FIEDLER-COMRADE AND FIEDLER-CHEBYSHEV PENCILS

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Abstract. Fiedler pencils are a family of strong linearizations for polynomials expressed in the monomial basis, that include the classical Frobenius companion pencils as special cases. We generalize the definition of a Fiedler pencil from monomials to a larger class of orthogonal polynomial bases. In particular, we derive Fiedler-comrade pencils for two bases that are extremely important in practical applications: the Chebyshev polynomials of the first and second kind. The new approach allows one to construct linearizations having limited bandwidth: a Chebyshev analogue of the pentadiagonal Fiedler pencils in the monomial basis. Moreover, our theory allows for linearizations of square matrix polynomials expressed in the Chebyshev basis (and in other bases), regardless of whether the matrix polynomial is regular or singular, and for recovery formulae for eigenvectors, and minimal indices and bases.

Key words. Fiedler pencil, Chebyshev polynomial, linearization, matrix polynomial, singular matrix polynomial, eigenvector recovery, minimal basis, minimal indices

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1. Motivation. In computational mathematics, many applications require to compute the roots of a polynomial expressed in a nonstandard basis. Particularly relevant in practice [28] are the Chebyshev polynomials of the first kind, that we denote by $T_k(x)$, see (3.1) for their formal definition. For example, suppose that we want to approximate numerically the roots of the polynomial

$$T_5(x) - 4T_4(x) + 4T_2(x) - T_1(x) \tag{1.1}$$

The roots of (1.1) are easy to compute analytically and they are $\pm 1/2$, ± 1, and 2. However, we know that in general a quintic (or higher degree) polynomial equation cannot be solved algebraically. A standard approach would be to solve the equivalent problem of computing the eigenvalues of the colleague matrix [17, 28] of (1.1):

$$\begin{bmatrix} 2 & 1/2 & -2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & 1/2 & 0 & 1/2 \\ & & & 1 & 0 \end{bmatrix},$$

where throughout the paper we occasionally omit some, or all, the zero elements of a matrix. Note *en passant* that (1.1) is monic in the Chebyshev basis, i.e., it is a degree 5 polynomial and its coefficient for $T_5(x)$ is 1. This is why we could linearize it with a standard eigenvalue problem. Had we considered a nonmonic polynomial, we could have used the colleague pencil instead, or we could have normalized it first: see [24] for a thorough discussion on the numerical implications of this choice.

The colleague matrix is an example of what is called a *linearization* in the theory of (matrix) polynomials. A linearization has the same elementary divisors of the

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original polynomial, and in particular it has the same eigenvalues. In the scalar case $m = 1$, this implies that the eigenvalues of the linearizations are precisely the roots of the linearized scalar polynomial. See Section 4.1 for more details.

When polynomials in the monomial basis are considered, many linearizations have been studied in recent years. One family of particular interest is Fiedler pencils (and Fiedler matrices), introduced in [13] and since then deeply studied and generalized in many directions, see for example [2, 4, 7, 11, 30] and the references therein. Among Fiedler pencils we find, for instance, companion linearizations (the monomial analogues of the colleague), the particular Fiedler pencil analyzed in [18] (particularly advantageous for the QZ algorithm), and pentadiagonal linearizations (also potentially advantageous numerically, although currently lacking an algorithm capable to fully exploit the small bandwidth).

On the other hand, many linearizations in nonmonomial bases exist and they have recently been studied under many points of view, see, e.g., [1, 3, 22, 23, 24, 25] and the references therein. One may wonder if these two research lines can be unified: Is it possible to construct Fiedler pencils for at least some nonmonomial bases, and in particular for the Chebyshev basis? The main goal of this paper is to answer this question in the affirmative. For the impatient reader, here is a pentadiagonal Fiedler-Chebyshev linearization of (1.1):

$$\begin{bmatrix} 2 & 1/2 & 1/2 & & \\ 1/2 & 0 & -4 & 1/2 & \\ 1/2 & 0 & 0 & 0 & 1/2 \\ & 1/2 & 1/2 & 0 & 2 \\ & & 1 & 0 & 0 \end{bmatrix}.$$

Additionally, matrix polynomials that arise in applications often have particular structures. Expanding a matrix polynomial $P(x)$ in a given polynomial basis $\{\phi_i(x)\}$, i.e.,

$$P(x) = \sum P_i \phi_i(x), \quad \text{with} \quad P_i \in \mathbb{C}^{m\times m},$$

the most relevant of theses structures are:

(i) symmetric: $P_i^T = P_i$;
(ii) palindromic: $P_i^T = P_{n-i}$;
(iii) skew-symmetric: $P_i^T = -P_i$;
(iv) alternating: $P(-x) = P(x)^T$ or $P(-x) = -P(x)^T$,

together with their variants involving conjugate-transposition instead of transposition. Since the structure of a matrix polynomial is reflected in its spectrum, numerical methods to solve polynomial eigenvalue problems should exploit to a maximal extent the structure of matrix polynomials [21]. For this reason, finding linearizations that retain whatever structure the matrix polynomial $P(x)$ might possess is a fundamental problem in the theory of linearizations (see, for example, [4, 5, 8, 21] and the references therein). The results in this work expand the arena in which to look for linearizations of matrix polynomials expressed in some orthogonal polynomial bases having additional useful properties. Furthermore, the Fiedler-Chebyshev pencils that we analyze in this paper may be used as a starting point to construct structure-preserving linearizations for some classes of structured matrix polynomials. In particular, one may use the theory that we are going to build to generalize the family of block symmetric Fiedler pencils with repetitions [4]. For example, if $P(x) = \sum_{k=0}^{7} C_k U_k(x)$ is a $m \times m$

matrix polynomial expressed in the Chebyshev polynomials of the second kind basis $\{U_k(x)\}$, the following pencil

$$\begin{bmatrix} 2xC_7+C_6 & -I_m & -C_7 & 0 & 0 & 0 & 0 \\ -I_m & 0 & 2xI_m & 0 & -I_m & 0 & 0 \\ -C_7 & 2xI_m & 2xC_5+C_4-C_6 & -I_m & -C_5 & 0 & 0 \\ 0 & 0 & -I_m & 0 & 2xI_m & 0 & -I_m \\ 0 & -I_m & -C_5 & 2xI_m & 2xC_3+C_2-C_4 & -I_m & -C_3 \\ 0 & 0 & 0 & 0 & -I_m & 0 & 2xI_m \\ 0 & 0 & 0 & -I_m & -C_3 & 2xI_m & 2xC_1+C_0-C_2 \end{bmatrix}$$

is a heptadiagonal block symmetric strong linearization of $P(x)$.

Apart from the preservation of structure, in order to be useful in the numerical applications a linearization of a matrix polynomial $P(x)$ must allow one to recover the eigenvectors, and minimal indices and bases of $P(x)$. We will show that this recovery property is satisfied by any of the linearizations presented in this work: eigenvectors and minimal bases of $P(x)$ can be recovered without any computational cost from those of the linearization, while the minimal indices of $P(x)$ are obtained from the minimal indices of the linearization by a uniform subtraction of a constant.

Our strategy is to first extend Fiedler pencils (and matrices, as Fiedler matrices are just the constant terms of the Fiedler pencils associated to polynomials that are monic in the considered basis) to a class of orthogonal nonmonomial bases, including among others Chebyshev polynomials of the *second* kind. Section 2 is devoted to this task. Then, in Section 3 we are going to show how to modify our construction to tackle Chebyshev polynomials of the first kind. For simplicity, we will first expose everything for scalar polynomials. In Section 4, we will discuss how to extend our theory to (square) matrix polynomials. To this goal, we review the concepts of (strong) linearization, minimal indices, minimal bases and duality of matrix pencils in Sections 4.1 and 4.3. Readers unfamiliar with the theory of matrix polynomials may find more details in [7, 9, 10, 14, 27] and the references therein. Finally, in Section 5 we are going to draw some conclusions and to say a few words about possible future applications of this work.

We have tried to keep the technical prerequisites to read this paper to the minimum. Nevertheless, in some instances we have found useful to apply certain techniques first invented by V. N. Kublanovskaya [19], and recently rediscovered and applied to the theory of Fiedler pencils [27]. Also, for simplicity we will state our results for the field $\mathbb{C}$. However, our theory is applicable to any field $\mathbb{F}$.

2. Fiedler pencils in orthogonal bases with constant recurrence relations. In this section we consider a family of orthogonal polynomials with a constant three-term recurrence relation, i.e., we set

$$\begin{aligned} &\phi_{-1}(x) = 0, \qquad \phi_0(x) = 1, \\ &\alpha\phi_{k+1}(x) = x\phi_k(x) - \beta\phi_k(x) - \gamma\phi_{k-1}(x), \qquad k = 0, \dots, n-1, \end{aligned} \tag{2.1}$$

where $0 \neq \alpha, \beta, \gamma \in \mathbb{C}$ do not depend on n.

Although the requirement of a constant recurrence relation unfortunately excludes many commonly used orthogonal polynomials, such as Legendre or Jacobi, some practically important polynomial bases that fit into the above defined category include the monomials ($\alpha = 1$, $\beta = \gamma = 0$) and the Chebyshev polynomials of the second kind ($\alpha = \gamma = 1/2$, $\beta = 0$).

Suppose now that we have a polynomial of degree n expressed in the basis

$\{\phi_0, \dots, \phi_n\}$,

$$p(x) = \sum_{j=0}^{n} c_j \phi_j(x), \quad \text{with } c_n \neq 0, \tag{2.2}$$

and where $c_0, \dots, c_n \in \mathbb{C}$. Then the following $n \times n$ pencil is known as the *comrade pencil* [3, 24] of (2.2):

$$C(x) = x \begin{bmatrix} c_n & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} - \begin{bmatrix} -d_{n-1} & -d_{n-2} & -d_{n-3} & \dots & -d_0 \\ \alpha & \beta & \gamma & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha & \beta & \gamma \\ & & & \alpha & \beta \end{bmatrix}, \tag{2.3}$$

where $d_{n-1} = \alpha c_{n-1} - \beta c_n$, $d_{n-2} = \alpha c_{n-2} - \gamma c_n$, and $d_k = \alpha c_k$ for $k = 0, \dots, n-3$. It is not hard to show that the characteristic polynomial of (2.3) is equal to $\alpha^n p(x)$ [1].

In the following, in the spirit of [13] we will construct a family of comrade pencils that contains as a particular case the comrade pencil (2.3). To this purpose, we now recall the definition of some special matrices that we denote by M_j and N_j. The discovery of the M_j is due to M. Fiedler [13] and was historically the first approach to Fiedler pencils (in the monomial basis).

Definition 2.1. *Given the polynomial* (2.2) *expressed in the orthogonal polynomial basis defined by* (2.1)*, define*

$$M_0 = \begin{bmatrix} I_{n-1} & \\ & -c_0 \end{bmatrix}, \quad N_0 = \begin{bmatrix} I_{n-1} & \\ & 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} c_n & \\ & I_{n-1} \end{bmatrix}$$

and for $k = 1, 2, \dots, n-1$

$$M_k = \begin{bmatrix} I_{n-k-1} & & & \\ & -c_k & 1 & \\ & 1 & 0 & \\ & & & I_{k-1} \end{bmatrix}, \quad N_k = M_k^{-1} = \begin{bmatrix} I_{n-k-1} & & & \\ & 0 & 1 & \\ & 1 & c_k & \\ & & & I_{k-1} \end{bmatrix}.$$

Importantly, the matrices N_k and M_k both satisfy the commutativity relations

$$[X_i, Y_j] = 0 \Leftrightarrow |i - j| \neq 1, \quad \text{for any } X, Y \in \{M, N\}. \tag{2.4}$$

In the following theorem we present a factorization of the comrade pencil $C(x)$ in terms of the matrices M_k and N_k introduced in Definition 2.1. This factorization can be seen as the comrade pencil analogue of the factorization of the companion matrix in [13, Lemma 2.1].

Theorem 2.2. *The comrade pencil* (2.3) *can be factorized as*

$$C(x) = M_n x - \alpha M_{n-1} \cdots M_1 M_0 - \beta M_n - \gamma M_n N_0 N_1 \cdots N_{n-1} M_n.$$

[1] In this work, the characteristic polynomial of a pencil $Ax + B$ refers to the polynomial $\det(Ax + B)$.

Proof. It is evident that the linear term in $C(x)$ is, by definition, the matrix M_n. We therefore only need to prove that equality holds for the constant term. From (2.3) we may write it, up to a minus sign, as the sum of three terms:

$$\alpha \begin{bmatrix} -c_{n-1} & -c_{n-2} & & \cdots & -c_0 \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix} + \beta M_n + \gamma \begin{bmatrix} 0 & c_n & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{bmatrix}.$$

That the first term is equal to $\alpha M_{n-1} \cdots M_1 M_0$ has been already proved in [13, Lemma 2.1]. It remains to show that the third term is equal to $\gamma M_n N_0 N_1 \cdots N_{n-1} M_n$. To see this, we claim that for $m = 0, \ldots, n-1$ it holds

$$N_0 N_1 \cdots N_m = \begin{bmatrix} I_{n-m-1} & \\ & J_{m+1} \end{bmatrix},$$

where J_k denotes a nilpotent Jordan block of size k, i.e., the matrix

$$J_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \in \mathbb{C}^{k \times k}.$$

We prove this result by induction. The claim is obviously true for $m = 0$. Now suppose that it holds for $m-1$ and note that

$$N_0 N_1 \cdots N_m = \begin{bmatrix} I_{n-m} & \\ & J_m \end{bmatrix} N_m = \begin{bmatrix} I_{n-m-1} & \\ & J_{m+1} \end{bmatrix},$$

concluding the inductive step. Then, in particular, $N_0 N_1 \cdots N_{n-1} = J_n$, and hence $\gamma M_n N_0 N_1 \cdots N_{n-1} M_n = \gamma M_n J_n M_n = \gamma M_n J_n$, concluding the proof. □

As in [13], our approach will be based in permuting the factors M_k in a different order. The important difference with respect to the monomial basis is that we will simultaneously permute the factors N_k in the *reverse* order. By this we mean that if the factors M_k appear as $M_{i_0} M_{i_1} \cdots M_{i_{n-1}}$, then the factors N_k will appear as $N_{i_{n-1}} \cdots N_{i_1} N_{i_0}$.

DEFINITION 2.3. *Let σ be a permutation of $\{0, 1, \ldots, n-1\}$, and let us define $M_\sigma := M_{\sigma(0)} \cdots M_{\sigma(n-1)}$, and $N_\sigma := N_{\sigma(n-1)} \cdots N_{\sigma(0)}$. Then the pencil*

$$F_\sigma(x) := M_n x - \alpha M_\sigma - \beta M_n - \gamma M_n N_\sigma M_n \tag{2.5}$$

is called the Fiedler-comrade pencil associated with the permutation σ.

The relations (2.4) imply that some Fiedler-comrade pencils associated with different permutations σ are equal. For example, for $n = 3$, the Fiedler-colleague pencils $xM_3 - \alpha M_0 M_2 M_1 - \beta M_3 - \gamma M_3 N_1 N_2 N_0 M_3$ and $xM_3 - \alpha M_2 M_0 M_1 - \beta M_3 - \gamma M_3 N_1 N_0 N_2 M_3$ are equal. From (2.4), we observe that the relative positions of the matrices M_i and M_{i+1} in the product $M_{\sigma(0)} \cdots M_{\sigma(n-1)}$ or, equivalently, the position of the matrices N_i and N_{i+1} in the product $N_{\sigma(n-1)} \cdots N_{\sigma(0)}$ are of fundamental importance. This motivates the definition of the consecutions and inversions of a permutation, introduced in [7], that we recall here.

DEFINITION 2.4. [7, Definition 3.3] *Let σ be a permutation of $\{0, 1, \ldots, n-1\}$. Then, for $i = 0, 1, \ldots, n-2$, the permutation σ has a consecution at i if $\sigma(i) < \sigma(i+1)$, and it has an inversion at i otherwise.*

The previous definition allows us to define a canonical form for the products $M_\sigma = M_{\sigma(0)} \cdots M_{\sigma(n-1)}$ and $N_\sigma = N_{\sigma(n-1)} \cdots N_{\sigma(0)}$ in (2.5).

LEMMA 2.5. *Let $F_\sigma(x)$ be the Fiedler-comrade pencil associated with the permutation σ, and let σ have precisely Γ consecutions at $c_1 - 1, \ldots, c_\Gamma - 1$. Denote $M_{j:i} = M_{j-1} \cdots M_i$ and $N_{j:i} = N_i \cdots N_{j-1}$. Then, $F_\sigma(x)$ can be written in the normal form*

$$F_\sigma(x) = M_n x - \alpha M_{c_1:0} M_{c_2:c_1} \cdots M_{n:c_\Gamma} - \beta M_n - \gamma M_n N_{n:c_\Gamma} \cdots N_{c_2:c_1} N_{c_1:0} M_n. \quad (2.6)$$

Proof. It is immediate from the commutativity properties of the matrices M_j and N_j. ∎

In the following theorem we show that any Fiedler-comrade pencil $F_\sigma(x)$ is strictly equivalent to $C(x)$, that is, there exist nonsingular matrices U and V such that $UF_\sigma(x)V = C(x)$. In addition, the theorem also shows that all Fiedler-comrade pencils associated with a polynomial $p(x)$ satisfy $\det(F_\sigma(x)) = \alpha^n p(x)$. In other words, the eigenvalues of $F_\sigma(x)$ are precisely the roots of $p(x)$.

THEOREM 2.6. *Any Fiedler-comrade pencil of a polynomial $p(x)$ as in* (2.2) *is strictly equivalent to the comrade pencil* (2.3). *Moreover, its characteristic polynomial is equal to $\alpha^n p(x)$.*

Proof. By Lemma 2.5, we may assume that any Fiedler pencil is in the normal form (2.6).

We now proceed by induction on the number of consecutions Γ in the permutation σ. If $\Gamma = 0$, we recover the comrade pencil (2.3), which is, obviously, strictly equivalent to itself. Additionally, as we said after the equation (2.3), we have $\det(C(x)) = \alpha^n p(x)$. Now suppose that we have proved the result for the sequence $c_2, \ldots, c_\Gamma$, $\Gamma \le n-1$, that is, for a Fiedler-comrade pencil with $\Gamma - 1$ consecutions, and prepend an extra element c_1. We now need to inductively prove the statement for $c_1, c_2, \ldots, c_\Gamma$. Let $Q = M_{c_2:c_1} \cdots M_{n:c_\Gamma}$, $P = M_{c_1:0}$, and $R = N_{c_1:0}$. Note that Q and M_n are invertible, while both P and R commute with M_n, as this will be important in the following.

By assumption, the pencil $M_n x - \alpha QP - \beta M_n - \gamma M_n R Q^{-1} M_n$ is strictly equivalent to the comrade pencil (2.3) since it is associated with a permutation that has $\Gamma - 1$ consecutions. So we just need to show that the pencils $M_n x - \alpha PQ - \beta M_n - \gamma M_n Q^{-1} R M_n$ and $M_n x - \alpha QP - \beta M_n - \gamma M_n R Q^{-1} M_n$ are strictly equivalent. Indeed,

$$\begin{aligned}
&QM_n^{-1}(M_n x - \alpha PQ - \beta M_n - \gamma M_n Q^{-1} R M_n) Q^{-1} M_n = \\
&M_n x - \alpha QP - \beta M_n - \gamma M_n R Q^{-1} M_n,
\end{aligned}$$

which shows that the result is true for any Fiedler-comrade pencil with Γ consecutions. The second statement of the theorem follows because $\det(QM_n^{-1}) \det(Q^{-1}M_n) = 1$. ∎

Interestingly, as in the monomial case, some of the Fiedler-comrade pencils have a pentadiagonal bandwidth. We say that σ is an even/odd permutation of $\{0, 1, \ldots, n-1\}$ if it either lists first all the even elements of $\{0, 1, \ldots, n-1\}$ and then all the odd ones, or *vice versa*.

THEOREM 2.7. *Let σ be an even/odd permutation. Then $F_\sigma(x)$ is a pentadiagonal pencil.*

Proof. The argument is very similar to the one in the monomial basis. Indeed, the key observation is that when we multiply the matrices M_k for only k even (or odd), we obtain a tridiagonal matrix because the non-identity blocks do not overlap. The very same observation holds for the N_k. We now only need the following facts: the product of two tridiagonal matrices is pentadiagonal, and the (left or right) product of a pentadiagonal matrix with a diagonal matrix is pentadiagonal. Therefore:

- The addend xM_n is diagonal;
- The addend $-\alpha M_\sigma$ is pentadiagonal;
- The addend $-\beta M_n$ is diagonal;
- The addend $-\gamma M_n N_\sigma M_n$ is pentadiagonal.

Hence, their sum is a pentadiagonal pencil. $\square$

We illustrate one of the pentadiagonal Fiedler-comrade pencils in two cases: for a polynomial with degree 7 (odd degree) and for a polynomial with degree 8 (even degree), so its general pattern can be discerned. First, for a polynomial $\sum_{k=0}^{7} c_k \phi_k(x)$ the Fiedler-comrade pencil associated with the permutation $(0, 2, 4, 6, 1, 3, 5)$ is equal to

$$\begin{bmatrix} xc_7 + \alpha c_6 - \beta c_7 & \alpha c_5 - \gamma c_7 & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & x - \beta & 0 & -\gamma & 0 & 0 & 0 \\ -\gamma c_7 & \alpha c_4 - \gamma c_6 & x - \beta & \alpha c_3 - \gamma c_5 & -\alpha & 0 & 0 \\ 0 & -\alpha & 0 & x - \beta & 0 & -\gamma & 0 \\ 0 & 0 & -\gamma & \alpha c_2 - \gamma c_4 & x - \beta & \alpha c_1 - \gamma c_3 & -\alpha \\ 0 & 0 & 0 & -\alpha & 0 & x - \beta & 0 \\ 0 & 0 & 0 & 0 & -\gamma & \alpha c_0 - \gamma c_2 & x - \beta \end{bmatrix},$$

and, second, for the polynomial $\sum_{k=0}^{8} c_k \phi_k(x)$ the pentadiagonal Fiedler-comrade pencil associated with the permutation $(0, 2, 4, 6, 1, 3, 5, 7)$ is equal to

$$\begin{bmatrix} xc_8 + \alpha c_7 - \beta c_8 & -\alpha & -\gamma c_8 & 0 & 0 & 0 & 0 & 0 \\ \alpha c_6 - \gamma c_8 & x - \beta & \alpha c_5 - \gamma c_7 & -\alpha & 0 & 0 & 0 & 0 \\ -\alpha & 0 & x - \beta & 0 & -\gamma & 0 & 0 & 0 \\ 0 & -\gamma & \alpha c_4 - \gamma c_6 & x - \beta & \alpha c_3 - \gamma c_5 & -\alpha & 0 & 0 \\ 0 & 0 & -\alpha & 0 & x - \beta & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma & \alpha c_2 - \gamma c_4 & x - \beta & \alpha c_1 - \gamma c_3 & -\alpha \\ 0 & 0 & 0 & 0 & -\alpha & 0 & x - \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & -\gamma & \alpha c_0 - \gamma c_2 & \beta \end{bmatrix}.$$

3. Fiedler pencils and Chebyshev polynomials of the first kind. We do not believe that our approach can be easily generalized to any (nonconstant) three-term recurrence relation, but these difficulties can be easily overcome when the recurrence is nonconstant only because of a small number of exceptions. The price that one pays is that there are fewer Fiedler-comrade pencils for a given degree n. We illustrate this by analyzing the important case of the Chebyshev polynomials of the first kind, that we denote by $T_k(x) := \cos(k \arccos(x))$[2].

Our motivation to focus on this particular case is that, among nonstandard polynomial bases, Chebyshev polynomials of the first kind are of great practical importance. To name but one reason, it is (mainly) Chebyshev technology that allows the software package `chebfun` [29] to graciously achieve its goal to deliver accurate numerical computations with continuous functions. Applications also exist for matrix polynomials expressed in the Chebyshev basis [12]. Unfortunately, the analysis of the previous section does not cover the Chebyshev polynomials of the first kind, since

[2]This defining formula holds on $[-1, 1]$.

they fail to satisfy a constant recurrence relation. Yet, they are very close to doing so. Indeed, the corresponding recurrence is

$$\begin{aligned} &T_0(x) = 1, \qquad T_1(x) = xT_0(x), \\ &\frac{1}{2}T_{k+1}(x) = xT_k(x) - \frac{1}{2}T_{k-1}(x), \qquad k = 1, \ldots, n-1. \end{aligned} \tag{3.1}$$

In other words, $\alpha = \gamma = \frac{1}{2}$, $\beta = 0$, with the *only* exception of $k = 0$, where $\alpha = 1$. This can be overcome by "melting" the matrices M_1 and M_0, as well as the matrices N_1 and N_0, in Definition 2.1, to accomodate the two different values that α can take. More explicitly, we can define the following factors:

Definition 3.1. *Given the polynomial $p(x) = \sum_{j=0}^{n} c_j T_j(x)$ expressed in the Chebyshev polynomial basis of the first kind defined by* (3.1)*, define*

$$M_1 = \begin{bmatrix} I_{n-2} & & \\ & -c_1 & -c_0 \\ & 2 & 0 \end{bmatrix}, \quad N_1 = \begin{bmatrix} I_{n-2} & & \\ & 0 & 1 \\ & 0 & 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} c_n & \\ & I_{n-1} \end{bmatrix}$$

and for $k = 2, 3, \ldots, n-1$

$$M_k = \begin{bmatrix} I_{n-k-1} & & & \\ & -c_k & 1 & \\ & 1 & 0 & \\ & & & I_{k-1} \end{bmatrix}, \quad N_k = M_k^{-1}.$$

Again, the matrices N_k and M_k satisfy the commutativity relations

$$[M_i, M_j] = 0 \Leftrightarrow |i - j| \neq 1, \qquad [N_i, N_j] = 0 \Leftrightarrow |i - j| \neq 1.$$

The Chebyshev version of the comrade pencil is known as the colleague pencil [3, 17, 24]. The colleague pencil of $p(x) = \sum_{j=0}^{n} c_j T_j(x)$ is

$$C_T(x) = x \begin{bmatrix} c_n & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} - \begin{bmatrix} -d_{n-1} & -d_{n-2} & -d_{n-3} & \ldots & -d_0 \\ \frac{1}{2} & & \frac{1}{2} & & \\ & \ddots & & \ddots & \\ & & \frac{1}{2} & & \frac{1}{2} \\ & & & 1 & \end{bmatrix}, \tag{3.2}$$

where $d_{n-2} = c_{n-2}/2 - c_n/2$ and $d_k = c_k/2$ for $k = 0, \ldots, n-3$ and $k = n-1$.

In Theorem 3.2 we present a factorization of the colleague pencil $C_T(x)$ in terms of the matrices M_k and N_k introduced in Definition 3.1.

Theorem 3.2. *The colleague pencil* (3.2) *can be factorized as*

$$C_T(x) = M_n x + \frac{1}{2}\Big(M_{n-1} \cdots M_2 M_1 + M_n N_1 N_2 \cdots N_{n-1} M_n\Big).$$

Proof. The proof follows closely that of Theorem 2.2, and we invite the reader to fill in the details. ◻

We now introduce the Fiedler-Chebyshev pencils of a polynomial $p(x)$ expressed in the Chebyshev basis. We have decided not to give the details of the proofs of the results in the rest of this section, because they follow very closely their Fiedler-comrade pencil analogues, that were explained in detail in the previous section.

DEFINITION 3.3. *Let σ be a permutation of $\{1, 2, \ldots, n-1\}$, and define $M_\sigma := M_{\sigma(1)} \cdots M_{\sigma(n-1)}$, and $N_\sigma := N_{\sigma(n-1)} \cdots N_{\sigma(1)}$. Then the pencil*

$$F_\sigma(x) = M_n x - \frac{1}{2}\left(M_\sigma + M_n N_\sigma M_n\right)$$

is called the Fiedler-Chebyshev pencil associated with the permutation σ.

Observe that, because of the one exceptional α in the recurrence relation, Fiedler-Chebyshev pencils of a polynomial of degree n are constructed from only $n-1$ building blocks M_i, in contrast with the situation of Section 2, where there were n such blocks. This implies that we only get 2^{n-2}, rather than 2^{n-1}, distinct Fiedler pencils. However, we can overcome this loss by defining a different family of Fiedler-Chebyshev pencils using a different M_1 and N_1, namely,

$$\widetilde{M}_1 = M_1^T = I_{n-2} \oplus \begin{bmatrix} -c_1 & 2 \\ -c_0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{N}_1 = N_1^T = I_{n-2} \oplus \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This second family is related to the Fiedler-Chebyshev pencils of Definition 3.3 by transposition.

Analogously to the normal form for Fiedler-comrade pencil, there is a normal form for Fiedler-Chebyshev pencils which follows immediately from the commutativity properties of the matrices M_k and N_k.

LEMMA 3.4. *Let $F_\sigma(x)$ be the Fiedler-Chebyshev pencil associated with the permutation σ, and let σ have precisely Γ consecutions, at $c_1 - 1, \ldots, c_\Gamma - 1$. Denote $M_{j:i} = M_{j-1} \cdots M_i$ and $N_{j:i} = N_i \cdots N_{j-1}$. Then, $F_\sigma(x)$ can be written in the normal form*

$$F_\sigma(x) = M_n x + \frac{1}{2}(M_{c_1:1} M_{c_2:c_1} \cdots M_{n:c_\Gamma} + M_n N_{n:c_\Gamma} \cdots N_{c_2:c_1} N_{c_1:1} M_n). \qquad (3.3)$$

The following theorem shows that all Fiedler-Chebyshev pencils associated with the same polynomial $p(x)$ are strictly equivalent to the colleague pencil $C_T(x)$ of $p(x)$, and that their characteristic polynomials are equal to $p(x)/2^{n-1}$.

THEOREM 3.5. *Every Fiedler-Chebyshev pencil of a polynomial $p(x)$ is strictly equivalent to the colleague pencil* (3.2) *of the polynomial $p(x)$. Moreover, its characteristic polynomial is equal to $p(x)/2^{n-1}$.*

Again, we obtain pentadiagonal pencils by taking even/odd permutations. As in the previous section, we illustrate one of the pentadiagonal Fiedler-Chebyshev pencils in two cases: for a polynomial with degree 7 (odd degree) and for a polynomial with degree 8 (even degree), so its general pattern can be discerned. First, for a polynomial $\sum_{k=0}^{7} c_k T_k(x)$ the Fiedler-Chebyshev pencil associated with the permutation $(1, 3, 5, 2, 4, 6)$ is equal to

$$\frac{1}{2}\begin{bmatrix} 2xc_7 + c_6 & -1 & -c_7 & 0 & 0 & 0 & 0 \\ c_5 - c_7 & 2x & c_4 - c_6 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2x & 0 & -1 & 0 & 0 \\ 0 & -1 & c_3 - c_5 & 2x & c_2 - c_4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 2x & 0 & -1 \\ 0 & 0 & 0 & -1 & c_1 - c_3 & 2x & c_0 - c_2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2x \end{bmatrix},$$

and, second, for the polynomial $\sum_{k=0}^{8} c_k T_k(x)$ the pentadiagonal Fiedler-Chebyshev pencil associated with the permutation $(1,3,5,7,2,4,6)$ is equal to

$$\frac{1}{2}\begin{bmatrix} 2xc_8+c_7 & c_6-c_8 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2x & 0 & -1 & 0 & 0 & 0 & 0 \\ -c_8 & c_5-c_7 & 2x & c_4-c_6 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2x & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & c_3-c_5 & 2x & c_2-c_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2x & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & c_1-c_3 & 2x & c_0-c_2 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2x \end{bmatrix}.$$

From these examples, the reader may get the generic version of the example we have used as a motivation in the introduction.

4. Matrix polynomials. The goal of this section is to extend our treatment to matrix polynomials. Being Chebyshev polynomials of the first kind the most important family of orthogonal polynomials in numerical applications, we will first focus on generalizing Fiedler-Chebyshev pencils to the matrix polynomial case. Later, we will argue that one can do the same with Fiedler-comrade pencils, obtaining similar results. To complete these tasks, we will make use of the concepts of (strong) linearizations, minimal bases e indices, and duality of matrix pencils. For readers not familiar with them, they are very briefly reviewed in Sections 4.1 and 4.3. See [7, 9, 10, 14, 27] for more complete summaries.

Also, we introduce the following notation. The degree of a matrix polynomial $A(x)$ is denoted by $\deg(A(x))$. For a matrix polynomial $A(x)$ of degree n, expressed in a certain polynomial basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$, we define the following map denoted by $\operatorname{row}(A)$:

$$A(x) = \sum_{k=0}^{n} A_k \phi_k(x) \mapsto \operatorname{row}(A) = \begin{bmatrix} A_n & \cdots & A_1 & A_0 \end{bmatrix}. \tag{4.1}$$

Clearly, (4.1) implicitly depends on the choice of basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$, which should be clear from the context.

4.1. Linearizations, eigenvectors, minimal bases and minimal indices of matrix polynomials. In this section we review the definitions and some basic properties of strong linearizations of matrix polynomials, eigenvectors of regular matrix polynomials, and minimal bases and minimal indices of singular matrix polynomials. The concept of strong linearization was introduced in [15, 16] for regular matrix polynomials, and then extended to the singular case in [6].

DEFINITION 4.1. *If $P(x)$ is an $m \times m$ square matrix polynomial of degree n, the pencil $Ax+B$ is a* linearization *of $P(x)$ if there exists unimodular (i.e., with nonzero constant determinant) matrix polynomials $U(x)$ and $V(x)$ such that*

$$U(x)(Ax+B)V(x) = \begin{bmatrix} P(x) & 0 \\ 0 & I_{mn-m} \end{bmatrix}.$$

Furthermore, a linearization $Ax+B$ is called a strong linearization *of $P(x)$ if the pencil $Bx+A$ is a linearization of the* reversal polynomial *of $P(x)$ (i.e., the matrix polynomial* $\operatorname{rev} P(x) := x^n P(x^{-1})$*).*

We recall that the key property of any strong linearization $Ax+B$ of a matrix polynomial $P(x)$ is that $Ax+B$ and $P(x)$ share the same finite and infinite elementary divisors [9, Theorem 4.1]. However, when $\det(P(x))$ is identically equal to 0 (i.e., the matrix polynomial $P(x)$ is singular), Definition 4.1 only guarantees that the number of left (resp. right) minimal indices of $Ax+B$ is equal to the number of left (resp. right) minimal indices of $P(x)$. The concepts of *minimal bases* and *minimal indices* of singular matrix polynomials are recalled in the following definitions.

A *vector polynomial* is a vector whose entries are polynomials in the variable x, and its degree is the greatest degree of its components. If an $m \times m$ matrix polynomial $P(x)$ is singular, then it has non-trivial left and right null spaces:

$$\begin{aligned} \mathcal{N}_\ell(P) &:= \{w(x)^T \in \mathbb{C}(x)^{1\times m} \quad \text{such that} \quad w(x)^T P(x) = 0\}, \\ \mathcal{N}_r(P) &:= \{v(x) \in \mathbb{C}(x)^{m\times 1} \quad \text{such that} \quad P(x)v(x) = 0\} \end{aligned} \tag{4.2}$$

where $\mathbb{C}(x)$ denotes the field of rational functions with complex coefficients. These null spaces are particular examples of *rational* subspaces, i.e., subspaces over the field $\mathbb{C}(x)$ formed by p-tuples whose entries are rational functions. It is not difficult to show that any rational subspace $\mathcal{V}$ has bases consisting entirely of vector polynomials. The *order* of a vector polynomial basis of $\mathcal{V}$ is defined as the sum of the degrees of its vectors [14, Definition 2]. Among all the possible polynomial bases of $\mathcal{V}$, those with least order are called *minimal bases* of $\mathcal{V}$ [14, Definition 3]. There are infinitely many minimal bases of $\mathcal{V}$, but the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same [14, Remark 4, p. 497]. This list of degrees is called the list of *minimal indices* of $\mathcal{V}$. With this definitions at hand, the right and left minimal indices of a singular matrix polynomial $P(x)$ are introduced in Definition 4.2.

Definition 4.2. *The left (resp. right) minimal indices and bases of a singular matrix polynomial $P(x)$ are defined as those of the subspace $\mathcal{N}_\ell(P)$ (resp. $\mathcal{N}_r(P)$).*

With a slight abuse of notation, we sometimes say that a matrix polynomial is a minimal basis to mean that its columns are a minimal basis of the subspace they span. With this convention, we recall that a matrix polynomial $M(x)$ is a minimal basis if and only if $M(x_0)$ has full column rank for all $x_0 \in \mathbb{C}$ and its high order coefficient matrix [10, Definition 2.11], denoted by $M(x)_{\mathrm{hc}}$, has full column rank (see [10, Theorem 2.14]).

When $\det(P(x))$ is not identically equal to 0 (i.e., the matrix polynomial $P(x)$ is regular), the finite eigenvalues of the matrix polynomial $P(x)$ are the zeros of the scalar polynomial $\det(P(x))$. Moreover, a column vector v (resp. a row vector w^T) is a right (resp. left) eigenvector of $P(x)$ associated with a finite eigenvalue x_* if $P(x_*)v = 0$ (resp. $w^T P(x_*) = 0$). Also, a regular matrix polynomial $P(x)$ has an infinite eigenvalue if and only if zero is an eigenvalue of the reversal polynomial $\operatorname{rev} P(x) = x^n P(x^{-1})$, and the corresponding left and right eigenvectors of $P(x)$ at the eigenvalue ∞ are just the left and right null vectors of $\operatorname{rev} P(0)$.

It is well known that strong linearizations may change right and left minimal indices arbitrarily, except by the constraints on their numbers (see, for example, [9, Theorem 4.11]), and that they do not preserve neither minimal basis or eigenvectors [7]. Therefore the recovery of the minimal indices, minimal basis, or eigenvectors of $P(x)$ from those of any of its linearizations is, in general, a non-trivial task. However, this recovery is very simple for the colleague pencil (4.4), as we show in the following section.

4.2. The colleague pencil and the generalized Horner shifts of a matrix polynomial. Let $C_j \in \mathbb{C}^{m \times m}$ and consider a matrix polynomial expressed in the Chebyshev basis (3.1):

$$P(x) = \sum_{j=0}^{n} C_j T_j(x). \tag{4.3}$$

The *colleague pencil* of the matrix polynomial (4.3) is:

$$C_T(x) = x \begin{bmatrix} C_n & & & & \\ & I_m & & & \\ & & \ddots & & \\ & & & I_m & \\ & & & & I_m \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -D_{n-1} & -D_{n-2} & -D_{n-3} & \dots & -D_0 \\ I_m & 0 & I_m & & \\ & \ddots & \ddots & \ddots & \\ & & I_m & 0 & I_m \\ & & & 2I_m & 0 \end{bmatrix}, \tag{4.4}$$

where the D_i are defined analogously to the d_i in Section 3.

The colleague pencil $C_T(x)$ is a remarkable pencil. It is a strong linearization for $P(x)$ regardless of whether $P(x)$ is regular or singular. Moreover, the eigenvectors (when $P(x)$ is regular) and the minimal indices and bases (when $P(x)$ is singular) of $C_T(x)$ and of $P(x)$ are related in simple ways. All these claims are proved in Theorem 4.4. But first, we need to introduce the notion of *generalized Horner shift* and *generalized Horner shift of the second kind*, which are matrix polynomials associated with (4.3). These matrix polynomials, denoted by $H_{k,h}(x)$ and $V_{k,h}(x)$, are generalizations of the Horner shifts of a matrix polynomial expressed in the monomial basis (see, for example, [7]).

DEFINITION 4.3. *Let $P(x)$ be a matrix polynomial as in* (4.3)*. Its* generalized Horner shift of order (k, h) *is*

$$H_{k,h}(x) = \sum_{j=0}^{k} C_{j+n-k} T_{j+h}(x),$$

and its generalized Horner shift of the second kind of order (k, h) *is*

$$V_{k,h}(x) = \sum_{j=0}^{k} C_{j+n-k} U_{j+h}(x),$$

where $U_0(x), \dots, U_n(x)$ are the Chebyshev polynomials of the second kind.

Note that the generalized Horner shifts in Definition 4.3 do not coincide with the Clenshaw shifts introduced in [26], although both families of matrix polynomials can be seen as a generalization of the Horner shifts.

In Theorem 4.4, we state and prove all the properties of the colleague pencil claimed at the beginning of this section. When the matrix polynomial $P(x)$ is regular, some of this properties have been already proved [1], however we extend the analysis to cover also the singular matrix polynomial case.

THEOREM 4.4. *Let $P(x)$ be a matrix polynomial as in* (4.3) *and let $C_T(x)$ be its colleague pencil* (4.4)*. Then:*

(a) *The colleague pencil $C_T(x)$ is a strong linearization of $P(x)$.*

(b) *Assume that $P(x)$ is singular.*

(b1) *If $M(x)$ is a right minimal basis of $P(x)$ with minimal indices $0 \leq \epsilon_1 \leq \epsilon_2 \leq \cdots \leq \epsilon_p$, then*

$$\begin{bmatrix} T_{n-1}(x)I_m & \cdots & T_1(x)I_m & T_0(x)I_m \end{bmatrix}^T M(x)$$

is a right minimal basis of $C_T(x)$ with minimal indices $0 \le \epsilon_1 + n - 1 \le \epsilon_2 + n - 1 \le \cdots \le \epsilon_p + n - 1$.

(b2) *If $N(x)^T$ is a left minimal basis of $P(x)$ with minimal indices $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_q$, then*

$$N(x)^T \begin{bmatrix} V_{0,0}(x) & V_{1,0}(x) & \cdots & V_{n-2,0}(x) & \frac{1}{2}V_{n-1,0}(x) \end{bmatrix}$$

is a left minimal basis of $C_T(x)$ with minimal indices $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_q$.

(c) *Assume that $P(x)$ is regular.*

(c1) *If v is a right eigenvector of $P(x)$ with finite eigenvalue x_*, then*

$$\begin{bmatrix} T_{n-1}(x_*)v^T & \cdots & T_1(x_*)v^T & T_0(x_*)v^T) \end{bmatrix}^T$$

is a right eigenvector of $C_T(x)$ with finite eigenvalue x_*.

(c2) *If w^T is a left eigenvector of $P(x)$ with finite eigenvalue x_*, then*

$$\begin{bmatrix} w^T V_{0,0}(x_*) & w^T V_{1,0}(x_*) & \cdots & w^T V_{n-2,0}(x_*) & \frac{1}{2}w^T V_{n-1,0}(x_*) \end{bmatrix}$$

is a left eigenvector of $C_T(x)$ with finite eigenvalue x_*.

(c3) *If v and w^T are, respectively, right and left eigenvectors of $P(x)$ for the eigenvalue ∞ then $\begin{bmatrix} v^T & 0^T_{(n-1)m\times 1} \end{bmatrix}^T$ and $\begin{bmatrix} w^T & 0_{1\times(n-1)m} \end{bmatrix}$ are, respectively, right and left eigenvectors of $C_T(x)$ for the eigenvalue ∞, where $0_{\ell_1\times\ell_2}$ denotes the zero matrix of size $\ell_1 \times \ell_2$.*

Proof. First, we prove part (a). Consider the vectors $\Lambda(x) = \begin{bmatrix} x^{n-1} & \cdots & x^0 \end{bmatrix}^T$ and $\Phi(x) = \begin{bmatrix} T_{n-1}(x) & \cdots & T_0(x) \end{bmatrix}^T$, and let B be the change of basis matrix such that $\Phi(x) = B\Lambda(x)$. Then, a direct computation gives

$$C_T(x)(B \otimes I_m)(\Lambda(x) \otimes I_m) = C_T(x)(\Phi(x) \otimes I_m) = \frac{1}{2}e_1 \otimes P(x),$$

which means that the pencil $C_T(x)(B \otimes I_m)$ belongs to the vector space $\mathbb{L}_1(P)$ (see [20, 27] for more details about the $\mathbb{L}_1(P)$ vector space). Since $B \otimes I_m$ is invertible, it is clear that $C_T(x)$ is a strong linearization of $P(x)$ if and only if $C_T(x)(B \otimes I_m)$ is. By [27, Theorem 8.3], the pencil $C_T(x)(B \otimes I_m)$ is a strong linearization of $P(x)$ if $\operatorname{row}(C_T(B \otimes I_m))$ has rank $mn - \mu$ where $\mu = n - \operatorname{rank}\operatorname{row}(P)$. Clearly, $\operatorname{row}(C_T(B \otimes I_m))$ and $\operatorname{row}(C_T)$ have the same rank. Similarly, $\operatorname{row}(P)$ has the same rank regardless of the choice of the basis in (4.1), as changing basis is equivalent to postmultiplying $\operatorname{row}(P)$ by an invertible square matrix.

The structure of $C_T(x)$ makes clear that the rank of $\operatorname{row}(C_T)$ is $m(n-1) + \nu$, where ν is the rank of the first block row of $\operatorname{row}(C_T)$. It remains to show that $\operatorname{rank}\operatorname{row}(P) = \nu$. To this goal observe that the rank of the first block row of $\operatorname{row}(C_T)$ is equal to $\operatorname{rank}\begin{bmatrix} C_n & D_{n-1} & D_{n-2} & D_{n-3} & \cdots & D_0 \end{bmatrix}$, and that

$$\begin{bmatrix} C_n & D_{n-1} & D_{n-2} & D_{n-3} & \cdots & D_0 \end{bmatrix} =$$
$$\begin{bmatrix} C_n & C_{n-1} & C_{n-2} & C_{n-3} & \dots & C_0 \end{bmatrix} \begin{bmatrix} I_m & 0 & -\frac{1}{2}I_m & & & \\ & \frac{1}{2}I_m & & & & \\ & & \frac{1}{2}I_m & & & \\ & & & \frac{1}{2}I_m & & \\ & & & & \ddots & \\ & & & & & \frac{1}{2}I_m \end{bmatrix},$$

which implies that $\operatorname{rank} \operatorname{row}(P) = \nu$.

Then, we prove part (b1). First, it is immediate to verify that $2C_T(x)(\Phi(x) \otimes M(x)) = e_1 \otimes (P(x)M(x)) = 0$. Since $\Phi(x) \otimes M(x)$ clearly has full column rank, we have that it is a basis of the right null space $\mathcal{N}_r(C_T(x))$. It remains to show that it is minimal. But this follows from the minimality of $M(x)$: indeed, for any $\mu \in \mathbb{C}$ $\operatorname{rank} \Phi(\mu) \otimes M(\mu) = \operatorname{rank} M(\mu)$, and denoting by $M(x)_{\mathrm{hc}}$ the high order coefficient matrix [14] of $M(x)$ we have that $(\Phi(x) \otimes M(x))_{\mathrm{hc}} = e_1 \otimes M(x)_{\mathrm{hc}}$. To complete the argument, note that all the blocks of $\Phi(x) \otimes M(x)$ are of the form $T_\ell(x)M(x)$, with $0 \le \ell \le n-1$, and that the maximum degree, which is equal to $n - 1 + \deg(M(x))$, is attained in the topmost block of $\Phi(x) \otimes Mx)$. The result now follows from [14, Main Theorem]. The proof of part (b2) follows very closely that of part (b1), so we omit it.

To prove part (c1), just note $2C_T(x)(\Phi(x) \otimes v) = e_1 \otimes (P(x)v)$, which implies that $C_T(x_*)(\Phi(x_*) \otimes v) = 0$ if and only if $P(x_*)v = 0$. Again, the proof for part (c2) is very similar, so we omit it.

Finally, notice that $\operatorname{rev} P(0) = 2^{n-1}C_n$. Since the leading coefficient of $C_T(x)$ is $\operatorname{diag}[C_n, I_{m(n-1)}]$ we get immediately part (c3). ▯

Before extending the notion of Fiedler-Chebyshev pencils to the matrix polynomial case and obtaining for them analogous results to those in Theorem 4.4 for the colleague pencil, we review the concept of duality of matrix pencils, which, together with Theorem 4.4, will be one of our main tools in completing these tasks.

4.3. Duality of matrix pencils. In this section we recall the concepts of pencil duality and column and row minimality [19, 27]. Duality will allow us to extend Theorem 4.4 to any Fiedler-Chebyshev pencil by slightly modifying the proofs of [27] for Fiedler pencils.

Definition 4.5. [27] *The $m \times n$ pencil $L(x) = xL_1 + L_0$ and the $n \times p$ pencil $R(x) = xR_1 + R_0$ are said to be dual pencils if the following two conditions hold:*

1. $L_1R_0 = L_0R_1$;
2. $\operatorname{rank} \begin{bmatrix} L_1 & L_0 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = 2n.$

In this case we say that $L(x)$ is a left dual of $R(x)$ and that $R(x)$ is a right dual of $L(x)$. Moreover, if $\operatorname{rank} \begin{bmatrix} L_1 & L_0 \end{bmatrix} = m$ *we say that $L(x)$ is row-minimal, and if* $\operatorname{rank} \begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = p$ *we say that $R(x)$ is column-minimal.*

The rest of the paper heavily uses Definition 4.5 specialized to the square case $m = n = p$.

We now recall two results that show how the concept of duality may be applied to the study of linearizations of matrix polynomials, and how right minimal indices and bases, and right eigenvectors of a pair of dual pencils are related.

Theorem 4.6. [27, Theorem 6.2] *Let $P(x)$ be a matrix polynomial and let $R(x)$ be a strong linearization of $P(x)$. If $R(x)$ is column-minimal, any row-minimal left dual pencil of $R(x)$ is also a strong linearization of $P(x)$.*

Theorem 4.7. [27, Theorems 3.8 and 4.14] *Let $L(x) = xL_1 + L_0$ and $R(x) = xR_1 + R_0$ be a pair of square row-minimal and column-minimal, respectively, pair of dual pencils.*

(a) *Assume that $L(x)$ and $R(x)$ are singular. If $M(x)$ is a right minimal basis for $R(x)$, then $N(x) = R_1M(x)$ is a right minimal basis for $L(x)$. Moreover, if $0 \le \epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_p$ are the right minimal indices of $M(x)$, then $0 \le \epsilon_1 - 1 \le \epsilon_2 - 1 \le \cdots \le \epsilon_p - 1$ are the right minimal indices of $N(x)$.*

(b) *Assume that $L(x)$ and $R(x)$ are regular. If v is a right eigenvector of $R(x)$ with finite eigenvalue x_*, then $R_1 v$ is a right eigenvector of $L(x)$ with finite eigenvalue x_*.*

4.4. Fiedler-Chebyshev pencils of a matrix polynomial. Analogously to Section 3, given a polynomial $P(x)$ as in (4.3), define

$$M_1 = \begin{bmatrix} I_{m(n-2)} & & \\ & -C_1 & -C_0 \\ & 2I_m & 0 \end{bmatrix}, \ N_1 = \begin{bmatrix} I_{m(n-2)} & & \\ & 0 & I_m \\ & 0 & 0 \end{bmatrix}, \ M_n = \begin{bmatrix} C_n & \\ & I_{m(n-1)} \end{bmatrix},$$

and for $k = 2, 3, \ldots, n-1$

$$M_k = \begin{bmatrix} I_{m(n-k-1)} & & & \\ & -C_k & I_m & \\ & I_m & 0 & \\ & & & I_{m(k-1)} \end{bmatrix}, \ N_k = \begin{bmatrix} I_{m(n-k-1)} & & & \\ & 0 & I_m & \\ & I_m & C_k & \\ & & & I_{m(k-1)} \end{bmatrix}.$$

Then, the Fiedler-Chebyshev pencil of $P(x)$ is defined as in Definition 3.3.

In Theorem 4.8 we extend the results in Theorem 4.4 for the colleague pencil to any Fiedler-Chebyshev pencil. We want to emphasize that to prove these results we rely heavily on Theorems 4.6 and 4.7.

THEOREM 4.8. *Let $P(x)$ be a matrix polynomial as in (4.3) and let $F_\sigma(x)$ be a Fiedler-Chebyshev pencil associated with a permutation σ with consecutions and inversions precisely at $c_1-1, c_2-1, \ldots, c_\Gamma-1$ and $i_1-1, i_2-1, \ldots, i_\Lambda-1$, and let $T_\sigma = N_{n:c_\Gamma} M_n \cdots N_{n:c_2} M_n N_{n:c_1} M_n$ and $S_\sigma = (M_n N_{n-1} \cdots N_{i_1})(M_n N_{n-1} \cdots N_{i_2}) \cdots (M_n N_{n-1} \cdots N_{i_\Lambda})$. Then:*

(a) *The pencil $F_\sigma(x)$ is a strong linearization of $P(x)$.*

(b) *Assume that $P(x)$ is singular.*

(b1) *If $M(x)$ is a right minimal basis of $P(x)$ with minimal indices $0 \le \epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_p$, then*

$$T_\sigma \begin{bmatrix} T_{n-1}(x) I_m & \cdots & T_1(x) I_m & T_0(x) I_m \end{bmatrix}^T M(x)$$

is a right minimal basis of $F_\sigma(x)$ with minimal indices $0 \le \epsilon_1 + n - 1 - \Gamma \le \epsilon_2 + n - 1 - \Gamma \le \cdots \le \epsilon_p + n - 1 - \Gamma$.

(b2) *If $N(x)^T$ is a left minimal basis of $P(x)$ with minimal indices $0 \le \eta_1 \le \eta_2 \le \cdots \le \eta_p$, then*

$$N(x)^T \begin{bmatrix} U_{n-2}(x) I_m & \cdots & U_1(x) I_m & U_0(x) I_m & \tfrac{1}{2} V_{n-1,0}(x) \end{bmatrix} S_\sigma$$

is a left minimal basis of $F_\sigma(x)$ with minimal indices $0 \le \eta_1 + n - 2 - \Lambda \le \eta_2 + n - 2 - \Lambda \le \cdots \le \eta_p + n - 2 - \Lambda$.

(c) *Assume that $P(x)$ is regular.*

(c1) *If v is a right eigenvector of $P(x)$ with finite eigenvalue x_*, then*

$$T_\sigma \begin{bmatrix} T_{n-1}(x_*) v^T & \cdots & T_1(x_*) v^T & T_0(x_*) v^T) \end{bmatrix}^T$$

is a right eigenvector of $F_\sigma(x)$ with finite eigenvalue x_*.

(c2) *If w^T is a left eigenvector of $P(x)$ with finite eigenvalue x_*, then*

$$\begin{bmatrix} w^T U_{n-2}(x_*) & \cdots & w^T U_1(x_*) & w^T U_0(x_*) & \tfrac{1}{2} w^T V_{n-1,0}(x_*) \end{bmatrix} S_\sigma$$

is a left eigenvector of $F_\sigma(x)$ with finite eigenvalue x_*.

(c3) *If v and w^T are, respectively, right and left eigenvectors of $P(x)$ for the eigenvalue ∞ then $\begin{bmatrix} v^T & 0_{1\times(n-1)m}\end{bmatrix}^T$ and $\begin{bmatrix} w^T & 0_{1\times(n-1)m}\end{bmatrix}$ are, respectively, right and left eigenvectors of $F_\sigma(x)$ for the eigenvalue ∞.*

Proof. We start proving parts (a), (b1) and (c1). The strategy of the proof follows closely the proof for the monomial basis given in [27, Theorem 7.2]; however, there are some differences that we here highlight. By Lemma 3.4, we may assume that any Fiedler pencil is in the normal form (3.3).

We now proceed by induction on the number of consecutions Γ in the permutation σ. If $\Gamma = 0$, we recover the colleague pencil (4.4), and, so, the results are true by Theorem 4.4. Suppose that we have proved the results in parts (a), (b1) and (c1) for the sequence $c_2, \ldots, c_\Gamma$, $\Gamma < n-1$, that is, for any Fiedler-Chebyshev pencil with $\Gamma-1$ consecutions, and prepend an extra element c_1. We now need to inductively prove the statement for $c_1, c_2, \ldots, c_\Gamma$. Let $Q = M_{c_2:c_1}\cdots M_{n:c_\Gamma}$, $P = M_{c_1:1}$, and $R = N_{c_1:1}$. Note that Q is invertible, while both P and R commute with M_n, as this will be important in the following.

By assumption, the pencil $F_{\widehat{\sigma}}(x) = M_n x - (QP - M_n R Q^{-1} M_n)/2$ is a strong linearization of $P(x)$ since the permutation $\widehat{\sigma}$ has $\Gamma - 1$ consecutions precisely at $c_2 - 1, \ldots, c_\Gamma - 1$. Moreover, $F_{\widehat{\sigma}}(x)$ is also a column-minimal pencil. To see this, consider the following two cases: (i) $P(x)$ is regular; and (ii) $P(x)$ singular. If $P(x)$ is regular, then it is obvious that $F_{\widehat{\sigma}}(x)$ is column-minimal, and if $P(x)$ is singular, the right minimal indices of $F_{\widehat{\sigma}}(x)$ are equal, by the inductive hypothesis, to $0 < \epsilon_1 + n - \Gamma \le \epsilon_2 + n - \Gamma \le \cdots \le \epsilon_p + n - \Gamma$ which are larger than 0. This implies that $F_{\widehat{\sigma}}(x)$ is column-minimal.

Now, observe that $F_{\widehat{\sigma}}(x)$ is strictly equivalent to the pencil $Q^{-1}M_n x - (P + Q^{-1}M_n R Q^{-1} M_n)/2$, which is still a column-minimal strong linearization of $P(x)$. We claim that the pencil $F_\sigma(x) = M_n x - (PQ + M_n Q^{-1} R M_n)/2$ is a row-minimal left dual of the latter pencil. To see this, we need to check the two conditions in Definition 4.5. For the first, note that

$$\begin{aligned} M_n(P + Q^{-1}M_nRQ^{-1}M_n) = & M_nP + M_nQ^{-1}M_nRQ^{-1}M_n = \\ & PM_n + M_nQ^{-1}RM_nQ^{-1}M_n = \\ & (PQ + M_nQ^{-1}RM_n)Q^{-1}M_n. \end{aligned}$$

For the second, we need to observe that by the inductive assumption

$$\operatorname{rank}\begin{bmatrix} -Q^{-1}M_n \\ P + Q^{-1}M_nRQ^{-1}M_n \end{bmatrix} = nm,$$

and hence, we only need to check that $\operatorname{rank}\begin{bmatrix} -M_n & PQ + M_nQ^{-1}RM_n \end{bmatrix} = nm$. By the structure of M_n, it is sufficient to argue that the $(1, n - c_\Gamma + 1)$th block element of $PQ + M_nQ^{-1}RM_n$ is equal to $I_m/2$. The latter claim follows from the following arguments. First, due to the structure of the matrices M_k, the matrix $PQ = M_{c_1:1}M_{c_2:c_1}\cdots M_{c_\Gamma:c_{\Gamma-1}}$ has $\begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}$ as its first block row, while, by direct multiplication, it may be checked that the matrix $M_{n:c_\Gamma}$ is equal to

$$\begin{bmatrix} -C_{n-1} & \cdots & -C_{c_\Gamma} & I_m \\ I_m & & & \\ & \ddots & & \\ & & I_m & \end{bmatrix} \oplus I_{m(c_\Gamma - 1)}.$$

Thus, the first block row of the matrix PQ is equal to $\begin{bmatrix} 0 & \cdots & 0 & I_m & 0 & \cdots & 0 \end{bmatrix}$, where the entry equal to I_m is in the block position $(1, n-c_\Gamma+1)$. Second, recall that the permutation σ has its last inversion at $i_\Lambda - 1$. This implies that we can rearrange the product $N_{0:c_1} \cdots N_{c_\Gamma:n}$ in the form $(N_{\rho(0)} N_{\rho(1)} \cdots N_{\rho(i_\Lambda-1)})(N_{n-1} \cdots N_{i_\Lambda+1} N_{i_\Lambda})$ for some permutation ρ of $(0, 1, \ldots, i_\Lambda - 1)$. Due to the structure of the matrices N_k, the matrix $N_{\rho(0)} N_{\rho(1)} \cdots N_{\rho(i_\Lambda-1)}$ has $\begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix}$ as its first block row, while, by direct multiplication, it may be checked that the matrix $N_{n-1} \cdots N_{i_\Lambda+1} N_{i_\Lambda}$ is equal to

$$\begin{bmatrix} & & & I_m \\ I_m & & & C_{n-1} \\ & \ddots & & \vdots \\ & & I_m & C_{i_\Lambda} \end{bmatrix} \oplus I_{m(i_\Lambda-1)}.$$

Thus, the first block row of the matrix $Q^{-1}R$ is equal to $\begin{bmatrix} 0 & \cdots & 0 & I_m & 0 & \cdots & 0 \end{bmatrix}$, where the entry equal to I_m is in the block position $(1, n-i_\Lambda+1)$. Since $n - i_\Lambda + 1 \neq n - c_\Gamma + 1$, we conclude that the $(1, n-c_\Gamma+1)$th block entry of $F_\sigma(x)$ is equal to $I_m/2$. By Theorem 4.6 we get finally that $F_\sigma(x)$ is a strong linearization of $P(x)$.

Now assume that $P(x)$ is singular and consider the vector $\Phi(x) = \begin{bmatrix} T_{n-1}(x) & \cdots & T_0(x) \end{bmatrix}^T$. By the induction hypothesis we have that a right minimal basis for $F_{\widehat{\sigma}}(x)$ and for $Q^{-1}F_{\widehat{\sigma}}(x)$ is given by

$$N_{n:c_\Gamma} M_n \cdots N_{n:c_2} M_n \Phi(x) \otimes M(x),$$

with minimal indices $0 \leq \epsilon_1 + n - \Gamma \leq \epsilon_2 + n - \Gamma \leq \cdots \leq \epsilon_p + n - \Gamma$. Since the pencils $F_\sigma(x)$ and $Q^{-1}F_{\widehat{\sigma}}(x)$ are related via a duality relation, from part (a) in Theorem 4.7 we get that a right minimal basis for $F_\sigma(x)$ is given by

$$\begin{aligned} &(Q^{-1}M_n) N_{n:c_\Gamma} M_n \cdots N_{n:c_2} M_n \Phi(x) \otimes M(x) = \\ &(N_{n:c_\Gamma} M_n)(N_{c_\Gamma:c_{\Gamma-1}} M_n) \cdots N_{c_2:c_1} N_{n:c_2} M_n \Phi(x) \otimes M(x) = T_\sigma \Phi(x) \otimes M(x), \end{aligned}$$

with minimal indices $0 \leq \epsilon_1 + n - 1 - \Gamma \leq \epsilon_2 + n - 1 - \Gamma \leq \cdots \leq \epsilon_p + n - 1 - \Gamma$. Therefore part (b1) is true for $F_\sigma(x)$. If $P(x)$ is regular, the argument to prove the result for the right eigenvectors of $F_\sigma(x)$ is similar to the one for part (b1) but using part (b) in Theorem 4.7 instead of part (a), so we omit it.

Next, we prove parts (b2) and (c2). We will get left eigenvectors, and left minimal indices and bases of a pencil from right eigenvectors, and right minimal indices and bases of its transpose pencil. Clearly, if a pencil $L(x)$ is a strong linearization of $P(x)$, then $L(x)^T$ is a strong linearization of $P(x)^T$.

Assume that $P(x)$ is singular. We need to consider first the following Fiedler-Chebyshev pencil

$$\widehat{C}(x) = xM_n - \frac{1}{2}\left(M_1 M_2 \cdots M_{n-1} + M_n N_{n-1} \cdots N_2 N_1 M_n\right).$$

Following closely the proof of Theorem 2.2, it may be checked that the pencil $\widehat{C}(x)^T$

is equal to

$$\frac{1}{2}\begin{bmatrix} 2xC_n^T + C_{n-1}^T & C_{n-2}^T - C_n^T & C_{n-3}^T & \cdots & C_2^T & C_1^T & -2I_m \\ -I_m & 2xI_m & -I_m & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & -I_m & 2xI_m & -I_m & & \\ & & & -I_m & 2xI_m & -I_m & \\ & & & & -I_m & 2xI_m & \\ -C_n^T & -C_{n-1}^T & \cdots & \cdots & -C_3^T & C_0^T - C_2^T & 2xI_m \end{bmatrix}.$$

We claim that a right minimal basis for the pencil above is given by

$$\begin{bmatrix} U_{n-2}(x)I_m & \cdots & U_1(x)I_m & U_0(x)I_m & \tfrac{1}{2}V_{n-1,0}(x)^T \end{bmatrix}^T N(x), \tag{4.5}$$

with minimal indices $0 \leq \eta_1 + n - 2 \leq \eta_2 + n - 2 \leq \cdots \leq \eta_q + n - 2$. The proof for the previous claim is similar to the one for the right minimal basis for the colleague pencil in Theorem 4.4, so we only sketch it. First, by direct multiplication and using the recurrence relations for the Chebyshev polynomials of the second kind ((2.1) with $\alpha = \gamma = 1/2$, $\beta = 0$), it may be checked that

$$\widehat{C}(x)^T \begin{bmatrix} U_{n-2}(x)I_m & \cdots & U_1(x)I_m & U_0(x)I_m & \tfrac{1}{2}V_{n-1,0}(x)^T \end{bmatrix}^T N(x) = \\ \frac{1}{2}e_n \otimes P(x)^T N(x) = 0,$$

so, using the equation above, it may be proved that (4.5) is, indeed, a right minimal basis for $\widehat{C}(x)^T$. To complete the argument, notice that there are only two different types of blocks in (4.5), namely, $U_\ell(x)N(x)$, with $0 \leq \ell \leq n-2$, and $V_{n-1,0}(x)N(x)/2$. Clearly, the maximum degree among all blocks of the form $U_\ell(x)N(x)$ is $\deg(N(x)) + n - 2$, attained only in the topmost block of (4.5). For the block $V_{n-1,0}(x)^T N(x)/2$, notice that

$$\begin{aligned} xV_{n-1,0}(x)^T N(x) = &P(x)^T N(x) + \\ &\left(C_n^T U_{n-2}(x) + \cdots + C_3^T U_1(x) + C_2^T U_0(x) - C_0^T U_0(x)\right) N(x) = \\ &\left(C_n^T U_{n-2}(x) + \cdots + C_3^T U_1(x) + C_2^T U_0(x) - C_0^T U_0(x)\right) N(x). \end{aligned}$$

Taking degrees in the equation above we get

$$1 + \deg(V_{n-1,0}(x)^T N(x)) \leq n - 2 + \deg(N(x)).$$

Thus, $\deg(V_{n-1,0}(x)^T N(x)) \leq n - 3 + \deg(N(x))$, and, therefore, the degree of (4.5) is $n - 2 + \deg(N(x))$.

Now let us consider the pencil $F_\sigma(x)^T$. With the notation $\widehat{M}_{j:i} = M_{j-1}^T \cdots M_i^T$ and $\widehat{N}_{j:i} = N_i^T \cdots N_{j-1}^T$, and using the commutativity properties of the matrices M_k and N_k, it is immediate to show that this pencil may be written as

$$F_\sigma(x)^T = xM_n^T - \frac{1}{2}\left(\widehat{M}_{i_1:1}\widehat{M}_{i_2:i_1} \cdots \widehat{M}_{i_\Lambda:n} + M_n^T \widehat{N}_{i_\Lambda:n} \cdots \widehat{N}_{i_2:i_1}\widehat{N}_{i_1:1}M_n^T\right).$$

We will prove part (b2) by induction on the number of inversions Λ in the permutation σ. The procedure is very similar to the one in the inductive argument for right eigenvectors, and right minimal indices and bases, so we only sketch it. For $\Lambda = 0$

we recover the pencil $\widehat{C}(x)$, so the result is true in this case as we just have seen. Now, let $\widehat{P} = \widehat{M}_{i_1:1}$, $\widehat{Q} = \widehat{M}_{i_2:i_1} \cdots \widehat{M}_{i_\Lambda:n}$ and $\widehat{R} = \widehat{N}_{i_1:1}$. Then, the pencil $F_\sigma(x)^T$ is a row-minimal left dual of the pencil $\widehat{Q}^{-1}\left(xM_n^T - (\widehat{Q}\widehat{P} + M_n^T\widehat{R}\widehat{Q}^{-1}M_n^T)/2\right)$, where the pencil $xM_n^T - (\widehat{Q}\widehat{P} + M_n^T\widehat{R}\widehat{Q}^{-1}M_n^T)/2$ is the transpose of a Fiedler-Chebyshev pencil associated with a permutation with $\Lambda - 1$ inversions at $i_2 - 1, \ldots, i_\Lambda - 1$. By the induction hypothesis, a right minimal basis for the previous pencil is given by

$$\widehat{N}_{n:i_\Lambda} M_n^T \cdots \widehat{N}_{n:i_2} M_n^T \begin{bmatrix} U_{n-2}(x)I_m & \cdots & U_1(x)I_m & U_0(x)I_m & \frac{1}{2}V_{n-1,0}(x)^T \end{bmatrix}^T N(x),$$

so, using Theorem 4.7, we get that a right minimal basis for $F_\sigma(x)^T$ is given by

$$\widehat{N}_{n:i_\Lambda} M_n^T \cdots \widehat{N}_{n:i_1} M_n^T \begin{bmatrix} U_{n-2}(x)I_m & \cdots & U_1(x)I_m & U_0(x)I_m & \frac{1}{2}V_{n-1,0}(x)^T \end{bmatrix}^T N(x).$$

Then, the result follows taking the transpose of the equation above. If $P(x)$ is regular, the argument to prove the result for the left eigenvectors of $F_\sigma(x)$ is similar to the one for part (b2) but using part-(b) in Theorem 4.7 instead of part-(a).

Finally we consider part (c3), that is, eigenvectors with eigenvalues at ∞. As we have seen in the proof of part (c3) in Theorem 4.4, these eigenvectors are the right and left null vectors of the leading coefficient C_n. Since the leading coefficient of every Fiedler-Chebyshev pencil is $\operatorname{diag}[C_n, I_{m(n-1)}]$ we get immediately part (c3). ☐

Observe that the matrix T_σ in Theorem 4.8 is symbolically the same[3] of [27, Theorem 7.6] for Fiedler pencils with an inversion at 0 (since the matrix M_1 never appears as a factor of the matrix T_σ). This means that the explicit form of the block vector

$$T_\sigma \begin{bmatrix} T_{n-1}(x)I_m & \cdots & T_1(x)I_m & T_0(x)I_m \end{bmatrix}^T, \tag{4.6}$$

mimics exactly the formulae already known for the monomial basis [7], with the *only* difference that any monomial x^j is replaced by $T_j(x)$, and that any product of x^h times a Horner shift of degree k is replaced by a generalized Horner shift of order (h, k). Similar observations can be made for the explicit form of the block vector

$$\begin{bmatrix} U_{n-2}(x)I_m & \cdots & U_1(x)I_m & U_0(x)I_m & \frac{1}{2}V_{n-1,0}(x) \end{bmatrix} S_\sigma. \tag{4.7}$$

After applying these modifications, *all* the results known for the monomial basis, see for instance [7, 27], translate almost verbatim.

THEOREM 4.9. *Let $P(x)$ be a matrix polynomial as in (4.3), let $F_\sigma(x)$ be a Fiedler-Chebyshev pencil of $P(x)$, and let $A(x) = \begin{bmatrix} A_1(x)^T & A_2(x)^T & \cdots & A_n(x)^T \end{bmatrix}^T$ and $B(x) = \begin{bmatrix} B_1(x) & B_2(x) & \cdots & B_n(x) \end{bmatrix}$ be, respectively, the block vectors in (4.6) and (4.7). Setting $c_\sigma(1:\ell)$ and $i_\sigma(1:\ell)$ for the number of consecutions and inversions, respectively, from 1 to ℓ, then the kth block entry of $A(x)$ is given by*

$$\begin{cases} T_{i_\sigma(1:n-2)+1}(x)I_m & \text{if } k = 1, \\ T_{i_\sigma(1:n-k-1)+1}(x)I_m & \text{if } 1 < k < n \text{ and there is an inversion at } n-k, \\ H_{k-1,i_\sigma(1:n-k-1)+1}(x) & \text{if } 1 < k < n \text{ and there is a consecution at } n-k, \text{ and} \\ T_0(x)I_m & \text{if } k = n, \end{cases}$$

[3]By symbolically the same we mean that it has the same formula, but of course it is built from the coefficients of (4.3) in the Chebyshev basis $T_0(x), \ldots, T_n(x)$ rather than those in the monomial basis.

and the kth block entry of $B(x)$ is given by

$$\begin{cases} U_{c_\sigma(1:n-2)}(x) I_m & \text{if } k=1, \\ U_{c_\sigma(1:n-k-1)}(x) I_m & \text{if } 1<k<n \text{ and there is a consecution at } n-k, \\ V_{k-1,c_\sigma(1:n-k-1)}(x) & \text{if } 1<k<n \text{ and there is an inversion at } n-k, \text{ and} \\ \frac{1}{2} V_{0,n-1}(x) & \text{if } k=n, \end{cases}$$

for $k = 1, 2, \ldots, n$.

Theorem 4.9 allows us to obtain, for example, explicit formulae for the left and right eigenvectors of a Fiedler-Chebyshev pencil $F_\sigma(x)$ associated with an eigenvalue x_*. Besides their intrinsic matrix theoretical interest, formulae for the eigenvectors of a linearization find applications in numerical analysis, e.g., for conditioning analysis [25]. As an example of the previous results, consider the pentadiagonal Fiedler-Chebyshev pencil

$$F_\sigma(x) = \frac{1}{2} \begin{bmatrix} 2xC_5 + C_4 & -I_m & -C_5 & 0 & 0 \\ C_3 - C_5 & 2xI_m & C_2 - C_4 & -I_m & 0 \\ -I_m & 0 & 2xI_m & 0 & -I_m \\ 0 & -I_m & C_1 - C_3 & 2xI_m & C_0 - C_2 \\ 0 & 0 & -2I_m & 0 & 2xI_m \end{bmatrix}.$$

Then, from Theorems 4.8 and 4.9 we obtain that its right and left eigenvectors with eigenvalue x_* are, respectively, of the form

$$\begin{bmatrix} T_2(x_*)v \\ (C_5 T_3(x_*) + C_4 T_2(x_*))v \\ T_1(x_*)v \\ (C_5 T_4(x_*) + C_4 T_3(x_*) + C_3 T_2(x_*) + C_2 T_1(x_*))v \\ T_0(x_*)v \end{bmatrix}$$

and

$$\left(\begin{bmatrix} U_2(x_*)w \\ U_1(x_*)w \\ (C_5^T U_3(x_*) + C_4^T U_2(x_*) + C_3^T U_1(x_*))w \\ U_0(x_*)w \\ \frac{1}{2}\left(C_5^T U_4(x_*) + C_4^T U_3(x_*) + C_3^T U_2(x_*) + C_2^T U_1(x_*) + C_1^T U_0(x_*)\right) w \end{bmatrix} \right)^T,$$

where v and w^T are, respectively, right and left eigenvectors of $P(x)$ with eigenvalue x_*.

Note that, since $T_0(x) = U_0(x) = 1$, the block vectors $A(x)$ and $B(x)$ in Theorem 4.9 have always one block entry equal to the identity matrix I_m: the nth block entry in the case of $A(x)$, and the $(n - c_1 + 1)$th entry if σ has its first consecution at $c_1 - 1$ (or the 1st entry if σ has no consecutions) in the case of $B(x)$. This fact allows one to recover eigenvalues and minimal bases of $P(x)$ from those of any of its Fiedler-Chebyshev linearizations.

THEOREM 4.10. *Let $P(x)$ be a matrix polynomial as in* (4.3) *and let $F_\sigma(x)$ be a Fiedler-Chebyshev pencil associated with a permutation σ with first consecution precisely at $c_1 - 1$ (if σ has no consecutions, set $c_1 := n$).*

(a) *Assume that $P(x)$ is singular.*

(a1) *Suppose $\{z_1(x), z_2(x), \ldots, z_p(x)\}$ is any right minimal basis of $F_\sigma(x)$, with vectors partitioned into blocks conformable to the blocks of $F_\sigma(x)$, and let $v_j(x)$ be the nth block of $z_j(x)$, for $j = 1, 2, \ldots, p$. Then $\{v_1(x), v_2(x), \ldots, v_p(x)\}$ is a right minimal basis of $P(x)$.*

(a2) *Suppose* $\{y_1(x)^T, y_2(x)^T, \ldots, y_p(x)^T\}$ *is any left minimal basis of* $F_\sigma(x)$*, with vectors partitioned into blocks conformable to the blocks of* $F_\sigma(x)$*, and let* $w_j(x)$ *be the* $(n-c_1+1)$*th block of* $y_j(x)$*, for* $j = 1, 2, \ldots, p$*. Then* $\{w_1(x)^T, w_2(x)^T, \ldots, w_p(x)^T\}$ *is a left minimal basis of* $P(x)$*.*

(b) *Assume that* $P(x)$ *is regular.*

(b1) *If* $z \in \mathbb{C}^{nm\times 1}$ *is a right eigenvector of* $F_\sigma(x)$ *with finite eigenvalue* x_* *partitioned into blocks conformable to the blocks of* $F_\sigma(x)$*, then the* n*th block of* z *is a right eigenvector of* $P(x)$ *with finite eigenvalue* x_**.*

(b2) *If* $y^T \in \mathbb{C}^{1\times nm}$ *is a left eigenvector of* $F_\sigma(x)$ *with finite eigenvalue* x_* *partitioned into blocks conformable to the blocks of* $F_\sigma(x)$*, then the* $(n-c_1+1)$*th block of* y^T *is a left eigenvector of* $P(x)$ *with finite eigenvalue* x_**.*

4.5. Fiedler-comrade pencils of a matrix polynomial. In this section we focus on matrix polynomials that are expressed using the orthogonal polynomials introduced in Section 2, that is, matrix polynomials of the form

$$P(x) = \sum_{k=0}^{n} C_k \phi_k(x), \quad \text{with} \quad C_k \in \mathbb{C}^{m\times m}, \tag{4.8}$$

where $\phi_0(x), \phi_1(x), \ldots, \phi_n(x)$ satisfy the constant recurrence relations (2.1).

Associated with the matrix polynomial (4.8), we define

$$M_0 = \begin{bmatrix} I_{m(n-1)} & \\ & -C_0 \end{bmatrix}, \quad N_0 = \begin{bmatrix} I_{m(n-1)} & \\ & 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} C_n & \\ & I_{m(n-1)} \end{bmatrix},$$

and, for $k = 1, 2, \ldots, n-1$,

$$M_k = \begin{bmatrix} I_{m(n-k-1)} & & & \\ & -C_k & I_m & \\ & I_m & 0 & \\ & & & I_{m(k-1)} \end{bmatrix}, \; N_k = \begin{bmatrix} I_{m(n-k-1)} & & & \\ & 0 & I_m & \\ & I_m & C_k & \\ & & & I_{m(k-1)} \end{bmatrix}.$$

Then, the Fiedler-comrade pencil $F_\sigma(x)$ of $P(x)$ is defined as in Definition 2.1.

For future references, in the following theorems we state the analogous results for Fiedler-comrade pencils to those for Fiedler-Chebyshev pencils in the previous section. The proofs of these results mimic almost exactly the proofs of Theorems 4.8, 4.9 and 4.10, so we omit them.

THEOREM 4.11. *Let* $P(x)$ *be a matrix polynomial as in* (4.8) *and let* $F_\sigma(x)$ *be a Fiedler-comrade pencil associated with a permutation* σ *with consecutions and inversions precisely at* $c_1-1, c_2-1, \ldots, c_\Gamma-1$ *and* $i_i-1, i_2-1, \ldots, i_\Lambda-1$*, and let* $T_\sigma = N_{n:c_\Gamma}M_n \cdots N_{n:c_2}M_n N_{n:c_1}M_n$ *and* $S_\sigma = (M_n N_{n-1}\cdots N_{i_1})(M_n N_{n-1}\cdots N_{i_2})\cdots (M_n N_{n-1}\cdots N_{i_\Lambda})$*. Then:*

(a) *The pencil* $F_\sigma(x)$ *is a strong linearization of* $P(x)$*.*

(b) *Assume that* $P(x)$ *is singular.*

(b1) *If* $M(x)$ *is a right minimal basis of* $P(x)$ *with minimal indices* $0 \le \epsilon_1 \le \epsilon_2 \le \cdots \le \epsilon_p$*, then*

$$T_\sigma \begin{bmatrix} \phi_{n-1}(x)I_m & \cdots & \phi_1(x)I_m & \phi_0(x)I_m \end{bmatrix}^T M(x)$$

is a right minimal basis of $F_\sigma(x)$ *with minimal indices* $0 \le \epsilon_1+n-1-\Gamma \le \epsilon_2+n-1-\Gamma \le \cdots \le \epsilon_p+n-1-\Gamma$*.*

(b2) *If* $N(x)^T$ *is a left minimal basis of* $P(x)$ *with minimal indices* $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$*, then*

$$N(x)^T \begin{bmatrix} \phi_{n-1}(x)I_m & \cdots & \phi_1(x)I_m & \phi_0(x)I_m \end{bmatrix} S_\sigma$$

is a left minimal basis of $F_\sigma(x)$ *with minimal indices* $0 \leq \eta_1+n-1-\Lambda \leq \eta_2+n-1-\Lambda \leq \cdots \leq \eta_p+n-1-\Lambda$.

(c) *Assume that* $P(x)$ *is regular.*

(c1) *If* v *is a right eigenvector of* $P(x)$ *with finite eigenvalue* x_**, then*

$$T_\sigma \begin{bmatrix} \phi_{n-1}(x_*)v^T & \cdots & \phi_1(x_*)v^T & \phi_0(x_*)v^T \end{bmatrix}^T$$

is a right eigenvector of $F_\sigma(x)$ *with finite eigenvalue* x_*.

(c2) *If* w^T *is a left eigenvector of* $P(x)$ *with finite eigenvalue* x_**, then*

$$\begin{bmatrix} w^T\phi_{n-1}(x_*) & \cdots & w^T\phi_1(x_*) & w^T\phi_0(x_*) \end{bmatrix} S_\sigma$$

is a left eigenvector of $F_\sigma(x)$ *with finite eigenvalue* x_*.

(c3) *If* v *and* w^T *are, respectively, right and left eigenvectors of* $P(x)$ *for the eigenvalue* ∞ *then* $\begin{bmatrix} v^T & 0_{1\times(n-1)m} \end{bmatrix}^T$ *and* $\begin{bmatrix} w^T & 0_{1\times(n-1)m} \end{bmatrix}$ *are, respectively, right and left eigenvectors of* $F_\sigma(x)$ *for the eigenvalue* ∞.

In Theorem 4.12 we give the explicit expressions of the block vectors

$$T_\sigma \begin{bmatrix} \phi_{n-1}(x)I_m & \cdots & \phi_1(x)I_m & \phi_0(x)I_m \end{bmatrix}^T, \tag{4.9}$$

and

$$\begin{bmatrix} \phi_{n-1}(x)I_m & \cdots & \phi_1(x)I_m & \phi_0(x)I_m \end{bmatrix} S_\sigma, \tag{4.10}$$

that appear in Theorem 4.11. To this purpose, we define the generalized Horner shift of order (h,k) associated with the matrix polynomial (4.8) as

$$H_{k,h}(x) = \sum_{j=0}^{k} C_{j+n-k}\phi_{j+h}(x).$$

THEOREM 4.12. *Let* $P(x)$ *be a matrix polynomial as in* (4.8)*, let* $F_\sigma(x)$ *be a Fiedler-comrade pencil of* $P(x)$*, and let* $A(x) = \begin{bmatrix} A_1(x)^T & A_2(x)^T & \cdots & A_n(x)^T \end{bmatrix}^T$ *and* $B(x) = \begin{bmatrix} B_1(x) & B_2(x) & \cdots & B_n(x) \end{bmatrix}$ *be, respectively, the block vectors in* (4.9) *and* (4.10)*. Setting* $c_\sigma(0:\ell)$ *and* $i_\sigma(0:\ell)$ *for the number of consecutions and inversions, respectively, from 0 to* ℓ*, then the* k*th block entry of* $A(x)$ *is given by*

$$\begin{cases} \phi_{i_\sigma(0:n-2)}(x)I_m & \text{if } k=1, \\ \phi_{i_\sigma(0:n-k-1)}(x)I_m & \text{if } 1<k<n \text{ and there is an inversion at } n-k, \quad \text{and} \\ H_{k-1,i_\sigma(0:n-k-1)}(x) & \text{if } 1<k<n \text{ and there is a consecution at } n-k, \end{cases}$$

and the k*th block entry of* $B(x)$ *is given by*

$$\begin{cases} \phi_{c_\sigma(0:n-2)}(x)I_m & \text{if } k=1, \\ \phi_{c_\sigma(0:n-k-1)}(x)I_m & \text{if } 1<k<n \text{ and there is a consecution at } n-k, \quad \text{and} \\ H_{k-1,c_\sigma(0:n-k-1)}(x) & \text{if } 1<k<n \text{ and there is an inversion at } n-k, \end{cases}$$

for $k=1,2,\ldots,n$.

Notice that, since $\phi_0(x) = 1$, the block vectors $A(x)$ and $B(x)$ in Theorem 4.12 have always one block entry equal to the identity matrix I_m: the $(n - i_1 + 1)$th entry if σ has its first inversion at $i_1 - 1$ (or the 1st entry if σ has no inversions) in the case of $A(x)$, and the $(n - c_1 + 1)$th entry if σ has its first consecution at $c_1 - 1$ (or the 1st entry if σ has no consecutions) in the case of $B(x)$. This fact allows one to recover eigenvalues, and minimal indices and bases of $P(x)$ from those of any of its Fiedler-comrade linearizations.

THEOREM 4.13. *Let $P(x)$ be a matrix polynomial as in (4.8) and let $F_\sigma(x)$ be a Fiedler-comrade pencil associated with a permutation σ with first consecution and first inverstion precisely at $c_1 - 1$ and $i_1 - 1$, respectively (if σ has no consecutions, set $c_1 := n$, and if σ has no inversions, set $i_1 := n$).*

(a) *Assume that $P(x)$ is singular.*
  (a1) *Suppose $\{z_1(x), z_2(x), \ldots, z_p(x)\}$ is any right minimal basis of $F_\sigma(x)$, with vectors partitioned into blocks conformable to the blocks of $F_\sigma(x)$, and let $v_j(x)$ be the $(n-i_1+1)$th block of $z_j(x)$, for $j = 1, 2, \ldots, p$. Then $\{v_1(x), v_2(x), \ldots, v_p(x)\}$ is a right minimal basis of $P(x)$.*
  (a2) *Suppose $\{y_1(x)^T, y_2(x)^T, \ldots, y_p(x)^T\}$ is any left minimal basis of $F_\sigma(x)$, with vectors partitioned into blocks conformable to the blocks of $F_\sigma(x)$, and let $w_j(x)$ be the $(n - c_1 + 1)$th block of $y_j(x)$, for $j = 1, 2, \ldots, p$. Then $\{w_1(x)^T, w_2(x)^T, \ldots, w_p(x)^T\}$ is a left minimal basis of $P(x)$.*

(b) *Assume that $P(x)$ is regular.*
  (b1) *If $z \in \mathbb{C}^{nm\times 1}$ is a right eigenvector of $F_\sigma(x)$ with finite eigenvalue x_* partitioned into blocks conformable to the blocks of $F_\sigma(x)$, then the $(n - i_1 + 1)$th block of z is a right eigenvector of $P(x)$ with finite eigenvalue x_*.*
  (b2) *If $y^T \in \mathbb{C}^{1\times nm}$ is a left eigenvector of $F_\sigma(x)$ with finite eigenvalue x_* partitioned into blocks conformable to the blocks of $F_\sigma(x)$, then the $(n - c_1 + 1)$th block of y^T is a left eigenvector of $P(x)$ with finite eigenvalue x_*.*

As a last example, consider the pentadiagonal Fiedler-comrade linearization

$$F_\sigma(x) = \begin{bmatrix} xC_5 + \alpha C_4 - \beta C_5 & \alpha C_3 - \gamma C_5 & -\alpha I_m & 0 & 0 \\ -\alpha & xI_m - \beta I_m & 0 & -\gamma I_m & 0 \\ -\gamma C_5 & \alpha C_2 - \gamma C_4 & xI_m - \beta I_m & \alpha C_1 - \gamma C_3 & -\alpha I_m \\ 0 & -\alpha I_m & 0 & xI_m - \beta I_m & 0 \\ 0 & 0 & -\gamma I_m & \alpha C_0 - \gamma C_2 & xI_m - \beta I_m \end{bmatrix},$$

of the matrix polynomial $P(x) = \sum_{k=0}^{5} C_k \phi_k(x)$. Then, its right and left eigenvectors with eigenvalue x_* are, respectively, of the form

$$\begin{bmatrix} \phi_2(x_*)v \\ \phi_1(x_*)v \\ (C_5\phi_3(x_*) + C_4\phi_2(x_*) + C_3\phi_1(x_*))v \\ \phi_0(x_*)v \\ (C_5\phi_4(x_*) + C_4\phi_3(x_*) + C_3\phi_2(x_*) + C_2\phi_1(x_*) + C_1\phi_0(x_*))v \end{bmatrix}$$

and

$$
\left(\begin{bmatrix} \phi_2(x_*)w \\ \left(C_5^T\phi_3(x_*) + C_4^T\phi_2(x_*)\right) w \\ \phi_1(x_*)w \\ \left(C_5^T\phi_4(x_*) + C_4^T\phi_3(x_*) + C_3^T\phi_2(x_*) + C_2^T\phi_1(x_*)\right) w \\ \phi_0(x_*)w \end{bmatrix} \right)^T ,
$$

where v and w^T are, respectively, right and left eigenvectors of $P(x)$ with eigenvalue x_*.

5. Future outlook. We see the present paper as a first step towards the understanding of Fiedler pencils for a class of nonmonomial bases. Fiedler pencils and their generalizations, such as generalized Fiedler pencils [2] or Fiedler pencils with repetitions [30], have been studied in the monomial basis for many years since their invention (for the monic case) by Fiedler [13]. Applications include, to name but a few, rootfinding [11], polynomial eigenvalue problems [18], and the design of structured linearizations [4]. We hope that Fiedler pencils in the Chebyshev and related bases, here introduced, will lead to a similarly fruitful research line in the next future. We believe that there is the potential for this to happen, as some of the nonmonomial bases considered here (namely, the Chebyshev basis of the first and second kind) are very relevant in several applications.

For instance, we display here, focusing for definiteness on degree $n = 5$, a block-symmetric linearization for the matrix polynomial $P(x) = \sum_{j=0}^{5} C_j\phi_j(x)$ expanded in an orthogonal basis with constant recurrence relations (as, for example, the Chebyshev polynomials of the second kind). We have constructed it as a "Fiedler-comrade pencil with repetitions", in the same spirit of [4]; an analogous example (for $n = 7$) was shown in the Introduction.

$$
xN_1N_3M_5 - \alpha N_0N_2N_4 - \beta N_1N_3M_5 - \gamma M_5N_3N_1N_4N_2N_0N_1N_3M_5 =
$$
$$
\begin{bmatrix} (x-\beta)C_5+\alpha C_4 & -\alpha I_m & -\gamma C_5 & 0 & 0 \\ -\alpha I_m & 0 & (x-\beta)I_m & 0 & -\gamma I_m \\ -\gamma C_5 & (x-\beta)I_m & (x-\beta)C_3+\alpha C_2-\gamma C_4 & -\alpha I_m & -\gamma C_3 \\ 0 & 0 & -\alpha I_m & 0 & (x-\beta)I_m \\ 0 & -\gamma I_m & -\gamma C_3 & (x-\beta)I_m & (x-\beta)C_1+\alpha C_0-\gamma C_2 \end{bmatrix} .
$$

We think that a systematic study of generalized Fiedler-Chebyshev pencils is an interesting potential future research line.

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