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005

GAMES TO TEACH MATHEMATICAL MODELLING*

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Abstract. We discuss the use of in-class games to create realistic situations for mathematical modelling. Two games are presented which are appropriate for use in post-calculus settings. The first game reproduces predator–prey oscillations and the second game simulates disease propagation in a mixing population. When used creatively these games encourage students to model realistic data and apply mathematical concepts to understanding the data.

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1. Introduction.

For the last two years I (J. Powell) have been involved with the SouthWestern Regional Institute for Mathematical Sciences (SWRIMS) at USU, an NSF-funded effort to involve mathematical researchers in educational issues. In the first year we focused on illustrating and enlivening mathematics through its application in biology and have evolved a Biology/Applied Mathematics Instruction Model (BAMIM). The central idea of BAMIM is that the mental state of a mathematical researcher creating mathematics for a biological application is the appropriate state of mind for a student who is learning mathematics. To this end, we have explored various collaborative educational arrangements between biological and mathematical sciences, both of which are traditionally taught in lecture-oriented environments.

At the end of my first decade in post-secondary education, I have become convinced that one of the biggest teaching challenges is the passive nature of lecture-oriented learning. While I can motivate, entertain, and illuminate in lectures, real learning occurs when students are forced to take responsibility for making the concepts make sense [1, 5, 6]. Traditionally this learning occurs outside the classroom, when students try to understand homework problems in the dark hours with nobody else to rely on. Survivors of this experience are invariably convinced of its utility and go on to construct curricula which are predicated on similar experiences for future students. The lonely and rarefied nature of this type of education, however, gives many students a needlessly negative perspective on mathematics and its applications [2, 3, 7].

Moreover, most students who survive this experience develop reliance on passive aids (books, calculators, examples, and text recipes) to see them through the midnight hours. They are talented in manipulating symbols and following textbook recipes but terrified of applying mathematical knowledge in unknown territory. To use a linguistic analogy, they have a rigorous knowledge of mathematical grammar but are barely conversational and certainly not colloquial [8]. Modelling and understanding science through mathematics is very hard for lecture-oriented students [1, 9].

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This is the context in which we have developed BAMIM. Our goal has been to overcome passivity and isolation in traditional mathematical education at one stroke. We confront students with biological circumstances and assist them in group construction and evaluation of mathematical concepts. Where possible, students are involved in observation and data collection, establishing a sense of ownership and responsibility for the science. Students emerge with a broader appreciation of mathematics and a greater ability to apply it. The greatest difficulty with BAMIM is finding "hands-on" biological applications for students which are convenient in a mathematical classroom and can be managed by a math instructor.

I have addressed this difficulty with "games," or simulations, with which students can generate surprising, but deterministic, data mimicking biological phenomena. Games have many advantages for teaching modelling. First, the mechanisms are known. They are the "rules" under which the game is played, and depending on the game construction these rules may be relatively easy to translate into mathematics. Second, all the state variables in a game are easily measurable. It can be very difficult to perform appropriate experiments outside a fully equipped biology lab. For example, experiments illustrating predator–prey oscillations between protozoan species have been performed, but they require the services of a media kitchen and finicky control of the protist environment. Game playing offers all the educational advantages of experimentation to the student, with none of the logistic drawbacks. Below are two games I have constructed and used to advantage in a variety of BAMIM contexts.

2. The predator–prey game.

2.1. Provenance of the game. The Lotka–Volterra and similar predator–prey equations [4] are old stand-by examples in mathematical ecology and provide very interesting ways to illustrate the influence and practice of nonlinear dynamics. At the modelling level, these equations illustrate commonly used assumptions in nonlinear modelling: continuous approximations to discrete systems and the "law of mass action." For all the frequent use and homage paid to the Lotka–Volterra model, few students (or mathematicians, for that matter) have observed oscillating populations or faced issues of parametrization. Also, students rarely understand where the law of mass action comes from, what its pitfalls are, or how to use it in modelling.

As an alternative to the experimental difficulties, I devised a predator–prey game. Students would simulate predator–prey interactions on a hexagonal grid, with the aim of reproducing oscillations like those in published data sets. An initial set of "rules" for the game was provided (see Appendix A). Students were then encouraged to alter the rules of the game, add refuges and territory, until they were able to reproduce oscillatory behavior.

2.2. The predator–prey experience. With the initial set of rules described in Appendix A and any initial values for coyotes and bunnies, the system collapses. First the prey species grows very dense, then the predators grow very dense and eat all the prey, then they starve and go extinct. Students typically attempt to create stability by changing the relative balance of predators and prey, trying for a happy equilibrium. This is frustrating, particularly since each iteration of the game takes five to ten minutes and the task is impossible anyway.

At this point (if they haven't already hit on the idea themselves), I suggest using modelling to figure out what the happy, natural balance should be. The students rapidly write down models and solve for equilibria. Upon trying the equilibrium values students find that oscillations grow and grow and the ecosystem collapses anyhow.

Generally students then go back to the drawing board, recheck their calculations, simulate again, and collapse the system again.

A stability analysis of the fixed point is suggested, which turns out to be oscillatory and unstable. Having just confirmed this directly through experimentation, students enjoy the mathematics predicting the oscillations they have simulated. This also provides an avenue for them to design a new "ecosystem"; by leaving model parameters free, they can seek interactions with desired stability properties. If the aim is to produce stable oscillations students must re-engineer the model, then reverse-engineer game rules reflecting the new model.

Variations of the game have been successfully used in a variety of contexts, with elementary through college students, and at workshops with local teachers. At various levels it can be used to introduce various ideas. For example, teachers suggested that it be used to motivate various algebraic skills at the high school level. At the collegiate level it provides illustration and application for many concepts in dynamical systems, as well as a microcosmic example of the interaction between theoretical and applied sciences through mathematics and modelling. I can lecture until I am blue in the face on the various imports and applications of stability and modelling, but the predator–prey game captures student interest. Since the students build the models, collect the data, and are responsible from start to finish, the lessons they learn are printed indelibly [10, 11, 12].

3. The disease game.

3.1. Why play with a disease? The behavior of diseases in our population fascinates students. I first discovered this in a series of lectures I gave in a science orientation class at Utah State University. The subject for the quarter was the human population, and lecturers from around the college of science were addressing the subject. I chose to discuss the predictions of various models, one of which was a disease model based on the work of Anderson and May. In a lecture hall of 250 students some degree of shuffling, whispering, and coming and going is generally unavoidable, particularly for the math lecturer. But when I started talking about disease modelling, "vertical and horizontal transmission," and the potential effects of disease on the human population, the audience was rapt.

In keeping with the philosophy of giving students as much responsibility for application and data collection as possible, it seemed reasonable to create a game which captured the elements of disease transfer. Disease grabs student imagination, and I hoped that this would emotionally involve students in applied mathematics and serve to illustrate the concepts I was teaching. For example, in second quarter calculus these were separation of variables, the logistic differential equation, and partial fractions integration. The design of the game and group objectives are presented in appendix B.

3.2. Implementing a disease game. Students implemented the discrete version of the game on a hex-grid system. I presented them with the rules and objectives described in appendix B. The students leapt immediately to doing the simulation, joking about the diseases they were modelling, the "splat factor" of the disease, the "sneeze radius," what direct overlap of diseased and susceptible populations meant, and so forth. With the initial set of basic rules, students found the population moving from susceptible to infected in only five to ten turns (five or so minutes of real time).

In collecting various data sets, students adjusted the "splat factor," how long diseased individuals stayed infectious, and the "sneeze radius." One group "removed" diseased individuals after a number of turns, modelling the effects of death or medical care. To an individual, my students were enthralled with the data collection and simulation and stayed after class to continue.

At first the theoretical end of the exercise was difficult. Even though I had worked through the logistic difference equation using separation of variables and integration by partial fractions, students found it very hard to apply the technique on their own, with free parameters. However, in group work only one student in a group has to begin understanding to infect the entire group. In short order everybody was comfortable applying the integration procedure to find an analytic solution.

The next theoretical difficulty was "fitting" analytic curves to simulated data. I encouraged students to consider the asymptotic limit of the solution, and the long-time behavior of the data, to determine one of the constants in terms of the other. Then they could experiment with the remaining parameter to produce a "good" fit.

This empowered the students enough to exceed my expectations, which were only that they turn the knobs on the model until getting a reasonable fit. Almost all of the groups went further, solving for the remaining parameter analytically, approximating derivatives from their data, and using an averaging procedure to achieve their final estimates (which were surprisingly good). The group who "removed" diseased individuals were completely unable to "fit" their solutions, but in the final report they gave a very nice discussion of where the model had failed and what might have to be done to improve it (which amounted to a verbal description of the susceptible-infected-removed disease model [4]). In the end, students remembered partial fraction integration, separation of variables, and the behavior of the logistic equation. They also learned a great deal about applying mathematics and its utility in surprising areas (not one of the students had initially believed that any math other than statistics would apply to biological questions).

4. Conclusion. The various successes of the disease game are indicative of the virtues of BAMIM in general. Game playing is a good technique for creating data sets which students firmly understand and feel responsible for. Intuitive understanding encourages students to apply mathematics and supplies an idea of when things work and when they don't. The sense of responsibility keeps them going when the going gets tough. As an instructor, I find this a welcome change from students quitting the moment a problem doesn't follow a recipe provided in their text. These advantages accrue to any personally gathered data (experiments, published data sets, whatever). Games, however, fit comfortably into a math classroom, encourage a group-work format, and also lead very directly to concepts in mathematical modelling. Depending on what kinds of questions students are asked, this provides good experience in mathematical modelling (as in the predator–prey game) or how models can be related to the real world (as in the disease game). In either case, students become responsible for the mathematics because they are creating it, as opposed to being victimized by it.

Appendix A. The predator–prey game. In this appendix I present the organization and objectives for the predator–prey game, as I presented it to junior-level college students in an Honors math-biology course.

A.1. Objectives. When this course was first conceived, one of our goals was to explore how some of the modern mathematical models for populations appear in the "real" world. To this end we have attempted to have students generate experimental data and use this to validate models. We were looking forward to predator–prey modelling, where it is possible to observe oscillations, chaos, dynamic equilibrium, etc. in real data. Unfortunately, it proved to be insurmountably difficult to get experimental predator–prey oscillations in a one-quarter class. Therefore, we hit on the idea of using games and model ecologies to get at the basic ideas and provide the "real" part of "real interdisciplinary math-science."

Students will develop a game which captures the major effects of predation between two species, altering the rules of the game to reflect increasingly complex interactions, and involve themselves in building models of their toy ecologies. Our objectives for this exercise are as follows:

- Produce model ecologies which reproduce bunny–coyote population oscillations in the intermountain region.
- Learn how to build various discrete and continuous models of "real" data sets.
- Explore the practical meaning of various changes to ecosystems and characteristics of species.

A.2. Organization.

A.2.1. Format of the meta-game. We will divide into groups of three. Each member of the group will be responsible for predators, prey, and administration by turns. The goals for the meta-game experience are to alter the rules of the game to produce the most "realistic" model ecology with a minimum set of rules, beginning with the simple rules described below. Here are the specific responsibilities to be considered in the meta-game:

1. Decide on and record the current rules for the game.
2. Iterate the game for at least ten turns, collecting data on the populations of predator and prey at each turn.
3. Evaluate the "reality" of this data set with the current set of rules. Record what the problems are.
4. Discuss what to change about the rules of the game. Options may include:
   (a) Rates of predation, reproduction, death....
   (b) Handling time, saturability, functional responses....
   (c) Structure of the environment: refuges, nearest neighbors, numbers per hex....
   (d) Step size, seasonality, Elvis sightings....

   Record what you have changed and why.
5. Change who does what and return to first step.

A.2.2. Rules of the initial game. Here are the beginning rules for the predator–prey game. There is no guarantee that they work at all, but they do give us a starting point. Let C_n and B_n denote the number of coyotes and bunnies at game turn n. Below are the rules for your first iteration of the predator–prey game. First, pick initial values for C_0 and B_0. Then:

1. Predator and prey players allocate their species, C_n and B_n, one per hex, on their game board, placing a red X for predator, a blue O for prey.
2. Overlay the transparencies (prey on top). Count the number of hexes which include both a predator and prey.
3. Calculate predator–prey interactions. For the bunnies:
   (a) The number of eaten bunnies is equal to the number of overlaps. Subtract eaten bunnies from population.
   (b) Surviving bunnies can reproduce. Double the population of surviving bunnies.

(c) This is now B_{n+1}.
(d) Record B_{n+1}.

For the coyotes:

(a) All of the eaten bunny-units become fuzzy baby coyotes.
(b) Lose one-quarter of the original population (round the losses down), then add in the FBCs.
(c) This is now C_{n+1}.
(d) Record C_{n+1}.

4. Return to the reallocation phase with the $(n+1)$st values available for distribution.

If either of the populations becomes zero or less the species has gone extinct.

A.2.3. Ongoing analysis. As you play the game, be alert to how the rules are affecting the population changes. Perhaps slow, lethargic bunnies just go extinct, and in the next set of rules you will need to have encounters between coyotes and bunnies resulting in fewer losses. Or perhaps the coyote birth rate is too slow, and you need to decide on a more fecund kind of coyote. Sometimes the rules are not at fault, but the population values just don't represent a sustainable ecosystem (e.g., 1 bunny and 100 coyotes).

Make sure to include in your data notes on what was wrong, why you changed things, and what you changed too—because next you have to model this game!

A.3. The initial model. An initial model for the initial game rules could be constructed as follows. The probability of a random hex being occupied by a bunny or coyote are, respectively,

$$p_c = \frac{C_n}{100} \qquad \text{and} \qquad p_b = \frac{B_n}{100}.$$

There are B_n chances for predation to occur, so the expected predation losses at turn n are

$$\beta B_n \frac{C_n}{100},$$

and since every overlap of a coyote and bunny results in a bunny loss, we choose $\beta = 1$. The total bunny breeding population is thus

$$B_n - \beta B_n \frac{C_n}{100},$$

which will grow at some rate α. Therefore, future bunnies should be given by:

$$B_{n+1} = \alpha B_n \left[1 - \beta \frac{C_n}{100}\right],$$

and since the breeding population is supposed to double over a turn, $\alpha = 2$.

A similar model for the changes in the coyote population is

$$C_{n+1} = \delta C_n + \gamma C_n \frac{B_n}{100},$$

where $\delta = .75$ and $\gamma = 1$. See if you can figure out where these terms come from, and why the parameters are what they are for the initial game step above. How would you alter this model to reflect alterations in your model ecosystem?

A.4. Modelling and the predator–prey game. Now that you know something about discrete modelling, analyzing discrete models, and basic predator–prey ecology, let's put it together. The following is the outline for a group-homework on discrete models.

1. Refine the rules of your predator–prey game to produce oscillations, including the following:
   (a) Nontrivial topography (perhaps altering the structure of the hex map so that only certain numbers of a species can live there, or to create internal "edges" to lower the number of "nearest neighbors."
   (b) A "refuge" for the prey.
   (c) A density-dependent response (for example, each bunny needs a certain number of hexes to live).
2. Build a discrete model for your ecology, and analyze the fixed points and their stability.
3. Try out the game with your rules; in particular, use some initial conditions near any fixed points in your model above.
4. How do the two compare? If they differ, can you tell why they differ? Are the oscillations due to the structure of your model, or to stochastic variations?
5. Prepare an informal group presentation to the rest of the class on your efforts. It need not be tremendously formal, but your group should be prepared to exhibit some overhead illustrations of both the game and model outputs.

Appendix B. Disease dynamics. Below appears the text of the disease game, which was used in second-quarter calculus classes.

B.1. Introduction. One of the biggest challenges for modern medical science will be dealing with diseases as the human population grows. On the one hand, we have lived in a golden age, with vaccines for the most common viral diseases and antibiotics for bacterial infections. However, bacteria have evolved to resist antibiotics, and recent estimates indicate that the most resistant forms of bacteria (tuberculosis and E. Coli variants) respond to only one antibiotic in the medical inventory. Perhaps even worse, as human populations increase humans come into contact with reservoirs of viral agents which we have no natural immunities to (HIV, Ebola, Marburg, Hantavirus...). With modern capitals all less than a day from one another in travel time, catastrophic amplification and spread of disease is one of the most realistic "doomsday" scenarios in the modern world.

The mathematical modelling of diseases began with Kermack and McKendrick studying the advent of plague in Bombay in 1927; it was put on firm theoretical setting by Anderson and May in the 1970s.

B.2. The disease game. To get an idea of how diseases propagate, we will try a simulation. Each group will get two transparent hex "playgrounds" for the disease simulation. We will model children at school who interact on the playground and infect one another. The population of children will be 50, of which one is initially infected. One team member will play the disease, one will play the susceptible, one will record data, and one will count new infections. Diseased and susceptible children will be placed separately on the playground, and then new infections will be assessed according to interaction rules. The transparencies will be superposed and interactions evaluated by the judge. These new infections are added to the infected population for the next day.

Each group will simulate two different cases: a disease with and without a particular infectious window. Decide how infectious you want your disease to be (e.g., if disease requires direct transferal of body fluids, perhaps infected and susceptible persons need to overlap exactly. A "spray" infection might occur if infected and susceptible members are within one "hex" of one another. A really infectious disease might occur if infected members left of "trail" of sites, which infect any susceptible population passing through the site). Simulate also a case where population members are infectious for only a set number of days, at which point they are "removed" from being infectious (and also from being susceptible).

Perform each simulation at least twice so that team members can rotate through different roles. Don't forget to collect your data!

B.3. Modelling diseases. Let p be the fraction of a mixing populace which is infected with a disease. Then $1 - p$ is the fraction of the population which is susceptible, and a model for p changes in time is

$$\frac{dp}{dt} = \lambda p(1-p) - \gamma p, \tag{1}$$

where λ captures the rate of transmission of the disease in the population and γ reflects the recovery of the infectious population. For different diseases and different mixing populations, λ and γ will vary.

B.4. Project goals. Each team will write up a group report, which will include the following:

- The general solution to equation (1), generated through separation of variables, partial fractions integration, and inversion.
- Choices of parameters which "best" fit the data generated in your simulations. What were your decision criteria?
- Plots of your simulation data and the solution curve which models it. Be sure to describe the rules you used to generate the data.
- What do you think about these approaches to disease modelling? Was the simulation experiment realistic? How good is the mathematical model?

Each team member should assume responsibility for some of the report, but the complete report should dovetail.

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