

# Covariant phase space formulation of parametrized field theories

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(Received 8 April 1992; accepted for publication 29 June 1992)

Parametrized field theories, which are generally covariant versions of ordinary field theories, are studied from the point of view of the covariant phase space: the space of solutions of the field equations equipped with a canonical (pre)symplectic structure. Motivated by issues arising in general relativity, we focus on phase space representations of the space-time diffeomorphism group, construction of observables, and the relationship between the canonical and covariant phase spaces.

## I. INTRODUCTION

One of the central features of Newtonian mechanics is the presence of an absolute time: a preferred foliation of Galilean space-time. Despite the presence of a universal notion of time, it is still possible to formulate dynamics in terms of an arbitrary time parameter. This is the “parametrized” formulation of mechanics, which is obtained by adjoining the Newtonian time to the configuration variables of the mechanical system.<sup>1</sup> The resulting formalism is often elegant but, given the existence of a preferred time, never really necessary.

The need for a field-theoretic formalism that includes arbitrary notions of time (and space) becomes apparent when one studies dynamical theories consistent with Einstein’s general theory of relativity. Here there is *no* preferred standard of time (or space), and it is usually best to keep this fact manifest by never selecting such a standard. This can be done by including the gravitational field, in the guise of the space-time metric, as a dynamical variable and keeping the resulting “general covariance”—more precisely: the space-time diffeomorphism covariance—of the theory manifest. However, it is not necessary to add new physics (gravitational dynamics) in order to achieve a generally covariant formulation of a field theory. It has been known for a long time<sup>2–4</sup> that any field theory on a fixed background space-time can be made generally covariant by adjoining suitable space-time variables to the configuration space of the theory in much the same way as one does in the parametrized formulation of mechanics. This diffeomorphism covariant formulation of field theory is likewise called “parametrized field theory.”

Parametrized field theory allows one to study field theory without prejudicing the choice of time (space), and for this reason alone it is a useful tool (see, e.g., Ref. 5). Because parametrized theories are generally covariant, they also serve as an important paradigm for the dynamics of gravitation.<sup>6</sup> Indeed, general relativity is often viewed as an “already parametrized” field theory; if this point of view could be explicitly implemented, then

one could solve some very basic problems<sup>7</sup> that are especially troublesome for the program of canonical quantization of the gravitational field.

A relatively unexplored formulation of Hamiltonian gravity is based on the “covariant phase space.”<sup>8–11</sup> The covariant phase space is defined as the space of solutions to the equations of motion and thus has the virtue of preserving manifest covariance. Because the space of solutions admits a (pre)symplectic structure, one can still employ sophisticated Hamiltonian methods to formulate the quantization problem. Thus it is of interest to try to apply covariant phase space methods to study the canonical quantum theory of gravity. Given the importance of parametrized field theory both as a paradigm for general relativity and as an elegant formulation of field theories, it is worth examining such theories from the point of view of the covariant phase space. In particular, how do the stubborn problems of time and observables<sup>7</sup> appear in the covariant phase space formulation of parametrized field theory? Can we use the parametrized field theory paradigm to better understand the covariant phase space of general relativity? This latter question is made more pressing since it has been shown recently that, strictly speaking, the canonical (as opposed to covariant) phase space structure of general relativity cannot be identified with that of any parametrized field theory.<sup>12</sup> As we shall see, the covariant and canonical approaches to the phase space of parametrized theories are quite different, and hence it is plausible that the parametrized field theory paradigm will be more suitable in the context of the covariant phase space formulation.

In this paper we will present the covariant phase space formulation of a general parametrized field theory. In particular we will address the issue of the action of the diffeomorphism group on the phase space, which is a delicate problem in the conventional Hamiltonian formulation,<sup>13</sup> as well as the related issue of how to construct “observables” in this formalism. In canonical gravity the construction of observables has so far proven intractable, so it is useful to see how the covariant phase space approach handles this question. Most important perhaps,

we will spell out in detail the (somewhat complicated) relationship between the covariant and canonical phase space approaches to parametrized field theory by comparing the phase spaces, group actions, and observables in each formulation. In particular, we would like to see how the reduced phase spaces of the canonical and covariant approaches manage to be the same, which is implied by the general results of Ref. 11. This is a nontrivial issue mainly because the covariant phase space admits an action of the space-time diffeomorphism group while the canonical phase space does not.<sup>13</sup> Presumably, these results will at least hint at the corresponding results in general relativity.

There are some disadvantages associated with trying to give an analysis that includes a “general field theory.” In particular, if one tries to give too broad a coverage of possible field theories, then the description becomes quite opaque if only because of the notational difficulties. Thus, for simplicity, we make some simplifying assumptions about the field theories being studied that, while perhaps violated in some very exceptional cases, are typically valid. One important exception to the previous statement is that we will not attempt to include parametrized gauge theories in our analysis. There are a couple of reasons for this. First, the structure of a parametrized gauge theory is rather different from that of a theory without any gauge invariances. This is because the parametrized formulation of nongauge theories leads to a phase space formulation that is well behaved with respect to the diffeomorphism group of the space-time manifold, while the parametrized gauge theory brings in the larger group of bundle automorphisms. It is an interesting problem to find a globally valid formulation of parametrized gauge theory, but we shall not do it here. At any rate, if one wants to use parametrized field theory to understand general relativity, then the relevant group is the diffeomorphism group and gauge theories can thus be played down in importance (however, see Ref. 14). One of the other main assumptions we will make is designed to simplify the task of relating the covariant and canonical phase spaces. Specifically, we will identify the space of Cauchy data for the field theory with the canonical phase space for the theory. Given a spacelike (Cauchy) hypersurface, the Cauchy data will be assumed to be the fields and their normal Lie derivatives on that surface. (There are, of course, important cases that violate this assumption, e.g., the Dirac field, but such field theories present no new features in the context of the present investigation.) In practice this identification, which is tantamount to identifying the tangent and cotangent bundles over the space of field configurations, is done in terms of metrics, both on space-time and on the manifold of fields, but it will be too cumbersome to try to make explicit the identification.

The plan of the paper is as follows. In Sec. II we deal with a brief summary of the salient features of param-

etrized field theories; this includes the usual canonical formulation. In Sec. III we provide a quick tour of the covariant phase space formalism and apply it to parametrized field theories. Next, in Secs. IV and V, we turn to the representations of the diffeomorphism group on phase space and the extraction of the observables; both of these issues are simply and neatly treated using the covariant phase space. The final section, Sec. VI, is in many ways the most interesting; in it we spell out the relationship between the covariant and canonical phase space formulations.

## II. PARAMETRIZED FIELD THEORIES

We consider a collection of fields,  $\psi^A$ , propagating on a globally hyperbolic space-time  $(\mathcal{M}, g_{\alpha\beta})$  according to the extrema of the action functional

$$S[\psi^A] = \int_{\mathcal{M}} \mathcal{L}(g; \psi^A, \partial\psi^A). \quad (2.1)$$

For simplicity we assume that the fields are nonderivatively coupled to the background geometry, and that the Lagrangian only depends on the fields and their first derivatives. The equations of motion are

$$\delta S / \delta \psi^A = 0. \quad (2.2)$$

Note that the solution space of this equation generally cannot admit an action of the space-time diffeomorphism group,  $\text{Diff}(\mathcal{M})$ , because the metric is fixed.

The parametrization process enlarges the configuration space by the space of diffeomorphisms from  $\mathcal{M}$  to itself. When dealing with these new dynamical variables, denoted  $X$ , it will be convenient to work with two copies of  $\mathcal{M}$ ,  $\mathcal{M}^\mu$  and  $\mathcal{M}^\alpha$ , and then view  $X \in \text{Diff}(\mathcal{M})$  as a map from  $\mathcal{M}^\mu$  to  $\mathcal{M}^\alpha$ ,

$$X: \mathcal{M}^\mu \rightarrow \mathcal{M}^\alpha.$$

Thus one can think of  $X$  as a field on  $\mathcal{M}^\mu$  taking values in  $\mathcal{M}^\alpha$ . Tensors on  $\mathcal{M}^\mu$  will be distinguished by Greek indices from the end of the alphabet, likewise tensors on  $\mathcal{M}^\alpha$  will have Greek indices from the beginning of the alphabet. As an important example, the metric on  $\mathcal{M}^\alpha$  is  $g_{\alpha\beta}$ ; given  $X \in \text{Diff}(\mathcal{M})$  this metric can be pulled back to  $\mathcal{M}^\mu$ :

$$g_{\mu\nu} = (X^*g)_{\mu\nu} = X_\mu^\alpha X_\nu^\beta g_{\alpha\beta} \circ X, \quad (2.3)$$

where  $X_\mu^\alpha$  is the differential of the map  $X$ .

Given  $X \in \text{Diff}(\mathcal{M})$ , the Lagrangian density defined on  $\mathcal{M}^\alpha$  can be pulled back to  $\mathcal{M}^\mu$ . This gives an action that can be considered as a functional of both  $\varphi^A := X^*\psi^A$  and  $X$ :

$$S[X, \varphi^A] = \int_{\mathcal{M}^\mu} \mathcal{L}(X^*g; \varphi^A, \partial\varphi^A). \tag{2.4}$$

Because the original action integral is unchanged by a diffeomorphism acting on both  $g_{\alpha\beta}$  and  $\psi^A$ , the action  $S[\varphi^A, X]$  is invariant with respect to changes of the diffeomorphism  $X$  provided one takes into account the change induced in  $\varphi^A = X^*\psi^A$ . This leads to the fact that if

$$\frac{\delta S[\varphi^A, X]}{\delta \varphi^A} = 0, \tag{2.5}$$

then the equations

$$\frac{\delta S[\varphi^A, X]}{\delta X^\alpha} = 0 \tag{2.6}$$

are automatically satisfied. This can be verified directly. The equations (2.6) are equivalent to

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.7}$$

where

$$T^{\mu\nu} = -2g^{-1/2} \frac{\delta S[\varphi^A, X]}{\delta g_{\mu\nu}}. \tag{2.8}$$

As is well known, (2.7) follows from (2.5).

The redundancy of the Euler–Lagrange equations associated with  $S[\varphi^A, X]$  is a consequence of the invariance of the action functional (2.4) with respect to the pull-back action of diffeomorphisms on its arguments. If  $\phi \in \text{Diff}(\mathcal{M})$ , then

$$S[\phi^*\varphi^A, X \circ \phi] = S[\varphi^A, X]. \tag{2.9}$$

Thus the space of solutions to (2.5) and (2.6) will admit an action (in fact more than one) of  $\text{Diff}(\mathcal{M})$ . This will be discussed in more detail in Sec. IV.

The canonical phase space formulation of parametrized field theory is developed in Ref. 4; here we simply summarize the needed results. If  $X$  and  $\varphi^A$  are viewed as a collection of fields on  $\mathcal{M}^\mu$ , then to pass to the Hamiltonian formulation we need a foliation  $Y$  of  $\mathcal{M}^\mu$ :

$$Y: R \times \Sigma \rightarrow \mathcal{M}^\mu,$$

where  $\Sigma$  is the three-manifold representing space. Tensors on  $\Sigma$  are represented via Latin indices. Derivatives along  $R$  are denoted with a dot. For each  $t \in R$ ,  $Y$  becomes an embedding,  $Y_{(t)}: \Sigma \rightarrow \mathcal{M}^\mu$ . We demand that the embedded hypersurface is spacelike, which means that the normal  $n_\mu$  to the hypersurface, defined by

$$Y_{(t)a}{}^\mu n_\mu = 0, \tag{2.10}$$

is timelike. Here  $Y_{(t)a}{}^\mu$  is the differential of the map  $Y_{(t)}$ . We will normalize  $n_\mu$  to unity:

$$g^{\mu\nu} n_\mu n_\nu = -1. \tag{2.11}$$

An equivalent way to express the requirement that the leaves of the foliation are spacelike is to demand that the metric induced on  $\Sigma$ ,

$$\gamma_{ab} := Y_{(t)a}{}^\mu Y_{(t)b}{}^\nu g_{\mu\nu} \circ Y_{(t)}, \tag{2.12}$$

is positive definite for each  $t$ . Note that  $n_\mu$  and  $\gamma_{ab}$  are functionals of  $Y_{(t)}$ .

The configuration space of the canonical formalism consists of pairs  $(q^A, Q)$ , where  $q^A := Y_{(t)}^* \varphi^A$  and  $Q := X \circ Y_{(t)}$ , which are just the fields pulled back to a slice. Note that  $Q$  represents an embedding of  $\Sigma$  into  $\mathcal{M}^\alpha$ :

$$Q: \Sigma \rightarrow \mathcal{M}^\alpha,$$

with normal

$$n_\alpha = X_\alpha^\mu n_\mu \circ X^{-1}. \tag{2.13}$$

In (2.13) we have defined  $X_\alpha^\mu$  via

$$\begin{aligned} X_\alpha^\mu X_\mu^\beta &= \delta_\alpha^\beta \\ X_\alpha^\mu X_\nu^\alpha &= \delta_\nu^\mu, \end{aligned} \tag{2.14}$$

i.e.,  $X_\alpha^\mu$  is the inverse to the differential of  $X \in \text{Diff}(\mathcal{M})$ , viewed as a map of the tangent space at  $p \in \mathcal{M}$  to that at  $X(p)$ . The embedding  $Q$  is spacelike; let  $Q_\alpha^\alpha$  be the differential of  $Q$ , then

$$\begin{aligned} (Q^*g)_{ab} &= Q_\alpha^\alpha Q_b^\beta g_{\alpha\beta} \circ Q \\ &= X_\mu^\alpha Y_{(t)a}{}^\mu X_\nu^\beta Y_{(t)b}{}^\nu g_{\alpha\beta} \circ (X \circ Y_{(t)}) \\ &= \gamma_{ab}. \end{aligned} \tag{2.15}$$

Thus the configuration space can be viewed as that of the fields  $q^A$  on  $\Sigma$  along with the set of spacelike embeddings of  $\Sigma$  into  $\mathcal{M}^\alpha$ .

The Hamiltonian form of the action is a functional of curves in the phase space  $\Upsilon$ , the phase space consisting of pairs  $(q^A, \Pi_A)$ ,  $(Q, P)$ , where  $\Pi_A$  and  $P$  are conjugate to  $q^A$  and  $Q$ , respectively; it takes the form

$$S[q, \Pi; Q, P, N] = \int_{R \times \Sigma} (\Pi_A \dot{q}^A + P_\alpha \dot{Q}^\alpha - N^\alpha H_\alpha). \tag{2.16}$$

Here  $\dot{Q}^\alpha$  is the derivative with respect to the parameter  $t$  of a one-parameter family of embeddings and is geometrically a vector field on  $\mathcal{M}^\alpha$ ;  $P_\alpha$  is therefore a one-form density of weight one on  $\mathcal{M}^\alpha$ .  $N^\alpha$  are Lagrange multipliers enforcing the first-class constraints

$$H_\alpha := P_\alpha + h_\alpha \approx 0. \tag{2.17}$$

The constraints identify the momentum conjugate to the embedding with  $h_\alpha$ , which represents the energy-momentum flux of the fields  $\psi^A$  through the hypersurface defined by  $Q: \Sigma \rightarrow \mathcal{M}^\alpha$ ;  $h_\alpha$  is a functional of  $q^A$ ,  $\Pi_A$ , and  $Q$ . The energy-momentum flux can be decomposed into its components normal and tangential to the hypersurface embedded in  $\mathcal{M}^\alpha$ :

$$h_\alpha = -n_\alpha h + Q_\alpha^a h_a, \tag{2.18}$$

where  $h$  is the energy density of  $\psi^A$  as measured by an observer instantaneously at rest in the hypersurface and  $h_a$  is the corresponding momentum density. Here  $Q_\alpha^a$  lifts one-forms on  $\Sigma$  to one-forms on  $\mathcal{M}^\alpha$  restricted to the embedded hypersurface and is defined as

$$Q_\alpha^a := \gamma^{ab} g_{\alpha\beta} Q_b^\beta. \tag{2.19}$$

### III. THE COVARIANT PHASE SPACE

The covariant phase space approach to dynamics exploits the point of view that the phase space of Hamiltonian mechanics is a symplectic manifold. It can be shown that *any* (local) action for a dynamical system contains within it the definition of a presymplectic structure on its critical points.<sup>11</sup> If there are no gauge transformations in the theory, then the presymplectic structure is a genuine symplectic structure and one can thus formulate Hamiltonian dynamics on the space of solutions to the equations of motion. In relativistic theories this leads to a manifestly covariant phase space description.

The action functional  $S$  can be viewed as a scalar function on the space  $\mathcal{A}$  of all field histories,  $S: \mathcal{A} \rightarrow \mathbb{R}$ . From this point of view, the variation of a field is a tangent vector  $\mathcal{V}$  to this space. The first variation of the action then can be viewed as the action on  $\mathcal{V}$  of the exterior derivative of  $S$ :

$$\delta S = dS(\mathcal{V}). \tag{3.1}$$

Now restrict attention to the submanifold  $\Gamma \subset \mathcal{A}$  of solutions to the equations of motion. (For simplicity we ignore the possibility that the space of solutions has singularities.) Then the first variation of the action reduces to a surface term at the (asymptotic) boundary of the space-time  $\mathcal{M}$  (for an explicit expression see Refs. 10 and 11):

$$i^* dS(\mathcal{V}) = \int_{\partial \mathcal{M}} j^a(\mathcal{V}) d\Sigma_a. \tag{3.2}$$

Here  $i: \Gamma \rightarrow \mathcal{A}$  is the natural embedding of the space of solutions into the space of all fields. The surface term defines the (pre)symplectic potential  $\Theta_\Sigma$ , which is a one-form on  $\Gamma$ , via

$$\Theta_\Sigma(\mathcal{V}) = \int_\Sigma j^a(\mathcal{V}) d\Sigma_a, \tag{3.3}$$

where  $\Sigma$  is a Cauchy surface in  $\mathcal{M}$ . For simplicity we will use the same notation ( $\Sigma$ ) to denote an abstract three-dimensional manifold as well as for its image after an embedding. Whenever it is necessary to distinguish the two we will work explicitly with the embedding. In (3.3)  $\mathcal{V}$  is a tangent vector to  $\Gamma$ , i.e., it is a solution of the linearized equations of motion. Note that, in general,  $\Theta_\Sigma$  depends on the choice of  $\Sigma$ . Denote as  $\Omega$  the closed two-form on  $\Gamma$  obtained as the exterior derivative of  $\Theta_\Sigma$ :

$$\Omega(\mathcal{V}, \mathcal{W}) = d\Theta_\Sigma(\mathcal{V}, \mathcal{W}). \tag{3.4}$$

Because  $d^2 S = 0$ , it can be seen from (3.2) that  $\Omega$  is independent of the choice of  $\Sigma$ . (If  $\Sigma$  is not compact this is true only with suitable boundary conditions at spatial infinity.)  $\Omega$  is the (pre)symplectic structure.

If there are no gauge symmetries in the theory, then the Hessian of the Lagrangian is nondegenerate and one can pass directly to the Hamiltonian form of the action. From this form of the action it can be seen that the (pre)symplectic structure defined on the covariant phase space  $\Gamma$  and the usual symplectic structure on the momentum phase space are equivalent. In particular  $\Omega$  is nondegenerate in this case and is thus a true symplectic structure.

If the action functional admits gauge transformations, then  $\Omega$  necessarily has degenerate directions. A detailed proof of this can be found in Ref. 9; it is worth sketching a simple proof here. First, we shall define a gauge transformation  $\mathcal{G}: \Gamma \rightarrow \Gamma$  as any suitably differentiable map of  $\Gamma$  onto itself that has arbitrary support on  $\mathcal{M}$ . By "support" we mean the region of space-time for which the transformation of field values in that region is not the identity. The requirement of arbitrary support is crucial; it guarantees that gauge transformations are, roughly speaking, parametrized by arbitrary functions on  $\mathcal{M}$ . Now consider a one-parameter family of gauge transformations  $\mathcal{G}_s$  beginning at the identity. If such families of transformations do not exist, then the symplectic structure need not be degenerate. Infinitesimal gauge transformations correspond to certain "pure gauge" tangent vectors  $\mathcal{L}$  to  $\Gamma$ ,

$$\mathcal{L} := \frac{d\mathcal{G}_s}{ds} \Big|_{s=0}, \tag{3.5}$$

which, thought of as fields on space-time, have arbitrary support on  $\mathcal{M}$ . We want to show that for each pure gauge tangent vector  $\mathcal{L}$ ,  $\Omega(\mathcal{V}, \mathcal{L}) = 0$  for all choices of  $\mathcal{V}$ . To do this consider  $\Omega(\mathcal{V}, \mathcal{L})$  and  $\Omega(\mathcal{V}, \mathcal{L}')$ , where the pure gauge solutions to the linearized equations,  $\mathcal{L}$  and  $\mathcal{L}'$ , are chosen to be identical in some (arbitrarily small) neighborhood of the hypersurface  $\Sigma$  in  $\mathcal{M}$  used to evaluate  $\Omega(\mathcal{V}, \mathcal{L})$ , but let  $\mathcal{L}'$  vanish on some other hypersurface. Such a pure gauge solution  $\mathcal{L}'$  can always be found because of the requirement that  $\mathcal{G}_s$  have arbitrary support. Because  $\Omega$  is defined by an integral involving the fields and their derivatives on  $\Sigma$ , it is clear that  $\Omega(\mathcal{V}, \mathcal{L}) = \Omega(\mathcal{V}, \mathcal{L}')$ . On the other hand, because  $\Omega$  is actually independent of the choice of the hypersurface and  $\mathcal{L}'$  vanishes on some hypersurface, we see that  $\Omega(\mathcal{V}, \mathcal{L}') = 0$ , and hence  $\Omega(\mathcal{V}, \mathcal{L}) = 0$  for all choices of  $\mathcal{V}$ . We see that to every infinitesimal gauge transformation corresponds a degenerate direction for  $\Omega$ .

As usual, one can show that in the degenerate case  $\Omega$  is the pull-back to  $\Gamma$  of a nondegenerate two-form,  $\omega$ , on the reduced phase space  $\hat{\Gamma}$ , which is the space of orbits in  $\Gamma$  of the group of gauge transformations. Thus  $\hat{\Gamma}$  is a symplectic manifold (possibly with singularities); functions on  $\hat{\Gamma}$  are the ‘‘observables’’ of the theory. As shown in Refs. 9 and 11, this definition of the reduced phase space and observables is formally equivalent to other standard definitions, e.g., that coming from the Hamiltonian formulation on the usual canonical momentum phase space.

Application of the covariant phase space formalism to parametrized field theory is relatively straightforward. The phase space  $\Gamma$  is the space of solutions to (2.5) and (2.6). A point in  $\Gamma$  is a pair  $(\varphi^A, X)$  satisfying these equations. Tangent vectors  $\mathcal{V}$  to  $\Gamma$  at  $(\varphi^A, X)$  are pairs  $(\delta\varphi^A, \delta X^\alpha)$ , which are solutions to the field equations which are linearized off  $(\varphi^A, X^\alpha)$ . Note  $\delta X^\alpha$  is a vector field on  $\mathcal{M}^\alpha$  generating a one-parameter family of diffeomorphisms.

The symplectic potential takes the form

$$\Theta_\Sigma(\mathcal{V}) = \int_\Sigma (\Pi_A \delta\varphi^A - h_\alpha \delta X^\alpha). \tag{3.6}$$

We can now take the exterior derivative (on  $\Gamma$ ) of  $\Theta$  to get the (pre)symplectic form  $\Omega$ . The explicit structure of  $\Omega$  depends on the specific form of the Lagrangian, but the general expression is of the form

$$\begin{aligned} \Omega(\mathcal{V}, \hat{\mathcal{V}}) = & - \int_{\Sigma \times \Sigma'} \left[ 2 \frac{\delta \Pi_A}{\delta \varphi'^B} \delta \varphi'^A \delta \hat{\varphi}'^B \right. \\ & + 2 \frac{\delta h_\alpha}{\delta X'^B} \delta X'^{\alpha A} \delta \hat{X}'^B + \frac{\delta h_\alpha}{\delta \varphi'^B} (\delta X'^\alpha \delta \hat{X}'^B \\ & - \delta \hat{X}'^\alpha \delta \varphi'^B) + \frac{\delta \Pi_A}{\delta X'^\alpha} (\delta \varphi'^A \delta \hat{X}'^\alpha \\ & \left. - \delta \hat{\varphi}'^A \delta X'^\alpha) \right], \end{aligned} \tag{3.7}$$

where we use the primes to distinguish fields at different spatial points (on the same hypersurface). Here  $\Omega$  is independent of the choice of  $\Sigma$ .

From the general argument presented above, we know that  $\Omega$  has a degenerate direction for each infinitesimal gauge transformation of the theory. Assuming the original (unparametrized) theory had no gauge invariances, the degeneracy of  $\Omega$  will stem from the action of infinitesimal diffeomorphisms on  $\Gamma$ . It is easily verified that given  $\phi \in \text{Diff}(\mathcal{M})$  and a solution  $(\varphi^A, X)$  to the equations (2.5) and (2.6), then  $(\phi^* \varphi^A, X \circ \phi)$  also satisfies these equations; this is simply the statement that the field equations are ‘‘covariant.’’ Now let  $\mathcal{L} = (L_\nu \varphi^A, L_\nu X)$  be the pure gauge vector field arising from the induced action on  $\Gamma$  of a one-parameter family of diffeomorphisms  $\phi_s$  of  $\mathcal{M}^\mu$  generated by the vector field  $\nu^\mu$ . [ $L_\nu$  denotes the Lie derivative and is defined as  $L_\nu \varphi^A = (d/ds) \phi_s^* \varphi^A|_{s=0}$ , and  $L_\nu X = (d/ds) X \circ \phi_s|_{s=0}$ .] Then it follows that  $\Omega(\mathcal{V}, \mathcal{L}) = 0 \forall \mathcal{V}$ . The reduced phase space  $\hat{\Gamma}$  is the space of orbits in  $\Gamma$  of  $\text{Diff}(\mathcal{M})$ . Actually, at this stage one has to make a choice. To obtain a reduced symplectic manifold, it is sufficient to pass to the space of orbits of the subgroup  $\text{Diff}_0(\mathcal{M}) \subset \text{Diff}(\mathcal{M})$  that is the connected component of the identity. From the point of view of dynamics as symplectic geometry, it requires additional physical input to identify points related by ‘‘large diffeomorphisms’’ in  $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$ . We will try to proceed in such a way that our results are independent of the choice made here. At any rate, a globally valid gauge which represents  $\hat{\Gamma}$  is to simply fix  $X$ , e.g.,  $X = \text{identity}$ , and we recover the original unparametrized description of the field theory on  $\mathcal{M}^\alpha$ .

#### IV. REPRESENTATIONS OF THE DIFFEOMORPHISM GROUP

There are two natural symplectic actions of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$ : one is a right action, the other is a left action. The right action of  $\phi \in \text{Diff}(\mathcal{M})$  is defined via

$$\Phi_{\text{right}}(\varphi^A, X) = (\phi^* \varphi^A, X \circ \phi). \tag{4.1}$$

Because  $\Omega$  as defined in (3.10) is independent of the choice of Cauchy surface, it is straightforward to verify that  $\Phi_{\text{right}}$  preserves the presymplectic structure.

Unlike the conventional Hamiltonian formulation of a generally covariant theory, the phase space representation of the Lie algebra  $\text{diff}(\mathcal{M})$  cannot be via the Poisson algebra of functions  $F$  on  $\Gamma$  because, in light of the degeneracy of the presymplectic form, the definition of such functions is trivial:

$$dF = \Omega(\mathcal{L}, \cdot) = 0. \tag{4.2}$$

The representation of  $\text{diff}(\mathcal{M})$  on  $\Gamma$  is via the one-parameter subgroups of  $\text{Diff}(\mathcal{M})$ , which are realized by vector fields on  $\mathcal{M}^\mu$ . These vector fields induce the pure gauge vector fields on  $\Gamma$ :

$$\mathcal{L} = (L_v \varphi^A, L_v X). \tag{4.3}$$

Note that because  $\Omega$  is closed and has degeneracy directions  $\mathcal{L}$  it follows that

$$L_{\mathcal{L}} \Omega = 0, \tag{4.4}$$

which is the infinitesimal version of the fact that  $\Phi_{\text{right}}$  preserves  $\Omega$ . Given a two-parameter family of symplectic diffeomorphisms generated by the two vector fields on  $\Gamma$ ,  $\mathcal{L} = (L_v \varphi^A, L_v X)$  and  $\mathcal{L}' = (L_w \varphi^A, L_w X)$ , it is a straightforward computation to show that the Lie bracket

$$[\mathcal{L}, \mathcal{L}'] = \mathcal{L}'' , \tag{4.5}$$

where

$$\mathcal{L}'' = (L_{[w,v]} \varphi^A, L_{[w,v]} X). \tag{4.6}$$

Thus the commutator algebra  $\text{vect}(\mathcal{M})$  of vector fields on  $\mathcal{M}^\mu$  is antihomomorphically mapped into the commutator algebra of ( $\Omega$  preserving) vector fields,  $\text{vect}(\Gamma)$  on  $\Gamma$ . Using the standard antihomomorphism from  $\text{diff}(\mathcal{M})$  into  $\text{vect}(\mathcal{M})$ , we obtain a homomorphism from  $\text{diff}(\mathcal{M})$  into  $\text{vect}(\Gamma)$ .

It is also possible to define a left action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  by letting the diffeomorphisms act on  $\mathcal{M}^\alpha$  and then using  $X$  to pull the results back to  $\mathcal{M}^\mu$ . Thus, given  $\phi \in \text{Diff}(\mathcal{M})$ , we obtain new points in  $\Gamma$  via

$$\Phi_{\text{left}} \cdot (\varphi^A, X) = (\tilde{\phi}^* \varphi^A, X \circ \tilde{\phi}), \tag{4.7}$$

where  $\tilde{\phi} = X^{-1} \circ \phi \circ X$ . Note that the left action of  $\text{Diff}(\mathcal{M})$  on  $X$  amounts to a new choice of  $X$  via  $X \rightarrow \phi \circ X$ , and this leads to a redefinition of  $\varphi^A$  in terms of  $\psi^A: \varphi^A = (\phi \circ X)^* \psi^A$ .

The left action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  is an antihomomorphism from  $\text{Diff}(\mathcal{M})$  into the group of (pre)symplectic diffeomorphisms of  $\Gamma$ . This can also be seen infinitesimally, i.e., at the level of Lie algebras. Fix a point

$(\varphi^A, X) \in \Gamma$ . A vector field  $v^\alpha$  on  $\mathcal{M}^\alpha$  generating a one-parameter family of diffeomorphisms of  $\mathcal{M}^\alpha$  defines a vector field on  $\mathcal{M}^\mu$  via

$$v^\mu = X^\mu_\alpha v^\alpha \circ X = (X^* v)^\mu. \tag{4.8}$$

Even though  $v^\mu$  so-defined is a “ $q$  number” (or in the language of Ref. 9 generates a field dependent local symmetry), it still leads to degenerate directions for  $\Omega$  through (4.4). As before, if we let  $\mathcal{L} = (L_{X^* v} \varphi^A, L_{X^* v} X)$  and  $\mathcal{L}' = (L_{X^* w} \varphi^A, L_{X^* w} X)$ , then the commutator of these two vector fields is given by

$$[\mathcal{L}, \mathcal{L}'] = \mathcal{L}'' , \tag{4.9}$$

where

$$\mathcal{L}'' = (L_{X^* [v,w]} \varphi^A, L_{X^* [v,w]} X). \tag{4.10}$$

The right action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  views  $\mathcal{M}^\mu$  as fundamental and  $(\varphi^A, X)$  as simply a collection of fields on  $\mathcal{M}^\mu$ . It is this action of the diffeomorphism group that is directly available on the covariant phase space of general relativity.<sup>10</sup> The key feature of the right action that makes it viable in general relativity is that it does not require a split of the phase space into nondynamical variables  $X$  and dynamical variables  $\varphi^A$ . The left action, on the other hand, stems from the action of  $\text{Diff}(\mathcal{M})$  on  $\mathcal{M}^\alpha$ , and it is only by identifying  $\mathcal{M}^\alpha$  as the image of  $\mathcal{M}^\mu$  under the map  $X$  that this action can be constructed. The left action is quite natural from the point of view of the parametrized field theory because it realizes the diffeomorphisms directly on  $\mathcal{M}^\alpha$ , which is essentially the goal of the parametrization process. It is unknown how to achieve such an action in general relativity. This would require a clean separation between gauge variables and dynamical variables, which is of course a long-standing problem in gravitation.

## V. OBSERVABLES

Because the symplectic structure is degenerate, in order to obtain a conventional phase space description one must pass to the reduced phase space  $\hat{\Gamma}$ , which can be identified with the space of diffeomorphism equivalence classes of the fields  $(\varphi^A, X)$  that satisfy the field equations. Functions on the reduced phase space are the “observables” of the theory. The observables can be represented as functions on  $\Gamma$  that are invariant under the (left or right) action of  $\text{Diff}(\mathcal{M})$  described in Sec. IV.

An important class of observables is obtained from any “constants of motion” that the the field theory for  $\psi^A$  may admit. More generally, if there exists a  $p$ -form  $\beta$  built from the fields  $\psi^A$  (and the metric  $g_{\alpha\beta}$ ) that is closed when  $\psi^A$  satisfies its equations of motion, then the integral  $Q_g[\psi]$  of  $\beta$  over a closed  $p$ -dimensional submanifold  $\sigma$ ,

$$Q_g[\psi] = \int_{\sigma} \beta, \tag{5.1}$$

is independent of the choice of  $\sigma$  (within its homology class). The subscript  $g$  indicates that  $Q$  will in general depend on the metric on  $\mathcal{M}^\alpha$ . Pulling  $\beta$  back to  $\mathcal{M}^\mu$  via  $X$  yields a closed  $p$ -form  $\beta' = X^*\beta$  on  $\mathcal{M}^\mu$  and an observable  $Q'[X, \varphi] := Q_{X^*g}[\varphi]$ . To see this, consider a diffeomorphism  $\phi$  of  $\mathcal{M}^\mu$  and let  $\sigma$  be some  $p$ -dimensional submanifold of  $\mathcal{M}^\mu$ . For simplicity, let us assume that we restrict our attention to orientation-preserving diffeomorphisms. For any integral we have the identity

$$\int_{\sigma} \phi^*\beta' = \int_{\phi(\sigma)} \beta', \tag{5.2}$$

where  $\phi(\sigma)$  is the image of  $\sigma$  under the diffeomorphism. Because  $\beta'$  is closed, the right-hand side of (5.2) is independent of the choice of closed  $p$ -dimensional submanifold within the homology class of  $\sigma$ , which is preserved by  $\text{Diff}(\mathcal{M})$ , hence we can replace  $\phi(\sigma)$  with  $\sigma$  to conclude:

$$\int_{\sigma} \phi^*\beta' = \int_{\sigma} \beta'. \tag{5.3}$$

Thus  $Q'[X \circ \phi, \phi^*\varphi] = Q'[X, \varphi]$ , and  $Q'$  is an observable.

Unfortunately, there is no guarantee that there are any such closed forms for a typical field theory, and even if they exist there will usually be only a finite number of them. What is usually desired is a *complete* set of observables that can serve (at least locally) as a set of coordinates on  $\hat{\Gamma}$ . For a field theory such a set is necessarily infinite dimensional.

One complete set of observables that is always available if the unparametrized field theory has no gauge symmetries are the fields  $\psi^A$  themselves. Let us exhibit these observables as functions on the covariant phase space  $\Gamma$ . Given a point  $(\varphi^A, X) \in \Gamma$ , we can obtain a collection of fields  $\theta^A$  on  $\mathcal{M}^\alpha$  via

$$\theta^A := X_*\varphi^A, \tag{5.4}$$

where  $X_*$  denotes the push forward of tensors on  $\mathcal{M}^\mu$  to tensors on  $\mathcal{M}^\alpha$  by  $X$ . By the way we constructed the parametrized formalism in Sec. II, it is clear that the fields  $\theta^A$ , defined by (5.4), satisfy the equations of motion (2.2) and hence are identifiable with the fields  $\psi^A$  of the unparametrized theory. Are the fields  $\theta^A$ , viewed as functions on  $\Gamma$ , observables? To see that they are we examine the right action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  and verify that  $\theta^A$  is invariant under this action. The right action of  $\phi \in \text{Diff}(\mathcal{M})$  on  $\theta^A$  is

$$\begin{aligned} \Phi_{\text{right}} \cdot \theta^A &= (X \circ \phi)_*(\phi^*\varphi^A) \\ &= (X^{-1})^*(\phi^{-1})^*(\phi^*\varphi^A) \\ &= (X^{-1})^*\varphi^A \\ &= X_*\varphi^A \\ &= \theta^A. \end{aligned} \tag{5.5}$$

It follows that  $\theta^A$  are also left invariant by the left action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$ .

### VI. RELATION TO THE CANONICAL THEORY

Let us now compare the canonical and covariant viewpoints on the phase space, the gauge group, and the observables.

#### A. Phase space

The covariant phase space  $\Gamma$  is built from space-time fields and space-time diffeomorphisms satisfying (2.5) and (2.6). The canonical phase space  $\Upsilon$  is built from spatial fields that are Cauchy data for (2.5), along with spacelike embeddings and their conjugate momenta. How can  $\Gamma$  and  $\Upsilon$  be related? Let us begin by answering the question at the level of the unparametrized field theory describing the fields  $\psi^A$  on  $\mathcal{M}^\alpha$ . Assuming the Cauchy problem is well posed, there is a bijection between the space of solutions to (2.2) and the set of Cauchy data for (2.2). In fact, there are an infinite number of ways to construct a bijection from the space of Cauchy data onto the space of solutions. This can be seen as follows. Introduce an arbitrary, but fixed, spacelike hypersurface  $\Sigma$  in  $\mathcal{M}^\alpha$ . Because the Cauchy problem is well posed, each set of Cauchy data on  $\Sigma$  leads to a unique solution of (2.2). Conversely, each solution to (2.2) induces a (unique) set of Cauchy data on  $\Sigma$ . For each choice of  $\Sigma$  such a correspondence can be made; each map between Cauchy data and space-time solutions is bijective provided the function spaces for the solution space and Cauchy data are appropriately chosen. The symplectic structures on the covariant phase space and on the space of Cauchy data are mapped into each other by the induced action of the bijection. More succinctly, the covariant phase space and the canonical phase space are symplectically diffeomorphic.

Now return to the parametrized theory. A point in  $\Gamma$  is determined by (i) picking a diffeomorphism, (ii) pulling back the prescribed metric on  $\mathcal{M}^\alpha$ , and (iii) solving the Euler-Lagrange equations (2.5), which are defined in terms of the pulled back metric. An allowed point in the canonical phase space lies in the constraint surface  $\bar{\Upsilon}$  defined via (2.17); a point in  $\bar{\Upsilon}$  is obtained by simply picking a spacelike embedding  $Q: \Sigma \rightarrow \mathcal{M}^\alpha$  and a set of Cauchy data on  $\Sigma$  [the embedding momenta are deter-

mined by the constraints (2.17)]. Corresponding to a given point in  $\Gamma$  there are an infinity of points in  $\bar{\Upsilon}$  because for every spacelike embedding there is a set of Cauchy data that generates the given solution. In the formalism based on  $\Upsilon$  it is precisely the canonical transformations generated by the constraint functions in (2.17) that map points in  $\bar{\Upsilon}$  to other points in  $\bar{\Upsilon}$  corresponding to the same spacetime solution. This redundancy in  $\Upsilon$  is somehow to be matched by the redundancy in  $\Gamma$ , which treats diffeomorphically related solutions as distinct.

The relation between  $\Gamma$  and  $\bar{\Upsilon}$  is again made by introducing an embedding  $Y: \Sigma \rightarrow \mathcal{M}^\mu$ ; for now we will not assume that the embedded hypersurface is spacelike. For each diffeomorphism  $X: \mathcal{M}^\mu \rightarrow \mathcal{M}^\alpha$  there is an embedding  $X \circ Y$  of  $\Sigma$  into  $\mathcal{M}^\alpha$ . In addition, the solutions to (2.5) (and their derivatives) can be pulled back to  $\Sigma$  using  $Y$ . Thus each point in  $\Gamma$  defines a point in the product of the space of Cauchy data for (2.5) [or (2.2)] and the space of embeddings of  $\Sigma$  into  $\mathcal{M}^\alpha$ . Let us denote this product space as  $\Upsilon'$  and the image of  $\Gamma$  in  $\Upsilon'$  as  $\Lambda_Y$ . Note that the map from  $\Gamma$  to  $\Upsilon'$  need not be surjective and certainly cannot be injective because two different diffeomorphisms  $X_1 \neq X_2$  can have the same action on a given hypersurface:  $X_1 \circ Y = X_2 \circ Y$ , and two distinct solutions to the field equations (2.5) can induce the same data on a slice provided the slice is not a Cauchy surface.

Because the space of *spacelike* embeddings is an open submanifold of the space of embeddings, it follows that  $\bar{\Upsilon}$  is an open submanifold of  $\Lambda_Y$ . The fact that the constraint surface arising in the canonical approach can be identified as a proper subset of the (image in  $\Upsilon'$  of the) covariant phase space has important repercussions for the action of the spacetime diffeomorphism group on the canonical phase space.

Let us denote the inverse image of  $\bar{\Upsilon}$  as  $\bar{\Gamma}$ , and  $\pi: \bar{\Gamma} \rightarrow \bar{\Upsilon}$  the surjection that assigns to a point  $(\varphi^A, X) \in \bar{\Gamma}$  the point  $(q^A, p^A, Q) \in \bar{\Upsilon}$ , where

$$\begin{aligned} q^A &= Y^* \varphi^A, \\ p^A &= Y^* L_n \varphi^A, \\ Q &= X \circ Y. \end{aligned} \tag{6.1}$$

The map  $\pi$  is not injective. To see why, let us think of  $\pi$  as taking a solution  $\varphi^A$  and a diffeomorphism  $X$  and constructing a spacelike embedding  $Q: \Sigma \rightarrow \mathcal{M}^\alpha$  and the Cauchy data on this hypersurface for the solution  $\psi^A = X_* \varphi^A$ . This interpretation is possible because (i)  $X \circ Y: \Sigma \rightarrow \mathcal{M}^\alpha$  is by assumption a spacelike embedding and (ii)  $Q_* X_* = Y^*$ . Now, if two points in  $\bar{\Gamma}$  are mapped to the same point in  $\bar{\Upsilon}$ , then, because the Cauchy problem is well posed, the two points in  $\bar{\Gamma}$  necessarily correspond to the same solution  $\psi^A$ . Because of the way the param-

etrized field theory is constructed from the field theory on  $\mathcal{M}^\alpha$ , or, equivalently, from our construction of observables in Sec. V, it is a simple exercise to see that this can happen if and only if the two points in  $\bar{\Gamma}$  are related by the (right) action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$ . Thus  $\pi$  fails to be injective whenever (i) one has two diffeomorphisms  $(X_1, X_2)$  that have the same action on the fiducial embedding  $Y$ , and (ii) the two diffeomorphisms and two corresponding solutions to (2.5),  $(\varphi_1^A, \varphi_2^A)$ , are related by the right action of (yet another) diffeomorphism  $\rho: \mathcal{M}^\mu \rightarrow \mathcal{M}^\mu$ . Notice that (i) and (ii) imply  $\rho$  must necessarily fix the embedding  $Y$ :

$$\rho \circ Y = Y. \tag{6.2}$$

Having spelled out the relationship between  $\bar{\Gamma}$  and  $\bar{\Upsilon}$ , let us relate the respective presymplectic structures. The presymplectic potential on the constraint surface (2.17)  $\bar{\Upsilon} \subset \Upsilon$  can be written as

$$\theta(\delta q^A, \delta \Pi_A, \delta Q) = \int_\Sigma (\Pi_A \delta q^A - h_\alpha \delta Q^\alpha). \tag{6.3}$$

The map  $\pi$  pushes forward a vector  $\mathcal{V} = (\delta \varphi^A, \delta X)$  tangent to  $\bar{\Gamma}$  at  $(\varphi^A, X)$  to a vector  $\pi_* \mathcal{V} = (Y^* \delta \varphi^A, Y^* L_n \delta \varphi^A, \delta X \circ Y)$  tangent to  $\bar{\Upsilon}$  at  $(Y^* \varphi^A, X \circ Y)$ . It follows from (3.6) that on  $\bar{\Gamma}$  we have  $\Theta_\Sigma(\mathcal{V}) = \theta(\pi_* \mathcal{V})$ , and hence  $\Theta_\Sigma = \pi^* \theta$ . This means that, on  $\bar{\Gamma}$ ,  $\Omega = d\Theta_\Sigma$  is the pull back by  $\pi$  of the presymplectic structure  $d\theta$  on the constraint surface in  $\Upsilon$ . Note that while the identification of  $\bar{\Gamma}$  with  $\bar{\Upsilon}$  is dependent on the choice of  $Y: \Sigma \rightarrow \mathcal{M}^\mu$ , the presymplectic structure itself is independent of the choice of  $Y$ .

### B. Gauge transformations

We have exhibited both a left and a right action of  $\text{Diff}(\mathcal{M})$  on the covariant phase space  $\Gamma$ . Because the map from  $\Gamma$  to  $\Upsilon'$  is neither one to one nor onto there is no reason to expect that we can push forward to  $\Upsilon'$  the Hamiltonian vector fields that generate the group action, and it is easy to check that, in fact, we cannot carry the group action from  $\Gamma$  to  $\Upsilon'$ . However, if we restrict attention to  $\pi: \bar{\Gamma} \rightarrow \bar{\Upsilon}$  the situation improves. It is still not possible to push forward the vector fields generating the right action, but it is possible to push forward the Hamiltonian vector fields generating the left action. To see this, we must check that the failure of  $\pi$  to be injective does not destroy the induced group action on  $\bar{\Upsilon}$ . Consider two points  $(\varphi_1^A, X_1)$  and  $(\varphi_2^A, X_2)$  in  $\bar{\Gamma}$  that map to the same point  $(Q, q^A, p^A)$  in  $\bar{\Upsilon}$ . Now consider an infinitesimal diffeomorphism  $\phi$  whose left action on  $\bar{\Gamma}$  gives two new points  $((X_1^{-1} \circ \phi \circ X_1)_* \varphi_1^A, \phi \circ X_1)$  and  $((X_2^{-1} \circ \phi \circ X_2)_* \varphi_2^A, \phi \circ X_2)$ . (We use an infinitesimal diffeomorphism so as to preserve the spacelike character of the embeddings; the infinitesimal action of one-parameter sub-



groups is sufficient for studying the Hamiltonian vector fields.) The infinitesimal group action carries over consistently to  $\bar{Y}$  because, as mentioned above, there must exist  $\rho \in \text{Diff}(\mathcal{M})$  that fixes  $Y$  and such that  $X_2 = X_1 \circ \rho$ ,  $\varphi_2^A = \rho^* \varphi_1^A$ . In detail

$$\begin{aligned} Q &\rightarrow Q'_1 = \phi \circ X_1 \circ Y, \\ q^A &\rightarrow q'^A_1 = Y^*(X_1^{-1} \circ \phi \circ X_1)^* \varphi^A_1, \\ p^A &\rightarrow p'^A_1 = Y^*(X_1^{-1} \circ \phi \circ X_1)^* L_n \varphi^A_1, \end{aligned} \tag{6.4}$$

is consistent with

$$\begin{aligned} Q &\rightarrow Q'_2 = \phi \circ X_2 \circ Y, \\ q^A &\rightarrow q'^A_2 = Y^*(X_2^{-1} \circ \phi \circ X_2)^* \varphi^A_2, \\ p^A &\rightarrow p'^A_2 = Y^*(X_2^{-1} \circ \phi \circ X_2)^* L_n \varphi^A_2, \end{aligned} \tag{6.5}$$

because (by assumption)

$$Q'_2 = \phi \circ X_2 \circ Y = \phi \circ X_1 \circ Y = Q'_1 \tag{6.6}$$

and

$$\begin{aligned} q'^A_2 &= Y^*(X_2^{-1} \circ \phi \circ X_2)^* \varphi^A_2 \\ &= ((X_1 \circ \rho)^{-1} \circ \phi \circ X_1 \circ Y)^* \rho^* \varphi^A_1 \\ &= Y^*(X_1^{-1} \circ \phi \circ X_1)^* \varphi^A_1 = q'^A_1, \end{aligned} \tag{6.7}$$

$$\begin{aligned} p'^A_2 &= Y^*(X_2^{-1} \circ \phi \circ X_2)^* L_n \varphi^A_2 \\ &= ((X_1 \circ \rho)^{-1} \circ \phi \circ X_1 \circ Y)^* \rho^* L_n \varphi^A_1 \\ &= Y^*(X_1^{-1} \circ \phi \circ X_1)^* L_n \varphi^A_1 = p'^A_1. \end{aligned} \tag{6.8}$$

In (6.8) we used the fact that  $\rho$  leaves invariant the hypersurface embedded by  $Y$  so that  $(\rho^* n)^\mu = n^\mu$ .

We see then that the infinitesimal left action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  can be carried over to the canonical phase space formalism, i.e., the Lie algebra  $\text{diff}(\mathcal{M})$  is realized on  $\bar{Y}$ , which was shown completely within the canonical approach by Isham and Kuchař.<sup>13</sup> However, the group action fails to carry over for two reasons. First, the failure of  $\pi$  to be 1-1 prevents the Hamiltonian vector fields generating the action of  $\text{Diff}(\mathcal{M})$  on  $\Gamma$  from being pushed forward to  $Y'$ . Second, while the vector fields on  $\bar{\Gamma}$  can be pushed forward to  $\bar{Y}$ , these vector fields cannot be complete because (as emphasized in Ref. 13)  $\text{Diff}(\mathcal{M})$  always maps some spacelike hypersurface into one that is not spacelike.

Often one does not exhibit the constraint functions in the form (2.17) as is natural when studying the representation of  $\text{diff}(\mathcal{M})$ , but rather in the projected form

$$\begin{aligned} H_1 &:= n^\alpha H_\alpha \approx 0, \\ H_a &:= Q_a^\alpha H_\alpha \approx 0. \end{aligned} \tag{6.9}$$

When smeared with a scalar function  $N^1$  and a vector  $\mathbf{N}$  on  $\Sigma$ , these constraint functions respectively generate canonical transformations corresponding to normal and tangential deformations of the embedding of  $\Sigma$  into  $\mathcal{M}^\alpha$ . This is also the meaning of the constraints occurring in general relativity although it is not known how to cast them into the parametrized form. In contrast with the deformation of a hypersurface along some arbitrary vector field, a *normal* deformation involves the metric and this leads to the well-known complication that the Poisson algebra of the projected constraint functions cannot represent a Lie algebra. Thus if we define

$$\begin{aligned} H(N^1) &:= \int_\Sigma N^1 H_1, \\ H(\mathbf{N}) &:= \int_\Sigma N^a H_a, \end{aligned} \tag{6.10}$$

then we have the Poisson brackets

$$\begin{aligned} [H(N^1), H(M^1)] &= H(\mathbf{J}), \\ [H(\mathbf{N}), H(\mathbf{M})] &= H(L_{\mathbf{N}} \mathbf{M}), \\ [H(N^1), H(\mathbf{M})] &= H(-L_{\mathbf{M}} N^1), \end{aligned} \tag{6.11}$$

where

$$\mathcal{J}^a = \gamma^{ab} (N^1 \partial_b M^1 - M^1 \partial_b N^1), \tag{6.12}$$

and hence the finite (as opposed to infinitesimal) canonical transformations generated by the projected constraint functions cannot realize a Lie group. The “open algebra” (6.11) can be summarized in terms of functions  $H(N^1, \mathbf{N})$  on  $Y$ , where

$$H(N^1, \mathbf{N}) := H(N^1) + H(\mathbf{N}) \tag{6.13}$$

satisfies

$$[H(N^1, \mathbf{N}), H(M^1, \mathbf{M})] = H(L_{\mathbf{N}} M^1 - L_{\mathbf{M}} N^1, L_{\mathbf{N}} \mathbf{M} + \mathbf{J}). \tag{6.14}$$

The projected constraint functions are distinguished by the fact that their Hamiltonian vector fields are complete; in particular, the finite transformations generated by these functions map spacelike hypersurfaces to spacelike hypersurfaces. For this reason, despite the technical complexity of the algebraic structure involved, one may choose to view the constraints (6.9) and the “hypersurface deformation algebra” (6.14) as fundamental. At any rate, there is no known alternative to this “open algebra”

in general relativity. Consequently, it is of interest to interpret this algebra from the perspective of the covariant phase space.

Because there are no hypersurfaces to be found in the realm of the covariant phase space, in order to make contact with the algebra (6.14) some spacelike hypersurfaces will have to be provided. So, for the purposes of the present discussion, let us assume that we identify  $\mathcal{M}^\mu = R \times \Sigma$  and require that the diffeomorphisms  $X$  are in fact spacelike foliations  $X: R \times \Sigma \rightarrow \mathcal{M}^\alpha$ . [If desired, a fixed spacelike foliation  $Y: R \times \Sigma \rightarrow \mathcal{M}^\mu$  can be introduced, and points in the covariant phase space identified with pairs  $(Y^* \varphi^A, X \circ Y)$ .] Each foliation  $X$  provides  $\mathcal{M}^\alpha$  with an adapted hypersurface basis built from the unit normal and tangent vectors to the hypersurfaces in  $\mathcal{M}^\alpha$ , which are defined as in Sec. II. In particular,

$$\begin{aligned} n_\alpha X_a^\alpha &= 0, \\ g^{\alpha\beta} n_\alpha n_\beta &= -1. \end{aligned} \tag{6.15}$$

A basis on  $\mathcal{M}^\mu$  can then be obtained by pull-back from that on  $\mathcal{M}^\alpha$ , the relationship being

$$\begin{aligned} n^\mu &= X_a^\mu n^\alpha \circ X, \\ X_a^\mu &= X_a^\mu X_a^\alpha \circ X. \end{aligned} \tag{6.16}$$

On  $\Gamma$ , the hypersurface deformations can be viewed as a modification of the *right* action of the infinitesimal diffeomorphisms that is available when one has a space-like foliation. We build a vector field  $N^\mu$  on  $\mathcal{M}^\mu = R \times \Sigma$  by specifying the amount of normal ( $N^\perp$ ) and tangential ( $N^\alpha$ ) deformation of each hypersurface  $\Sigma$ :

$$N^\mu = N^\perp n^\mu + N^\alpha X_a^\mu. \tag{6.17}$$

To compute the induced action of these vector fields on the covariant phase space we consider the pure gauge vector fields  $\mathcal{L}, \mathcal{L}'$  on  $\Gamma$  defined by  $\mathcal{L} = (L_N \varphi^A, L_N X)$ ,  $\mathcal{L}' = (L_M \varphi^A, L_M X)$  ( $M^\mu$  is defined similarly to  $N^\mu$ ) and compute the commutator  $[\mathcal{L}, \mathcal{L}']$ . The variations in  $n^\mu$  are computed using (6.15) and (6.16); after a straightforward computation we find

$$[\mathcal{L}, \mathcal{L}'] = \mathcal{L}'', \tag{6.18}$$

where

$$\begin{aligned} \mathcal{L}'' &= (L_{N \cdot M} \varphi^A, L_{N \cdot M} X), \\ (N \cdot M)^\mu &= (M^\alpha X_a^\nu \nabla_\nu N^\perp - N^\alpha X_a^\nu \nabla_\nu M^\perp) n^\mu \\ &\quad + (\gamma^{ab} X_b^\nu (M^\perp \nabla_\nu N^\perp - N^\perp \nabla_\nu M^\perp) \\ &\quad + (M^b \nabla_b N^\alpha - N^b \nabla_b M^\alpha) X_a^\mu, \end{aligned} \tag{6.19}$$

and  $\gamma^{ab}$  is the inverse metric induced on the leaves of the foliation. Comparing (6.19) with (6.14) it follows that the construction  $(N^\perp, N^\alpha) \rightarrow N^\mu \rightarrow \mathcal{L}$  described above represents an antihomomorphism from the algebra of hypersurface deformations into the algebra of ( $\Omega$  preserving) vector fields on  $\Gamma$ .

### C. Observables

In the canonical Hamiltonian formulation of dynamical systems with (first class) constraints “observables” are defined as functions on the phase space that have a vanishing Poisson bracket with the constraint functions modulo the constraints. More geometrically, observables are functions on the phase space which project to the space of orbits of the Hamiltonian vector fields in the constraint surface. These abstract ways of defining observables are meant to capture the notion of observables as “gauge invariant” functions on the physically accessible portion of the phase space.

In the canonical formulation of parametrized field theories the constraint functions generate canonical transformations corresponding to the change in the phase space data as the hypersurface they are on is deformed through space-time. The embeddings change according to the deformation, the truly dynamical variables  $(q^A, \Pi_A)$  change according to the dynamical equations, and this induces the change in the embedding momenta via the constraints (2.17) that are preserved in the course of the dynamical evolution. Because this motion on  $\Upsilon$  can be viewed as the infinitesimal action of  $\text{Diff}(\mathcal{M})$ ,<sup>13</sup> it follows that in the canonical formalism the observables can be equivalently characterized as either constants of motion or invariants under infinitesimal diffeomorphisms. The latter characterization makes direct contact with the covariant phase space notion of observables, but the observables constructed in Sec. V differ somewhat from the constant of motion observables on  $\Upsilon$ .

To see this we must spell out the construction of observables in canonical parametrized field theory, which is essentially an application of Hamilton–Jacobi theory. Imagine solving the Hamilton equations of motion for the canonical variables. This can be done by solving the many-fingered time functional differential equations<sup>6</sup>

$$\begin{aligned} \frac{\delta q^A(x)}{\delta Q^\alpha(y)} &= [q^A(x), H_\alpha(y)], \\ \frac{\delta \Pi_A(x)}{\delta Q^\alpha(y)} &= [\Pi_A(x), H_\alpha(y)], \end{aligned} \tag{6.20}$$

along with the constraints (2.17). Here the brackets are the Poisson brackets, and we allow the usual abuse of notation which identifies the solutions to the equations of motion with the canonical variables themselves. The so-

lution is thus specified by giving the canonical data as a functional of the embeddings  $Q$  and a set of initial data  $(\hat{q}^A, \hat{\Pi}_A)$  on an initial embedding  $\hat{Q}$ :

$$\begin{aligned} q^A &= q^A[Q, \hat{q}^A, \hat{\Pi}_A], \\ \Pi_A &= \Pi_A[Q, \hat{q}^A, \hat{\Pi}_A], \end{aligned} \quad (6.21)$$

$$\begin{aligned} q^A[\hat{Q}, \hat{q}^A, \hat{\Pi}_A] &= \hat{q}^A, \\ \Pi_A[\hat{Q}, \hat{q}^A, \hat{\Pi}_A] &= \hat{\Pi}_A. \end{aligned} \quad (6.22)$$

For each  $Q$  (and  $\hat{Q}$ ) Eqs. (6.21) specify a diffeomorphism  $(\hat{q}^A, \hat{\Pi}_A) \rightarrow (q^A, \Pi_A)$  that preserves the natural symplectic structure on the space of pairs  $(q^A, \Pi_A)$ . In other words, dynamical evolution is a canonical transformation. Inverting the map (6.21) amounts to expressing the initial data as a functional of the solution:

$$\begin{aligned} \hat{q}^A &= \hat{q}^A[Q, q^A, \Pi_A], \\ \hat{\Pi}_A &= \hat{\Pi}_A[Q, q^A, \Pi_A]. \end{aligned} \quad (6.23)$$

Because initial data are always “constants of the motion,” the functionals on  $\Upsilon$  specified in (6.23) will have (strongly) vanishing Poisson brackets with the constraint functions,

$$[\hat{q}^A, H_\alpha] = 0 = [\hat{\Pi}_A, H_\alpha], \quad (6.24)$$

and therefore represent a set of observables. Obviously, this set is complete. Therefore, in canonical parametrized theory the natural observables correspond to the freely specifiable Cauchy data  $(\hat{q}^A, \hat{\Pi}_A)$  on a hypersurface determined by  $\hat{Q}$ . Note that this means that there will always be a symplectic diffeomorphism that identifies the observables with points in the canonical phase space for the unparametrized theory.

Our construction of the observables  $\mathcal{O}^A$  on the covariant phase space also led back to the unparametrized theory: the space of observables is equivalent to the space

of solutions to (2.2). As mentioned above, the space of solutions to (2.2) is symplectically diffeomorphic to the space of Cauchy data for (2.2) which is, in turn, (assumed) equivalent to the canonical phase space of the unparametrized theory. Thus the reduced phase spaces in each case coincide:  $\hat{\Gamma} \simeq \hat{\Upsilon}$ .

Notice however that it is *only* the reduced phase spaces that coincide. The relation between  $\Gamma$  and  $\Upsilon$  is not entirely simple; indeed, one can at best identify  $\bar{\Upsilon}$  with an open subset of  $\Gamma$ . In particular,  $\Gamma$  admits an action of  $\text{Diff}(\mathcal{M})$  but  $\Upsilon$  (or  $\bar{\Upsilon}$ ) does not. Thus, while  $\hat{\Gamma}$  arises as  $\hat{\Gamma} = \Gamma / \text{Diff}(\mathcal{M})$ ,  $\hat{\Upsilon}$  is obtained as the space of orbits of a much more complicated structure than a Lie group and these orbits are in a rather different space than  $\Gamma$ . In this sense it is perhaps remarkable that  $\hat{\Gamma} \simeq \hat{\Upsilon}$ .

#### ACKNOWLEDGMENT

I would like to thank Karel Kuchař for discussions.

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