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Asymptotic Conservation Laws in Classical Field Theory

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A new, general, field theoretic approach to the derivation of asymptotic conservation laws is presented. In this approach asymptotic conservation laws are constructed directly from the field equations according to a universal prescription which does not rely upon the existence of Noether identities or any Lagrangian or Hamiltonian formalisms. The resulting general expressions of the conservation laws enjoy important invariance properties and synthesize all known asymptotic conservation laws, such as the Arnowitt-Deser-Misner energy in general relativity. [S0031-9007(96)01656-0]

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In nonlinear gauge theories such as Yang-Mills theory and general relativity, conserved quantities such as charge and energy-momentum are computed from the limiting values of two-dimensional surface integrals in asymptotic regions. Such *asymptotic conservation laws* are most often derived by one of two, rather distinct, methods. One method, applicable to gauge theories such as Yang-Mills and general relativity, relies upon the construction of identically conserved currents furnished by Noether's theorem [1–3] and subsequent extraction of a "superpotential" to define a conserved surface integral. Unfortunately, there is no general field-theoretic criterion to select the appropriate current: For *any* field theory there are infinitely many currents that can be expressed as the divergence of a skew-symmetric super-potential, modulo the field equations. This method of finding asymptotic conservation laws is thus somewhat *ad hoc*. An alternative approach to finding asymptotic conservation laws in gauge theories is based upon the Hamiltonian formalism. In this approach, asymptotic conservation laws arise as surface term contributions to symmetry generators [2,4–6]. The Hamiltonian approach to finding asymptotic conservation laws lacks the *ad hoc* flavor of the superpotential formalism but is somewhat indirect: To construct asymptotic conservation laws using this method one must have the Hamiltonian formalism well in hand and one must know *a priori* the general form of the putative symmetry generator in order to find the appropriate surface integral.

The purpose of this Letter is to describe a new, general construction of asymptotic conservation laws for classical field theories which provides a viable alternative to existing methods. Our construction is based upon a remarkable new spacetime differential form Ψ, which we derive for *any* system of field equations. We show, by virtue of an identity involving the exterior derivative $D\Psi$ and the linearized field equations, that Ψ defines asymptotic conservation laws (when they exist) for any field theory. Thus we are able to establish that asymptotic conservation laws for field theories can be viewed as arising from (asymptotically) closed differential forms canonically associated to the field equations. We also show that Ψ possesses a number of attractive field theoretical properties. In particular, this differential form is constructed directly from the field equations according to a universal prescription which does not rely upon the existence of Noether identities or any Lagrangian or Hamiltonian formalisms. The form enjoys important invariance properties and synthesizes all known asymptotic conservation laws; for example, it is easily seen to reduce to the Arnowitt-Deser-Misner (ADM) energy in general relativity. We emphasize that in obtaining these results no *a priori* assumptions are made concerning the space of gauge symmetries of the theory. The symmetries required for the existence of asymptotic conservation laws are automatically derived as a consequence of our general formalism. Thus our formalism contains a number of distinct field theoretical advantages. With regard to specific applications, we are able to extend existing results on asymptotic conservation laws in a number of ways. For example, we obtain covariant expressions for asymptotic conservation laws which can be used to generalize the conservation laws in asymptotically flat general relativity and Yang-Mills theory to other asymptotic structures. We also obtain a formula which generalizes the asymptotic conservation laws of the Einstein equations to arbitrary second-order metric theories. To illustrate this latter point, we consider a class of "string-generated gravity models" [7]. It follows immediately from our formalism that the standard ADM formulas still describe conservation of energy, momentum, and angular momentum for these theories, at least in the asymptotically flat context.

To describe our results in more detail, let us fix an n-dimensional spacetime manifold M, with local coordinates x^i, $i = 0, 1, \dots, n-1$, and let us label the totality of fields for our classical theory by $\varphi^A, A = 1, \dots, N$. These fields are subject to a system of field equations which we write as

$$\Delta_B(\varphi^A, \varphi^A{}_{,i}, \varphi^A{}_{,ij}) = 0\,. \tag{1}$$

We have postulated that the field equations are of second order only for the sake of simplicity. The general theory which we outline here is developed in [8] for field equations of arbitrary order. To address the problem of finding asymptotic conservation laws for (1), we begin by first broadening the usual notion of a local conservation law. We say that a *conservation law of degree p* for (1) is a spacetime p form

$$\omega = \omega_{k_1 \dots k_p}[x^i, \varphi^A] dx^{k_1} \wedge \cdots \wedge dx^{k_p}\,, \tag{2}$$

depending locally on the fields φ^A and their derivatives to some finite order, such that the exterior derivative

$$D\omega = (D_k \omega_{k_1 \dots k_p}) dx^k \wedge dx^{k_1} \wedge \cdots \wedge dx^{k_p}\,, \tag{3}$$

vanishes on all solutions to the field equations. In (3), D_k is the usual total derivative operator. When $p = n-1$, we may express (2) in the form $\omega = \varepsilon_{kk_1 \cdots k_{n-1}} J^k dx^{k_1} \wedge \cdots \wedge dx^{k_{n-1}}$, in which case the vanishing of (3) coincides with the vanishing of the divergence $D_k J^k$ of the current J^k. Accordingly, we shall say that the p-form conservation law (2) is an *ordinary or classical conservation law* in the case $p = n - 1$ and a *lower-degree conservation law* when $p < n - 1$.

Recently ([8–13]) a number of methods have been developed for the systematic computation of all lower-degree conservation laws for field equations such as (1). *The central premise of this note is that the techniques introduced in [8] can be successfully adapted to the analysis of asymptotic conservation laws.* The principal ideas are as follows. If ω is a lower-degree conservation law for $\Delta_B = 0$, then it is readily checked that the variation $\delta_h \omega$ of ω with respect to field variations $h^A = \delta\varphi^A$ is closed by virtue of the field equations $\Delta_B = 0$ and their formal linearization $\delta_h \Delta_B = 0$. Such linearized conservation laws play a pivotal role in the general theory of lower-degree conservation laws, as presented in [8] (see also [14]). In [8] it is shown that every closed p form $\omega[h]$, depending upon the fields φ^A and their derivatives and linearly on the field variations h^A and their derivatives, can be cast into a *universal normal form* $\Psi_\rho[h]$, which depends upon certain auxiliary fields ρ that are subjected to a set of algebraic and differential constraints. This normal form, which forms the basis for our construction of asymptotic conservation laws, is obtained as follows. Details, generalizations, and further examples of this construction will appear in [8].

To begin, it is helpful to write the formal linearization of the field equations (1) as

$$\delta_h \Delta_B = \sigma^{ij}_{AB} h^A{}_{,ij} + \sigma^i_{AB} h^A{}_{,i} + \sigma_{AB} h^A\,.$$

We then define a *linear lower-degree conservation law* for the field equations (1) to be a p form $\omega[h]$, where $p < n - 1$, of the type

$$\omega[h] = M_A h^A + M^{i_1}_A h^A{}_{,i_1} + \cdots + M^{i_1 i_2 \cdots i_k}_A h^A{}_{,i_1 i_2 \cdots i_k}\,, \tag{4}$$

which satisfies

$$D\omega[h] = \rho^B \delta_h \Delta_B + \rho^{Bi_1} D_{i_1} \delta_h \Delta_B + \cdots + \rho^{Bi_1 i_2 \cdots i_k} D_{i_1 i_2 \cdots i_k} \delta_h \Delta_B\,. \tag{5}$$

In Eqs. (4) and (5) the coefficients $M^{i_1 i_2 \cdots i_l}_A$ and $\rho^{Bi_1 i_2 \cdots i_l}$ are spacetime p forms and $(p + 1)$ forms, respectively, which depend on the fields φ^A and their derivatives. Equation (5) is an identity in the field variations h^A and their derivatives but is still subject to the field equations $\Delta_B = 0$. The next step is to derive equations for these multipliers ρ from the integrability condition $D^2\omega[h] = 0$. It is a remarkable fact that the highest order multiplier $\rho^{Bi_1 i_2 \cdots i_k}$ is thus constrained by purely algebraic condition

$$\rho^{B(i_1 i_2 \cdots i_k} \wedge dx^l \sigma^{jh)}_{AB} = 0\,. \tag{6}$$

We call this fundamental equation *the algebraic Spencer equation for the linear conservation law $\omega[h]$ for the field equations (1)* [15]. For the field equations that one typically considers, it is not too difficult to apply standard methods from tensor algebra to solve the Spencer equation (6) [8]. The solutions to (6) often allow one to simplify the identity (5), by repeated "integration by parts," to the reduced form

$$D\omega[h] = \rho^B \delta_h \Delta_B\,. \tag{7}$$

We remark that for Lagrangian theories, full knowledge of the gauge symmetries of the theory often allows one to pass directly to the reduced form (7).

Define $\rho^{Bi} = dx^i \wedge \rho^B$ and define μ^B_i to be the interior product $\mu^B_i = \partial/dx^i \lrcorner \rho^B$. Assuming that the reduced equation (7) holds, we then have the following complete characterization of all linear lower-degree conservation laws of degree $p < n - 1$ [8].

Theorem (classification of linear lower degree conservation laws).—Let $\Delta_B(\varphi^A, \varphi^A{}_{,i}, \varphi^A{}_{,ij}) = 0$ be a system of second-order field equations and suppose the linear p form $\omega[h]$ satisfies the reduced equations (7). Then (i) the conservation law multiplier ρ^B satisfies

$$\rho^{B(i}\sigma^{jk)}_{AB} = 0, \qquad \rho^{B(i}\sigma^{j)}_{AB} + D_k\rho^{B(i}\sigma^{jk)}_{AB}] = 0\,,$$
$$\rho^{Bi}\sigma_{AB} + D_j[\rho^{B(i}\sigma^{j)}_{AB}] = 0\,; \qquad \text{(8a,b,c)}$$

(ii) *the p form*

$$\Psi_\rho[h] = \frac{1}{q}\,h^A\left[\mu^B_i\sigma^i_{AB} - \frac{2}{q+1}D_j(\mu^B_i\sigma^{ij}_{AB})\right] + \frac{2}{q+1}\,h^A_j[\mu^B_i\sigma^{ij}_{AB}], \qquad (9)$$

where $q = n - p$, is closed by virtue of the equations $\Delta_B = 0$, $\delta_h\Delta_B = 0$, and (8); and (iii) *every linear p form $\omega[h]$ satisfying (7) differs from $\Psi_\rho[h]$ by an exact linear $(p - 1)$ form.*

We remark that this theorem can be readily generalized to the case where (5) holds, rather than (7).

To use this theorem for computing asymptotic conservation laws we proceed as follows. Suppose the spacetime M is non-compact and admits an asymptotic region which is diffeomorphic to $\mathbf{R} \times C'$, where C' is the complement of a compact set in $\mathbf{R}^{n-1}$. Fix local coordinates $(t, x^1, \ldots, x^{n-1})$ in the asymptotic region. We consider a fixed solution $\mathring{\varphi}^A$ to the field equations and then, given a second solution φ^A, we set $h^A = \varphi^A - \mathring{\varphi}^A$. Let $\omega[\mathring{\varphi}, h]$ be a spacetime $(n - 2)$ form depending locally on the fields $\mathring{\varphi}^A$ and h^A and their derivatives. We call ω *an asymptotic conservation law for the field equations* $\Delta_B = 0$ *relative to the background* $\mathring{\varphi}$ if, whenever h^A satisfies the appropriate asymptotic decay condition as $r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2} \longrightarrow \infty$, the p form $\omega[\mathring{\varphi}, h]$ satisfies $\omega \sim O(1)$ and $D\omega \sim O(1/r)$. Under these conditions the limit $\lim_{r\to\infty} \times \int_{S_{(r,t_0)}} \omega[\mathring{\varphi}, h]$, where $S_{(r,t)}$ is an $(n - 2)$-dimensional sphere of radius r in the $t = t_0$ hypersurface, exists and is independent of t_0. We therefore can deduce from our classification theorem that if the asymptotic boundary conditions are such that

(i) $\Psi_\rho[h] \sim O(1)$,

(ii) the equations $\delta_h\Delta_B(\mathring{\varphi}) = 0$ hold asymptotically to an order such that $\rho^B\delta_h\Delta_B(\mathring{\varphi}) \sim O(1/r)$, and

(iii) the conservation law multiplier equations (8) hold, not exactly and not for all field values, but only at $\mathring{\varphi}$ and only to the appropriate asymptotic order, then *the normal form $\Psi_\rho[h]$ will be an asymptotic conservation law for the field equations* $\Delta_B = 0$ *relative to the background solution* $\mathring{\varphi}$. We note that the conditions (i), (ii), and (iii) can be relaxed so long as the relevant integrals arising from the application of Stokes Theorem to (7) vanish asymptotically.

To demonstrate the utility of our general theory of asymptotic conservation laws, and to expose some of its novel features, we consider the vacuum Einstein field equations. Here, as a consequence of (8a), the auxiliary fields ρ are determined by a single vector field X and the normal form $\Psi_\rho[h]$ becomes

$$\Psi_X[h] = \frac{1}{16\pi}\,\varepsilon_{ijhk}\left[-\frac{2}{3}(\nabla_u h_{rs})X_t\sigma^{ti,rsuj} + \frac{1}{3}\,h_{rs}(\nabla_u X_t)\sigma^{ti,rsuj}\right] \times dx^h \wedge dx^k\,, \qquad (10)$$

where ∇ is the covariant derivative defined by the Chritoffel symbols of $\mathring{g}$ and $\sigma^{rs,ijhk} = \partial G^{rs}/\partial g_{ij,hk}(g_0)$ is the symbol of the Einstein tensor. In expanded form, $\Psi_X[h]$ becomes

$$\Psi_X[h] = \frac{1}{32\pi}\,\varepsilon_{ijhk}[h^{li}\nabla_l X^j - \frac{1}{2}\,h\nabla^i X^j - X_l\nabla^i h^{lj} + X^i(\nabla_l h^{jl} - \nabla^j h)]dx^h \wedge dx^k\,, \qquad (11)$$

where $h = \mathring{g}^{ij}h_{ij}$, and indices are raised and lowered with respect to the background metric. The differential form (11) recently appeared in [2] [see Eqs. (61), (63), and (75)], where it was derived from the Noether identities and used in the study of black hole entropy.

The spacetime form $\Psi_X[h]$ possesses the following desirable properties which reflect general features of our formalism [8].

(1) The form $\Psi_X[h]$ is closed by virtue of the Einstein equation for the background $\mathring{g}$, and the linearized Einstein equations for the variation h, provided that X is a Killing vector field of the background. By using the classification theorem above it can be shown that, modulo 2-forms which are exact by virtue of these equations, $\Psi_X[h]$ is the *only* linear lower-degree conservation law for the vacuum Einstein equations.

(2) It is readily established that under the gauge transformation $h_{ij} \longrightarrow h_{ij} + \Delta_{(i}Y_{j)}$ the form $\Psi_X[h]$ changes by an exterior derivative of a canonically defined one-form. This implies that $\Psi_X[h]$ is suitably "gauge invariant."

(3) The form $\Psi_X[h]$ is constructed directly from the field equations in a covariant fashion and with no reference made to the Bianchi identities or to any Lagrangian or Hamiltonian. Indeed, most properties of $\Psi_X[h]$ can be inferred directly from properties of the symbol of the Einstein tensor.

(4) If the vector field X is an asymptotic Killing vector field for $\mathring{g}$ and if $h = g - \mathring{g}$ satisfies appropriate decay conditions, then $\Psi_X[h]$ is always an asymptotic conservation law for the Einstein field equations.

(5) In terms of the $(3 + 1)$ formalism, a lengthy but straightforward computation shows that the pullback ω of

$\Psi_X[h]$ to a leaf of a foliation of spacetime by spacelike hypersurfaces becomes

$$\omega = \frac{1}{16\pi}[G^{abcd}(X^{\perp}\tilde{\nabla}_b h_{cd} - h_{cd}\tilde{\nabla}_b X^{\perp}) + 2X^b \delta_h \pi^a_b - X^a h_{cd}\pi^{cd} - \tilde{\nabla}_b(h^a_{\perp}X^b - h^b_{\perp}X^a)]d^2S_a . \tag{12}$$

In this equation (i) $X^{\perp}$ and X^b are the normal and tangential components of a Killing vector with respect to the spacelike hypersurface and $h^a_{\perp}$ is the normal-tangential component of h^{ij}; (ii) the derivative operator $\tilde{\nabla}_b$ is compatible with the background metric γ_{ab} induced on the hypersurface; (iii) we have defined

$$G^{abcd} = \frac{1}{2}\gamma^{1/2}(\gamma^{ac}\gamma^{bd} + \gamma^{ad}\gamma^{bc} - 2\gamma^{ab}\gamma^{cd});$$

and (iv) $\pi^a_b = \gamma_{bc}\pi^{ac}$, where π^{ac} is the canonical field momentum. With asymptotically flat or asymptotically anti-de Sitter boundary conditions the integral of (12) over the sphere at infinity coincides with the standard ADM-type formulas ([4,6,16]) for the asymptotic conservation laws in this context

Thus our construction is general enough to capture the usual conservation laws in general relativity. Moreover, Ψ_X provides a means of generalization of these conservation laws to other asymptotic structures. In particular, the presence of the term $X^a h_{cd}\pi^{cd}$ in (12) coupling h_{ab} to the canonical momentum of the background, which is dictated by our derivation of ω from a covariant spacetime two-form $\Psi_X[h]$, has to our knowledge not appeared in previous ADM-type formulas. While this term does not contribute to the surface integrals at infinity for the asymptotically flat and asymptotically anti-de Sitter boundary conditions, we believe that its inclusion should be necessary for other asymptotic conditions.

Let us briefly describe other applications of our formalism. First of all, the classification theorem we have derived for linear lower-degree conservation laws is used in [8] to classify all conservation laws for the Einstein equations, Yang-Mills equations, and similar equations. With regard to asymptotic conservation laws, in Yang-Mills theory our classification theorem leads to the standard surface integral formulas in the asymptotically flat context [8]. Work is in progress to investigate Yang-Mills conservation laws in the presence of other asymptotic conditions. As another application, we consider the asymptotic conservation laws for any covariant metric theory described by second-order field equations. We can show that the normal form (10) generalizes without change to all such theories (provided, of course, that one uses the symbol appropriate to the field equations of interest). To illustrate this point, we consider the string-generated gravity models, which are based upon the Lovelock Lagrangian density [7]. Using the modified symbol of the string-corrected Einstein equations, it is straightforward to compute the asymptotic conservation laws for string generated gravity in the asymptotically flat content from (10). Because the string-corrected symbol reduces to that of the Einstein equations on a flat background, it follows immediately that the conservation laws (11), (12) still apply in the presence of the corrections to general relativity suggested by string theory.

In summary, the theory of linearized conservation laws leads to a new, efficient, systematic, and covariant method for obtaining asymptotic conservation laws in field theory. We expect that this approach will prove useful for a wide range of field theories and asymptotic boundary conditions which have been heretofore unexplored and we anticipate that the spacetime form $\Psi_\rho[h]$ will be a valuable tool in the further study of conservation laws in field theory.

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partial differential equations. But, to paraphase Bryant and Griffiths [10](p. 578), one is lead inevitably to (6) and the resulting theory of Spencer cohomology once one agrees to study lower-degree conservation laws in field theory.

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