

## On Hack's law

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**Abstract.** Hack's law is reviewed, emphasizing its implications for the elongation of river basins as well as its connections with their fractal characteristics. The relation between Hack's law and the internal structure of river basins is investigated experimentally through digital elevation models. It is found that Hack's exponent, elongation, and some relevant fractal characters are closely related. The self-affine character of basin boundaries is shown to be connected to the power law decay of the probability of total contributing areas at any link and to Hack's law. An explanation for Hack's law is derived from scaling arguments. From the results we suggest that a statistical framework referring to the scaling invariance of the entire basin structure should be used in the interpretation of Hack's law.

### 1. Introduction

Hack [1957] demonstrated the applicability of a power function relating length and area for streams of the Shenandoah Valley and adjacent mountains in Virginia. He found the equation

$$L = 1.4A^{0.6} \quad (1)$$

where  $L$  is the length of the longest stream in miles from the outlet to the divide and  $A$  is the corresponding area in square miles. Hack also corroborated his equation through the measurements of *Langbein* [1947], who had measured  $L$  and  $A$  for nearly 400 sites in the northeastern United States. *Gray* [1961] later refined the analysis, finding a relationship  $L \propto A^{0.568}$ . Many other researchers have corroborated Hack's original study, and, although the exponent in the power law may slightly vary from region to region, it is generally accepted to be slightly below 0.6. Equation (1) rewritten as  $L \propto A^h$  with  $h > 0.5$  is usually termed "Hack's law."

*Muller* [1973], on the basis of extensive data analysis of several thousand basins, found that the exponent in Hack's equation was not constant but that it changed from 0.6 for basins less than 8,000 square miles (20,720 km<sup>2</sup>) to 0.5 for basins between 8,000 and 10<sup>5</sup> square miles (20,720–259,000 km<sup>2</sup>), and to 0.47 for basins larger than 10<sup>5</sup> square miles (259,000 km<sup>2</sup>).

As *Mesa and Gupta* [1987] point out, *Muller's* empirical observations are not consistent with the implications of the random topology model of channel network structure first in-

troduced in the classic paper of *Shreve* [1966]. In fact, they theoretically derived the value of Hack's exponent,  $h$ , for the random topology model of channel networks as

$$h(n) = \frac{1}{2} \left( \frac{\pi + (\pi/n)^{1/2}}{\pi - 1/n} \right) \quad (2)$$

where  $n$  is the basin's magnitude. Equation (2) implies a continuously decreasing  $h(n)$  with an increasing  $n$ . For  $n = 10, 100$ , and 500 the exponent  $h(n)$  is 0.68, 0.530, and 0.513, respectively. When  $n$  tends to infinity,  $h$  tends to the asymptotic value of 0.5. This result makes clear the importance of the magnitude of the network in the exponent  $h$  under the premises of the random topology model. Further and more general results on random trees can also be found in work by *Durrett et al.* [1991].

The classical explanation for the exponent  $h$  being larger than 0.5 was to conjecture that basins have anisotropic shapes and tend to become narrower as they enlarge or elongate. The hypothesis of basin elongation was verified by *Ijjasz-Vasquez et al.* [1993] under the framework of optimal channel networks (OCNs), which are the result of the search of fluvial systems for a drainage configuration whose total energy expenditure is minimized [*Rodriguez-Iturbe et al.*, 1992a; *Rinaldo et al.*, 1992]. Thus Hack's relationship may result from the competition and minimization of energy in river basins.

*Mandelbrot* [1983] suggested that an exponent larger than 0.5 in  $L \propto A^h$  could arise from the fractal characters of river channels which cause the measured length to vary with the spatial scale of the object. Thus (1) would be a reflection of a fractal dimension of river channels close to  $d_f = 2 \times 0.6 = 1.2$  [e.g., *Feder*, 1988]. However, as explained in detail by *Feder* [1988], to derive the fractal dimension of a river from the previous argument, one needs to assume that the shapes of the river basins are self-similar. This assumption does not seem to be verified in nature [*Ijjasz-Vasquez et al.* 1994; *Nikora and Sapozhnikov*, 1993].

*Peckham* [1995] derived a relationship between Hack's exponent  $h$  and a fractal dimension, say  $D_{SST}$ , of a self-similar tree as  $h = 1/D_{SST}$ . In the case of a space-filling tree,  $D_{SST} = 2$  and  $h = 0.5$ , while for  $D_{SST} \leq 2$  Hack's exponent  $h$  can be higher. Similar arguments connecting Hack's exponent to the global fractal characters of the network have been proposed by *Nikora and Sapozhnikov* [1993].

One goal of this paper is to suggest a new statistical inter-

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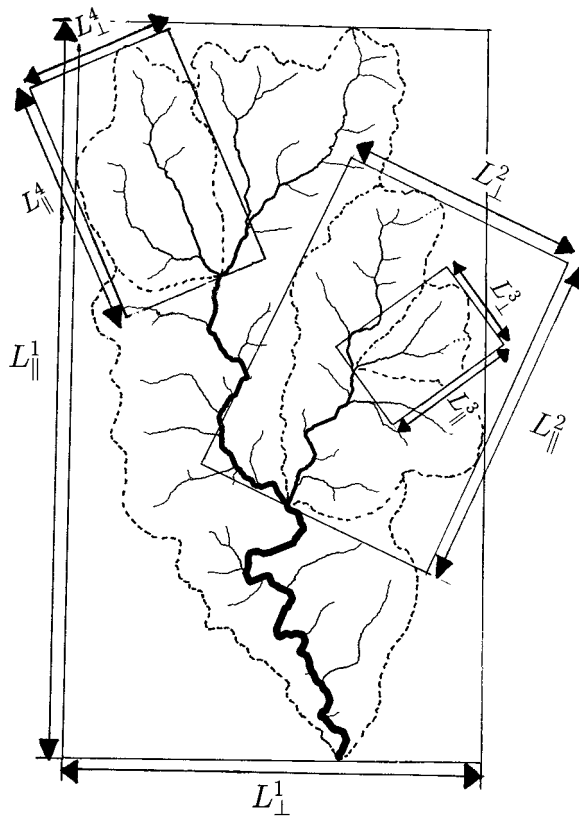
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**Figure 1.** Sketch of a river basin; its diameter,  $L_{\parallel}$ ; and its width,  $L_{\perp}$ . Some subbasins are also drawn. For any subbasin the longest sides of the rectangle enclosing the network are parallel to the diameter  $L_{\parallel}$ , defined as the straight line from the outlet to the farthest point in the basin. The shortest sides are  $L_{\perp}$ .

pretation of the empirical evidence. Specifically, we focus on the internal structure of basins whose extension is in the range of 50–2000 km<sup>2</sup>. Theoretical and experimental motivations justify this choice. At lower scales, diffusive processes interact with concentrative erosive processes responsible for concave landforms, and area-length relationships are altered. At very large scales geologic controls dominate. We expect instead that at medium to small scales, self-organization plays a predominant role, yielding the observed recurrent characters of river basins. Furthermore, *Montgomery and Dietrich's* [1992] collection of data shows that a composite data set, from 100 m<sup>2</sup> up to 10<sup>7</sup> km<sup>2</sup>, can reasonably be fitted with an exponent of 0.5 in Hack's relation, and hence a large span of orders of magnitude in basin area is not the most adequate to fit as a whole when investigating Hack's equation.

## 2. Does Hack's Law Imply Elongation?

This section considers the connection between Hack's law, the fractal sinuosity of stream channels, and the elongation of river basins. The meaning of the terms "elongation" and "fractal sinuosity" first needs to be defined.

The planar projection of river basins may be characterized by

$$A = sL_{\perp}L_{\parallel} \quad (3)$$

where  $L_{\perp}$  and  $L_{\parallel}$  are the appropriate measures of length and width scales shown in Figure 1 and  $s$  depends on the shape.

Shapes will be called self-similar if the ratio  $L_{\perp}/L_{\parallel}$  is constant for all areas,  $A$ , and if the shape of the basins,  $s$ , is a constant. Alternatively, if  $L_{\perp}/L_{\parallel}$  decreases as  $A$  increases, with  $s$  being constant, basins are self-affine and elongate with  $A$ . With  $s$  constant we now define

$$a(L_{\parallel}) = A/L_{\parallel}^2 \propto L_{\perp}/L_{\parallel} \quad (4)$$

Constant  $a(L_{\parallel})$  then implies self-similarity, while  $a(L_{\parallel})$  decreasing with  $A$  implies self-affinity.

One interpretation of Hack's law is that the length measured along channels from the outlet to the divide,  $L$ , scales as  $L_{\parallel}$ , while  $s$  remains constant. Then using (1) in (4), one obtains for  $h = 0.57$ :

$$a(L_{\parallel}) \propto A^{1-2h} \sim A^{-0.14} \quad (5)$$

This suggests that  $a(L_{\parallel})$  decreases as  $A$  increases, implying that according to Hack's law, basins elongate with increasing  $A$ .

Another interpretation of Hack's law suggested by *Mandelbrot* [1983] is that stream channels are fractals and therefore stream length,  $L$ , is scaling as

$$L \propto L_{\parallel}^{\phi_L} \quad (6)$$

where  $\phi_L$  is a fractal scaling exponent. If river basins are assumed to be self-similar in shape ( $a(L_{\parallel}) \sim \text{const}$ ), one has

$$A/L_{\parallel}^2 = A/L_{\parallel}^{2/\phi_L} \sim \text{const} \quad (7)$$

and thus  $L \propto A^{\phi_L/2}$ , which implies

$$\phi_L = 2h \quad (8)$$

The more general case compatible with Hack's law is that streams are fractals and basins are self-affine. Self-affinity of watershed shapes may be quantified through the Hurst exponent  $H$  [e.g., *Meakin et al.*, 1991; *Nikora and Sapozhnikov*, 1993]:

$$L_{\perp} \propto L_{\parallel}^H \quad (9)$$

where  $H < 1$ , because by definition  $L_{\perp} < L_{\parallel}$ . Then

$$A = sL_{\parallel}^{H+1} \quad (10)$$

and  $a(L_{\parallel})$  becomes

$$a(L_{\parallel}) = L_{\perp}/L_{\parallel} \propto L_{\parallel}^{H-1} \quad (11)$$

For  $H < 1$ ,  $a(L_{\parallel})$  decreases as  $L_{\parallel}$  increases, implying elongation. Using (1) in (4), one obtains

$$a(L_{\parallel}) \propto L_{\parallel}^{1/h}/L_{\parallel}^2 \quad (12)$$

which combined with (6) yields

$$a(L_{\parallel}) \propto L_{\parallel}^{\phi_L/h-2} \quad (13)$$

Thus we have [*Maritan et al.*, 1996]

$$H = (\phi_L/h) - 1 \quad (14)$$

which relates self-affinity,  $H$ , the fractal sinuosity of streams,  $\phi_L$ , and Hack's exponent,  $h$ ; with reference to the notation of *Maritan et al.* [1996], we have  $\phi_L = d_i = \varphi$ . Notice that (14) differs from previous relationships [*Meakin et al.*, 1991; *Nikora and Sapozhnikov*, 1993] for the key introduction of the scaling exponent  $\phi_L$ . Self-similarity, on the other hand, implies (8). One observes that (14) imposes constraints between  $\phi_L$  and  $h$ . Thus Hack's exponents between 0.57 and 0.6 require values of

$\phi_L$  smaller than 1.14 and 1.2, respectively, for elongation to occur. Higher values of  $\phi_L$  will imply contraction of river basins when increasing with size.

The relationships derived above were first tested in river channel networks extracted from digital elevations models (DEMs) using standard procedures [e.g., *Jenson and Domingue, 1988; Tarboton et al., 1991*]. The basins used in the analysis are given in Table 1.

Figure 2a shows the result of Hack's analysis. While the best estimate of the slope is close to the commonly found value of 0.6, we notice that the detailed study of the morphology of the fluvial basins allowed by DEMs permits the analysis of the statistical fluctuations present in Hack's type of diagrams. These fluctuations were neither considered in the original work by Hack nor in the experimental studies following it. From Figure 2b we have

$$L_{\parallel} \propto A^{0.52} \tag{15}$$

which gives a Hurst exponent  $H = 0.92$ . The relationship between  $L$  and  $L_{\parallel}$  is presented in Figure 2c:

$$L \propto L_{\parallel}^{1.15} \tag{16}$$

implying a fractal scaling exponent for channel sinuosity of  $\phi_L = 1.15$ . With  $\phi_L = 1.15$  and  $H = 0.92$  in (14), one obtains  $h = 0.60$  that matches the value obtained by direct fitting in Figure 2a.

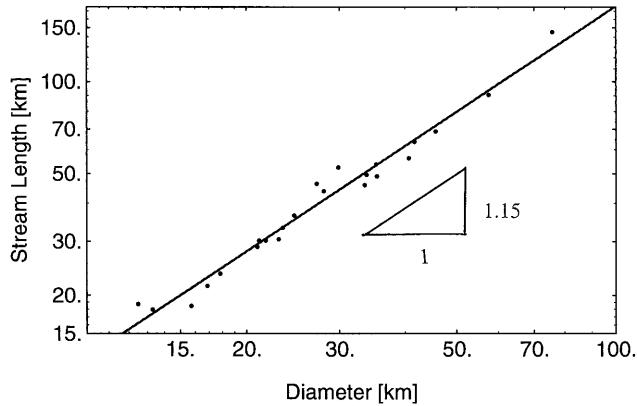
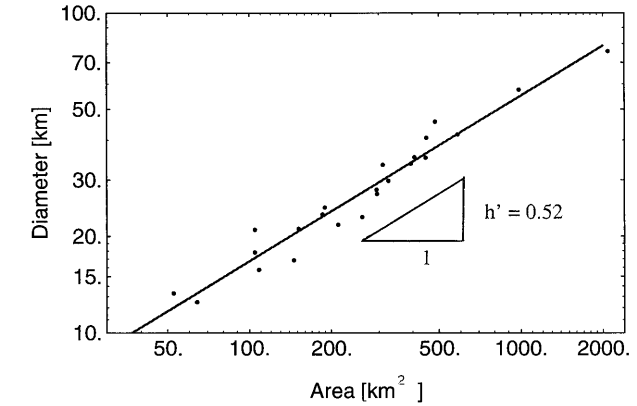
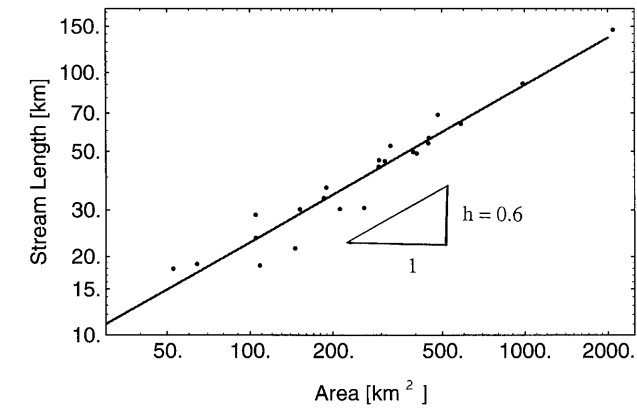
Thus assuming self-similar shapes and neglecting elongation would have resulted in  $h = \phi_L/2 = 0.575$ , and taking streams as nonfractal would have resulted instead in  $h = 1/(0.92 + 1) = 0.52$ . In both cases one underestimates the value of  $h$ . The main contribution to the exponent  $h$  comes from fractal sinuosity, with a smaller contribution coming from elongation.

The above analysis gives standard results and also clarifies the conditions which define geometrical elongation. Nevertheless, the requirement that the shape coefficient  $s$  is a constant is too restrictive to be fulfilled exactly. To account for the

**Table 1.** Geometrical Data Relative to the River Basins Analyzed

River Basin	Location	$A$ , km <sup>2</sup>	$L$ , km	$L_{\parallel}$ , km
Guyandotte	W. Va.	2088	145.1	75.8
Little Coal	W. Va.	984	90.5	57.5
Tug Dry Fork	W. Va.	586	63.7	41.6
Johns	Ky.	484	68.8	45.7
Big Coal	W. Va.	449	56.2	40.7
Racoon	Pa.	446	53.6	35.2
Pigeon	W. Va.	405	49.1	35.3
Moshannon	Pa.	393	49.7	33.8
Brushy	Ala.	325	52.4	29.9
Rock Castle	Ky.	310	45.9	33.5
Sturgeon	Ky.	295	46.3	27.1
Island	W. Va.	260	30.5	23.0
Wolf	Ky.	212	30.3	21.7
Sexton	Ky.	186	33.3	23.4
Big	Ky.	152	30.2	21.1
Indian	W. Va.	148	41.0	28.3
Big	Idaho	146	21.5	16.9
Schoarie Creek Headwater	N. Y.	108	18.5	15.8
Pound	Ky.	105	23.5	17.8
Tipton Run	Pa.	64	18.8	12.5
Blackberry Fork	Ky.	53	18.0	13.3

Data were obtained by processing U.S. Geological Survey 7.5' DEMs.

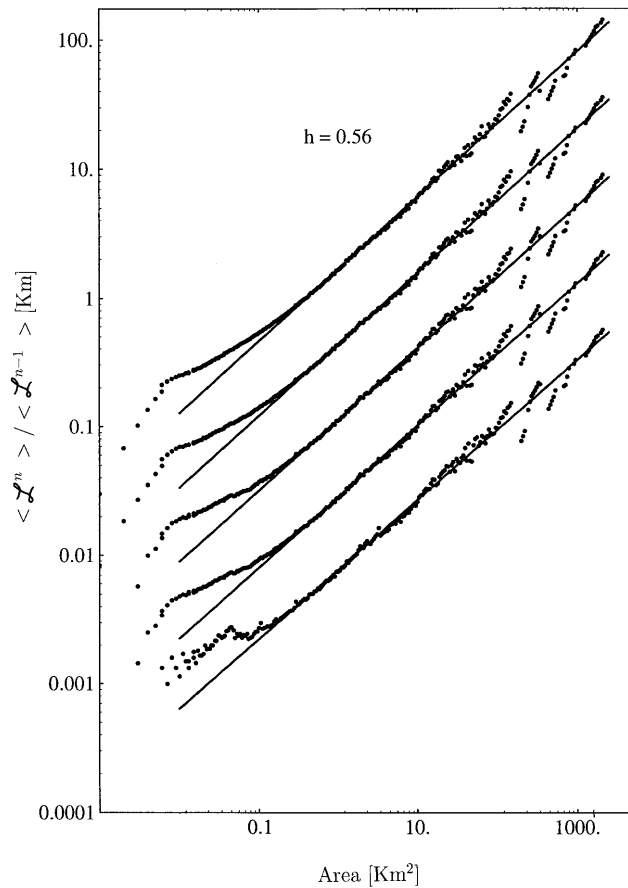


**Figure 2.** (a) Hack's law for the basins in Table 1. (b) Elongation for the same set of data as in Figure 2a. An exponent greater than 0.5 means that basins elongate. (c) Stream length versus diameter for the basins in Table 1.

natural variability of river network structures, it is necessary to relax the deterministic definitions in favor of statistical ones. This will be pursued in the following sections, together with a generalization of Hack's law that preserves the validity of (14).

### 3. Hack's Law and Scaling Arguments

Hack's original work dealt with nonoverlapping basins. In this section we will start with the assumption that a Hack-type relationship is valid for the different subbasins embedded in a larger basin. Thus a statistical framework is developed for the



**Figure 3.** Analysis of the scaling of the moments of stream lengths with area for Guyandotte River. The top curve shows the first moment; the following four curves show the ratio  $\langle \mathcal{L}^n \rangle_a / \langle \mathcal{L}^{n-1} \rangle_a \propto a^h$  for  $n = 2, 3, 4,$  and  $5,$  respectively, shifted downward. The plot shows a well defined simple scaling for at least 2 decades. At the very small scales the scaling is broken owing to the transition from fluvial to hillslope geometry.

different moments of such a relationship and the resulting equations are validated through data from DEMs.

The stream length at any point, say  $\mathcal{L}$ , is defined as the main distance measured through the network from the point to the boundary of the basin. Technically, one defines the mainstream path upstream of any junction following the site having a larger contributing area until a source is reached. In the case of equal contributing areas at a junction one chooses the longest path and, if necessary, makes a random choice. To relate to the notation adopted throughout this paper, we emphasize that for subbasins of area  $a$ , we denote the mainstream length as  $\mathcal{L}$  and, at the closure of the basin, where  $a = A$ , one has  $\mathcal{L} = L$ . Denote now by  $\pi(\mathcal{L}, a)$  the conditional probability distribution of the stream lengths,  $\mathcal{L}$ , from points with a given contributing area  $a$ . The length distribution is then given by

$$\hat{\pi}(\mathcal{L}) = \int da \pi(\mathcal{L}, a) p(a) \quad (17)$$

where  $p(a)$  is the probability density function (pdf) of the contributing areas. We shall further denote with  $\Pi(\mathcal{L}, a)$  and  $\hat{\Pi}(\mathcal{L})$  the corresponding probabilities of exceedence, for exam-

ple,  $\Pi(\mathcal{L}, a) = \int_{\mathcal{L}}^{\infty} \pi(x, a) dx$ . The distribution  $\pi(\mathcal{L}, a)$  will be assumed to obey a scaling form of the following type:

$$\pi(\mathcal{L}, a) = (1/\mathcal{L}^\xi) g[\mathcal{L}/l_c(a)] \quad (18)$$

where  $l_c(a)$  is a characteristic length and  $g(x)$  is a scaling function that accounts for the finite size of any network. It satisfies the following properties:

$$\lim_{x \rightarrow \infty} g(x) = 0 \quad (19)$$

$$\lim_{x \rightarrow 0} g(x) = c \quad (20)$$

where  $c$  is a suitable constant. Equation (18) is a common assumption when analyzing scaling behavior [e.g., Meakin et al., 1991; Maritan et al., 1996]. The first condition, (19), ensures the convergence at infinity of the integral in (17), while (20) gives a power law behavior for  $\pi(\mathcal{L}, a)$  in the infinite size limit, that is,  $\pi(\mathcal{L}, \infty) \sim \mathcal{L}^{-\xi}$  when  $l_c(a) \rightarrow \infty$ . Assuming that a Hack-type relationship is valid throughout a basin, whatever characteristic length we use will obey

$$l_c(a) \sim a^h \quad (21)$$

Thus (18) becomes

$$\pi(\mathcal{L}, a) = \mathcal{L}^{-\xi} g(\mathcal{L}/a^h) \quad (22)$$

The mean value of  $\mathcal{L}$  with a common contributing area  $a$  is given by

$$\langle \mathcal{L} \rangle_a = a^{h(2-\xi)} \int_{\varepsilon}^{\infty} x^{1-\xi} g(x) dx \quad (23)$$

where the lower limit of the integration is irrelevant because  $\xi < 2$ . We then have

$$\langle \mathcal{L} \rangle_a \sim a^{h(2-\xi)} \quad (24)$$

If Hack's law is considered valid for any subbasins embedded in a basin, then the original relationship is recovered from (24) when  $\xi = 1$ . It now has an explicit statistical character referring to the mean value of (22). Equation (22) explicitly postulates fractality as it embeds a basic self-affinity in the distribution of lengths when these are rescaled by a factor  $a^h$ . Notice that under this framework any moment of the distribution of lengths with a common contributing area  $a$  is given by

$$\langle \mathcal{L}^n \rangle_a \propto a^{h(n-\xi+1)} \quad (25)$$

and thus

$$\langle \mathcal{L}^n \rangle_a / \langle \mathcal{L}^{n-1} \rangle_a \propto a^h \quad (26)$$

We have tested the above framework by plotting in double logarithmic scales the first moment and the ratio of the first five consecutive moments versus area (e.g., equation (26)) for the Guyandotte River, the Tug Dry Fork River and the Johns River. These are shown in Figures 3, 4, and 5. We observe that clear straight lines emerge whose slopes are  $0.56 \pm 0.01$ ,  $0.54 \pm 0.01$ , and  $0.60 \pm 0.01$ . Deviation from the straight line at the very small scales is due to the transition from a fluvial to a hillslope type of topography. We have also tested other basins reported in Table 1 with analogous results. First moments of the length distribution for the largest basins of Table 1 are reported in the third column of Table 2. None of the basins shows a consistent difference between the value of the exponent of the first moment in (24),  $h(2 - \xi)$ , and the outcome

of the moment ratios,  $h$ , in (26), suggesting that  $\xi$  is equal to 1. Thus the above assessment of Hack's exponent defined through scaling arguments yields values close to the original ones in which suitable nonoverlapping basins were analyzed with very robust statistics.

Now define  $\mathcal{L}_\parallel$  as the random variable that represents the diameter of a subbasin whose outlet is a randomly chosen point inside the basin. Its distribution can be formally analyzed in complete analogy with the one of the random variable  $\mathcal{L}$ . In particular, its first moment scales with contributing area as

$$\langle \mathcal{L}_\parallel \rangle_a = ba^{h'} \quad (27)$$

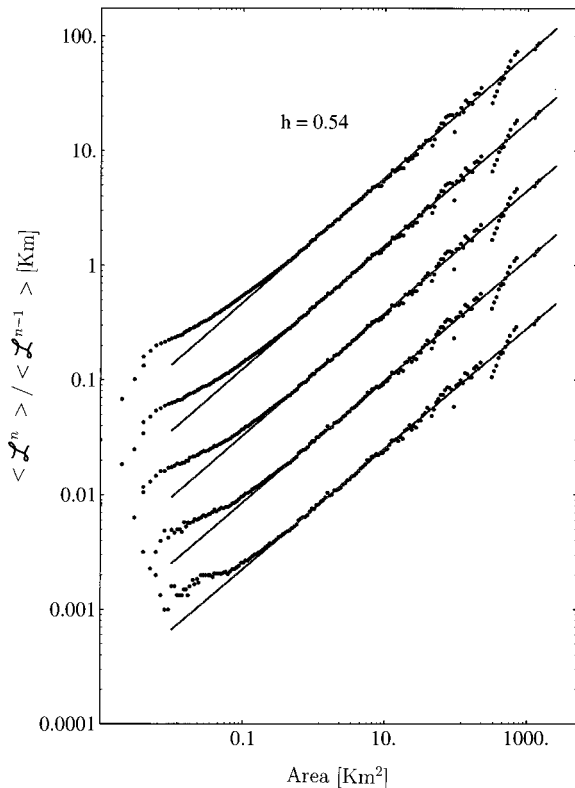
where  $h' \geq 0.5$  and  $b$  is a numeric coefficient. In a way analogous to (10) we introduce  $H = 1/h' - 1$  where  $H < 1$  implies self-affinity. The fourth column of Table 2 reports the value of the Hurst exponent  $H$  evaluated as described above, and the fifth column gives the evaluation of  $H$  as in (14). Most of the values are below 1, and thus the range of  $H$  reported by Maritan *et al.* [1996] is underestimated.

#### 4. Hack's Law and the Fractal Structure of River Basins

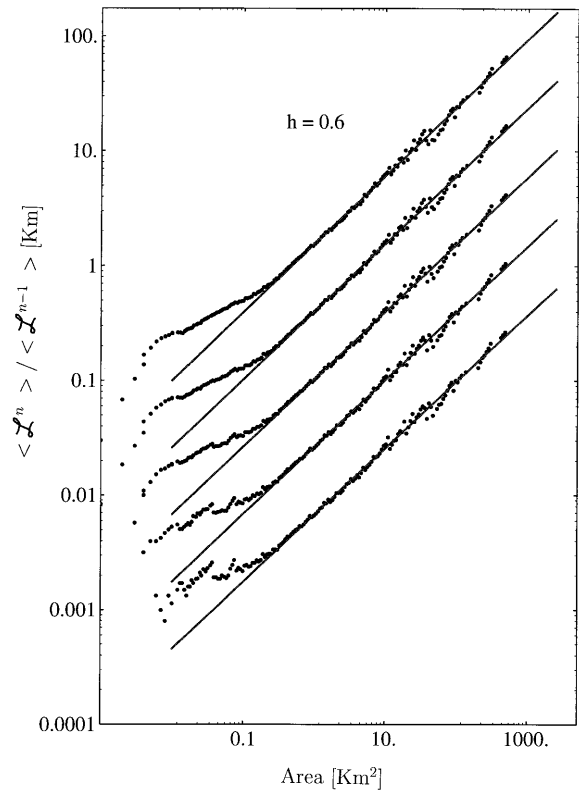
Hack's law generalized as above is now related to the fractal structure of river basins through the distribution of contributing areas and the distribution of stream lengths.

Rodriguez-Iturbe *et al.* [1992b] found that the probability distribution for the contributing area to a randomly chosen point in a drainage network follows the power law

$$P[A > a] \propto a^{-\beta} \quad (28)$$



**Figure 4.** Analysis of the scaling of the moments of stream lengths with area for Tug Fork River. See caption of Figure 3 for details.



**Figure 5.** Analysis of the scaling of the moments of stream lengths with area for Johns River. See caption of Figure 3 for details.

where  $\beta$  is most frequently in the range between 0.43 and 0.45 for basins of very different geologic conditions, climate, vegetation, and soils. The values of  $\beta$  for the basins in Table 1 whose areas are larger than 200 km<sup>2</sup> are reported in Table 2.

Maritan *et al.* [1996] showed that the exponent  $\beta$  can be expressed as

$$\beta = (H + 1 - \phi_L)/(H + 1) \quad (29)$$

From (14) we observe also that  $h = \phi_L/(H + 1)$ . Thus (29) implies

$$\beta + h = 1 \quad (30)$$

As shown in Table 2, this is in relatively good agreement with measurements from DEMs.

Ijjasz-Vasquez *et al.* [1994] studied the scaling properties of the main channel and the boundaries of river basins. They concluded that both are self-affine objects. The boundary of the basin is composed of two self-affine fractals starting from the outlet and merging in the farthest point from it. The straight line joining the outlet and the merging point coincides with  $\mathcal{L}_\parallel$  and can be regarded as the analogy of the first collision time of two self-affine fractals starting from a common origin. The self-affinity of the two fractal trails is characterized by a scaling relation similar to (9) between the maximum distance measured orthogonally from the diameter to the boundary, which is proportional to  $L_\perp$ , and  $\mathcal{L}_\parallel$  itself. The basin boundaries studied by Ijjasz-Vasquez *et al.* [1994] had a well-defined Hurst exponent,  $H$ , between 0.74 and 0.80, in agreement with the measurements reported by Nikora and Sapozhnikov [1993] and with the measurements in Table 2, where  $H \geq 0.75$ .

**Table 2.** Values Relative to the Rivers in Table 1 Whose Area Is Larger Than 200 km<sup>2</sup>

River Basin	$\phi_L$	$h$	$H$	$(\phi_L/h) - 1$	$\beta$	$\gamma$	$\beta/h$
Guyandotte	1.06	0.56	0.92	0.89	0.43	0.74	0.77
Little Coal	1.06	0.56	0.92	0.89	0.41	0.74	0.73
Tug Dry Fork	1.07	0.54	1.00	0.98	0.41	0.77	0.76
Johns	1.06	0.60	0.75	0.77	0.40	0.67	0.67
Big Coal	1.05	0.56	0.89	0.88	0.42	0.75	0.75
Racoon	1.02	0.52	1.00	0.96	0.45	0.89	0.86
Pigeon	1.06	0.55	0.92	0.93	0.45	0.75	0.82
Moshannon	1.02	0.52	0.96	0.96	0.45	0.86	0.85
Brushy	1.04	0.54	0.96	0.93	0.43	0.82	0.80
Rock Castle	1.06	0.55	0.92	0.93	0.43	0.79	0.78
Sturgeon	1.03	0.52	1.01	0.98	0.46	0.92	0.88
Island	1.07	0.54	0.96	0.98	0.43	0.78	0.80
Wolf	1.07	0.55	0.92	0.96	0.41	0.75	0.75

Here  $\phi_L$ , sinuosity exponent;  $h$ , Hack's exponent;  $H$ , Hurst's exponent;  $\beta$ , power law exponent for areas;  $\gamma$ , power law exponent for lengths. Data were obtained after extracting the drainage networks from DEMs using the threshold  $A_t = 50$  pixels as the maximum contributing area necessary to maintain a channel. The exponent  $h$  was estimated by fitting  $\langle \mathcal{L} \rangle_a$  versus the contributing area in a log-log plot. The value of  $H$  was obtained from  $\langle \mathcal{L} \rangle_a \sim a^{(1/H)+1}$  where  $\langle \mathcal{L} \rangle_a$  is the expected value of diameters with the contributing area,  $a$ . In column 5, (14) is used to estimate  $H$  from the values of  $\phi_L$  and  $h$  whose values are reported in columns 2 and 3, respectively. In column 8,  $\gamma$  is estimated according to (31) being  $\beta$  and  $h$  as given in columns 6 and 3, respectively.

This argument, (29), and (30) clearly show that the self-affine characteristics (e.g., Hurst exponent) of the basin boundaries are intimately related to Hack's exponent and the distribution law of contributing areas.

Now consider the random variable defined as the distance,  $\mathcal{L}$ , measured through the network from a randomly chosen point to the boundary of the basin. The exceedence probability distribution of this distance may be derived from Hack's law and the power law distribution of contributing areas.

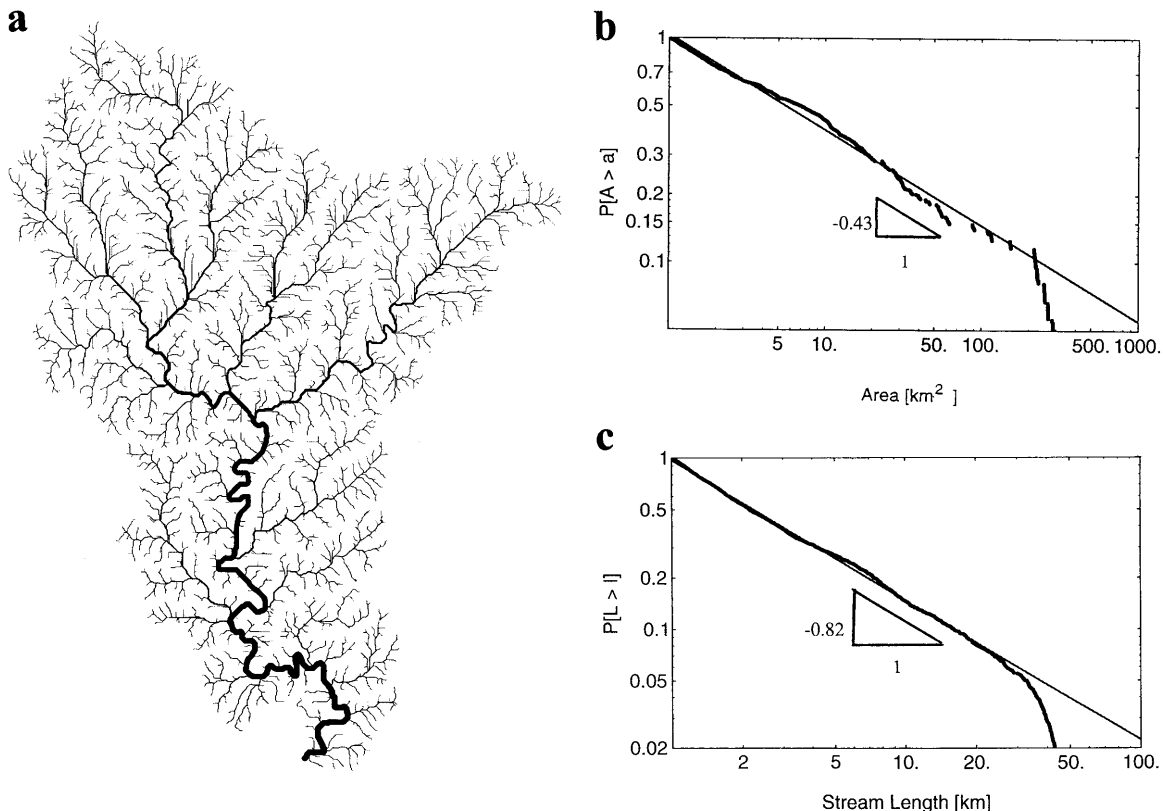
Calling  $\mathcal{L}$  the random stream length to the divide, we have

$$P[\mathcal{L} \geq l] \propto P[A^h \geq l] = P[A \geq l^{1/h}] \propto l^{-\beta/h} \equiv l^{-\gamma} \quad (31)$$

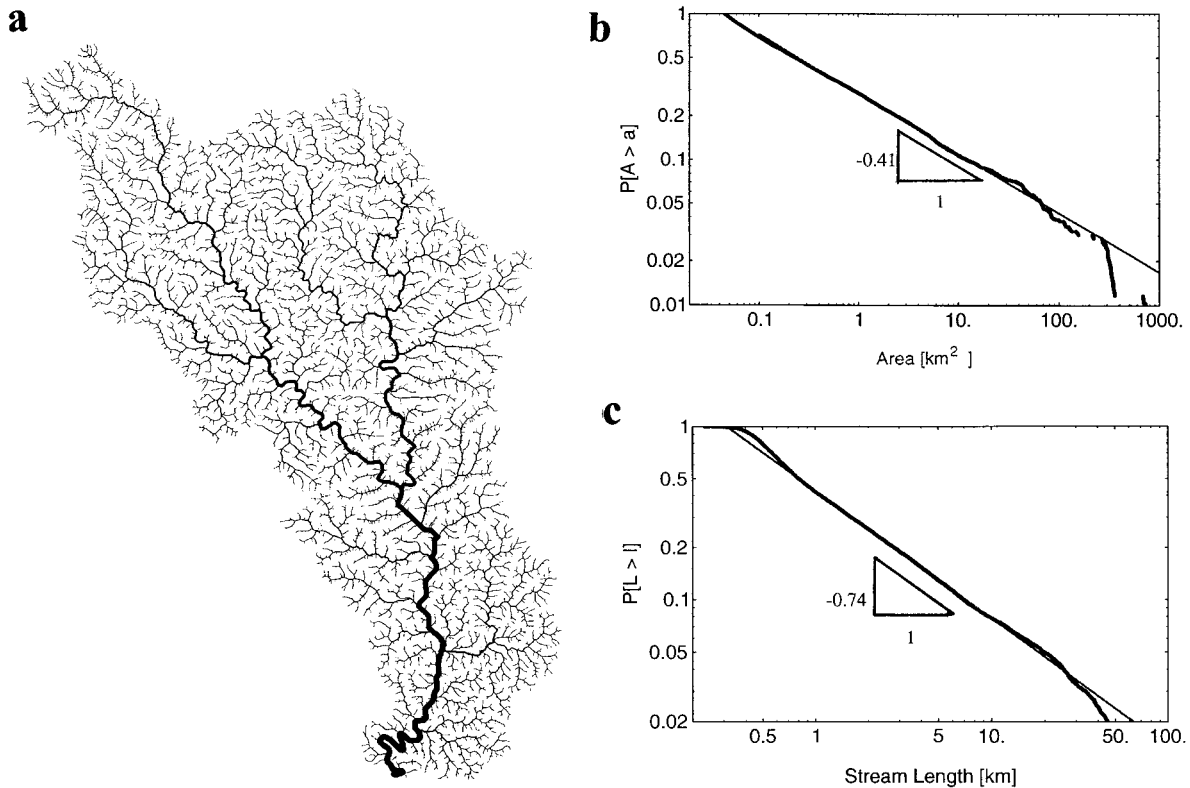
where  $\beta$  is defined in (28) and  $h$  is Hack's exponent, assumed here to hold also for the internal structure of a basin. This holds for  $\xi = 1$  in (22). The power law of lengths defined in (31) has been verified in numerous basins with excellent fitting throughout several logarithmic scales. It links the aggregation pattern of areas with the stream length structure of the network and Hack's law.

Table 2 shows a synthesis of the experimental analyses conducted on the DEMs of the basins in Table 1 with area larger than 200 km<sup>2</sup>. The exponent  $\gamma$  falls between 0.70 and 0.90 for values of  $\beta$  between 0.41 to 0.46 and  $h$  between 0.52 and 0.60 with an average of 0.55. The average exponents of the probability distributions are 0.79 for the lengths and 0.43 for total contributing areas.

Figures 6 and 7 show examples of  $P[\mathcal{L} \geq l]$  for the river basins whose DEM features have been described in Table 1. In particular, Figure 6 shows the detailed results obtained for the Brushy Creek basin where we have used a support area of 50 pixels to define the drainage networks and the lengths are measured in kilometers. In this case, the exponent of the power law of lengths is 0.8, the value of  $\beta$  is 0.45, and the value of  $h$  resulting from (31) is 0.56. Figure 7 shows the same results for the Little Coal River basin.



**Figure 6.** (a) Brushy Creek with support area 50; (b)  $P[A > a]$  versus  $a$  ( $\beta \sim 0.45$ ); (c)  $P[\mathcal{L} \geq L]$  versus  $L$  ( $\gamma \sim 0.82$ ).



**Figure 7.** (a) Little Coal River with support area 50; (b)  $P[A > a]$  versus  $a$  ( $\beta \sim -0.41$ ); (c)  $P[L \geq L]$  versus  $L$  ( $\gamma \sim 0.74$ ).

The consistency of the measured exponents suggests that the scaling theory developed in this paper is a comprehensive description of the planar organization of basins.

## 5. Conclusions

The results of this paper suggest the following conclusions:

1. Hack's law appears to hold, in a statistical sense, for any point inside a basin.

2. The statistical framework developed here predicts that the ratio of consecutive moments of the distribution of stream lengths to the divide from points with the same contributing area should follow a power law as a function of area with the exponent being close to Hack's  $h$ . This prediction is confirmed in natural basins.

3. Hack's exponent, the self-affinity of the river basin, and the fractal sinuosity of its channels are related by  $H = \phi_L/h - 1$ . For elongation to occur,  $H$  needs to be smaller than 1. This imposes constraints between Hack's exponent and the fractal characters of rivers. The values of  $h$  and  $\phi_L$  obtained in natural river basins yield  $H < 1$  and thus confirm the tendency to elongate.

4. Hack's relationship and the self-affine character of basin boundaries are related. The values obtained for  $H$  and  $\phi_L$  in real basins yield excellent agreement with the commonly accepted values of  $h$ .

5. The exponents  $\beta$  and  $h$  are found to be dependent variables through  $\beta + h = 1$ . Estimations of  $\beta$  and  $h$  in natural basins tend to confirm this dependency.

6. The stream length from any point to the divide follows a power law distribution whose exponent,  $\gamma$ , is the ratio of the exponent of the power law describing the contributing area at

any randomly chosen point,  $\beta$ , and Hack's exponent,  $h$ . Analyses of DEMs yield an excellent fit to the power law for the stream lengths and moreover confirm the relation between  $\gamma$ ,  $\beta$ , and  $h$ .

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