COLLABORATIVE SPECTRUM SENSING BASED ON UPPER BOUND ON JOINT PDF OF EXTREME EIGENVALUES

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ABSTRACT

Detection based on eigenvalues of received signal covariance matrix is currently one of the most effective solution for spectrum sensing problem in cognitive radios. However, the results of these schemes always depend on asymptotic assumptions since the distribution of ratio of extreme eigenvalues is exceptionally mathematically complex to compute in practice. In this paper, a new approach to determine the distribution of ratio of the largest and the smallest eigenvalues is introduced to calculate the decision threshold and sense the spectrum. In this context, we derive a simple and analytically tractable expression for the distribution of the ratio of the largest and the smallest eigenvalues based on upper bound on the joint probability density function (PDF) of the largest and the smallest eigenvalues of the received covariance matrix. The performance analysis of proposed approach is compared with the empirical results. The decision threshold as a function of a given probability of false alarm is calculated to illustrate the effectiveness of the proposed approach.

1. INTRODUCTION

In cognitive radio networks, the spectrum resources are available for a secondary user (SU) only when they are not occupied by the primary user (PU), which aims at avoidance of intolerable interference. Thus, the SU should be able to detect the presence of the PU. In this context, high probability of accurate detection in spectrum sensing becomes extremely important in the implementation of cognitive radio networks.

The detection algorithms have their mathematical foundations in the random matrix theory. One of these algorithms is the eigenvalue ratio (ER) detector [1], which uses the ratio of the largest eigenvalue to the smallest eigenvalue as a test statistic and then gives the decision threshold. ER based detection has been considered as an effective and efficient solution to this problem [1–5]. The key advantage of this detection method lies in the fact that there is no need to obtain a prior information of the PU. The main idea is to decide whether the spectrum resources are being used by a PU or not. In other words, these eigenvalues behave differently, which depends on the presence or absence of PU. Due to these properties, the eigenvalue ratio based detection outperforms the energy detector (ED), which is a traditional approach for spectrum sensing, especially when the noise uncertainty increases larger [5, 6].

The decision threshold is precalculated, which is determined by the distribution of the test statistics T_N. However, the exact distribution of the test statistics of the ER detector is generally a mathematically intractable function. Some semi-analytical approaches for the distribution are presented in [5, 7] where the computational complexity becomes intractable with the increase in number of SUs K and received samples N. The exact expression for this ratio has also been derived in [3], however the distribution can only be evaluated numerically. The complexity of the exact expression may become computationally cumbersome with the increase in K and N. As a consequence, a Gaussian approximation is introduced in [8] to derive the analytical distribution of T_N such that the decision threshold γ can be calculated. The derived decision threshold is based on the asymptotic Gaussianity of the extreme eigenvalue distribution obtained by fitting asymptotic moments of Tracy-Widom distribution of order 2 [9]. Despite the simplicity of the decision threshold, the proposed approximation is only valid under the assumption that the distribution of the largest and the smallest eigenvalues converges to the Trace-Widom distribution of order two [9]. It has been also shown that such convergence only occurs when $K \to \infty$, $N \to \infty$ and $\frac{K}{N} \to c \in (0, 1)$. However, the resulting cumulative distribution function (CDF) of the Trace-Widom random variable involve matrix determinants with function entries that are difficult to evaluate when K and N are larger. Therefore, the most of the previous approaches not only based on asymptotic assumptions but also leads to mathematically complicated solutions. In this paper, our main concern is to develop a mathematically tractable solution to solve the problem of sensing the presence or absence of the PU with the collaboration of finite number of SUs. However, we consider a case when few SUs are collaborating to sense the presence PU.

The contributions of this paper are described as follows: we introduce a mathematically tractable approach to derive the distribution of ratio of the largest and the smallest eigenvalues. The derived distribution is based on the upper bound on the joint probability density function (PDF) of the largest and the smallest eigenvalues. The proposed approach gives useful results which are applicable to the scenarios when few SUs are collaborating to sense the presence or absence of the PU in a given sensing time.

The rest of this paper is organized as follows. Section 2 defines the system model to explain detection problem of PU; section 3 introduces the upper bound on the joint PDF of the largest and the smallest eigenvalues. Next in section 4, we derive the distribution of the ratio of extreme eigenvalues based on the upper bound. This is followed by the calculation of decision threshold for a given probability of false alarm. Later, we present numerical results to validate the simulation and analytical results and illustrate the usefulness of the proposed upper bound approach for realistic cognitive scenarios. Finally in section 5, we conclude the results.

2. EIGENVALUE RATIO DETECTION PROBLEM

Consider there are K collaborating SUs such that each user collects N samples during the sensing time to detect a PU. The SUs may be considered as a K receive antennas in one secondary terminal or K secondary terminals each with single antenna, or any combinations of these. The collected samples from K collaborating SUs will be forwarded to a fusion center for combined processing [9].

The aim of the SU cognitive phase is to construct and analyze tests associated with the following hypothesis testing problem:

$$\mathcal{H}_0 : \boldsymbol{y}(n) = \boldsymbol{w}(n) \tag{1}$$

$$\mathcal{H}_1 : \boldsymbol{y}(n) = \boldsymbol{h}(n)\, s(n) + \boldsymbol{w}(n) \tag{2}$$

where $\boldsymbol{y}(n) = [y_1(n), \cdots, y_K(n)]^T$ is the $K \times 1$ observed complex time series, $\boldsymbol{w}(n)$ represents a $K \times 1$ circularly symmetric complex Gaussian (CSCG) noise with zero mean and variance σ_w^2. In (2), the vector $\boldsymbol{h}(n) \in \mathbb{C}^{K\times 1}$ typically represents the propagation channel between the PU and K collaborating SUs and the signal $s(n)$ denotes a standard scalar i.i.d circular complex Gaussian process w.r.t samples $n = 1, 2, \cdots, N$ and stands for the source signal to be detected with $\mathbb{E}\left[s^2(n)\right] = \sigma_s^2 \neq 0$. We stack the observed data into $K \times N$ data matrix $\boldsymbol{Y}$ which may be expressed as

$$\boldsymbol{Y} = \begin{pmatrix} y_1(1) & y_1(2) & \cdots & y_1(N) \\ y_2(1) & y_2(2) & \cdots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_K(1) & y_K(2) & \cdots & y_K(N) \end{pmatrix} \tag{3}$$

As the sample number $N \to \infty$, the sample covariance matrix, $\boldsymbol{R}(N) = \frac{1}{N}\boldsymbol{Y}\boldsymbol{Y}^*$, converges to $\mathbb{E}[\boldsymbol{y}\boldsymbol{y}^*]$, where $\boldsymbol{y}$ is the column vector of N samples collected by a single receiver. From the eigenvalues of $\boldsymbol{R}(N)$, it is possible to infer the absence or presence of the primary signal. Denote the normalized covariance matrix as $\boldsymbol{R}_N = \frac{N}{\sigma_w^2}\boldsymbol{R}(N) = \frac{1}{\sigma_w^2}\boldsymbol{Y}\boldsymbol{Y}^*$. Under the hypothesis $\mathcal{H}_0$, $\boldsymbol{R}_N$ is a complex white Wishart matrix subject to $\mathcal{CW}_K(N, \boldsymbol{I}_K)$, where $\boldsymbol{I}_K$ is a $K \times K$ identity matrix, while it turns out to be the class of spiked population models under the hypothesis $\mathcal{H}_1$ [10].

Suppose the ordered eigenvalues of $\boldsymbol{R}(N)$ and $\boldsymbol{R}_N$ are $0 < l_K < l_{K-1} < \cdots < l_1$ and $0 < \lambda_K < \lambda_{K-1} < \cdots < \lambda_1$ respectively, with the relationship

$$\lambda_k = \frac{\sigma_w^2}{N} l_k \tag{4}$$

where $k = 1, 2, \cdots, K$.

Let us define the test statistics for the for ER based detection as

$$T_N = \frac{\lambda_1}{\lambda_K} \tag{5}$$

and denoting γ as the decision threshold employed by the detector such that

$$T_N \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma \tag{6}$$

to decide if the target spectrum resource is occupied or not.

3. UPPER BOUND ON JOINT PDF OF EXTREME EIGENVALUES

In this section, we present the upper bound on the joint PDF of the largest and the smallest eigenvalues. If $g_{\lambda_1,\lambda_K}(x,y)$ denotes the joint PDF of the largest and the smallest eigenvalues such that $g_{\lambda_1,\lambda_K}(x,y)$ satisfies [11]:

$$\begin{aligned} g_{\lambda_1,\lambda_K}(x,y) &\leq f_{\lambda_1,\lambda_K}(x,y) \\ &= C_{K,N} e^{-\frac{1}{2}(x+y)}\, x^{\frac{1}{2}(N+K-3)}\, y^{\frac{1}{2}(N-K-1)} \end{aligned} \tag{7}$$

where $C_{K,N} = \frac{1}{4\Gamma(K-1)\Gamma(N-K+1)}$ with $\Gamma(\cdot)$ is the Gamma function [12]. The associated marginal PDF of the largest and the smallest eigenvalues can be respectively calculated as

$$\begin{aligned} f_{\lambda_1}(x) &= \int_0^\infty f_{\lambda_1,\lambda_K}(x,y)dy \\ &= C_{K,N}\, 2^{\frac{1}{2}(N-K+1)}\, \Gamma\left(\frac{1}{2}(N-K+1)\right) \\ &\quad \times e^{-\frac{x}{2}}\, x^{\frac{1}{2}(N+K-3)} \end{aligned} \tag{8}$$

$$\begin{aligned} f_{\lambda_K}(y) &= \int_0^\infty f_{\lambda_1,\lambda_K}(x,y)dx \\ &= C_{K,N}\, 2^{\frac{1}{2}(N+K-1)}\, \Gamma\left(\frac{1}{2}(N+K-1)\right) \\ &\quad \times e^{-\frac{y}{2}}\, y^{\frac{1}{2}(N-K-1)} \end{aligned} \tag{9}$$

Also, the associated marginal CDFs of the largest and the smallest eigenvalues can be respectively calculated as

$$\begin{aligned} F_{\lambda_1}(x) &= \int_0^x f_{\lambda_1}(x')dx' \\ &= C_{K,N}\, 2^N\, \Gamma\left(\frac{1}{2}(N-K+1)\right) \\ &\quad \left(\Gamma\left(\frac{1}{2}(N+K-1)\right) - \Gamma\left(\frac{1}{2}(N+K-1), \frac{x}{2}\right)\right) \end{aligned} \tag{10}$$

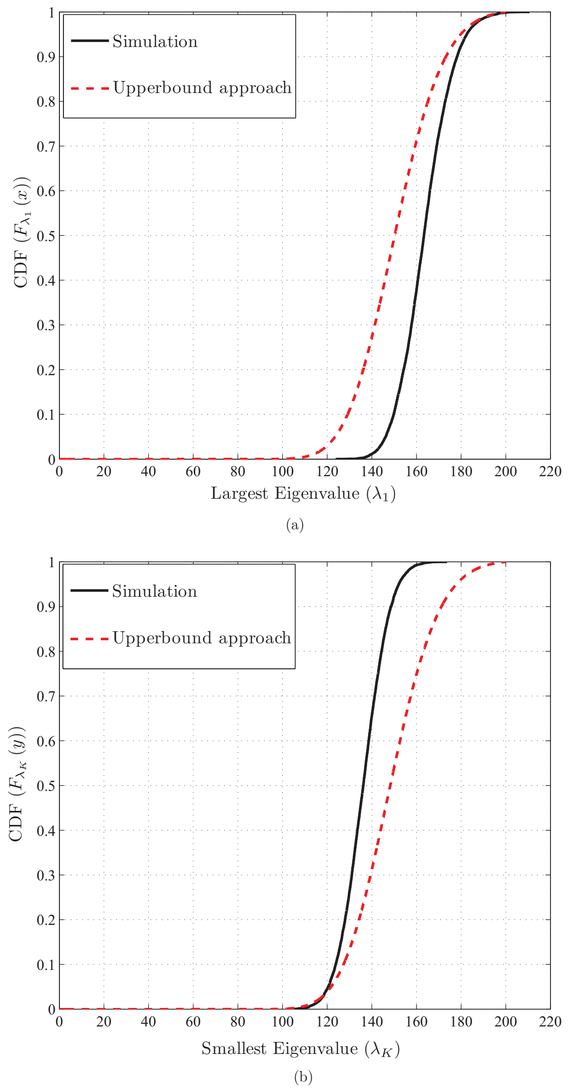

Fig. 1: CDFs of the largest and the smallest eigenvalues for $K = 2$ and $N = 150$.

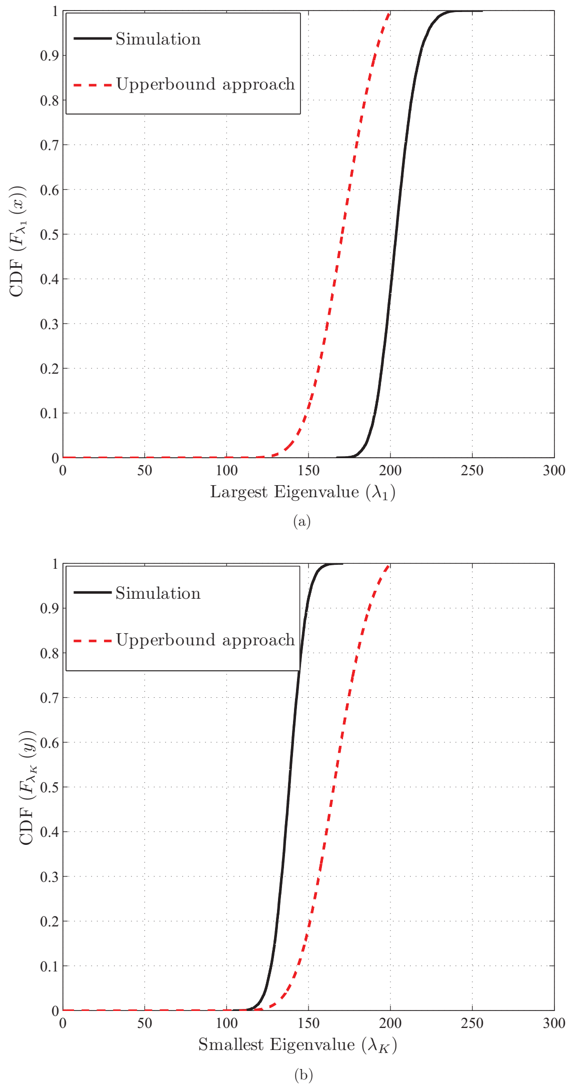

Fig. 2: CDFs of the largest and the smallest eigenvalues for $K = 4$ and $N = 150$.

$$
\begin{aligned}
F_{\lambda_K}(y) &= \int_0^y f_{\lambda_K}(y')dy' \\
&= C_{K,N}\, 2^N\, \Gamma\left(\frac{1}{2}(N+K-1)\right) \\
&\quad \left(\Gamma\left(\frac{1}{2}(N-K+1)\right) - \Gamma\left(\frac{1}{2}(N-K+1), \frac{y}{2}\right)\right)
\end{aligned} \tag{11}
$$

where $\Gamma(.,.)$ is the incomplete Gamma function [12].

Fig. 1(a) and Fig. 1(b) shows the CDFs of the largest eigenvalue and the smallest eigenvalue using (10) and (11) respectively for $K = 2$ and $N = 150$. The analytical CDFs are compared with the empirical CDFs (compare the solid curve with the dashed curve). It can be seen that the marginal CDFs of the largest and the smallest eigenvalues are behaving same as the empirical CDFs. However, the known difference between the two CDFs is due to the obvious reason that the analytical CDFs are derived from the upper bound on the joint density or distribution. It is to note that the results are still useful to calculate the decision threshold numerically as a function of probability of false alarm. Similarly, in Fig. 2(a) and Fig. 2(b) the CDFs of the largest and the smallest are calculated for $K = 4$ and $N = 150$. It is observed that the intrinsic difference between the empirical and upper bound CDFs (due to upper bound on joint density or distribution) increases with the increase in number of collaborating SUs. However, a tighter upper bound results is expected to be reported at a later date which is calculated using [13]. Despite the fact that our results are based on the upper bound, the decision threshold can be easily calculated numerically to decide the occupancy of the spectrum.

4. DISTRIBUTION OF EIGENVALUE RATIO (T_N)

In this section, we introduce a novel application of upper bound on the joint PDF of the largest and smallest

Table 1: Numerical values of (15) for selected values of K and N

$F = F_Z(z)$	$F = 0.1$	$F = 0.2$	$F = 0.3$	$F = 0.4$	$F = 0.5$	$F = 0.6$	$F = 0.7$	$F = 0.8$	$F = 0.9$
$K = 2, N = 150$	0.8200	0.8800	0.9300	0.9700	1.0100	1.0600	1.1000	1.1600	1.2500
$K = 4, N = 100$	0.8200	0.9000	0.9600	1.0100	1.0600	1.1200	1.1800	1.2600	1.3700
$K = 4, N = 150$	0.8400	0.9100	0.9600	1.0000	1.0400	1.0900	1.1300	1.1900	1.2800
$K = 8, N = 100$	0.8900	0.9700	1.0400	1.0900	1.1500	1.2100	1.2800	1.3600	1.4900

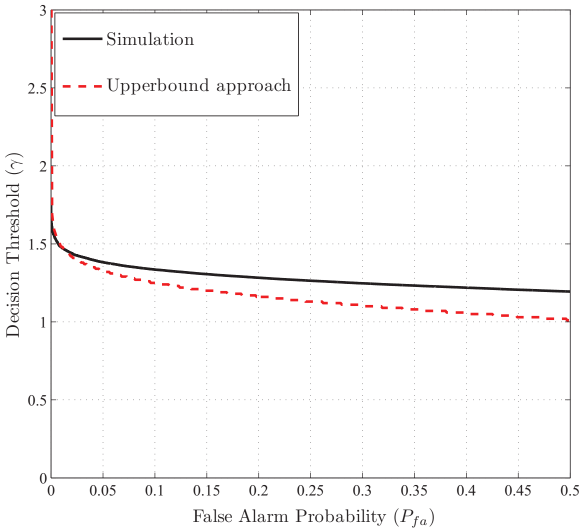

Fig. 3: Decision threshold as a function of probability of false alarm for $K = 2$ and $N = 150$

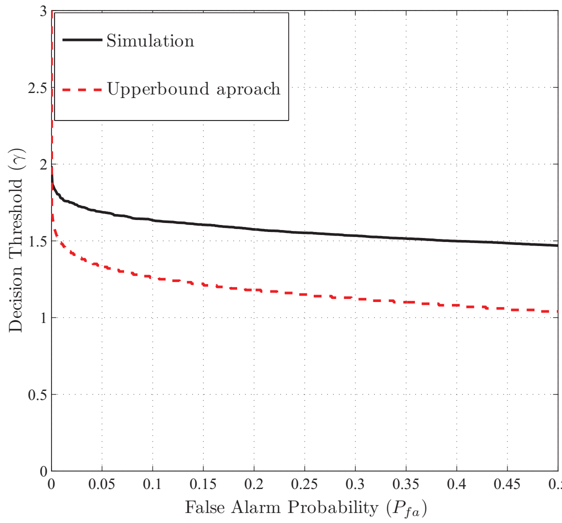

Fig. 4: Decision threshold as a function of probability of false alarm for $K = 4$ and $N = 150$

eigenvalues to calculate the decision threshold as a function of probability of false alarm.

Using (7), we first derive PDF of the ratio of the largest and the smallest eigenvalues T_N and then derive CDF of the ratio. The CDF of the ratio between λ_1 and λ_K i.e., $z = \frac{x}{y}$ can be expressed as

$$F_Z(z) = P\{\frac{x}{y} \leq z\} = P\{x \leq yz, y > 0\} \quad (12)$$

$$= \int_0^\infty \int_{-\infty}^{yz} f_{\lambda_1,\lambda_K}(x, y)dxdy \quad (13)$$

Using (13), the PDF of the ratio can be calculated as

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}F_Z(z) = \int_0^\infty y f_{\lambda_1,\lambda_K}(yz, y)dy \\ &= \int_0^\infty y f_{\lambda_1,\lambda_K}(yz, y)dy \\ &= C_{K,N} \int_0^\infty y e^{-\frac{y}{2}(z+1)} z^{\frac{1}{2}(N+K-3)} y^{N-2}\, dy \\ &= C_{K,N}\, z^{\frac{1}{2}(N+K-3)} \int_0^\infty e^{-\frac{y}{2}(z+1)} y^{N-1}\, dy \end{aligned}$$

Finally, we arrived at

$$f_z(z) = C_{K,N}\Gamma(N)\, 2^N\, z^{\frac{1}{2}(N+K-3)}(1 + z)^{-N} \quad (14)$$

The CDF of the ratio T_N can be calculated as

$$\begin{aligned} F_Z(z) &= C_{K,N}\, \Gamma(N)\, 2^N \int_0^z z'^{\frac{1}{2}(N+K-3)}(1 + z')^{-N} dz' \\ &= C_{K,N}\, 2^N \left(\Gamma\left(\frac{1}{2}(N - K + 1)\right)\right. \\ &\times \Gamma\left(\frac{1}{2}(N + K - 1)\right) + \frac{2z^{\frac{1}{2}(K-N-1)}\Gamma(N)}{(K - N - 1)} \\ &\times {}_2F_1\left.\left(N, \frac{1}{2}(N - K + 1); \frac{1}{2}(N - K + 3); -\frac{1}{z}\right)\right) \end{aligned} \quad (15)$$

where ${}_2F_1(a, b; c; z)$ is the Hypergeometric Function [12].

Using (15), the numerical values of CDFs of ratio of the largest and the smallest eigenvalues for selected values of K and N are available in Table. 1. As desired, the resulting decision threshold γ for a given probability of false alarm (P_{fa}) is calculated numerically by solving

$$P_{fa} = 1 - F(\gamma). \quad (16)$$

It is to note that in (16), $F(\gamma)$ is solved numerically in Matlab using the trapezoidal integration method [14]. For a given target P_{fa}, the corresponding the decision threshold for $K = 2$ and $N = 150$ is shown in Fig. 3. The analytically calculated decision threshold is compared with the simulation based approach (compare the

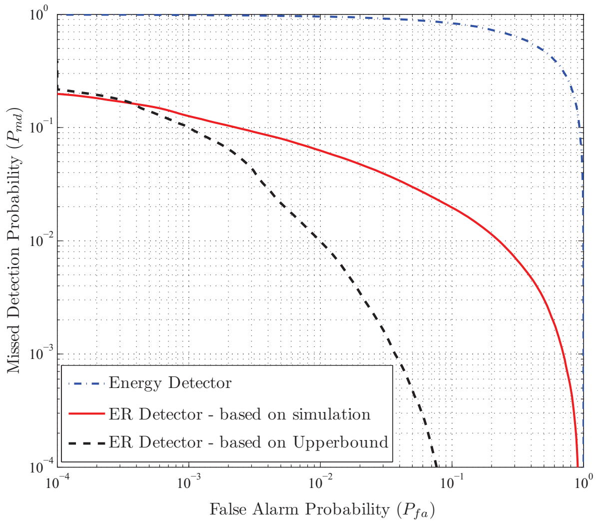

Fig. 5: Receiver operating characteristic curves for $K = 20$ and $N = 50$.

solid curve with the dashed curve). It can be seen that the results of our proposed approach is acceptable for low probability of false alarm. However, there is a trade off between the acceptable performance and mathematical tractability with the calculation of decision threshold. Similarly, Fig. 4 shows the calculated decision threshold as a function of range of probability of false alarm for $K = 4$ and $N = 150$. It can be seen that decision threshold based on our approach is useful for reasonably small number of collaborating SUs to detect the occupancy of spectrum.

Finally, the receiver operating characteristics (ROC) curves are given in Fig. 3 which shows achieved probability of missed detection under $\mathcal{H}_0$ versus the probability of false alarm under $\mathcal{H}_1$. In this figure, we assume a constant modulation transmitted signal with $K = 20$ collaborating SUs and $N = 50$ samples during the given sensing time. The SNR is set to be -10 dB, while the noise uncertainty is set to 1/2 dB. It is apparent that ER detector behaves better than energy detector. However, our approach provides a closed lower bound to the simulation result (compare the solid curve with the dashed curve).

5. CONCLUSIONS

In this paper, we used upper bound on joint PDF of the largest and the smallest eigenvalues of the received covariance matrix to derive a simple and analytical tractable expression for distribution of ratio of eigenvalues. It has been shown that the upper bound can be exploited effectively to approximate the decision threshold for a given probability of false alarm. The proposed approach is useful specially for reasonably small number of collaborating SUs. However, a trade off is required to be defined between acceptable performance and mathematical tractability of the detection problem.

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