, we have chosen to prove it directly, because it illustrates more clearly how the exceptional fusion systems (*i.e.* those with more than two Alperin subgroups) found by Ruiz and Viruel arise.

Proof: We regard X as a group of permutations on the set $\mathcal{P}$. For any $\theta \in \mathcal{P}$, let $X_\theta = \mathrm{Stab}_X(\theta)$. Fix $\alpha \in \mathcal{P}$ and suppose that there is a $\beta \in \mathcal{P}$ such that $X_\alpha \neq X_\beta$.

Since we assumed $|X|$ is coprime to p, we have that any subgroup of X corresponds to a p'-subgroup Y of $\mathrm{GL}_2(q)$. By Maschke's Theorem (see, for example, Gorenstein [20, Theorem 3.3.1]), the representation of Y on $\mathbb{F}_q^2$ is completely reducible. Thus if Y normalizes a 1-dimensional subspace, it must also normalize a complementary 1-dimensional subspace. This means that if a subgroup of X fixes a point then it must, in fact, fix two points. Hence X_α fixes some other point $\alpha' \in \mathcal{P}$, X_β fixes another point $\beta' \in \mathcal{P}$. This implies that $\{\alpha, \alpha'\} \cap \{\beta, \beta'\} = \emptyset$.

By 4.3.2, a point stabilizer in $\mathrm{PGL}_2(q)$ has order $q(q-1)$. This means that point stabilizers in X have order dividing $(q-1)$ (since $|X|$ is coprime to p). We also have, by

4.3.2, that point stabilizers in X are cyclic, thus X_α and X_β are cyclic. Assume without loss of generality that $|X_\alpha| \geq |X_\beta|$. Since $|X_\beta|$ is divisible by $(q-1)/(3, q-1)$, we are left with two cases: $|X_\alpha| = q-1$ or $|X_\alpha| = |X_\beta| = (q-1)/3$.

Case 1: $|X_\alpha| = q-1$. It was shown above that X_α fixes the two points α, α'. By 4.3.1, no non-identity element of X_α can fix any other points in $\mathcal{P}$, and so the stabilizer in X_α of any point in $\mathcal{P} \setminus \{\alpha, \alpha'\}$ is trivial. By the Orbit-Stabilizer Theorem, this means that X_α has an orbit of length $q-1$. Hence X_α has two orbits on $\mathcal{P}$ of length 1 (namely $\{\alpha\}$ and $\{\alpha'\}$) and an orbit of length $q-1$ (namely $\mathcal{P} \setminus \{\alpha, \alpha'\}$).

Now consider the orbits of X. Since $X_\beta \leq X$ fixes neither α nor α', we have that either X is transitive on $\mathcal{P}$ or has an orbit $\{\alpha, \alpha'\}$ of length 2. Suppose the latter case holds. Then X_β stabilizes $\{\alpha, \alpha'\}$, and if $x \in X_\beta$ is non-trivial, then x swaps α and α', and so $x^2 = 1$ by 4.3.1, since x^2 fixes α, α', β and β'. Since X_β is cyclic this means that $|X_\beta| = 2$. This means that either $(q-1)/3$ divides 2 or $q-1$ divides 2; *i.e.* either $q = 7$ or $q = 3$.

Now assume that X is transitive on $\mathcal{P}$. This means that $|X| = |X_\alpha||\mathrm{Orb}_X(\alpha)| = (q-1)(q+1)$, by the Orbit-Stabilizer Theorem. Let $\mathcal{B} = \{\{\theta, \theta'\} \subseteq \mathcal{P} | \theta \neq \theta',\ X_\theta = X_{\theta'}\}$. Then $\mathcal{B}$ is an X-invariant block system. To see it is a block system, note that if $\{\gamma, \gamma'\}$ and $\{\delta, \delta'\} \in \mathcal{B}$ with $\{\gamma, \gamma'\} \cap \{\delta, \delta'\} \neq \emptyset$ then $1 \neq X_\gamma = X_{\gamma'} = X_\delta = X_{\delta'}$. Therefore by Lemma 4.3.1 $\{\gamma, \gamma'\} = \{\delta, \delta'\}$. Also note that $\mathcal{B}$ is X-invariant because by transitivity all point stabilizers are conjugate in X and $X_\alpha = X_{\alpha'}$.

Consider the action of the group X_α on $\mathcal{B}$. The group X_α fixes the block $\{\alpha, \alpha'\}$, and so it must act on the remaining $(q-1)/2$ blocks. Consider the stabilizer in X_α of a block $\{\gamma, \gamma'\}$. We have that $|X_\alpha|/|\mathrm{Stab}_{X_\alpha}(\{\gamma, \gamma'\})| \leq (q-1)/2$, and so $\mathrm{Stab}_{X_\alpha}(\{\gamma, \gamma'\}) \neq 1$. Let $x \in \mathrm{Stab}_{X_\alpha}(\{\gamma, \gamma'\})$, $x \neq 1$. Then x must swap γ and γ' because otherwise $|\mathrm{Fix}(x)| \geq 3$ (where $\mathrm{Fix}(x) = \{\theta \in \mathcal{P} | \theta^x = \theta\}$). But then x^2 must fix γ and γ', and therefore $x^2 = 1$. Since this is true for every block, we can deduce that there must be an involution in the kernel of the action of X_α on the blocks. But X_α is cyclic, and so it contains only one involution, z_1 say. This element must act trivially on $\mathcal{B}$, but not fix any point other than α and α' (otherwise it will fix 3 points). Therefore it must swap the two points of any block which is not $\{\alpha, \alpha'\}$. Similarly, there is an involution $z_2 \in X_\beta$ which swaps the points of any block which is not $\{\beta, \beta'\}$. Now, the product $z_1 z_2 \in X$ fixes every point except the four points α, α', β and β', *i.e.* $q + 1 - 4 = q - 3$ points. Since $z_1 z_2 \neq 1$, it must be the case that $q - 3 < 3$, that is, $q = 3$ or 5.

Case 2: $|X_\alpha| = |X_\beta| = (q - 1)/3$. Consider the action of X on $\mathcal{P}$. The group X_α has two fixed points α and α' and three orbits Ω_1, Ω_2 and Ω_3, each of length $(q - 1)/3$. Let us consider the possibilities for the X-orbit of α.

We shall say that subsets $A, B \subseteq \mathcal{P}$ are *fused* by X if $A \cup B$ is contained in the X-orbit of some point in $\mathcal{P}$.

The group X_β also has two fixed points β and β' (distinct from both α and α') and three orbits each of length $(q - 1)/3$. Since X_β does not fix α, we have that the X_β-orbit of α has length $(q - 1)/3$. Therefore if $(q - 1)/3 > 2$ then α is fused to (without loss of generality) Ω_1 by X. If $(q - 1)/3 \leq 2$ then there is also the possibility that $\mathrm{Orb}_{X_\beta}(\alpha) = \{\alpha, \alpha'\}$. But $(q - 1)/3 \leq 2$ if and only if $q \leq 7$, so this case only arises when $q = 7$.

Thus we may assume without loss of generality that α is fused to Ω_1 by X, and that α' is fused to one of Ω_1 and Ω_2. We have the following possibilities; up to a permutation of the labelling of Ω_1, Ω_2 and Ω_3:

(i) α' is fused to Ω_2, and Ω_1 is fused to neither Ω_2 nor Ω_3; thus $|\mathrm{Orb}_X(\alpha)| = 1 + (q-1)/3$;

(ii) α' is fused to Ω_2, and Ω_1 is fused to Ω_3; thus $|\mathrm{Orb}_X(\alpha)| = 1 + 2(q - 1)/3$;

(iii) α' is fused to Ω_2, and Ω_1 is fused to Ω_2; thus $|\mathrm{Orb}_X(\alpha)| = 2 + 2(q - 1)/3$;

(iv) α' is fused to Ω_1, and Ω_1 is fused to neither Ω_2 nor Ω_3; thus $|\mathrm{Orb}_X(\alpha)| = 2 + (q-1)/3$;

(v) α' is fused to Ω_1, and Ω_1 is fused to Ω_2; thus $|\mathrm{Orb}_X(\alpha)| = 2 + 2(q-1)/3$;

(vi) Ω_1, Ω_2 and Ω_3 are all fused; thus $|\mathrm{Orb}_X(\alpha)| = |\mathcal{P}| = q+1$;

In case (i), we have an orbit of length $1 + (q-1)/3 = (q+2)/3$. Since X is a p'-subgroup of $\mathrm{PGL}_2(q)$, we have that each divisor of $(q+2)/3$ divides either $(q+1)$ or $(q-1)$. Suppose $a|(q+2)/3$. If $a|(q-1)$ then $a|(q+2-(q-1)) = 3$ and $a = 3$, and if $a|(q+1)$ then $a|(q+2-(q+1)) = 1$. Hence $(q+2)/3 = 3$ and so $q = 7$.

In case (ii), we have an orbit of length $1 + 2(q-1)/3 = (2q+1)/3$. Suppose that $a|(2q+1)/3$. Then by the same reasoning as above, either $a|(q-1)$ or $a|(q+1)$. If $a|(q-1)$ then $a|2q-2$ and $a|2q+1$ and so $a|2q+1-2q+2 = 3$, hence $a = 3$ and $q = 4$. If $a|(q+1)$ then $a|2q+2$ and so $a|2q+2-2q-1 = 1$ and hence $a = 1$ and $q = 1$. Both of these outcomes contradict our assumptions about q.

In case (iii), we have an orbit of length $2+2(q-1)/3 = 2(q+2)/3$. Suppose $a|2(q+2)/3$. If $a|(q-1)$ then $a|6$, and if $a|(q+1)$ then $a|2$. Therefore $2(q+2)/3|12$ and so $(q+2)|18$. From this we can deduce that the only possibility is that $q = 7$.

In case (iv), we have an orbit of length $2 + (q-1)/3 = (q+5)/3$. Consider the action of X on $\mathcal{O} = \mathrm{Orb}_X(\alpha)$. It acts transitively on $\mathcal{O}$ and X_α has two fixed points. Let $\mathcal{B}' = \{\{\theta, \theta'\} \subseteq \mathcal{O} | \theta \neq \theta', X_\theta = X_{\theta'}\}$. It is easy to see that $\mathcal{B}'$ forms an X-invariant block system. The subgroup X_α fixes one of these blocks, and so acts on the remaining $(q-1)/6$ blocks. This means that if B is one of the remaining blocks, then $(q-1)/3|\mathrm{Stab}_{X_\alpha}(B)| \leq (q-1)/6$ and so $\mathrm{Stab}_{X_\alpha}(B) \neq 1$. Now, using the same argument as in case 1, we see that the involution in X_α acts by swapping the points in each block except for the points it fixes. Now take an element $y \in X_\gamma$ for some $\gamma \in \mathcal{O} \setminus \{\alpha, \alpha'\}$. Again, it can be shown that the involution in X_γ acts by swapping the points in each block except for the points it fixes. The product $xy \in X$ is non-trivial and fixes $(q-7)/3$ points. Therefore $(q-7)/3 < 3$, *i.e.* $q < 16$. This leaves only the possibilities $q = 7$ or 13.

In case (iv), we must have that $(q-1)/3 = 2$ since the smallest possible orbit length is $(q-1)/3$. Hence this case only occurs for $q = 7$.

Now suppose that X is transitive. This means that $(q+1)||X|$. Thus we can deduce that $|X| = (q+1)(q-1)/3$. By transitivity all point stabilizers are conjugate, and so have order $(q-1)/3$. By a similar argument to that in case 1, the set $\mathcal{B}$ defined earlier is an X-invariant block system.

Consider the action of the group X_α on $\mathcal{B} \setminus \{\alpha, \alpha'\}$. Let $x \in \mathrm{Stab}_{X_\alpha}(\{\gamma, \gamma'\})$, where $\{\gamma, \gamma'\}$ is a block not equal to $\{\alpha, \alpha'\}$, and suppose $x \neq 1$. Then x cannot fix γ otherwise it will fix 4 points in its action on $\mathcal{P}$. Therefore x must swap γ and γ'. But then x^2 fixes γ and γ', and so $x^2 = 1$. So if all the block stabilizers are non-trivial then we can use a similar argument to that in case 1 to show that $q-3 < 3$, which is impossible if $3|(q-1)$.

We may now assume that there exists a block with a trivial stabilizer. This means that there is an orbit of length $(q-1)/3$ on $\mathcal{B}$. The blocks outside this orbit have non-trivial stabilizers, and as we have seen before, non-trivial block stabilizers have order 2. But this means that there is only one other orbit of blocks, of length $(q-1)/6$. Let B_α denote this smaller orbit under the action of X_α. Since all the point stabilizers are conjugate, we can similarly define B_θ for any point θ as being the orbit of length $(q-1)/6$ in the action of X_θ on $\mathcal{B}$. Now, suppose that $\{\lambda, \lambda'\}, \{\mu, \mu'\} \in B_\theta \cap B_\phi$. Let z_θ and z_ϕ be the unique involutions in X_θ and X_ϕ respectively. The kernel of the action of X_θ on the $(q-1)/6$ blocks in B_θ contains z_θ, and similarly for z_ϕ. This means that z_θ acts by swapping the two elements in each block in B_θ (and similary for z_ϕ). Thus z_θ and z_ϕ both swap λ to λ' and μ to μ'. This means that $z_\theta z_\phi$ fixes 4 points and so $z_\theta z_\phi = 1$. Therefore $z_\theta = z_\phi$ and so $B_\theta = B_\phi$. This shows that if B_θ and B_ϕ are distinct then they intersect in at most one block.

We now prove that $\bigcap_{\{\theta,\theta'\} \in B_\alpha} B_\theta = \{\alpha, \alpha'\}$. To see this, let $\{\theta, \theta'\} \in B_\alpha$. Then z_α swaps θ and θ', and so z_α normalizes X_θ. Since there is only one involution in X_θ, we have that z_α and z_θ commute. But then $(\alpha^{z_\theta})^{z_\alpha} = \alpha^{z_\alpha z_\theta} = \alpha^{z_\theta}$ and so $\alpha^{z_\theta} \in \mathrm{Fix}(z_\alpha) = \{\alpha, \alpha'\}$. Similarly $\alpha'^{z_\theta} \in \mathrm{Fix}(z_\alpha)$. Thus z_θ must stabilize the block $\{\alpha, \alpha'\}$, which means that $\mathrm{Stab}_{\langle z_\theta \rangle}(\{\alpha, \alpha'\})$ is non-trivial, and so $\{\alpha, \alpha'\} \in B_\theta$. This proves our claim.

By counting the elements in each of the B_θ for $\theta \in B_\alpha$, we can obtain a bound for q. We have:

$$\begin{aligned} \frac{q+1}{2} \geq \left| \bigcup_{\{\theta,\theta'\} \in B_\alpha} B_\theta \right| &= (|B_\theta| - 1)|B_\alpha| + 1 \\ &= \left(\frac{q-1}{6} - 1\right)\left(\frac{q-1}{6}\right) + 1, \end{aligned}$$

which implies that

$$q^2 - 26q + 25 = (q-1)(q-25) \leq 0,$$

and hence that $q \leq 25$. The prime powers $r \leq 25$ for which $3|(r-1)/3$ are just 7, 13, 19 and 25. Therefore q is one of 7, 13, 19 and 25. We have assumed that X is transitive and therefore that $|X| = (q-1)(q+1)/3$. But by consulting the Atlas [16], we see that $\mathrm{PGL}_2(25)$ has no subgroup of order $208 = (25-1)(25+1)/3$, $\mathrm{PGL}_2(19)$ has no subgroup of order $120 = (19-1)(19+1)/3$ and $\mathrm{PGL}_2(13)$ has no subgroup of order $56 = (13-1)(13+1)/3$. Thus the only possibility in this case is $q = 7$. This completes the proof. ∎

4.4 The case $|\mathrm{Alp}(\mathcal{F})| \geq 2$

Now we are ready to tackle the case $|\mathrm{Alp}(\mathcal{F})| \geq 2$.

Theorem 4.4.1 *Let $\mathcal{F}$ be a saturated fusion system over S, and suppose that $|\mathrm{Alp}(\mathcal{F})| > 2$. Then $q = 3$, 5, 7 or 13.*

Proof: Let $V, W \in \mathrm{Alp}(\mathcal{F})$ with $V \neq W$. By Corollary 2.2.8, we may assume that $V = \{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right) | a, c \in \mathbb{F}_q \}$ and $W = \{ \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{smallmatrix}\right) | b, c \in \mathbb{F}_q \}$. We know that $\mathrm{Aut}_{\mathcal{F}}(V)$

contains an element ϕ_V which acts as $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right)$ when V is considered as a 2-dimensional $\mathbb{F}_q$-space, which by Lemma 4.1.2 extends to an element of the form $\overline{\phi_V} \in \mathrm{Aut}_{\mathcal{F}}(S) = \mathfrak{z}\left(\begin{smallmatrix} \omega & 0 \\ \gamma & \omega^{-2} \end{smallmatrix}\right)$ where $\mathfrak{z}$ is a central automorphism of S. Considering $\overline{\phi_V}$ as a linear map on $S/Z(S)$, we see that one eigenvector generates $V/Z(S)$ and the other eigenvector must generate another one-dimensional subspace of the form $V'/Z(S)$, where V' is an elementary abelian subgroup of order q^2. Thus we have that $\overline{\phi_V}$ (acting on S) normalizes V and V'. Also note that $\overline{\phi_V}$ has order $q-1$. Similarly we get an element $\overline{\phi_W} \in \mathrm{Aut}_{\mathcal{F}}(S)$ of order $q-1$ whose $\mathrm{GL}_2(q)$-part is an upper triangular matrix.

We have that $G_V := \langle\overline{\phi_V}\rangle\mathrm{Inn}(S)/\mathrm{Inn}(S)$ and $G_W := \langle\overline{\phi_W}\rangle\mathrm{Inn}(S)/\mathrm{Inn}(S)$ are subgroups of $\mathrm{Out}_{\mathcal{F}}(S)$ (of order $q-1$), and so are isomorphic to subgroups of $\mathrm{GL}_2(q)$ (see Lemma 4.1.4). Let F_1 denote the subgroup of $\mathrm{Out}_{\mathcal{F}}(S)$ isomorphic to $Z(\mathrm{GL}_2(q))$. Note that the groups G_VF_1/F_1 and G_WF_1/F_1 have order $(q-1)/(3,q-1)$. By saturation $p \nmid |\langle G_V, G_W\rangle|$, and so the group $X := \langle G_V, G_W\rangle F_1/F_1$ is isomorphic to a subgroup of $\mathrm{PGL}_2(q)$ which satisfies the conditions of Lemma 4.3.3. Now, since $|\mathrm{Alp}(\mathcal{F})| > 2$, there must exist a pair V, W such that $\{V, V'\} \neq \{W, W'\}$. Thus not every element of X has the same two fixed points. Hence by Lemma 4.3.3, $q = 3$, 5, 7 or 13. ∎

We recall the following fact from group theory:

Lemma 4.4.2 *Let G be a finite group with $Z \leq Z(G)$ a subgroup contained in the centre of G. If G/Z is cyclic then G is abelian.*

Proof: Let $aZ \in G/Z$ be a generator of G/Z and let $x, y \in G$. Then $x = a^n z_1$ for some $z_1 \in Z$ and some n and $y = a^m z_2$ for some $z_2 \in Z$ and some m. So

$$\begin{aligned} x^{-1}y^{-1}xy &= z_1^{-1}a^{-n}z_2^{-1}a^{-m}a^n z_1 a^m z^2 \\ &= 1 \end{aligned}$$

since $z_1, z_2 \in Z$. Hence G is abelian. ∎

Proposition 4.4.3 *Suppose that $\mathcal{F}$ is a saturated fusion system over S with $|\mathrm{Alp}(\mathcal{F})| = 2$. Then, after possibly adjusting $\mathcal{F}$ by an isotypical equivalence, $\mathrm{Aut}_{\mathcal{F}}(S) \geq \mathrm{Inn}(S)W$ where W is the group of automorphisms which act as an element of the set $\{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right) | \lambda, \mu \in \mathbb{F}_q\}$ if $3 \nmid (q-1)$, and W is the group of automorphisms of shape $(q-1) \times \frac{(q-1)}{3}$ generated by the automorphisms which act as $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-2} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right)$ if $3|(q-1)$.*

Proof: Let $\mathrm{Alp}(\mathcal{F}) = \{V, V'\}$. By Lemma 2.2.7, we may assume that $V = \left\{\left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right) | a, c \in \mathbb{F}_q\right\}$ and $V' = \left\{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d & b & 1 \end{smallmatrix}\right) | b, d \in \mathbb{F}_q\right\}$. We know that $\mathrm{Aut}_{\mathcal{F}}(V)$ contains an element ϕ_V which acts as $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right)$ where ω is a primitive element of $\mathbb{F}_q$. As above, we see that this element lifts to an element $\overline{\phi_V} \in \mathrm{Aut}_{\mathcal{F}}(S)$ of the form $\mathfrak{z}\left(\begin{smallmatrix} \omega & 0 \\ \gamma & \omega^{-2} \end{smallmatrix}\right)$ where $\gamma \in \mathbb{F}_q$. Since $\overline{\phi_V}$ normalizes V, it must also normalize V', and so $\gamma = 0$. Note that the map $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-2} \end{smallmatrix}\right) \in \mathrm{Aut}(S)$ agrees with $\overline{\phi_V}$ on V. Hence they must differ by an element of $C_{\mathrm{Aut}(S)}(V)$.

Let ψ be the element of $\mathrm{Aut}(S)$ which acts as the matrix $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-2} \end{smallmatrix}\right)$. Let $A_1 = \langle \overline{\phi_V} \rangle \mathrm{Inn}(S)$. Since $\mathcal{F}$ is saturated, $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(S)$ is a p'-group, therefore so is $A_1/\mathrm{Inn}(S)$. Now, since $\overline{\phi_V}|_V = \psi|_V$, we have that $\overline{\phi_V} C_{\mathrm{Aut}(S)}(V) = \psi C_{\mathrm{Aut}(S)}(V)$. Let $A_2 = \langle \psi \rangle \mathrm{Inn}(S)$ and let $J = C_{\mathrm{Aut}(S)}(V)\mathrm{Inn}(S)$. Then $A_1 J = A_2 J$.

Now consider the group $A_1 J/\mathrm{Inn}(S) = A_2 J/\mathrm{Inn}(S)$. We have that $J/\mathrm{Inn}(S)$ is a normal subgroup, and $A_1/\mathrm{Inn}(S)$ and $A_2/\mathrm{Inn}(S)$ are both complements to $J/\mathrm{Inn}(S)$ in $A_1 J/\mathrm{Inn}(S)$. By 2.3.9, J is a p-group and so $(|A_1 J/\mathrm{Inn}(S)|, |A_1/\mathrm{Inn}(S)|) = 1$. Hence by the Schur-Zassenhaus Theorem, there exists a $g \in A_1 J/\mathrm{Inn}(S)$ such that $(A_1/\mathrm{Inn}(S))^g = A_2/\mathrm{Inn}(S)$. Thus there exists an $\alpha \in C_{\mathrm{Aut}(S)}(V)$ such that $A_1^\alpha = A_2$. Therefore we have that the fusion system $\mathcal{F}^\alpha$ has $\mathrm{Aut}_{\mathcal{F}^\alpha}(V) = \mathrm{Aut}_{\mathcal{F}}(V)$ and $\mathrm{Aut}_{\mathcal{F}^\alpha}(S)$ contains ψ. By an abuse of notation, we shall assume that $\mathcal{F}$ contains ψ as an $\mathcal{F}$-morphism. By another abuse of notation we shall set $\overline{\phi_V} = \psi$.

Using the notation of Theorem 4.4.1, we have that $X = \langle G_V, G_W \rangle F_1/F_1$ is cyclic by 4.3.3. Let $G = \langle G_V, G_W \rangle$. Then $G \leq \mathrm{Out}_{\mathcal{F}}(S) \leq \mathrm{GL}_2(q)$ and so $F_1 \cap G \leq Z(G)$. But

$X = GF_1/F_1 \cong G/(F_1 \cap G)$ is cyclic and hence by Lemma 4.4.2, G is abelian. Therefore $\langle \overline{\phi_{V'}} \rangle \leq \mathrm{Inn}(S) C_{\mathrm{Aut}(S)}(\overline{\phi_V})$.

By 4.1.2 we have that $\overline{\phi_{V'}}$ is of the form $\mathfrak{z} \left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right)$ where $\mathfrak{z}$ is a central automorphism. But $\overline{\phi_{V'}} \in \mathrm{Inn}(S) C_{\mathrm{Aut}(S)}(\overline{\phi_V})$ and so by 4.1.3 we have that $\mathfrak{z} \in \mathrm{Inn}(S)$. Since $\mathrm{Inn}(S) \leq \mathrm{Aut}_{\mathcal{F}}(S)$, the morphism $\mathfrak{z} \left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathrm{Aut}_{\mathcal{F}}(S)$ if and only if the morphism $\left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathrm{Aut}_{\mathcal{F}}(S)$. Hence we may assume, without loss of generality, that $\left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathrm{Aut}_{\mathcal{F}}(S)$. By an abuse of notation we shall say that $\overline{\phi_{V'}} = \left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathrm{Aut}_{\mathcal{F}}(S)$.

Now consider $\langle \overline{\phi_V}, \overline{\phi_{V'}} \rangle$. Note that $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-2} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mu^{-2} & 0 \\ 0 & \mu \end{smallmatrix}\right)$ if and only if $\lambda = \mu^{-2}$ and $\mu = \lambda^{-2}$ *i.e.* only if $\lambda^3 = \mu^3 = 1$ and $\lambda = \mu$. Therefore $\langle \overline{\phi_V}, \overline{\phi_{V'}} \rangle \cong (q-1)^2$ unless $3|(q-1)$. So suppose $3|(q-1)$. The only elements of order 3 in $\mathbb{F}_q$ are $\omega^{(q-1)/3}$ and $\omega^{2(q-1)/3}$. Therefore $\langle \overline{\phi_V} \rangle \cap \langle \overline{\phi_{V'}} \rangle$ has order 3, and so $\langle \overline{\phi_V}, \overline{\phi_{V'}} \rangle$ has order $(q-1)^2/3$. But the element $\theta := \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega^3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega^4 \end{smallmatrix}\right) \left(\begin{smallmatrix} \omega^2 & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right) \in \langle \overline{\phi_V}, \overline{\phi_{V'}} \rangle$ has order $(q-1)/3$, and $\langle \theta \rangle \cap \langle \overline{\phi_V} \rangle = 1$. Hence $\langle \overline{\phi_V}, \overline{\phi_{V'}} \rangle \cong (q-1) \times \frac{(q-1)}{3}$.

We have shown that $\mathrm{Aut}_{\mathcal{F}}(S)$ contains $\mathrm{Inn}(S)$ and a group of diagonal matrices of shape $(q-1) \times \frac{(q-1)}{(3,q-1)}$, as claimed. ∎

Proposition 4.4.4 *The fusion system* $\mathcal{F}_S(\mathrm{PSL}_3(q))$ *has exactly two Alperin subgroups and* $\mathrm{Aut}_{\mathcal{F}}(S)$ *is of the form* $\mathrm{Inn}(S)W$ *where* W *is a subgroup of the collection of automorphisms generated by* $\langle \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-2} \end{smallmatrix}\right), \left(\begin{smallmatrix} \omega^{-2} & 0 \\ 0 & \omega \end{smallmatrix}\right) \rangle$ *of order* $(q-1)^2/(3, q-1)$.

Proof: Let $G := \mathrm{PSL}_3(q)$ and let $\mathcal{F} = \mathcal{F}_S(G)$. Note that the action by conjugation of the matrix $\left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{smallmatrix}\right)$ on S induces the automorphism $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha\beta^3 \end{smallmatrix}\right)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$ where $\alpha = ab^{-1}$ and $\beta = b$, in the sense of Lemma 2.2.3. The set of all such matrices forms a group of the required shape.

Now we show that the subgroups V and V' defined in the previous proposition are the $\mathcal{F}$-Alperin subgroups. Matrix calculations show that V and V' are self-centralizing. We now show that the $\mathcal{F}$-conjugacy classes of V and V' have size 1. To see this, consider the natural action of $\mathrm{SL}_3(q)$ on a 3-dimensional $\mathbb{F}_q$-space W. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ ax+y \\ cx+z \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ cx+by+z \end{pmatrix}.$$

So if $\overline{V}$ and $\overline{V'}$ are the subgroups of $\mathrm{SL}_3(q)$ corresponding to V and V' respectively, then $C_W(\overline{V})$ is 2-dimensional and $[W,\overline{V}]$ is 1-dimensional, and $C_W(\overline{V'})$ and $[W,\overline{V'}]$ are 1-dimensional. But we also have

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ ax+y \\ cx+by+z \end{pmatrix},$$

and so if U is any other elementary abelian subgroup of S of order q^2 then $C_W(\overline{U})$ and $[W,\overline{U}]$ are both 2-dimensional. Hence V, V' and U are mutually not conjugate in $\mathrm{PSL}_3(q)$. Since V and V' are also self-centralizing, this means that V and V' are $\mathcal{F}$-centric.

Regard V as a 2-dimensional vector space over $\mathbb{F}_q$ with basis $\left\{ \left(\begin{smallmatrix} 1&0&0\\1&1&0\\0&0&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&0&0\\0&1&0\\1&0&1 \end{smallmatrix}\right) \right\}$. Then the elements of $\mathrm{PSL}_3(q)$ which correspond to the matrices $\left(\begin{smallmatrix} 1&0&0\\0&1&0\\0&\lambda&1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1&0&0\\0&1&\mu\\0&0&1 \end{smallmatrix}\right)$ act on V as the automorphisms $\left(\begin{smallmatrix} 1&0\\-\lambda&1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1&-\mu\\0&1 \end{smallmatrix}\right)$ respectively. These generate a subgroup of $\mathrm{Aut}_G(V)$ isomorphic to $\mathrm{SL}_2(q)$. Hence by Lemma 3.1.1, V is $\mathcal{F}$-radical. Similarly V' is $\mathcal{F}$-radical. This shows that V and V' are $\mathcal{F}$-Alperin subgroups of S. By Proposition 4.4.1, there cannot be any more $\mathcal{F}$-Alperin subgroups unless $q = 3$, 5, 7 or 13. The theorem for these cases has already been proved by Ruiz and Viruel in [36, Lemma 4.9]. Hence the theorem is proved. ∎

The previous two results combine to give the following theorem:

Theorem 4.4.5 *If $\mathcal{F}$ is a saturated fusion system over S with $|\mathrm{Alp}(\mathcal{F})| = 2$ then $\mathcal{F}$ contains a fusion system isotypically equivalent to the fusion system $\mathcal{F}_S(\mathrm{PSL}_3(q))$.*

4.5 Statement of the classification

Let us summarise what has been proved so far in these first few chapters.

Theorem 4.5.1 *If $\mathcal{F}$ is a saturated fusion system over S then one of the following holds:*

(i) *$\mathcal{F}$ has no proper Alperin subgroups and $\mathcal{F} \cong \mathcal{F}_S(S : W)$ where W is isomorphic to any p'-subgroup of $\Gamma\mathrm{L}_2(q)$;*

(ii) *$\mathcal{F}$ has exactly one proper Alperin subgroup and $\mathcal{F} \cong \mathcal{F}_S(q^2 : W)$ where $\mathrm{SL}_2(q) \leq W \leq \Gamma\mathrm{L}_2(q)$ and $W/\mathrm{SL}_2(q)$ has p' order;*

(iii) *$\mathcal{F}$ has exactly two Alperin subgroups and $\mathcal{F} \cong \mathcal{F}_S(\mathrm{PSL}_3(q).W)$ where $W \leq \mathrm{Out}(\mathrm{PSL}_3(q))$ and $p \nmid |W|$;*

(iv) *$\mathcal{F}$ has $|\mathrm{Alp}(\mathcal{F})| > 2$ and is one of the exceptional cases of Ruiz and Viruel.*

In particular, $\mathcal{F}$ is not exotic in cases (i), (ii) and (iii).

Note that if $\mathcal{F}$ is a non-exotic saturated fusion system over a finite p-group, with p odd, then $\mathcal{F}$ has a unique associated linking system. This fact follows from [28, Theorem 4.5] and [11, Proposition 3.1]. Hence the theorem above comprises a classification not just of saturated fusion systems over S, but of p-local finite groups over S.

Chapter 5

Construction of a Saturated Fusion System

In this chapter we shall construct some saturated fusion systems over certain p-groups of shape $p^r : p$ where $r \in \mathbb{N}$. In fact the p-groups we shall be considering are Sylow p-subgroups of semidirect products that arise naturally from considering the so-called *basic* irreducible $\mathbb{F}_p\mathrm{GL}_2(p)$-modules as described by Brauer and Nesbitt in [7].

It turns out that by extending the action of $\mathrm{GL}_2(p)$ on these p-groups, we can generate a saturated fusion system using a theorem of Broto, Levi and Oliver from [12]. In fact, in the case when the dimension of the modules is 3, these turn out to be exactly the fusion systems considered by Broto, Levi and Oliver [12, Example 5.5].

In a later chapter we shall show that these fusion systems are exotic, extending the result of Broto *et al.*, *op cit.*.

5.1 Construction of the p-group

The fusion system we create will contain the fusion system from a semidirect product of a certain modular $\mathrm{GL}_2(p)$-module, where p is an odd prime. The irreducible $\mathbb{F}_p\mathrm{GL}_2(p)$-modules were originally described by Brauer and Nesbitt [7], although our exposition shall follow that of Parker and Rowley [30].

Let $\mathbb{F}_p$ be the field with p elements (p an odd prime) and consider the ring $\overline{A} = \mathbb{F}_p[x, y]$ of polynomials in two commuting indeterminates x, y with coefficients in the field $\mathbb{F}_p$. Note that we can make $\overline{A}$ into an $\mathbb{F}_p\mathrm{GL}_2(p)$-module as follows: let $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(p)$ and define an action of X on $\overline{A}$ by setting $x \mapsto (\alpha x + \beta y)$ and $y \mapsto (\gamma x + \delta y)$, and then extending linearly to all of $\overline{A}$. The kernel of this action is clearly trivial, and so this makes $\overline{A}$ into a faithful $\mathbb{F}_p\mathrm{GL}_2(p)$-module.

Now consider the subspaces $A(n, p) = \{f \in \overline{A} | \deg(f) = n\}$ of $\overline{A}$ consisting of all homogeneous polynomials of degree n. Then, as proved in [7, p588], for each $0 \leq n \leq p-1$ the subspace $A(n, p)$ is an irreducible $\mathbb{F}_p\mathrm{GL}_2(p)$-module of dimension $n + 1$. Throughout the remainder of this chapter and the next, we shall exclusively use the notation $A(n, p)$ to denote this subspace, although we shall normally consider $A(n, p)$ as an abelian group. If the values of n and p are clear from the context then we may drop this part of the notation. We shall always assume that $n \geq 2$.

We can also define an action of the multiplicative group $\mathbb{F}_p^*$ on $A(n, p)$ as follows: let $\lambda \in \mathbb{F}_p^*$ and let $f \in A$. Then define $f^\lambda = \lambda f$. Now set $\Gamma = \mathrm{GL}_2(p) \times \mathbb{F}_p^*$ and make A into a $\mathbb{F}_p\Gamma$-module by setting $f^{(X,\lambda)} = \lambda(f^X)$.

Fix $n \in \mathbb{N}$ and an odd prime p. Let $\mathcal{G}(n, p)$ be the semidirect product of the elementary abelian group $A(n, p)$ with the group Γ with the action as described above *i.e.* $\mathcal{G} = A \rtimes \Gamma$. The group $\mathcal{G}$ has order $|A||\mathrm{GL}_2(p)||\mathbb{F}_p^*| = p^{n+2}(p-1)^3(p+1)$. Let $S(n, p) = A(n, p) \rtimes \langle (\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1) \rangle$. It is easy to see that $S(n, p)$ has order p^{n+2}, and so $S(n, p)$ is a Sylow p-subgroup of $\mathcal{G}(n, p)$. Throughout the remainder of this chapter and the next, we shall reserve the notation $S(n, p)$ to denote this group, although if the values of n and p are clear from the context then we may drop this part of the notation. We shall always assume that $n \geq 2$.

Also, let R be the subgroup of S generated by the polynomial x^n and the element of

S corresponding to the element $(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right),1) \in \Gamma$, which acts on A as $x \mapsto x$ and $y \mapsto x+y$.

For a general element $f \in A$, define the *weight* of $f = \sum_{i=0}^{n} a_i x^{n-i} y^i$ by

$$\mathrm{wt}(f) = \max\{i | a_i \neq 0\}$$

and define $\mathrm{wt}(0) = -1$. Now for $-1 \leq i \leq n$ define

$$C_i = \{f \in A | \mathrm{wt}(f) \leq i\}.$$

Note that for each $-1 \leq i \leq n$, the set C_i is a subgroup of $A(n,p)$, and $C_n = A(n,p)$.

In what follows we shall make use of the following general formula for conjugation in semidirect products. Let H, K be finite groups where K acts on H. Let (h,k), (r,g) be elements of the semidirect product $H \rtimes K$. Then

$$\begin{aligned}(h,k)^{(r,g)} &= (r,g)^{-1}(h,k)(r,g)\\ &= ((-r)^{g^{-1}}, g^{-1})(h^g + r, kg)\\ &= ((-r)^{g^{-1}kg} + h^g + r, g^{-1}kg).\end{aligned} \tag{5.1}$$

Lemma 5.1.1 *We have the following:*

(i) $Z(S(n,p)) = C_0 = \langle x^n \rangle$;

(ii) for $0 \leq i \leq n$, $[C_i, S(n,p)] = C_{i-1}$. *In particular, for* $i \leq n-1$, C_i *is the* $(n-i)$*th term of the lower central series of* $S(n,p)$;

(iii) for $0 \leq i \leq n$, C_i *is a characteristic subgroup of* $S(n,p)$;

(iv) R *is elementary abelian of rank* 2.

Proof: First, we let $H = A$ and $K = \langle(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right),1)\rangle$ and consider S as the semidirect product $H \rtimes K$. Suppose that $(r,g) \in Z(S)$. Then (r,g) commutes with every element of the form $(0,k)$ where $k \in K$. Referring to 5.1 we see that $(0,k)^{(r,g)} = ((-r)^{k^g} + r, k^g)$. By considering the second coordinate, it is clear that $k^g = k$ for all $k \in K$, and therefore $(0,k)^{(r,g)} = ((-r)^k + r, k)$. Hence (r,g) centralizes $(0,k)$ for all $k \in K$ if and only if $r^k - r = 0$ for all $k \in K$. Suppose that $r = \sum_{i=0}^{n} a_i x^{n-i} y^i$ where each $a_i \in \mathbb{F}_p$ and that $k = \left(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1\right)$. Then

$$\begin{aligned} r^k &= \sum_{i=0}^{n} a_i x^{n-i} (x+y)^i \\ &= \sum_{i=0}^{n} \sum_{j=0}^{i} a_i \binom{i}{j} x^{n-j} y^j, \end{aligned}$$

so that

$$\begin{aligned} r^k - r &= \sum_{i=0}^{n} \sum_{j=0}^{i} a_i \binom{i}{j} x^{n-j} y^j - \sum_{l=0}^{n} a_l x^{n-l} y^l \\ &= a_0 x^n + \sum_{i=1}^{n} \sum_{j=0}^{i} a_i \binom{i}{j} x^{n-j} y^j - a_0 x^n - \sum_{l=1}^{n} a_l x^{n-l} y^l \\ &= \sum_{i=1}^{n} \sum_{j=0}^{i-1} a_i \binom{i}{j} x^{n-j} y^j \\ &= a_1 x^n + \sum_{i=2}^{n} \sum_{j=0}^{i-1} a_i \binom{i}{j} x^{n-j} y^j. \end{aligned} \tag{5.2}$$

Thus $r^k - r = 0$ if and only if $a_i = 0$ for all $i \geq 1$, *i.e.* $r \in \langle x^n \rangle$.

We also have that (r,g) must commute with every element of the form $(h,1)$ where $h \in H$. Again referring to 5.1 we see that $(h,1)^{(r,g)} = (-r+r+h^g, 1) = (h^g, 1)$. Thus (r,g) centralizes $(h,1)$ for all $h \in H$ if and only if $h^g = h$ for all $h \in H$. In particular, we must have $(y^n)^g = y^n$, but the only element of K for which this holds is the identity. Hence $g = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$, and so $Z(S) \subseteq \langle x^n \rangle$. It is easy to see that $\langle x^n \rangle \subseteq Z(S)$, and therefore $Z(S) = \langle x^n \rangle$, proving (i).

The first part of (ii) is simply a special case of [31, Proposition 1]. For the second part, we shall show that $[S,S] = C_{n-1}$. The statement then follows by induction, the inductive step being the first part of the statement. By consulting 5.1, we see that for any elements $(h,k),(r,g) \in S = H \rtimes K$, the commutator $[(h,k),(r,g)]$ is given by

$$[(h,k),(r,g)] = (h^g - h + r - r^k, 1).$$

The calculations in the proof of (i) show that $h^g - h$ and $r - r^k$ are both elements of C_{n-1}, and therefore so is $(h^g - h + r - r^k, 1)$. Hence $[S,S] \leq C_{n-1}$. But $[S,S] \geq [A,S] = [C_n, S] = C_{n-1}$, and therefore $[S,S] = C_{n-1}$. This shows that for $0 \leq i \leq n-1$, C_i is the $(n-i)$th term of the lower central series. This also means that these C_i are all characteristic subgroups of S. It remains to show that $C_n = A$ is characteristic.

Suppose that $C' \leq S$ is $\mathrm{Aut}(S)$-conjugate to C_n. Since C_{n-1} is characteristic in S, $C_{n-1} \leq C \cap C'$. Let $(r,g) \in C'$ with $r \in A$ and $g \in K$. Since C' is abelian, (r,g) must commute with elements of the form $(h,1)$ where $h \in C_{n-1}$. Since $n \geq 2$, we have that $C_{n-1} \gneqq Z(S)$. In particular, C_{n-1} contains the element $x^{n-1}y$ and so by using the conjugation formula 5.1 we see that (r,g) commutes with every $(x^{n-1}y, 1)$ only if $g = 1$. Hence $C' = C_n$, and therefore C_n is characteristic in S, proving (iii).

To see that R is elementary abelian, note that the generators x^n and $(\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), 1)$ commute, and so by the properties of semidirect products, $R = \langle x^n \rangle \rtimes \langle (\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), 1) \rangle$ is just the direct product $R = \langle x^n \rangle \times \langle (\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right), 1) \rangle$, which is clearly elementary abelian. ∎

Now we prove a general lemma from which we can deduce that $S(n,p)$ has exponent p for $2 \leq n \leq p-2$ and p^2 for $n = p-1$.

Lemma 5.1.2 *Let T be a finite p-group with exponent p and let V be a finite dimensional $\mathbb{F}_p T$-module. There exists an $0 < n < \infty$ such that $[V,T;n] = 1$; let m be the smallest such $n \in \mathbb{N}$. If $m \leq p-1$ then the exponent of the semidirect product $V \rtimes T$ is p.*

Proof: Let $X = V \rtimes T$. Then we may write any element of X as ta where $t \in T$ and $a \in V$. Note that $at = ta[a,t]$ and hence

$$\begin{aligned}(ta)^2 = tata &= t^2a[a,t]a \\ &= t^2a^2[a,t],\end{aligned}$$

since $[a,t] \in V$, and V is an abelian group. We shall now use induction to show that

$$(ta)^n = t^n a^n \prod_{i=0}^{n-2} \left[a^{\binom{n}{i}}, t; n-1-i\right].$$

Suppose that $(ta)^k - 1 = t^{k-1}a^{k-1}\prod_{i=0}^{k-3}\left[a^{\binom{k-1}{i}}, t; k-2-i\right]$. Then

$$\begin{aligned}(ta)^k &= t^{k-1}a^{k-1}\prod_{i=0}^{k-3}\left[a^{\binom{k-1}{i}}, t; k-2-i\right] ta \\ &= t^k a^k \prod_{i=0}^{k-3}\left[a^{\binom{k-1}{i}}, t; k-2-i\right]\left[a^{k-1}\prod_{j=0}^{k-3}\left[a^{\binom{k-1}{j}}, t; n-2-j\right], t\right].\end{aligned} \tag{5.3}$$

Note that if $x, y \in V$ and $z \in T$ then $[xy, z] = [x,z]^y[y,z] = [x,z][y,z]$ (see, for example, [20, Theorem 2.2.1(i)]). Hence

$$\begin{aligned}\left[a^{k-1}\prod_{j=0}^{k-3}\left[a^{\binom{k-1}{j}}, t; n-2-j\right], t\right] &= \left[a^{k-1}, t\right]\prod_{j=0}^{k-3}\left[a^{\binom{k-1}{j}}, t; k-1-j\right] \\ &= \prod_{j=0}^{k-2}\left[a^{\binom{k-1}{j}}, t; k-1-j\right].\end{aligned}$$

By substituting this back into 5.3 we see that

$$\begin{aligned}(ta)^k &= t^k a^k \prod_{i=0}^{k-3} \left[a^{\binom{k-1}{i}}, t; k-2-i\right] \prod_{j=0}^{k-2} \left[a^{\binom{k-1}{j}}, t; k-1-j\right] \\ &= \prod_{i=1}^{k-2} \left[a^{\binom{k-1}{i-1}}, t; k-1-i\right] \prod_{j=0}^{k-2} \left[a^{\binom{k-1}{j}}, t; k-1-j\right] \qquad (5.4) \\ &= \prod_{i=1}^{k-2} \left[a^{\binom{k-1}{i-1}+\binom{k-1}{i}}, t; k-1-i\right] [a, t; k-1]\end{aligned}$$

Recall that $\binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i}$, and so

$$(ta)^k = t^k a^k \prod_{i=0}^{k-2} \left[a^{\binom{k}{i}}, t; k-1-i\right].$$

Now, T and V have exponent p, and p divides $\binom{p}{i}$ for all $1 \le i \le p-2$, so $t^p = a^p = \left[a^{\binom{p}{i}}, t; p-1-i\right] = 1$ for all $1 \le i \le p-2$. Therefore

$$\begin{aligned}(ta)^p &= t^p a^p \prod_{i=0}^{p-2} \left[a^{\binom{p}{i}}, t; p-1-i\right] \\ &= [a, t; p-1].\end{aligned}$$

But $m \le p-1$ and therefore $[a, t; p-1] = 1$. Thus ta has order dividing p; since p is prime, we have that the order of ta is p. We have now shown that every element of X has order p, and so the exponent of X is p. ∎

Corollary 5.1.3 *The group $S(n,p)$ has exponent p if $2 \le n \le p-2$ and has exponent p^2 if $n = p-1$.*

Proof: The case when $n \le p-2$ follows directly from 5.1.1 and 5.1.2. If $n = p-1$ then the proof of 5.1.2 shows that for any element $s \in S$, $s^p \in [A, S; p-1] = C_0 = Z(S)$. Consider the element $Y = (y^n, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))$. When $n = p-1$ it is routine to check that $Y^p \ne 1$.

Since $Z(S)$ is a cyclic group of order p, we have that $Y^{p^2} = 1$. Hence S has exponent p^2. ∎

5.2 Construction of a saturated fusion system

Before constructing the fusion system, we shall describe a theorem of Broto, Levi and Oliver which provides a sufficient condition for a fusion system to be saturated. We shall use this theorem to show that the fusion system we construct is saturated.

First we need a few definitions.

Definition 5.2.1 *Let G be a finite group with T a Sylow p-subgroup of G.*

(i) A subgroup P of T is p-centric *in G if $Z(P) \in \mathrm{Syl}_p(C_G(P))$.*

(ii) A proper subgroup H of G is strongly p-embedded *if $p \mid |H|$ but $p \nmid |H \cap H^g|$ for all $g \in G \setminus H$.*

(iii) A subgroup P of T is essential *if either $P = T$ or P is p-centric in G and $\mathrm{Out}_G(P)$ has a strongly p-embedded subgroup.*

Lemma 5.2.2 *Let G be a finite group with a strongly p-embedded subgroup. Then $O_p(G) = 1$.*

Proof: Let H be a strongly p-embedded subgroup of G, let $T \in \mathrm{Syl}_p(H)$ and let $R \in \mathrm{Syl}_p(G)$ with $R \geq T$. Suppose that $R \gneqq T$; then $N_R(T) \gneqq T$ and so there exists a $g \in N_R(T) \setminus T$ such that $H \cap H^g \supseteq T$. But $N_R(T) \cap H = T$ since $T \in \mathrm{Syl}_p(H)$, and so $g \notin H$. This contradicts the assumption that H is strongly p-embedded.

Therefore $T \in \mathrm{Syl}_p(G)$ and so $O_p(G) \leq H$. But then $O_p(G) \leq H \cap H^g$ for all $g \in G$. Since H is strongly p-embedded, we have that $p \nmid |H \cap H^g|$ for all $g \in G$, and so $O_p(G) = 1$ as claimed. ∎

The following is an immediate corollary of the previous result:

Corollary 5.2.3 *If P is essential in G then P is also p-radical in G,* i.e. $O_p(\mathrm{Out}_G(P)) = 1$.

We are now ready to state the theorem.

Proposition 5.2.4 [12, Proposition 5.1] *Fix a finite group G, a Sylow p-subgroup T of G and subgroups $Q_1, \ldots, Q_m \leq T$ such that no Q_i is G-conjugate to a subgroup of Q_j for $i \neq j$. For each i set $K_i = \mathrm{Aut}_G(Q_i)$ and fix subgroups $\Delta_i \leq \mathrm{Aut}(Q_i)$ which contain K_i. Set $\mathcal{F} = \langle \mathcal{F}_T(G), \Delta_1, \ldots, \Delta_m \rangle$ and assume for each i that*

(i) $p \nmid |\Delta_i : K_i|$;

(ii) Q_i is p-centric, but no proper subgroup of Q_i is $\mathcal{F}$-centric or an essential p-subgroup of G; and

(iii) for all $\alpha \in \Delta_i \setminus K_i$, we have $O_p(K_i \cap K_i^{\alpha}) = \mathrm{Inn}(Q_i)$.

Then $\mathcal{F}$ is a saturated fusion system over T.

Let p be prime. As in § 5.1, set $\mathcal{G}(n,p) = A(n,p) \rtimes \Gamma$ and let $S(n,p)$ be the Sylow p-subgroup of $\mathcal{G}$ given by $S = A \rtimes \langle \left(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right), 1 \right) \rangle$, and let R be the elementary abelian subgroup of S described in 5.1.1.

Now let $\mathcal{E}(n,p)$ be the fusion system $\langle \mathcal{F}_S(\mathcal{G}), \mathrm{Aut}_{\mathcal{E}}(R) := \mathrm{Aut}(R) \cong \mathrm{GL}_2(p) \rangle$. Note that $\mathrm{Aut}_{\mathcal{E}}(A) = \Gamma / C_\Gamma(A)$ with the action stated above. We have the following:

Lemma 5.2.5 *The set of elements of Γ which act trivially on $A(n,p)$ is given by*

$$C_\Gamma(A) = \left\{ \left(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right), a^{-n} \right) | a \in \mathbb{F}_p^{\times} \right\}$$

Proof: Let $X = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(p)$ and suppose that $(X, \lambda) \in C_\Gamma(A)$. Then in particular, $x^n = (x^n)^{(X,\lambda)} = \lambda(ax + by)^n$, therefore $b = 0$. Similarly, $y^n = (y^n)^{(X,\lambda)} = \lambda(cx + dy)^n$, so $c = 0$ and $\lambda = d^{-n}$. Furthermore, $xy^{n-1} = (xy^{n-1})^{(X,\lambda)} = \lambda ad^{n-1}xy^{n-1}$, and so $\lambda ad^{n-1} = 1$. Hence $\lambda ad^n = d$ and therefore $a = d$ since $\lambda = d^{-n}$. We have shown that $C_\Gamma(A)$ contains the set $\left\{\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), a^{-n}\right) | a \in \mathbb{F}_p^\times\right\}$.

To see that the reverse inclusion holds, let $X = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in \mathrm{GL}_2(p)$. Then $(x^iy^{n-i})^{(X,a^{-n})} = a^{-n}(ax)^i(ay)^{n-i} = a^{(n-i)+i-n}x^iy^{n-i} = x^iy^{n-i}$, hence $(X, a^{-n}) \in C_\Gamma(A)$. ∎

Let $\Omega = \Gamma/C_\Gamma(A) = \mathrm{Aut}_\mathcal{G}(A)$. Note that $|\Omega| = |\mathrm{GL}_2(p)|$.

We show that the fusion systems $\mathcal{E}(n, p)$ satisfy the hypotheses of Proposition 5.2.4 with $m = 1$ and $Q_1 = R$, and are therefore saturated.

Lemma 5.2.6 *Let $\mathcal{G} = \mathcal{G}(n, p)$ as before. Then the following hold:*

(i) $N_\mathcal{G}(R) = \langle\{\lambda x^n + \mu x^{n-1}y | \lambda, \mu \in \mathbb{F}_p\}, \left\{\left(\left(\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right), \zeta\right) | \alpha, \gamma, \zeta \in \mathbb{F}_p^\times, \beta \in \mathbb{F}_p\right\}\rangle$;

(ii) $C_\mathcal{G}(R) = \langle R, \left(\left(\begin{smallmatrix} \alpha & 0 \\ \beta & \alpha \end{smallmatrix}\right), 1\right)\rangle$.

In particular, R is p-centric in $\mathcal{G}$, $\mathrm{Aut}_\mathcal{G}(R)$ has order $p(p-1)^2$ and is isomorphic to the set of lower triangular matrices in $\mathrm{GL}_2(p)$.

Proof: Consider $\mathcal{G} = A \rtimes \Gamma$ as the set $A \times \Gamma$ with multiplication given by $(r_1, g_1)(r_2, g_2) = (r_1^{g_1} + r_2, g_1g_2)$.

Let $H = \langle x^n \rangle = R \cap A$ and let $K = \{\left(\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right), 1\right) | \alpha \in \mathbb{F}_p\}$. It is clear that $R = H \rtimes K$.

Let $(r, g) \in \mathcal{G}$ and let $(h, k) \in R$. Suppose that $r = \sum_{i=0}^n a_ix^{n-i}y^i$ where each $a_i \in \mathbb{F}_p$. By consulting formula 5.1, we see that if (r, g) normalizes R then $g \in N_\Gamma(K)$, and $(-r)^{g^{-1}kg} + h^g + r \in H$ for all $h \in H$. In particular, we must have that $(-r)^k + r \in H$, for all $k \in K$. But this holds if and only if $r^k - r \in H$ for all $k \in K$. But $H = \{\lambda x^n \in A | \lambda \in \mathbb{F}_p\}$ and so by consulting formula 5.2 we see that $r^k - r \in H$ if and only if $a_i = 0$ for all $i > 1$, *i.e.* if and only if $r \in \{\lambda x^n + \mu x^{n-1}y | \lambda, \mu \in \mathbb{F}_p\}$. Thus we have shown that $N_{\mathcal{G}}(R) = \{\lambda x^n + \mu x^{n-1}y | \lambda, \mu \in \mathbb{F}_p\} \rtimes N_\Gamma(K)$. It is easy to see that $N_\Gamma(K) = \left\{\left(\left(\begin{smallmatrix}\alpha & 0\\ \beta & \gamma\end{smallmatrix}\right), \zeta\right) | \alpha, \beta, \gamma, \zeta \in \mathbb{F}_p\right\}$ and so (i) follows.

To see (ii), note that if (r, g) centralizes (h, k) then $r^k - r = 0$, and so $a_i = 0$ for all $i \neq 0$ (by formula 5.2 again), *i.e.* $r \in H$. We also require that $g \in C_\Gamma\left(\left(\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right), 1\right)\right)$; but this is easily seen to be equal to $\left\{\left(\left(\begin{smallmatrix}\alpha & 0\\ \beta & \alpha\end{smallmatrix}\right), \lambda\right) \mid \alpha \in \mathbb{F}_p^\times, \beta \in \mathbb{F}_p\right\}$. It is now clear that R is p-centric in $\mathcal{G}$

In particular, we have shown that $\mathrm{Aut}_{\mathcal{G}}(R)$ is generated by the automorphisms $c(x^{n-1}y)$ and $c\left(\left(\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right), \lambda\right)\right)$ where $c(g)$ denotes conjugation by the element g. If we consider R as a 2-dimensional vector space over $\mathbb{F}_p$ with basis $\mathcal{B} = \{x^n, (\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right), \lambda)\}$, then we may produce an explicit embedding of $\mathrm{Aut}_{\mathcal{G}}(R)$ into $\mathrm{GL}_2(p) \cong \mathrm{Aut}(R)$ by considering the action of the generating automorphisms on the $\mathbb{F}_p$-basis.

Since x^n and $x^{n-1}y$ are both elements of the abelian group A, we have that $c(x^{n-1}y)$ acts trivially on x^n, and so with respect to the basis $\mathcal{B}$, $c(x^{n-1}y)$ maps the vector $(1, 0)$ to $(1, 0)$. Considering the conjugation formula 5.1, we see that $c(x^{n-1}y)$ maps $(\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right), 1) \in H \rtimes K$ to $(-x^n, (\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right), 1)) \in H \rtimes K$, *i.e.* with respect to the basis $\mathcal{B}$, $c(x^{n-1}y)$ maps the vector $(0, 1)$ to $(-1, 1)$. Hence $c(x^{n-1}y)$ corresponds (with respect to $\mathcal{B}$) to the matrix $\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$. Similarly it can easily be shown that $c\left(\left(\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right), \lambda\right)\right)$ corresponds to the matrix $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$. But the matrices $\left(\begin{smallmatrix}1 & 0\\ -1 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right) \in \mathrm{GL}_2(p)$ generate the subgroup of lower triangular matrices. Hence $\mathrm{Aut}_{\mathcal{G}}(R)$ is isomorphic to the set of lower triangular matrices in $\mathrm{GL}_2(p)$ and has order $p(p-1)^2$. ∎

Lemma 5.2.7 *Let* $K = \mathrm{Aut}_{\mathcal{G}}(R)$ *and* $\Delta = \mathrm{Aut}_{\mathcal{E}}(R) = \mathrm{Aut}(R)$. *Then*

(i) $p \nmid |\Delta : K|$;

(ii) R *is* p*-centric in* $\mathcal{G}$ *but no proper subgroup* $P \subsetneq R$ *is* $\mathcal{E}$*-centric or an essential* p*-subgroup of* $\mathcal{G}$*; and*

(iii) for all $\alpha \in \Delta \setminus K$*, we have* $O_p(K \cap K^\alpha) = 1$.

Proof: By Lemma 5.2.6, K has order $p(p-1)^2$ and so clearly $p \nmid |\Delta : K|$. It was also shown in Lemma 5.2.6 that R is p-centric in $\mathcal{G}$, and since R is abelian, any proper subgroup of R must contain R in its centralizer, and therefore is not $\mathcal{E}$-centric or an essential p-subgroup of $\mathcal{G}$. This proves (i) and (ii). To prove (iii), note that $\mathrm{Aut}_{\mathcal{G}}(R)$ is the normalizer of a Sylow p-subgroup of $\mathrm{GL}_2(p)$, and so every $\mathrm{GL}_2(p)$-conjugate of $K = \mathrm{Aut}_{\mathcal{G}}(R)$ is also the normalizer of a Sylow p-subgroup. Sylow p-subgroups of $\mathrm{GL}_2(p)$ have order p, hence any two distinct Sylow p-subgroups intersect trivially. Now, $K = N_\Delta(K)$ and so for all $\alpha \in \Delta \backslash K$ we have $K \neq K^\alpha$. Let $\alpha \in \Delta \backslash K$ such that $K \neq K^\alpha$. Let $T_1, T_2 \in \mathrm{Syl}_p(\mathrm{GL}_2(p))$ be such that $K = N_{\mathrm{GL}_2(p)}(T_1)$ and $K^\alpha = N_{\mathrm{GL}_2(p)}(T_2)$. Then $1 = T_1 \cap T_2 \in \mathrm{Syl}_p(K \cap K^\alpha)$. Hence $|K \cap K^\alpha|_p = 1$ and so $O_p(K \cap K^\alpha) = 1$.

Since any two Sylow p-subgroups generate all of $\mathrm{GL}_2(p)$, your conjugate is either equal to $\mathrm{Aut}_{\mathcal{G}}(R)$ or intersects it with trivial p-part. By observing that $\mathrm{Aut}_{\mathcal{G}}(R)$ is self-normalizing in $\mathrm{GL}_2(p)$, statement (iii) follows. ∎

Corollary 5.2.8 *Let $n \geq 2$ and p be an odd prime. Then the fusion systems $\mathcal{E}(n,p)$ over $S(n,p)$ are saturated.*

Proof: This now follows from Proposition 5.2.4 and Lemma 5.2.7. ∎

5.3 Properties of $\mathcal{E}(n,p)$

In this section we collect some result about the properties of the fusion systems $\mathcal{E}(n,p)$ which we shall use in the next chapter to prove that most of the fusion systems are exotic.

Recall that $\Omega = \mathrm{Aut}_{\mathcal{E}}(A(n,p)) \cong (\mathrm{GL}_2(p) \times \mathbb{F}_p^*)/\{(\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right), u^{-n})\}$. We shall require the following lemma about Ω in the next chapter.

Lemma 5.3.1 $[\Omega, \Omega] \cong \begin{cases} \mathrm{PSL}_2(p) & n \text{ } \textit{even}; \\ \mathrm{SL}_2(p) & n \text{ } \textit{odd}. \end{cases}$

Proof:

$$\begin{aligned}
[\Omega, \Omega] &\cong \frac{([\mathrm{GL}_2(p), \mathrm{GL}_2(p)] \times [\mathbb{F}_p, \mathbb{F}_p])\, C_\Gamma(A)}{C_\Gamma(A)} \\
&\cong \frac{\mathrm{SL}_2(p) \times 1}{(\mathrm{SL}_2(p) \times 1) \cap C_\Gamma(A)}.
\end{aligned}$$

Now, $(\mathrm{SL}_2(p) \times 1) \cap C_\Gamma(A) = \{(\left(\begin{smallmatrix} u & 0 \\ 0 & u \end{smallmatrix}\right), u^{-n})\, |u = \pm 1, u^{-n} = 1\}$. But $(-1)^{-n} = 1$ if and only if n is even, otherwise $(-1)^{-n} = -1$. Hence if n is even we have

$$\begin{aligned}
[\Omega, \Omega] &\cong \frac{\mathrm{SL}_2(p) \times 1}{\left\{(I, 1), \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), 1\right)\right\}} \\
&\cong \mathrm{PSL}_2(p),
\end{aligned}$$

and if n is odd then we have

$$\begin{aligned}
[\Omega, \Omega] &\cong \frac{\mathrm{SL}_2(p) \times 1}{(I, 1)} \\
&\cong \mathrm{SL}_2(p).
\end{aligned}$$

∎

Now let $P \leq S(n, p)$. We shall now show that $\mathrm{Aut}_{\mathcal{E}}(P)$ does not contain a subgroup isomorphic to $\mathrm{Sym}(m)$ for any $m \geq 6$.

Proposition 5.3.2 *Let $P \leq S(n, p)$ with $|P| > p^2$ and suppose that P is not contained in $A(n, p)$. Then $\mathrm{Aut}_{\mathcal{E}}(P)$ does not contain a subgroup isomorphic to $\mathrm{Alt}(m)$ for any $m \geq 4$.*

Proof: The only $\mathcal{E}$-Alperin subgroup of S that contains P is S itself. Hence by Alperin's Fusion Theorem, $\mathrm{Aut}_{\mathcal{E}}(P)$ just contains restrictions of elements of $\mathrm{Aut}_{\mathcal{E}}(S)$ which normalize P. Therefore $\mathrm{Aut}_{\mathcal{E}}(P)$ is isomorphic to a quotient of a subgroup of $\mathrm{Aut}_{\mathcal{E}}(S)$, namely $N_{\mathrm{Aut}_{\mathcal{E}}(S)}(P)/C_{\mathrm{Aut}_{\mathcal{E}}(S)}(P)$. But $\mathrm{Aut}_{\mathcal{E}}(S) \cong C_{p-1} \times C_{p-1}$ is abelian, and so every quotient of every subgroup of $\mathrm{Aut}_{\mathcal{E}}(S)$ is also abelian. In particular, no quotient of any subgroup of $\mathrm{Aut}_{\mathcal{E}}(S)$ contains a subgroup isomorphic to a alternating group of degree greater than or equal to 4, since such alternating groups are non-abelian. ∎

Recall that a group K is called a *section* of a group G if K is isomorphic to a homomorphic image of a subgroup of G.

Lemma 5.3.3 *Let X be a section of a group G.*

(i) Let $Y \leq X$. Then Y is a section of G.

(ii) $[X, X]$ is a section of $[G, G]$.

(iii) Suppose further that L is a section of X. Then L is a section of G.

Proof: (i) Since X is a section of G, there exists a subgroup $H \leq G$ and a surjective homomorphism $\theta : H \to X$. Denote by $\widehat{\theta}$ the induced isomorphism $H/\ker\theta \to X$. By the Correspondence Theorem, there exists a subgroup $K \leq H$ with $\ker\theta \leq K$ such that $K/\ker\theta = \widehat{\theta}^{-1}(Y)$. Now $\theta\mid_K$ is a surjective homomorphism $K \to Y$, and so Y is a section of G.

(ii) Let $H \leq G$ and $\theta : H \to X$ be a surjective homomorphism. For all $h_1, h_2 \in H$, $\theta(h_1^{-1}h_2^{-1}h_1h_2) \in [X, X]$, and so $\theta([H, H]) \leq [X, X]$. Conversely, if $x_1, x_2 \in X$ then there exist elements $h_1, h_2 \in H$ such that $\theta(h_1) = x_1$ and $\theta(h_2) = x_2$. Therefore every commutator $x_1^{-1}x_2^{-1}x_1x_2 \in [X, X]$ has a preimage $h_1^{-1}h_2^{-1}h_1h_2 \in [H, H]$. Therefore $[X, X] \leq \theta([H, H])$. Thus $\theta\mid_{[H,H]}$ is a surjective homomorphism $[H, H] \to [X, X]$, and so $[X, X]$ is a section of $[G, G]$.

(iii) Since L is a section of X, there exists a subgroup $Y \leq X$ and a surjective homomorphism $\phi : Y \to L$. Now by (i), Y is a section of G and so there exists a subgroup $H \leq G$ and a surjective homomorphism $\theta : H \to Y$. Hence $\theta\phi : H \to L$ is a surjective homomorphism, and so L is a section of G. ∎

Lemma 5.3.4 *The Sylow* 3*-subgroups of* $\mathrm{SL}_2(p)$ *are cyclic for* $p \neq 3$.

Proof: Note that the order of $\mathrm{SL}_2(p)$ is $(p-1)p(p+1)$. Since $p \neq 3$, we have that either $3 \mid (p-1)$ or $3 \mid (p+1)$ (not both). Now, by [39, 6.23], $\mathrm{SL}_2(p)$ contains cyclic subgroups of order $p-1$ and of order $p+1$. Hence every Sylow 3-subgroup of $\mathrm{SL}_2(p)$ is contained in a cyclic subgroup, and is therefore cyclic. ∎

Corollary 5.3.5 *The alternating group* $\mathrm{Alt}(m)$ *is not a section of* $\mathrm{SL}_2(p)$*, for any* $m \geq 6$ *and any prime* $p \neq 3$.

Proof: By Lemma 5.3.4, the Sylow 3-subgroups of $\mathrm{SL}_2(p)$ are cyclic. Every section of a cyclic group is cyclic, therefore every section of $\mathrm{SL}_2(p)$ has cyclic Sylow 3-subgroups. However, $\mathrm{Alt}(m)$ does not have cyclic Sylow 3-subgroups since it contains the elementary abelian group generated by the 3-cycles $(1\ 2\ 3)$ and $(4\ 5\ 6)$. Therefore $\mathrm{Alt}(m)$ is not a section of $\mathrm{SL}_2(p)$. ∎

Remark Note that the alternating group $\mathrm{Alt}(5)$ is a subgroup of $\mathrm{PSL}_2(p)$ whenever $p(p^2-1) \equiv 0 \bmod 5$ (see [39, Theorem 6.26]). This means that in general, we cannot prove that $\mathrm{Alt}(m)$ is not a section of $\mathrm{SL}_2(p)$ for $m \leq 5$.

Proposition 5.3.6 *Let* $Q \leq S(n,p)$ *(where* $p > 3$*) with* $|Q| > p^2$ *and suppose that* $Q \leq A(n,p)$*. Then* $\mathrm{Aut}_{\mathcal{E}}(Q)$ *does not contain a subgroup isomorphic to* $\mathrm{Alt}(m)$ *for any* $m \geq 6$.

Proof: The only $\mathcal{E}$-Alperin subgroups of S containing Q are S itself and A. By Alperin's Fusion Theorem, this means that every morphism in $\mathrm{Aut}_{\mathcal{E}}(Q)$ can be written as a composition of restrictions of $\mathcal{E}$-morphisms of S and A. But A is a characteristic subgroup of S, so every morphism in $\mathrm{Aut}_{\mathcal{E}}(S)$ restricts to a morphism in $\mathrm{Aut}_{\mathcal{E}}(A)$. Therefore every $\mathcal{E}$-morphism of Q is the restriction of an $\mathcal{E}$-morphism of A. Hence $\mathrm{Aut}_{\mathcal{E}}(Q)$ is a section of $\mathrm{Aut}_{\mathcal{E}}(A)$, namely $N_{\mathrm{Aut}_{\mathcal{E}}(A)}(Q)/C_{\mathrm{Aut}_{\mathcal{E}}(A)}(Q)$.

Now suppose that $\mathrm{Aut}_{\mathcal{E}}(Q)$ has a subgroup $K \cong \mathrm{Alt}(m)$. Then K is a section of $\Omega \cong \mathrm{Aut}_{\mathcal{E}}(A)$, and so $[K, K] \cong \mathrm{Alt}(m)$ is a section of $[\Omega, \Omega]$ by 5.3.3. Now, $[\Omega, \Omega] = \mathrm{PSL}_2(p)$ or $\mathrm{SL}_2(p)$ by Lemma 5.3.1; in particular, (by Lemma 5.3.3) every section of $[\Omega, \Omega]$ is a section of $\mathrm{SL}_2(p)$.

Since we have shown that $[K, K] \cong \mathrm{Alt}(m)$ is a section of $[\Omega, \Omega]$, we have that $\mathrm{Alt}(m)$ is a section of $\mathrm{SL}_2(p)$. Therefore by 5.3.5, $m \leq 5$. ∎

Chapter 6

The proof that $\mathcal{E}$ is exotic

In this chapter we shall prove that the fusion systems $\mathcal{E}(n,p)$ do not come from finite groups when $p \geq 13$ and $n \geq 5$. In particular, this gives us an infinite family of exotic fusion systems. Indeed, it may be possible to show that the fusion systems $\mathcal{E}(n,p)$ are exotic for smaller values of p and n, but our main goal in this thesis is to produce an infinite family of exotic fusion systems. As such, we have introduced these minor restrictions on the values of n and p in order to minimize technical difficulties.

It would seem, *a priori*, that in order to do show that $\mathcal{E}(n,p)$ is exotic we shall need to check that every finite group which contains a Sylow p-subgroup isomorphic to $S(n,p)$, and show that it does not give rise to the fusion system $\mathcal{E}(n,p)$. This would be an impossible task. However, we can significantly reduce the problem by appealing to another important result of Broto, Levi and Oliver. This will show that we only need to check the almost simple groups. The Classification of Finite Simple Groups then allows us to prove exoticness.

6.1 Reducing the problem

We shall start by stating the theorem of Broto *et al.* that allows us to make this reduction. First we need a few definitions.

Definition 6.1.1 *Let $\mathcal{F}$ be a fusion system over a finite p-group T and let P be a subgroup of T. Then*

(i) P is strongly closed *in $\mathcal{F}$ if no element of P is $\mathcal{F}$-conjugate to an element of $T \setminus P$;*

(ii) P is normal *in $\mathcal{F}$ if $N_{\mathcal{F}}(P) = \mathcal{F}$.*

Recall that a finite group G is *almost simple* (with respect to K) if G has a normal non-abelian simple subgroup K with $C_G(K) = 1$, *i.e.* G is a group of automorphisms of a non-abelian simple group K such that G contains all of the inner automorphisms of K.

Theorem 6.1.2 [12, Lemma 5.2] *Let $\mathcal{F}$ be a fusion system over a nonabelian p-group T. For each subgroup $1 \neq P \leq T$ which is strongly closed in $\mathcal{F}$, assume that*

(i) P is p-centric in T;

(ii) P is not normal in $\mathcal{F}$; and

(iii) P does not factorize as a direct product of two or more distinct subgroups which are permuted transitively by $\mathrm{Aut}_{\mathcal{F}}(P)$.

If $\mathcal{F}$ is the fusion system of a finite group, then there exists a finite group G, which is almost simple with respect to a non-abelian simple group K, such that $\mathcal{F} \cong \mathcal{F}_T(G)$. Furthermore $p \mid |K|$.

Remark The statement given here differs very slightly from that given in [12]. Here we emphasise that P is assumed not to factorize as a *direct* product. We reproduce the proof here for the convenience of the reader.

Proof: Suppose that $\mathcal{F} \cong \mathcal{F}_T(G)$ where G is a group of minimal order with this property. Since $\mathcal{F}_T(G) \cong \mathcal{F}_T(G/O_{p'}(G))$, we have that $O_{p'}(G) = 1$. Let N be a minimal normal subgroup of G. Then by [20, Theorem 2.1.5], N is either an elementary abelian p-group or a direct product of isomorphic non-abelian simple groups.

Let $P = T \cap N$. If $x \in P$ and $x^g \in T$ for some $g \in G$ then $x^g \in P$ since $N \trianglelefteq G$. Hence P is strongly closed in $\mathcal{F}$.

If N is an elementary abelian p-group then $P = N$ and so P is normal in G, and hence normal in $\mathcal{F}$. This contradicts (ii). Therefore $N = X_1 \times \ldots \times X_m$, a direct product of m isomorphic non-abelian simple groups. Note that the factors in the direct product are permuted transitively by the conjugation action of G (if the action were not transitive then N would have a proper normal subgroup consisting of a direct product of the factors in a G-orbit). But $G = KN_G(P)$ by the Frattini Lemma, and so the X_i (for $1 \leq i \leq m$) are permuted transitively by $N_G(P)$. Furthermore, $P = (X_1 \cap P) \times \ldots \times (X_m \cap P)$ and the factors in this direct product are permuted transitively by $\mathrm{Aut}_G(P) = \mathrm{Aut}_{\mathcal{F}}(P)$. By assumption, strongly closed subgroup cannot be written as such a direct product where $m > 1$; hence $m = 1$ and so N is a simple group.

Now let $N_0 = C_G(N)$; note that $N_0 \trianglelefteq G$. If $N_0 > 1$ then $p \mid |N_0|$ since $O_{p'}(G) = 1$. But $[T \cap N_0, P] \leq [T \cap N_0, N] = 1$. Now, P is p-centric and so $Z(P) \in \mathrm{Syl}_p(C_G(P))$. The group $T \cap N_0$ is a p-subgroup of $C_G(P)$ and therefore $T \cap N_0 \leq P$. But $1 \neq T \cap N_0 \leq N_0 \cap P \leq N_0 \cap N \leq Z(N) = 1$, which is a contradiction. This shows that $C_G(N) = 1$ and so G is almost simple. In particular, $p \mid |N|$ since $O_{p'}(G) = 1$. ∎

We now set about showing that the fusion system $\mathcal{E}(n,p)$ satisfies the hypothesis of Theorem 6.1.2.

Lemma 6.1.3 *Let $\mathcal{E} = \mathcal{E}(n,p)$. Then S itself is the only non-trivial subgroup of S which is strongly closed in $\mathcal{E}$.*

Proof: Let $1 \neq P \triangleleft S(n,p)$ be strongly closed in $\mathcal{E}$. Since S is a p-group, P intersects non-trivially with $Z(S)$ (see, for example, [35, 5.8]). But $Z(S)$ is a cyclic group of order

p, and is therefore simple, so P contains $Z(S)$. Note that $\mathrm{Aut}_{\mathcal{E}}(A) = \mathrm{Aut}_G(A) = \Omega$, and so we may regard A as an $\mathbb{F}_p\Omega$-module. In fact, A is irreducible as an $\mathbb{F}_p\Omega$-module (see [7, p588]) and so the Γ-submodule of A generated by $Z(S) \leq A$ is just A. But P is strongly closed in $\mathcal{E}$, so it must contain every $\mathcal{E}$-conjugate of $Z(S)$, and therefore P contains the submodule generated by $Z(S)$, namely A. Also, since $\mathrm{Aut}_{\mathcal{E}}(R) = \mathrm{Aut}(R)$, we have that the element $x^n \in A$ is $\mathcal{E}$-conjugate to $(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1) \in S$, and therefore P contains the group $\langle A, (\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1)\rangle = S$. Hence $P = S$. ∎

Lemma 6.1.4 *Let $\mathcal{E} = \mathcal{E}(n, p)$. Then*

(i) S is not normal in $\mathcal{E}$; and

(ii) S does not factorize as a direct product of two or more distinct subgroups which are permuted transitively by $\mathrm{Aut}_{\mathcal{E}}(S)$.

Proof: First we show that S is not normal in $\mathcal{E}$. Note that $x^n \in Z(S)$ and $(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1) \in S \setminus Z(S)$ are generators of the elementary abelian group R, and so there is an automorphism in $\mathrm{Aut}(R) = \mathrm{Aut}_{\mathcal{E}}(R)$ such that $x^n \mapsto (\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right), 1)$. This morphism cannot extend to a morphism of S because $Z(S)$ is a characteristic subgroup of S. Therefore S is not normal in $\mathcal{E}$.

Now suppose that $S = Q_1 \times \ldots \times Q_m$ for distinct isomorphic subgroups $Q_i \leq S$. Then $Z(S) = Z(Q_1) \times \ldots \times Z(Q_2)$. Therefore $p = |Z(S)| = |Z(Q_1)|^m$. But p is prime and so $m = 1$. ∎

Corollary 6.1.5 *Let $\mathcal{E}$ be the fusion system over $S(n, p)$ described above. If $\mathcal{E}$ is the fusion system of a finite group then it is the fusion system of a finite almost simple group.*

Proof: It is clear that S is p-centric in S, so we can apply Lemma 6.1.2 to $\mathcal{E}$. This gives the desired conclusion. ∎

In fact we can reduce our problem still further. We show that if G is an almost simple group such that $\mathcal{F}_S(G) \cong \mathcal{E}(n,p)$ then $S(n,p)$ is a Sylow p-subgroup of a simple subgroup of G.

Proposition 6.1.6 *Let G be an almost simple group with $\mathrm{Inn}(K) \leq G \leq \mathrm{Aut}(K)$ where K is a non-abelian finite simple group. Suppose that G contains $S(n,p)$ as a Sylow p-subgroup and that $\mathcal{F}_S(G) \cong \mathcal{E}(n,p)$. Then $S(n,p)$ is a Sylow p-subgroup of K.*

Proof: If $x \in S \cap K$ and $x^g \in S$ for some $g \in G$ then $x^g \in S \cap K$ since $K \trianglelefteq G$. Hence $S \cap K$ is strongly closed in $\mathcal{F}_S(G) \cong \mathcal{E}(n,p)$. Therefore, by Lemma 6.1.3, $S \cap K = S$. Thus $S \leq K$. ∎

In the sections which follow, we consider every finite almost simple group G in turn and show that if G has a Sylow p-subgroup isomorphic to $S(n,p)$ for $p \geq 13$ and $n \geq 5$ then $\mathcal{F}_S(G) \ncong \mathcal{E}(n,p)$. In particular, we will have an infinite family of exotic fusion systems.

By the Classification of Finite Simple Groups, the almost simple groups fall into three classes: those which are groups of automorphisms of an abelian groups, those which are groups of automorphisms of an alternating group, those which are groups of automorphisms of a Lie type group, and those which are groups of automorphisms of a sporadic group. We shall refer to these as almost simple groups of *abelian*, *alternating*, *Lie*, or *sporadic* type respectively.

Note that we do not need to consider the simple groups of abelian type since S is not abelian.

6.2 Alternating type groups

In this section, let us suppose that G is an almost simple group with a simple normal subgroup N isomorphic to $\mathrm{Alt}(m)$ $(m \geq 5)$ and with $C_G(N) = 1$. Then G is isomorphic to a subgroup of Aut(Alt(m)). It is well-known that for $m \geq 5$ and $m \neq 6$, Aut(Alt(m)) $\cong$ Sym(n), and that Aut(Alt(6)) contains a subgroup of index 2 isomorphic to Sym(6) (see, for example, [39, 3.2.17, 3.2.19(iii)]). Since we only consider the case when p is odd, we have that the Sylow p-subgroups of G are isomorphic to those of Alt(m).

The following is a well-known result about symmetric groups. See, for example, [17], [24] or [34].

Proposition 6.2.1 *The order of a Sylow p-subgroup of* Sym(m) *is p^α, where $\alpha = [m/p] + [m/p^2] + \cdots = \sum_{i=1}^{\infty}[m/p^i]$.*

Proof: The order of a Sylow p-subgroup of Sym(m) is the highest power of p dividing $m!$. The number of positive integers divisible by p is $[m/p]$, by p^2 is $[m/p^2]$, and by p^i is $[m/p^i]$. Therefore the highest power of p dividing $m!$ is

$$\begin{aligned}
\left(\left[\frac{m}{p}\right] - \left[\frac{m}{p^2}\right]\right) + 2\left(\left[\frac{m}{p^2}\right] - \left[\frac{m}{p^3}\right]\right) + \cdots &= \sum_{i=1}^{\infty} i\left(\left[\frac{m}{p^i}\right] - \left[\frac{m}{p^{i+1}}\right]\right) \\
&= \left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \cdots \\
&= \sum_{i=1}^{\infty}\left[\frac{m}{p^i}\right].
\end{aligned}$$

∎

We shall also require the following fact about alternating groups. For a proof, we refer the reader to [22, 5.2.10].

Proposition 6.2.2 [22, 5.2.10] *If p is an odd prime, then the symmetric group* Sym(m) *has p-rank equal to $[m/p]$.*

Proposition 6.2.3 *Let T be a Sylow p-subgroup of* Sym(m)*, and suppose that T is non-abelian and has an elementary abelian subgroup of index p. Then $|T| \geq p^{p+1}$.*

Proof: Suppose that X is an elementary abelian subgroup of T of index p. Then $1 = \log_p|T : X| = \log_p|T| - \log_p|X| = (\sum_{i=1}^{\infty}[m/p^i]) - [m/p] = \sum_{i=2}^{\infty}[m/p^i]$. Therefore $[m/p^2] = 1$ since $[m/p^i] = 0$ implies that $[m/p^j] = 0$ for all $j \geq i$. But this means that $[m/p] \geq p$, and so $\log_p|T| = [m/p] + [m/p^2] \geq p+1$, as required. ∎

Corollary 6.2.4 *If G is an almost simple group which is isomorphic to a subgroup of* $\mathrm{Aut}(\mathrm{Alt}(m))$*, and G contains a Sylow p-subgroup isomorphic to $S(n,p)$ for some n, then $n = p$ and $|S| = p^{p+1}$.*

Proof: Every Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of $\mathrm{Sym}(m)$. So if $S(n,p) \in \mathrm{Syl}_p(G)$ then $S(n,p) \in \mathrm{Syl}_p(\mathrm{Sym}(m))$. The group $S(n,p)$ is non-abelian and contains an elementary abelian p-subgroup of index p, namely $A(n,p)$. Hence Proposition 6.2.3 applies and we have the desired conclusion. ∎

Lemma 6.2.5 *Let $G = \mathrm{Sym}(m)$ for some $p^2 \leq m \leq p^2+p-1$ and let X be an elementary abelian p-subgroup of G of maximal rank. Then $\mathrm{rank}_p(X) = p$ and $|Aut_G(X)| = p!(p-1)^p$.*

Proof: The fact that $\mathrm{rank}_p(X) = p$ follows immediately from 6.2.2. We may assume, without loss of generality, that $X = \langle (1\ 2\ \ldots\ p), (p+1\ \ldots\ 2p), \ldots, (p^2-p+1\ \ldots\ p^2)\rangle$.

Consider $C_G(\langle (1\ 2\ \ldots\ p)\rangle)$. It is easy to see that this is equal to the direct product $\langle (1\ 2\ \ldots\ p)\rangle \times \mathrm{Sym}(\{p+1, \ldots, m\})$, and similarly for the other p-cycles. Therefore

$$\begin{aligned} C_G(X) &= C_G(\langle (1\ 2\ \ldots\ p)\rangle) \cap \ldots \cap C_G(\langle (p^2-p+1\ \ldots\ p^2)\rangle) \\ &= \langle (1\ 2\ \ldots\ p)\rangle \times \ldots \times \langle (p^2-p+1\ \ldots\ p^2)\rangle \times \mathrm{Sym}(\{p^2+1, \ldots, m\}). \end{aligned}$$

Hence $|C_G(X)| = p^p|\mathrm{Sym}(\{p^2+1, \ldots, m\})|$.

Now let us consider $N_G(X)$. Let $\overline{G} = \mathrm{Sym}(\{1, \ldots, p^2\}) \leq \mathrm{Sym}(m)$. Then $N_G(X) = N_{\overline{G}}(X) \times \mathrm{Sym}(\{p^2+1, \ldots, m\})$. Note that $N_{\overline{G}}(X)$ is imprimitive with system of imprimitivity given by

$$\{\{1, \ldots, p\}, \ldots, \{p^2 - p + 1, \ldots, p^2\}\}.$$

Hence $N_{\overline{G}}(X)$ is a wreath product where the base group is a direct product of normalizers of p-cycles.

Consider the normalizer N in $\overline{G}$ of the p-cycle $(1 \ \ldots \ p)$. If $g \in N$ then $1^g = a$ where $1 \leq a \leq p$, and $2^g \equiv a + b(\text{mod } p)$, where $1 \leq b \leq p - 1$. The images of 1 and 2 uniquely determine g. This means that $|N| = p(p - 1)$.

We have that $N_{\overline{G}}(X) \cong N \wr \mathrm{Sym}(p)$ and so $|N_{\overline{G}}(X)| = |N||\mathrm{Sym}(p)| = p^p(p-1)^p p!$. Hence $|N_G(X)| = p^p(p-1)^p p! |\mathrm{Sym}(\{1, \ldots, p^2\})|$. Thus $|\mathrm{Aut}_G(X)| = |N_G(X)|/|C_G(X)| = p!\,(p-1)^p$. ∎

Proposition 6.2.6 *Let G be an almost simple group which is isomorphic to a subgroup of* $\mathrm{Aut}(\mathrm{Alt}(m))$*, and suppose that G has a Sylow p-subgroup isomorphic to $S(n, p)$, where $p \geq 13$. Then $\mathcal{F}_S(G)$ is not isomorphic to $\mathcal{E}$.*

Proof: Note that either $G \cong \mathrm{Alt}(m)$, $G \cong \mathrm{Sym}(m)$ or $m = 6$ and $G \cong \mathrm{Aut}(\mathrm{Alt}(m))$ contains a subgroup of index 2 isomorphic to $\mathrm{Sym}(m)$. Now, since $S(n, p) \in \mathrm{Syl}_p(G)$, we have that $A(n, p)$ is an elementary abelian p-subgroup of G of maximal rank. Therefore by 6.2.5, $|\mathrm{Aut}_{\mathrm{Sym}(m)}(A)| = p!\,(p-1)^p$. Note that $\mathrm{Aut}_G(A) \geq \mathrm{Aut}_{\mathrm{Alt}(m)}(A)$. Hence $|\mathrm{Aut}_G(A)|$ is divisible by $\frac{1}{2}p!\,(p-1)^p$. We have that $|\mathrm{Aut}_{\mathcal{E}}(A)| = |\mathrm{GL}_2(p)| = (p-1)^2 p(p+1)$.

Note that, since $p \geq 13$,

$$(p-1)!\,(p-1)^{p-2} - 2(p+1) \geq 2(p-1)^2 - 2(p+1) = 2p(p-3) > 0;$$

a routine argument now shows that $\frac{1}{2}p!\,(p-1)^p > (p-1)^2 p(p+1)$ for $p \geq 13$. In particular we have that $(p-1)^2 p(p+1)$ is not divisible by $\frac{1}{2}p!\,(p-1)^p$. Hence $\mathcal{E}(n, p)$ is not isomorphic to $\mathcal{F}_S(G)$. ∎

Corollary 6.2.7 *The fusion system $\mathcal{E}(n,p)$ does not appear as a fusion system in any almost simple group of alternating type for $p \geq 13$ and for any n.*

6.3 Lie type groups in defining characteristic

Let G be a finite simple Chevalley group or a finite simple twisted Lie type group defined over a field of characteristic p, where p is an odd prime. We shall show that G does not contain a Sylow p-subgroup isomorphic to $S(n,p)$ for any $n \geq 5$. It will then follow by Theorem 6.1.6 that no almost simple group of Lie type which is defined over a field of characteristic p gives rise to a fusion system isomorphic to $\mathcal{E}(n,p)$.

Proposition 6.3.1 *Let p be an odd prime, and let $q = p^a$ for some a. Let G be a finite simple Chevalley group or a finite simple twisted Lie type group defined over a field of order q and let U be a Sylow p-subgroup of G. Suppose that U is non-abelian and let X be an elementary abelian subgroup of U with maximal possible rank in U. Then* $\mathrm{rank}(X) > 3$ *implies that $|U : X| > p$.*

Proof: The orders of the Sylow p-subgroups and the p-ranks of the finite simple Lie type groups are well-known and can be found in [22, Tables 2.2 and 3.3.1] or [30, Table 13.17, 13.19]. For convenience, we reproduce Tables 13.17 and 13.19 from [30].

Type of G		$\log_q\lvert U\rvert$	$\log_q\lvert X\rvert$	$\log_q\lvert U\rvert - \log_q\lvert X\rvert$
$\mathrm{A}_l(q)$	$l \geq 1$, odd	$\binom{l+1}{2}$	$(l+1)^2/4$	$\frac{1}{4}(l^2-1)$
	$l \geq 1$, even	$\binom{l+1}{2}$	$l(l+2)/4$	$\frac{1}{4}l^2$
$\mathrm{B}_l(q)$	$l \geq 4$, q odd	l^2	$1 + l(l-1)/2$	$\frac{1}{2}l(l-1)-1$
	$l \leq 3$, q odd	l^2	$2l-1$	l^2-2l+1
$\mathrm{C}_l(q)$	$l \geq 3$	l^2	$l(l+1)/2$	$\frac{1}{2}l(l-1)$
$\mathrm{D}_l(q)$	$l \geq 4$	$l(l-1)$	$l(l-1)/2$	$\frac{1}{2}l(l-1)$
$\mathrm{E}_6(q)$		36	16	20
$\mathrm{E}_7(q)$		63	27	36
$\mathrm{E}_8(q)$		120	36	84
$\mathrm{F}_4(q)$	q odd	24	9	15
$\mathrm{G}_2(q)$	$p \leq 3$	6	3	3
	$p = 3$	6	4	2
${}^2\mathrm{A}_l(q)$	l odd	$\binom{l+1}{2}$	$(l+1)^2/4$	$\frac{1}{4}(l^2-1)$
	l even	$\binom{l+1}{2}$	$l^2/4+1$	$\frac{1}{4}l^2+l-1$
${}^2\mathrm{D}_l(q)$	$l \geq 4$	$l(l-1)$	$(l-1)(l-2)/2+2$	$\frac{1}{2}l(l+1)-3$
${}^2\mathrm{E}_6(q)$		36	12	24
${}^3\mathrm{D}_4(q)$		12	5	7
${}^2\mathrm{G}_2(3^{2m+1})$		3	2	1

Let us suppose that $\mathrm{rank}(X) > 3$ and that $|U : X| > p$. By considering each of the cases listed in the above table in turn, we shall show that either $\log_p|U : X| = \log_p|U| - \log_p|X| > 1$ or that $\mathrm{rank}_p(X) = \log_p|X| \leq 3$. If the former conclusion holds, then we have contradicted the assumption that $|U : X| > p$, and if the latter holds then we have contradicted the assumption that $\mathrm{rank}(X) > 3$.

If $G \cong \mathrm{A}_l(q)$ where $l = 1$ then $\log_q|U| - \log_q|X| = \frac{1}{4}(1-1) = 0$ and so U is abelian, in contradiction to our assumptions. If $l = 2$ and $q = p$ then $\log_q|X| = \log_p|X| = 2 \leq 3$. If $l = 2$ and $q = p^a$ for $a > 1$ then $\log_q|U| - \log_q|X| = 1$ and so $\log_p|U| - \log_p|X| > 1$ as required. If l is odd and $l \geq 3$ then $\log_q|U| - \log_q|X| \geq 2 > 1$, as required. If l is even and $l \geq 4$ then $\log_q|U| - \log_q|X| \geq 4 > 1$.

Suppose $G \cong \mathrm{B}_l(q)$, and let $l \geq 4$. Then $\log_q|U| - \log_q|X| \geq 5 > 1$ and if $l = 3$ then $\log_q|U| - \log_q|X| = 4 > 1$ as required. If $l \leq 2$ and $q = p$ then $\log_q|X| = \log_p|X| \leq 3$, and if $l \leq 2$ and $q = p^a$ where $a > 1$ then $\log_q|U| - \log_q|X| = 1$ and so $\log_p|U| - \log_p|X| > 1$.

Now assume that $G \cong \mathrm{C}_l(q)$. Since $l \geq 3$ we have that $\log_q|U| - \log_q|X| \geq 3 > 1$, and so $\log_p|U| - \log_p|X| > 1$ as required.

So suppose that $G \cong \mathrm{D}_l(q)$. Since $l \geq 4$ we have that $\log_q|U| - \log_q|X| \geq 3 > 1$ as required.

Now suppose that $G \cong^2 \mathrm{A}_l(q)$. If $l = 1$ then $\log_q|U| - \log_q|X| = 0$ and so U is abelian, in contradiction to our assumptions. If l is odd and $l \geq 3$ then $\log_q|U| - \log_q|X| \geq 2$ and so $\log_p|U| - \log_p|X| > 1$. If l is even then $\log_q|U| - \log_q|X| = \frac{1}{4}l^2 + l - 1 \geq \frac{1}{4}2^2 + 2 - 1 = 2$ (since $\frac{1}{4}l^2 + l - 1$ is an increasing function of l for $l \geq 2$).

Now let $G \cong^2 \mathrm{D}_l(q)$. Since $l \geq 4$ we have $\log_q|U| - \log_q|X| \geq 7$.

Consider the case when $G \cong^2 \mathrm{G}_2(3^{2m+1})$. If $q = p$ then $\log_q|X| = \log_p|X| = 2 \leq 3$, and if $q = p^a$ where $a > 1$ then $\log_q|U| - \log_q|X| = 1$ and so $\log_p|U| - \log_p|X| > 1$.

It is clear from the table that the finite simple Chevalley groups of exceptional types and the simple groups of twisted Lie types ${}^2\mathrm{E}_6(q)$ and ${}^3\mathrm{D}_4(q)$ satisfy the conclusion. ∎

Theorem 6.3.2 *No almost simple group of Lie type which is defined over a field of characteristic p gives rise to a fusion system isomorphic to $\mathcal{E}(n,p)$ for $n \geq 5$ and $p \geq 13$.*

Proof: Let G be an almost simple group of Lie type, with G a group of automorphisms of the group K, where K is a finite simple Chevalley group or a simple twisted group of Lie type defined over a finite field of characteristic p. Suppose that G has a Sylow p-subgroup isomorphic to $S(n,p)$ and that $\mathcal{F}_S(G) \cong \mathcal{E}(n,p)$. Then by Theorem 6.1.6, K has a Sylow p-subgroup isomorphic to $S(n,p)$. However, $S(n,p)$ is non-abelian and contains an elementary abelian subgroup A, of rank $n+1 \geq 6$, at index p. But by Proposition 6.3.1, K has no such Sylow p-subgroup; this is a contradiction. ∎

6.4 Exceptional groups in non-defining characteristic

We now turn our attention to the case when G is an almost simple group of Lie type defined over a field of characteristic $r \neq p$; we aim to show that if G contains a Sylow p-subgroup isomorphic to $S(n,p)$ then $\mathcal{F}_S(G) \ncong \mathcal{E}(n,p)$.

In this section we shall consider the subcase when the almost simple group G is a subgroup of $\mathrm{Aut}(K)$, where K is a finite Chevalley group of exceptional type (*i.e.* of type E_6, E_7, E_8, F_4 or G_2), or a twisted simple group of type ${}^2\mathrm{B}_2$, ${}^3\mathrm{D}_4$, ${}^2\mathrm{E}_6$, ${}^2\mathrm{F}_4$ or ${}^2\mathrm{G}_2$ defined over a finite field of characteristic $r \neq p$. We refer to all these groups K as *simple groups of exceptional Lie type*. We shall show that none of these groups has a Sylow p-subgroup isomorphic to $S(n,p)$. First we require the following well-known fact from number theory:

Lemma 6.4.1 [26, Lemma 5] *Let p be an odd prime, let $r \neq p$ be a prime and let q be a power of r. Let k be the multiplicative order of q modulo p, that is, k is the smallest integer m such that $q^m \equiv 1$ mod p. Then p divides the cyclotomic polynomial $\Phi_n(q)$ if and only if $n = kp^i$ for some $i \geq 0$.*

Now fix a finite simple group K of exceptional Lie type, whose associated Dynkin diagram has m nodes and where the root system of K has N positive roots. By [22, Theorem 2.2.9], the order of the universal central extension of K can be written as

$$q^N \prod_{i=0}^{m} \Phi_i(q)^{r_i},$$

where the r_i are non-negative integers. The values of the r_i are given explicitly in [21, Tables 10.1, 10.2]; we reproduce Table 10.2 here, for convenience. It shows the cyclotomic polynomial factors of the orders of the groups of exceptional Lie type:

Type of K	$\prod_i \Phi_i^{r_i}$
${}^2\mathrm{B}_2$	$\Phi_1\Phi_4$
${}^3\mathrm{D}_4$	$\Phi_1^2\Phi_2^2\Phi_3^2\Phi_6^2\Phi_{12}$
G_2	$\Phi_1^2\Phi_2^2\Phi_3\Phi_6$
${}^2\mathrm{G}_2$	$\Phi_1\Phi_2\Phi_6$
F_4	$\Phi_1^4\Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$
${}^2\mathrm{F}_4$	$\Phi_1^2\Phi_2^2\Phi_4^2\Phi_6\Phi_{12}$
E_6	$\Phi_1^6\Phi_2^4\Phi_3^3\Phi_4^2\Phi_5\Phi_6^2\Phi_8\Phi_9\Phi_{12}$
${}^2\mathrm{E}_6$	$\Phi_1^4\Phi_2^6\Phi_3^2\Phi_4^2\Phi_6^3\Phi_8\Phi_{10}\Phi_{12}\Phi_{18}$
E_7	$\Phi_1^7\Phi_2^2\Phi_3^3\Phi_4^2\Phi_5\Phi_6^3\Phi_7\Phi_8\Phi_9\Phi_{10}\Phi_{12}\Phi_{14}\Phi_{18}$
E_8	$\Phi_1^8\Phi_2^8\Phi_3^4\Phi_4^4\Phi_5^2\Phi_6^4\Phi_7\Phi_8^2\Phi_9\Phi_{10}^2\Phi_{12}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$

Theorem 6.4.2 [22, Theorem 4.10.2] *Let G be a finite group of Lie type, defined over $\mathbb{F}_q$, and let $p \geq 13$ be a prime with $p \nmid q$. Denote by k the multiplicative order of q modulo p. Let P be a Sylow p-subgroup of G. Then there exists a subgroup P_T of P such that P_T has the following properties:*

(i) P_T is an abelian normal subgroup of P;

(ii) P_T is homocyclic of exponent $|\Phi_k(q)|_p$ and rank r_k;

(iii) P_T has index p^b in P, where $b = r_{pk} + r_{p^2k} + \cdots$.

Since the p-rank of K is r_k by [22, Theorem 4.10.3], we have that the subgroup P_T described above contains an elementary abelian subgroup of K of maximal rank.

Proposition 6.4.3 *Let K be a simple group of exceptional Lie type defined over $\mathbb{F}_q$. Let $p \geq 11$ with $p \nmid q$. Then K has abelian Sylow p-subgroups. In particular, K does not contain a Sylow p-subgroup isomorphic to $S(n,p)$ for any $n \geq 2$.*

Proof: Let $P \in \mathrm{Syl}_p(K)$. Then by 6.4.2, P contains an abelian subgroup P_T with $\log_p|P : P_T| = r_{pk} + r_{p^2k} + \cdots$. But by consulting Table 6.4 we see that every group of exceptional Lie type has $r_{p^ik} = 0$ for all $i > 0$ and all $p \geq 11$. Hence $\log_p|P : P_T| = 0$ and so $P = P_T$. In particular, P is abelian and therefore not isomorphic to $S(n,p)$. ∎

Theorem 6.4.4 *No almost simple group of exceptional Lie type which is defined over a field of characteristic $q \neq p$ gives rise to a fusion system isomorphic to $\mathcal{E}(n,p)$ for $n \geq 2$ and $p \geq 11$.*

Proof: Let G be an almost simple group of exceptional Lie type which is a group of automorphisms of the group K, where K is a simple group of exceptional Lie type. Suppose that G has a Sylow p-subgroup isomorphic to $S(n,p)$ for some $n \geq 2$ and $p \geq 11$. Then by Theorem 6.1.6, K has a Sylow p-subgroup isomorphic to $S(n,p)$. But this is impossible by Proposition 6.4.3. Therefore G cannot give rise to a fusion system isomorphic to $\mathcal{E}(n,p)$. ∎

Note that in particular, the theorem holds for $n \geq 5$ and $p \geq 13$.

6.5 Classical groups in non-defining characteristic

In this section we shall consider the case that G is an almost simple group of Lie type which is a group of automorphisms of a simple classical group defined over a field of characteristic $q \neq p$.

We start by recalling some facts from the theory of spaces with forms. As a general reference, see [2, Chapter 7].

Lemma 6.5.1 **[Witt's Lemma]** *Let V be a linear, orthogonal, symplectic or unitary space, and let U, W be subspaces of V. Then any isometry $\phi : U \to W$ extends to an isometry $\widehat{\phi} : V \to V$.*

Proof: See [2, Section 20]. ∎

Let V be a 2-dimensional orthogonal, symplectic or unitary $\mathbb{F}_q$-space. Recall that V is called a *hyperbolic plane* if V has a basis $\{e, f\}$ such that $(e, e) = (f, f) = 0$ (and if V is an orthogonal space with quadratic form Q then $Q(e) = Q(f) = 0$) and $(e, f) = 1$. By [2, 19.13], there is, up to isometry, a unique hyperbolic plane for each type of space. The space V is called *definite* if there are no non-trivial *singular* vectors; that is, no vectors $v \in V$ such that $(v, v) = 0$ and $Q(v) = 0$ when V is an orthogonal space. As a consequence of Witt's Lemma we have the following.

Lemma 6.5.2 [2, 20.8.2] *Let V be an orthogonal, symplectic or unitary space. Then V is the orthogonal direct sum of hyperbolic planes and at most one definite plane. Moreover, this decomposition is unique up to an isometry of V.*

The plan in this section is to use Witt's Lemma to show that classical groups contain elementary abelian p-groups which contain a group isomorphic to an alternating group in their automizer. We then show that this does not occur in the fusion systems $\mathcal{E}(n, p)$, and hence that the classical groups cannot give rise to these fusion systems.

Firstly we consider the linear and unitary groups.

Lemma 6.5.3 *Let (V, f) be a linear or unitary space of dimension l over $\mathbb{F}_q$ where $l \geq k$. Let G_k denote the isomorphism type of the isometry group of a k-dimensional linear (respectively unitary) space over $\mathbb{F}_q$. Then* $\mathrm{Isom}(V)$ *contains a subgroup isomorphic to a wreath product $G_k \wr \mathrm{Sym}([l/k])$.*

Proof: By [2, 21.6], V admits an orthonormal basis $\{v_1, \ldots, v_l\}$. Now define subspaces $U_1 = \{v_1, \ldots, v_k\}$, $U_2 = \{v_{k+1}, \ldots, v_{2k}\}$, $\ldots$, $U_{[l/k]} = \{v_{[l/k]k-(k-1)}, \ldots, v_{[l/k]k}\}$. Each U_i is a k-dimensional non-degenerate subspace of V, for every $1 \leq i \leq [l/k]$. The U_i are clearly all isometric spaces and so $\mathrm{Isom}(U_i) \cong G_k$ for every i.

Given an isometry $\phi \in \mathrm{Isom}(U_i)$ for some i, define a linear map $\widehat{\phi}$ of V by setting $\widehat{\phi}(v_j) = v_j$ for $j \notin \{(i-1)k + 1, \ldots, ik\}$, and $\widehat{\phi}(v_j) = \phi(v_j)$ for $j \in \{(i-1)k + 1, \ldots, ik\}$.

Then $\widehat{\phi}$ is an isometry since $\{v_1, \ldots, v_l\}$ is an orthonormal basis. It is also easy to see that for $i \neq j$, isometries of U_i and U_j extended in this way will commute as isometries of V. Hence $\mathrm{Isom}(V)$ contains a subgroup isomorphic to a direct product $G_k \times \ldots \times G_k$ of $[l/k]$ copies of G_k.

By Witt's Lemma, there is a subgroup of $\mathrm{Isom}(V)$ isomorphic to $\mathrm{Sym}([l/k])$ which permutes the $[l/k]$ subspaces U_i. Thus $\mathrm{Isom}(V)$ contains the wreath product $G_k \wr \mathrm{Sym}([l/k])$. ∎

Next we consider the symplectic groups.

Lemma 6.5.4 *Let (V, f) be a $2l$-dimensional symplectic space over $\mathbb{F}_q$, where $l \geq k$. Let G_k denote the isomorphism type of the isometry group of a $2k$-dimensional symplectic space over $\mathbb{F}_q$. Then* $\mathrm{Isom}(V)$ *contains a subgroup isomorphic to a wreath product $G_k \wr$* $\mathrm{Sym}([l/k])$*.*

Proof: By [2, 19.16], V admits a hyperbolic basis $\{e_1, \ldots, e_l, f_1, \ldots, f_l\}$. Now define subspaces $U_1 = \{e_1, \ldots, e_k, f_1, \ldots, f_k\}$, $U_2 = \{e_{k+1}, \ldots, e_{2k}, f_{k+1}, \ldots, f_{2k}\}$, $\ldots$, $U_{[l/k]} = \{e_{([l/k]-1)k+1}, \ldots, e_{[l/k]k}, f_{([l/k]-1)+1}, \ldots, f_{[l/k]k}\}$. Each of the U_i is a $2k$-dimensional non-degenerate hyperbolic subspace of V. The U_i are clearly all isometric and so $\mathrm{Isom}(U_i) \cong G_k$ for each $1 \leq i \leq [l/k]$.

Given an isometry $\phi \in \mathrm{Isom}(U_i)$ for some i, define a linear map $\widehat{\phi}$ by setting $\widehat{\phi}(e_j) = e_j$ and $\widehat{\phi}(f_j) = f_j$ for $j \notin \{(i-1)k+1, \ldots, ik\}$, and $\widehat{\phi}(e_j) = \phi(e_j)$ and $\widehat{\phi}(f_j) = \phi(f_j)$ for $j \in \{(i-1)k+1, \ldots, ik\}$. Then $\widehat{\phi}$ is an isometry since $\{e_1, \ldots, e_l, f_1, \ldots, f_l\}$ is a hyperbolic basis of V. It is also easy to see that for $i \neq j$, isometries of U_i and U_j extended in this way will commute as isometries of V. Hence $\mathrm{Isom}(V)$ contains a subgroup isomorphic to a direct product $G_k \times \ldots \times G_k$ of $[l/k]$ copies of G_k.

By Witt's Lemma, there is a subgroup of $\mathrm{Isom}(V)$ isomorphic to $\mathrm{Sym}([l/k])$ which permutes the $[l/k]$ subspaces U_i. Thus $\mathrm{Isom}(V)$ contains the wreath product $G_k \wr \mathrm{Sym}([l/k])$.

$\blacksquare$

Before tackling the orthogonal groups, we recall some more facts about orthogonal spaces.

Lemma 6.5.5 [2, 21.1] *There is, up to isometry, a unique* 2*-dimensional orthogonal definite space over* $\mathbb{F}_q$*.*

Let us define some notation. Given subspaces A and B of an orthogonal space, write $A \perp B$ for the orthogonal direct sum of A and B. We also write A^i for the orthogonal direct sum of i isometric copies of the space A. In this section we shall also write $A \cong B$ to mean that A and B are isometric as orthogonal spaces. It should be clear from the context whether this symbol denotes an isometry of orthogonal spaces or an isomorphism of groups.

Lemma 6.5.6 [2, 21.6] *Let H denote the isometry type of the orthogonal hyperbolic plane and let D denote the isometry type of the unique* 2*-dimensional orthogonal definite space. Let V be an l-dimensional orthogonal space over* $\mathbb{F}_q$*.*

(i) If l is odd then q is odd and V is uniquely determined up to isometry.

(ii) If l is even then V is isometric to exactly one of $H^{(n/2)}$ or $H^{(n/2)-1} \perp D$.

If l is even, then if $V \cong H^{(n/2)}$ we say that V is of $+$*-type* and if $V \cong H^{(n/2)-1} \perp D$ then we say that V is of $-$*-type*. If l is odd we say that V is of 0*-type*.

Lemma 6.5.7 *Let V^+ be a $+$-type space and let V^- be a $-$-type space. Then*

(i) $V^+ \perp V^+$ is of $+$-type;

(ii) $V^+ \perp V^-$ is of $-$-type;

(iii) $V^- \perp V^-$ *is of* $+$*-type.*

Proof: This is clear from the definitions of $+$- and $-$-type, and the fact that $D^2 \cong H^2$ (for a proof of this fact see [2, 21.2]). ∎

Proposition 6.5.8 *Let* V *be an* l*-dimensional orthogonal space over* $\mathbb{F}_q$ *of* ϵ*-type, where* $\epsilon \in \{-, 0, +\}$*. Let* W *be a subspace of* V *of dimension* $k \leq l/2$*. Then* $\mathrm{Isom}(V)$ *contains a subgroup of the form* $\mathrm{Isom}(W) \wr \mathrm{Sym}([l/k] - 1)$*.*

Proof: By Witt's Lemma, the theorem follows if we can show that V contains a subspace isometric to the orthogonal direct sum of $[l/k] - 1$ isometric copies of W.

Let $\eta \in \{-, 0, +\}$ denote the type of W. There are 9 possible combinations of the types ϵ and η; we shall label each of these possibilities by the ordered pair (η, ϵ). Thus case $(+, -)$ shall refer to the case when W is of $+$-type and V is of $-$-type.

First we consider the cases when W is of $+$-type; that is, the cases $(+, *)$. Note that k is even and $k \geq 2$. We have $W \cong H^{k/2}$ and so $Y := W^{[l/k]-1} \cong H^{(k/2)([l/k]-1)}$. But $(k/2)([l/k] - 1) \leq l/2 - k/2 \leq l/2 - 1$. Now, in case $(+, +)$ $V \cong H^{l/2}$, in case $(+, -)$ $V \cong H^{l/2-1} \perp D$ and in case $(+, 0)$ $V \cong H^{(l-1)/2} \perp L$ where L is a 1-dimensional space. It is therefore clear that in all of these cases, V contains a subspace isometric to Y.

Now consider the cases when W is of $-$-type; that is, the cases $(-, *)$. Again we have that k is even. We have that $W \cong H^{k/2-1} \perp D$ and so $Y := W^{[l/k]-1} \cong H^{(k/2-1)([l/k]-1)} \perp D^{[l/k]-1}$. Since $D^2 \cong H^2$, we have another bifurcation of cases; depending on whether $[l/k]$ is odd or even. If $[l/k]$ is odd then $D^{[l/k]-1} \cong H^{(1/2)([l/k]-1)}$ and if $[l/k]$ is even then $D^{[l/k]-1} \cong H^{(1/2)([l/k]-2)} \perp D$. We label these extra cases as $(-, *, +)$ and $(-, *, -)$ respectively. This notation refers to the fact that if $[l/k]$ is odd then Y is a $+$-type space, and if $[l/k]$ is even then Y is a $-$-type space. Let us count the total number of isometric copies of H and D there are in Y. If $[l/k]$ is odd we have

$$\left(\frac{k}{2}-1\right)\left(\left[\frac{l}{k}\right]-1\right)+\frac{1}{2}\left(\left[\frac{l}{k}\right]-1\right)=\left(\frac{k-1}{2}\right)\left(\left[\frac{l}{k}\right]-1\right)$$

copies of H and no copies of D. If $[l/k]$ is even then we have

$$\left(\frac{k}{2}-1\right)\left(\left[\frac{l}{k}\right]-1\right)+\frac{1}{2}\left(\left[\frac{l}{k}\right]-2\right)=\left(\frac{k-1}{2}\right)\left(\left[\frac{l}{k}\right]-1\right)-\frac{1}{2}$$

copies of H and 1 copy of D. Now let us check that V contains enough copies of H to contain Y. In each of the cases $(-,+)$, $(-,-)$ and $(-,0)$, V contains at least $l/2-1$ copies of H. But

$$\begin{aligned}\left(\frac{k-1}{2}\right)\left(\left[\frac{l}{k}\right]-1\right) &\leq \left(\frac{k-1}{2}\right)\left(\frac{l}{k}-1\right)=\frac{l}{2}-\frac{k}{2}-\frac{1}{2}\left(\frac{l}{k}-1\right)\\ &\leq \frac{l}{2}-\frac{k}{2}\\ &\leq \frac{l}{2}-1,\end{aligned}$$

(since $l/k-1\geq 2-1=1$) and so V does indeed contain enough copies of H to contain Y. In particular, in the cases $(-,*,+)$ and $(-,-,-)$, we have that V contains Y. So suppose that $[l/k]$ is even and that we are in one of the cases $(-,+,-)$ or $(-,0,-)$. Thus Y is a $-$-type space and so contains a term isometric to D. We have that $V=Y\perp Y^{\perp}$. Note that

$$\begin{aligned}\dim(V)-\dim(Y) &= l-(k-1)([l/k]-1)\\ &\geq l-(k-1)(l/k-1)=k+l/k-1\\ &\geq k\geq 2.\end{aligned}$$

Thus $Y\perp D$ is a $+$-type space of dimension less than or equal to l. Hence in case

$(-,+,-)$, V contains a subspace isometric to Y. Also note that in case $(-,0,-)$, $\dim(V)$ is odd, and so $\dim(V)-\dim(Y)$ is odd. So in this case we have $\dim(V)-\dim(Y)\geq 3$. Thus $Y\perp D\perp L$ (where L is a 1-dimensional space) is a 0-type space of dimension less than or equal to l. Hence in case $(-,0,-)$, V contains a subspace isometric to Y.

Now we consider the cases when W is odd-dimensional; that is, the cases $(0,*)$. We have $W\cong H^{(k-1)/2}\perp L$ where L is a 1-dimensional space, and so $Y:=W^{[l/k]-1}\cong H^{\left(\frac{k-1}{2}\right)\left(\left[\frac{l}{k}\right]-1\right)}\perp L^{[l/k]-1}$. We have a further splitting of cases: Y can be either of $+$-type, of $-$-type, or odd dimensional. We shall label these extra cases as $(0,*,+)$, $(0,*,-)$ and $(0,*,0)$ respectively. In all of these cases, the maximum number of mutually orthogonal copies of H in Y is given by:

$$\begin{aligned}
\left(\frac{k-1}{2}\right)\left(\left[\frac{l}{k}\right]-1\right)+\frac{1}{2}\left(\left[\frac{l}{k}\right]-1\right) &\leq \frac{k}{2}\left(\frac{l}{k}-1\right)\\
&= \frac{l}{2}-\frac{k}{2}\\
&\leq \frac{l}{2}-1.
\end{aligned}$$

Hence in the cases $(0,*,+)$, $(0,-,-)$ and $(0,0,0)$ we are done. The remaining cases are $(0,+,-)$, $(0,0,-)$, $(0,+,0)$ and $(0,-,0)$. In all of these cases, note that

$$\begin{aligned}
\dim(V)-\dim(Y) &= l-(k-1)([l/k]-1)-([l/k]-1)\\
&= l-k([l/k]-1)\\
&\geq l-(l-k)=k\geq 1.
\end{aligned}$$

Consider the case $(0,+,-)$. Then $\dim(V)-\dim(Y)$ is even and so $\dim(V)-\dim(Y)\geq 2$, therefore $Y\perp D$ is a $+$-type space of dimension less than or equal to l, and so we are done. In the case $(0,0,-)$ we have that $\dim(V)-\dim(Y)\geq 1$ and so $Y\perp L$ (where L is some 1-dimensional space) is an odd dimensional space of dimension less than or equal to l, and so again we are done. It is easy to see in the cases $(0,+,0)$ and $(0,-,0)$ that V contains a subspace isometric to Y.

We have thus shown that every orthogonal space V of dimension l with a subspace W of dimension $k < l/2$ contains a subspace isometric to $W^{[l/k]-1}$. As noted, the proposition now follows from Witt's Lemma. ∎

Lemma 6.5.9 *Let p be an odd prime and q any prime power with $p \nmid q$. Let $k = \mathrm{ord}_p(q)$. Let G be a simple classical group of dimension l defined over $\mathbb{F}_q$. We have the following:*

(i) if G is a linear group, then $p \mid |G|$ if and only if $k \leq l$;

(ii) if G is symplectic, unitary or orthogonal then $p \mid |G|$ if and only if $k \leq l/2$.

Proof: This follows fairly easily after looking at the formulas for the orders of the classical groups, as may be found in [13, Chapter 1]. ∎

Lemma 6.5.10 *Let $p \geq 13$ and q be any prime power with $p \nmid q$. Let $k = \mathrm{ord}_p(q)$. Let G be the isometry group of an l-dimensional linear, orthogonal or unitary space V over $\mathbb{F}_q$, or the isometry group of a $2l$-dimensional symplectic space U over $\mathbb{F}_q$. If $p \mid |G|$ then G contains an elementary abelian p-group T, with $\mathrm{rank}_p(T) = [l/k]-1$, such that G contains a subgroup of the form $T \rtimes \mathrm{Sym}([l/k]-1)$.*

Proof: Let G_k denote the isomorphism type of the isometry group of a k-dimensional subspace of V, or of the isometry group of a $2k$-dimensional subspace of U. Since $p \mid |G|$ and $p > 11$, Lemma 6.5.9 shows that $k \leq l$ if V is a linear space and $k \leq l/2$ otherwise. Hence by 6.5.3, 6.5.4 and 6.5.8, G contains a subgroup $X \cong G_k \wr \mathrm{Sym}([l/k]-1)$. Now by [42], G_k has a non-trivial cyclic Sylow p-subgroup. Hence the base group of X has a Sylow p-subgroup U which is a direct product of $[l/k]-1$ cyclic p-groups. Now by the Frattini Lemma, $N_X(U)$ contains a subgroup isomorphic to $\mathrm{Sym}([l/k]-1)$ which acts on U. Now, in each direct summand of U there is a unique subgroup of order p. Let

T be the elementary abelian group obtained by taking the direct product of all of these. Since subgroups of cyclic groups are uniquely determined by their order, the subgroup isomorphic to $\mathrm{Sym}([l/k]-1)$ must permute the direct summands of T. Hence $N_X(T)$ contains a subgroup isomorphic to $\mathrm{Sym}([l/k]-1)$ and therefore G contains a subgroup of the form $T \wr \mathrm{Sym}([l/k]-1)$. ∎

In the following lemma, let $T \leq G$ be the elementary abelian subgroup described in Lemma 6.5.10. Let $m = [l/k]-1$. We shall consider T as an $\mathbb{F}_q\mathrm{Sym}(m)$-module.

Lemma 6.5.11 *Suppose that $m \geq 4$, and let T be the $\mathrm{Sym}(m)$-module described above. Choose a basis $\{t_1, \ldots, t_m\}$ for T. Then:*

(i) $C_T(\mathrm{Sym}(m)) = \langle t_1 + \cdots + t_m \rangle$;

(ii) $[T, \mathrm{Sym}(m)] = \langle t_i - t_j | i, j \in \{1, \ldots, m\} \rangle$.

(iii) $C_T(\mathrm{Alt}(m)) = C_T(\mathrm{Sym}(m))$.

In particular, $\dim(C_T(\mathrm{Sym}(m))) = 1$ *and* $\dim([T, \mathrm{Sym}(m)]) = m-1$.

Proof: Suppose that $t = \lambda_1 t_1 + \cdots + \lambda_m t_m \in C_T(\mathrm{Sym}(m))$. Fix $i, j, \in \{1, \ldots, m\}$ and let $\sigma = (i\ j) \in \mathrm{Sym}(m)$. Then $t = t^\sigma = \lambda_1 t_1 + \cdots + \lambda_j t_i + \cdots + \lambda_i t_j + \cdots + \lambda_m t_m$. In particular, we have that $\lambda_i = \lambda_j$. Since this holds for all choices of i, j, we have $\lambda_1 = \ldots = \lambda_m$.

This shows that $C_T(\mathrm{Sym}(m)) = \langle t_1 + \cdots + t_m \rangle$; it is therefore 1-dimensional. This proves (i).

Now let $i \in \{1, \ldots, m\}$ and let $\sigma \in \mathrm{Sym}(m)$. Suppose that $i^\sigma = j$ Then $[t_j, \sigma] = t_j^\sigma - t_j = t_i - t_j$. Hence $\langle t_i - t_j | i, j \in \{1, \ldots, m\} \rangle = [T, \mathrm{Sym}(m)]$.

Note that $\{t_1 - t_m, t_2 - t_m, \ldots, t_{m-1} - t_m\}$ is a basis for $[T, \mathrm{Sym}(m)]$; hence $\dim([T, \mathrm{Sym}(m)]) = m-1$.

To see that $C_T(\mathrm{Alt}(m)) = C_T(\mathrm{Sym}(m))$, simply note that the proof given for (i) still holds when $\sigma = (i\ j)(i'\ j')$ where $\{i', j'\} \cap \{i, j\} = \emptyset$. ∎

Proposition 6.5.12 *Let G be the isometry group of an l-dimensional orthogonal or unitary space over $\mathbb{F}_q$, or the isometry group of a $2l$-dimensional symplectic space over $\mathbb{F}_q$. Suppose that G contains a Sylow p-subgroup isomorphic $S(n,p)$ where $n \geq 5$. Let $k = \mathrm{ord}_p(q)$. Then $l/k \geq p/2$.*

Proof: By Lemma 6.4.2, the Sylow p-subgroups of G are abelian (and therefore not isomorphic to $S(n,p)$) unless $r_{pk} + r_{p^2k} + \ldots \neq 0$. The specific values of r_m are given in [21, Table 10.1], which we partially reproduce here for convenience:

Lie type of G	r_m	
A_j	$[(j+1)/m]$	for $m > 1$;
$\mathrm{B}_j, \mathrm{C}_j$	$[2j/\mathrm{lcm}(2,m)]$	for $m \geq 1$
D_j	$[2j/\mathrm{lcm}(2,m)]$	unless $m \mid 2j$ and $m \nmid j$
	$(2j/m) - 1$	if $m \mid 2j$ and $m \nmid j$
${}^2\mathrm{A}_j$	$[(j+1)/\mathrm{lcm}(2,m)]$	if $m \not\equiv 2 \bmod 4$;
	$[2(j+1)/m]$	if $m \equiv 2 \bmod 4$, $m > 2$;
${}^2\mathrm{D}_j$	$[2j/\mathrm{lcm}(2,m)]$	unless $m \mid j$
	$[2j/\mathrm{lcm}(2,m)] - 1$	if $m \mid j$.

It is clear from this table that if $r_{pk} = 0$ then $r_{p^ik} = 0$ for all $i \geq 1$. Therefore $r_{pk} \geq 1$. Recall that the Chevalley groups of the types in Table 6.5 are in one-to-one correspondence with the simple classical groups, as given by the following table. The information in the table, and the proofs, can be found in [13, Theorems 11.3.2, 14.5.1, 14.5.2].

$\mathrm{A}_j(q) \cong \mathrm{PSL}_{j+1}(q)$	$\mathrm{D}_j(q) \cong \mathrm{P}\Omega^+_{2j}(q)$
$\mathrm{B}_j(q) \cong \mathrm{P}\Omega_{2j+1}(q)$	${}^2\mathrm{A}_j(q) \cong \mathrm{PSU}_{j+1}(q)$
$\mathrm{C}_j(q) \cong \mathrm{PSp}_{2j}(q)$	${}^2\mathrm{D}_j(q) \cong \mathrm{P}\Omega^-_{2j}(q)$

Thus if G has type $\mathrm{A}_{l-1}(q)$ then $r_{pk} = [l/pk] \geq 1$, and so $l/k \geq p$. If G has type $\mathrm{B}_{(2l-1)/2}(q)$ then $r_{pk} = [(l-1)/\mathrm{lcm}(2,pk)] \geq 1$, and so $l/k \geq p + 1/k \geq p$. If G has type $\mathrm{C}_{l/2}(q)$ then $r_{pk} = [l/\mathrm{lcm}(2,pk)] \geq 1$ and so $l/k \geq p$. If G has type $\mathrm{D}_{l/2}(q)$ then either $r_{pk} = [l/\mathrm{lcm}(2,pk)] \geq 1$ or $r_{pk} = (l/pk) - 1 \geq 1$. In the former case, $l/k \geq p$ as before, and in the latter case $l/pk \geq 2$ and so $l/k \geq 2p \geq p$. If G has type ${}^2\mathrm{A}_{l-1}(q)$ then either $r_{pk} = [l/\mathrm{lcm}(2, pk)] \geq 1$ or $r_{pk} = [2l/pk] \geq 1$. In the former case, $l/k \geq p$ as before, and in the latter case $2l \geq pk$ and so $l/k \geq p/2$. If G has type ${}^2\mathrm{D}_{l/2}(q)$ then either $r_{pk} = [l/\mathrm{lcm}(2, pk)] \geq 1$ or $r_{pk} = [l/\mathrm{lcm}(2, pk)] - 1 \geq 1$. In the former case $l/k \geq p$ as before, and in the latter case $l \geq 2\mathrm{lcm}(2, pk)$ and so $l/k \geq 2p \geq p$. This completes the proof. ∎

For any group G, let $G' = [G, G]$ denote the commutator subgroup of G. We note the following fact about classical groups:

Lemma 6.5.13 *Every simple classical group is of the form $G'/Z(G')$, where G is the isometry group of an linear, orthogonal, symplectic or unitary space.*

Proof: See, for example, [2, 43.12]. ∎

Proposition 6.5.14 *Let G be the isometry group of an l-dimensional orthogonal or unitary space over $\mathbb{F}_q$, or the isometry group of a $2l$-dimensional symplectic space over $\mathbb{F}_q$. Suppose that G contains a Sylow p-subgroup isomorphic to $S(n, p)$ for some n. Then $\overline{G} = G'/Z(G')$ contains a subgroup of the form $W \rtimes X$ where W is an elementary abelian group with $\mathrm{rank}_p(W) = [l/k] - 2$ and $X \cong \mathrm{Alt}([l/k] - 1)$ acts faithfully on W.*

Proof: Let $m = [l/k] - 1$. By Lemma 6.5.10, G contains a subgroup isomorphic to $T \rtimes \mathrm{Sym}(m)$. Thus we may regard $C_T(\mathrm{Sym}(m))$ and $[T, \mathrm{Sym}(m)]$ as subgroups of G. Note that

$$\frac{[T, \mathrm{Sym}(m)]Z(G')}{Z(G')} \cong \frac{[T, \mathrm{Sym}(m)]}{[T, \mathrm{Sym}(m)] \cap Z(G')}.$$

But if $t \in [T, \mathrm{Sym}(m)] \cap Z(G')$ then $t \in C_T(\mathrm{Alt}(m)) = C_T(\mathrm{Sym}(m))$. Therefore $\dim([T, \mathrm{Sym}(m)] \cap Z(G')) \leq 1$, hence $\dim([T, \mathrm{Sym}(m)]Z(G')/Z(G')) \geq m-2$. Let $W = [T, \mathrm{Sym}(m)]Z(G')/Z(G')$.

Since G contains a subgroup isomorphic to $\mathrm{Sym}(m)$ which normalizes T (by Lemma 6.5.10), we have that G' contains a subgroup $X \cong \mathrm{Alt}(m)$ which normalizes $[T, G]$. Now,

$$\frac{XZ(G')}{Z(G')} \cong \frac{X}{X \cap Z(G')}.$$

But $m + 1 \geq p/2$ by Lemma 6.5.12, and since we assumed that $p \geq 13$, we have that $m \geq 5$. Therefore X is a simple group, and so $X \cap Z(G') = 1$. Hence $XZ(G')/Z(G') \cong X$ acts on W. Thus $\overline{G}$ contains the group $W \rtimes X$, which is of the form required. Since X is a simple group, the action of X on W is either faithful or trivial. Since the action is not trivial, it must be faithful. ∎

Theorem 6.5.15 *Let p be a prime with $p \geq 13$ and let $q \neq p$ be prime. Let G be a simple classical group defined over $\mathbb{F}_q$. Suppose that G contains a Sylow p-subgroup isomorphic to $S(n, p)$ where $n \geq 5$. Then $\mathcal{F}_S(G) \ncong \mathcal{E}$.*

Proof: Suppose the theorem is false. Let the dimension of G be l if G is linear, orthogonal or unitary and $2l$ if G is symplectic. By Proposition 6.5.14 G contains a p-subgroup W such that $\mathrm{Aut}_G(W)$ contains a subgroup isomorphic to $\mathrm{Alt}([l/k] - 1)$. Since we have assumed that $\mathcal{F}_S(G) \cong \mathcal{E}$, this means that $[l/k] \leq 6$ by Propositions 5.3.2 and 5.3.6. But $[l/k] \geq p/2$ by Proposition 6.5.12, so $p/2 \leq 6$, therefore $p \leq 12$. But we assumed that $p > 11$, *i.e.* $p \geq 13$, and so we have a contradiction. ∎

Corollary 6.5.16 *Let p be a prime with $p > 11$ and let $q \neq p$ be prime. Let G be an almost simple group of Lie type, which is a group of automorphisms of a simple classical group defined over $\mathbb{F}_q$. Suppose that G contains a Sylow p-subgroup isomorphic to $S(n, p)$ where $n \geq 5$. Then $\mathcal{F}_S(G) \ncong \mathcal{E}$.*

Proof: This follows from the fact that $P \leq H \leq G$ then $\mathrm{Aut}_H(P) \leq \mathrm{Aut}_G(P)$. ∎

6.6 Final remarks

Finally, it remains to consider the case that G is an almost simple group with a normal subgroup N isomorphic to a sporadic group, with $C_G(N) = 1$. By consulting [22, Table 5.6.1] we see that no such group G has an elementary abelian p-subgroup of rank greater than 3 for $p \geq 13$. This means that G does not contains a Sylow p-subgroup isomorphic to $S(n,p)$ for $n \geq 5$ and $p \geq 13$. In particular, no such group G gives rise to a fusion system isomorphic to $\mathcal{E}(n,p)$.

We are now ready to state the conclusion of the last two chapters.

Theorem 6.6.1 *The fusion system $\mathcal{E}$ over $S(n,p)$ is exotic for all $n \geq 5$ and $p \geq 13$.*

Proof: In this chapter we have shown that $\mathcal{E}$ is not the fusion system of any alternating, Lie type, or sporadic almost simple group. Hence by the Classification of Finite Simple Groups, $\mathcal{E}$ is not the fusion system of any almost simple group. Therefore by Theorem 6.1.2, $\mathcal{E}$ is not the fusion system of any finite group. Hence $\mathcal{E}$ is exotic. ∎

Thus we have constructed an infinite family of exotic fusion systems.

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