2016

Leaky Bucket Lab

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Laboratory Experiences in Mathematical Biology

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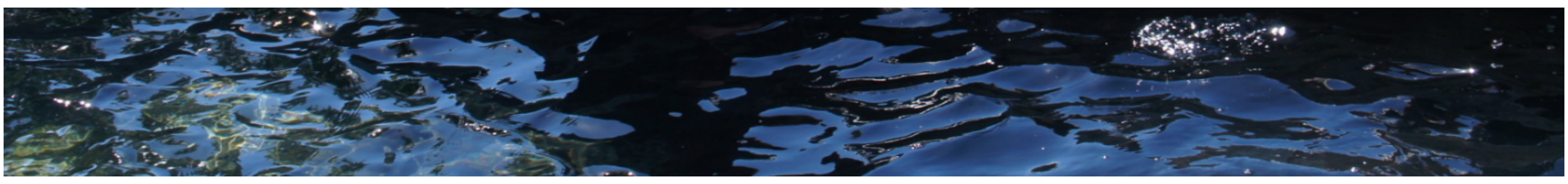
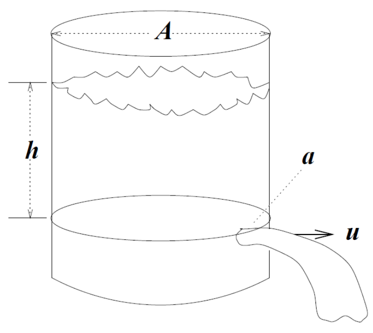
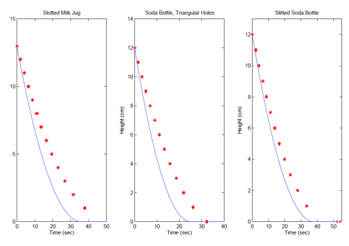




Overview: Students measure and record the height of fluid remaining in a container as it exits through a small hole over time. Torricelli's law is used as a base model to illustrate simple concepts (quadratic polynomials and their roots) for college algebra students, as well as complex concepts (modeling container shapes mathematically and integrating separable differential equations) for more advanced students. At all levels students are encouraged to explore alternate models since the classic model performs poorly in comparison with data.

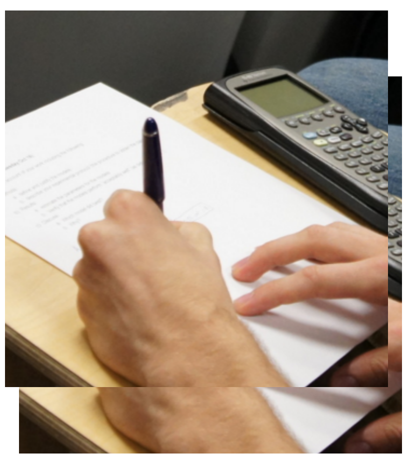
Lesson Outline: Students attempt to explain and predict the time trajectory of fluid exiting a container through a small aperture. In algebra and statistics courses, the lab requires students to comprehend and parameterize the classic Torricelli model. More advanced students must also formulate an alternate model of their own to explain drainage dynamics. See Pedagogical Resources for additional teaching and scaffolding suggestions.

Lab Setup: Students cut an aperture of $a \approx .25\ \mathrm{cm}^2$ and inscribe horizontal marks every centimeter above. Bucket is filled to twelve cm above the hole while the hole is covered with duct tape; students remove the tape and time the bucket's drainage, recording the dynamics of changing height.



Data and Examples: Data along with some student approaches are presented to illustrate the range of student creativity and to help prepare teachers to scaffold student thinking.

Background and Extensions: To build biological context and facilitate in lab presentation, a brief discussion of leaky buckets in nature and Torricelli's Law and is presented here.



Assessment Items: Primary assessment of student learning is taken from students' written reports additional assessment items targeting lab objectives are included here.

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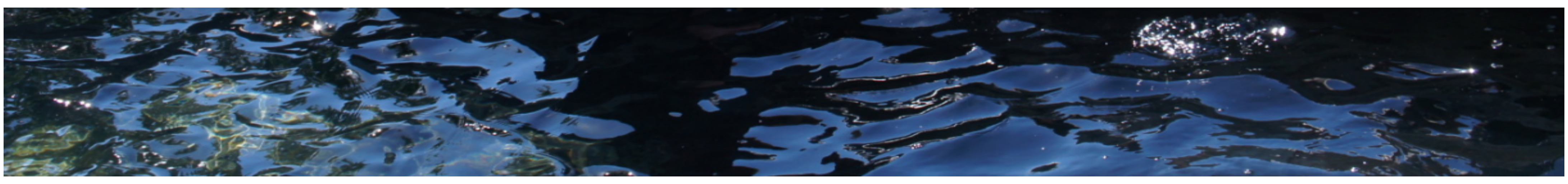

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Expectations

The expectations and lab agenda that follow were written for a mathematical biology class consisting of upper-level mathematics, statistics, biology and engineering students. The lab should be adjusted to fit your students' level of mathematical expertise.

The general objectives for students are:

- Accurately predict the rate of drainage of fluid from a leaky bucket, given knowledge of the bucket's geometry and the size/shape of drainage aperture.
- Create two models (one of which may be the Torricelli model or a close relative) which will predict the emptying time of a leaky bucket which can only be measured, not tested in advance. The models must be ``significantly different'' from each other.
- Calibrate models (i.e., estimate parameters) using data collected from buckets teams construct and test.
- Develop protocol by which team models can be applied to similar, but independent, containers which can only be measured before validation begins.

We ask students (or student groups) to produce a short written report or present their findings via PowerPoint/Beamer. The reports should include:

- Define and justify the models (Methods)
- Define the experimental protocol used to estimate the parameters (Methods)
- Perform measurements and estimate the parameters (Results)
- Verify that the models perform ``acceptably well'' (as justified and defined by the modelers) on the original containers (Results)
- Apply the models (with parameters determined by calibration and measurement of validation bucket geometry) to the new containers supplied for strong validation (Results)
- Answer the questions: ``Which model did best? Why?'' (Discussion and Conclusion)

Lab Agenda

The in-class portion of the Leaky Bucket Lab proceeds as follows:

1. Lecture: Introduction to Leaky Bucket Lab, initial data collection [15 minutes]
2. Lecture: Derivation of Torricelli model [20 minutes]
3. Group Time: Design and creation of initial buckets and protocol, drainage observations and initial comparison with Torricelli predictions [60 minutes]
4. Class Discussion: Groups sketch data, comparison w/ Torricelli model, share ideas on what's wrong [20 minutes]
5. Group Time: Discussion and development of alternate models. Collection of additional calibration data [120 minutes]
6. Class Discussion: Groups present alternate models, calibration strategy, scheme for addressing validation [45 minutes]
7. Validation Buckets Revealed: Groups measure relevant geometry from new buckets [15 minutes]
8. Validation Challenge: Each group does one or two validation runs, contributes to public data pool [15 minutes]

This agenda is covered over a few lab/lecture days, with the expectation that student groups should be meeting, discussing their models, parameterizing and comparing with data. In classes which have less scheduling freedom many details can be streamlined; e.g. buckets with holes and benchmarks can simply be provided to students, or the data collection done as a demo in front of the class. In classes where the point is more that applications exist (e.g. of non-polynomial integration in calculus, or separation of variables in ODEs) the class can be provided with one of the models discussed below and allowed to work with it and class-collected data.

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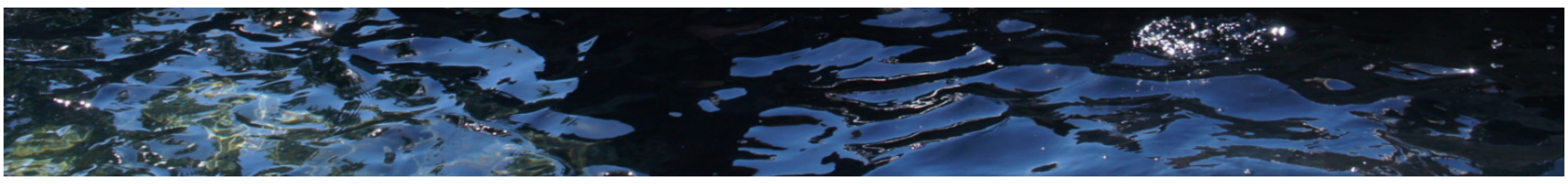

Lab Setup: Students cut an aperture of $a \approx .25\ \mathrm{cm}\uparrow 2$ and inscribe horizontal marks every centimeter above. Bucket is filled to twelve cm above the hole while the hole is covered with duct tape; students remove the tape and time the bucket's drainage, recording the dynamics of changing height.

Materials

The following materials are needed (for each group of 3-4 students):

- 1-2 quart translucent or clear plastic jugs such as those containing milk, soda or juice for use as leaky buckets
- Scalpels or X-Acto knives for cutting apertures and removing burrs (a drill with bits is useful for circular holes, but not necessary)
- Waterproof marker
- Stop watch
- Duct tape (just on general principles)
- Ruler with at least millimeter scale
- Graduated cylinders or kitchen measuring cups for measuring metric volumes
- Access to tap water
- Plastic dish washing tub to capture drained water if a large sink is not available

Methods

When time is not available for groups of students to develop and refine their own procedures, or if instructors wish to offer a starting point to get things rolling, we provide the
following procedure (based on using a 1/2 gallon milk jug):

1. Divide into groups of 3-4. Each group will need at least one person to manage the stopwatch (Timer), spot fluid levels (Spotter) and record data (Recorder).
2. Set up the bucket. Where the jug begins to have regular horizontal cross sections (2-4 cm above base for a standard US plastic half gallon milk jug) cut a horizontal slit 1-2 mm tall and 1-2 cm wide, being careful that the top and bottom of the slit are parallel to the base of the jug. Every cm vertically from the bottom of the slit make a horizontal mark, up to between 10 and 15 cm above the bottom of the slit (depending on how far the jug maintains a relatively consistent cross section).
3. Measure the bucket. At a minimum, students need to estimate the cross-sectional area of the bucket and the area of the aperture. Students may wish to measure the cross section volumetrically, adding a known volume to the bucket and dividing by a measured vertical height.
4. Observe drainage trajectories.
   a) Fill the bucket to the desired initial height (12 or 13 cm are used in this paper), as measured by the bottom of the fluid meniscus. The aperture will need to be covered either with a piece of duct tape or a convenient finger. If using a finger be careful not to press hard enough to deform the container.
   b) Position the bucket so that it can drain into a sink or basin.
   c) Spotter removes tape and says “Start!” Timer starts stopwatch.
   d) As fluid passes each vertical mark, Spotter calls “Mark!” and Timer gives the time of the split, which Recorder records next to the appropriate vertical level.
   e) Continue until the bottom of the fluid meniscus is level with the top of the slit. Timer records final emptying time. For a 1/2 gallon container with aperture of .4 cm2 filled to 12 cm above the slit this will be between 30 and 60 sec.
5. Repeat the observation sequence at least three times for the same initial height of fluid to assess variability.

One of the biggest issues is determining when to stop; depending on the size and shape of both bucket and aperture the flow may transition from a free stream to an attached dribble to periodic drips. Ideally students should discover and address this on their own; if time is tight instructors can experiment with the bucket in advance to determine a stopping rule for the observation sequence.

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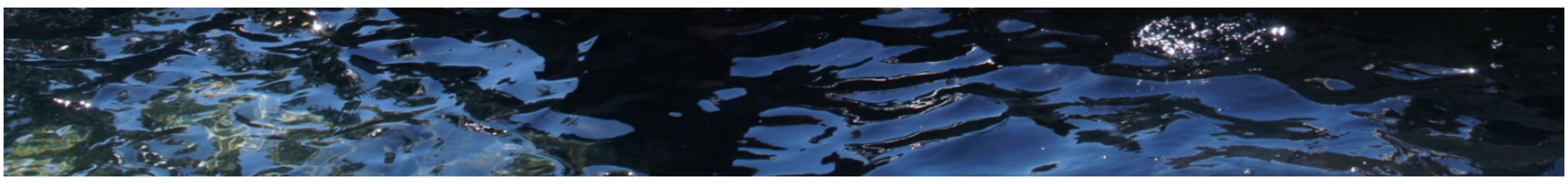

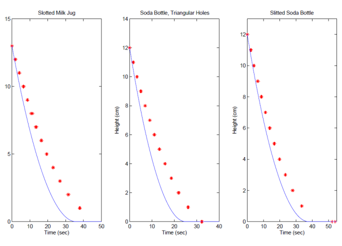

Data and Examples: Data along with some student approaches are presented to illustrate the range of student creativity and to help prepare teachers to scaffold student thinking.

h (cm)	Slitted Milk Jug $A = 86, a = .39\text{cm}^2$				Soda Jug, $\Delta + \Delta$ $A = 92, a = .575\text{cm}^2$				Slitted Soda Jug $A = 91.95, a = .38\text{cm}^2$		
13	0	0	0	0							
12	1.72	2.17	1.87	1.72	0	0	0	0	0	0	0
11	3.99	4.31	3.99	3.99	1.62	1.55	1.61	1.74	2.17	2.43	2.1
10	6.7	6.4	6.17	6.7	3.37	3.27	3.36	3.3	3.98	4.21	4.13
9	8.63	8.8	8.58	8.63	4.93	5.08	5.12	5.12	6.28	6.78	6.32
8	11.1	11.6	10.63	11.1	7.02	6.89	6.99	6.93	8.7	8.91	8.5
7	13.54	13.75	13.09	13.54	9.12	8.99	9.05	9.05	11.12	11.26	11.14
6	16.47	16.64	16.2	16.47	11.18	10.92	11.08	11.08	13.65	14.05	13.76
5	19.52	19.56	19.33	19.52	13.43	13.33	13.42	13.21	16.45	16.78	16.59
4	23.19	23.12	22.78	23.19	15.93	15.8	15.83	15.77	19.74	20.00	19.82
3	26.97	26.96	26.67	26.97	18.74	18.58	18.64	18.68	23.38	23.52	23.44
2	31.73	31.68	31.24	31.73	22.08	21.96	22.11	21.68	27.7	28.17	27.82
1	38.11	37.73	37.85	38.11	26.21	26.08	26.17	26.24	33.59	33.66	33.39
0	92.2	81.84	62.14	92.2	32.12	32.39	32.36	32.49	54.08	55.83	52.12

Table 1: Data collected by students from three buckets with differing apertures. The `buckets' are two two-liter soda bottles and a 1/2 gallon milk jug. One milk jug and one soda bottle were drained through a rectangular slit (with areas, a, indicated above) while the remaining soda bottle was drained through two triangular holes (bases horizontal to ground level) with total area $a = .575$. Student estimates for the cross-sectional area, A, of the container are also given above.

Examples

A purely empirical approach makes no attempt to respect underlying mechanisms, although it should reflect observed dependencies among parameters and variables (e.g. emptying time increases as aperture size decreases). Students, particularly from biological and/or statistical backgrounds, are often inclined to fit decreasing, concave functions of time to observed height trajectories. The most popular candidates are exponential models $h = h_0 e^{-\lambda t}$

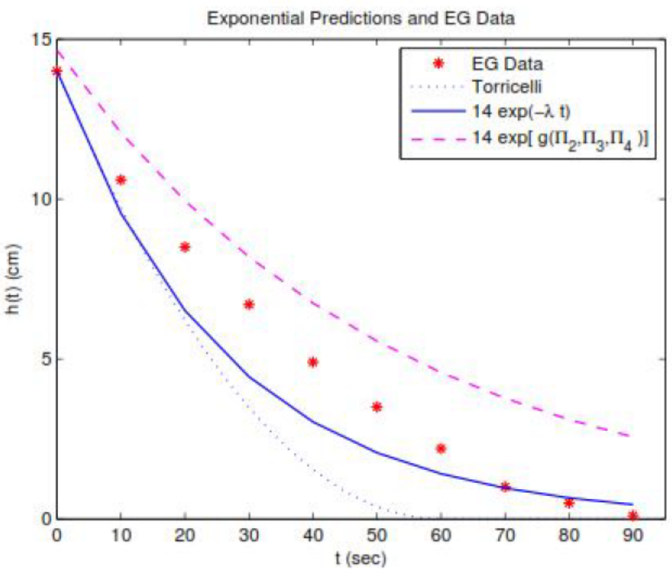

Figure 1: Comparison of two exponential fits and validation data (*). Torricelli predictions appear for reference (dotted curve). The solid curve depicts exponential predictions generated by fitting exponentials to calibration data individually, then using linear regression to extrapolate to a and A values needed for the validation bucket. The dashed curve depicts the use of a Pi Theorem approach to generating exponential predictions for the validation data; in this case the Pi Theorem approach is vastly inferior.

The most common student correction to the Torricelli model is to include a term reflecting fluid friction at the aperture, generally assuming that the amount of fluid leaving is a fraction, α, of the volumetric flow predicted by Torricelli's law. Students give a variety of reasons for including α. The velocity field at the aperture could be uniform, so that the amount of fluid leaving is less than the peak velocity times the area of the hole; flow could be impeded by the edges of the aperture, so that the effective area is smaller than measured, or the peak fluid velocity itself could be lower than expected. Each of these could lower the total flow rate at the aperture by some fraction, α.

The Torricelli model with the α is

$$A\,dh/dt = -\alpha a \sqrt{2gh}, \quad h(0) = h_0. \qquad (1)$$

The solution follows directly,

$$h = h_0 \left(1 - \alpha \sqrt{2}/2 \; a/A \sqrt{g/h_0} \; t\right)^2.$$

The parameter α is found by using the data to approximate dh/dt and then estimate α using (1).

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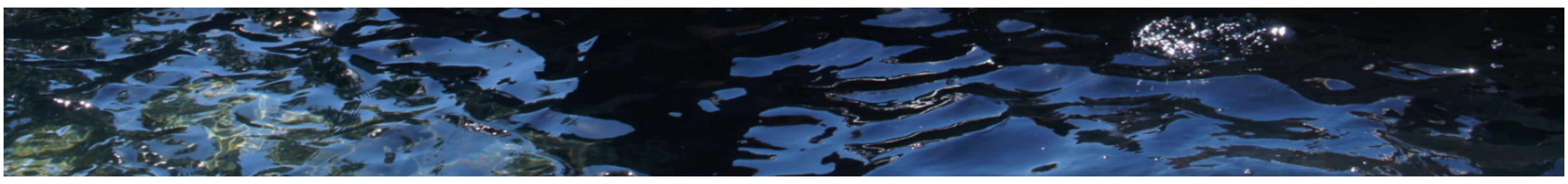
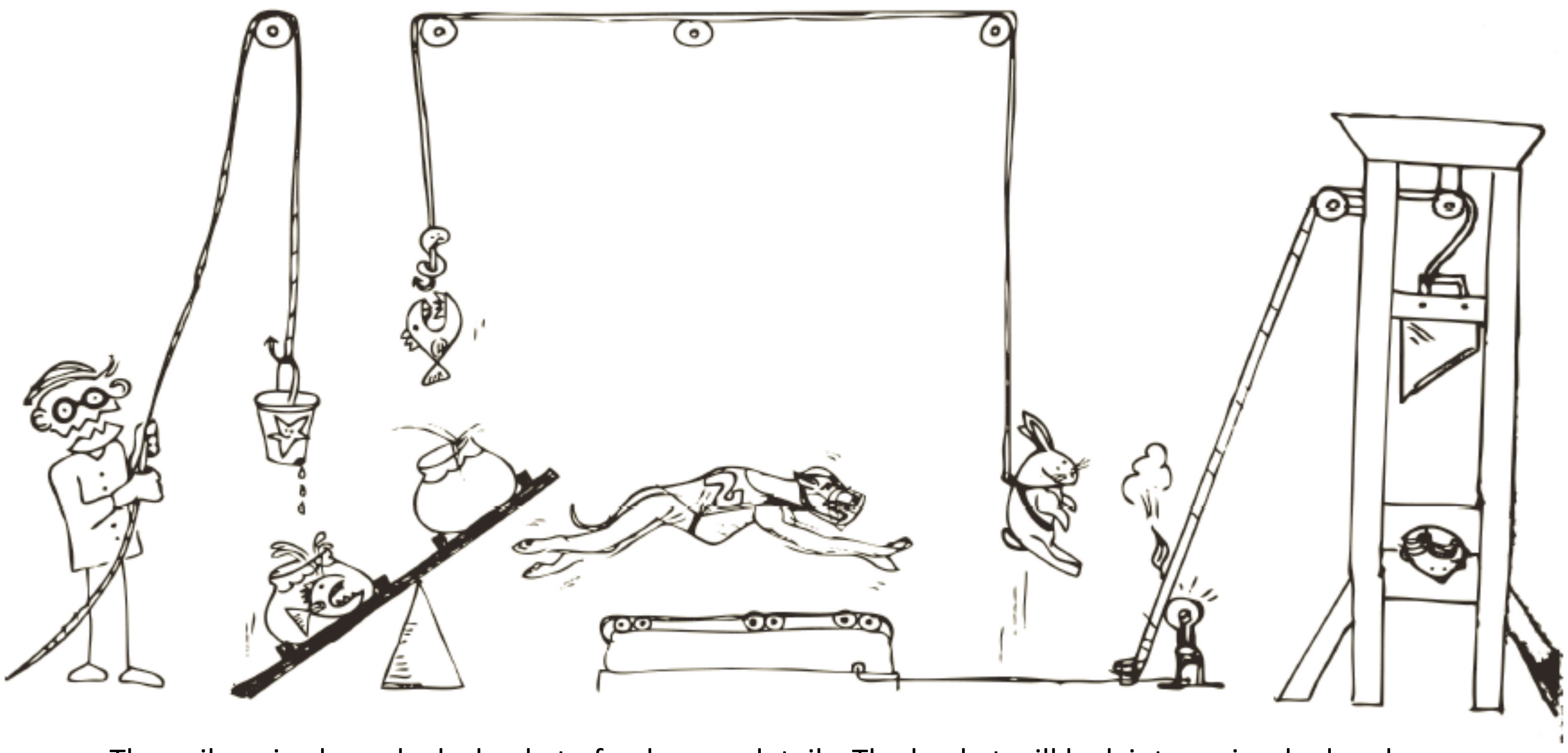


Background and Extensions: Many modern biological applications require some knowledge of fluid mechanics. Examples include individual-based flight or swimming models, microbes in a chemostat, nutrient cycling dynamics in mountain lakes, mathematical physiology, to name only a few. The Leaky Bucket works well as a transition from discrete modeling to the more obviously biological labs (Yeast Lab, Brine Shrimp Lab), where it serves to pave the way to continuous models. Finally, the Leaky Bucket lab provides a good introduction to many mathematical tools that students will need for other biological applications.

We often begin the Leaky Bucket Lab with the following scenario for students to build context. Imagine, if you will, that you are taken captive by an "evil genius" (AKA your teacher). This genius truly is evil, and has quite a diabolical plan for you.

> "I have a container of liquid." says the Evil Genius. "If you are to make it out of here alive you must tell me how long it will take for the liquid to drain out of my container. After you have made your guess we will start the flow of the liquid and see if you will survive. Are you up to the challenge?"

In an attempt to survive you will be allowed to work with fellow captives in an initial ``testing'' phase where you will measure data from a basic experiment before you go up against the Evil Genius. It is up to you to ensure that you have plans to measure all the parameters needed in your model. This may involve different levels of ingenuity, flexibility, and special equipment from the instructors, depending on the models used. The Evil Genius has agreed to play by a few rules. Holes on more than one level will not be used, however multiple holes may be used. The shape and size of the holes will also be freely adjusted. Can you survive?



The evil genius has a leaky bucket of unknown details. The bucket will leak into a piranha bowl on a lever. The second piranha will then be launched into the air catching a worm. Thus causing a rabbit to be lifted into the view of a greyhound. The greyhound will run, powering a light, which will burn the rope holding the guillotine in the air. Good Luck! (Artist: Jeta Renna)

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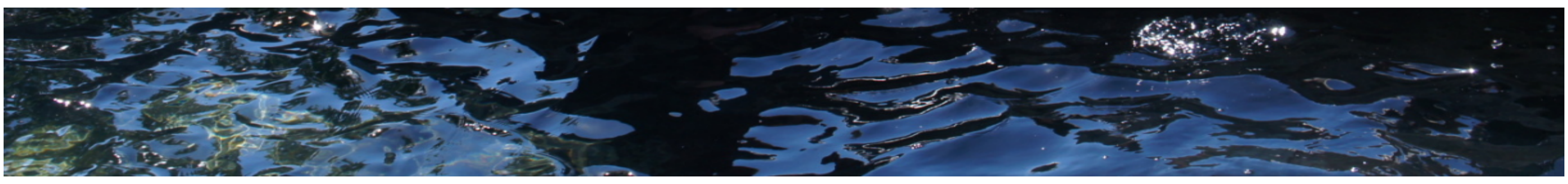

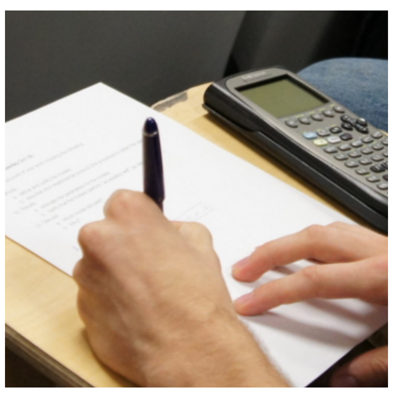

Assessment Items: The following assessment items were written to target learning objectives in the Leaky Bucket Lab for students in an ODE setting and are typically appropriate for students with at least some calculus experience

1. Comprehension and Communication: In your own words, compare and contrast a scientific *law* (like Torricelli's), a mathematical *theory* and a mathematical *model*.

2. Algorithmic Skill: Describe the shape of the leaky buckets with the following cross-sectional areas and solve Torricelli's Model (below) analytically for each.

$$dh/dt = -a\sqrt{2g}/A(t)\,\sqrt{h}$$

   a) $A(t) = t^2$
   b) $A(t) = \pi\csc^2 t$
   c) $A(t) = \pi\, t^4$

3. Comprehension and Communication: Describe how you would fit the following data with a quadratic function.

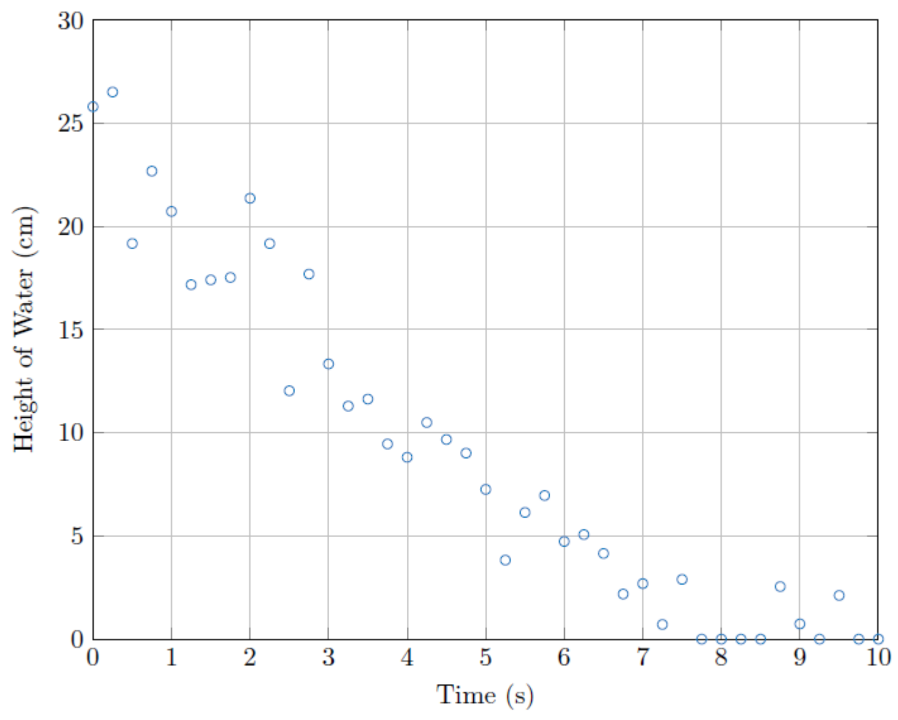

Figure 1: Height of water column leaking from a bucket over time}.

4. Comprehension and Communication: Members of your group provided the following alternate model for the Leaky Bucket lab, but failed to mechanistically describe the terms in their model. You make the assumption that h represents height and t represents time and determine that you can figure it out.

$$dh/dt = -a\sqrt{2b}/A\,\sqrt{h} + \beta h/t^2$$

   a) What are the units of the model's parameters?
   b) Provide a mechanistic interpretation for each term of the model.

5. Application: In the construction of Torricelli's Model for the Leaky Bucket Lab we used Bernoulli's principle that states:

$$v^2/2 + gh + p/\rho = \text{constant}$$

   where v is fluid speed, g is the gravitational acceleration (9.81 m/s^2), h is the fluid's height above a reference point, p is pressure, and ρ is density. In the end, this leads to $v = \sqrt{2gh}$ in Torricelli's Model.

   a) What assumptions were made in Bernoulli's principle that lead to Torricelli's Model?
   b) How would the model change if you challenged or adapted those assumptions to better fit the Leaky Bucket Lab setup?