Superposition Formulas for Exterior Differential Systems

Ian Anderson
Dept of Math. and Stat.
Utah State University

Mark Fels
Dept of Math. and Stat.
Utah State University

Peter Vassiliou
Dept. of Math and Stat.
University of Canberra

June 11, 2008

1 Introduction

In this paper we use the method of symmetry reduction for exterior differential systems to obtain a far-reaching generalization of Vessiot's integration method [27], [28] for Darboux integrable, partial differential equations. This group-theoretic approach provides deep insights into this classical method; uncovers the fundamental geometric invariants of Darboux integrable systems; provides for their algorithmic integration; and has applications well beyond those currently found in the literature. In particular, our integration method is applicable to systems of hyperbolic PDE such as the Toda lattice equations, [19], [20], [22], 2-dimensional wave maps [3] and systems of overdetermined PDE such as those studied by Cartan [8].

Central to our generalization of Vessiot's work is the novel concept of a ***superposition formula*** for an exterior differential system $\mathcal{I}$ on a manifold M. A superposition formula for $\mathcal{I}$ is a pair of differential systems $\hat{\mathcal{W}}$, $\check{\mathcal{W}}$, defined on manifolds M_1 and M_2, and a mapping

$$\Sigma\colon M_1 \times M_2 \to M \tag{1.1}$$

such that

$$\Sigma^*(\mathcal{I}) \subset \pi_1^*(\hat{\mathcal{W}}) + \pi_2^*(\check{\mathcal{W}}). \tag{1.2}$$

Here $\pi_1^*(\hat{\mathcal{W}} + \pi_2^*(\check{\mathcal{W}})$ is the differential system generated by the pullbacks of $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ to the product manifold $M_1 \times M_2$ by the canonical projection maps π_1 and π_2. It is then clear that if $\hat{\phi}\colon N_1 \to M_1$ and $\check{\phi}\colon N_2 \to M_2$ are integral manifolds for $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$, then

$$\phi = \Sigma \circ (\hat{\phi},\, \check{\phi})\colon N_1 \times N_2 \to M \tag{1.3}$$

is an (possibly non-immersed) integral manifold of $\mathcal{I}$.

In this article we shall

[i] establish general sufficiency conditions (in terms of geometric invariants of the differential system $\mathcal{I}$) for the existence of a superposition formula;

[ii] establish general sufficiency conditions under which the superposition formula gives *all* local integral manifolds of $\mathcal{I}$ in terms of the integral manifolds of $\pi_1^*(\hat{\mathcal{W}}) + \pi_2^*(\check{\mathcal{W}})$ on $M_1 \times M_2$ (in which case we say that the superposition formula is surjective);

[iii] provide an algorithmic procedure for finding the superposition formula; and

[iv] demonstrate the effectiveness of our approach with an extensive number of examples and applications.

Differential systems admitting superposition formula are easily constructed by symmetry reduction. To briefly describe this construction, let G be a symmetry group of a differential system $\mathcal{W}$ on a manifold N. We assume that the quotient space $M = N/G$ of N by the orbits of G has a smooth manifold structure for which the projection map $\mathbf{q}\colon N \to M$ is smooth. We then define the ***G reduction of $\mathcal{W}$ or quotient of $\mathcal{W}$ by G*** as the differential system on M given by

$$\mathcal{W}/G = \{\, \omega \in \Omega^*(M) \,|\, \mathbf{q}^*(\omega) \in \mathcal{W} \,\}. \tag{1.4}$$

The traditional application of symmetry reduction has been to integrate $\mathcal{W}$ by integrating $\mathcal{W}/G$. See, for example, [1].

But now suppose that differential systems $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ on manifolds M_1 and M_2 have a common symmetry group G. Define the differential system $\mathcal{W} = \pi_1^*(\hat{\mathcal{W}}) + \pi_2^*(\check{\mathcal{W}})$ on $M_1 \times M_2$ and let G act on $M_1 \times M_2$ by the diagonal action. Then, by definition, the quotient map $\mathbf{q}\colon M_1 \times M_2 \to M = (M_1 \times M_2)/G$ defines a superposition formula for the quotient differential system $\mathcal{W}/G$ on M. In this paper we discover the means by which the inverse process to symmetry reduction is possible, that is, *we show how certain general classes of differential systems $\mathcal{I}$ on M can be identified with a quotient system $\mathcal{W}/G$ in which case the integral manifolds of $\mathcal{I}$ can then be found from those of $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$.*

To intrinsically describe the class of differential systems for which we shall construct superposition formulas, we first introduce the definition of a decomposable differential system.

Definition 1.1. *An exterior differential system $\mathcal{I}$ on M is* ***decomposable of type*** $[p, q]$, *where $p, q \geq 2$, if about each point $x \in M$ there is a coframe*

$$\tilde{\theta}^1,\ \ldots,\ \tilde{\theta}^r,\ \hat{\sigma}^1,\ \ldots,\ \hat{\sigma}^p,\ \check{\sigma}^1,\ \ldots,\ \check{\sigma}^q, \tag{1.5}$$

such that $\mathcal{I}$ is algebraically generated by 1-forms and 2-forms

$$\mathcal{I} = \{\, \tilde{\theta}^1,\ \ldots,\ \tilde{\theta}^r,\ \hat{\Omega}^1,\ \ldots,\ \hat{\Omega}^s,\ \check{\Omega}^1,\ \ldots,\ \check{\Omega}^t \,\}, \tag{1.6}$$

where $s, t \geq 1$, $\hat{\Omega}^a \in \Omega^2(\hat{\sigma}^1, \ldots, \hat{\sigma}^p)$, and $\check{\Omega}^\alpha \in \Omega^2(\check{\sigma}^1, \ldots, \check{\sigma}^q)$. The differential systems algebraically generated by

$$\hat{\mathcal{V}} = \{\, \tilde{\theta}^i,\, \hat{\sigma}^a,\, \check{\Omega}^\alpha \,\} \quad \textit{and} \quad \check{\mathcal{V}} = \{\, \tilde{\theta}^i,\, \check{\sigma}^\alpha,\, \hat{\Omega}^a \,\} \tag{1.7}$$

*are called the associated **singular differential systems** for $\mathcal{I}$ with respect to the decomposition* (1.6)*.*

With the goal of constructing superposition formulas, we have found it most natural to focus on the case where $\mathcal{I}$ is decomposable (but not necessarily Pfaffian) and $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ are (constant rank) Pfaffian[1]. All of the examples we consider are of this type. Note that any class r hyperbolic differential system, as defined in [6], is a decomposable differential system and that the associated characteristic Pfaffian systems coincide, for $r > 0$, with the singular systems(1.7).

The definition of a Darboux integrable, decomposable differential system is given in terms of its singular systems. For any Pfaffian system V, let $V^{(\infty)}$ denoted the largest integrable subbundle of V. The rank of $V^{(\infty)}$ gives the number of functionally independent first integrals for V. By definition, a scalar second order partial differential equation in the plane is Darboux integrable if the associated singular Pfaffian systems $\hat{V}$ and $\check{V}$ each admit at least 2 (functionally independent) first integrals. Thus, in order to generalize the definition of Darboux integrablity, we must determine the required number of functionally independent first integrals necessary to integrate a general decomposable Pfaffian system. We do this with the following definitions.

Definition 1.2. *A pair of Pfaffian systems $\hat{V}$ and $\check{V}$ define a **Darboux pair** if the following conditions hold.*

[i]
$$\hat{V} + \check{V}^{(\infty)} = T^*M \quad \textit{and} \quad \check{V} + \hat{V}^{(\infty)} = T^*M. \tag{1.8}$$

[ii]
$$\hat{V}^{(\infty)} \cap \check{V}^{\infty} = \{\, 0 \,\}. \tag{1.9}$$

[iii]
$$d\omega \in \Omega^2(\hat{V}) + \Omega^2(\check{V}) \quad \textit{for all} \quad \omega \in \Omega^1(\hat{V} \cap \check{V})\ . \tag{1.10}$$

Definition 1.3. *Let $\mathcal{I}$ be a decomposable differential system and assume that the associated singular systems $\hat{V}$ and $\check{V}$ are Pfaffian. Then $\mathcal{I}$ is said to be **Darboux integrable** if $\{\, \hat{V},\, \check{V} \,\}$ define a Darboux pair.*

Property **[i]** of Definition 1.2 is the critical one – it will insure that there are a sufficient number of first integrals to construct a superposition formula. Property **[ii]** is a technical condition which states simply that $\hat{V}$ and $\check{V}$ share no

[1] We use the term Pfaffian system to designate either a constant rank subbundle V of T^*M or the differential system, denoted by the corresponding calligraphic letter $\mathcal{V}$, generated by the sections of V.

common integrals – this condition can always be satisfied by restricting $\hat{V}$ and $\check{V}$ to a level set of any common integrals. The form of the structure equations for $\hat{V} \cap \check{V}$ required by property **[iii]** is always satisfied when $\hat{V}$ and $\check{V}$ are the singular Pfaffian systems for a decomposable differential system $\mathcal{I}$.

Our main result can now be stated.

Theorem 1.4. *Let $\mathcal{I}$ be a decomposable differential system on M whose associated singular Pfaffian systems $\{\,\hat{V},\, \check{V}\,\}$ define a Darboux pair. Then there are Pfaffian systems W_1 and W_2 on manifolds M_1 and M_2 which admit a common Lie group G of symmetries and such that*

[i] *the manifold M can be identified (at least locally) as the quotient of $M_1 \times M_2$ by the diagonal action of the group G;*

[ii]
$$\mathcal{I} = (\pi_1^*(\hat{\mathcal{W}} + \pi_2^*(\check{\mathcal{W}}))/G; \quad \textit{and} \tag{1.11}$$

[iii] *the quotient map $\mathbf{q}\colon M_1 \times M_2 \to M$ defines a surjective superposition formula for $\mathcal{I}$.*

The manifolds M_1 and M_2 in Theorem 1.4 are simply any maximal integral manifolds for $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$ and the Pfaffian systems W_1 and W_2 are just the restrictions of $\check{V}$ and $\hat{V}$ to these manifolds. But the proper identification of the Lie group G and its action on M_1 and M_2 is not so easy to uncover. This is done through a sequence of non-trivial coframe adaptations and represents the principle technical achievement of the paper (See Theorem 4.1 and Definition 5.7) We call G the ***Vessiot group*** for the Darboux integrable, differential system $\mathcal{I}$.

The paper is organized as follows. In Section 2 we obtain some simple sufficiency conditions for a differential system to be decomposable and we give necessary and sufficient conditions for a Pfaffian system to be decomposable (Theorem 2.3). We also introduce the initial adapted coframes for a Darboux pair (Theorem 2.9). In Section 3 we answer the question of when the symmetry reduction of a Darboux pair is also a Darboux pair (Theorem 3.2) and we use this result to give a general method for constructing Darboux integrable differential systems (Theorem 3.3). Section 4 establishes the sequence of coframe adaptations leading to the definition of the Vessiot group G and its action on M and hence on M_1 and M_2 (Theorem 4.1). In Section 5 we construct the superposition formula (Theorem 5.10) and prove that it may be identified with the quotient for the diagonal action of the Vessiot group. This proves Theorem 1.4.

A substantial portion of the paper is devoted to examples. These examples, given in Section 6, provide closed-form general solutions to a wide range of differential equations, explicitly illustrate the various coframe adaptations of Section 4, and underscore the importance of the Vessiot group as the fundamental invariant for Darboux integrable differential systems. The first two examples are taken from the classical literature. The equation considered in Example 6.1 is specifically chosen to illustrate in complete detail all the various coframe adaptations used to determine the superposition formula. In Example 6.2 we integrate all non-linear Darboux equations of the type $u_{xx} = f(u_{yy})$ (with second order invariants) and relate the integration of these equations to Cartan's classification of rank 3 Pfaffian systems in 5 variables [7]. This connection between the method of Darboux and Cartan's classification is apparent from our interpretation of the method of Darboux in terms of symmetry reduction of differential systems.

In Examples 6.3 and 6.4 we present a number of examples of Darboux integrable equations where the unknown function takes values in a group or in a non-commutative algebra. Example 6.3 provides us with a system whose Vessiot group is an arbitrary Lie group G.

Some of the simplest examples of systems of Darboux integrable partial differential equations can be constructed by the coupling of a nonlinear Darboux integrable scalar equation to a linear or Moutard-type equation. These are presented in Example 6.5. It is noteworthy that for these equations the Vessiot group is a semi-direct product of the Vessiot group for the non-linear equation with an Abelian group. The representation theoretical implications of this observation will be further explored elsewhere.

In Example 6.6 we illustrate the computational power of our methods by explicitly integrating the B_2 Toda lattice system. While the general solution to the A_n Toda systems were known to Darboux, this is, to the best of our knowledge, the first time explicit general solutions to other Toda systems have been given. Based on this work we conjecture that the Vessiot group for the $\mathfrak{g}$ Toda lattice system, where $\mathfrak{g}$ is any semi-simple Lie group, is the associated Lie group G itself. In Example 6.7 we integrate a wave map system – this system admits a number of interesting geometric properties which will be explored in detail in a subsequent paper. Finally, in Example 6.8, we present some original examples of Darboux integrable, non-linear, over-determined systems in 3 independent variables.

The extensive computations required by all these examples were done using the DifferentialGeometry package in Maple 11. This research was supported by NSF grant DMS-0410373 and DMS-0713830.

2 Preliminaries

2.1 Decomposable Differential Systems

In this section we give some simple necessary conditions, in terms of the notion of singular vectors, for a differential system[2] $\mathcal{I}$ to be decomposable (see Definition 1.1); we give sufficient conditions for a Pfaffian system to be decomposable; and we address the problem (see Theorem 2.6) of re-constructing a decomposable differential system from its associated singular systems.

Let $\mathcal{I}$ be a differential system on M. Fix a point x in M and let

$$E^1_x(\mathcal{I}) = \{\, X \in T_xM \mid \theta(X) = 0 \quad \text{for all 1-forms } \theta \in \mathcal{I}\,\}.$$

The ***polar equations*** determined by a non-zero vector $X \in E^1_x(\mathcal{I})$ are, by definition, the linear system of equations for $Y \in E^1_x(\mathcal{I})$ given by

$$\theta(X, Y) = 0 \quad \text{for all 2-forms } \theta \in \mathcal{I}.$$

Then $X \in E^1_x(\mathcal{I})$ is said to be ***regular*** if the rank of its polar equations is maximal and ***singular*** otherwise.

Lemma 2.1. *Let $\mathcal{I}$ be a decomposable differential system of type $[p, q]$. Then $E^1_x(\mathcal{I})$ decomposes into a direct sum of p and q dimensional subspaces*

$$E^1_x(\mathcal{I}) = S_1 \oplus S_2 \tag{2.1}$$

such that

[i] *every vector in S_1 and every vector in S_2 is singular; and*

[ii] *every 2 plane spanned by any pair of vectors $X \in S_1$ and $Y \in S_2$ is an integral 2-plane for $\mathcal{I}$.*

Proof. Let $\{\, \partial_{\tilde{\theta}^i}, \partial_{\hat{\sigma}^a}, \partial_{\check{\sigma}^\alpha} \,\}$ denote the dual frame to (1.5). Then $E^1_x(\mathcal{I}) = \operatorname{span}\{\, \partial_{\hat{\sigma}^a}, \partial_{\check{\sigma}^\alpha} \,\}$ and the polar equations for $X = X_1 + X_2$, where $X_1 = t^a \partial_{\hat{\sigma}^a}$ and $X_2 = s^\alpha \partial_{\check{\sigma}^\alpha}$, are

$$\hat{\Omega}^a(X_1, Y) = 0 \quad \text{and} \quad \check{\Omega}^\alpha(X_2, Y) = 0.$$

Then, clearly, the vectors X_1 and X_2 are singular vectors and, moreover, the plane spanned by X_1, X_2 is an integral 2-plane. ∎

[2] We assume that all exterior differential systems are of constant rank.

Remark 2.2. The differential system $\hat{\mathcal{V}}$ defined by (1.7) is the smallest differential system containing $\mathcal{I}$ and $\mathrm{Ann}(S_2)$ and it is for this reason that we have opted to called $\hat{\mathcal{V}}$ (and $\check{\mathcal{V}}$) the singular differential systems associated to the decomposition of $\mathcal{I}$.

The necessary conditions for decomposability given in Lemma 2.1 are seldom sufficient. However, if $\mathcal{I}$ is a Pfaffian system and properties **[i]** and **[ii]** of Lemma 2.1 are satisfied, then there exists a local coframe

$$\theta^1, \dots, \theta^r,\ \hat{\sigma}^1, \dots, \hat{\sigma}^p, \check{\sigma}^1, \dots, \check{\sigma}^q$$

on M with $I = \{\, \theta^i \,\}$ and with structure equations

$$d\theta^i \equiv A^i_{ab}\, \hat{\sigma}^a \wedge \hat{\sigma}^b + B^i_{\alpha\beta}\, \check{\sigma}^\alpha \wedge \check{\sigma}^\beta \qquad \mathrm{mod}\ I. \tag{2.2}$$

At each point x_0 of M define linear maps

$$A = [A^i_{ab}(x_0)] \colon \mathbf{R}^r \to \Lambda^2(\mathbf{R}^p) \quad \text{and} \quad B = [B^i_{\alpha\beta}(x_0)] \colon \mathbf{R}^r \to \Lambda^2(\mathbf{R}^q). \tag{2.3}$$

Theorem 2.3. *Let I be a Pfaffian system. Then I is decomposable if properties* **[i]** *and* **[ii]** *of Lemma 2.1 are satisfied; the matrices A^i_{ab} and $B^i_{\alpha\beta}$ given by* (2.2), *are non-zero, constant rank; and*

$$\dim\big(\ker(A) + \ker(B)\big) = \mathrm{rank}\, I. \tag{2.4}$$

Proof. Equation (2.4) implies that we can find r linearly independent r dimensional column vectors $T^1, \dots, T^{r_1}, \dots, T^{r_2}, \dots, T^r$ such that

$$\begin{aligned} \ker(A) \cap \ker(B) &= \mathrm{span}\{\, T^1, \dots, T^{r_1} \,\}, \\ \ker(B) &= \mathrm{span}\{\, T^1, \dots, T^{r_1}, T^{r_1+1}, \dots, T^{r_2} \,\}, \quad \text{and} \\ \ker(A) &= \mathrm{span}\{\, T^1, \dots, T^{r_1}, T^{r_2+1}, \dots, T^r \,\}. \end{aligned}$$

The 1-forms $\tilde{\theta}^i = t^i_j\, \theta^j$, where $T^i = [t^i_j]$, then satisfy

$$\begin{aligned} d\tilde{\theta}^i &\equiv 0 && \text{for } i = 1 \dots r_1, \\ d\tilde{\theta}^i &\equiv t^i_j A^j_{ab}\, \hat{\sigma}^a \wedge \hat{\sigma}^b && \text{for } i = r_1 + 1 \dots r_2, \\ d\tilde{\theta}^i &\equiv t^i_j B^j_{\alpha\beta}\, \check{\sigma}^\alpha \wedge \check{\sigma}^\beta && \text{for } i = r_2 + 1 \dots r, \end{aligned} \tag{2.5}$$

and the decomposability of I follows by taking $\hat{\Omega}^i = t^i_j A^j_{ab}\, \hat{\sigma}^a \wedge \hat{\sigma}^b$ for $i = r_1 + 1 \dots r_2$ and $\check{\Omega}^i = t^i_j B^j_{\alpha\beta}\, \check{\sigma}^a \wedge \check{\sigma}^b$ for $i = r_2 + 1 \dots r$. That the integers s and t in Definition 1.1 satisfy $s, t \geq 1$ follows from the fact that A^i_{ab} and $B^i_{\alpha\beta}$ are non-zero and that $[t^i_j]$ is invertible. ∎

Corollary 2.4. *If $\mathcal{I}$ is a decomposable Pfaffian system, then the singular systems $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ are also Pfaffian.*

Proof. This is immediate from (1.7) and (2.5). ∎

To each decomposable exterior differential system $\mathcal{I}$ we have associated a pair of differential systems $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ and, when these are Pfaffian, we have defined what it means for $\{\hat{V}, \check{V}\}$ to define a Darboux pair. In our subsequent analysis, the Darboux pair $\{\hat{V}, \check{V}\}$ will be taken as the fundamental object of study. From this viewpoint, it becomes important to address the problem of reconstructing the EDS $\mathcal{I}$ from its singular systems. To this end, we introduce the following novel construction.

Definition 2.5. *Let $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ be two differential systems on a manifold M. Define a new differential system $\mathcal{K} = \hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ to be the exterior differential system whose integral elements are precisely the integral elements of $\hat{\mathcal{V}}$, the integral elements of $\check{\mathcal{V}}$, and the sum of integral elements of $\hat{\mathcal{V}}$ with integral elements of $\check{\mathcal{V}}$.*

This definition is motivated by two observations. First we note that if $\mathcal{I}$ is a decomposable exterior differential system with singular Pfaffian systems $\hat{V}$ and $\check{V}$ then in general $\mathcal{I}$ properly contains the Pfaffian system with generators $\hat{V} \cap \check{V}$ and is properly contained in the EDS $\hat{\mathcal{V}} \cap \check{\mathcal{V}}$ so that neither of these set-theoretic constructions reproduce $\mathcal{I}$.

Secondly, one can interprete the EDS $\mathcal{K}$ in Definition 2.5 as satisfying an ***infinitesimal superposition principle*** with respect to $\{\hat{\mathcal{V}}, \check{\mathcal{V}}\}$ at the level of integral elements. Of course, this by itself does not imply that $\mathcal{K}$ admits a superposition principle for its integral manifolds. The main results of this article can then be re-formulated as follows: if the EDS $\mathcal{K}$ admits a superposition principle for its integral elements with respect to a Darboux pair $\{\hat{V}, \check{V}\}$, then $\mathcal{K}$ admits a superposition principle for its integral manifolds.

Theorem 2.6. *If $\mathcal{I}$ is a decomposable exterior differential system with singular differential systems $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$, then $\mathcal{I} = \hat{\mathcal{V}} \Cap \check{\mathcal{V}}$.*

Proof. We first remark that the differential system $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ is always contained in $\hat{\mathcal{V}} \cap \check{\mathcal{V}}$. Let $\{\tilde{\theta}^i, \hat{\sigma}^a, \check{\sigma}^\alpha\}$ be the coframe given by Definition 1.1. By definition, the algebraic generators for $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ are

$$\hat{\mathcal{V}} = \{\tilde{\theta}^i, \hat{\sigma}^a, \check{\Omega}^\beta\} \quad \text{and} \quad \check{\mathcal{V}} = \{\tilde{\theta}, \check{\sigma}^\alpha, \hat{\Omega}^b\}$$

from which it is not difficult to show that

$$\hat{\mathcal{V}} \cap \check{\mathcal{V}} = \{\, \tilde{\theta}^i,\, \hat{\Omega}^b,\, \check{\Omega}^\beta,\, \hat{\sigma}^a \wedge \check{\sigma}^\alpha \}.$$

Since $\partial_{\check{\sigma}^\alpha}$ defines a 1-integral element for $\hat{\mathcal{V}}$ and $\partial_{\hat{\sigma}^\alpha}$ defines a 1-integral element for $\check{\mathcal{V}}$, the 2-plane spanned by $\{\partial_{\check{\sigma}^\alpha},\, \partial_{\hat{\sigma}^a}\}$ must be a integral element for $\hat{\mathcal{V}} \boxplus \check{\mathcal{V}}$ and therefore the forms $\hat{\sigma}^a \wedge \check{\sigma}^\alpha \notin \hat{\mathcal{V}} \boxplus \check{\mathcal{V}}$. In view of (1.6), we conclude that $\hat{\mathcal{V}} \boxplus \check{\mathcal{V}} \subset \mathcal{I}$.

To complete the proof, it suffices to check that every $(r+s)$-form $\omega \in \mathcal{I}$ vanishes on every plane $\hat{E}_r + \check{E}_s$, where $\hat{E}_r$ is an r-integral plane of $\hat{\mathcal{V}}$ and $\check{E}_s$ is an s-integral plane of $\check{\mathcal{V}}$. The form ω is a linear combination of the wedge product of the 1-forms in (1.6) with arbitrary $(r+s-1)$-forms ρ and the wedge product of the 2-forms in (1.6) with arbitrary $(r+s-2)$-forms τ. It is clear that $(\tilde{\theta}^i \wedge \rho)_{|(\hat{E}_r + \check{E}_s)} = 0$. The values of $(\hat{\Omega}^b \wedge \tau)_{|(\hat{E}_r + \check{E}_s)}$ can be calculated as linear combinations of terms involving

$$\hat{\Omega}^b(\hat{X}, \hat{Y}), \quad \hat{\Omega}^b(\hat{X}, \check{Y}), \quad \hat{\Omega}^b(\check{X}, \hat{Y}), \quad \hat{\Omega}^b(\check{X}, \check{Y}),$$

where $\hat{X}, \hat{Y} \in \hat{E}_r$ and $\check{X}, \check{Y} \in \check{E}_s$. These terms all vanish and in this way we may complete the proof of the theorem. ∎

2.2 The first adapted coframes for a Darboux pair

Let $\{\, \hat{V}, \check{V}\,\}$ be a Darboux pair on a manifold M. In addition to the properties listed in Definition 1.2, we shall assume that the derived systems $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$, as well as the intersections $\hat{V}^{(\infty)} \cap \check{V}$, $\hat{V} \cap \check{V}^{(\infty)}$ and $\hat{V} \cap \check{V}$ are all (constant rank) subbundles of T^*M.

To define our initial adapted coframe for the Darboux pair $\{\, \hat{V}, \check{V}\,\}$, we first chose tuples of independent 1-forms $\hat{\boldsymbol{\eta}}$ and $\check{\boldsymbol{\eta}}$ which satisfy

$$\hat{V}^{(\infty)} \cap \check{V} = \operatorname{span}\{\hat{\boldsymbol{\eta}}\} \quad \text{and} \quad \hat{V} \cap \check{V}^{(\infty)} = \operatorname{span}\{\check{\boldsymbol{\eta}}\}. \tag{2.6a}$$

Property (1.9) implies that the forms $\{\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\}$ are linearly independent. Then chose tuples of 1-forms $\boldsymbol{\theta}$, $\hat{\boldsymbol{\sigma}}$ and $\check{\boldsymbol{\sigma}}$ such that

$$\hat{V}^{(\infty)} = \operatorname{span}\{\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\},\ \check{V}^{(\infty)} = \operatorname{span}\{\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\},\ \hat{V} \cap \check{V} = \operatorname{span}\{\, \boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\} \tag{2.6b}$$

and such that the sets of 1-forms $\{\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,\}$, $\{\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}$ and $\{\, \boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\,\}$ are linearly independent.

Lemma 2.7. *Let* $\{\hat{V}, \check{V}\}$ *be a Darboux pair on a manifold* M*. Then the 1-forms* $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ *satisfying* (2.6) *define a local coframe for* T^*M *and*

$$\hat{V} = \operatorname{span}\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\} \quad \textit{and} \quad \check{V} = \operatorname{span}\{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}. \tag{2.6c}$$

Proof. From the definitions (2.6a) and (2.6b) it is clear that $\operatorname{span}\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\} \subset \hat{V}$ and $\operatorname{span}\{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\} \subset \check{V}$. Property (1.8) then implies that the forms $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ span T^*M.

To prove the first of (2.6c), let $\omega = \mathbf{a}\,\boldsymbol{\theta} + \mathbf{b}\,\hat{\boldsymbol{\sigma}} + \mathbf{c}\,\hat{\boldsymbol{\eta}} + \mathbf{d}\,\check{\boldsymbol{\sigma}} + \mathbf{e}\,\check{\boldsymbol{\eta}}$. If $\omega \in \hat{V}$, then the 1-forms $\mathbf{d}\,\check{\boldsymbol{\sigma}} \in \hat{V}$ and therefore, by (2.6b), $\mathbf{d}\,\check{\boldsymbol{\sigma}} \in \hat{V} \cap \check{V}^{(\infty)}$. Since the forms $\check{\boldsymbol{\eta}}$ and $\check{\boldsymbol{\sigma}}$ are independent, we must have $\mathbf{d} = 0$ and (2.6c) is established.

To prove the independence of the forms $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$, set $\omega = 0$. Then the argument just given implies that $\mathbf{d} = 0$ so that $\mathbf{b}\hat{\boldsymbol{\sigma}} \in \hat{V} \cap \check{V}$. Therefore $\mathbf{b}\hat{\boldsymbol{\sigma}} \in \hat{V}^{(\infty)} \cap \check{V}$ and this forces $\mathbf{b} = 0$. The independence of the forms $\{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\}$ then gives $\mathbf{a} = \mathbf{c} = \mathbf{e} = 0$ and the lemma is proved. ∎

Any local coframe $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ satisfying (2.6) is called a ***0-adapted coframe for the Darboux pair*** $\{\hat{V}, \check{V}\}$**.**

We shall also need the definition of a 0-adapted frame as the dual of the 0-adapted coframe. Let

$$\begin{gathered} \hat{H} = \operatorname{ann} \hat{V}, \quad \check{H} = \operatorname{ann} \check{V}, \quad \hat{H}^{(\infty)} = \operatorname{ann} \hat{V}^{(\infty)}, \quad \check{H}^{(\infty)} = \operatorname{ann} \check{V}^{(\infty)}, \\ \text{and} \qquad K = \hat{H}^{(\infty)} \cap \check{H}^{(\infty)}. \end{gathered} \tag{2.7}$$

The definition of Darboux integrability can be re-formulated in terms of the distributions $\hat{H}, \check{H}$ ([26]). If we introduce the dual basis $\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\}$ to the 0-adapted coframe $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$, that is,

$$\boldsymbol{\theta}(\boldsymbol{\partial}_{\boldsymbol{\theta}}) = \mathbf{1}, \quad \hat{\boldsymbol{\sigma}}(\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}) = \mathbf{1}, \quad \hat{\boldsymbol{\eta}}(\boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}) = \mathbf{1}, \quad \check{\boldsymbol{\sigma}}(\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}) = \mathbf{1}, \quad \check{\boldsymbol{\eta}}(\boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}) = \mathbf{1}, \tag{2.8}$$

with all others pairings yielding 0, then it readily follows that

$$\begin{gathered} \hat{H} = \operatorname{span}\{\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\}, \qquad \hat{H}^{(\infty)} = \operatorname{span}\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\}, \\ \check{H} = \operatorname{span}\{\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}\}, \qquad \check{H}^{(\infty)} = \operatorname{span}\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}\}, \\ \text{and} \quad K = \operatorname{span}\{\boldsymbol{\partial}_{\boldsymbol{\theta}}\}. \end{gathered} \tag{2.9}$$

Any local frame $\{\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\}$ satisfying (2.9) is called a 0-adapted frame.

We shall use the following lemma repeatedly.

Lemma 2.8. *Let f be a real-valued function on M. If $X(f) = 0$ for all vector fields in $\hat{H}$, then $df \in \hat{V}^{(\infty)}$. Likewise, if $X(f) = 0$ for all vector fields in $\check{H}$, then $df \in \check{V}^{(\infty)}$.*

Proof. It suffices to note that if $X(f) = 0$ for all vector fields $X \in \hat{H}$, then $Z(f) = Z \lrcorner\, df = 0$ for all vector fields $Z \in \hat{H}^{(\infty)}$ and therefore $df \in \operatorname{ann} \hat{H}^{(\infty)} = \hat{V}^{(\infty)}$. ∎

Real-valued functions on M with $df \in V$ (or $V^{(\infty)}$) are called ***first integrals*** for V. We denote the algebra of all first integrals for V by $\mathrm{Int}(V)$.

Our 1-adapted coframe for a Darboux pair is easily constructed from complete sets of functionally independent first integrals for $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$, that is, locally defined functions $\{\,\hat{\boldsymbol{I}}\,\}$ and $\{\,\check{\boldsymbol{I}}\,\}$ such that

$$\hat{V}^{(\infty)} = \mathrm{span}\{\, d\hat{\boldsymbol{I}}\,\} \quad \text{and} \quad \check{V}^{(\infty)} = \mathrm{span}\{\, d\check{\boldsymbol{I}}\,\}.$$

Complete the 1-forms $\{\,\hat{\boldsymbol{\eta}}\,\}$ in (2.6a) to the local basis (2.6b) for $\hat{V}^{(\infty)}$ using forms

$$\hat{\boldsymbol{\sigma}} = d\hat{\boldsymbol{I}}_1 \tag{2.10}$$

chosen from the $\{\, d\hat{\boldsymbol{I}}\,\}$. Next let $\{\,\hat{\boldsymbol{I}}_2\,\}$ be a complementary set of invariants to the set $\{\,\hat{\boldsymbol{I}}_1\,\}$, that is, $\{\,\hat{\boldsymbol{I}}\} = \{\,\hat{\boldsymbol{I}}_1,\, \hat{\boldsymbol{I}}_2\,\}$. Because the forms $\hat{\boldsymbol{\eta}}$ belong to $\hat{V}^{\infty}$ we can write

$$\hat{\boldsymbol{\eta}} = \hat{\mathbf{R}}_0\, d\hat{\boldsymbol{I}}_1 + \hat{\boldsymbol{S}}_0\, d\hat{\boldsymbol{I}}_2.$$

Since the 1-forms $\hat{\boldsymbol{\eta}}$ are independent of the 1-forms $\hat{\boldsymbol{\sigma}}$, the coefficient matrix $\hat{\boldsymbol{S}}_0$ must be invertible and we can therefore adjust our local basis of 1-forms $\{\,\hat{\boldsymbol{\eta}}\,\}$ for $\hat{V}^{(\infty)} \cap \check{V}$ by setting

$$\hat{\boldsymbol{\eta}} = d\hat{\boldsymbol{I}}_2 + \hat{\mathbf{R}}\, d\hat{\boldsymbol{I}}_1 = d\hat{\boldsymbol{I}}_2 + \hat{\mathbf{R}}\, \hat{\boldsymbol{\sigma}}. \tag{2.11}$$

The exterior derivatives of these forms are

$$d\hat{\boldsymbol{\eta}} = d\hat{\mathbf{R}} \wedge \hat{\boldsymbol{\sigma}} = \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}(\hat{\mathbf{R}})\, \check{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \cdots$$

and therefore, on account of (1.10), the fact that $\hat{\boldsymbol{\eta}} \in \hat{V}^{(\infty)} \cap \check{V}$, and (2.6c), we must have $\boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}(\hat{\mathbf{R}}) = 0$. Lemma 2.8 implies that $d(\hat{\mathbf{R}}) \in \hat{V}^{(\infty)}$ and consequently

$$d\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{E}}\, \hat{\boldsymbol{\eta}} \wedge \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{F}} \hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}. \tag{2.12}$$

Similar arguments are used to modify the forms $\check{\boldsymbol{\sigma}}$ and $\check{\boldsymbol{\eta}}$. These adaptations are summarized in the following theorem.

Theorem 2.9. *Let* $\{\hat{V}, \check{V}\}$ *be a Darboux pair on a manifold* M*. Then about each point of* M *there exists a 0-adapted coframe* $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ *with structure equations*

$$\begin{aligned} d\hat{\boldsymbol{\sigma}} &= 0, \qquad d\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{E}}\,\hat{\boldsymbol{\eta}} \wedge \hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{F}}\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}, \\ d\check{\boldsymbol{\sigma}} &= 0, \qquad d\check{\boldsymbol{\eta}} = \check{\boldsymbol{E}}\,\check{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}} + \check{\boldsymbol{F}}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}, \end{aligned} \tag{2.13}$$

$$d\boldsymbol{\theta} \equiv \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}} \qquad \bmod \{\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\}. \tag{2.14}$$

We remark that the structure equations for the 1-forms $\boldsymbol{\theta} \in \hat{V} \cap \check{V}$ are a consequence of (1.10) and the properties (2.6) of a 0-adapted coframe. A ***1-adapted coframe*** for the Darboux pair $\{\hat{V}, \check{V}\}$ is a 0-adapted coframe $\{\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\}$ satisfying the structure equations (2.13) and (2.14).

Corollary 2.10. *If* $\{\hat{V}, \check{V}\}$ *define a Darboux pair, then* $\hat{\mathcal{V}} \boxplus \check{\mathcal{V}}$ *is always a decomposable exterior differential system for which* $\hat{V}$ *and* $\check{V}$ *are singular Pfaffian systems.*

Proof. It follows from Theorem 2.9 that the algebraic generators for $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ are

$$\hat{\mathcal{V}} = \{\boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \hat{\boldsymbol{\sigma}},\ \check{\boldsymbol{F}}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\} \quad \text{and}$$

$$\check{\mathcal{V}} = \{\boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \check{\boldsymbol{\sigma}},\ \hat{\boldsymbol{F}}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}\}.$$

From these equations it is not difficult to argue that the differential system $\hat{\mathcal{V}} \cap \check{\mathcal{V}}$ has generators

$$\hat{\mathcal{V}} \cap \check{\mathcal{V}} = \{\boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \hat{\boldsymbol{F}}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \check{\boldsymbol{F}}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\}.$$

A repetition of the arguments used in the proof of Theorem 2.6 shows that

$$\hat{\mathcal{V}} \boxplus \check{\mathcal{V}} = \{\boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \hat{\boldsymbol{F}}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \check{\boldsymbol{F}}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\}. \tag{2.15}$$

It is then clear that $\hat{\mathcal{V}} \boxplus \check{\mathcal{V}}$ is a decomposable differential system (Definition 1.1) and that $\hat{V}$ and $\check{V}$ are singular Pfaffian systems. ∎

Example 2.11. There are many examples of differential equations which can be described either by Pfaffian differential systems or by differential systems generated by 1-forms and 2-forms. Our definition of Darboux integrability in terms of decomposable differential system is such that theseequations will be

Darboux integrable irrespective of their formulation as an exterior differential system. The simplest example, but by no means the only example, is the wave equation $u_{xy} = 0$. The wave equation can be encoded as adifferential system on manifold of dimensions 5, 6 and 7 by

$$\begin{aligned} &\mathcal{I}_1 = \{\,\theta, dp\wedge dx, dq\wedge dy\,\}, && \hat{V}_1 = \{\,\theta, dp, dx\,\}, && \check{V} = \{\,\theta, dq, dy\,\},\\ &\mathcal{I}_2 = \{\,\theta, \theta_x, dq\wedge dy\,\}, && \hat{V}_2 = \{\,\theta, \theta_x, dp, dx\,\}, && \check{V}_2 = \{\,\theta, \theta_x, dq, dy\,\},\\ &\mathcal{I}_3 = \{\,\theta, \theta_x, \theta_y\,\}, && \hat{V}_3 = \{\,\theta, \theta_x, \theta_y, dp, dx\,\}, && \check{V}_3 = \{\,\theta, \theta_x, \theta_y, dq, dy\,\}, \end{aligned}$$

where $\theta = du - p\,dx - q\,dy$, $\theta_x = dp - r\,dx$ and $\theta_y = dq - t\,dy$. Each of these satisfy the definition of a decomposable, Darboux integrable differential system. Consistent with Corollary 2.6, one easily checks that $\mathcal{I}_i = \hat{\mathcal{V}}_i \Cap \check{\mathcal{V}}_i$.

We remark that for the standard contact system on $J^1(\mathbf{R},\mathbf{R})$, that is, $\mathcal{I}_4 = \{\,\theta,\, dp\wedge dx + dq\wedge dy\,\}$, the Pfaffian systems $\hat{V}_4 = \{\,\theta,\, dp,\, dx\,\}$ and $\check{V} = \{\,\theta,\, dq,\, dy\,\}$, are singular systems which form a Darboux pair but $\hat{V}_4 \Cap \check{V}_4 \neq \mathcal{I}_4$. Equality fails because $\mathcal{I}_4$ is not decomposable. ∎

Remark 2.12. In very special cases, such as Liouville's equation $u_{xy} = e^u$, one finds that $\hat{V} \cap \check{V}^{(\infty)} = \hat{V}^{(\infty)} \cap \check{V} = \{\,0\,\}$ so that the vector space sums appearing in (1.8) are direct sum decompositions. Under these circumstances, our 0-adapted coframe is simply $\{\,\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \check{\boldsymbol{\sigma}}\,\}$. This initial coframe is then automatically 2-adapted (see Theorem 4.3) and the sequence of coframe adaptations needed to prove Theorem 4.1 can begin with the third coframe adaptation (Section 4.2).

Remark 2.13. We define an ***involution of a Darboux pair*** $\{\,\hat{V}, \check{V}\,\}$ on M to be a diffeomorphism $\Phi\colon M \to M$ such that $\Phi^2 = \mathrm{id}_M$ and $\Phi^*(\hat{V}) = \check{V}$. For such maps $\Phi^*(\hat{V}^{(\infty)}) = \check{V}^{(\infty)}$ and therefore, given 1 adapted co-frame elements $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\sigma}}$, one may take

$$\check{\boldsymbol{\eta}} = \Phi^*(\hat{\boldsymbol{\eta}}) \quad \text{and} \quad \check{\boldsymbol{\sigma}} = \Phi^*(\hat{\boldsymbol{\sigma}}).$$

Such involutions are quite common, especially for differential equations arising in geometric and physical applications, where the diffeomorphism Φ arises from a symmetry of the differential equations in the independent variables.

3 The group theoretic construction of Darboux pairs

3.1 Direct Sums of Pfaffian Systems and Trivial Darboux Pairs

Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be differential systems on manifolds M_1 and M_2. Then the ***direct sum*** $\mathcal{W}_1+\mathcal{W}_2$ is the EDS on $M_1\times M_2$ generated by the pullbacks of $\mathcal{W}_1$ and $\mathcal{W}_2$ by the canonical projection maps $\pi_i\colon M_1\times M_2\to M_i$. The following theorem shows that if $\mathcal{W}_1$ and $\mathcal{W}_2$ are Pfaffian, then $\mathcal{W}_1+\mathcal{W}_2$ is Darboux integrable in the sense of Definition 1.3.

Theorem 3.1. *Let W_1 and W_2 be Pfaffian systems on manifolds M_1 and M_2 such that $W_1^{(\infty)}=W_2^{(\infty)}=0$. Then the direct sum $J=W_1+W_2$ on $M_1\times M_2$ is a decomposable Pfaffian system. The associated singular Pfaffian systems*

$$\hat{W}=W_1\oplus\Lambda^1(M_2)\quad\textit{and}\quad\check{W}=\Lambda^1(M_1)\oplus W_2 \tag{3.1}$$

form a Darboux pair and

$$\mathcal{W}_1+\mathcal{W}_2=\hat{\mathcal{W}}\boxplus\check{\mathcal{W}}, \tag{3.2}$$

where $\mathcal{W}_1,\mathcal{W}_2,\hat{\mathcal{W}},\check{\mathcal{W}}$ are the differential systems generated by $W_1,W_2,\hat{W},\check{W}$.

Proof. It is easy to check that J is decomposable with singular systems (3.1). Properties **[i]** and **[ii]** of Definition 1.2 follow directly from (3.1) and the simple observations that

$$\hat{W}^{(\infty)}=\Lambda^1(M_2)\quad\text{and}\quad\check{W}^{(\infty)}=\Lambda^1(M_1). \tag{3.3}$$

Let $\omega=\sum_i(f_i\,\theta_1^i+g_i\,\theta_2^i)$, where the coefficients $f_i,g_i\in C^\infty(M_1\times M_2)$ and the $\theta_\ell^i\in W_\ell$ are 1-forms in J. Then to check property **[iii]**, it suffices to note that every summand in the exterior derivative

$$d\omega=\sum_i d_1f_i\wedge\theta_1^i+d_2f_i\wedge\theta_1^i+f_i\,d_1\theta_1^i+d_1g_j\wedge\theta_2^j+d_2g_j\wedge\theta_2^j+g_i\,d_2\theta_2^i,$$

where $d_i=d_{M_i}$, lies either in $\Omega^2(\hat{W})$ or $\Omega^2(\check{W})$. Equation (3.2) is a direct consequence of Theorem 2.6. ∎

Darboux pairs of the form (3.1) are said to be trivial.

3.2 Symmetry Reduction of Darboux Pairs

A Lie group G is a ***regular symmetry group of a differential system*** $\mathcal{I}$ on a manifold M if there is a regular action of G on M which preserves $\mathcal{I}$. By the definition of regularity, the quotient space M/G of M by the orbits of G has a smooth manifold structure such that **[i]** the projection map $\mathbf{q}\colon M \to M/G$ is a smooth submersion, and **[ii]** local, smooth cross-sections to $\mathbf{q}$ exist on some open neighborhood of each point of M/G. The G ***reduction of*** $\mathcal{I}$ ***or the quotient of*** $\mathcal{I}$ ***by*** G is the differential system on $\overline{M} = M/G$ given by

$$\mathcal{I}/G = \{\, \omega \in \Omega^*(\overline{M}) \,|\, \mathbf{q}^*(\omega) \in \mathcal{I} \,\}. \tag{3.4}$$

Details of this construction and basis facts regarding the G reduction of differential systems are given in [1].

To obtain the main results of this section, we shall require that G be ***transverse*** to $\mathcal{I}$ in the sense which we now define. Let I be any G invariant sub-bundle of T^*M and let Γ be the infinitesimal generators for G. We say that G is transverse to I if for each $x \in M$

$$\Gamma_x \cap \operatorname{ann} I_x = \{\, 0 \,\}. \tag{3.5}$$

We say that G is transverse to the differential system $\mathcal{I}$ if G is transverse to the sub-bundle of T^*M whose sections are 1-forms in $\mathcal{I}$. With this transversality condition, $\mathcal{I}/G$ is assured to be a constant rank differential system whose local sections can be determined as follows. If A is any sub-bundle of $\Lambda^k(M)$ or sub-algebra of $\Lambda^*(M)$, then the sub-bundle or sub-algebra of ***semi-basic forms in*** A is defined as

$$A_{\mathbf{sb}} = \{\, \omega \in A \,|\, X \lrcorner\, \omega = 0 \text{ for all } X \in \Gamma \,\}. \tag{3.6}$$

Denote by $\mathcal{I}_{\mathbf{sb}}$ the sections of $\mathcal{I}$ which are semi-basic, that is, which take values in $\Lambda^*_{\mathbf{sb}}(M)$. Then one can show (see Lemma 4.1 of [1]) that

$$(\mathcal{I}/G)_{|\overline{U}} = \xi^*(\mathcal{I}_{\mathbf{sb}|U}), \tag{3.7}$$

where $\overline{U}$ is any (sufficiently small) open set of $\overline{M}$, $\xi\colon \overline{U} \to M$ is a smooth cross-section of $\mathbf{q}$, and $U = \mathbf{q}^{-1}(\overline{U})$. We remark that the G reduction $\mathcal{I}/G$ of a Pfaffian system $\mathcal{I}$ need not be Pfaffian (but see **[c]** below).

For the proof of the next theorem on the G reduction of Darboux pairs, we shall require the following elementary facts regarding the reduction of Pfaffian systems. Here I and J are Pfaffian systems with regular symmetry group G, we assume that G is transverse to I, and the G bundle reduction of I is defined as

$$I/G = \{\,\omega \in \Lambda^1(\overline{M}) \,|\, \mathbf{q}^*(\omega) \in I\,\}. \tag{3.8}$$

By transversality, this is a sub-bundle of $T^*\overline{M}$ of dimension $\dim I - q$, where q is the dimension of the orbits of G.

[**a**] If $I \subset J$, then G is transverse to J.

[**b**] If G is transverse to $I \cap J$, then $(I/G) \cap (J/G) = (I \cap J)/G$.

[**c**] If G is transverse to the derived system I', then the quotient differential system $\mathcal{I}/G$ is the constant rank Pfaffian system determined by I/G and $(I/G)' = I'/G$.

[**d**] If G is transverse to $I^{(\infty)}$, then $(I/G)^{(\infty)} = I^{(\infty)}/G$.

Facts [**a**] and [**b**] are trivial, while [**c**] is Theorem 5.1 in [1]. Fact [**d**] follows from [**a**] and [**c**] by induction.

Finally, we shall also use the following technical observation. If Z is a sub-bundle of I, then the canonical pairing $(X, \omega) = \omega(X)$ on $\Gamma_x \times Z_x$, where $x \in M$, is non-degenerate if and only if

$$I = \mathbf{q}^*(I/G) \oplus Z. \tag{3.9}$$

Such bundles Z can always be constructed locally on G invariant open subsets of M.

Theorem 3.2. *Let $\{\,\hat{W}, \check{W}\,\}$ be a Darboux pair on M. Suppose that G is a regular symmetry group of $\hat{W}$ and $\check{W}$ and that, in addition, G is transverse to $\hat{W} \cap \check{W}^{(\infty)}$ and $\hat{W}^{(\infty)} \cap \check{W}$. Then the quotient differential systems $\{\,\hat{\mathcal{W}}/G, \check{\mathcal{W}}/G\,\}$ on $\overline{M}$ are (constant rank) Pfaffian systems and form a Darboux pair.*

Proof of Theorem 3.2. Since $\hat{W}^{(\infty)} \cap \check{W} \subset \hat{W}^{(\infty)} \subset \hat{W}' \subset \hat{W}$ and G is transverse to $\hat{W}^{(\infty)} \cap \check{W}$, it follows that G is transverse to $\hat{W}^{(\infty)}$, $\hat{W}'$ and $\hat{W}$. Thus, by [**c**], $\hat{\mathcal{V}} = \hat{\mathcal{W}}/G$ and, similarly, $\check{\mathcal{V}} = \check{\mathcal{W}}/G$ are the Pfaffian systems for $\hat{V} = \hat{W}/G$ and $\check{V} = \check{W}/G$. By [**d**]

$$(\hat{W}/G)^{(\infty)} = \hat{W}^{(\infty)}/G \quad \text{and} \quad (\check{W}/G)^{(\infty)} = \check{W}^{(\infty)}/G. \tag{3.10}$$

We now verify the three conditions in Definition 1.2 for $\{\hat{V}, \check{V}\}$ to be a Darboux pair. From (3.8), (3.10) and fact **[b]** we deduce that

$$\begin{aligned}
\dim(\hat{V}) &= \dim \hat{W} - q, \\
\dim(\hat{V}^{(\infty)}) &= \dim(\hat{W}^{(\infty)}/G) = \dim \hat{W}^{(\infty)} - q, \quad \text{and} \\
\dim(\hat{V} \cap \hat{V}^{(\infty)}) &= \dim\big((\hat{W}/G) \cap (\check{W}^{(\infty)}/G)\big) = \dim\big((\hat{W} \cap \check{W}^{(\infty)})/G\big) \\
&= \dim(\hat{W} \cap \check{W}^{(\infty)}) - q.
\end{aligned}$$

From these equations and property **[i]** in the definition of Darboux pair (applied to $\{\hat{W}, \check{W}\}$) we calculate

$$\begin{aligned}
\dim\big(\hat{V} + \check{V}^{(\infty)}\big) &= \dim(\hat{V}) + \dim(\check{V}^{(\infty)}) - \dim(\hat{V} \cap \check{V}^{(\infty)}) \\
&= \dim \hat{W} + \dim \check{W}^{(\infty)} + \dim(\hat{W} \cap \check{W}^{(\infty)}) - q \\
&= \dim M - q = \dim \overline{M}.
\end{aligned}$$

This proves that $\hat{V} + \check{V}^{(\infty)} = T^*\overline{M}$. Similar arguments yield $\hat{V}^{(\infty)} + \check{V} = T^*\overline{M}$ and so $\{\hat{V}, \check{V}\}$ satisfy property **[i]** of a Darboux pair.

To check that $\{\hat{V}, \check{V}\}$ satisfy property **[ii]** of a Darboux pair, we use **[b]** and (3.10) to calculate

$$\hat{V}^{(\infty)} \cap \check{V}^{(\infty)} = (\hat{W}^{(\infty)}/G) \cap (\check{W}^{(\infty)}/G) = (\hat{W}^{(\infty)} \cap \check{W}^{(\infty)})/G = 0. \tag{3.11}$$

To prove property **[iii]** of a Darboux pair, we first prove that

$$[\Lambda^2(\hat{W}) + \Lambda^2(\check{W})]_{\mathbf{sb}} = \Lambda^2(\hat{W}_{\mathbf{sb}}) + \Lambda^2(\check{W}_{\mathbf{sb}}). \tag{3.12}$$

It suffices to check this locally. Since G acts transversally to $\hat{W}^{(\infty)} \cap \check{W}$, there is a locally defined sub-bundle Z of $\hat{W}^{(\infty)} \cap \check{W}$ such that the canonical pairing on $\Gamma \times Z$ is pointwise non-degenerate. For any such sub-bundle Z we have that $\hat{W} = \hat{W}_{\mathbf{sb}} \oplus Z$ and $\check{W} = \check{W}_{\mathbf{sb}} \oplus Z$ and therefore

$$\Lambda^2(\hat{W}) + \Lambda^2(\check{W}) = \Lambda^2(\hat{W}_{\mathbf{sb}}) + \Lambda^2(\check{W}_{\mathbf{sb}}) + T^*M \wedge Z.$$

This (local) decomposition immediately leads to (3.12). To complete the proof of property **[iii]**, let $\omega \in \hat{V} \cap \check{V}$. Then $\mathbf{q}^*(\omega) \in \hat{W} \cap \check{W}$ and hence, by the properties of the Darboux pair $\{\hat{W}, \check{W}\}$ and (3.12), $\mathbf{q}^*(d\omega) \in \Omega^2(\hat{W}_{\mathbf{sb}}) + \Omega^2(\check{W}_{\mathbf{sb}})$. We pullback this last equation by a local cross-section of $\mathbf{q}$ and invoke (3.7) to conclude that $d\omega \in \Omega^2(\hat{V}) + \Omega^2(\check{V})$. ∎

Theorem 3.3. *Let $\{\hat{W}, \check{W}\}$ be a Darboux pair on M. Suppose that G is a regular symmetry group of $\hat{W}$ and $\check{W}$ and that, in addition, G is transverse to $\hat{W}^{(\infty)} \cap \check{W}$ and $\hat{W} \cap \check{W}^{(\infty)}$. Then*

$$(\hat{\mathcal{W}} \Cap \check{\mathcal{W}})/G = (\hat{\mathcal{W}}/G) \Cap (\check{\mathcal{W}}/G). \tag{3.13}$$

Proof. We first remark that Theorem 3.2 implies that $\hat{V} = \hat{W}/G$ and $\check{V} = \check{W}/G$ define a Darboux pair and, second, that it suffices to check (3.13) locally.

Transversality implies that there are locally defined sub-bundles $\hat{Z} \subset \hat{W}^{(\infty)} \cap \check{W}$ and $\check{Z} \subset \hat{W} \cap \check{W}^{(\infty)}$ on which the natural pairing with Γ (the infinitesimal generators for G) is non-degenerate at each point $x \in M$. It follows from (3.9) that

$$\hat{W}^{(\infty)} \cap \check{W} = \mathbf{q}^*(\hat{V}^{(\infty)} \cap \check{V}) \oplus \hat{Z}, \quad \hat{W}^{(\infty)} = \mathbf{q}^*(\hat{V}^{(\infty)}) \oplus \hat{Z}, \tag{3.14a}$$

$$\hat{W} \cap \check{W}^{(\infty)} = \mathbf{q}^*(\hat{V} \cap \check{V}^{(\infty)}) \oplus \check{Z}, \quad \check{W}^{(\infty)} = \mathbf{q}^*(\check{V}^{(\infty)}) \oplus \check{Z}, \tag{3.14b}$$

$$\hat{W} = \mathbf{q}^*(\hat{V}) \oplus \hat{Z}, \qquad \check{W} = \mathbf{q}^*(\hat{V}) \oplus \check{Z}, \quad \text{and} \tag{3.14c}$$

$$\check{W} \cap \hat{W} = \mathbf{q}^*(\check{V} \cap \hat{V}) + \hat{Z} = \mathbf{q}^*(\check{V} \cap \hat{V}) + \check{Z}. \tag{3.14d}$$

If $\hat{\boldsymbol{\zeta}}$ and $\check{\boldsymbol{\zeta}}$ are local basis of sections for $\hat{Z}$ and $\check{Z}$ then, because $\hat{\boldsymbol{\zeta}} \in \hat{W}^{(\infty)}$ and $\check{\boldsymbol{\zeta}} \in \check{W}^{(\infty)}$, $d\hat{\boldsymbol{\zeta}} \equiv 0 \mod \hat{W}^{(\infty)}$ and $d\check{\boldsymbol{\zeta}} \equiv 0 \mod \check{W}^{(\infty)}$, or

$$d\hat{\boldsymbol{\zeta}} \equiv 0 \mod \{\mathbf{q}^*(\hat{V}^{(\infty)}),\, \hat{\boldsymbol{\zeta}}\} \quad \text{and} \quad d\check{\boldsymbol{\zeta}} \equiv 0 \mod \{\mathbf{q}^*(\check{V}^{(\infty)}),\, \check{\boldsymbol{\zeta}}\}.$$

These equations, together with (3.14c), show that the Pfaffian systems $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ are algebraically generated as

$$\hat{\mathcal{W}} = \{\mathbf{q}^*(\hat{\mathcal{V}}),\, \hat{\boldsymbol{\zeta}}\} \quad \text{and} \quad \check{\mathcal{W}} = \{\mathbf{q}^*(\check{\mathcal{V}}),\, \check{\boldsymbol{\zeta}}\}. \tag{3.15}$$

To complete the proof of the theorem, we now make the key observation that since $\check{\boldsymbol{\zeta}} \in \hat{W} \cap \check{W}$, (3.14c) implies that we may express these forms as linear combinations of the $\hat{\boldsymbol{\zeta}}$ and the forms in $\mathbf{q}^*(\hat{V} \cap \check{V})$. Hence $\check{\mathcal{W}}$ is also algebraically generated as $\check{\mathcal{W}} = \{\mathbf{q}^*(\check{\mathcal{V}}), \check{\boldsymbol{\zeta}}\}$. We may list an explicit set of local generators for $\mathbf{q}^*(\hat{\mathcal{V}})$ and $\mathbf{q}^*(\check{\mathcal{V}})$ using the 1-adapted coframe constructed for the Darboux pair $\{\hat{V}, \check{V}\}$ in Theorem 2.9. The arguments given in the proof of Theorem 2.10 can then be repeated to show that $\hat{\mathcal{W}} \Cap \check{\mathcal{W}} = \{\mathbf{q}^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}}), \hat{\boldsymbol{\zeta}}\}$ and hence $(\hat{\mathcal{W}} \Cap \check{\mathcal{W}})_{\mathbf{sb}} = \mathbf{q}^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}})$, as required. ∎

Corollary 3.4. *Let W_1 and W_2 be Pfaffian systems on manifolds M_1 and M_2 with $W_1^{(\infty)} = W_2^{(\infty)} = 0$. Suppose that*

[i] *a Lie group G acts freely on M_1 and M_2 and as symmetry groups of both W_1 and W_2;*

[ii] *G is transverse to both W_1 and W_2; and*

[iii] *the diagonal action of G on $M_1 \times M_2$ is regular.*

Then the quotient differential system

$$\mathcal{J} = (\mathcal{W}_1 + \mathcal{W}_2)/G \tag{3.16}$$

is a decomposable, Darboux integrable, differential system with respect to the Darboux pair

$$\hat{U} = (W_1 \oplus \Lambda^1(M_2))/G \quad \textit{and} \quad \check{U} = (\Lambda^1(M_1) \oplus W_2)/G. \tag{3.17}$$

Proof. As in Theorem 3.1, define the Darboux pair $\hat{W} = W_1 \oplus \Lambda^1(M_2)$ and $\check{W} = \Lambda^1(M_1) \oplus W_2$. To conclude from Theorem 3.2 that $\hat{U} = \hat{W}/G$, $\check{U} = \check{W}/G$ are also a Darboux pair, we observe that **[i]** and **[ii]** imply that the diagonal action of G on $M_1 \times M_2$ is transverse to the Pfaffian systems W_1 and W_2 pulled back to $M_1 \times M_2$ (a fact which is not true without **[i]**. We also note that, by (3.3),

$$\hat{W} \cap \check{W}^{(\infty)} = W_1 \quad \text{and} \quad \hat{W}^{(\infty)} \cap \check{W} = W_2. \tag{3.18}$$

We now use (3.2) and Theorem 3.3 to deduce that

$$\mathcal{J} = (\mathcal{W}_1 + \mathcal{W}_2)/G = (\hat{\mathcal{W}} \Cap \check{\mathcal{W}})/G = (\hat{\mathcal{W}}/G) \Cap (\check{\mathcal{W}}/G) = \hat{\mathcal{U}} \Cap \check{\mathcal{U}}.$$

An application of Corollary 2.10 completes the proof. ∎

4 The coframe adaptations for a Darboux pair

In this section we present a series of coframe adaptations for any Darboux pair $\{\hat{V}, \check{V}\}$. These coframe adaptations lead to the proof of the following theorem.

Theorem 4.1. *Let $\{\hat{V}, \check{V}\}$ be a Darboux pair of Pfaffian differential systems on a manifold M. Then around each point of M there are 2 coframes*

$$\{\hat{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha\} \quad \textit{and} \quad \{\check{\theta}^i, \hat{\pi}^a, \check{\pi}^\alpha\} \tag{4.1}$$

such that

$$\hat{V} = \{\hat{\theta}^i, \hat{\pi}^a\}, \quad \hat{V}^\infty = \{\hat{\pi}^a\}, \quad \check{V} = \{\check{\theta}^i, \check{\pi}^a\}, \quad \check{V}^\infty = \{\check{\pi}^a\} \tag{4.2}$$

and with structure equations

$$\begin{aligned} d\hat{\theta}^i &= \frac{1}{2}G^i_{\alpha\beta}\check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}C^i_{jk}\hat{\theta}^j \wedge \hat{\theta}^k \quad \textit{and} \\ d\check{\theta}^i &= \frac{1}{2}H^i_{ab}\hat{\pi}^a \wedge \hat{\pi}^b - \frac{1}{2}C^i_{jk}\check{\theta}^j \wedge \check{\theta}^k. \end{aligned} \tag{4.3}$$

The coefficients C^i_{jk} are the structure constants of a Lie algebra whose isomorphism class is an invariant of the Darboux pair $\{\hat{V}, \check{V}\}$.

We have already constructed the first coframe adaptation in Section 2. The second coframe (Section 4.1) adaptation provides us with a stronger form for the decomposition of the structure equations than that initially given by (2.5). The third and fourth coframe adaptations (Section 4.2 and 4.3) lead to the definition of the Vessiot Lie algebra $\mathfrak{vess}(\hat{V}, \check{V})$, with structure constants C^i_{jk}, as a fundamental invariant for any Darboux integrable differential system. The importance of this algebra was first observed by Vassiliou [26] who introduced the notion of the tangential symmetry algebra for certain special classes of Darboux integrable systems. The tangential symmetry algebra is a Lie algebra of vector fields Γ on M which is isomorphic (as an abstract Lie algebra) to the Vessiot Lie algebra. The tangential symmetry algebra also plays a significant role in Eendebak's projection method [10] for Darboux integrable equations.

Of course, the tangential symmetry algebra Γ determines a local transformation group on M but this transformation group will generally not be the correct one required for constructing the superposition formula (See Example 6.1). One final coframe adaptation (Section 4.4) is needed to arrive at the proper transformation groups (See Definition 5.7) for this construction – this is the coframe that satisfies the structure equations given in Theorem 4.1.

The coframe adaptations 1 – 4 involve only differentiations and elementary linear algebra operations. For the fifth coframe adaptation, in the case where the Vessiot algebra is not semi-simple, one must integrate a linear system of total differential equations (see (4.61)) and also solve the exterior derivative equation $d(\alpha) = \beta$, where β is a closed 1 or 2-form (see (4.64) and (4.73)).

4.1 The second adapted coframe for a Darboux pair.

Let $\{\hat{V}, \check{V}\}$ be a Darboux pair and let $\{\,\boldsymbol{\theta}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\,\}$ be a 1-adapted coframe with dual frame $\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\,\}$. From this point forward, we are solely interested in adjustments to the forms $\boldsymbol{\theta}$ that will simplify the 1-adapted structure equations

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}} + \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}} \mod \{\,\boldsymbol{\theta}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\eta}}\,\}. \tag{4.4}$$

Of the "mixed" wedge products

$$\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}, \quad \hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\eta}}, \quad \hat{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\sigma}}, \quad \text{and} \quad \hat{\boldsymbol{\eta}} \wedge \check{\boldsymbol{\eta}} \tag{4.5}$$

the products $\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}$ are the only ones definitely not present in (4.4). We shall now show that it is possible to make an adjustment to the 1-forms $\boldsymbol{\theta}$ of the type

$$\boldsymbol{\theta}' = \boldsymbol{\theta} + \boldsymbol{P}\,\hat{\boldsymbol{\eta}} + \boldsymbol{Q}\,\check{\boldsymbol{\eta}} \tag{4.6}$$

so that the structure equations for the modified forms $\boldsymbol{\theta}'$ are free of all the wedge products (4.5). We begin with the following simple observation.

Lemma 4.2. *If* $\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\,\}$ *is the dual frame to a 1-adapted coframe for the Darboux pair* $\{\,\hat{V}, \check{V}\,\}$*, then*

$$[\,\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}\,] = 0. \tag{4.7}$$

Proof. It suffices to note that none of the structure equations for a 1-adapted coframe contain any of the wedge products $\hat{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}$. ∎

The construction of the second adapted coframe is completely algebraic in that only differentiations and linear algebraic operations are involved. Let $\hat{S}_a =$

$\partial_{\hat\sigma^a}$ and $\hat S_\alpha = \partial_{\check\sigma^\alpha}$ and define two sequences of vector fields inductively by

$$\hat S_{a_1a_2\cdots a_\ell} = [\,\hat S_{a_1a_2\cdots a_{\ell-1}},\,\hat S_{a_\ell},\,] \quad \text{and} \quad \check S_{\alpha_1\alpha_2\cdots\alpha_\ell} = [\,\check S_{\alpha_1\alpha_2\cdots\alpha_{\ell-1}},\,\check S_{\alpha_\ell}\,]. \tag{4.8}$$

Because the vector fields $\hat S_{a_\ell} \in \hat H$, the vector fields $S_{a_1a_2\cdots a_\ell}$ belong to $\hat H^{(\infty)}$ and are therefore linear combinations of the vector fields $\{\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\sigma}},\,\boldsymbol{\partial_{\hat\eta}}\}$. However, because $d\hat{\boldsymbol\sigma} = \mathbf{0}$, one can deduce by a straightforward induction that in fact

$$S_{a_1a_2\cdots a_\ell} \in \operatorname{span}\{\,\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\eta}}\,\} \quad \text{and, likewise,} \quad \check S_{\alpha_1\alpha_2\cdots\alpha_\ell} \in \operatorname{span}\{\,\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\check\eta}}\,\}. \tag{4.9}$$

The Jacobi identity also shows that

$$\hat H^{(\infty)} = \operatorname{span}\{\,\hat S_{a_1a_2\cdots a_\ell}\,\}_{\ell\ge1} \quad \text{and} \quad \check H^{(\infty)} = \operatorname{span}\{\,\check S_{\alpha_1\alpha_2\cdots\alpha_\ell}\,\}_{\ell\ge1}. \tag{4.10}$$

From (4.7) it also follows that

$$[\,\hat S_{a_1a_2\cdots a_k}, \check S_{\alpha_1\alpha_2\cdots\alpha_\ell}\,] = 0. \tag{4.11}$$

for $k, l \ge 1$. By virtue of (4.10) we can choose a specific collection of iterated Lie brackets $\hat S_{a_1a_2\cdots a_\ell}$ and $\check S_{\alpha_1\alpha_2\cdots\alpha_\ell}$, $\ell \ge 2$, which complete $\{\,\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\sigma}}\,\}$ and $\{\,\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\check\sigma}}\,\}$ to local bases for $\hat H^{(\infty)}$ and $\check H^{(\infty)}$ respectively. Denote the iterated brackets so chosen by $\{\,\hat{\boldsymbol s}'\,\}$ and $\{\,\check{\boldsymbol s}'\,\}$. Since $\operatorname{span}\{\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\sigma}},\,\boldsymbol{\partial_{\hat\eta}}\} = \operatorname{span}\{\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\sigma}},\,\hat{\boldsymbol s}'\}$ and $\operatorname{span}\{\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\check\sigma}},\,\boldsymbol{\partial_{\check\eta}}\} = \operatorname{span}\{\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\check\sigma}},\,\check{\boldsymbol s}'\}$, this gives us a preferred 0-adapted frame $\{\,\boldsymbol{\partial_\theta}, \boldsymbol{\partial_{\hat\sigma}}, \hat{\boldsymbol s}', \boldsymbol{\partial_{\check\sigma}}, \check{\boldsymbol s}'\,\}$ where, on account of (4.11),

$$[\,\boldsymbol{\partial_{\hat\sigma}},\,\boldsymbol{\partial_{\check\sigma}}\,] = 0, \quad [\,\boldsymbol{\partial_{\hat\sigma}},\,\check{\boldsymbol s}'\,] = 0 \quad [\,\hat{\boldsymbol s}',\,\boldsymbol{\partial_{\check\sigma}}\,] = 0, \quad [\,\hat{\boldsymbol s}',\,\check{\boldsymbol s}'\,] = 0. \tag{4.12}$$

It follows that the coframe $\{\,\boldsymbol\theta',\,\hat{\boldsymbol\sigma}',\,\hat{\boldsymbol\eta}',\,\check{\boldsymbol\sigma}',\,\check{\boldsymbol\eta}'\,\}$ dual to $\{\,\boldsymbol{\partial_\theta},\,\boldsymbol{\partial_{\hat\sigma}},\,\hat{\boldsymbol s}',\,\boldsymbol{\partial_{\check\sigma}},\,\check{\boldsymbol s}'\,\}$ is a 0-adapted coframe for the Darboux pair $\{\hat V, \check V\}$. Equations (4.9) imply (4.6).

On account of (4.12), the structure equations for the forms $\{\,\boldsymbol\theta'\,\}$ are free of all the wedge products $\hat{\boldsymbol\sigma}' \wedge \check{\boldsymbol\sigma}'$, $\hat{\boldsymbol\sigma}' \wedge \check{\boldsymbol\eta}'$, $\hat{\boldsymbol\eta}' \wedge \check{\boldsymbol\sigma}'$, and $\hat{\boldsymbol\eta}' \wedge \check{\boldsymbol\eta}'$. We can express this result by writing

$$d\boldsymbol\theta' \in \Omega^2(\hat V^{(\infty)}) + \Omega^2(\check V^{(\infty)}) \mod \{\,\boldsymbol\theta'\,\}.$$

The coframe $\{\,\boldsymbol\theta', \hat{\boldsymbol\sigma},\,\hat{\boldsymbol\eta},\,\check{\boldsymbol\sigma},\,\check{\boldsymbol\eta}\,\}$ therefore satisfies the criteria of the following theorem.

Theorem 4.3. *Let $\{\,\hat V, \check V\}$ be a Darboux pair on a manifold M. Then about each point of M there exists a 1-adapted coframe $\{\,\boldsymbol\theta,\,\hat{\boldsymbol\sigma},\,\hat{\boldsymbol\eta},\,\check{\boldsymbol\sigma},\,\check{\boldsymbol\eta}\,\}$ with structure equations* (2.13) *and*

$$d\boldsymbol\theta = \boldsymbol A\,\hat{\boldsymbol\pi} \wedge \hat{\boldsymbol\pi} + \boldsymbol B\,\check{\boldsymbol\pi} \wedge \check{\boldsymbol\pi} \qquad \mod \{\,\boldsymbol\theta\,\}, \tag{4.13}$$

where $\hat{\boldsymbol\pi}$ and $\check{\boldsymbol\pi}$ denote the tuples of forms $(\,\hat{\boldsymbol\sigma},\,\hat{\boldsymbol\eta}\,)$ and $(\,\check{\boldsymbol\sigma},\,\check{\boldsymbol\eta}\,)$.

Coframes which satisfy the conditions of Theorem 2.8 are call ***2-adapted coframes***.

4.2 The third adapted coframes for a Darboux pair

Written out in full, the structure equations (4.13) are

$$d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{B}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{C}\,\boldsymbol{\theta} \wedge \boldsymbol{\theta} + \boldsymbol{M}\,\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta} + \boldsymbol{N}\,\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}. \tag{4.14}$$

We obtain the third coframe reduction for a Darboux pair by showing that a change of coframe $\boldsymbol{\theta}' = \boldsymbol{P}\,\boldsymbol{\theta}$ can be made so as to eliminate either all the wedge products $\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}$ <u>or</u> all the wedge products $\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta}$ in (4.14). The construction of this coframe uses another set of iterated Lie brackets.

We start with a 2-adapted coframe $\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}},\}$, where $\hat{\boldsymbol{\pi}} = (\,\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}\,)$ and $\check{\boldsymbol{\pi}} = (\,\check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,)$, and introduce the provisional frame $\{\,\boldsymbol{\theta},\, \hat{\boldsymbol{\iota}}, \check{\boldsymbol{\iota}}\}$, where $\hat{\boldsymbol{\iota}} = d\hat{\boldsymbol{I}}$ and $\check{\boldsymbol{\iota}} = d\check{\boldsymbol{I}}$. This is not a 0-adapted coframe although it is the case that

$$\operatorname{span}\{\,\hat{\boldsymbol{\iota}}\,\} = \operatorname{span}\{\,\hat{\boldsymbol{\pi}}\,\} \quad \text{and} \quad \operatorname{span}\{\,\check{\boldsymbol{\iota}}\,\} = \operatorname{span}\{\,\check{\boldsymbol{\pi}}\,\}. \tag{4.15}$$

From (4.14) and (4.15) we find that the structure equations for this coframe are

$$\begin{gathered} d\hat{\boldsymbol{\iota}} = 0, \quad d\check{\boldsymbol{\iota}} = 0, \qquad \text{and} \\ d\boldsymbol{\theta} = \boldsymbol{A}\,\hat{\boldsymbol{\iota}} \wedge \hat{\boldsymbol{\iota}} + \boldsymbol{B}\,\check{\boldsymbol{\iota}} \wedge \check{\boldsymbol{\iota}} + \boldsymbol{C}\,\boldsymbol{\theta} \wedge \boldsymbol{\theta} + \boldsymbol{M}\,\hat{\boldsymbol{\iota}} \wedge \boldsymbol{\theta} + \boldsymbol{N}\,\check{\boldsymbol{\iota}} \wedge \boldsymbol{\theta}. \end{gathered} \tag{4.16}$$

Denote the dual to this provisional coframe by $\{\,\boldsymbol{\partial}_{\boldsymbol{\theta}}, \hat{\boldsymbol{U}}, \check{\boldsymbol{U}}\}$. Since

$$\operatorname{span}\{\,\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}\,\} = \operatorname{ann}\check{V} = \operatorname{ann}\{\boldsymbol{\theta}, \hat{\boldsymbol{\iota}}, \check{\boldsymbol{\sigma}}\} \subset \operatorname{ann}\{\boldsymbol{\theta}, \hat{\boldsymbol{\iota}}\} = \operatorname{span}\{\,\check{\boldsymbol{U}}\,\},$$

we have that

$$\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}} \subset \operatorname{span}\{\,\check{\boldsymbol{U}}\,\} \quad \text{and} \quad \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}} \subset \operatorname{span}\{\,\hat{\boldsymbol{U}}\,\}. \tag{4.17}$$

From (4.16) it follows that the structure equations for the vectors fields $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{U}}$ are

$$\begin{gathered} [\,\hat{\boldsymbol{U}},\, \hat{\boldsymbol{U}}\,] = -\boldsymbol{A}\boldsymbol{\partial}_{\boldsymbol{\theta}}, \qquad [\,\hat{\boldsymbol{U}},\, \boldsymbol{\partial}_{\boldsymbol{\theta}}\,] = -\boldsymbol{M}\boldsymbol{\partial}_{\boldsymbol{\theta}}, \\ [\,\check{\boldsymbol{U}},\, \check{\boldsymbol{U}}\,] = -\boldsymbol{B}\boldsymbol{\partial}_{\boldsymbol{\theta}}, \qquad [\,\check{\boldsymbol{U}},\, \boldsymbol{\partial}_{\boldsymbol{\theta}}\,] = -\boldsymbol{N}\boldsymbol{\partial}_{\boldsymbol{\theta}}, \\ \text{and} \quad [\,\check{\boldsymbol{U}},\, \hat{\boldsymbol{U}}\,] = 0. \end{gathered} \tag{4.18}$$

As in Section 4.2, define two sequences of vector fields inductively by

$$\hat{U}_{a_1 a_2 \cdots a_\ell} = [\,\hat{U}_{a_1 a_2 \cdots a_{\ell-1}},\, \hat{U}_{a_\ell}\,] \quad \text{and} \quad \check{U}_{\alpha_1 \alpha_2 \cdots \alpha_\ell} = [\,\hat{U}_{\alpha_1 \alpha_2 \cdots \alpha_{\ell-1}}\, \hat{U}_{\alpha_\ell}\,]. \tag{4.19}$$

A simple induction argument, based upon the last of (4.18) and the Jacobi identity, shows that

$$[\,\hat{U}_{a_1 a_2 \cdots a_k}, \check{U}_{\alpha_1 \alpha_2 \cdots \alpha_\ell}\,] = 0. \tag{4.20}$$

for $k, l \geq 1$. On account of (2.7), (2.9) and (4.17) the iterated brackets $\hat{U}_{a_1 a_2 \cdots a_\ell}$ will span all of $\check{H}$ and therefore, by the first two equations in (2.9), we can choose a basis $\boldsymbol{X}$ for $K = \hat{H}^{(\infty)} \cap \check{H}^{(\infty)}$ (see (2.7)) consisting of a linear independent set of the vector fields $\hat{U}_{a_1 a_2 \cdots a_\ell}$. Alternatively, we can choose a basis $\boldsymbol{Y}$ for K consisting of a linear independent set of the vector fields $\check{U}_{a_1 a_2 \cdots a_\ell}$. Due to (4.20) these two bases for K satisfy

$$[\,\boldsymbol{X}, \check{\boldsymbol{U}}\,] = 0, \quad [\,\boldsymbol{X}, \boldsymbol{Y}\,] = 0, \quad [\,\boldsymbol{Y}, \hat{\boldsymbol{U}}\,] = 0. \tag{4.21}$$

Denote the dual of the coframe $\{\boldsymbol{X}, \hat{\boldsymbol{U}}, \check{\boldsymbol{U}}\,\}$ by $\{\,\boldsymbol{\theta_X}, \hat{\boldsymbol{\iota}}\,, \check{\boldsymbol{\iota}}\,\}$ and the dual of the coframe $\{\boldsymbol{Y}, \hat{\boldsymbol{U}}, \check{\boldsymbol{U}}\,\}$ by $\{\,\boldsymbol{\theta_Y}, \hat{\boldsymbol{\iota}}\,, \check{\boldsymbol{\iota}}\,\}$.

Theorem 4.4. *Let $\{\,\hat{V}, \check{V}\}$ be a Darboux pair on a manifold M. Then about each point of M there are two 2-adapted coframes*

$$\{\,\boldsymbol{\theta_X}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\,\} \quad \textit{and} \quad \{\,\boldsymbol{\theta_Y}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}}, \check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}}\,\}$$

with structure equations (2.13),

$$d\boldsymbol{\theta_X} = \boldsymbol{A}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{B}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{C}\,\boldsymbol{\theta_X} \wedge \boldsymbol{\theta_X} + \boldsymbol{M}\,\hat{\boldsymbol{\pi}} \wedge \boldsymbol{\theta_X}, \tag{4.22}$$

and

$$d\boldsymbol{\theta_Y} = \boldsymbol{E}\,\hat{\boldsymbol{\pi}} \wedge \hat{\boldsymbol{\pi}} + \boldsymbol{F}\,\check{\boldsymbol{\pi}} \wedge \check{\boldsymbol{\pi}} + \boldsymbol{K}\,\boldsymbol{\theta_Y} \wedge \boldsymbol{\theta_Y} + \boldsymbol{N}\,\check{\boldsymbol{\pi}} \wedge \boldsymbol{\theta_Y}. \tag{4.23}$$

Moreover, $\operatorname{span}\{\,\boldsymbol{\theta_X}\} = \operatorname{span}\{\,\boldsymbol{\theta_Y}\}$ *and the vector fields $\boldsymbol{X}$, $\boldsymbol{Y}$ belonging to the dual frames $\{\,\boldsymbol{X}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\,\}$ and $\{\,\boldsymbol{Y}, \boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\hat{\boldsymbol{\eta}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\sigma}}}, \boldsymbol{\partial}_{\check{\boldsymbol{\eta}}}\,\}$ satisfy*

$$[\,\boldsymbol{X}, \boldsymbol{Y}] = 0. \tag{4.24}$$

Coframes which satisfy the conditions of Theorem 4.4 are called ***3-adapted coframes***.

Remark 4.5. In the special case where the pair $\{\,\hat{V}, \check{V}\}$ admits an involution (see Remark 2.13) the forms $\boldsymbol{\theta_Y}$ may be defined by $\boldsymbol{\theta_Y} = \Phi^*(\boldsymbol{\theta_X})$. Then the structure equations (4.23) can be immediately inferred from (4.22).

4.3 The fourth adapted coframe and the Vessiot algebra for a Darboux pair.

We now show that any 3-adapted coframe may be adjusted so that the structure functions C^i_{jk} and K^i_{jk} (see (4.22)–(4.24)) are constants and satisfy $C^i_{jk} = -K^i_{jk}$.

Theorem 4.6. *Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on M.*

[i] *Then, about each point of M, there exist local coframes $\{\, \theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha \,\}$ and $\{\, \theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha \,\}$ which are 3-adapted and with structure equations*

$$d\theta^i_X = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2}C^i_{jk}\theta^j_X \wedge \theta^k_X + M^i_{aj}\hat{\pi}^a \wedge \theta^j_X \tag{4.25}$$

and

$$d\theta^i_Y = \frac{1}{2}E^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}F^i_{\alpha\beta}\,\check{\pi}^\alpha \wedge \check{\pi}^\beta - \frac{1}{2}C^i_{jk}\theta^j_Y \wedge \theta^k_Y + N^i_{\alpha j}\check{\pi}^\alpha \wedge \theta^j_Y. \tag{4.26}$$

The structure functions $C^i_{jk} = -C^i_{kj}$ are constant on M and are the structure constants of a real Lie algebra.

[ii] *The isomorphism class of the Lie algebra defined by the structure constants C^i_{jk} is an invariant of the Darboux pair $\{\, \hat{V}, \check{V}\}$.*

Coframes which satisfy the conditions of Theorem 4.6 are called ***4-adapted coframes***. As with the 2 and 3-adapted coframes defined previously, $\hat{\boldsymbol{\pi}} = (\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\eta}})$ and $\check{\boldsymbol{\pi}} = (\check{\boldsymbol{\sigma}}, \check{\boldsymbol{\eta}})$, with structure equations (2.13). In passing from the 3-adapted coframes to the 4-adapted coframes only the forms θ^i_X and θ^i_Y are modified. The (abstract) Lie algebra defined by the structure constants C^i_{jk} is called the ***Vessiot algebra*** for the Darboux pair $\{\, \hat{V}, \check{V}\,\}$ and is denoted by $\mathfrak{vess}(\hat{V}, \check{V})$. The vectors fields X_i in the dual basis $\{\, X_i, \hat{U}_a, \check{U}_\alpha \,\}$ to the 4-adapted coframe $\{\, \theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha \,\}$ define a realization for $\mathfrak{vess}(\hat{V}, \check{V})$, with structure equations $[\, X_j, X_k \,] = -C^i_{jk}X_i$.

Proof of Theorem 4.6 . Theorem 4.4 provides us with two locally defined 3-adapted frames $\{\, X_i, \hat{U}_a, \check{U}_\alpha \,\}$ and $\{\, Y_i, \hat{U}_a, \check{U}_\alpha \,\}$ and dual coframes $\{\, \theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha \,\}$ and $\{\, \theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha \,\}$. The structure equations are (4.22)–(4.24). We begin the proof of part **[i]** by showing that the C^i_{jk} are functions only of the first integrals $\hat{I}^a$ while the K^i_{jk} are functions only of the first integrals $\check{I}^\alpha$.

Since span$\{\, \boldsymbol{\theta_X} \}$ = span$\{\, \boldsymbol{\theta_Y} \}$, there is an invertible matrix Q such that

$$\theta^i_X = Q^i_j\theta^j_Y. \tag{4.27}$$

On account of the structure equations (4.22) and the fact that the vector fields X_i and Y_j commute, the identity

$$d\theta^i_X(X_j, Y_k) = X_j(\theta^i_X(Y_k)) - Y_k(\theta^i_X(X_j)) - \theta^i_X([X_j, Y_k])$$

leads to

$$X_j(Q^i_k) = C^i_{j\ell} Q^\ell_k. \tag{4.28}$$

We shall use this result repeatedly in what follows – it is equivalent to the fact that the vector fields X_i and Y_j commute, a property of the two 3-adapted coframes which is not encoded in the structure equations (4.22) and (4.23).

We next substitute (4.27) into (4.22) and equate the coefficients of $\theta^j_Y \wedge \theta^k_Y$ to deduce that

$$X_\ell(Q^i_k)Q^\ell_j - X_l(Q^i_j)Q^l_k + Q^i_l K^l_{jk} = C^i_{\ell m} Q^\ell_j Q^m_k.$$

By virtue of (4.28), this equation simplifies to

$$Q^i_l K^l_{jk} = -C^i_{lm} Q^l_j Q^m_k. \tag{4.29}$$

Also, by equating to zero the coefficients of $\check{\pi}^\alpha \wedge \theta^i_X \wedge \theta^j_X$ and $\hat{\pi}^a \wedge \theta^i_Y \wedge \theta^j_Y$ in the expansions of the identities $d^2\theta^i_X = 0$ and $d^2\theta^i_Y = 0$, we find that

$$\check{U}_a(C^i_{jk}) = 0 \quad \text{and} \quad \hat{U}_\alpha(K^i_{jk}) = 0.$$

By Lemma 2.8, these equations imply that

$$C^i_{jk} \in \operatorname{Int}(\hat{V}) \quad \text{and} \quad K^i_{jk} \in \operatorname{Int}(\check{V}). \tag{4.30}$$

Our next goal is to show that the coframes θ^i_X and θ^i_Y may be adjusted so that $K^i_{jk} = -C^i_{jk}$ while still preserving (4.22)–(4.24). To this end we introduce local coordinates $(\hat{I}^a, \check{I}^\alpha, z^m)$ on M satisfying (2.10) and (2.11). Then, on account of (4.30), equation (4.29) becomes

$$Q^\ell_k(\hat{I}^a, \check{I}^\alpha, z^m)\, K^k_{ij}(\check{I}^\alpha) = -C^\ell_{hk}(\hat{I}^a)\, Q^h_i(\hat{I}^a, \check{I}^\alpha, z^m)\, Q^k_j(\hat{I}^a, \check{I}^\alpha, z^m). \tag{4.31}$$

Evaluate this equation, first at a fixed point $(\hat{I}^a_0, \check{I}^\alpha_0, z^m_0)$ and then at the point $(\hat{I}^a, \check{I}^\alpha_0, z^m_0)$. With

$$\overset{\circ}{K}{}^k_{ij} = K^k_{ij}(\check{I}^\alpha_0), \quad \overset{\circ}{C}{}^k_{ij} = C^k_{ij}(\hat{I}^a_0), \quad \overset{\circ}{Q}{}^j_i = Q^j_i(\hat{I}^a_0, \check{I}^\alpha_0, z^m_0), \quad \text{and}$$

$$Q^j_i(\hat{I}^a) = Q^j_i(\hat{I}^a, \check{I}^\alpha_0, z^m_0)$$

the results are

$$\overset{\circ}{Q}{}^{\ell}_{k}\overset{\circ}{K}{}^{k}_{ij} = -\overset{\circ}{C}{}^{\ell}_{hk}\,\overset{\circ}{Q}{}^{h}_{i}\overset{\circ}{Q}{}^{k}_{j} \quad \text{and} \quad Q^{\ell}_{k}(\hat{I}^{a})\,\overset{\circ}{K}{}^{k}_{ij} = -C^{\ell}_{hk}(\hat{I}^{a})\,Q^{h}_{i}(\hat{I}^{a})\,Q^{k}_{j}(\hat{I}^{a}). \tag{4.32}$$

It then readily follows that the matrix

$$P^{i}_{j}(\hat{I}^{a}) = Q^{i}_{\ell}(\hat{I}^{a})(\overset{\circ}{Q}{}^{-1})^{\ell}_{j}. \tag{4.33}$$

satisfies

$$P^{\ell}_{k}(\hat{I}^{a})\overset{\circ}{C}{}^{k}_{ij} = C^{\ell}_{hk}(\hat{I}^{a})\,P^{h}_{i}(\hat{I}^{a})\,P^{k}_{j}(\hat{I}^{a}). \tag{4.34}$$

The 1-forms $\overset{\circ}{\theta}{}^{i}_{X}$ defined by

$$\overset{\circ}{\theta}{}^{j}_{X}P^{i}_{j} = \theta^{i}_{X} \tag{4.35}$$

then satisfy structure equations of the required form (4.25), where the structure functions C^{i}_{jk} coincide with the constants $\overset{\circ}{C}{}^{i}_{jk}$. Finally, by evaluating (4.31) at $(\hat{I}^{a}_{0}, \check{I}^{\alpha}, z^{m}_{0})$, it follows that

$$R^{i}_{j}(\check{I}^{\alpha}) = (Q^{-1}(\hat{I}^{a}_{0}, \check{I}^{\alpha}, z^{m}_{0}))^{i}_{j} \tag{4.36}$$

satisfies

$$-R^{\ell}_{k}(\check{I}^{\alpha})\overset{\circ}{C}{}^{k}_{ij} = K^{\ell}_{hk}(\check{I}^{\alpha})\,R^{h}_{i}(\check{I}^{\alpha})\,R^{k}_{j}(\check{I}^{\alpha}) \tag{4.37}$$

and the 1-forms $\overset{\circ}{\theta}{}^{i}_{Y}$ defined by

$$\overset{\circ}{\theta}{}^{j}_{Y}R^{i}_{j} = \theta^{i}_{Y} \tag{4.38}$$

satisfy the required structure equations (4.26), again with $C^{i}_{jk} = \overset{\circ}{C}{}^{i}_{jk}$.

The next step in the proof of **[i]** requires us to check that the coframes we have just constructed are still 3-adapted. We must therefore show that the dual vector fields $\overset{\circ}{X}_{i}$ and $\overset{\circ}{Y}_{j}$ commute and for this it suffices to show, because of our remarks at the beginning of this proof, that

$$\overset{\circ}{X}_{j}(\overset{\circ}{Q}{}^{i}_{k}) = \overset{\circ}{C}{}^{i}_{jl}\overset{\circ}{Q}{}^{l}_{k} \quad \text{where} \quad \overset{\circ}{\theta}{}^{i}_{X} = \overset{\circ}{Q}{}^{i}_{j}\,\overset{\circ}{\theta}{}^{j}_{Y}, \tag{4.39}$$

The proof of this formula follows from three simple observations. We first note that (4.35) and (4.38) imply $\overset{\circ}{Q}{}^{i}_{k} = (P^{-1})^{i}_{\ell}\,Q^{\ell}_{m}\,R^{m}_{k}$. Secondly, because $\{\,\overset{\circ}{X}{}^{i}, \hat{\pi}^{a}, \check{\pi}^{\alpha}\,\}$ is the dual frame to $\{\,\overset{\circ}{\theta}{}^{i}_{X}, \hat{\pi}^{a}, \check{\pi}^{\alpha}\,\}$, we have $\overset{\circ}{X}_{i} = P^{j}_{i}X_{j}$. Finally, because $P^{i}_{j} = P^{i}_{j}(\hat{I}^{a})$ and $R^{i}_{j} = R^{i}_{j}(\check{I}^{\alpha})$, it follows that $\overset{\circ}{X}_{j}(P^{i}_{k}) = \overset{\circ}{X}_{j}(R^{i}_{k}) = 0$. A straightforward calculation based upon these three observations and equations (4.28) and (4.34) leads to (4.39), as required.

Finally, by equating to zero the coefficients of $\theta^i_X \wedge \theta^j_X \wedge \theta^k_X$ in the expansion of the equations $d^2\theta^i_X = 0$ one finds that the C^i_{jk} satisfy the Jacobi identity and are therefore the structure constants of a real r-dimensional Lie algebra.

To prove **[ii]**, let $\{\,\theta'^i_X, \hat{\pi}'^a, \check{\pi}'^\alpha\,\}$ be another coframe adapted to the Darboux pair $\{\,\hat{V}, \check{V}\,\}$ with structure equations

$$d\theta'^i_X = \frac{1}{2}A'^i_{ab}\,\hat{\pi}'^a \wedge \hat{\pi}'^b + \frac{1}{2}B'^i_{\alpha\beta}\,\check{\pi}'^\alpha \wedge \check{\pi}'^\beta + \frac{1}{2}C'^i_{jk}\theta'^j \wedge \theta'^k + M'^i_{aj}\hat{\pi}'^a \wedge \theta'^j. \tag{4.40}$$

From the definition of a 0-adapted coframe (see (2.6b)) we have

$$\theta'^i_X = R^i_j\theta^j + \hat{S}^i_p\hat{\eta}^p + \check{S}^i_q\check{\eta}^q,$$

where the matrix R^i_j is invertible. Substitute this equation into (4.40) and then substitute from (2.13) and (4.25). From the coefficients of $\hat{\boldsymbol{\sigma}} \wedge \boldsymbol{\theta}$ we deduce that $\boldsymbol{\partial}_{\hat{\boldsymbol{\sigma}}}R^i_j = 0$. By Lemma 2.8 this implies that $R^i_j \in \mathrm{Inv}(\hat{V})$ in which case one finds from the coefficients of $\theta^j \wedge \theta^k$ that

$$C'^i_{lm}R^l_jR^m_k = R^i_lC^l_{jk}.$$

Hence the structure constants C^i_{jk} and C'^i_{jk} define the same abstract Lie algebra and the proof of part **[ii]** is complete. ∎

Corollary 4.7. *Let $\{\,\hat{V}, \check{V}\,\}$ and $\{\,\hat{W}, \check{W}\,\}$ be Darboux pairs on manifolds M and N respectively and suppose that $\phi\colon M \to N$ is smooth map satisfying*

$$\phi^*(\hat{W}) \subset \hat{V} \quad \textit{and} \quad \phi^*(\check{W}) \subset \check{V}. \tag{4.41}$$

Then ϕ induces a Lie algebra homomorphism

$$\tilde{\phi}\colon \mathfrak{vess}(\hat{V}, \check{V}) \to \mathfrak{vess}(\hat{W}, \check{W}). \tag{4.42}$$

Proof. The proof of this corollary is similar to that of part **[ii]** of Theorem 4.6. Let $\{\,\theta'^s_X, \hat{\pi}'^c, \check{\pi}'^\gamma\,\}$ be a 4-adapted coframe on N for the Darboux pair $\{\,\hat{W}, \check{W}\,\}$, with structure equations

$$d\theta'^s_X = \frac{1}{2}A'^s_{cd}\,\hat{\pi}'^c \wedge \hat{\pi}'^d + \frac{1}{2}B'^s_{\gamma\delta}\,\check{\pi}'^\gamma \wedge \check{\pi}'^\delta + \frac{1}{2}K^s_{rt}\theta'^r \wedge \theta'^t + M'^s_{at}\hat{\pi}'^a \wedge \theta'^t. \tag{4.43}$$

The constants K^s_{rt} are the structure constants for the Lie algebra $\mathfrak{vess}(\hat{W}, \check{W})$. Denote the dual frame by $\{\,X'_r, U'_c, U'_\gamma\,\}$.

Let $\{\,\theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha\,\}$ be a 4-adapted coframe on M for the Darboux pair$\{\,\hat{V}, \check{V}\,\}$ with the structure equations (4.25). The inclusions (4.41) imply that

$$\phi^*(\hat{W} \cap \check{W}) \subset \hat{V} \cap \check{V}, \quad \phi^*(\hat{W}^{(\infty)}) \subset \hat{V}^{(\infty)} \quad \text{and} \quad \phi^*(\check{W}^{(\infty)}) \subset \check{V}^{(\infty)}$$

and therefore there are functions R_i^s, $\hat{S}_p^s$, $\check{S}_q^s$ $\hat{T}_a^c$ and $\check{T}_\alpha^\gamma$ on M such that

$$\phi^*(\theta_X'^s) = R_i^s\theta_X^i + \hat{S}_p^s\hat{\eta}^p + \check{S}_q^s\check{\eta}^q, \quad \phi^*(\hat{\pi}'^c) = \hat{T}_a^c\hat{\pi}^a \quad \text{and} \quad \phi^*(\check{\pi}'^\gamma) = \check{T}_\alpha^\gamma\check{\pi}^\alpha.$$

We pullback (4.43) using these equations and substitute from (4.25). The same arguments as in the proof of **[ii]** of Theorem 4.6 now show that $R_i^s \in \mathrm{Inv}(\hat{V})$ and $R_i^r C_{jk}^i = K_{st}^r R_j^s R_k^t$. This proves that the Jacobian mapping $\phi_*(X_i) = R_i^t X_t'$ induces a Lie algebra homomorphism of Vessiot algebras. ∎

Remark 4.8. Let $\{\, \theta_X^i,\, \hat{\pi}^a,\, \check{\pi}^\alpha \,\}$ and $\{\, \theta_Y^i,\, \hat{\pi}^a,\, \check{\pi}^\alpha \,\}$ be two coframes which are 4-adapted to the Darboux pair $\{\, \hat{V}, \check{V} \,\}$ and which satisfy the structure equations (4.25) and (4.26). Then it is not difficult to check that the commutativity of the dual vector fields X_i and Y_j is equivalent to the supposition that the change of frame (4.27) defines an automorphism of the Vessiot algebra, that is,

$$Q_\ell^i C_{jk}^\ell = C_{lm}^i Q_j^\ell Q_k^m. \tag{4.44}$$

In the next section we shall need all the derivatives of the matrix Q_j^i. The $\hat{\pi}^a$ and $\check{\pi}^\alpha$ components of dQ_j^i are easily determined by substituting (4.27) into (4.25) and comparing the result with (4.26). If we then take into account (4.28) and (4.44) we find that

$$dQ_j^i = Q_j^\ell M_\ell^i - Q_\ell^i N_j^\ell + C_{j\ell}^i Q_k^\ell \theta_X^j, \tag{4.45}$$

where $M_j^i = M_{aj}^i\hat{\pi}^a$ and $N_j^i = N_{\alpha j}^i\check{\pi}^\alpha$. We note, also for future use, that

$$A_{ab}^i = Q_j^i E_{ab}^j \quad \text{and} \quad B_{\alpha\beta}^i = Q_j^i F_{\alpha\beta}^j. \tag{4.46}$$

where E_{ab}^j and $F_{\alpha\beta}^j$ are defined by (2.13).

4.4 The fifth adapted coframe for a Darboux pair and the proof of Theorem 4.1

Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on M and let $\{\theta^i_X, \hat{\pi}^a, \check{\pi}^\alpha\}$ and $\{\theta^i_Y, \hat{\pi}^a, \check{\pi}^\alpha\}$ be local coframes on M which are 4-adapted to the Darboux pair $\{\hat{V}, \check{V}\}$ and which therefore satisfy the structure equations (4.25) and (4.26). In this section we shall prove that it is possible to define forms

$$\hat{\theta}^i = \hat{R}^i_j \theta^j_X + \phi^i_a\, \hat{\pi}^a \quad \text{and} \quad \check{\theta}^i = \check{R}^i_j \theta^j_Y + \psi^i_a\, \check{\pi}^a, \tag{4.47}$$

$$\text{where} \quad \hat{R}^i_j,\ \phi^i_a \in \mathrm{Int}(\hat{V}) \quad \text{and} \quad \check{R}^i_j,\ \psi^i_\alpha \in \mathrm{Int}(\check{V}),$$

which satisfy structure equations

$$d\hat{\theta}^i = \frac{1}{2} G^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k \tag{4.48}$$

and

$$d\check{\theta}^i = \frac{1}{2} H^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b - \frac{1}{2} C^i_{jk} \check{\theta}^j \wedge \check{\theta}^k. \tag{4.49}$$

These are the structure equations for the Darboux pair $\{\hat{V}, \check{V}\}$ announced in Theorem 4.1.

We shall focus on (4.48) and simply note that the proof of (4.49) is similar. Our starting point are the equations (4.25) and (4.26), in particular,

$$d\theta^i_X = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \theta^j_X \wedge \theta^k_X + M^i_{aj} \hat{\pi}^a \wedge \theta^j_X. \tag{4.50}$$

In what follows it will be useful to introduce the 1-forms and 2-forms

$$M^\ell_j = M^\ell_{aj} \hat{\pi}^a,\ N^\ell_j = N^\ell_{\alpha j} \check{\pi}^\alpha,\ \ A^\ell = A^\ell_{ab} \hat{\pi}^a \wedge \hat{\pi}^b\ F^\ell = F^\ell_{\alpha\beta} \check{\pi}^\alpha \wedge \check{\pi}^\beta. \tag{4.51}$$

By setting to zero the coefficients of $\hat{\pi}^a \wedge \check{\pi}^\alpha \wedge \theta^i_X$ and $\check{\pi}^\alpha \wedge \hat{\pi}^a \wedge \hat{\pi}^b$ in the equations $d^2\theta^i_X = 0$, we find that $\check{U}_\alpha(M^i_{aj}) = 0$ and $\check{U}_\alpha(A^i_{ab}) = 0$. These equations imply that $M^i_{aj} \in \mathrm{Int}(\hat{V})$ and $A^i_{ab} \in \mathrm{Int}(\hat{V})$ and therefore (see Lemma 2.8)

$$dM^i_{aj} = \hat{U}_b(M^i_{aj})\, \hat{\pi}^b \quad \text{and} \quad dA^i_{ab} = \hat{U}_c(A^i_{ab})\, \hat{\pi}^c. \tag{4.52}$$

Bearing these two results in mind, we then respectively deduce from the coefficients of $\hat{\pi}^a \wedge \theta^j_X \wedge \theta^k_X$, $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \theta^j_X$ and $\hat{\pi}^c \wedge \hat{\pi}^a \wedge \hat{\pi}^b$ in $d^2\theta^i_X = 0$ that

$$M^\ell_{aj} C^i_{\ell k} + M^\ell_{ak} C^i_{j\ell} - M^i_{a\ell} C^\ell_{jk} = 0, \tag{4.53}$$

$$dM^i_j - M^i_\ell \wedge M^\ell_j + C^i_{\ell j} A^\ell = 0, \quad \text{and} \tag{4.54}$$

$$dA^i - A^\ell M^i_\ell = 0. \tag{4.55}$$

The proof of Theorem 4.1 hinges upon a detailed analysis of equations (4.53)–(4.55). We deal first with (4.53) since this is a purely algebraic constraint. It states that for each fixed value of a, the linear transformation $M_a\colon \mathfrak{vess} \to \mathfrak{vess}$ defined by

$$M_a(X_j) = M^i_{aj} X_i$$

is a derivation or infinitesimal automorphism of the Vessiot Lie algebra $\mathfrak{vess} = \mathfrak{vess}(\hat{V}, \check{V})$. Consequently, to analyze this equation we shall invoke some basic structure theory for Lie algebras. Specifically, we shall consider separately the cases where $\mathfrak{vess}$ is semi-simple, where $\mathfrak{vess}$ is abelian, and where $\mathfrak{vess}$ is solvable. Then we shall make use of the Levi decomposition of $\mathfrak{vess}$ to solve the general case.

Case I. We first consider the case where the Lie algebra $\mathfrak{vess}$ is semi-simple. Here the proof of (4.48) is relatively straightforward and is based upon the fact that every derivation of a semi-simple Lie algebra is an inner derivation (see, for example, Varadarajan[25], page 215). In fact, because the proof of this result is constructive, we can deduce from (4.53) that there are uniquely defined smooth functions $\phi^\ell_a \in \mathrm{Int}(\hat{V})$ such that

$$M^i_{aj} = \phi^\ell_a C^i_{\ell j}. \tag{4.56}$$

The forms $\hat{\theta}^i$ defined by

$$\hat{\theta}^i = \theta^i_X + \phi^i_a \hat{\pi}^a \tag{4.57}$$

then satisfy structure equations of the form

$$d\hat{\theta}^i = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k. \tag{4.58}$$

For these structure equations, equations (4.54) reduce to $C^i_{\ell j} A^\ell_{ab} = 0$. Since we are assuming that $\mathfrak{vess}$ is semi-simple, the center of $\mathfrak{vess}$ is trivial and therefore this equation implies that $A^\ell_{ab} = 0$. The structure equations (4.58) then reduce to the form (4.50), as desired.

Case II. Now we consider the other extreme case, namely, the case where $\mathfrak{vess}$ is abelian. Strictly speaking, we need not treat this as a separate case but the analysis here will simplify our subsequent discussion of the case where $\mathfrak{vess}$ is solvable. When $\mathfrak{vess}$ is abelian the structure equations (4.50) are

$$d\theta^i_X = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + M^i_{aj} \hat{\pi}^a \wedge \theta^j_X, \tag{4.59}$$

equation (4.53) is vacuous and, by virtue of (4.52), (4.54) simplifies to

$$d M^i_j - M^i_\ell \wedge M^\ell_j = 0, \quad \text{where} \quad M^i_j = M^i_{aj} \hat{\pi}^a. \tag{4.60}$$

By the Frobenius theorem we conclude that there are locally defined functions $R^i_j \in \mathrm{Int}(\hat{V})$ such that

$$d(R^i_j) + R^i_\ell M^\ell_j = 0 \quad \text{and} \quad \det(R^i_j) \neq 0. \tag{4.61}$$

(For this precise application of the Frobenius theorem see, for example, Flanders [11], page 102.) We remark that this is the first (and only) step in all our coframe adaptations that require the solution to systems of linear ordinary differential equations

The forms $\hat{\theta}^i_0 = R^i_j \theta^j_X$ are then easily seen to satisfy

$$d\hat{\theta}^i_0 = \frac{1}{2} \overset{0}{A}{}^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} \overset{0}{B}{}^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta. \tag{4.62}$$

Since the structure functions $\overset{0}{A}{}^i_{ab} = R^i_j\, A^j_{ab} \in \mathrm{Int}(\hat{V})$ it follows, either directly from (4.62) or from (4.55) (with $M^i_{a\ell} = 0$) that the 2-forms

$$\chi^i = \frac{1}{2} \overset{0}{A}{}^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b \tag{4.63}$$

are all d-closed. If we pick 1-forms $\phi^i = \phi^i_a\, \hat{\pi}^a$, with $\phi^i_a \in \mathrm{Int}(\hat{V})$, such that $d\phi^i = -\chi^i$, then the forms

$$\hat{\theta}^i = \hat{\theta}^i_0 + \phi^i_a\, \hat{\pi}^a = R^i_j \theta^j_X + \phi^i_a\, \hat{\pi}^a \tag{4.64}$$

will satisfy the required structure equations (4.48), with $C^i_{jk} = 0$. By applying the usual homotopy formula for the de Rham complex, we see that this last coframe adaptation can be implemented by quadratures.

Case III. Now we suppose that $\mathfrak{vess}$ is solvable. Recall that a Lie algebra $\mathfrak{g}$ is said to be p-step solvable if the derived algebras $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$ satisfy

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \cdots \supset \mathfrak{g}^{(p-1)} \supset \mathfrak{g}^{(p)} = \{0\}.$$

The annihilating subspaces $\Lambda^{(i)} = \mathrm{ann}(\mathfrak{g}^{(i)})$ therefore satisfy

$$\{0\} = \Lambda^{(0)} \subset \Lambda^{(1)} \subset \Lambda^{(2)} \cdots \subset \Lambda^{(p-1)} \subset \Lambda^{(p)} = \mathfrak{g}^* \tag{4.65}$$

and, because the $\mathfrak{g}^{(i)}$ are all ideals,

$$d\Lambda^{(i)} \subset \Lambda^{(i-1)} \otimes \Lambda^{(i)}. \tag{4.66}$$

Let $s = \dim(\mathfrak{g})$ and $s_i = \dim \Lambda^{(i)}$. If $M\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation, then a simple induction shows that $M\colon \mathfrak{g}^{(i)} \to \mathfrak{g}^{(i)}$ and therefore

$$M^*\colon \Lambda^{(i)} \to \Lambda^{(i)}. \tag{4.67}$$

Thus the matrix representing M^*, with respect to any basis for $\mathfrak{g}^*$ adapted to the flag (4.65), is block triangular.

We apply these observations to the forms $\{\, \theta^i_X \,\}$ and the structure equations (4.50). The forms $\{\, \theta^i_X \,\}$, restricted to the vectors X_i, define a basis for $\mathfrak{vess}^*$ and consequently, by a *constant* coefficient change of basis, we can suppose that the basis $\{\, \theta^i_X \,\}$ is adapted to the derived flag (4.65). Since the notation for this adaptation becomes rather cumbersome in the general case of a p-step Lie algebra, we shall consider just the cases where $\mathfrak{vess}$ is a 2-step or a 3-step solvable Lie algebra. The construction of the coframe with structure equations (4.48) in these two special cases will make the nature of the general construction apparent.

In the case where $\mathfrak{vess}$ is a 2-step solvable Lie algebra, we begin with a 4-adapted coframe

$$\{\, \theta^1_1, \dots, \theta^{s_1}_1, \theta^{s_1+1}_2, \dots, \theta^{s_2}_2, \hat{\pi}^a, \check{\pi}^\alpha \,\},$$

where

$$\operatorname{span}\{\, \theta^1_X, \dots, \theta^s_X \,\} = \operatorname{span}\{\, \theta^1_1, \dots, \theta^{s_1}_1, \theta^{s_1+1}_2, \dots, \theta^{s_2}_2 \,\} \quad \text{over } \mathbf{R},$$

and where $\{\, \theta^1_1, \dots, \theta^{s_1}_1\}$ is a basis for $\Lambda^{(1)}(\mathfrak{vess})$. In this basis the structure equations (4.50) become

$$d\theta^r_1 = \frac{1}{2} A^r_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^r_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + M^r_{as}\, \hat{\pi}^a \wedge \theta^s_1, \quad \text{and} \tag{4.68a}$$

$$d\theta^i_2 = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{rs}\, \theta^r_1 \wedge \theta^s_1 + C^i_{rj}\, \theta^r_1 \wedge \theta^j_2 \tag{4.68b}$$
$$+ M^i_{ar}\, \hat{\pi}^a \wedge \theta^r_1 + M^i_{aj}\, \hat{\pi}^a \wedge \theta^j_2.$$

In these equations the indices r, s range from 1 to s_1 and the indices i, j range from $s_1 + 1$ to s_2. On account of (4.66), there are no quadratic θ terms in (4.68a) because $\theta^r_1 \in \Lambda^{(1)}(\mathfrak{vess})$ and there are no $\theta^i_2 \wedge \theta^j_2$ terms in (4.68b) because $\theta^2_1 \in \Lambda^{(2)}(\mathfrak{vess})$. There are no $\hat{\pi}^a \wedge \theta^i_2$ terms in (4.68a) by virtue of (4.67).

Since the structure equations (4.68a) are identical in form to the structure equations (4.59) for the abelian case, we can invoke the arguments there to define new forms

$$\hat{\theta}^r_1 = R^r_s \theta^s_1 + \phi^r_a \hat{\pi}^a \tag{4.69}$$

so that the structure equations (4.68) reduce to

$$d\hat{\theta}^r_1 = \frac{1}{2}\overset{0}{B}{}^r_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta, \quad \text{and} \tag{4.70a}$$

$$\begin{aligned} d\theta^i_2 = {} & \frac{1}{2}\overset{0}{A}{}^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}\overset{0}{B}{}^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2}\overset{0}{C}{}^i_{rs}\,\hat{\theta}^r_1 \wedge \hat{\theta}^s_1 + \overset{0}{C}{}^i_{rj}\,\hat{\theta}^r_1 \wedge \theta^j_2 \\ & + \overset{0}{M}{}^i_{ar}\,\hat{\pi}^a \wedge \hat{\theta}^r_1 + \overset{0}{M}{}^i_{aj}\,\hat{\pi}^a \wedge \theta^j_2. \end{aligned} \tag{4.70b}$$

It is important to track the coordinate dependencies of the coefficients in these structure equations as we perform these frame changes. Since the coefficients A^i_{ab}, M^i_{ar} and M^i_{aj} in (4.68b) and the coefficients R^r_s and ϕ^r_a in (4.69) are all $\hat{V}$ first integrals, it is easily checked that the same is true of the coefficients $\overset{0}{A}{}^i_{ab}$, $\overset{0}{C}{}^i_{rs}$, $\overset{0}{C}{}^i_{rj}$, $\overset{0}{M}{}^i_{ar}$ and $\overset{0}{M}{}^i_{aj}$ in (4.70b).

The coefficients of $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \theta^i_2$ and $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \hat{\pi}^c$ in $d^2\theta^i_2 = 0$ give the same equations as we had in the abelian case and consequently we can define

$$\hat{\theta}^i_2 = R^i_j\theta^j_2 + \phi^i_a\hat{\pi}^a \tag{4.71}$$

so as to eliminate the $\hat{\pi}^a \wedge \hat{\pi}^b$ and $\hat{\pi}^a \wedge \theta^j_2$ terms in (4.70b). The structure equations are now

$$d\hat{\theta}^r_1 = \frac{1}{2}\overset{0}{B}{}^r_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta, \quad \text{and} \tag{4.72a}$$

$$d\hat{\theta}^i_2 = \frac{1}{2}\overset{1}{B}{}^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2}\overset{1}{C}{}^i_{rs}\,\hat{\theta}^r_1 \wedge \hat{\theta}^s_1 + \overset{1}{C}{}^i_{rj}\,\hat{\theta}^r_1 \wedge \hat{\theta}^j_2 + \overset{1}{M}{}^i_{ar}\,\hat{\pi}^a \wedge \hat{\theta}^r_1. \tag{4.72b}$$

Finally, from the coefficients of $\hat{\pi}^a \wedge \hat{\pi}^b \wedge \hat{\theta}^r_1$ in $d^2\hat{\theta}^i_2 = 0$ we find that

$$d\overset{1}{M}{}^i_b = 0, \quad \text{where} \quad \overset{1}{M}{}^i_b = \overset{1}{M}{}^i_{br}\,\hat{\pi}^r, \tag{4.73}$$

and therefore, again by the de Rham homotopy formula, there are functions $R^i_r \in \mathrm{Int}(\hat{V})$ such that $M^i_{ar} = \hat{U}_a(R^i_r)$. The change of frame

$$\hat{\theta}^i_2 = \hat{\hat{\theta}}{}^i_2 + R^i_r\hat{\theta}^r_1 \tag{4.74}$$

then leads to the desired structure equations

$$d\hat{\theta}^r_1 = \frac{1}{2}\overset{0}{B}{}^r_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta, \quad \text{and} \tag{4.75a}$$

$$d\hat{\hat{\theta}}{}^i_2 = \frac{1}{2}\overset{2}{B}{}^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2}\overset{2}{C}{}^i_{rs}\hat{\theta}^r_1 \wedge \hat{\theta}^s_1 + \overset{2}{C}{}^i_{rj}\hat{\theta}^r_1 \wedge \hat{\hat{\theta}}{}^j_2. \tag{4.75b}$$

At this point it is a simple matter to check that the equations $d^2\hat{\hat{\theta}}{}^i_2 = 0$ force the coefficients $\overset{2}{C}{}^i_{rs}$ and $\overset{2}{C}{}^i_{rj}$ (which belong to $\mathrm{Int}(\hat{V})$) to be constant. Equations (4.75) establish (4.48) for the case of a 2-step solvable Vessiot algebra.

When $\mathfrak{vess}$ is a 3-step solvable Lie algebra we assume that the 4-adapted coframe $\{\, \theta_1^r,\, \theta_2^u,\, \theta_3^i,\, \hat{\pi}^a,\, \check{\pi}^\alpha \,\}$ is adapted to the flag (4.65) in the sense that

$$\Lambda^{(1)} = \operatorname{span}\{\, \theta_1^r \,\}, \quad \Lambda^{(2)} = \operatorname{span}\{\, \theta_1^r,\, \theta_2^u \,\} \quad \text{and} \quad \Lambda^{(3)} = \operatorname{span}\{\, \theta_1^r,\, \theta_2^u,\, \theta_3^i \,\}.$$

The structure equations (4.76b) are then of the form

$$d\theta_1^r = \frac{1}{2} A^r_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^r_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + M^r_{as}\, \hat{\pi}^a \wedge \theta_1^s, \tag{4.76a}$$

$$\begin{aligned} d\theta_2^u = {} & \frac{1}{2} A^u_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^u_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^u_{rs}\, \theta_1^r \wedge \theta_1^s + C^u_{rv}\, \theta_1^r \wedge \theta_2^v \\ & + M^u_{ar}\, \hat{\pi}^a \wedge \theta_1^r + M^u_{av}\, \hat{\pi}^a \wedge \theta_2^v, \quad \text{and} \end{aligned} \tag{4.76b}$$

$$\begin{aligned} d\theta_3^i = {} & \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta \\ & + \frac{1}{2} C^i_{rs}\, \theta_1^r \wedge \theta_1^s + C^i_{ru}\, \theta_1^r \wedge \theta_2^u + C^i_{rj}\, \theta_1^r \wedge \theta_3^j + \frac{1}{2} C^i_{uv}\, \theta_2^u \wedge \theta_2^v + C^i_{uj}\, \theta_2^u \wedge \theta_3^j \\ & + M^i_{ar}\, \hat{\pi}^a \wedge \theta_1^r + M^i_{au}\, \hat{\pi}^a \wedge \theta_2^u + M^i_{aj}\, \hat{\pi}^a \wedge \theta_3^j. \end{aligned} \tag{4.76c}$$

In these equations r, s range from 1 to s_1, u, v from $s_1 + 1$ to s_2, and i, j from $s_2 + 1$ to s_3. Note that the form of the structure equations (4.76a)–(4.76c) is preserved by changes of coframe of the type

$$\begin{gathered} \boldsymbol{\theta}_1 \to \boldsymbol{R}_{11}\boldsymbol{\theta}_1 + \boldsymbol{\phi}_1 \hat{\boldsymbol{\pi}}, \quad \boldsymbol{\theta}_2 \to \boldsymbol{R}_{12}\boldsymbol{\theta}_1 + \boldsymbol{R}_{22}\boldsymbol{\theta}_2 + \boldsymbol{\phi}_2 \hat{\boldsymbol{\pi}}, \quad \text{and} \\ \boldsymbol{\theta}_3 \to \boldsymbol{R}_{13}\boldsymbol{\theta}_1 + \boldsymbol{R}_{23}\boldsymbol{\theta}_2 + \boldsymbol{R}_{33}\boldsymbol{\theta}_3 + \boldsymbol{\phi}_3 \hat{\boldsymbol{\pi}}, \end{gathered}$$

where the coefficients $\boldsymbol{R}_{ij} \in \operatorname{Int}(\hat{V})$. Exactly as in the previous case of a 2-step solvable algebra, we use such a change of coframe for $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ to reduce the structure equations (4.76) to

$$d\hat{\theta}_1^r = \frac{1}{2} B^r_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta, \tag{4.77a}$$

$$d\hat{\theta}_2^u = \frac{1}{2} B^u_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^u_{rs}\, \hat{\theta}_1^r \wedge \hat{\theta}_1^s + C^u_{rv}\, \hat{\theta}_1^r \wedge \hat{\theta}_2^v, \quad \text{and} \tag{4.77b}$$

$$\begin{aligned} d\theta_3^i = {} & \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta \\ & + \frac{1}{2} C^i_{rs}\, \hat{\theta}_1^r \wedge \hat{\theta}_1^s + C^i_{ru}\, \hat{\theta}_1^r \wedge \hat{\theta}_2^u + C^i_{rj}\, \hat{\theta}_1^r \wedge \theta_3^j + \frac{1}{2} C^i_{uv}\, \hat{\theta}_2^u \wedge \hat{\theta}_2^v + C^i_{uj}\, \hat{\theta}_2^u \wedge \theta_3^j \\ & + M^i_{ar}\, \hat{\pi}^a \wedge \hat{\theta}_1^r + M^i_{au}\, \hat{\pi}^a \wedge \hat{\theta}_2^u + M^i_{aj}\, \hat{\pi}^a \wedge \theta_3^j. \end{aligned} \tag{4.77c}$$

Again, as in Case **II**, a change of frame $\hat{\boldsymbol{\theta}}_3 = \boldsymbol{R}_{33}\boldsymbol{\theta}_3 + \boldsymbol{\phi}_3 \hat{\boldsymbol{\pi}}$ leads to the simplification of (4.77c) to

$$d\hat{\theta}_3^i = \frac{1}{2}B^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + M^i_{ar}\,\hat{\pi}^a \wedge \hat{\theta}_1^r + M^i_{au}\,\hat{\pi}^a \wedge \hat{\theta}_2^u \\ + \frac{1}{2}C^i_{rs}\,\hat{\theta}_1^r \wedge \hat{\theta}_1^s + C^i_{ru}\,\hat{\theta}_1^r \wedge \hat{\theta}_2^u + C^i_{rj}\,\hat{\theta}_1^r \wedge \hat{\theta}_3^j + \frac{1}{2}C^i_{uv}\,\hat{\theta}_2^u \wedge \hat{\theta}_2^v + C^i_{uj}\,\hat{\theta}_2^u \wedge \hat{\theta}_3^j. \tag{4.78}$$

Finally, just as in the reduction from (4.72) to (4.75), we use a change of frame $\hat{\boldsymbol{\theta}}_3 \to \hat{\boldsymbol{\theta}}_3 + \boldsymbol{R}_{13}\hat{\boldsymbol{\theta}}_1 + \boldsymbol{R}_{23}\hat{\boldsymbol{\theta}}_2$ to transform (4.78) to

$$d\hat{\theta}_3^i = \frac{1}{2}B^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta \\ + \frac{1}{2}C^i_{rs}\,\hat{\theta}_1^r \wedge \hat{\theta}_1^s + C^i_{ru}\,\hat{\theta}_1^r \wedge \hat{\theta}_2^u + C^i_{rj}\,\hat{\theta}_1^r \wedge \hat{\theta}_3^j + \frac{1}{2}C^i_{uv}\,\hat{\theta}_2^u \wedge \hat{\theta}_2^v + C^i_{uj}\,\hat{\theta}_2^u \wedge \hat{\theta}_3^j. \tag{4.79}$$

Equations (4.77a), (4.77b) and (4.79) give the desired result. We note once more that the structure functions C^*_{**} in (4.77c) and(4.78) are not necessarily constant but they are constant in (4.79).

The reduction of the structure equations for a general p-step solvable algebra follows this construction and can be formally defined by induction on p.

Case IV: Finally, we consider the case where $\mathfrak{vess}$ is a generic Lie algebra. We use the fact that every real Lie algebra $\mathfrak{g}$ admits a Levi decomposition into a semi-direct sum $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and $\mathfrak{s}$ is a semi-simple subalgebra of $\mathfrak{g}$. The radical $\mathfrak{r}$ is the unique maximal solvable ideal in $\mathfrak{g}$ – the semi-simple component $\mathfrak{s}$ in the Levi decomposition is not unique. The dual space $\mathfrak{g}^*$ then decomposes according to

$$\mathfrak{g}^* = \mathrm{ann}(\mathfrak{r}) \oplus \mathrm{ann}(\mathfrak{s}). \tag{4.80}$$

The fact that $\mathfrak{r}$ is an ideal implies that

$$d\,\mathrm{ann}(\mathfrak{r}) \subset \Lambda^2(\mathrm{ann}(\mathfrak{r})) \quad \text{and} \quad d\,\mathrm{ann}(\mathfrak{s}) \subset \mathfrak{g}^* \otimes \mathrm{ann}(\mathfrak{s}). \tag{4.81}$$

If $M\colon \mathfrak{g} \to \mathfrak{g}$ is a derivation, then M preserves $\mathfrak{r}$ and hence $M^*\colon \mathrm{ann}(\mathfrak{r}) \to \mathrm{ann}(\mathfrak{r})$.

Now choose, by a constant coefficient change of basis, 1-forms $\{\,\theta_1^r, \theta_2^i\,\}$ adapted to the decomposition (4.80) with $\mathrm{ann}(\mathfrak{r}) = \mathrm{span}\{\,\theta_1^r\,\}$ and $\mathrm{ann}(\mathfrak{s}) = \mathrm{span}\{\,\theta_2^i\,\}$. In view of (4.81), the structure equations (4.50) become

$$d\theta_1^r = \frac{1}{2}A^r_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^r_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2}C^r_{st}\,\theta_1^s \wedge \theta_1^t + M^r_{as}\,\hat{\pi}^a \wedge \theta_1^s, \tag{4.82a}$$

$$d\theta_2^i = \frac{1}{2}A^i_{ab}\,\hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2}B^i_{\alpha\beta}\,\breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2}C^i_{jk}\,\theta_2^j \wedge \theta_2^k + C^i_{rj}\,\theta_1^r \wedge \theta_2^j \\ + M^i_{ar}\,\hat{\pi}^a \wedge \theta_1^r + M^i_{aj}\,\hat{\pi}^a \wedge \theta_2^j. \tag{4.82b}$$

In these equations the indices r, s, t range from 1 to $s_0 = \dim(\mathrm{ann}(\mathfrak{r}))$ and i, j, k range from s_0+1 to $s = \dim(\mathfrak{g})$.

Since the structure constants C^r_{st} are those for the semi-simple Lie algebra $\mathfrak{s}$, we can return to the analysis presented in Case **I** to prove that there is a change of coframe $\hat{\theta}^r_1 = \theta^r_1 + \phi^r_a \hat{\pi}^a$ (see (4.57)) which reduces the structure equations (4.82) to

$$d\hat{\theta}^r_1 = \frac{1}{2} B^r_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^r_{st}\, \hat{\theta}^s_1 \wedge \hat{\theta}^t_1, \quad \text{and} \tag{4.83a}$$

$$\begin{aligned} d\theta^i_2 = \frac{1}{2} A^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b + \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk}\, \theta^j_2 \wedge \theta^k_2 + C^i_{rj}\, \hat{\theta}^r_1 \wedge \theta^j_2 \\ + M^i_{ar}\, \hat{\pi}^a \wedge \hat{\theta}^r_1 + M^i_{aj}\, \hat{\pi}^a \wedge \theta^j_2. \end{aligned} \tag{4.83b}$$

The structure functions C^r_{st}, C^i_{jk} and C^i_{rj} in (4.83) are identical to the corresponding structure constants in (4.82). One now checks that the $\hat{\theta}^r_1$ terms in (4.83b) do not effect the arguments made in Cases **II** and **III**. Thus, by a change of frame $\hat{\boldsymbol{\theta}}_2 = \boldsymbol{R}\boldsymbol{\theta}_2 + \boldsymbol{\phi}\hat{\boldsymbol{\pi}}$, one can reduce (4.83) to

$$d\hat{\theta}^r_1 = \frac{1}{2} B^r_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^r_{st}\, \hat{\theta}^s_1 \wedge \hat{\theta}^t_1, \quad \text{and} \tag{4.84a}$$

$$d\hat{\theta}^i_2 = \frac{1}{2} B^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk}\, \hat{\theta}^j_2 \wedge \hat{\theta}^k_2 + C^i_{rj}\, \hat{\theta}^r_1 \wedge \theta^j_2 + M^i_{ar}\, \hat{\pi}^a \wedge \hat{\theta}^r_1. \tag{4.84b}$$

Finally, by a now familiar computation, one sees that a change of frame $\hat{\boldsymbol{\theta}}_2 \to \hat{\boldsymbol{\theta}}_2 + \boldsymbol{R}\hat{\boldsymbol{\theta}}_1$ allows one to eliminate the $\hat{\pi}^a \wedge \hat{\theta}^r_1$ terms in (4.84b).

This completes our derivation of the structure equations (4.48) and the proof of Theorem 4.1 is, at last, finished.

Remark 4.9. We collect together a few additional formulas which will be needed in the next section for the proof of Lemma 5.8 and the construction of the superposition formula. If the forms $\hat{\theta}^i$ and $\check{\theta}^i$ in (4.47) satisfy (4.48) and (4.49), then it easy to check that coefficients $\hat{R}^i_j$, $\check{R}^i_j$, $\phi^i = \phi^i_a \hat{\pi}^a$ and $\psi^i = \psi^i_\alpha \check{\pi}^\alpha$ satisfy (see (4.51))

$$d\hat{R}^i_j = -C^i_{\ell k}\, \hat{R}^\ell_j\, \phi^k - \hat{R}^i_\ell\, M^\ell_j, \quad d\check{R}^i_j = C^i_{\ell k}\, \check{R}^\ell_j \psi^k - \check{R}^i_\ell N^\ell_j, \tag{4.85a}$$

$$\hat{R}^i_\ell B^\ell_{\alpha\beta} = G^i_{\alpha\beta}, \quad \check{R}^i_\ell\, E^\ell_{ab} = H^i_{ab}, \tag{4.85b}$$

$$d\phi^i = \frac{1}{2} C^i_{jk} \phi^j \wedge \phi^k - \frac{1}{2} \hat{R}^i_\ell\, A^\ell, \quad d\psi^i = -\frac{1}{2} C^i_{jk} \psi^j \wedge \psi^k - \frac{1}{2} \check{R}^i_\ell\, F^\ell. \tag{4.85c}$$

The computations leading to (4.85) also show that the matrices $\hat{R}^i_j$ and $\check{R}^i_j$ define automorphisms of the Vessiot algebra, that is

$$\hat{R}^i_\ell\, C^\ell_{jk} = C^i_{\ell m}\, \hat{R}^\ell_j\, \hat{R}^m_k, \quad \text{and} \quad \check{R}^i_\ell\, C^\ell_{jk} = C^i_{\ell m}\, \check{R}^\ell_j\, \check{R}^m_k. \tag{4.86}$$

Finally, a series of straightforward computations, based on (4.44), (4.45), (4.85a) and (4.86), shows that the matrix

$$\lambda = \hat{R} Q \check{R}^{-1} \quad \text{satisfies} \tag{4.87}$$

$$\lambda^i_\ell C^\ell_{jk} = C^i_{\ell m} \lambda^\ell_j \lambda^m_k \quad \text{and} \quad d\,\lambda^i_j = C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell + \lambda^i_h\, C^h_{mj}\, \psi^m. \tag{4.88}$$

∎

We conclude this section by determining the residual freedom in the determination of the 1-forms $\hat{\theta}^i$. Specifically, we compute the group of coframe transformations which fix the forms $\hat{\pi}^a$ and $\check{\pi}^\alpha$ and which transform the $\hat{\theta}^i$ by

$$\tilde{\theta}^i = \Lambda^i_j \hat{\theta}^i + \sigma^i, \quad \text{where} \quad \sigma^i = S^i_a \check{\pi}^a \tag{4.89}$$

in such a manner as to preserve the form of the structure equations (4.48).

If we take the exterior derivative of (4.89) and substitute into the structure equations for $\tilde{\theta}^i$ we find first, from the $\check{\pi}^\alpha \wedge \theta^i$ and the $\hat{\pi}^a \wedge \check{\pi}^\alpha$ terms, that $\Lambda^i_j, S^i_a \in \operatorname{Int}(\hat{V})$ and then that

$$d\Lambda^i_j = C^i_{\ell k}\, \Lambda^\ell_j \sigma^k, \quad \Lambda^i_\ell\, C^\ell_{jk} = C^i_{\ell m}\, \Lambda^\ell_j\, \Lambda^m_k, \quad d\sigma^i = \frac{1}{2} C^i_{jk} \sigma^j \wedge \sigma^k. \tag{4.90}$$

To integrate these equations, let G be a local Lie group whose Lie algebra is the Vessiot algebra $\mathfrak{vess}$ with structure constants C^i_{jk}. On G construct a coframe η^i of invariant 1-forms with structure equations $d\eta^i = \frac{1}{2} C^i_{jk} \eta^j \wedge \eta^k$. Then there exists ([17], Proposition 1.3) a map $\sigma\colon \operatorname{Int}(\hat{V}) \to G$ such that $\sigma^i = \sigma^*(\eta^i)$. Define $S(\hat{I}) = \operatorname{Ad}(\sigma(\hat{I})$. Then

$$d\,S^i_j = C^i_{\ell k} \sigma^\ell S^k_j \quad \text{and} \quad d\,(\Lambda^i_\ell (S^{(-1)})^\ell_j) = 0.$$

and hence the general solution to (4.90) is

$$\sigma^i = \sigma^*(\eta^i), \quad \Lambda^i_j = \overset{0}{\Lambda}{}^i_\ell\, S^\ell_j, \quad S = \operatorname{Ad}(\sigma(\hat{I})), \tag{4.91}$$

where the matrix $\overset{0}{\Lambda}{}^i_\ell$ is a constant automorphism of the Vessiot algebra. When σ is a constant map, S is constant inner automorphism so that as far as the general solution (4.91) is concerned, one can restrict the $\overset{0}{\Lambda}{}^i_\ell$ to a set of representatives of the group of outer automorphisms of the Vessiot algebra.

Remark 4.10. The bases for the infinitesimal Vessiot transformation groups $\hat{\Gamma} = \{ \hat{X}_i \}$ and $\check{\Gamma} = \{ \check{X}_i \}$ are related by $\check{X}_j = \lambda^i_j \hat{X}_i$, where λ is the matrix (4.87). Let $\hat{X}_r$, $r = 1 \dots m$, be a basis for the center of $\hat{\Gamma}$ and pick a complementary set of vectors $\hat{X}_u$, $u = m + 1 \dots s$, which complete the $\hat{X}_r$ to a basis for $\hat{\Gamma}$. Choose a similar basis for $\check{\Gamma}$. Then, because λ defines a Lie algebra automorphism, we have that

$$\check{X}_r = \lambda^t_r \hat{X}_t \quad \text{and} \quad \check{X}_u = \lambda^v_u \hat{X}_v + \lambda^t_u \hat{X}_t,$$

where $t = 1 \dots m$ and $v = m+1 \dots s$. The second equation in (4.88) now implies that $d\,\lambda^r_s = 0$. Consequently, we may always pick bases for the infinitesimal Vessiot transformation groups $\hat{\Gamma}$ and $\check{\Gamma}$ so that the infinitesimals generators for the center coincide. In turn, this implies that any vector in the center of either infinitesimal Vessiot transformation group is a infinitesimal symmetry of *both* $\hat{V}$ and $\check{V}$. ∎

5 The Superposition Formula for Darboux Pairs

5.1 A preliminary result

The proof of the superposition formula for a Darboux pair will depend upon the following technical result concerning group actions and Maurer-Cartan forms.

Theorem 5.1. *Let M be a manifold and G a connected Lie group. Let ω_L^i and ω_R^i be the left and right invariant Maurer-Cartan forms on G, with structure equations*

$$d\omega_L^i = \frac{1}{2} C^i_{jk}\, \omega_L^j \wedge \omega_L^k \quad \text{and} \quad d\omega_R^i = -\frac{1}{2} C^i_{jk}\, \omega_R^j \wedge \omega_R^k. \tag{5.1}$$

Let X_i^L and X_i^R be the dual basis of left and right invariant vector fields on G.

[**i**] *Suppose that there are right and left commuting, regular, free group actions of G on M,*

$$\hat{\mu}\colon G \times M \to M \quad \text{and} \quad \check{\mu}\colon G \times M \to M, \tag{5.2}$$

with common orbits. Denote the infinitesimal generators of these actions by $(\hat{\mu}_x)_*(X_i^L) = \hat{X}_i$ and $(\check{\mu}_x)_*(X_i^R) = \check{X}_i$.

[**ii**] *Suppose there are 1-forms $\hat{\omega}^i$ and $\check{\omega}^i$ on M such that* [**a**] *$\hat{\omega}^i(\hat{X}_j) = \delta^i_j$ and $\check{\omega}^i(\check{X}_j) = \delta^i_j$;* [**b**] *$span\{\, \hat{\omega}^i \,\} = span\{\, \check{\omega}^i \,\}$ with $\hat{\omega}^i(x_0) = \check{\omega}^i(x_0)$ at some fixed point of $x_0 \in M$ and;* [**c**]

$$d\hat{\omega}^i = \frac{1}{2} C^i_{jk}\; \hat{\omega}^j \wedge \hat{\omega}^k \quad \text{and} \quad d\check{\omega}^i = -\frac{1}{2} C^i_{jk}\; \check{\omega}^j \wedge \check{\omega}^k. \tag{5.3}$$

Then about each point x_0 of M there is an open G bi-invariant neighborhood $\mathcal{U}$ of M and a mapping $\rho\colon \mathcal{U} \to G$ such that

$$\rho^*(\omega_L^i) = \hat{\omega}^i, \quad \rho^*(\omega_R^i) = \check{\omega}^i, \quad \rho(\hat{\mu}(x, g)) = \rho(x) \cdot g, \quad \rho(\check{\mu}(g, x)) = g \cdot \rho(x). \tag{5.4}$$

If M/G is simply connected, then one can take $\mathcal{U} = M$.

Remark 5.2. This theorem can be viewed as a simple extension (or refinement) of three different, well-known results. First, suppose that [**i**] holds. Then, because the actions $\hat{\mu}$ and $\check{\mu}$ are free and regular, we may construct $\hat{\mu}$ and $\check{\mu}$ invariant sets $\hat{\mathcal{U}}$ and $\check{\mathcal{U}}$ and maps

$$\hat{\rho}\colon \hat{\mathcal{U}} \to G \quad \text{and} \quad \check{\rho}\colon \check{\mathcal{U}} \to G \tag{5.5}$$

which are G equivariant (see, for example, [12]) with respect to the actions $\hat{\mu}$ and $\check{\mu}$, respectively. Theorem 5.1 states that these maps can be chosen such that they are equal on a common bi-invariant domain, $\hat{\rho}^*(\omega_L^i) = \hat{\omega}^i$ and $\check{\rho}^*(\omega_R^i) = \check{\omega}^i$. Secondly, assume **[ii]**. Then a fundamental theorem in Lie theory (see Griffiths [17] or Sternburg [23], page 220) asserts that there always exists maps $\hat{\varphi} : M \to G$ and $\check{\varphi} : M \to G$ such that

$$\hat{\varphi}^*(\omega_L^i) = \hat{\omega}^i \quad \text{and} \quad \check{\varphi}^*(\omega_R^i) = \check{\omega}^i. \tag{5.6}$$

From this viewpoint, Theorem 5.1 states that these maps can be taken to be equal and equivariant with respect to the actions $\hat{\mu}$ and $\check{\mu}$. Thirdly, suppose that just one of the actions in **[i]** is given, that G acts regularly on M and the corresponding forms (say $\hat{\omega}^i$) in **[ii]** are given. Then $\pi\colon M \to M/G$ may be viewed as a principal G bundle with (by **[ii]**) flat connection 1-forms $\hat{\omega}^i$. Then Kobayashi and Nomizu [18](Theorem 9.1, Volume I, page 92) state that there is an open set U in M/G and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi^*(\omega_L^i) = \hat{\omega}^i$ (where the Maurer-Cartan forms ω_L^i define the canonical flat connection on $U \times G$). Moreover when M/G is simply connected, then M is globally the trivial G bundle with canonical flat connection. Compare this result with Corollary 5.6 to Theorem 5.1.

We establish Theorem 5.1 with the following sequence of lemmas.

Lemma 5.3. *Under the hypothesis of Theorem 5.1 there is a bi-invariant, connected neigborhood $\mathcal{U}$ around x_0 and a mapping $\rho\colon \mathcal{U} \to G$ such that $\rho^*(\omega_L^i) = \hat{\omega}^i$ and $\rho^*(\omega_R^i) = \check{\omega}^i$. The map ρ is uniquely defined for a given domain $\mathcal{U}$.*

Proof. As is customary, we suppose that $\omega_L^i(e) = \omega_R^i(e)$, where e is the identity of G. In accordance with the aforementioned Theorem 9.1 in [18] there is a $\hat{\mu}$ invariant open set $\mathcal{U}$ in M and a unique smooth map $\rho\colon \mathcal{U} \to G$ such that

$$\rho^*(\omega_L^i) = \hat{\omega}^i \quad \text{and} \quad \rho(x_0) = e. \tag{5.7}$$

Since the $\hat{\mu}$ and $\check{\mu}$ orbits coincide, the set $\mathcal{U}$ is actually bi-invariant. Because G is connected, we may assume that $\mathcal{U}$ is connected. The uniqueness of ρ is established in Theorem 2.3 (page 220) of [23].

In order to prove that ρ satisfies $\rho^*(\omega_R^i) = \check{\omega}^i$, let $\Lambda_j^i(g)$ be the matrix for the linear transformation

$$\mathrm{Ad}^*(g) = R_g^* \circ L_{g^{-1}}^* \colon T_e^*G \to T_e^*G$$

with respect to the basis $\hat{\omega}^i(e)$. Then an easy computation shows that

$$\omega_L^i(g) = \Lambda_j^i(g)\,\omega_R^j(g) \quad \text{and} \quad \Lambda_k^i(g_1 g_2) = \Lambda_j^i(g_2)\Lambda_k^j(g_1). \tag{5.8}$$

The Jacobian of the multiplication map $m\colon G \times G \to G$ may be computed in terms of Λ_j^i as

$$m^*(\omega_L^i)(g_1 g_2) = \Lambda_j^i(g_2)\,\omega_L^j(g_1) + \omega_L^i(g_2). \tag{5.9}$$

We take the Lie derivative of the first equation in (5.8) with respect to X_k^L to find that $L_{X_k^L}\Lambda_j^i = C_{k\ell}^i \Lambda_j^\ell$ and hence

$$d\Lambda_j^i = C_{k\ell}^i\,\Lambda_j^\ell\,\omega_L^k. \tag{5.10}$$

Similarly, by **[ii][b]**, we may write $\hat{\omega}^i = \lambda_j^i \check{\omega}^j$ and calculate from (5.3)

$$d\lambda_j^i = C_{k\ell}^i\,\lambda_j^\ell\,\hat{\omega}^k. \tag{5.11}$$

The combination of equations (5.7), (5.10) and (5.11) (and the fact that $\Lambda_j^i(e) = \lambda_j^i(x_0)$) now shows that

$$d\left(\rho^*(\Lambda_j^i)(\lambda^{-1})_k^j\right) = 0 \quad \text{and hence} \quad \rho^*(\Lambda_j^i) = \lambda_j^i. \tag{5.12}$$

(Here we use the connectivity of $\mathcal{U}$). The equations $\rho^*(\omega_R^i) = \check{\omega}^i$ now follows immediately from (5.7), (5.8), and (5.12). ∎

We note, for future use, that (5.11) also implies

$$\mathcal{L}_{\hat{X}_i}\check{\omega}^j = \mathcal{L}_{\hat{X}_i}(\lambda^{-1})_k^j\,\hat{\omega}^k + (\lambda^{-1})_k^j\,C_{i\ell}^k\,\hat{\omega}^\ell = 0. \tag{5.13}$$

This shows, by virtue of the connectivity of G, that

$$(\hat{\mu}_g)^*(\check{\omega}^j) = \check{\omega}^j. \tag{5.14}$$

Lemma 5.4. *With $\rho : \mathcal{U} \to G$ as in Lemma 5.3, the maps $\hat{\varphi} = \rho \circ \hat{\mu}_{x_0} : G \to G$ and $\check{\varphi} = \rho \circ \check{\mu}_{x_0} : G \to G$ satisfy*

$$\hat{\varphi}^*(\omega_L^i) = \omega_L^i \quad \textit{and} \quad \check{\varphi}^*(\omega_R^i) = \omega_R^i \tag{5.15}$$

and hence $\hat{\varphi} = \check{\varphi} = id_G$.

Proof. The definitions $(\hat{\mu}_x)_*(X_i^L) = \hat{X}_i$ and $(\check{\mu}_x)_*(X_i^R) = \check{X}_i$, together with **[ii](a)**, imply that $\hat{\mu}_{x_0}^*(\hat{\omega}^i) = \omega_L^i$ and $\check{\mu}_{x_0}^*(\check{\omega}^i) = \omega_R^i$. Equations (5.15) then follow directly from Lemma 5.3. These equations imply that $\hat{\varphi}$ and $\check{\varphi}$ are translations in G and therefore, since G is connected and $\hat{\varphi}(e) = \check{\varphi}(e) = e$, we have that $\hat{\varphi} = \check{\varphi} = \mathrm{id}_G$. ∎

Lemma 5.5. *The maps $\rho\colon \mathcal{U} \to G$ satisfies $\rho(\hat{\mu}(x,g)) = \rho(x)\cdot g$ and $\rho(\check{\mu}(x,g)) = g \cdot \rho(x)$ for all $x \in \mathcal{U}$ and $g \in G$.*

Proof. To prove these equivariance properties of ρ, define maps

$$\hat{\rho} = R_{g^{-1}} \circ \rho \circ \hat{\mu}_g \colon M \to G \quad \text{and} \quad \check{\rho} = L_{g^{-1}} \circ \rho \circ \check{\mu}_g \colon M \to G,$$

where $\hat{\mu}_g(x) = \hat{\mu}(g,x)$ and $\check{\mu}_g(x) = \check{\mu}(g,x)$. We wish to prove that $\hat{\rho} = \check{\check{\rho}} = \rho$.

Lemma 5.4 shows that $\hat{\rho}(x_0) = \check{\rho}(x_0) = e$ so that it suffices to show that

$$\hat{\rho}^*(\omega_R^i) = \check{\omega}^i \quad \text{and} \quad \check{\rho}^*(\omega_L^i) = \check{\omega}^i \tag{5.16}$$

and then to appeal to the uniqueness statement in Lemma 5.3. The first equation (5.16) follows from the fact that the forms ω_R^i are right invariant, the second equation in (5.4) and (5.14). The proof of the second equation in (5.16) is similar. ∎

Corollary 5.6 (Corollary to Theorem 5.1)**.** *Let $\mathcal{S}$ be the submanifold of M defined by $\mathcal{S} = \rho^{-1}(e)$, where ρ is the map* (5.4)*. Then the map*

$$\Psi \colon \mathcal{U} \to \mathcal{S} \times G \quad \textit{defined by} \quad \Psi(x) = (\hat{\mu}(\rho(x)^{-1}, x), \rho(x))$$

is a G bi-equivariant diffeomorphism to an open subset of $\mathcal{S} \times G$ and satisfies

$$\Psi_*(\hat{X}_i) = X_i^R, \quad \Psi_*(\check{X}_i) = X_i^L, \quad \Psi^*(\omega_L^i) = \hat{\omega}^i, \quad \Psi^*(\omega_R^i) = \check{\omega}^i. \tag{5.17}$$

If M/G is simply-connected, then $\Psi\colon M \to \mathcal{S} \times G$ is a global diffeomorphism

Proof. It is easy to check that Ψ sends the intersection of $\mathcal{U}$ with the domain of $\hat{\mu}$ in M into $\mathcal{S} \times G$. To check the equivariance of Ψ, it is convenient to write $\hat{\mu}(g,x) = x \mathbin{\hat{*}} g$ and $\check{\mu}(g,x) = g \mathbin{\check{*}} x$. Because the orbits for the two actions coincide, we know that for each $g \in G$ there is a g' such that $g \mathbin{\check{*}} x = x \mathbin{\hat{*}} g'$. We then use (5.4) to calculate

$$\begin{aligned}
\Psi(x \mathbin{\hat{*}} g) &= \big((x \mathbin{\hat{*}} g) \mathbin{\hat{*}} \rho(x \mathbin{\hat{*}} g)^{(-1)},\, \rho(x \mathbin{\hat{*}} g)\big) = \big(x \mathbin{\hat{*}} \rho(x)^{(-1)},\, \rho(x) \mathbin{\hat{*}} g\big) \\
&= \Psi(x) * g, \quad \text{and} \\
\Psi(g \mathbin{\check{*}} x) &= \big((g \mathbin{\check{*}} x) \mathbin{\hat{*}} \rho(g \mathbin{\check{*}} x)^{(-1)},\, \rho(g \mathbin{\check{*}} x)\big) = \big(x \mathbin{\hat{*}} g') \mathbin{\hat{*}} \rho(x \mathbin{\hat{*}} g')^{(-1)},\, g \mathbin{\check{*}} \rho(x)\big) \\
&= \big((x \mathbin{\hat{*}} \rho(x)^{(-1)},\, g \mathbin{\check{*}} \rho(x)\big) = g * \Psi(x),
\end{aligned}$$

as required. ∎

5.2 The Superposition Formula for Darboux Pairs

We are now ready to establish, using the 5-adapted coframe of Section 4.4 and the technical results of Section 5.1, the superposition formula for any Darboux pair.

By way of summary, let us recall that our starting point is the local (0-adapted) coframe $\{\,\boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\, \hat{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}}\,\}$ which is adapted to a given Darboux pair $\{\,\hat{V}, \check{V}\,\}$ in the sense that

$$I = \operatorname{span}\{\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\}, \quad \hat{V} = \operatorname{span}\{\boldsymbol{\theta},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\}, \quad \check{V} = \operatorname{span}\{\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\}. \tag{5.18}$$

In our first coframe adaption we adjusted the forms $\hat{\boldsymbol{\eta}}$ and $\check{\boldsymbol{\eta}}$ to the form

$$\hat{\boldsymbol{\eta}} = d\hat{\boldsymbol{I}}_2 + \hat{\mathbf{R}}\,\hat{\boldsymbol{\sigma}} \quad \text{and} \quad \check{\boldsymbol{\eta}} = d\check{\boldsymbol{I}}_2 + \check{\boldsymbol{R}}\,\check{\boldsymbol{\sigma}}. \tag{5.19}$$

The fourth coframe adaptations $\{\,\boldsymbol{\theta_X},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}$ and $\{\,\boldsymbol{\theta_Y},\, \hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}\,\}$ allowed us to associate to the Darboux pair $\{\,\hat{V}, \check{V}\,\}$ an abstract Lie algebra $\mathfrak{vess}(\hat{V}, \check{V})$. We have $\theta^i_X = Q^i_j \theta^j_Y$ and we can write

$$\hat{V} \cap \check{V} = \operatorname{span}\{\boldsymbol{\theta_X},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\}, \tag{5.20}$$

From the 4-adapted coframes we then constructed the final coframes

$$\hat{\theta}^i = \hat{R}^i_j \theta^j_X + \phi^i_a \hat{\pi}^a, \quad \check{\theta}^i = \check{R}^i_j \theta^j_Y + \psi^i_\alpha \check{\pi}^\alpha, \tag{5.21}$$

where $\hat{\boldsymbol{\pi}} = [\hat{\boldsymbol{\sigma}},\, \hat{\boldsymbol{\eta}}]$, and $\check{\boldsymbol{\pi}} = [\check{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}}]$, with structure equations

$$d\hat{\theta}^i = \frac{1}{2} G^i_{\alpha\beta}\, \check{\pi}^\alpha \wedge \check{\pi}^\beta + \frac{1}{2} C^i_{jk} \hat{\theta}^j \wedge \hat{\theta}^k, \quad d\check{\theta}^i = \frac{1}{2} H^i_{ab}\, \hat{\pi}^a \wedge \hat{\pi}^b - \frac{1}{2} C^i_{jk} \check{\theta}^j \wedge \check{\theta}^k. \tag{5.22}$$

The coefficients $G^i_{\alpha\beta}$ are functions of the first integrals $\check{I}^\alpha$ while the H^i_{ab} are functions of the $\hat{I}^a$. The two 5-adapted coframes

$$\{\,\hat{\boldsymbol{\theta}},\ \hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\sigma}},\ \check{\boldsymbol{\eta}}\,\} \quad \text{and} \quad \{\,\check{\boldsymbol{\theta}},\ \hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\sigma}},\ \check{\boldsymbol{\eta}}\,\} \tag{5.23}$$

are *not* adapted to the Darboux pair $\{\,\hat{V}, \check{V}\,\}$ (in the sense of (2.6)) although we do have

$$\hat{V} = \operatorname{span}\{\,\hat{\boldsymbol{\theta}},\ \hat{\boldsymbol{\sigma}},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}}\,\} \quad \text{and} \quad \check{V} = \operatorname{span}\{\,\check{\boldsymbol{\theta}},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\sigma}},\ \check{\boldsymbol{\eta}}\,\}. \tag{5.24}$$

Definition 5.7. *Let $\{\,\hat{V}, \check{V},\,\}$ be a Darboux pair with 5-adapted coframes* (5.23). *Let $\{\,\hat{X}_i, \hat{V}_a, \check{V}_\alpha\,\}$ and $\{\,\check{X}_i, \check{W}_a, \check{W}_\alpha\,\}$ be the dual frames corresponding to the 5 adapted coframes so that, in particular,*

$$\hat{\theta}^i(\hat{X}_j) = \delta^i_j,\ \hat{\pi}^a(\hat{X}_j) = 0,\ \check{\theta}^i(\check{X}_j) = \delta^i_j,\ \check{\pi}^a(\check{X}_j) = 0. \tag{5.25}$$

The ***infinitesimal Vessiot group actions*** *for the Darboux pair* $\{\hat{V}, \check{V}\}$ *are defined by the Lie algebras of vector fields* $\hat{X}_i$ *and* $\check{X}_i$*. These define (local) groups actions*

$$\hat{\mu}\colon G \times M \to M \quad \textit{and} \quad \check{\mu}\colon G \times M \to M. \tag{5.26}$$

We take $\hat{\mu}$ *to be a right action on* M *and* $\check{\mu}$ *to be a left action.*

The vectors fields $\hat{X}_i$ and $\check{X}_i$ are related by (see (4.87) and (4.88))

$$\hat{X}_i = \lambda_i^j \check{X}_j \quad \text{and satisfy} \quad [\,\hat{X}_i,\, \check{X}_j\,] = 0. \tag{5.27}$$

From the structure equations (5.22) we find that

$$\mathcal{L}_{\hat{X}_j} \hat{\theta}^i = C^i_{jk} \hat{\theta}^k \quad \text{and} \quad \mathcal{L}_{\check{X}_j} \check{\theta}^i = C^i_{jk} \check{\theta}^k. \tag{5.28}$$

$$\mathcal{L}_{\hat{X}_j} \hat{\pi}^a = \mathcal{L}_{\hat{X}_j} \check{\pi}^\alpha = \mathcal{L}_{\check{X}_j} \hat{\pi}^a = \quad \mathcal{L}_{\check{X}_j} \check{\pi}^\alpha = 0. \tag{5.29}$$

The actions $\hat{\mu}$ and $\check{\mu}$ commute because $[\,\hat{X}_i, \check{X}_j] = 0$. In what follows we shall assume that these actions are regular.

Now let G be a Lie group with Lie algebra $\mathfrak{vess}(\hat{V}, \check{V})$. Let ω^i_L and ω^i_R be the left and right invariant Maurer-Cartan forms on G, with structure equations (5.1). We shall assume that these two coframes on G coincide at the identity e. Let X_i^L and X_i^R be the dual basis of left and right invariant vector fields.

To apply Theorem 5.1 we need to identify the forms $\hat{\omega}^i$ and $\check{\omega}^i$.

Lemma 5.8. *The forms* $\hat{\omega}^i$ *and* $\check{\omega}^i$ *defined in terms of the 5-adapted coframes ((4.47) or (5.21)) by*

$$\hat{\omega}^i = \hat{\theta}^i + \lambda^i_j \psi^j_\alpha \check{\pi}^\alpha \quad \textit{and} \quad \check{\omega}^i = \check{\theta}^i + \mu^i_j \phi^j_a \hat{\pi}^a, \tag{5.30}$$

where $\lambda = \hat{R} Q \check{R}^{-1}$ *and* $\mu = \lambda^{-1}$ *(see Remark 4.9), have the same span and satisfy the structure equations*

$$d\hat{\omega}^i = \frac{1}{2} C^i_{jk}\, \hat{\omega}^j \wedge \hat{\omega}^k \quad \textit{and} \quad d\check{\omega}^i = -\frac{1}{2} C^i_{jk}\, \check{\omega}^j \wedge \check{\omega}^k. \tag{5.31}$$

Proof. To check that the forms $\hat{\omega}^i$ and $\check{\omega}^i$ have the same span we simply use the various definitions given here to calculate

$$\begin{aligned}
\hat{\omega}^i &= \hat{\theta}^i + \lambda^i_j\, \psi^j_\alpha\, \check{\pi}^\alpha = \hat{R}^i_j\, \theta^j_X + \phi^i_a \hat{\pi}^a + \lambda^i_j \psi^j_\alpha \check{\pi}^\alpha \\
&= \hat{R}^i_j\, Q^j_k\, \theta^k_Y + \phi^i_a \hat{\pi}^a + \lambda^i_j \psi^j_\alpha \check{\pi}^\alpha = (\hat{R}\, Q\, \check{R}^{-1})^i_j\, \check{R}^j_k\, \theta^k_Y + \phi^i_a\, \hat{\pi}^a + \lambda^i_j\, \psi^j_\alpha\, \check{\pi}^\alpha \\
&= \lambda^i_j\, (\check{R}^j_k\, \theta^k_Y + \psi^j_a \check{\pi}^a + (\lambda^{-1})^j_k\, \phi^k_\alpha\, \hat{\pi}^\alpha) = \lambda^i_j\, (\check{\theta}^j + \mu^j_k\, \phi^k_\alpha\, \hat{\pi}^\alpha) = \lambda^i_j\, \check{\omega}^j.
\end{aligned} \tag{5.32}$$

We now calculate $d\hat{\omega}^i$ using (4.48), (4.85), (4.86) and (4.88) to find that

$$\begin{aligned} d\hat{\omega}^i &= d\hat{\theta}^i + d\lambda^i_j \wedge \psi^j + \lambda^i_j \wedge d\psi^j \\ &= \frac{1}{2} G^i_{\alpha\beta}\, \breve{\pi}^\alpha \wedge \breve{\pi}^\beta + \frac{1}{2} C^i_{jk}\, \hat{\theta}^j \wedge \hat{\theta}^k + (C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell + \lambda^i_h\, C^h_{mj}\, \psi^m) \wedge \psi^j \\ &\quad + \lambda^i_j\, (-\frac{1}{2} C^j_{hk} \psi^h \wedge \psi^k - \frac{1}{2} \check{R}^j_\ell\, F^\ell_{\alpha\beta}\, \breve{\pi}^\alpha \wedge \breve{\pi}^\beta) \\ &= \frac{1}{2} C^i_{jk}\, \hat{\theta}^j \wedge \hat{\theta}^k + C^i_{\ell m}\, \lambda^m_j\, \hat{\theta}^\ell \wedge \psi^j + \frac{1}{2} \lambda^i_\ell\, C^\ell_{hk}\, \psi^h \wedge \psi^k = \frac{1}{2} C^i_{jk} \hat{\omega}^j \wedge \hat{\omega}^k. \end{aligned}$$

The derivation of the structure equations for $\breve{\omega}^i$ is similar. ∎

Let $x_0 \in M$. By a change of frame at x_0 we may suppose that $\hat{\omega}^i(x_0) = \breve{\omega}^i(x_0)$. At this point all the hypothesis for Theorem 5.1 are satisfied and, in accordance with this theorem and Corollary 5.6, we can construct maps

$$\rho \colon \mathcal{U} \to G \quad \text{and} \quad \Psi \colon \mathcal{U} \to \mathcal{S} \times G \tag{5.33}$$

which satisfy (5.4), (5.17) and $\rho(x_0) = e$.

Define $\mathcal{S}_1$ and $\mathcal{S}_2$ to be the integral manifolds through x_0 for the restriction of $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$ to $\mathcal{S}$. Because

$$T^*\mathcal{S}_x = \hat{V}^{(\infty)}_x \oplus \check{V}^{(\infty)}_x \quad \text{for all } x \in \mathcal{S}, \tag{5.34}$$

we may choose $\mathcal{S}_1$ and $\mathcal{S}_2$ small enough so as to be assured of the existence of a local diffeomorphism

$$\chi \colon \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_0, \tag{5.35}$$

where S_0 is an open set in $\mathcal{S}$. The map χ satisfies (with $s = \chi(s_1, s_2)$ and $s_i \in \mathcal{S}_i$)

$$\chi^*(\hat{V}^{(\infty)}_s) = T^*_{s_2}\mathcal{S}_2 \quad \text{and} \quad \chi^*(\check{V}^{(\infty)}_s) = T^*_{s_1}\mathcal{S}_1. \tag{5.36}$$

Set $\chi(x^0_1, x^0_2) = x_0$. Hence, for any first integrals $\hat{I}$ and $\check{I}$ of $\hat{V}$ and $\check{V}$, we have

$$\hat{I}(\chi(s_1, s_2) = \hat{I}(\chi(x^0_1, s_2)) \quad \text{and} \quad \check{I}(\chi(s_1, s_2)) = \check{I}(\chi(s_1, x^0_2)). \tag{5.37}$$

Put $\mathcal{U}_0 = \Phi^{-1}(\mathcal{S}_0)$.

To define the superposition formula for the Darboux pair $\{\hat{V}, \check{V}\}$, we take M_1 to be the (maximal, connected) integral manifold of $\hat{V}^{(\infty)}$ through x_0 in $\mathcal{U}_0$

and M_2 to be the (maximal, connected) integral manifold of $\check{V}^{(\infty)}$ through x_0 in $\mathcal{U}_0$. The group actions $\hat{\mu}$ and $\check{\mu}$ and the maps ρ and Ψ all restrict to maps

$$\begin{aligned} &\hat{\mu}_i\colon G \times M_i \to M_i, \quad \check{\mu}_i\colon G \times M_i \to M_i, \\ &\rho_i\colon M_i \to G, \quad \text{and} \quad \Psi_i\colon M_i \to \mathcal{S}_i \times G \end{aligned} \tag{5.38}$$

for $i = 1, 2$.

With the maps Ψ and χ defined by (5.4) and (5.35) and with these choices for M_1 and M_2, we define

$$\Sigma\colon M_1 \times M_2 \to M \quad \text{by} \quad \Sigma(x_1, x_2) = \hat{\mu}\big(g_1 \cdot g_2, \chi(s_1, s_2)\big), \tag{5.39}$$

where $x_i \in M_i$, and $\Psi(x_i) = (s_i, g_i)$.

To prove that Σ defines a superposition formula, we must calculate the pullback by Σ of an adapted coframe on M. To this end, we introduce the following notation. For any differential form α on M, denote the restriction of α to M_i by α_i. If f is a function on M and $x_i \in M_i$, then $f(x_i)$ denotes the value of f at x_i, viewed as a point of M. We also remark that the equivariance of ρ with respect to the group action $\hat{\mu}$ and $\check{\mu}$ implies that Σ satisfies

$$\begin{aligned} (\rho \circ \Sigma)(x_1, x_2) &= \rho(\hat{\mu}(g_1 \cdot g_2, \chi(s_1, s_2)) = g_1 \cdot g_2 \cdot \rho(\chi(s_1, s_2)) \\ &= g_1 \cdot g_2 = \rho_1(x_1) \cdot \rho_2(x_2) = (m \circ (\rho_1 \times \rho_2))(x_1, x_2). \end{aligned} \tag{5.40}$$

Lemma 5.9. *The pullback by Σ of the 4-adapted coframe (see Theorm 4.6) on M is given in terms of the restrictions of the 5-adapted coframes* (5.23) *on M_1 and M_2 by*

$$\Sigma^*(\hat{\boldsymbol{\sigma}}) = \hat{\boldsymbol{\sigma}}_2, \quad \Sigma^*(\check{\boldsymbol{\sigma}}) = \check{\boldsymbol{\sigma}}_1, \quad \Sigma^*(\hat{\boldsymbol{\eta}}) = \hat{\boldsymbol{\eta}}_2, \quad \Sigma^*(\check{\boldsymbol{\eta}}) = \check{\boldsymbol{\eta}}_1, \tag{5.41}$$

$$\Sigma^*(\boldsymbol{R}\boldsymbol{\theta}_{\boldsymbol{X}}) = \boldsymbol{\lambda}\,(\hat{\boldsymbol{\theta}}_1 + \check{\boldsymbol{\theta}}_2). \tag{5.42}$$

Proof. We begin with the observation that the first integrals $\hat{I}^a$ and $\check{I}^\alpha$ are invariants of $\hat{\mu}$ and therefore, by (5.37),

$$\begin{aligned} \Sigma^*(\hat{I}^a)(x_1, x_2) &= \hat{I}^a(\hat{\mu}(g_1 g_2, \chi(s_1, s_2))) = \hat{I}^a(\chi(s_1, s_2)) \\ &= \hat{I}^a(\chi(x_{01}, s_2)) = \hat{I}^a(\hat{\mu}(g_2, \chi(x_{01}, s_2))) \\ &= \hat{I}^a(x_2), \qquad \text{and similarly} \end{aligned} \tag{5.43}$$

$$\Sigma^*(\check{I}^\alpha)(x_1, x_2) = \check{I}^\alpha(x_1). \tag{5.44}$$

For future reference, we note (see (5.21)) that

$$\Sigma^*(G^i_{\alpha\beta})(x_1,x_2) = G^i_{\alpha\beta}(x_1) \quad \text{and} \quad \Sigma^*(H^i_{ab})(x_1,x_2) = G^i_{ab}(x_2). \tag{5.45}$$

From equations (5.43) and (5.44) it follows directly that

$$\Sigma^*(\hat{\boldsymbol{\sigma}}) = \hat{\boldsymbol{\sigma}}_2, \qquad \Sigma^*(\check{\boldsymbol{\sigma}}) = \check{\boldsymbol{\sigma}}_1, \tag{5.46}$$

$$\Sigma^*(\hat{\boldsymbol{\eta}}) = \Sigma^*(d\hat{\boldsymbol{I}} + \hat{\mathbf{R}}\,\hat{\boldsymbol{\sigma}}) = \hat{\boldsymbol{\eta}}_2 \quad \text{and} \quad \Sigma^*(\check{\boldsymbol{\eta}}) = \Sigma^*(d\check{\boldsymbol{I}} + \check{\mathbf{R}}\,\check{\boldsymbol{\sigma}}) = \check{\boldsymbol{\eta}}_1. \tag{5.47}$$

Equations (5.41) are therefore established.

Equations (5.21) and (5.30) imply that

$$\hat{R}^i_j\theta^j_X = \hat{\omega}^i - \phi^i_a\,\hat{\pi}^a - \lambda^i_k\psi^k_a\,\check{\pi}^\alpha. \tag{5.48}$$

We use this equation to calculate $\Sigma^*(\hat{R}^i_j\theta^j_X)$.

Since $\phi^i_a \in \mathrm{Int}(\hat{V})$ and $\psi^i_\alpha \in \mathrm{Int}(\check{V})$, equations (5.43) and (5.44) imply that

$$\Sigma^*(\phi^i_a\,\hat{\pi}^a)(x_1,x_2) = \phi^i_a(x_2)\,\hat{\pi}^a_2 \quad \text{and} \quad \Sigma^*(\psi^i_a\,\check{\pi}^\alpha)(x_1,x_2) = \psi^i_\alpha(x_1)\,\check{\pi}^\alpha_1. \tag{5.49}$$

We deduce from equations (5.8), (5.12) and (5.40) that

$$\begin{aligned}\Sigma^*(\lambda^i_k)(x_1,x_2) = \Sigma^*(\rho^*\Lambda^i_k)(x_1,x_2) = \Lambda^i_k(g_1\cdot g_2) = \\ = \Lambda^i_j(g_2)\Lambda^j_k(g_1) = \lambda^i_j(x_2)\lambda^j_k(x_1).\end{aligned} \tag{5.50}$$

To calculate $\Sigma^*(\hat{\omega}^i)$ it is helpful to introduce the projection maps $\pi_i\colon G\times G \to G$ and to note that $\pi_i(\rho_1\times\rho_2)(x_1,x_2) = g_i$, $i=1,2$. We can then re-write (5.9) as

$$m^*(\omega^i_L) = \pi^*_2(\lambda^i_j)\pi^*_1(\omega^j_L) + \pi^*_2(\omega^i_L).$$

This equation, together with (5.4) and (5.40) leads to

$$\begin{aligned}\Sigma^*(\hat{\omega}^i)(x_1,x_2) &= \Sigma^*(\rho^*(\omega^i_L))(x_1,x_2) = (\rho^*_1\times\rho^*_2)(m^*(\omega^i_L))(x_1,x_2) \\ &= (\rho^*_1\times\rho^*_2)(\pi^*_2(\lambda^i_j)\pi^*_1(\omega^j_L) + \pi^*_2(\omega^i_L))(x_1,x_2) = \\ &= \lambda^i_j(x_2)\,\hat{\omega}^j_1 + \hat{\omega}^i_2.\end{aligned} \tag{5.51}$$

Finally, the combination of equations (5.32) and (5.48) – (5.51) allows us to calculate

$$\begin{aligned}\Sigma^*(\hat{R}^i_j\,\theta^j_X) &= \Sigma^*\big(\hat{\omega}^i - \phi^i_a\,\hat{\pi}^a - \lambda^i_k\,\psi^k_\alpha\,\check{\pi}^\alpha\big) \\ &= \lambda^i_j(x_2)\,\hat{\omega}^j_1 + \hat{\omega}^i_2 - \phi^i_a(x_2)\,\pi^a_2 - \lambda^i_j(x_2)\lambda^j_k(x_1)\psi^k_\alpha(x_1)\,\check{\pi}^\alpha_1 \\ &= \lambda^i_j(x_2)\,\big(\hat{\omega}^j_1 - \lambda^j_k\,(x_1)\psi^k_\alpha(x_1)\,\check{\pi}^\alpha_1\,\big) + \lambda^i_j(x_2)\,\big(\check{\omega}^j_2 - \mu^j_k(x_2)\,\phi^k_a(x_2)\,\hat{\pi}^a_2\,\big) \\ &= \lambda^i_j(x_2)\,\hat{\theta}^j_1 + \lambda^i_j(x_2)\,\check{\theta}^j_2,\end{aligned} \tag{5.52}$$

which proves (5.42). ∎

Recall that for any Darboux pair $\{\hat{V}, \check{V}\}$, the EDS $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ is given by Definition 2.5.

Theorem 5.10 (THE SUPERPOSITION FORMULA). *Let $\{\hat{V}, \check{V}\}$ define a Darboux pair on M. Define M_1 and M_2 as above and let W_1 and W_2 be the restrictions of $\check{V}$ and $\hat{V}$ to M_1 and M_2, respectively. Then the map $\Sigma\colon M_1 \times M_2 \to M$, defined by* (5.39)*, satisfies*

$$\Sigma^*(\hat{V} \cap \check{V}) \subset W_1 + W_2 \quad \textit{and} \quad \Sigma^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}}) \subset \mathcal{W}_1 + \mathcal{W}_2. \tag{5.53}$$

Thus Σ defines a superposition formula for $\hat{V} \Cap \check{V}$ with respect to the Pfaffian systems $\mathcal{W}_1$ and $\mathcal{W}_2$.

Proof. To prove this theorem we need only explicitly list the generators for $W_1 + W_2$, $\mathcal{W}_1 + \mathcal{W}_2$, $\hat{V} \cap \check{V}$ and $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ and check that, using Lemma (5.17), that the latter pullback into the former by Σ. For this we use the definitions of $\hat{V}$ and $\check{V}$ in terms of the 4-adapted and 5-adapted coframes by(5.20) and (5.24).

From the definition of M_1 and M_2 as integral manifolds of $\hat{V}^{(\infty)}$ and $\check{V}^{(\infty)}$, we have that

$$\hat{\boldsymbol{\eta}}_1 = 0,\ \hat{\boldsymbol{\sigma}}_1 = 0 \text{ on } M_1 \quad \text{and} \quad \check{\boldsymbol{\eta}}_2 = 0,\ \check{\boldsymbol{\sigma}}_2 = 0 \text{ on } M_2$$

and hence the two 5-adapted coframes on M restrict to coframes

$$\{\hat{\boldsymbol{\theta}}_1, \check{\boldsymbol{\eta}}_1, \check{\boldsymbol{\sigma}}_1\} \text{ for } M_1 \quad \text{and} \quad \{\check{\boldsymbol{\theta}}_2, \hat{\boldsymbol{\eta}}_2, \hat{\boldsymbol{\sigma}}_2\} \text{ for } M_2. \tag{5.54}$$

These naturally combine to give a coframe on $M_1 \times M_2$. The 1-form generators for the Pfaffian systems W_1 and W_2 are then given by

$$W_1 = \operatorname{span}\{\hat{\boldsymbol{\theta}}_1,\ \check{\boldsymbol{\eta}}_1\} \quad \text{and} \quad W_2 = \operatorname{span}\{\check{\boldsymbol{\theta}}_2,\ \hat{\boldsymbol{\eta}}_2\}. \tag{5.55}$$

The differential system $\mathcal{W}_1 + \mathcal{W}_2$ is algebraically generated by the 1-forms (5.55) and their exterior derivatives (see (2.13), (5.22)), that is,

$$\mathcal{W}_1 + \mathcal{W}_2 = \{\hat{\boldsymbol{\theta}}_1,\ \check{\boldsymbol{\eta}}_1,\ \check{\boldsymbol{\theta}}_2,\ \hat{\boldsymbol{\eta}}_2,\ d\check{\boldsymbol{\eta}}_1,\ d\hat{\boldsymbol{\eta}}_2,\ d\hat{\boldsymbol{\theta}}_1,\ d\check{\boldsymbol{\theta}}_2\}.$$

The restrictions of (5.22) to M_1 and M_2 lead to the structure equations

$$d\hat{\boldsymbol{\theta}}_1 = \frac{1}{2}\boldsymbol{G}_1\check{\boldsymbol{\pi}}_1 \wedge\ \check{\boldsymbol{\pi}}_1 \mod \{\hat{\boldsymbol{\theta}}_1\} = \frac{1}{2}\boldsymbol{G}_1\check{\boldsymbol{\sigma}}_1 \wedge\ \check{\boldsymbol{\sigma}}_1 \mod \{\hat{\boldsymbol{\theta}}_1, \check{\boldsymbol{\eta}}_1\} \quad \text{and}$$

$$d\check{\boldsymbol{\theta}}_2 = \frac{1}{2}\boldsymbol{H}_2\hat{\boldsymbol{\pi}}_2 \wedge\ \hat{\boldsymbol{\pi}}_2 \mod \{\check{\boldsymbol{\theta}}_2\} = \frac{1}{2}\boldsymbol{H}_2\hat{\boldsymbol{\sigma}}_2 \wedge\ \hat{\boldsymbol{\sigma}}_2 \mod \{\hat{\boldsymbol{\theta}}_2, \hat{\boldsymbol{\eta}}_2\}$$

from which it follows that $\mathcal{W}_1 + \mathcal{W}_2$ is algebraically generated by

$$\mathcal{W}_1 + \mathcal{W}_2 = \{\, \hat{\boldsymbol{\theta}}_1,\ \check{\boldsymbol{\eta}}_1,\ \check{\boldsymbol{\theta}}_2,\ \hat{\boldsymbol{\eta}}_2,\ \check{\boldsymbol{F}}_1\check{\boldsymbol{\sigma}}_1 \wedge \check{\boldsymbol{\sigma}}_1,\ \hat{\boldsymbol{F}}_2\hat{\boldsymbol{\sigma}}_2 \wedge \hat{\boldsymbol{\sigma}}_2, \boldsymbol{H}_2\hat{\boldsymbol{\sigma}}_2 \wedge\ \hat{\boldsymbol{\sigma}}_2\}. \tag{5.56}$$

The Pfaffian system $\hat{V}\cap\check{V}$ is generated by the forms $\{\boldsymbol{\theta},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\}$ or equivalently (see (5.20)) by $\{\boldsymbol{\theta}_{\boldsymbol{X}},\, \hat{\boldsymbol{\eta}},\, \check{\boldsymbol{\eta}}\}$. The first inclusion in (5.53) follows immediately from the Lemma 5.9 and (5.55).

By definition, the differential system $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ is algebraically generated, in terms of the 1-adapted coframe (see Theorem 2.9), as

$$\hat{\mathcal{V}} \Cap \check{\mathcal{V}} = \{\, \boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ d\hat{\boldsymbol{\eta}},\ d\check{\boldsymbol{\eta}},\ \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\,\}$$

or, equally as well, in terms of the 4-adapted coframe by

$$\hat{\mathcal{V}} \Cap \check{\mathcal{V}} = \{\, \boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \check{\boldsymbol{F}}\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \hat{\boldsymbol{F}}\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}, \boldsymbol{A}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{B}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\,\}.$$

In this latter equation the coefficients $\boldsymbol{A}$ and $\boldsymbol{B}$ are those appearing in (4.25). By (4.46) and (4.85), we also have that

$$\hat{\mathcal{V}} \Cap \check{\mathcal{V}} = \{\, \boldsymbol{R}\boldsymbol{\theta}_{\boldsymbol{X}},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \check{\boldsymbol{F}}\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}},\ \hat{\boldsymbol{F}}\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}}, \boldsymbol{H}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\ \boldsymbol{G}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\,\}. \tag{5.57}$$

With the algebraic generators for $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ in this form, it is easy to calculate the pullback of $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ to $M_1 \times M_2$ by Σ. By using (5.41), (5.42), (5.52) we find that

$$\Sigma^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}}) = \{\, \hat{\boldsymbol{\theta}}_1{+}\check{\boldsymbol{\theta}}_2,\ \hat{\boldsymbol{\eta}}_2,\ \check{\boldsymbol{\eta}}_1,\ \check{\boldsymbol{F}}_1\check{\boldsymbol{\sigma}}_1{\wedge}\check{\boldsymbol{\sigma}}_1,\ \hat{\boldsymbol{F}}_2\hat{\boldsymbol{\sigma}}_2{\wedge}\hat{\boldsymbol{\sigma}}_2,\ \boldsymbol{H}_2\,\hat{\boldsymbol{\sigma}}_2{\wedge}\hat{\boldsymbol{\sigma}}_2,\ \boldsymbol{G}_1\,\check{\boldsymbol{\sigma}}_1{\wedge}\check{\boldsymbol{\sigma}}_1\,\} \tag{5.58}$$

and the second inclusion in (5.53) is firmly established. ∎

One may use the first integrals of $\hat{V}$ and $\check{V}$ and the map $\rho\colon \mathcal{U} \to G$, and local coordinates z^i on G to define local coordinates

$$(\hat{I}^a = \hat{I}^a(x),\ \check{I}^\alpha = \check{I}^\alpha(x),\ z^i = \rho^i(x)) \tag{5.59}$$

on M and induced coodinates $(\check{I}^\alpha_1, z^i_1)$ on M_1 and $(\hat{I}^a_2, z^i_2)$ on M_2. In these coordinates the superposition formula becomes

$$\Sigma\big((\check{I}^\alpha_1, z^j_1), (\hat{I}^a_2, z^k_2)\big) = \big(\hat{I}^a = \hat{I}^a_2, \check{I}^\alpha = \check{I}^\alpha_1, z^i = (z^j_1)\cdot(z^k_2)\big). \tag{5.60}$$

We shall use this formula extensively in the examples in Section 6.

To prove that the superposition map Σ is subjective, at the level of integral manifolds, we use the concept of an integrable extension of a differential system

$\mathcal{I}$ on M [5]. This is a differential system $\mathcal{J}$ on a manifold N together with a submersion $\varphi\colon N \to M$ such that $\varphi^*(\mathcal{I}) \subset \mathcal{J}$ and such that the quotient differential system $\mathcal{J}/\varphi^*\mathcal{I}$ is a completely integrable Pfaffian system. This means that there are 1-forms $\vartheta^\ell \in \mathcal{J}$ such that $\mathcal{J}$ is algebraically generated by the forms $\{\,\vartheta^\ell\,\} \cup \varphi^*\mathcal{I}$ and

$$d\vartheta^\ell \equiv 0 \qquad \text{mod } \{\,\vartheta^\ell\,\} \cup \varphi^*\mathcal{I} \tag{5.61}$$

or, equivalently, modulo the 1-forms in $\mathcal{J}$ and the 2-forms in $\varphi^*\mathcal{I}$. Under these conditions the map φ is guaranteed to be a local surjection from the integral manifolds of $\mathcal{J}$ to the integral elements of $\mathcal{I}$. Indeed, as argued in [5], let $P \subset M$ be an integral manifold of $\mathcal{I}$ defined in a neighborhood of $x \in M$. Choose a point $y \in N$ such that $\varphi(y) = x$. Then $\tilde{P} = \varphi^{-1}(P)$ is a submanifold on N containing y and the restriction of $\mathcal{J}$ to $\tilde{P}$ is a completely integrable Pfaffian system $\tilde{\mathcal{J}}$. By the Frobenius theorem (applied to $\tilde{\mathcal{J}}$ as a Pfaffian system on $\tilde{P}$) there is locally a unique integral manifold Q for $\tilde{\mathcal{J}}$ through y. The manifold Q is then an integral manifold of $\mathcal{J}$ which projects by φ to the original integral manifold P on some open neighborhood of x.

Corollary 5.11. *Let $\{\hat{V}, \check{V}\}$ be a Darboux pair on M. Then the EDS $\mathcal{W}_1 + \mathcal{W}_2$ on $M_1 \times M_2$ is an integrable extension of $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ on M with respect to the superposition formula $\Sigma\colon M_1 \times M_2 \to M$.*

Proof. In view of (5.56) and (5.58) we may take the *differential ideal* generated by $\mathcal{F} = \{\,\hat{\boldsymbol{\theta}}_1\}$ as a complement to $\Sigma^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}})$ in $\mathcal{W}_1 + \mathcal{W}_2$. The 1-forms $\hat{\boldsymbol{\theta}}_1$ are closed modulo $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\eta}}_1$ and $\boldsymbol{G}_1\check{\boldsymbol{\sigma}}_1 \wedge \check{\boldsymbol{\sigma}}_1$ and therefore the generators for $\mathcal{F}$ are closed, modulo $\mathcal{F}$ and modulo the 1-forms and 2-forms in $\Sigma^*(\hat{\mathcal{V}} \Cap \check{\mathcal{V}})$. Hence $\mathcal{W}_1 + \mathcal{W}_2$ is an integral extension of $\hat{\mathcal{V}} \Cap \check{\mathcal{V}}$. ∎

Corollary 5.12. *Let $\mathcal{I}$ be a decomposable, Darboux integrable Pfaffian system. Then the map $\Sigma\colon M_1 \times M_2 \to M$ is a superposition formula for $\mathcal{I}$ with respect to the EDS $\mathcal{W}_1 + \mathcal{W}_2$ which is locally surjective on integral manifolds.*

Proof. If $\{\,\hat{V}, \check{V}\,\}$ is the Darboux pair for $\mathcal{I}$, then $\mathcal{I} = \hat{\mathcal{V}} \Cap \check{\mathcal{V}}$ and the corollary follows from Theorem 5.10 and Corollary 5.11. ∎

5.3 Superposition Formulas and Symmetry Reduction of Differential Systems

In this section we shall use the superposition formula established in Theorem 5.10 to prove that if $\mathcal{I}$ is any decomposable differential system for which the singular Pfaffian systems $\{\hat{V}, \check{V}\}$ form a Darboux pair, then $\mathcal{I}$ can be realized as a quotient differential system, with respect to its Vessiot group, by the construction given in Corollary 3.4.

To precisely formulate this result, it is useful to first summarize the essential results of Sections 4 and 5.1–5.2. We have constructed, through the coframe adaptation of Section 4, a local Lie group G and local right and left group actions $\hat{\mu}, \check{\mu} : G \times M \to M$. For the sake of simplicity, let us suppose that G is a Lie group and that these actions are globally defined. The infinitesimal generators for these actions are the vector fields $\hat{X}_i$ and $\check{X}_i$, defined by the duals of the 5-adapted coframes.

[i] The actions $\hat{\mu}$ and $\check{\mu}$ are free actions with the same orbits (the vector fields $\hat{X}_i$ and $\check{X}_i$ are pointwise independent and related by $\hat{X}_i = \lambda_i^j \check{X}_j$).

[ii] The actions $\hat{\mu}$ and $\check{\mu}$ commute (see (5.27)).

[iii] The actions $\hat{\mu}$ and $\check{\mu}$ are symmetry groups of $\hat{V}$ and $\check{V}$, respectively (see (5.28)).

[iv] Each integral manifold of $\hat{V}^\infty$ or $\check{V}^\infty$ is fixed by both actions $\hat{\mu}$ and $\check{\mu}$ (see (5.29)).

[v] We have defined $\iota_1 \colon M_1 \to M$ and $\iota_2 \colon M_2 \to M$ to be fixed integral manifolds of $\hat{V}^\infty$ and $\check{V}^\infty$ and W_1 and W_2 to be the restrictions of $\hat{V}$ and $\check{V}$ to these integral manifolds.

[vi] Properties **[i]** – **[iv]** imply that the actions $\hat{\mu}$ and $\check{\mu}$ restrict to actions on $\hat{\mu}_i$ and $\check{\mu}_i$ on M_i. These restricted actions are free, $\hat{\mu}_1$ and $\check{\mu}_2$ are symmetries of W_1 and W_2 respectively. The actions $\hat{\mu}_1$ and $\check{\mu}_2$ are transverse to W_1 and W_2 (see (5.25)).

The diagonal action $\delta \colon G \times (M_1 \times M_2) \to M_1 \times M_2$ is defined as the left action

$$\delta(h, (x_1, x_2)) = \big(\hat{\mu}_1(h^{-1}, x_1),\, \check{\mu}_2(h, x_2)\big). \tag{5.62}$$

Note that the infinitesimal generators for δ are

$$Z_i = -\hat{X}_{1i} + \check{X}_{2i}. \tag{5.63}$$

Granted that the action δ is regular, we then have that all the hypothesis of Corollary 3.4 are satisfied. We can therefore construct the quotient manifold $\mathbf{q}\colon M_1 \times M_2 \to (M_1 \times M_2)/G$, the quotient differential system $\mathcal{J} = (\mathcal{W}_1 + \mathcal{W}_2)/G$ and the Darboux pairs $\{\,\hat{U}, \check{U}\}$ (see (3.17)).

We use the superposition formula (5.39) to identify the manifold M with $(M_1 \times M_2)/G$ and the original differential system $\mathcal{I}$ with the quotient system $\mathcal{J}$.

Theorem 5.13. *Let $\mathcal{I}$ be a decomposable differential system on M whose singular Pfaffian systems $\{\,\hat{V},\, \check{V}\,\}$ define a Darboux pair. Let G be the Vessiot group for $\{\,\hat{V}, \check{V}\,\}$ and define Pfaffian systems W_1 on M_1 and W_2 on M_2 as above. Then the manifold M can be identified as the quotient of $M_1 \times M_2$ by the diagonal action δ of the Vessiot group G, the superposition formula Σ is the quotient map, and $\mathcal{I} = (\mathcal{W}_1 + \mathcal{W}_2)/G$.*

All the manifolds, actions, and differential systems appearing in Theorem 5.13 are presented in the following diagram:

Proof of Theorem 5.13. The following elementary facts are needed.

[i] The superposition map $\Sigma : M_1 \times M_2 \to M$ is invariant with respect to the diagonal action δ of the Vessiot group G on $M_1 \times M_2$.

[ii] For each point $x = (x_1, x_2) \in M_1 \times M_2$, $\ker \Sigma_*(x) = \ker \mathbf{q}_*(x)$.

[iii] $\Sigma^*(\hat{V}) = [W_1 \oplus \Lambda^1(M_2)]_{\mathbf{sb}}$ and $\Sigma^*(\check{V}) = [\Lambda^1(M_1) \oplus W_2]_{\mathbf{sb}}$.

Facts **[i]** and **[ii]** show that $\Sigma\colon M_1 \times M_2 \to M$ can be identified with $\mathbf{q}\colon M_1 \times M_2 \to (M_1 \times M_2)/G$. Fact **[iii]** shows that $\hat{\mathcal{V}} = \hat{\mathcal{U}}$ and $\check{\mathcal{V}} = \check{\mathcal{U}}$ (in the notation of Corollary 3.4) so that $\mathcal{I} = \hat{\mathcal{V}} \Cap \check{\mathcal{V}} = \hat{\mathcal{U}} \Cap \check{\mathcal{U}} = \mathcal{J}$.

To prove **[i]**, let $x_1 \in M_1$, $x_2 \in M_2$ and $h \in G$. If $\Psi(x_1) = (s_1, g_1)$ and $\Psi(x_2) = (s_2, g_2)$ then it is a simple matter to check, using the G bi-equivariance of ρ, that $\hat{\mu}_1(h^{-1}, x_1) = (s_1, g_1 \cdot h^{-1})$ and $\check{\mu}_2(h, x_2) = (s_2, h \cdot g_2)$. The invariance of Σ then follows immediately from its definition (5.39).

Lemma 5.9 and (5.63) show that

$$\ker \Sigma_* = \operatorname{span}\{\partial_{\hat{\theta}_1} - \partial_{\check{\theta}_2}\} = \operatorname{span}\{-\hat{X}_{1i} + \check{X}_{2i}\} = \operatorname{span}\{\, Z_i\,\} \tag{5.64}$$

which proves [**ii**].

Lemma 5.9 and equations (5.18) show that

$$\Sigma^*(\hat{V}) = \Sigma^*(\{\,\hat{\boldsymbol{\theta}},\, \hat{\boldsymbol{\sigma}},\, \check{\boldsymbol{\eta}},\, \hat{\boldsymbol{\eta}}\,\} = \{\,\hat{\boldsymbol{\theta}}_1 + \check{\boldsymbol{\theta}}_2,\, \hat{\boldsymbol{\sigma}}_2,\, \hat{\boldsymbol{\eta}}_2,\, \check{\boldsymbol{\eta}}_1\,\} \tag{5.65}$$

while (5.54) and (5.55) give

$$W_1 \oplus \Lambda^1(M_2) = \{\,\hat{\boldsymbol{\theta}}_1,\, \check{\boldsymbol{\eta}}_1,\, \check{\boldsymbol{\theta}}_2,\, \hat{\boldsymbol{\eta}}_2,\, \hat{\boldsymbol{\sigma}}_2\,\} \tag{5.66}$$

in which case [**iii**] follows immediately from (5.64).

To make precise the identification of M with the quotient manifold $\overline{M} = (M_1 \times M_2)/G$, define a map $\Upsilon \colon \overline{M} \to M$ as follows. For each point $\bar{x} \in \overline{M}$, pick a point $(x_1, x_2) \in M_1 \times M_2$ such that $\mathbf{q}(x_1, x_2) = \bar{x}$ and let $\Upsilon(\bar{x}) = \Sigma(x_1, x_2)$.

The diagram

$M_1 \times M_2$ Error: Incorrect label spe

$\overline{M}$

commutes and, on the domain of any (local) cross-section ζ of $\mathbf{q}$, one has $\Upsilon = \Sigma \circ \zeta$. This observation and facts **[i]** and **[ii]** then suffice to show that Υ is a well-defined, smooth diffeomorphism. Moreover, the same computations used to establish fact **[i]** show that Σ, and hence Υ, is G equivariant with respect to the action $\check{\mu}_1$ on $M_1 \times M_2$ and $\check{\mu}$ on M and also G equivariant with respect to the action $\hat{\mu}_2$ on $M_1 \times M_2$ and $\hat{\mu}$ on M.

Finally, we recall that $\hat{U}$ and $\check{U}$ may be calculated from the cross-section ζ

and the δ semi-basis forms by

$$\hat{U} = \zeta^*([W_1 \oplus T^*(M_2)]_{\mathbf{sb}}) \quad \text{and} \quad \check{U} = \zeta^*(T^*(M_1) \oplus W_2]_{\mathbf{sb}}) \tag{5.67}$$

in which case **[iii]** implies that $\Upsilon^*(\hat{\mathcal{V}}) = \hat{\mathcal{U}}$ and $\Upsilon^*(\check{\mathcal{V}}) = \check{\mathcal{U}}$. ∎

Remark 5.14. Finally we remark that all the results of this section remain valid in so long as the singular systems for $\mathcal{I}$ have algebraic generators

$$\hat{\mathcal{V}} = \{\, \boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \hat{\boldsymbol{\sigma}},\ \boldsymbol{P}\,\check{\boldsymbol{\sigma}} \wedge \check{\boldsymbol{\sigma}}\,\} \quad \text{and} \quad \check{\mathcal{V}} = \{\, \boldsymbol{\theta},\ \hat{\boldsymbol{\eta}},\ \check{\boldsymbol{\eta}},\ \check{\boldsymbol{\sigma}},\ \boldsymbol{Q}\,\hat{\boldsymbol{\sigma}} \wedge \hat{\boldsymbol{\sigma}},\,\}. \tag{5.68}$$

where $\boldsymbol{P} \in \operatorname{Int}(\check{V})$ and $\boldsymbol{Q} \in \operatorname{Int}(\hat{V})$. Such systems need not be Pfaffian.

6 Examples

In this section we illustrate our general theory with a variety of examples. Examples 6.1 and 6.2 are taken from the classical literature and are simple enough that most of the computations can be explicitly given. We consider PDE where the unknown functions take values in a group or in a non-commutative algebra in Examples 6.3 and 6.4. In Example 6.5 we present some novel examples of Darboux integrable systems constructed by the coupling of a nonlinear, Darboux integrable scalar equation to a linear or Moutard-type equation. A Toda lattice system and a wave map system are explicitly integrated in Examples 6.6 and 6.7. In Example 6.8, we solve some non-linear, over-determined systems in 3 independent variables.

Example 6.1. As our first example we shall find the closed-form, general solution to

$$u_{xy} = \frac{u_x u_y}{u - x}. \tag{6.1}$$

This example is taken from Goursat's well-known classification (Equation VI) of Darboux integrable equations [16] and is simple enough that all the steps leading to the solution can be explicitly given. See also Vessiot [28] (pages 9–22) or Stomark [24] (pages 350–356).

The canonical Pfaffian system for (6.1) is $I = \{\ \alpha^1, \alpha^2, \alpha^3\ \}$, where

$$\alpha^1 = du - p\,dx - q\,dy, \quad \alpha^2 = dp - r\,dx - vpq\,dy, \quad \alpha^3 = dq - vpq\,dx - t\,dy$$

and $v = 1/(u - x)$. The associated singular Pfaffian systems are

$$\hat{V} = \{\,\alpha^i,\ dx,\ dr - q(vr + v^2p)\,dy\,\} \quad \text{and} \quad \check{V} = \{\,\alpha^i,\ dy,\ dt - tpv\,dy\,\}. \tag{6.2}$$

The first integrals for $\hat{V}$ and $\check{V}$ are

$$\hat{I}^1 = x, \quad \hat{I}^2 = vp, \quad \hat{I}^3 = vr + v^2p, \quad \check{I}^1 = y, \quad \check{I}^2 = \frac{t}{q} - y \tag{6.3}$$

and we easily calculate that $\hat{V}^{(\infty)} \cap \check{V} = \{\,\hat{\eta}^1\,\}$ and $\hat{V} \cap \check{V}^{(\infty)} = \{\,0\,\}$, where

$$\hat{\eta}^1 = d\hat{I}^2 + ((\hat{I}^2)^2 - \hat{I}^3)\,d\hat{I}^1 = v\alpha^2 - v^2p\alpha^1. \tag{6.4}$$

A 1-adapted coframe (2.6) is therefore given by

$$\theta^1 = \alpha^1,\ \theta^2 = \alpha^3,\ \hat{\sigma}^1 = d\hat{I}^1,\ \hat{\sigma}^2 = d\hat{I}^3,\ \hat{\eta}^1,\ \check{\sigma}^1 = d\check{I}^1,\ \check{\sigma}^2 = d\check{I}^2. \tag{6.5}$$

We relabel the coframes elements by $\hat{\pi}^1 = \sigma^1$, $\hat{\pi}^2 = \sigma^2$, $\hat{\pi}^3 = \hat{\eta}^1$, $\check{\pi}^1 = \check{\sigma}^1$, $\check{\pi}^2 = \check{\sigma}^2$ and calculate

$$\begin{aligned} d\hat{\pi}^3 &= -2\,\hat{I}^2\,\hat{\pi}^1 \wedge \hat{\pi}^3 + \hat{\pi}^1 \wedge \hat{\pi}^2, \\ d\theta^1 &= (u-x)\,\hat{\pi}^1 \wedge \hat{\pi}^3 + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^1 + \check{\pi}^1 \wedge \theta^2, \\ d\theta^2 &= \ q\,\hat{\pi}^1 \wedge \hat{\pi}^3 + q\,\check{\pi}^1 \wedge \check{\pi}^2 + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^2 + \check{I}^2\,\check{\pi}^1 \wedge \theta^2. \end{aligned} \tag{6.6}$$

This coframe satisfies the structure equations (4.13) and is therefore 2-adapted.

The next step, described in Section 4.2, is to eliminate the $\check{\pi}^\alpha \wedge \theta^i$ terms from (6.6). We calculate the distributions $\hat{U}$ and $\check{U}$ (see (4.15)) and their derived flags to be

$$\begin{aligned} \hat{U} &= \{\partial_{\hat{\pi}^1} + ((\hat{I}^2)^2 - \check{I}^3)\partial_{\hat{\pi}^3},\ \partial_{\hat{\pi}^3},\ \partial_{\hat{\pi}^2}\}, \qquad \check{U} = \{\partial_{\check{\pi}^1},\ \partial_{\check{\pi}^2}\}, \\ \hat{U}^{(\infty)} &= \hat{U} \cup \{\,(u-x)\,\partial_{\theta^1} + q\partial_{\theta^2},\ \partial_{\theta^1}\}, \qquad \check{U}^{(\infty)} = \check{U} \cup \{\,q\,\partial_{\theta^2},\ q\,\partial_{\theta^1}\}. \end{aligned} \tag{6.7}$$

and then, in accordance with (4.19)-(4.21), define

$$X_1 = (u-x)\,\partial_{\theta^1} + q\partial_{\theta^2}, \quad X_2 = \partial_{\theta^1}, \quad Y_1 = q\,\partial_{\theta^2}, \quad Y_2 = \partial_{\theta^1}, \tag{6.8}$$

The coframes dual to the vector fields $\{\,X_i, \hat{U}, \check{U}\,\}$ and $\{\,Y_i, \hat{U}, \check{U}\,\}$ are the 3-adapted coframes

$$\theta^1_X = \frac{1}{q}\theta^2, \quad \theta^2_X = \theta^1 - \frac{u-x}{q}\theta^2, \quad \theta^1_Y = \frac{1}{q}\theta^2, \quad \theta^2_Y = \frac{1}{q}\theta^1 \quad \text{with} \tag{6.9}$$

$$\begin{aligned} d\theta^1_X &= \hat{\pi}^1 \wedge \hat{\pi}^3 + \check{\pi}^1 \wedge \check{\pi}^2 = d\theta^1_Y, \\ d\theta^2_X &= -(u-x)\,\check{\pi}^1 \wedge \check{\pi}^2 + \theta^1_X \wedge \theta^2_X + \hat{\pi}^1 \wedge \theta^1_X + \hat{I}^2\,\hat{\pi}^1 \wedge \theta^2_X, \\ d\theta^2_Y &= \frac{u-x}{q}\,\hat{\pi}^1 \wedge \hat{\pi}^3 - \theta^1_Y \wedge \theta^2_Y + \check{\pi}^1 \wedge \theta^1_Y - \check{I}^2\,\check{\pi}^1 \wedge \theta^2_Y. \end{aligned} \tag{6.10}$$

The coframe (6.9) is in fact 4-adapted and hence the Vessiot algebra for (6.1) is a 2 dimensional non-abelian Lie algebra.

We may skip the adaptations given in Section 4.3 and move on to the final adaptations given in Section 4.4. The Vessiot algebra is 1-step solvable and the structure equations (6.10) are precisely of the form (4.68). The change of coframe $\hat{\theta}^1 = \theta^1_X + \hat{I}^2\pi^1$ transforms the structure equations (6.10) to the form (4.70). The change of coframe $\hat{\theta}^2 = \theta^2_X - x\hat{\theta}^1$ leads to the $\hat{5}$-adapted coframe $\{\hat{\theta}^1, \hat{\theta}^2\}$ with structure equations

$$d\hat{\theta}^1 = \check{\pi}^1 \wedge \check{\pi}^2, \quad d\hat{\theta}^2 = -u\,\check{\pi}^1 \wedge \check{\pi}^2 + \hat{\theta}^1 \wedge \hat{\theta}^2. \tag{6.11}$$

Similarly, the $\check{5}$-adapted coframe $\check{\theta}^1 = \theta^1_Y + \check{I}^2\check{\pi}^1$, $\check{\theta}^2 = \theta^2_Y - y\check{\theta}^1$ satisfies

$$d\check{\theta}^1 = \hat{\pi}^1 \wedge \hat{\pi}^3, \quad d\check{\theta}^2 = \frac{u - x - yq}{q}\,\hat{\pi}^1 \wedge \hat{\pi}^3 - \check{\theta}^1 \wedge \check{\theta}^2. \tag{6.12}$$

Before continuing we remark that the vector fields X_1, X_2, defined by (6.8), are given in terms of the dual vector fields $\hat{X}_1$ and $\hat{X}_2$, computed from the $\hat{5}$-adapted coframe, by $X_1 = \hat{X}_1 - x\hat{X}_2$ and $X_2 = \hat{X}_2$. These vector field systems have the same orbits and structure equations but the actions are evidently different and it is the latter action that is needed to properly construct the superposition formula.

The forms (5.30) are

$$\begin{aligned} \hat{\omega}^1 &= \hat{\theta}^1 + \check{I}^2\,\check{\pi}^1 = \frac{dq}{q}, \quad \hat{\omega}^2 = \hat{\theta}^2 - u\check{I}^2\,\check{\pi}^1 = du - q dy - u\frac{dq}{q}, \\ \check{\omega}^1 &= \check{\theta}^1 + \hat{I}^2\,\hat{\pi}^1 = \frac{dq}{q}, \quad \check{\omega}^2 = \check{\theta}^2 + (\frac{p}{q} + yvp)\,\hat{\pi}^1 = \frac{du}{q} - dy - y\frac{dq}{q}. \end{aligned} \tag{6.13}$$

The Vessiot group for (6.1) is the matrix group $\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}$ with Maurer-Cartan forms

$$\omega^1_L = \frac{da}{a},\ \omega^1_L = db - \frac{b}{a}da,\ \omega^1_R = \frac{da}{a},\ \omega^2_R = \frac{db}{a}.$$

The map $\rho\colon M \to G$ defined by $a = q$ and $b = u - yq$ satisfies (5.4).

Finally, if we introduce coordinates $y_1 = y$, $u_1 = u$, $q_1 = q$, $t_1 = t$ on the $\hat{V}^{(\infty)}$ integral manifold $M_1 = \{\, \hat{I}^1 = 0,\ \hat{I}^2 = 0,\ \hat{I}^3 = 0 \,\}$ and $x_2 = x$, $u_2 = u$, $p_2 = p$, $q_2 = u$, $r_2 = r$ on the $\check{V}^{(\infty)}$ integral manifold $M_2 = \{\, \check{I}^1 = 0, \check{I}^2 = 0 \,\}$, then the superposition formula (5.60) is

$$\begin{gathered} x = x_2, \quad \frac{p}{u - x} = \frac{p_2}{u_2 - x_2}, \quad \frac{r}{u - x} + \frac{p}{(u - x)^2} = \frac{r_2}{u_2 - x_2} + \frac{p_2}{(u_2 - x_2)^2} \\ y = y_1, \quad \frac{t}{q} = \frac{t_1}{q_1}, \quad q = q_1 q_2, \quad u - yq = u_2 + (u_1 - y_1 q_1) q_2, \end{gathered}$$

or, explicitly in terms of the original coordinates $\{x, y, u, p, q, r, t\}$ on M,

$$\begin{gathered} x = x_2, \quad y = y_1, \quad u = u_2 + q_2 u_1, \quad p = (1 + \frac{u_1 q_2}{u_2 - x_2})p_2, \quad q = q_1 q_2, \\ r = (1 + \frac{u_1 q_2}{u_2 - x_2})r_2 + \frac{u_1 p_2 q_2}{(u_2 - x_2)^2}, \quad t = t_1 q_2. \end{gathered} \tag{6.14}$$

It remains to find the integral manifolds for $\hat{W}$ and $\check{W}$. Restricted to M_1, the Pfaffian system $\hat{V}$ becomes $\hat{W} = \{\, du_1 - q_1\, dy_1, dq_1 - t_1 dy_1 \,\}$ with integral manifolds

$$y_1 = \beta,\ u_1 = f(\beta),\ q_1 = f'(\beta),\ t_1 = f''(\beta). \tag{6.15}$$

Restricted to M_2, the Pfaffian system $\check{V}$ becomes $\check{W} = \{\, du_2 - p_2 dx_2,\ dp_2 - r_2 dx_2,\ dq_2 - \dfrac{p_2 q_2}{u_2 - x_2} dx_2 \}$. To find the integral manifolds of $\check{W}$ we calculate the second derived Pfaffian system to be $\check{W}^{(2)} = \{\, dq_2 - \dfrac{q_2}{u_2 - x_2}\, du_2 \,\}$ which leads to the equation $q_2\, du_2 - u_2\, dq_2 + x_2\, dq_2 = 0$ or

$$d\left(\frac{u_2}{q_2}\right) - x_2\, d\left(\frac{1}{q_2}\right) = 0 \quad \text{or} \quad d\left(\frac{u_2 - x_2}{q_2}\right) + \frac{1}{q_2}\, dx_2 = 0.$$

The integral manifolds for $\check{W}$ are therefore given by

$$x_2 = \alpha, \quad u_2 = x_2 - g(\alpha)/g'(\alpha), \quad q_2 = -1/g'(\alpha), \tag{6.16}$$

with p_2 and r_2 determined algebraically from the vanishing of the first and second forms in $\check{W}$. The substitution of (6.15) and (6.16) into the superposition formula (6.14) leads to the closed form general solution

$$u = \frac{-f(y) - g(x)}{g'(x)} + x \tag{6.17}$$

for (6.1).

Example 6.2. In this example we shall construct the superposition formula for the Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3\}$, where

$$\begin{aligned} &\alpha^1 = du - p\,dx - q\,dy, \quad \alpha^2 = dp + \frac{1}{b^3}\,(\tan b\tau - b\tau)\,dx - s\,dy, \\ &\alpha^3 = dq - s\,dx - b(b\tau + \cot b\tau)\,dy. \end{aligned} \tag{6.18}$$

The coordinates for this example are (x, y, u, p, q, s, τ) and b is a parameter. This example nicely illustrates the various coframe adaptations in Sections 4 and has a surprising connection with some of Cartan's results in the celebrated 5 variables paper [7]. The values $b = 1$, $b = \sqrt{-1}$ (the Pffaffian system (6.18) remains real) and the limiting value $b = 0$ give the three Pfaffian systems for the equations

$$u_{xx} = f(u_{yy})$$

which are Darboux integrable at the 2-jet level ([2], pages 373-374) and [4], pages 400–411). The case $b = 0$ is treated in Goursat([15] Vol 2, page 130-132.

One easily calculates $\hat{V}^\infty \cap \check{V} = \hat{V} \cap \check{V}^\infty = \{\,0\,\}$ and that the first integrals for $\hat{V}$ and $\check{V}$ are

$$\begin{aligned} &\hat{I}^{\,1} = s + \tau, \quad \hat{I}^{\,2} = -(x + b^2 y)\hat{I}^{\,1} + q + b^2 p, \\ &\check{I}^{\,1} = s - \tau, \quad \check{I}^{\,2} = -(x - b^2 y)\check{I}^{\,1} + q - b^2 p. \end{aligned} \tag{6.19}$$

We immediately arrive at the 2-adapted coframe

$$\begin{aligned} &\hat{\pi}^1 = d\hat{I}^{\,1}, \quad \hat{\pi}^2 = d\hat{I}^{\,2}, \quad \check{\pi}^1 = d\hat{I}^{\,1}, \quad \check{\pi}^2 = d\check{I}^{\,2}, \quad \theta^1 = 2\alpha^1, \\ &\theta^2 = -b\cot b\tau\,\alpha^2 + \frac{1}{b}\tan b\tau\,\alpha^3, \quad \theta^3 = -b\cot b\tau\,\alpha^2 - \frac{1}{b}\tan b\tau\,\alpha^3, \end{aligned} \tag{6.20}$$

with structure equations

$$\begin{aligned} d\theta^1 &= (x + b^2 y)\,\hat{\pi}^1 \wedge \theta^2 + \hat{\pi}^2 \wedge \theta^2 - (x - b^2 y)\,\check{\pi}^1 \wedge \theta^3 - \check{\pi}^2 \wedge \theta^3, \\ d\theta^2 &= \hat{\pi}^1 \wedge \hat{\pi}^2 - b\cot 2b\tau\,\hat{\pi}^1 \wedge \theta^2 + b\csc 2b\tau\,\check{\pi}^1 \wedge \theta^3, \\ d\theta^3 &= \check{\pi}^1 \wedge \check{\pi}^2 - b\csc 2b\tau\,\hat{\pi}^1 \wedge \theta^2 + b\cot 2b\tau\,\check{\pi}^1 \wedge \theta^3. \end{aligned} \tag{6.21}$$

To compute the 3-adapted coframe θ^i_X, we simply calculate the derived flag for the 2 dimensional distribution $\hat{U} = \{\,\partial_{\hat{\pi}^1}, \partial_{\hat{\pi}^2}\,\}$ (see Section 4.2). From the structure equations (6.21) we find

$$\begin{aligned} &\hat{U}^{(1)} = \hat{U} \cup \{\,-\partial_{\theta^2}\,\} \quad \text{and} \\ &\hat{U}^{(2)} = \hat{U} \cup \{\,-\partial_{\theta^2}, (x + b^2 y)\,\partial_{\theta^1} - b\cot 2b\tau\,\partial_{\theta^2} - b\csc 2b\tau\,\partial_{\theta^3}, \partial_{\theta^1}\,\}. \end{aligned} \tag{6.22}$$

We take $\{ X_1, X_2, X_3 \}$ to be the last 3 vectors in $\hat{U}^{(2)}$ and calculate

$$[X_1, X_2] = b^2 X_3. \tag{6.23}$$

Therefore the Vessiot algebra for the Pfaffian system (6.18) is abelian if $b = 0$ and nilpotent otherwise. The 3-adapted coframe $\boldsymbol{\theta_X}$ (see Theorem 4.4) is

$$\theta_X^1 = -\theta^2 + \cos 2b\tau\, \theta^3,\ \theta_X^2 = -\frac{1}{b} \sin 2b\tau\, \theta^3,\ \theta_X^3 = \theta^1 + \frac{1}{b}(x + b^2 y) \sin 2b\tau\, \theta^3,$$

and the structure equations (4.22) are

$$\begin{aligned} d\theta_X^1 &= -\hat{\pi}^1 \wedge \hat{\pi}^2 + \cos 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 + b^2 \hat{\pi}^1 \wedge \theta_X^2, \\ d\theta_X^2 &= -\frac{1}{b} \sin 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 - \hat{\pi}^1 \wedge \theta_X^1, \\ d\theta_X^3 &= \frac{1}{b}(x + b^2 y)\, \sin 2b\tau\, \check{\pi}^1 \wedge \check{\pi}^2 - \hat{\pi}^2 \wedge \theta_X^1 - b^2 \theta_X^1 \wedge \theta_X^2. \end{aligned} \tag{6.24}$$

This coframe is actually 4-adapted.

We labeled the vectors in the derived flag (6.22) so that the last vector X_3 spans the derived algebra of the Vessiot algebra. By doing so we are assured that the structure equations (6.24) are of the precise form (4.68a)-(4.68b). The analysis at this point refers back to Case II and the structure equations (4.59), as applied to just the first two equations in (6.24). The matrices M and R (see equations (4.60) and (4.61)) are found to be

$$M = \begin{bmatrix} 0 & b^2 \hat{\pi}^1 \\ -\hat{\pi}^1 & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \cos b\hat{I}^{\,1} & -b \sin b\hat{I}^{\,1} \\ \frac{1}{b} \sin b\hat{I}^{\,1} & \cos b\hat{I}^{\,1} \end{bmatrix}. \tag{6.25}$$

We then compute the 2-forms χ^i and the 1-forms ϕ^i (see (4.63)) to be

$$\begin{aligned} \chi^1 &= -\cos b\hat{I}^{\,1}\, \hat{\pi}^1 \wedge \hat{\pi}^2, \quad \chi^2 = -\frac{1}{b} \sin b\hat{I}^{\,1}\, \hat{\pi}^1 \wedge \hat{\pi}^2, \\ \phi^1 &= -\hat{I}^{\,2} \cos b\hat{I}^{\,1}\, \hat{\pi}^1, \quad \phi^2 = -\frac{1}{b} \hat{I}^{\,2} \sin b\hat{I}^{\,1}\, \hat{\pi}^1, \end{aligned} \tag{6.26}$$

so that the forms (4.69) are

$$\begin{aligned} \hat{\theta}_1^1 &= \ \cos b\hat{I}^{\,1}\, \theta_X^1 - b \sin b\hat{I}^{\,1}\, \theta_X^2 - \ \hat{I}^2 \cos b\hat{I}^{\,1}\, \hat{\pi}^1, \\ \hat{\theta}_1^2 &= \frac{1}{b} \sin b\hat{I}^{\,1}\, \theta_X^1 + \ \cos b\hat{I}^{\,1} \theta_X^2 - \frac{1}{b} \hat{I}^{\,2} \sin b\hat{I}^{\,1}\, \hat{\pi}^1. \end{aligned} \tag{6.27}$$

The structure equations (6.24) are now reduced to the form (4.70). The final required frame change (4.74) leads to the $\hat{5}$-adapted coframe

$$\hat{\theta}^1 = \hat{\theta}_1^1, \quad \hat{\theta}^2 = \hat{\theta}_1^2, \quad \hat{\theta}^3 = \theta_X^3 + \hat{I}^{\,2} \cos b\hat{I}^{\,1}\, \hat{\theta}_1^1 + b\hat{I}^{\,2} \sin b\hat{I}^1\, \theta_1^2 + \frac{1}{2}(\hat{I}^{\,2})^2\, \hat{\pi}^2 \tag{6.28}$$

with structure equations

$$d\hat{\theta}^1 = \cos b\check{I}^{\,1}\,\check{\pi}^1 \wedge \check{\pi}^2, \quad d\hat{\theta}^2 = \frac{1}{b}\sin b\check{I}^{\,1}\,\check{\pi}^1 \wedge \check{\pi}^2,$$
$$d\hat{\theta}^3 = \frac{1}{b}\big((x + b^2 y)\sin 2b\tau + b\hat{I}^{\,2}\cos 2b\tau\big)\,\check{\pi}^1 \wedge \check{\pi}^2 - b^2\hat{\theta}^1 \wedge \hat{\theta}^2. \tag{6.29}$$

The diffeomorphism $\Phi(x, y, u, p, q, s, \tau) = (x, -y, u, p, -q, -s, \tau)$ is an involution for (6.18) in the sense of Remark 2.13 and can therefore be used to find the $\check{5}$ adapted coframe.

The forms (5.30) are

$$\hat{\omega}^1 = \hat{\theta}^1 + \check{I}^{\,2}\cos b\check{I}^{\,1}\check{\pi}^1, \quad \hat{\omega}^2 = \hat{\theta}^2 + \check{I}^{\,2}\frac{1}{b}\sin b\check{I}^{\,1}\check{\pi}^1, \quad \hat{\omega}^3 = \hat{\theta}^3 + A\check{\pi}^1, \tag{6.30}$$

where $A = \check{I}^2(\hat{I}^{\,2}\cos 2b\tau + \frac{1}{b}(x + b^2 y)\sin 2b\tau - \frac{1}{2}\check{I}^{\,2})$.

The Vessiot group for this example is the matrix group

$$\begin{bmatrix} 1 & z^1 & z^3 + b^2 z^1 z^2/2 \\ 0 & 1 & b^2 z^2 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.31}$$

with multiplication

$$z^1 = z^1_1 + z^1_2, \quad z^2 = z^2_1 + z^2_2, \quad z^3 = z^3_1 + z^3_2 - \frac{b^2}{2}(z^1_1 z^2_2 - z^2_1 z^1_2) \tag{6.32}$$

and left invariant forms

$$\hat{\omega}^1 = dz^1, \quad \hat{\omega}^2 = dz^2, \quad \hat{\omega}^3 = dz^3 + \frac{b^2}{2}(z^1 dz^2 - z^2 dz^1). \tag{6.33}$$

The map ρ (Theorem 5.1) is then found to be

$$z^1 = \frac{1}{b^2}\,(-2x\sin bs\sin b\tau + 2b^2 y\cos bs\cos b\tau - b\hat{I}^{\,2}\sin b\hat{I}^{\,1} + b\check{I}^{\,2}\sin b\check{I}^{\,1}),$$
$$z^2 = \frac{1}{b^3}\,(\ 2bx\cos bs\sin b\tau + 2yb^2\sin bs\cos b\tau\ + \hat{I}^{\,2}b\cos b\hat{I}^{\,1} - \check{I}^{\,2}b\cos\check{I}^{\,1}),$$
$$z^3 = 2u - \frac{1}{2b^3}(\sin 2b\tau - 2b\tau\cos 2b\tau)(x^2 - b^4 y^2) - 2px\cos^2 2b\tau$$
$$- 2qy\sin^2 2b\tau - \frac{1}{2b}\hat{I}^2\check{I}^2\sin 2b\tau. \tag{6.34}$$

Finally, the combination of equations (6.19), (6.32) and (6.34) leads to the superposition formula

$$\begin{aligned}
x &= \frac{1}{2}(x_1 + x_2 + b^2(y_2 - y_1)) - \frac{\sin b(\tau_1 - \tau_2)}{\sin b(\tau_1 + \tau_2)}\xi,\\
y &= \frac{1}{2b^2}(x_2 - x_1 + b^2(y_1 + y_2)) - \frac{\cos b(\tau_1 - \tau_2)}{b^2 \cos b(\tau_1 + \tau_2)}\xi,\\
u &= u_1 + u_2 + 2\,\frac{p_1 \sin 2b\tau_2 - p_2 \sin 2b\tau_1)}{\sin 2b(\tau_1 + \tau_2}\,\xi\\
&\quad + \frac{1}{b^2}\left(\frac{2\tau_1 \sin^2(2b\tau_2)}{\sin^2(2b(\tau_1+\tau_2))} + \frac{2\tau_2 \sin^2(2b\tau_1)}{\sin^2(2b(\tau_1+\tau_2))} - \frac{\sin 2b\tau_1 \sin 2b\tau_2}{b \sin 2b(\tau_1+\tau_2)}\right)\xi^2, \qquad (6.35)\\
p &= p_1 + p_2 + 2\,\frac{\tau_1 \sin 2b\tau_2 - \tau_2 \sin 2b\tau_1}{b^2 \sin 2b(\tau_1 + \tau_2)}\,\xi,\\
q &= -b^2 p_1 + b^2 p_2 - 2\frac{\tau_1 \sin 2b\tau_2 + \tau_2 \sin 2b\tau_1}{\sin 2b(\tau_1 + \tau_2)}\,\xi,\\
s &= -\tau_1 + \tau_2, \quad \tau = \tau_1 + \tau_2, \quad \text{where} \quad \xi = \frac{1}{2}(x_2 - x_1 - b^2(y_1 + y_2)).
\end{aligned}$$

The restriction of $\hat{V}$ to the manifold $M_1 = \{\, \hat{I}^1 = \hat{I}^2 = 0 \,\}$ gives

$$\begin{aligned}
\hat{W} = \{\, & du_1 - p_1\, dx_1 + b^2\, dy_1,\ dp_1 - 1/b^3(\tan b\tau_1 - b\tau_1)dx_1 + \tau_1 dy_1, && (6.36)\\
& \tau_1\, dx_1 - b(b\tau_1 + \cot(b\tau_1))\, dy_1 - b^2 dp_1 \,\}. && (6.37)
\end{aligned}$$

This is a rank 3 Pfaffian system on a 5 manifold whose derived flag has dimensions $[3, 2, 0]$. The equivalence problem for such systems was analyzed in detail by Cartan [7], where it is established that the fundamental invariant for such systems is a certain rank 4 symmetric tensor T in two variables. For $b = 0$, this tensor vanishes while for $b \neq 0$ we find that T is the 4-th symmetric power of a 1-form. In accordance with Cartan's result the symmetry algebra of $\hat{W}$ when $b = 0$ is the 14 dimensional exceptional Lie algebra g_2 and, indeed, it is not difficult to transform $\hat{W}$ to the canonical Pfaffian system for the Hilbert-Cartan equation $z' = (y'')^2$. For $b \neq 0$ the symmetry algebra of $\hat{W}$ is the 7 dimensional solvable Lie algebra with infinitesimal generators

$$\{\, \partial_{x_1}, \partial_{y_1}, \partial_{u_1}, x_1\partial_{x_1} + y_1\partial_{y_1} + 2u_1\partial_{u_1} + p_1\partial_{p_1}, (x_1 - b^2 y_1)\partial_{u_1} - \partial_{p_1}, Y_1, Y_2 \,\}, \qquad (6.38)$$

where $Y_2 = [\partial_{y_1}, Y_1]$ and

$$\begin{aligned}
Y_1 =& (x_1 + b^2 y_1)\big(b \cot b\tau_1 \partial_{x_1} - \frac{1}{b}\tan b\tau_1 \partial_{y_1} + 2p_1 b \csc b\tau_1 \partial_{u_1} + 2\tau_1 \csc 2b\tau_1 \partial_{p_1}\big)\\
&+ 2x_1 y_1 \partial_{u_1} + (y_1 - \frac{x_1}{b^2})\partial_{p_1} - \partial_{\tau_1}. \qquad (6.39)
\end{aligned}$$

The Pfaffian (6.37) with $b \neq 0$ may be transformed into Cartan [7], page 170, equation (5').

Example 6.3. For our next example, let G be an n-parameter matrix group and, for the mapping $(x, y) \to U(x, y) \in G$, consider the system of differential equations

$$U_{xy} = U_x U^{-1} U_y. \tag{6.40}$$

The general solution to these equations is well-known to be $U(x, y) = A(x)B(y)$, with $A(x), B(y) \in G$. In the case when U is a 1×1 matrix, this system reduces to the wave equation $v_{xy} = 0$ under the change of variable $u = \exp(v)$. We show how our integration method leads directly to the general solution and, in the process, we calculate the Vessiot algebra of (6.40) to be the Lie algebra of G.

The Pfaffian system for (6.40) is $I = \{\Theta, \Theta^1, \Theta^2\}$, where

$$\begin{aligned}
\Theta &= dU - U_x\,dx - U_y\,dy,\\
\Theta^1 &= dU_x - U_{xx}\,dx - U_x U^{-1} U_y\,dy,\\
\Theta^2 &= dU_y - U_x U^{-1} U_y\,dx - U_{yy}\,dy.
\end{aligned} \tag{6.41}$$

The first integrals for the singular systems are

$$\begin{aligned}
&\hat{I}^1 = y, \quad \hat{I}^2 = U^{-1}U_y, \quad \hat{I}^3 = D_y(\hat{I}^2) = U^{-1}U_{yy} - U^{-1}U_y U^{-1} U_y,\\
&\check{I}^1 = x, \quad \check{I}^2 = U_x U^{-1}, \quad \check{I}^3 = D_x(\check{I}^2) = U_{xx}U^{-1} - U_x U^{-1} U_x U^{-1},
\end{aligned} \tag{6.42}$$

and our 0-adapted coframe for I is $\{\Theta, d\hat{I}^1, d\hat{I}^3, \hat{\eta}, d\check{I}^1, d\check{I}^3, \check{\eta}\}$, where

$$\begin{aligned}
\hat{\eta} &= d\hat{I}^2 - \hat{I}^3 d\hat{I}^1 = U^{-1}\Theta^2 - U^{-1}\,\Theta\,\hat{I}^2, \quad \text{and}\\
\check{\eta} &= d\check{I}^2 - \check{I}^3 d\check{I}^1 = \Theta^1 U^{-1} - \check{I}^2\,\Theta\,U^{-1}.
\end{aligned}$$

The structure equations are

$$d\Theta = d\hat{I}^1 \wedge (U\,\hat{\eta} + \Theta\,\hat{I}^2) + d\check{I}^1 \wedge (\check{\eta}\,U + \check{I}^2\Theta). \tag{6.43}$$

This coframe satisfies (4.13) and is therefore 2-adapted.

The next step is to eliminate either the $d\check{I}^1 \wedge (\hat{I}^2\Theta)$ or the $d\hat{I}^1 \wedge (\Theta\,\check{I}^2)$ terms in (6.43). By inspection, we see that the forms $\Theta_X = U^{-1}\Theta$ and $\Theta_Y = \Theta\,U^{-1}$ provide us with the required 4-adapted coframes, with structure equations

$$\begin{aligned}
d\Theta_X &= d\hat{I}^1 \wedge \check{\eta} + d\check{I}^1 \wedge (U^{-1}\hat{\eta}\,U) - \Theta_X \wedge \Theta_X + d\hat{I}^1 \wedge (\Theta_X \hat{I}^2 - \hat{I}^2\Theta_X),\\
d\Theta_Y &= d\hat{I}^1 \wedge (U\,\check{\eta}\,U^{-1}) + d\check{I}^1 \wedge \check{\eta} + \Theta_Y \wedge \Theta_Y - d\check{I}^1 \wedge (\Theta_Y \check{I}^2 - \check{I}^2\Theta_Y).
\end{aligned} \tag{6.44}$$

Since the forms Θ_X and Θ_Y are Lie algebra valued, these structure equations show that the Vessiot algebra for (6.40) is the Lie algebra of G (Theorem 4.6).

The final coframe adaptation in Section 4.4 is given by

$$\begin{aligned} \hat{\Theta} &= \Theta_X + \hat{I}^2 d\hat{I}^1 = U^{-1}dU - U^{-1}U_x dx \quad \text{and} \\ \check{\Theta} &= \Theta_Y + \check{I}^2 d\check{I}^1 = dU\,U^{-1} - U_y\,U^{-1}dy, \end{aligned} \tag{6.45}$$

with structure equations

$$d\hat{\Theta} = d\check{I}^1 \wedge (U^{-1}\check{\eta}\,U) - \hat{\Theta}\wedge\hat{\Theta} \quad \text{and} \quad d\check{\Theta} = d\hat{I}^1 \wedge (U\;\hat{\eta}\;U^{-1}) + \check{\Theta}\wedge\check{\Theta}. \tag{6.46}$$

The form (5.30) are then found to be precisely the left and right invariant forms on G, that is,

$$\hat{\omega} = U^{-1}\,dU \quad \text{and} \quad \check{\omega} = dU\,U^{-1}, \tag{6.47}$$

so that the map ρ constructed in Theorem 5.1 is simply $\rho(x, y, U, \ldots) = U$.

With respect to coordinates x, U_1, U_{1x}, U_{1xx} on the $M_1 = \{\,\hat{I}^a = 0\,\}$ and coordinates y, U_{2y}, U_{2yy} on the level set $M_2 = \{\,\check{I}^a = 0\,\}$, the Pfaffian systems $\hat{W}$ and $\check{W}$ are

$$\hat{W} = \{\,dU_1 - U_{1x}dx,\; dU_{1x} - U_{1xx}dx\} \quad \text{and} \quad \check{W} = \{\,dU_2 - U_{2y}dy,\; dU_{2y} - U_{2yy}dy\}$$

and the superposition formula is

$$U = U_1U_2,\; U_x = U_{1x}U_2,\; U_y = U_1U_{2y},\; U_{xx} = U_{1xx}U_2,\; U_{yy} = U_1U_{2yy}.$$

Example 6.4. It is an open problem to determine which scalar Darboux integrable equations admit generalizations wherein the dependent variable U takes values in an arbitrary non-commutative, finite dimensional algebra $\mathcal{A}$. Here are two such examples which provide us with many Darboux integrable systems amenable to the methods presented in this paper.

I. $U_{xy} = (U_x + I)\,U^{-1}\,U_y$

II. $U_{xy} = U_x\,(U-y)^{-1}\,U_y + U_y\,(U-x)^{-1}\,U_x.$

The first integrals for the singular systems (excluding x and y) and general solutions are [3]

[3] We remark that the solution to **II** given by Vessiot [28] (equations C^2_{II} and (70), pages 6 and 45) in the scalar case is incorrect.

$$\textbf{I}.\quad \hat{I}^1 = (U_x + I)\,U^{-1},\quad \hat{I}^2 = D_x(\hat{I}^1),\quad \check{I}^2 = U_y^{-1}\,U_{yy},$$
$$U = (F')^{-1}\,(-F + G),$$

$$\textbf{II}.\quad \hat{I}^1 = (U - x)^{-1}\,[\,U_{xx}\,U_x^{-1}\,(U - x) - 2U_x + I\,],$$
$$\check{I}^1 = (U - y)^{-1}\,[\,U_{yy}\,U_y^{-1}\,(U - y) - 2U_y + I\,],$$
$$U = (xF' + yG' - F - G)\,(F' + G')^{-1},$$

where $F = F(x)$ and $G = G(y)$ take values in $\mathcal{A}$. For both systems **I** and **II** the Vessiot algebra is the tensor product of $\mathcal{A}$ with the Vessiot algebra for the corresponding scalar equation. We conjecture that all equations of Moutard type ([15] Volume II, page 250, equation 19) admit non-commutative generalizations.

Example 6.5. Some of the simplest examples of Darboux integrable systems can be constructed by the coupling of a Darboux integrable scalar equation to a linear or Moutard-type equation. As examples, we give

$$\textbf{I}.\quad u_{xy} = e^{2u},\qquad v_{xy} = n(n+1)e^{2u}v,$$
$$\textbf{II}.\quad u_{xy} = e^u u_y,\quad v_{xy} + ((n-\alpha)e^u + \alpha u_x)v_x = 0,$$
$$\textbf{III}.\quad u_{xy} = e^u u_y,\quad v_{xy} - e^u v_y + (n+1)\,u_y v_x + (n+1)!\,e^u u_y = 0,$$
$$\textbf{IV}.\quad u_{xy} = e^u u_x\quad v_{xy} + (e^v)_x - (nBe^{-v})_y + (n-2)B = 0,$$

where $B = e^u u_x$ and n is a positive integer. The system **I** appears in [19](page 116); systems **II**–**IV** do not seem to have appeared in the literature.

For each of these systems the restricted Pfaffian systems $\hat{W}$ and $\check{W}$ are jet spaces for two functions of a single variable (x or y). The Vessiot algebra for **I** is the semi-direct product of $\mathfrak{sl}(2)$ and an Abelian Lie algebra of dimension $2n+1$, as determined by the (unique) $(2n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2)$. The infinitesimal action of the Vessiot group for **I**, restricted to $\hat{W}$ or $\check{W}$, is the action listed in [14] as number 27 (where now the variables x, y in [14] serve as the dependent variables for the jet spaces $\hat{W}$ and $\check{W}$).

For **II** the Vessiot algebra is a semi-direct product of the 2-dimensional solvable algebra with an $(n+1)$-dimensional Abelian algebra. The infinitesimal action of the Vessiot group, restricted to $\hat{W}$ or $\check{W}$, is number 24 in [14]. The infinitesimal Vessiot groups for **III** and **IV** have dimensions $n+3$ and $n+4$ and coincide, respectively, with numbers 25 and 26 in [14].

For $n = 1$ the general solutions to these systems are

$$\textbf{I}.\quad u = \frac{1}{2}\ln\frac{F_1' G_1'}{(F_1+G_1)^2}, \quad v = 2\frac{F_2-G_2}{F_1+G_1} - \frac{F_2'}{F_1'} + \frac{G_2'}{G_1'}$$

$$\textbf{II}.\quad u = \ln\frac{F_1'}{G_1-F_1}, \quad v = \frac{1}{F_1^{\alpha}}\big(F_2 - G_2 - (F_1-G_1)\frac{G_2'}{G_2'}\big),$$

$$\textbf{III}.\quad u = \ln\frac{F_1'}{G_1-F_1}, \quad v = \frac{F_2-G_2}{(F_1-G_1)^2} - \frac{G_2'}{(F_1-G_1)G_2'} - \ln(G_1-F_1),$$

$$\textbf{IV}.\quad u = \ln\frac{G_1'}{F_1-G_1},$$

$$v = \ln\big(\frac{(G_1'(F_2-G_2) - G_2'(F_1-G_1))F_1'}{(F_1'(F_2-G_2) - F_2'(F_1-G_1))(F_1-G_1)}\big).$$

The general solutions for arbitrary n can be obtained in closed compact form by the method of Laplace.

In addition, any non-linear Darboux integrable system can be coupled to its formal linearization to obtain another Darboux integrable system. For example, if we prolong the partial differential equations

$$\textbf{V}.\quad 3u_{xx}u_{yy}^3 + 1 = 0, \quad v_{xx} - \frac{1}{u_{yy}^4}v_{yy} = 0$$

to order 3 in the derivatives of u, we obtain a rank 8 Pfaffian system on a 14-dimensional manifold which is Darboux integrable, with 4 first integrals for each associated singular Pfaffian system.

The Vessiot algebra is Abelian and of dimension 6. The two Lie algebras of vector fields dual to the forms $\hat{\theta}$ and $\check{\theta}$ for the 5-adapted coframe coincide and are given by

$$\{\partial_y,\ \partial_u,\ x\partial_u + \partial_{u_x},\ \partial_v,\ x\partial_v + \partial_{v_x}, u_y\partial_v + u_{xy}\partial_{v_x} + u_{yy}\partial_{v_y} + u_{xyy}\partial_{v_{xy}} + u_{yyy}\partial_{v_{yy}}\}.$$

In accordance with Remark 4.10, these vector fields are also infinitesimal symmetries for $\mathbf{V}$. In terms of the arbitrary functions $\phi(\alpha)$ and $\psi(\beta)$ appearing in the general solution to $3u_{xx}u_{yy}^3 + 1 = 0$ (Goursat([15] Vol. 2, page 130), the general solution for v is given implicitly as

$$x = \frac{1}{2}\frac{\phi'' - \psi''}{\alpha - \beta}, \quad y = \frac{1}{2}(\beta-\alpha)(\phi''+\psi'') + \phi' - \psi', \quad t = \frac{1}{\alpha-\beta}, \tag{6.48}$$

$$v = F + G - \frac{\phi''-\psi''+(\beta-\alpha)\phi'''}{(\beta-\alpha)\phi''''}F' - \frac{\phi''-\psi''+(\beta-\alpha)\psi'''}{(\beta-\alpha)\psi''''}G', \tag{6.49}$$

where $F = F(\alpha)$ and $G = G(\beta)$.

Finally, we remark that the Laplace transformation (not the integral one, Darboux [9] , Forsyth [13], pages 45–59.) can be applied to the linear components of any of the above systems to obtain new Darboux integrability systems. As well, the linear component in any of these systems can be replaced by their formal adjoint to obtain yet other Darboux integrable systems.

Example 6.6. Although we are unaware of an explicit general proof it is generally acknowledged that the Toda lattice systems (see, for example, [19], [22]) are Darboux integrable. In this example we shall check that the B_2 Toda lattice equations

$$u_{xy} = 2e^u - 2e^v, \quad v_{xy} = -e^u + 2e^v \tag{6.50}$$

are Darboux integrable and find the closed-form, general solution. We use this example to illustrate a slightly different computational approach, one based upon the symmetry reduction interpretation of the superposition formula given in Sections 3 and 5.3.

The canonical Pfaffian system for (6.50) satisfies our definition of Darboux integrable upon prolongation to 4-th order, that is, as a rank 14 Pfaffian system I on a 20 dimensional manifold. The diffeomorphism $x \leftrightarrow y$ interchanges the singular Pfaffian systems $\hat{V}$ and $\check{V}$. The first integrals for the singular Pfaffian system $\hat{V}$ (containing dx) are

$$\hat{I}^1 = x,\ \hat{I}^2 = v_{xx} + \frac{2}{3}u_{xx} - \frac{1}{3}u_x v_x - \frac{1}{6}u_x^2 - \frac{1}{3}v_x^2,\ \hat{I}^3 = D_x\hat{I}^2,\ \hat{I}^4 = D_x\hat{I}^3,\ \text{and}$$

$$\hat{I}^5 = u_{xxxx} + 2v_{xxxx} - 2v_x v_{xxx} - u_{xx}u_x^2 - 2u_x u_{xx} v_x + \frac{1}{8}u_x^4 + \frac{1}{2}u_{xx}^2 + \frac{1}{2}v_x^2 u_x^2 - v_x^2 u_{xx} - \frac{1}{2}v_{xx}u_x^2 + v_{xx}u_{xx} + \frac{1}{2}u_x^3 v_x - u_x v_{xxx}.$$

Let $\hat{W}$ be the restriction of $\hat{V}$ (or, equivalently, I) to $M_1 = \{\,\hat{I}^a = 0\}$. We find that $\hat{W}$ is a rank 12 Pfaffian system on a 15 manifold. The derived flag of $\hat{W}$ has dimensions $[12, 10, 8, 6, 4, 2, 1, 0]$ while the dimensions of the space of Cauchy characteristics for these derived Pfaffian sytems are [0, 2, 4, 6, 8, 10, 12, 15]. By using the invariants of these Cauchy characteristics as new coordinates, we are able to write $\hat{W}$ in the canonical form

$$\hat{W} = \{\, du_2 - \dot{u}_2 du_1,\ d\dot{u}_2 - \ddot{u}_2 du_1,\ \ldots,\ du_1 - u_1' dx,\ du_1' - du_1'' dx,\ \ldots\}. \tag{6.51}$$

Here there are 7 contact forms for u_2 and 5 for u_1. In these coordinates the integral manifolds of $\hat{W}$ are given by $u_1 = F_1(x)$, $u_1' = F_1'(x)$, …and $u_2 = F_2(F_1(x))$, $\dot{u}_2 = (\dot{F}_2)(F_1(x))$, ….

The 10 dimensional infinitesimal Vessiot group, restricted to M_1, now takes a remarkably simple and well-known form – it is the infinitesimal conformal group $o(3,2)$ acting on the 3-dimensional space $(u_1, u_2, \dot{u}_2)$ by contact transforms. (See, for example, Olver [21] page 473.) Explicitly, the generating functions for the infinitesimal action of the Vessiot group on M_1 are

$$\begin{aligned} Q = [&u_2, -u_2 + u_1\dot{u}_2, -u_1\dot{u}_2^2 + 2u_2\dot{u}_2, \frac{1}{2}, \dot{u}_2, \frac{1}{2}\dot{u}_2^2, u_1, \\ &-\frac{1}{2}u_1^2\dot{u}_2^2 - 2u_2^2 + 2u_1u_2\dot{u}_2, -2u_1u_2 + u_1^2\dot{u}_2, \frac{1}{2}u_1^2]. \end{aligned} \tag{6.52}$$

To obtain the infinitesimal generator $\hat{X}_q$ corresponding to a function $q \in Q$, first construct the vector field $X_q^0 = -q_{\dot{u}_2}\partial_{u_1} + (q - \dot{u}_2 q_{\dot{u}_2})\partial_{u_2} + (q_u + \dot{u}_2 q_{u_2})\partial_{u_2}$ and then prolong X_q^0 to the vector field $\hat{X}_q$ on M_1 by requiring it to be a symmetry of $\hat{W}$. We remark that the basis for $o(3,2)$ so obtained is the canonical Chevalley basis in the sense that the first 2 vectors define the Cartan subalgebra, the next 4 correspond to the positive roots, and the last 4 to the negative roots.

By Theorem 5.13, the superposition formula for the B_2 Toda lattice can therefore be constructed from the joint invariants for the diagonal action of the conformal algebra $\mathfrak{o}(3,2)$ on $M_1 \times M_2$. We use coordinates $[y, v_1, v_1', v_1'' \ldots, v_2, \dot{v}_2, \ddot{v}_2, \ldots]$ on M_2. To compactly describe these joint invariants we first calculate the joint differential invariants in the variables

$$\{u_1,\ u_1',\ v_1,\ v_1',\ \dot{u}_2,\ \dot{v}_2,\ \ddot{u}_2,\ \ddot{v}_2,\ \dddot{u}_2,\ \dddot{v}_2\ \}$$

for the 7 dimensional subalgebra of $o(3,2)$ generated by Q_1, Q_2, Q_4, Q_5, Q_6, Q_7, Q_{10}. These are

$$\begin{aligned} &J_1 = v_1'(\ddot{u}_2)^{1/3}(\ddot{v}_2)^{2/3}/(\ddot{u}_2 - \ddot{v}_2), \quad J_2 = u_1'(\ddot{u}_2)^{2/3}(\ddot{v}_2)^{1/3}/(\ddot{u}_2 - \ddot{v}_2), \\ &J_3 = (\ddot{u}_2 v_1 - \ddot{u}_2 u_1 + \dot{u}_2 - \dot{v}_2)(\ddot{u}_2)^{1/3}(\ddot{v}_2)^{2/3}/(\ddot{v}_2 - \ddot{u}_2)^2, \\ &J_4 = (\ddot{v}_2 u_1 - \ddot{v}_2 v_1 + \dot{v}_2 - \dot{u}_2)(\ddot{u}_2)^{2/3}(\ddot{v}_2)^{1/3}/(\ddot{v}_2 - \ddot{u}_2)^2, \quad \text{and} \end{aligned} \tag{6.53}$$

$$\begin{aligned} J_5 = -(&\ddot{u}_2\ddot{v}_2 v_1^2 - 2\ddot{u}_2\ddot{v}_2 v_1 u_1 + \ddot{u}_2\ddot{v}_2 u_1^2 - 2\ddot{u}_2\dot{u}_2 u_1 + 2\ddot{u}_2\dot{u}_2 v_1 + 2\ddot{u}_2 u_2 \\ &- 2\ddot{u}_2 v_2 - 2\ddot{v}_2\dot{v}_2 v_1 + 2\ddot{v}_2\dot{v}_2 u_1 + 2\ddot{v}_2 v_2 - 2\ddot{v}_2 u_2 \\ &+ \dot{u}_2^2 - 2\dot{u}_2\dot{v}_2 + \dot{v}_2^2)\ddot{u}_2\ddot{v}_2/(2(\ddot{v}_2 - \ddot{u}_2)^4). \end{aligned} \tag{6.54}$$

Then, in terms of these partial invariants the (lowest) order joint differential invariants for $o(3,2)$ are

$$K_1 = -\frac{J_1J_2(J_3J_4 - 2J_5)^2}{(J_3J_4 - J_5)^2} \quad \text{and} \quad K_2 = -\frac{J_1J_2(J_3J_4 - J_5)}{(J_3J_4 - 2J_5)^2} \tag{6.55}$$

and the solutions to the B_2 Toda lattice are

$$u = \ln(K_1/4) \quad \text{and} \quad v = \ln(2K_2), \quad \text{where} \tag{6.56}$$

$$u_1 = F_1(x),\ u_2 = F_2(F_1(x)),\ v_1 = G_1(y),\ v_2 = G_2(G_1(y)). \tag{6.57}$$

It is hoped that a more transparent representation of these solutions, similar to that available for the A_n Toda lattice will be be obtained.

Example 6.7. Let P denote the 2-dimensional Minkowski plane with metric $dx \odot dy$ and let N be a pseudo-Riemannian manifold with metric g. A mapping $\varphi : P \to N$ which is a solution to the Euler-Lagrange equations for the Lagrangian

$$L = g(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y})\, dx \wedge dy \tag{6.58}$$

is called a wave map. There are precisely two inequivalent, non-flat metrics (up to constant scaling) in 2 dimensions, namely

$$g_1 = \frac{1}{1+e^{-u}}(du^2 + dv^2) \quad \text{and} \quad g_2 = \frac{1}{1-e^{-u}}(du^2 + dv^2) \tag{6.59}$$

which define Darboux integrable wave maps at the 2-jet level (that is, without prolongation). Surprizingly, these metrics are not constant curvature. It is not difficult to check that under the change of coordinates

$$x = x - y, \quad t = x + y, \quad \theta = \arctan(\sqrt{e^u - 1}), \quad \chi = v/2 \tag{6.60}$$

the differential equations (4.13) in [3] become the wave map equations for the metric g_2. The Vessiot algebras for the wave map equations for g_1 and g_2 are $\mathfrak{sl}(2) \times R$ and $\mathfrak{so}(3) \times R$ respectively.

The wave map equations for g_1 are

$$u_{xy} = \frac{v_x v_y - u_x u_y}{2e^u + 2}, \quad v_{xy} = -\frac{u_x v_y + u_y v_x}{2e^u + 2}. \tag{6.61}$$

The standard encoding of these equations as a Pfaffian system results in a rank 6 Pfaffian system on a 12 manifold. There are 4 first integrals for each singular Pfaffian system – for the singular Pfaffian system $\hat{V}$ containing dx the first integrals are $\hat{I}^1 = x$, $\hat{I}^2 = \dfrac{e^u(u_x^2 + v_x^2)}{1+e^u}$, $\hat{I}^3 = D_x \hat{I}^2$, and

$$\hat{I}^4 = \frac{v_x}{u_x^2 + v_x^2} u_{xx} - \frac{u_x}{u_x^2 + v_x^2} v_{xx} - \frac{(1+2e^u)v_x}{2+2e^u}. \tag{6.62}$$

After considerable computation, the superposition is obtained and we find the general solution, in terms of the four arbitrary functions $F_1(x)$, $F_2(x)$, $G_1(y)$, $G_2(y)$ to be

$$2e^u = -1 + \sqrt{1+A^2}\sqrt{1+B^2} + AB\sin(\Delta) \tag{6.63}$$

and

$$\begin{aligned} v &= F_1(x) - F_2(x) + G_1(y) - G_2(y) \qquad (6.64)\\ &+ \arctan\left(\frac{AF_2'\sqrt{1+A^2}}{A'}\right) + \arctan\left(\frac{BG_2'\sqrt{1+B^2}}{B'}\right)\\ &+ \arctan\left(\frac{AB'\cos(\Delta) + G_2'B^2\sqrt{1+A^2}\sqrt{1+B^2} + G_2'AB(1+B^2)\sin(\Delta)}{G_2'AB\sqrt{1+B^2}\cos(\Delta) - BB'\sqrt{1+A^2} - AB'\sqrt{1+B^2}\sin(\Delta)}\right)\\ &+ \arctan\left(\frac{A'B\cos(\Delta) - F_2'A^2\sqrt{1+A^2}\sqrt{1+B^2} - F_2'AB(1+A^2)\sin(\Delta)}{F_2'AB\sqrt{1+A^2}\cos(\Delta) + AA'\sqrt{1+B^2} + A'B\sqrt{1+A^2}\sin(\Delta)}\right), \end{aligned}$$

where

$$A(x) = \sqrt{\left(\frac{F_1'}{F_2'}\right)^2 - 2\frac{F_1'}{F_2'}}, \quad B(y) = \sqrt{\left(\frac{G_1'}{K'}\right)^2 - 2\frac{G_1'}{G_2'}}, \quad \Delta = F_2 - G_2. \tag{6.65}$$

Note that

$$F_1'(x) = F_2'(x)(1 + \sqrt{1+A(x)^2}), \quad G_1'(y) = G_2'(y)(1 + \sqrt{1+B(y)^2}). \tag{6.66}$$

Example 6.8. We turn now to some simple examples of overdetermined systems for a single unknown function of 3 independent variables, beginning with the system

$$u_{xz} = uu_x, \quad u_{yz} = uu_y. \tag{6.67}$$

The structure equations for the canonical encoding of this system as a rank 4 Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ on an 11-manifold are (modulo I), $d\alpha^1 \equiv 0$,

$$d\alpha^2 \equiv \hat{\pi}^1 \wedge \hat{\pi}^3 + \pi^2 \wedge \pi^4, \quad d\alpha^3 \equiv \hat{\pi}^1 \wedge \hat{\pi}^4 + \pi^2 \wedge \pi^5, \quad d\alpha^4 \equiv \check{\pi}^1 \wedge \check{\pi}^2, \tag{6.68}$$

where $\hat{\pi}^1 = dx$, $\hat{\pi}^2 = dy$, $\check{\pi}^1 = dz$,

$$\begin{aligned} &\hat{\pi}^3 = du_{xx} - (u_{xx}u + u_x^2)\,dz, \quad \hat{\pi}^4 = du_{xy} - (u_{xy}u + u_y u_x)\,dz, \\ &\hat{\pi}^5 = du_{yy} - (u_{yy}u + u_y^2)\,dz, \quad \check{\pi}^2 = du_{zz} - (u_z + u^2)(u_x\,dx + u_y\,dy). \end{aligned} \tag{6.69}$$

The first integrals for the singular Pfaffian systems $\hat{V} = I \cup \{\hat{\pi}^1, \ldots, \hat{\pi}^5\}$ and $\check{V} = I \cup \{\check{\pi}^1, \check{\pi}^2\}$ are $\hat{I}^1 = x$, $\hat{I}^2 = y$, $\check{I}^1 = z$,

$$\hat{I}^3 = \frac{u_y}{u_x}, \quad \hat{I}^4 = D_x\hat{I}^3, \quad \hat{I}^5 = D_y\hat{I}^3, \quad \check{I}^2 = u_z - u^2/2, \quad \check{I}^3 = D_z\check{I}^2. \tag{6.70}$$

The form $\hat{\pi}^3$ is not in $\hat{V}^\infty + \check{V}$ and therefore (6.67) is not Darboux integrable on the 2-jet. The prolongation of (6.67) defines a decomposable rank 8 Pfaffian system $I^{[1]} = \{\alpha^1, \ldots, \alpha^8\}$ on a 16 dimensional manifold. In addition to the first integrals (6.70), we now also have

$$\hat{I}^6 = D_x^2\hat{I}^3,\ \hat{I}^7 = D_{xy}\hat{I}^3,\ \hat{I}^8 = D_y^2\hat{I}^3,\ \hat{I}^9 = \frac{u_{xxx}}{u_x} - \frac{3u_{xx}^2}{2u_x^2},\ \check{I}^4 = D_z^2\check{I}^2 \tag{6.71}$$

and the conditions (1.8)–(1.10) for Darboux integrability of $I^{[1]}$ are now satisfied.

A 1-adapted coframe is $\hat{\sigma}^1 = d\hat{I}^1$, $\hat{\sigma}^2 = d\hat{I}^2$, $\hat{\sigma}^6 = d\hat{I}^6$, $\hat{\sigma}^7 = d\hat{I}^7$, $\hat{\sigma}^8 = d\hat{I}^8$, $\hat{\sigma}^9 = d\hat{I}^9$, $\check{\sigma}^1 = d\check{I}^1$, $\check{\sigma}^2 = d\check{I}^4$,

$$\hat{\eta}^1 = \frac{1}{u_x}\,\alpha^3 - \frac{u_y}{u_x^2}\,\alpha^2 = d\hat{I}^3 - \hat{I}^4 d\hat{I}^1 - \hat{I}^5 d\hat{I}^2,$$

$$\hat{\eta}^2 = \frac{1}{u_x}\,\alpha^6 - \frac{u_y}{u_x^2}\,\alpha^5 - \frac{u_{xx}}{u_x^2}\,\alpha^3 - \frac{u_{xy}u_x - 2u_y u_{xx}}{u_x^3}\,\alpha^2 = d\hat{I}^4 - \hat{I}^6 d\hat{I}^1 - \hat{I}^7 d\hat{I}^2,$$

$$\hat{\eta}^3 = \frac{1}{u_x}\,\alpha^7 - \frac{u_y}{u_x^2}\,\alpha^6 - \frac{u_{xy}}{u_x^2}\,\alpha^3 - \frac{-2u_y u_{xy} + u_{yy}u_x}{u_x^3}\alpha^2 = d\check{I}^5 - \hat{I}^7 d\hat{I}^1 - \hat{I}^8 d\hat{I}^2,$$

$$\check{\eta}^1 = \alpha^4 - u\,\alpha^1 = d\check{I}^2 - \check{I}^3 d\check{I}^1, \quad \check{\eta}^2 = \alpha^8 - u\,\alpha^4 - u_z\alpha^1 = d\check{I}^3 - \check{I}^4 d\check{I}^1,$$

$$\theta^1 = \alpha^1 = du - u_x\,dx - u_y\,dy - u_z\,dz,$$

$$\theta^2 = \alpha^2 = du_x - u_{xx}\,dx - u_{xy}\,dy - uu_x\,dz,$$

$$\theta^3 = \alpha^3 = du_{xx} - u_{xxx}\,dx - u_{xxy}\,dy - (uu_{xx} + u_x^2)\,dz.$$

This frame is in fact 2-adapted. We next calculate $\hat{U}^{(\infty)} = \hat{U} \cup \{X_1, X_2, X_3\}$ and $\check{U}^{(\infty)} = \check{U} \cup \{Y_1, Y_2, Y_3\}$, where

$$
\begin{aligned}
X_1 &= -u_x \hat{I}^4 \partial_{\theta^1} - (u_{xx}\hat{I}^4 + u_x \hat{I}^6)\partial_{\theta^2} - \frac{3u_{xx}^2 \hat{I}^4 + 4u_{xx}u_x\hat{I}^6 - 2u_x^2 \hat{I}^4\hat{I}^9}{2u_x}\partial_{\theta^3},\\
X_2 &= -u_x\partial_{\theta^3},\ X_3 = -u_x\partial_{\theta^1} - u_{xx}\partial_{\theta^2} - \frac{3u_{xx}^2 + 2u_x^2\hat{I}^9}{2u_x}\partial_{\theta^3},\ Y_1 = -\partial_{\theta^1}, \quad (6.72)\\
Y_2 &= u\partial_{\theta^1} + u_x\partial_{\theta^2} + u_{xx}\partial_{\theta^3},\ Y_3 = (-\frac{u^2}{2} + \check{I}^2)\partial_{\theta^1} - uu_x\partial_{\theta^2} - (u_x^2 + uu_{xx})\partial_{\theta^3}.
\end{aligned}
$$

For the computations of Section 4.3 we use the base point defined by setting $u = 0$, $u_x = 1$, $u_{xx} = 0$, $\hat{I}^6 = 1$, and all other first integrals (6.70)–(6.71) to 0, The matrices (4.33) and (4.36) are then given by

$$
P = \begin{bmatrix} \frac{1}{\hat{I}^6} & \frac{2\hat{I}^4\hat{I}^9}{\hat{I}^6} & -\frac{\hat{I}^4}{\hat{I}^6} \\ 0 & 1 & 0 \\ 0 & -\hat{I}^9 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & -1 & 0 \\ \check{I}^2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \qquad (6.73)
$$

From these matrices we calculate the 4 adapted coframes

$$
\begin{aligned}
\theta_X^1 &= \frac{u_{xx}}{u_x^2}\theta^1 - \frac{1}{u_x}\theta^2, & \theta_Y^1 &= -\frac{u_x^2 + uu_{xx}}{u_x^3}\theta^2 + \frac{u}{u_x^2}\theta^3,\\
\theta_X^2 &= -\frac{u_{xx}^2}{2u_x^3}\theta^1 + \frac{2u_{xx}}{u_x^2}\theta^2 - \frac{1}{u_x}\theta^3, & \theta_Y^2 &= \frac{u_{xx}}{u_x^3}\theta^2 - \frac{1}{u_x^2}\theta^3, \qquad (6.74)\\
\theta_X^3 &= -\frac{1}{u_x}\theta^1, & \theta_Y^3 &= -\theta^1 + \frac{u(2u_x^2 + uu_{xx})}{2u_x^3}\theta^2 - \frac{u^2}{2u_x^2}\theta^3.
\end{aligned}
$$

The Vessiot algebra is $\mathfrak{sl}(2)$. Because this algebra is semi-simple, we can use Case **I** of Section 4.4, Theorem 5.1, and (6.74) to directly determine the map $\rho : M \to \mathrm{Aut}(\mathfrak{sl}(2)$ as

$$
\rho(u, u_x, u_{xx}) = \lambda = \begin{bmatrix} \frac{u_x^2 - uu_{xx}}{u_x^2} & -\frac{u}{u_x} & -\frac{u_{xx}(uu_{xx} - 2u_x^2)}{2u_x^3} \\ \frac{u_{xx}}{u_x^2} & \frac{1}{u_x} & \frac{u_{xx}^2}{2u_x^3} \\ \frac{u(uu_{xx} - 2u_x^2)}{2u_x^2} & \frac{u^2}{2u_x} & \frac{(uu_{xx} - 2u_x)^2}{4u_x^3} \end{bmatrix}. \qquad (6.75)
$$

To obtain the superposition formula we introduce local coordinates (z, w, w_x, w_{xx}, w_z, w_{zz}, w_{zzz}) for M_1 and (x, y, v, v_x, v_y, v_z, v_{xx}, v_{xy}, v_{yy}, v_{yz}, v_{xxx},

$v_{xxy},\ v_{xyy},\ v_{yyy},\ v_{yyz})$ for M_2. The inclusions $\iota_1 \colon M_1 \to M$ and $\iota_2 \colon M_2 \to M$ are fixed by

$$\iota_1(w_I) = u_I, \quad \iota_1^*(\hat{I}^a) = 0, \quad \iota_2(v_I) = u_I, \quad \iota_2^*(\check{I}^a) = 0.$$

The superposition formula is then found by solving the equations

$$\iota_1^*(\check{I}^a) = \check{I}^a, \quad \iota_2^*(\hat{I}^a) = \hat{I}^a, \quad \rho(u, u_x, u_{xx}) = \rho(w, w_x, w_{xx}) \cdot \rho(v, v_x, v_{xx}) \tag{6.76}$$

for the coordinates of M. We find that

$$u = w - \frac{2vw_x^2}{-2w_x + vw_{xx}}. \tag{6.77}$$

Finally, we calculate the integral manifolds for $\hat{W}$ and $\check{W}$. It is immediate that $\check{W}$ is the canonical Pfaffian system on $J^3(\mathbf{R}, \mathbf{R}^2)$ and hence the integral manifolds are defined by $v = V(x, y)$. The last non-zero form in the derived flag for

$$\begin{aligned} \hat{W} = \{\, & dw - w_z dz,\ dw_z - w_{zz}\, dz,\ dw_{zz} - w_{zzz}\, dz,\ dw_x - w_x w dz, \\ & dw_{xx} + (-w_x^2 - ww_{xx}) dz \,\} \end{aligned}$$

yields the Pfaffian equation

$$dw_{xx} - \frac{w_{xx}}{w_x} dw_x - w_x^2 dz = 0$$

from which it follows that $w_x = G'(z)$ and $w_{xx} = G(z)G'(z)$. One of the remaining equations in $\hat{W}$ then gives $w = G''(z)/G'(z)$. On replacing $V(x, y)$ by $-2/F(x, y)$, the superposition formula (6.77) becomes

$$u(x, y, z) = \frac{G''(z)}{G'(z)} - \frac{2G'(z)}{F(x, y) + G(z)}, \tag{6.78}$$

which gives the general solution to (6.67).

We continue this example by considering two variations on (6.67). First we observe that to (6.67) we may add any equation of the form

$$F(x, y, \frac{u_y}{u_x}, \frac{u_y u_{xx} - u_x u_{xy}}{u_x}, \frac{u_y u_{xy} - u_x u_{tyy}}{u_x^2}) = 0 \tag{6.79}$$

to obtain a rank 4 Pfaffian system $I = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\}$ on a 10 dimensional manifold. The structure equations become ($\bmod I$) $d\alpha^1 \equiv 0$,

$$d\alpha^2 \equiv \hat{\pi}^1 \wedge \hat{\pi}^2, \quad d\alpha^3 \equiv \hat{\pi}^3 \wedge \hat{\pi}^4, \quad d\alpha^4 \equiv \check{\pi}^1 \wedge \check{\pi}^2 \tag{6.80}$$

and hence all such systems are involutive with Cartan character $s_1 = 3$.

For example, consider the system of 3 equations

$$u_y u_{xy} - u_x u_{yy} = 0, \quad u_{xz} = u u_x, \quad u_{yz} = u u_y \tag{6.81}$$

The foregoing calculations can be repeated, almost without modification, to arrive at the same superposition formula (6.77) – the only difference is that now $\check{W}$ is the (prolonged) canonical Pfaffian system for the equation $v_y v_{xy} - v_x v_{yy} = 0$, an equation which is itself Darboux integrable. Thus, in more complicated situations, the method of Darboux, can be used to integrate the Pfaffian systems $\hat{W}$ and $\check{W}$ and the superposition formula for the original system is given by a composition of superposition formulas. In the case of the present example, the calculation of the first integrals for $\check{W}$ reveals that this system is contact equivalent to the wave equation (via $X = x$, $Y = v$, $V = y$, $V_X = -v_x/v_y$, $V_Y = 1/v_y$) and leads to the parametric solution

$$x = \sigma, \quad y = f(\sigma) + g(\tau), \quad u = \frac{G''(z)}{G'(z)} - \frac{2G'(z)}{\tau + G(z)}. \tag{6.82}$$

Our second variation of (6.67) is obtained by the differential substitution $u_x = \exp(v)$. This leads to the equations

$$v_{xz} = \exp(v) \quad \text{and} \quad v_{yzz} = v_{yz} v_z. \tag{6.83}$$

It is surprising that the canonical Pfaffian system for these equations (obtained by the restriction of the contact ideal on $J^3(\mathbf{R}, \mathbf{R}^2)$) does *not* define a decomposable Pfaffian system. The following theorem resolves this difficulty.

Theorem 6.9. *The system of differential equations*

$$\begin{aligned} u_{xz} &= F(x, y, z, u, u_z, u_{yz}, u_{zz}), \\ u_{yzz} &= G(x, y, z, u, u_x, u_y, u_z, u_{yz}, u_{zz}, u_{yyz}, u_{zzz}) \end{aligned} \tag{6.84}$$

determines the rank 4 Pfaffian system

$$\begin{aligned} &\alpha^1 = du - u_x dx - u_y dy - u_z dz, \quad \alpha^2 = du_z - F dx - u_{yz} dy - u_{zz} dz, \\ &\alpha^3 = du_{yz} - D_y(F) dx - u_{yyz} dy - G dz, \\ &\alpha^4 = du_{zz} - D_z(F) dx - G dy - u_{zzz} dz, \end{aligned} \tag{6.85}$$

on an 11 dimensional manifold. If the compatiblity conditions for (6.84) *hold, then this Pfaffian system is decomposable and involutive with Cartan characters* $s_1 = 1$ *and* $s_2 = 1$.

We use Theorem 6.9 to write (6.83) as a rank 4 Pfaffian system on an 11-manifold. The prolongation of this system is Darboux integrable and calculations, virtually identical to those provided for (6.67), lead directly to the general solution

$$v(x, y, z) = \ln\Big(\frac{2F_x(x, y)G'(z)}{(F(x, y) + G(z))^2}\Big). \tag{6.86}$$

References

[1] I. M. Anderson and M. E. Fels, *Exterior Differential Systems with Symmetry*, Acta. Appl. Math. **87** (2005), 3–31.

[2] I. M. Anderson and M. Juráš, *Generalized Laplace Invariants and the Method of Darboux*, Duke J. Math **89** (1997), 351–375.

[3] B. M. Barbashov, V. V. Nesterenko, and A. M. Chervyakov, *General solutions of nonlinear equations in the geometric theory of the relativistic string*, Commun. Math Physics **84** (1982), 471–481.

[4] F. De Boer, *Application de la méthode de Darboux à l'intégration de l'équation différentielle s = f(r,t).*, Archives Neerlandaises **27** (1893), 355–412.

[5] R. L Bryant and P. A. Griffiths, *Characteristic cohomology of differential systems (I)*, Selecta Math. (N.S.) **1** (1995), 21–112.

[6] R. L. Bryant, P. A. Griffiths, and L. Hsu, *Hyperbolic exterior differential systems and their conservation laws, Parts I and II*, Selecta Math., New series **1** (1995), 21–122 and 265–323.

[7] É Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. École Norm. **3** (1910), no. 27, 109–192.

[8] ———, *Sur les systèmes en involution d'équations aux dériv'ees partielles du second ordre à une fonction inconnue de trois variable indèpendantes*, Bull Soc. Math France **39** (1911), 352-443.

[9] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Gauthier-Villars, Paris, 1896.

[10] P. T. Eendebak, *Contact Structures of Partial Differential Equations*, 2007.

[11] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Dover, New York, 1963.

[12] M. E. Fels and P. J. Olver, *Moving coframes I. A practical algorithm*, Acta. Appl. Math. **51** (1998), 161-213.

[13] A. Forsyth, *Theory of Differential Equations, Vol 6*, Dover Press, New York, 1959.

[14] A. González-López, N. Kamran, and P. J. Olver, *Lie algebras of vector fields in the real plane*, Proc. London Math. Soc. **64** (1992), 339-368.

[15] E. Goursat, *Lecon sur l'intégration des équations aux dériées partielles du second ordre á deux variables indépendantes, Tome 1, Tome 2*, Hermann, Paris, 1897.

[16] E. E. Goursat, *Recherches sur quelques équations aux dériées partielles du second ordre*, Ann. Fac. Sci. Toulouse **1** (1899), 31–78 and 439–464.

[17] P. Griffiths, *On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. **41** (1974), 775–814.

[18] S.S. Kobyashi and K. Nomizu, *Foundations of Differential Geometry*, John Wiley, 1963.

[19] A. N. Leznov and M. V. Saleliev, *Representation theory and integration of nonlinear spherically symmetric equations to gauge theories*, Commun. Math. Phys. **74** (1980), 111-118.

[20] ———, *Two-dimensional exactly and completely integrable dynamical systems*, Commun. Math. Phys. **89** (1983), 59–75.

[21] P. J. Olver, *Equivalence, Invariants, and Symmetry* (1995).

[22] V. V. Sokolov and A. V. Ziber, *On the Darboux integrable hyperbolic equations*, Phys Lett. A **208**, 303–308.

[23] S Sternberg, *Lectures on Differential Geometry*, Second, Chelsa, New York, 1984.

[24] O. Stormark, *Lie's structural approach to PDE systems* **80** (2000).

[25] V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Springer-Verlag, New York, 1984.

[26] P. J. Vassiliou, *Vessiot structure for manifolds of* (p, q)*-hyperbolic type: Darboux integrability and symmetry*, Trans. Amer. Math. Soc. **353** (2001), 1705–1739.

[27] E. Vessiot, *Sur les équations aux dérivées partielles du second ordre, F(x,y,z,p,q,r,s,t)=0, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **18** (1939), 1–61.

[28] ———, *Sur les équations aux dérivées partielles du second ordre, F(x,y,z,p,q,r,s,t)=0, intégrables par la méthode de Darboux*, J. Math. Pure Appl. **21** (1942), 1–66.