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LOGARITHMIC IMPROVEMENT OF REGULARITY CRITERIA FOR THE NAVIER-STOKES EQUATIONS IN TERMS OF PRESSURE

CHUONG V. TRAN, XINWEI YU

ABSTRACT. In this article we prove a logarithmic improvement of regularity criteria in the multiplier spaces for the Cauchy problem of the incompressible Navier-Stokes equations in terms of pressure. This improves the main result in [S. Benbernou, A note on the regularity criterion in terms of pressure for the Navier-Stokes equations, Applied Mathematics Letters 22 (2009) 1438–1443].

1. INTRODUCTION

At the center stage of mathematical fluid mechanics are the incompressible Navier-Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad (x, t) \in \Omega \times (0, \infty) \quad (1)$$

$$\operatorname{div} u = 0, \quad (x, t) \in \Omega \times (0, \infty) \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \quad (3)$$

with appropriate boundary conditions. Here $\Omega \subseteq \mathbb{R}^d$ is a domain with certain regularity, $u : \Omega \mapsto \mathbb{R}^d$ is the velocity field, $p : \Omega \mapsto \mathbb{R}$ is the pressure, and $\nu > 0$ is the (dimensionless) viscosity. The system (1)–(3) on one hand describes the motion of viscous Newtonian fluids, while on the other hand serve as the starting point of mathematical modeling of many other types of fluids, such as non-Newtonian fluids, magnetic fluids, electric fluids, and ferro-fluids. In this article we focus on the Cauchy problem of (1)–(3), where $\Omega = \mathbb{R}^d$.

As (1)–(3) serve as the foundation of the modern quantitative theory of incompressible fluids, it is important to have complete mathematical understanding of these equations. However the achievement of this goal is still out of the question. In particular, there is still no satisfactory answer to the question of well-posedness of the Cauchy problem of (1)–(3).

The first systematic study of this well-posedness problem (for the case $d = 3$) was carried out by Jean Leray in [19], where it is shown that for arbitrary $T \in (0, \infty]$ there is at least one function $u(x, t)$ satisfying the following:

- i. $u \in L^\infty(0, T; L^2(\mathbb{R}^d)) \cap L^2(0, T; H^1(\mathbb{R}^d))$;
- ii. u satisfies (1) and (2) in the sense of distributions;
- iii. u takes the initial value in the L^2 sense: $\lim_{t \searrow 0} \|u(\cdot, t) - u_0(\cdot)\|_{L^2} = 0$;

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iv. u satisfies the energy inequality

$$\|u(\cdot, t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2 \quad (4)$$

for all $0 \leq t \leq T$.

Such a function $u(x, t)$ is called a Leray-Hopf weak solution for (1)–(3) in $\mathbb{R}^d \times [0, T)$.

It is easy to show that if a Leray-Hopf weak solution is smooth, then it satisfies (1)–(3) in the classical sense. One could further show that such a smooth Leray-Hopf solution must be the unique solution to (1)–(3). Therefore the well-posedness problem would be settled if all Leray-Hopf solutions could be shown to be smooth. However such a general result has not been established up to now. On the other hand, various additional assumptions guaranteeing the smoothness of Leray-Hopf solutions have been discovered. For example, it has been shown that if a Leray-Hopf solution $u(x, t)$ further satisfies

$$u \in L^r(0, T; L^s(\mathbb{R}^d)) \quad \text{with} \quad \frac{2}{r} + \frac{d}{s} \leq 1, \quad d < s \leq \infty, \quad (5)$$

then $u(x, t)$ is smooth and is thus a classical solution, see e.g. [8], [20], [22]. The borderline case $u \in L^\infty(0, T; L^3)$ is much more complicated and requires a totally different approach. It was settled much later by Escauriaza, Seregin, and Sverak in [7]. Many generalizations and refinements of (5) have been proved, see e.g. [3], [5], [9], [25], [26], [27].

If we formally take divergence of (1) we obtain the following relation between u and p :

$$-\Delta p = \operatorname{div}(\operatorname{div}(u \otimes u)) \quad (6)$$

where $u \otimes u$ is a $d \times d$ matrix with i - j entry $u_i u_j$. Thus intuitively we have $p \sim u^2$. Transforming (5) via this relation, we expect that

$$p \in L^r(0, T; L^s(\mathbb{R}^d)) \quad \text{with} \quad \frac{2}{r} + \frac{d}{s} \leq 2, \quad \frac{d}{2} < s \leq \infty \quad (7)$$

should guarantee the smoothness of u . This is indeed the case and was confirmed in [2], [4].

Many efforts have been made to refine (7), see e.g. [1], [6], [10], [14], [15], [17], [23], [24]. It is worth mentioning that the relation (6) has also played crucial roles in the proofs of other regularity criteria not of the Prodi-Serrin type. For example, in [21] it is used to show that Leray-Hopf weak solutions are regular as long as either $|u|^2 + 2p$ is bounded above or p is bounded below. Among generalizations of (7), in [1] it is shown that u is smooth as long as $p \in L^{2/(2-r)}(0, T; \dot{X}_r(\mathbb{R}^d)^d)$ for $0 < r \leq 1$ where $\dot{X}_r(\mathbb{R}^d)$ is the multiplier space. Multiplier spaces are defined for $0 \leq r < d/2$ and functions $f \in L^2_{\text{loc}}(\mathbb{R}^d)$ through the norm

$$\|f\|_{\dot{X}_r} := \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty, \quad (8)$$

where $\dot{H}^r(\mathbb{R}^d)$ is the completion of the space $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{r/2} u\|_{L^2}$, see e.g. [12] for properties of such spaces. Among its properties we would emphasize the following two.

- $L^{d/r} \subset \dot{X}_r$ for $0 \leq r < d/2$,
- This inclusion is strict. For example by the Hardy inequality for fractional Laplacians (see e.g. [11], [16]) we have $|x|^{-r} \in \dot{X}_r(\mathbb{R}^d)$.

Thus the above criterion refines (7).

In this article we will present the following logarithmic improvement of this criterion.

Theorem 1. *Let $u_0 \in L^2(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for some $q > d$, and $\nabla \cdot u_0 = 0$. Let $u(t, x)$ be a Leray-Hopf solution of NSE in $[0, T)$. If the pressure p satisfies*

$$\int_0^T \frac{\|p\|_{\dot{X}_r}^{2/(2-r)}}{\log(e + \|p\|_{W^{m,\infty}})} dt < \infty \quad (9)$$

for some $m \in \mathbb{N} \cup \{0\}$ and $r \in (0, 1]$, then $u(t, x)$ is smooth up to T and could be extended beyond T .

2. PROOF OF THEOREM

Without loss of generality, we take $\nu = 1$ in (1) to simplify the presentation. We apply the following result from [13], [18] to guarantee short-time smoothness of the solution and thus relieving us from worrying about the legitimacy of the various integral and differential manipulations below.

Theorem 2. *Let $u_0 \in L^s(\mathbb{R}^d)$, $s \geq d$. Then there exists $T > 0$ and a unique classical solution $u \in BC(0, T; L^s(\mathbb{R}^d))$. Moreover, let $(0, T_*)$ be the maximal interval such that the solution u stays in $C(0; T_*; L^s(\mathbb{R}^d))$, $s > d$. Then for any $t \in (0, T_*)$,*

$$\|u(\cdot, t)\|_{L^s} \geq \frac{C}{(T_* - t)^{\frac{s-3}{2s}}} \quad (10)$$

where the constant C is independent of T_* and s .

We also recall that (6) implies

$$p = \sum_{i,j=1}^d R_i R_j(u_i u_j) \quad (11)$$

where R_j , $j = 1, \dots, d$ are the Riesz transforms. As a consequence of the standard theory of singular integrals, the following holds: For any $s \in (1, \infty)$ and $m \in \mathbb{N}$, $\alpha \in (0, 1)$,

$$\|p\|_{L^s} \leq C \|u\|_{L^{2s}}^2, \quad \|p\|_{C^{m,\alpha}} \leq C \max_{i,j=1,2,\dots,d} \|u_i u_j\|_{C^{m,\alpha}} \quad (12)$$

where the constant C depends on s, m, α but not on p or u .

Proof(of Theorem 1).

Assume the contrary. Let $T^* \leq T$ be the first ‘‘blow-up’’ time. By Theorem 2 we must have $\limsup_{t \nearrow T^*} \|u(\cdot, t)\|_{L^s} = \infty$ for all $s \geq d$. In the following we will prove in two steps that under such assumption $\|u(\cdot, t)\|_{H^k}$ stays bounded up to T^* for $k > \frac{d}{2} + m$, thus reaching contradiction as $H^k \hookrightarrow L^s$. Note that again by Theorem 2 we can assume u to be smooth in $(0, T^*)$ and freely manipulate all functions in integration and differentiation.

- (1) L^s estimate. Pick any $s > \max\{4, d\}$. We multiply (1) by $|u|^{s-2}u$ and integrate in \mathbb{R}^d to obtain

$$\begin{aligned} \|u\|_{L^s}^{s-1} \frac{d}{dt} \|u\|_{L^s} &= - \int_{\mathbb{R}^d} |u|^{s-2} u \cdot \nabla p dx + \int_{\mathbb{R}^d} |u|^{s-2} u \cdot \Delta u dx \\ &= \int_{\mathbb{R}^d} p u \cdot \nabla (|u|^{s-2}) dx + \int_{\mathbb{R}^d} |u|^{s-2} u \cdot \Delta u dx \\ &= (s-2) \int_{\mathbb{R}^d} p |u|^{s-2} (\hat{u} \cdot \nabla |u|) dx + \int_{\mathbb{R}^d} |u|^{s-2} u \cdot \Delta u dx. \end{aligned} \quad (13)$$

where $\hat{u} := \frac{u}{|u|}$ (if $u = 0$, just define $\hat{u} = 0$ too).

Recalling the identity

$$u \cdot \Delta u = \nabla \cdot (|u| \nabla |u|) - |\nabla u|^2, \quad (14)$$

we easily derive

$$\int_{\mathbb{R}^d} |u|^{s-2} u \cdot \Delta u dx = -\frac{4(s-2)}{s^2} \|\nabla |u|^{s/2}\|_{L^2}^2 - \|\nabla u\| |u|^{s/2-1}\|_{L^2}^2, \quad (15)$$

and reach the following estimate

$$\frac{d}{dt} \|u\|_{L^s}^s + \|\nabla u\| |u|^{s/2-1}\|_{L^2}^2 + \|\nabla |u|^{s/2}\|_{L^2}^2 \lesssim \int_{\mathbb{R}^d} p |u|^{s-2} (\hat{u} \cdot \nabla |u|) dx. \quad (16)$$

From here on we will use $A \lesssim B$ to denote $A \leq cB$ for some constant c whose value does not depend on u .

Since

$$\left| \int_{\mathbb{R}^d} p |u|^{s-2} (\hat{u} \cdot \nabla |u|) dx \right| \leq \int_{\mathbb{R}^d} \left(|p| |u|^{\frac{s-2}{2}} \right) \left| |u|^{\frac{s-2}{2}} \nabla |u| \right| dx, \quad (17)$$

application of Young's inequality turns (16) into

$$\frac{d}{dt} \|u\|_{L^s}^s + \|\nabla u\| |u|^{s/2-1}\|_{L^2}^2 + \|\nabla |u|^{s/2}\|_{L^2}^2 \lesssim \int_{\mathbb{R}^d} |p|^2 |u|^{s-2} dx. \quad (18)$$

Now let $w := |u|^{s/2}$. From (18) it follows that

$$\frac{d}{dt} \|u\|_{L^s}^s + \frac{4+s^2}{s^2} \|\nabla w\|_{L^2}^2 \lesssim \int_{\mathbb{R}^d} |p|^2 |w|^{2(1-2/s)} dx. \quad (19)$$

We have

$$\begin{aligned} \int_{\mathbb{R}^d} |p|^2 |w|^{2(1-2/s)} dx &\leq \|pw\|_{L^2} \|p\|_{L^{s/2}} \|w^{1-4/s}\|_{L^{2s/(s-4)}} \\ &\leq \|p\|_{\dot{X}_r} \|w\|_{\dot{H}^r} \|u\|_{L^s}^2 \|w\|_{L^2}^{1-4/s} \\ &= \|p\|_{\dot{X}_r} \|w\|_{\dot{H}^r} \|w\|_{L^2} \\ &\leq \|p\|_{\dot{X}_r} \|w\|_{L^2}^{(2-r)} \|w\|_{\dot{H}^1}^r \\ &\leq C \|p\|_{\dot{X}_r}^{2/(2-r)} \|w\|_{L^2}^2 + \frac{1}{2} \|w\|_{\dot{H}^1}^2 \end{aligned} \quad (20)$$

where we have applied Holder's inequality and the definition of \dot{X}_r norm (8).

From (20) we conclude

$$\frac{d}{dt} \|u\|_{L^s} \lesssim \|p\|_{\dot{X}_r}^{\frac{2}{2-r}} \|u\|_{L^s} = \frac{\|p\|_{\dot{X}_r}^{2/(2-r)}}{\log(e + \|p\|_{W^{m,\infty}})} \log(e + \|p\|_{W^{m,\infty}}) \|u\|_{L^s}. \quad (21)$$

Now take $k > \frac{d}{2} + m$. By Sobolev embedding theorems we have $\|u\|_{C^{m,\alpha}} \lesssim \|u\|_{H^k}$ for some $\alpha > 0$. Application of (12) now gives

$$\|p\|_{W^{m,\infty}} \leq \|p\|_{C^{m,\alpha}} \lesssim \max_{i,j=1,2,\dots,d} \|u_i u_j\|_{C^{m,\alpha}} \lesssim \|u\|_{C^{m,\alpha}}^2 \lesssim \|u\|_{H^k}^2. \quad (22)$$

Consequently (21) yields

$$\frac{d}{dt} \|u\|_{L^s} \lesssim \frac{\|p\|_{\dot{X}_r}^{2/(2-r)}}{\log(e + \|p\|_{W^{m,\infty}})} \log(e + \|u\|_{H^k}) \|u\|_{L^s}. \quad (23)$$

Let $\epsilon > 0$ be small and $T^* - \epsilon < t < T^*$. Integrating (23) from $T^* - \epsilon$ to t we see that

$$\|u\|_{L^s}(t) \leq \|u_0\|_{L^s} \exp \left[\left(\int_{T^*-\epsilon}^{T^*} \frac{\|p\|_{\dot{X}^r}^{2/(2-r)}}{\log(e + \|p\|_{W^{m,\infty}})} dt \right) \max_{[0,T]} \log(e + \|u\|_{H^k}) \right]. \quad (24)$$

Thanks to the integrability assumption (9), for any $\delta > 0$, we can take $\epsilon > 0$ small enough to have

$$\|u(t)\|_{L^p} \leq C(\epsilon)(e + \max_{t' \in [T^*-\epsilon, T_1]} \|u(t')\|_{H^k})^\delta \quad (25)$$

for all $T_1 \in (T^* - \epsilon, T^*)$ and $t \in [T^* - \epsilon, T_1]$. Note that by our assumption $C(\epsilon) \rightarrow \infty$ as $\delta \searrow 0$. In the following we will see that it is possible to take a fixed positive value of δ and thus exclude this possibility.

- (2) H^k estimate. Fix a natural number $k > \frac{d}{2} + m$. Let $\Lambda := (-\Delta)^{1/2}$. We multiply (1) by $\Lambda^{2k}u$ and integrate:

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u\|_{\dot{H}^k}^2 + \|u\|_{\dot{H}^{k+1}}^2 &\lesssim \left| \int_{\mathbb{R}^d} u \cdot \nabla u \cdot \Lambda^{2k} u dx \right| \\ &= \left| \int_{\mathbb{R}^d} \Lambda^{k-1} \nabla \cdot (u \otimes u) \cdot (\Lambda^{k+1} u) dx \right| \\ &\leq C \|u\|_{L^\infty} \|u\|_{\dot{H}^k} \|u\|_{\dot{H}^{k+1}} \\ &= C \|u\|_{L^\infty} \|u\|_{\dot{H}^k}^\alpha \|u\|_{\dot{H}^k}^{1-\alpha} \|u\|_{\dot{H}^{k+1}}. \end{aligned} \quad (26)$$

Here we have used the calculus inequality $\|D^m(uv)\|_{L^2} \lesssim \|u\|_{L^\infty} \|D^m v\|_{L^2} + \|v\|_{L^\infty} \|D^m u\|_{L^2}$. The parameter $\alpha \in [0, 1]$ will be determined in a short while.

Now we interpolate

$$\|u\|_{L^\infty} \lesssim \|u\|_{L^s}^\theta \|\Lambda^{k+1} u\|_{L^2}^{1-\theta}, \quad (27)$$

$$\|u\|_{\dot{H}^k} \lesssim \|u\|_{L^s}^\mu \|\Lambda^{k+1} u\|_{L^2}^{1-\mu} \quad (28)$$

where $\theta := \frac{k+1-d/2}{k+1-d/2+d/s}$, $\mu := \frac{1}{k+1-d/2+d/s}$. Application of (27) and (28) to (26) ((28) to $\|u\|_{\dot{H}^k}^\alpha$ only) yields

$$\|u\|_{L^\infty} \|u\|_{\dot{H}^k} \|u\|_{\dot{H}^{k+1}} \lesssim \|u\|_{L^s}^\gamma \|u\|_{\dot{H}^k}^{1-\alpha} \|u\|_{\dot{H}^{k+1}}^{[1-\theta+\alpha(1-\mu)]+1} \quad (29)$$

where γ is some positive number which is finite for all values of α .

As $d < s$ we observe that $\theta + \mu > 1$. Thus there is $\alpha \in (0, 1)$ such that

$$1 - \alpha + [1 - \theta + \alpha(1 - \mu)] + 1 = 2 + (1 - \theta - \alpha\mu) < 2. \quad (30)$$

Take this α in (26). We obtain

$$\left| \int u \cdot \nabla u \cdot \Lambda^{2k} u \right| \lesssim \|u\|_{L^s}^\gamma \|u\|_{\dot{H}^k}^{\kappa'} \|u\|_{\dot{H}^{k+1}}^{\kappa'} \lesssim \|u\|_{L^s}^{2\gamma/(2-\kappa')} \|u\|_{\dot{H}^k}^{2\kappa/(2-\kappa')} + \frac{1}{2} \|u\|_{\dot{H}^{k+1}}^2 \quad (31)$$

We set $\delta_0 := \left(2 - \frac{2\kappa}{2-\kappa'}\right)$. As $\kappa + \kappa' < 2$, $\delta_0 > 0$. Fix $\epsilon_0 > 0$ such that (25) holds for

$\left(\frac{2\gamma}{2-\kappa'}\right)^{-1} \frac{\delta_0}{2}$ and ϵ_0 . Then we have

$$\frac{d}{dt} \|u(t)\|_{\dot{H}^k}^2 \leq C(\epsilon_0)(e + \max_{t' \in [T^*-\epsilon_0, T_1]} \|u(t')\|_{\dot{H}^k})^{\delta_0/2} \|u(t)\|_{\dot{H}^k}^{2-\delta_0} \quad (32)$$

for all $t \in [T^* - \epsilon_0, T_1]$. Integrating from $T^* - \epsilon_0$ to $t \in (T^* - \epsilon_0, T_1)$, we arrive

$$(e + \|u(t)\|_{\dot{H}^k})^2 \leq C_1(\epsilon_0) + C(\epsilon_0)\epsilon_0(e + \max_{t' \in [T^* - \epsilon_0, T_1]} \|u(t')\|_{\dot{H}^k})^{2 - \frac{\delta_0}{2}} \quad (33)$$

and consequently

$$(e + \max_{t' \in [T^* - \epsilon_0, T_1]} \|u(t')\|_{\dot{H}^k})^{\delta_0/2} \leq C_2(\epsilon_0) < \infty. \quad (34)$$

As both $C_2(\epsilon_0)$ and δ_0 are independent of T_1 , setting $T_1 \nearrow T^*$ we have

$$\max_{t' \in [T^* - \epsilon_0, T^*)} \|u(t')\|_{\dot{H}^k} < \infty \quad (35)$$

which contradicts the assumption that T^* is a blow-up time.

Remark 1. *It is clear that we can replace $\|p\|_{W^{m,\infty}}$ by $\|u\|_{W^{m,\infty}}$.*

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