Gravitational waves: just plane symmetry

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Abstract. We present some remarkable properties of the symmetry group for gravitational plane waves. Our main observation is that metrics with plane wave symmetry satisfy every system of generally covariant vacuum field equations *except* the Einstein equations. The proof uses the homothety admitted by metrics with plane wave symmetry and the scaling behavior of generally covariant field equations. We also discuss a mini-superspace description of spacetimes with plane wave symmetry.

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1. Introduction

We would like to present some remarkable properties of the symmetry group of gravitational plane waves. Our main observation is that metrics with plane wave symmetry satisfy every system of generally covariant vacuum field equations *except* the Einstein equations. Put slightly differently, the only non-trivial field equations that can be imposed on a metric with plane wave symmetry are equivalent to the vacuum Einstein equations. Let us call a Ricci-flat metric that admits plane wave symmetry a *gravitational plane wave.* Gravitational plane waves satisfy every generally covariant system of vacuum field equations and so can be called "universal solutions" of the Einstein equations.

Related results can be found in work by Horowitz and Steif [1]. In [1] it is shown that metrics which (1) are Ricci-flat, (2) admit a constant (null) vector field*, satisfy all other field equations that are symmetric rank two tensors covariantly constructed from scalar invariants and polynomials in the curvature and their covariant derivatives. By contrast, our hypothesis is only that the metric admits plane wave symmetry, and the field equations being considered are any symmetric rank two tensors which are covariantly constructed as smooth, local functions of the metric and its derivatives to any order (with no polynomiality or analyticity assumptions). Our proof of the "universality" of metrics with plane wave symmetry is based upon an interplay between the homothety admitted by metrics with plane wave symmetry and the scaling properties of generally covariant field equations. The proof can be viewed, to some extent, as a generalization of an idea of Schmidt [2], who used the homothety admitted by gravitational plane waves to show that all generally covariant scalars are constant. Our arguments are valid in any spacetime dimension and can also be used to show that a number of other covariantly constructed tensor fields must vanish when evaluated on metrics with plane wave symmetry.

In the next section we specify what we mean by "plane wave symmetry", and construct the most general metric with that symmetry. We work in an arbitrary number of spacetime dimensions. It follows easily that metrics with plane wave symmetry always admit a continuous homothety. In §3 we define "generally covariant field equations" and indicate their behavior under scaling of the metric. In §4 we combine the results of §2 and §3 to show that metrics with plane wave symmetry satisfy every system of field equations except the Einstein equations. In §5 we mention some other results along these lines. We also comment on the construction of a mini-superspace description of spacetimes with plane wave symmetry.

2.   Plane wave symmetry

The spacetime corresponding to a gravitational plane wave [3,4] can be defined as the

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* Metrics satisfying (1) and (2) are the "plane fronted" waves, a class of vacuum metrics that includes the gravitational plane waves.

manifold $M = \mathbf{R}^n$, with standard global coordinates $x^\alpha = (u, v, x^i)$, $i = 1, 2, \ldots, n-2$, and metric

$$g_{plane} = -2du \otimes dv + \delta_{ij}dx^i \otimes dx^j + f_{ij}(u)x^i x^j du \otimes du, \tag{1}$$

where $f_{ij} = f_{ji}$ are any smooth functions of the null coordinate u such that

$$f_i^i \equiv \delta^{ij}f_{ij} = 0. \tag{2}$$

Condition (2) renders g_{plane} Ricci-flat. Henceforth we raise and lower the Latin indices using the Kronecker delta as in (2). The wave profile (amplitude, polarization, *etc.*) is determined by the choice of the functions f_{ij} since the components of the Weyl tensor in the (u, v, x^i) chart are proportional to the functions f_{ij}. For any smooth choice of wave profile f_{ij}, the plane wave spacetime is geodesically complete, and stably causal [4]. Thus the plane wave spacetimes provide a relatively rare class of examples of non-singular, causally tame vacuum solutions with a rather simple physical interpretation. One somewhat pathological feature of these spacetimes is that they are not globally hyperbolic, as was first noticed by Penrose [5].

The metric (1) admits a $(2n-3)$-dimensional group G of isometries generated by the vector fields

$$\frac{\partial}{\partial v} \quad \text{and} \quad Y = S^i(u)\frac{\partial}{\partial x^i} + S_i'(u)x^i\frac{\partial}{\partial v}, \tag{3}$$

where a prime denotes a u derivative, and $S^i(u)$ is any smooth solution of the linear system

$$S^{i\prime\prime} = f_j^i S^j. \tag{4}$$

The solution space to (4) is $2(n-2)$-dimensional, being labeled by the initial data $S^i(u_0)$ and $S^{i\prime}(u_0)$. Thus one can view Y, as defined in (3), as representing $2(n-2)$ Killing vector fields, corresponding to any choice of basis for the solution space to (4).

Basic properties of the isometry group of gravitational plane waves can be found in [3,4]. We mention here some properties of the isometry group that will feature in what follows.

The group orbits are the null hypersurfaces $u = $ constant. Any function invariant under the plane wave symmetry group is a function of u only. We call such functions *G-invariant functions*. The components of the Killing vector fields Y depend upon the invariant u. This dependence cannot be removed by a coordinate transformation and reflects the fact that, roughly speaking, the group action varies from orbit to orbit. More precisely, the orbits of the symmetry group, while diffeomorphic, are distinct as homogeneous spaces. Related to this is the fact that the transformation group generated by the Killing vector fields depends upon the choice of the functions f_{ij} in (1) so that, strictly speaking, different plane wave spacetimes have different symmetry transformations (although the abstract $(2n-3)$-dimensional Lie group is the same

for all choices of the wave profile). The dependence of the symmetry transformations on the choice of wave profile is through the functions $S^i(u)$, which are functionals of $f_{ij}(u)$ via (4).

Our first task is to find the general form of a metric admitting a plane wave symmetry group. We do this by considering a fixed (but arbitrary) set of functions $f_{ij}(u)$ (satisfying (2), although this is not essential) and then defining the vector fields Y as in (3), (4). We then find all metrics with Lorentz signature whose Lie derivative along the vector fields Y and $\frac{\partial}{\partial v}$ vanish. The general form of a metric g admitting plane wave symmetry is then found to be

$$g = \alpha g_{plane} + \beta du \otimes du, \tag{5}$$

where α and β are G-invariant functions, *i.e.*, $\alpha = \alpha(u)$, $\beta = \beta(u)$, and

$$\alpha > 0 \tag{6}$$

is required to give g the Lorentz signature. If we drop condition (6), then (5) is the general form of a symmetric rank-2 tensor field invariant under the plane wave symmetry group characterized by f_{ij}.

Next, we point out that the G-invariant metric given in (5), (6) admits a continuous homothety for any choice of the G-invariant functions α and β. This means that there exists a one-parameter family of diffeomorphisms $\Psi_s : M \to M$ such that

$$\Psi_s^* g = s^2 g, \quad s > 0. \tag{7}$$

The homothety is given by the transformation

$$u \to u$$

$$v \to s^2 v + \frac{1}{2}(1 - s^2) \int \frac{\beta(u)}{\alpha(u)} \, du \tag{8}$$

$$x^i \to s x^i.$$

Note that the homothetic transformation preserves the orbits of the plane wave symmetry group, that is, u is invariant under the homothety.

We summarize this section as follows.

Definition 1. A tensor field on $\mathbf{R}^4$ admits a *plane wave symmetry* with wave profile f_{ij} if it is invariant under the group of diffeomorphisms generated by $\frac{\partial}{\partial v}$ and Y, given in (3), (4).

Proposition 1. If a symmetric tensor field g of type $\binom{0}{2}$ has plane wave symmetry, then it takes the form (5) for some choice of the G-invariant functions α and β. If g is a Lorentz metric with plane wave symmetry then it takes the form (5) with $\alpha > 0$.

Proposition 2. If a metric has plane wave symmetry, then it admits a continuous homothety which preserves the orbits of the plane wave symmetry group.

3. Generally Covariant Field Equations

We now characterize the set of field equations that we want to consider. We will consider field equations for a metric that take the form of an equality between "generally covariant" tensor fields on a given manifold M. Such tensor fields are often called just "tensor fields", or "natural tensor fields", or "invariant tensor fields", or "metric concomitants". Whatever the name, the point is that such tensor fields are globally defined by the metric, with no other structures being needed. If the manifold M is orientable (as it is for the plane wave spacetimes), it is sensible to fix an orientation and to enlarge the class of generally covariant field equations by allowing the orientation of the manifold to be used in their construction (via the volume form defined by the metric). All the results that follow are valid with or without the use of an orientation on M. The precise implementation of our general covariance criteria is as follows.

Definition 2. A *generally covariant tensor* of type $\binom{p}{q}$ built from a metric, denoted T, is a mapping that assigns to each metric g a tensor field $T[g]$ of type $\binom{p}{q}$ on any manifold M. This rule must be smooth and local, that is, in any chart about any point $x \in M$, the components of $T[g]$ are smooth functions of the components of the metric and their derivatives (to some finite order) at x. Finally, we require for any (orientation-preserving) diffeomorphism, $\phi \colon M \to M$, that

$$T[\phi^* g] = \phi^* T[g]. \tag{9}$$

Because we are considering metric field theories only, we have restricted our notion of generally covariant tensors to those that are constructed from a metric. If other fields (*e.g.,* electromagnetic) were to be considered, we would of course enlarge the definition of generally covariant tensors accordingly. It is a standard result [6] that generally covariant tensors can always be constructed as smooth functions of the metric, the volume form of the metric (in the orientation-preserving case), the curvature tensor, and covariant derivatives of the curvature tensor to some finite order, all of which are examples of generally covariant tensors. Note that we use the symbol T to denote the mapping from metrics to tensor fields, and we use the symbol $T[g]$ to denote a specific tensor field on M defined by applying the rule T to a given metric tensor field g on M.

Definition 3. A set of *generally covariant field equations* for a metric is defined by partial differential equations of the form

$$T[g] = 0, \tag{10}$$

where T is a generally covariant symmetric tensor of type $\binom{0}{2}$.

The first result we need is that generally covariant tensor fields inherit the symmetries of the metric used to construct them. This follows directly from the equivariance requirement (9) when ϕ is an isometry of the metric (*i.e.*, $\phi^* g = g$). In particular, let T be a generally covariant symmetric tensor of type $\binom{0}{2}$, and let g be a metric on M with plane wave symmetry, then the tensor field $T[g]$ on M takes the form (5). With a simple redefinition of the functions α and β, we have the following result.

Proposition 3. Let T be a generally covariant symmetric tensor of type $\binom{0}{2}$, and g a metric with plane wave symmetry, then there exist G-invariant functions ρ and σ such that

$$T[g] = \rho g + \sigma \, du \otimes du. \tag{11}$$

We remark that the functions ρ and σ, while showing up as functions on M in (11), also can be viewed as generally covariant scalar fields, that is, they are obtained by evaluating generally covariant tensors of type $\binom{0}{0}$ on g.

The other result we need concerns the behavior of generally covariant tensors with respect to scaling of the metric. From the work of Anderson [7] and Gilkey [8] we have the following result.

Proposition 4. Let T be a generally covariant tensor of type $\binom{p}{q}$ and let g be any metric tensor field, then $T[g]$ can be written as

$$T[g] = T_0[g] + T_1[g] + T_2[g] + \ldots + T_N[g] + R_N[g], \tag{12}$$

where each of T_i, $i = 1, 2, \ldots, N$, and R_N are generally covariant tensors of type $\binom{p}{q}$ that enjoy the scaling behavior:

$$\begin{aligned} T_j[s^2 g] &= s^{q-p-j} T_j[g], \\ R_N[s^2 g] &= \mathcal{O}(s^{q-p-N-1}). \end{aligned} \tag{13}$$

Here the notation $A = \mathcal{O}(s^r)$ means that $s^{-r} A$ has a limit as $s \to 0$. Using the results of [7] it is not hard to show that, when T is symmetric and of type $\binom{0}{2}$,

$$(T_0)_{\mu\nu} = a g_{\mu\nu}, \tag{14}$$

$$(T_1)_{\mu\nu} = 0, \tag{15}$$

$$(T_2)_{\mu\nu} = b R_{\mu\nu} + c R g_{\mu\nu}, \tag{16}$$

where a, b, c are constants, $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature.

4. Universality

We now show that metrics with plane wave symmetry are universal in the sense that they satisfy "almost all" generally covariant field equations. Given a set of

generally covariant field equations (10), we can expand $T[g]$ as in (12); each term in the expansion is a natural tensor field with scaling behavior (13). We suppose that g is a metric with plane wave symmetry. Using the homothety Ψ_s, given in (8), we have that

$$\Psi_s^* T_j[g] = T_j[\Psi_s^* g] = T_j[s^2 g] = s^{2-j} T_j[g], \tag{17}$$

where the first equality comes from (9) and the last equality comes from (13). On the other hand, Proposition 3 allows us to conclude that there exist G-invariant functions ρ and σ such that

$$\Psi_s^* T_j[g] = \Psi_s^* \left(\rho g + \sigma \, du \otimes du \right) = s^2 \rho g + \sigma \, du \otimes du, \tag{18}$$

where we used the fact that the 1-form du is invariant under the homothety as is any G-invariant function. Therefore, for all $s > 0$,

$$s^2 \rho g + \sigma \, du \otimes du = s^{2-j} \left(\rho g + \sigma \, du \otimes du \right), \tag{19}$$

which implies that either $j = 2$ and $\rho = 0$, or that $j = 0$ and $\sigma = 0$, or that $\rho = \sigma = 0$. Similarly, it follows that $R_N[g] = 0$ for $N > 2$. Thus, at most,

$$T[g] = T_0[g] + T_2[g]. \tag{20}$$

Furthermore, either from direct computation or by an application of a scaling argument analogous to that just described (see Theorem 2, below), it is easily seen that the scalar curvature vanishes for any metric with plane wave symmetry. Therefore we have the following result.

Theorem 1. Let g be a metric with plane wave symmetry, and suppose that T is a generally covariant symmetric tensor of type $\binom{0}{2}$. Then $T[g]$ is a linear combination of g and the Ricci tensor of g.

Thus the only generally covariant field equations that are *not* automatically satisfied by a metric with plane wave symmetry are of the form

$$a g_{\mu\nu} + b R_{\mu\nu} = 0, \tag{21}$$

with a and b constants. More explicitly, if

$$g = \alpha(u) g_{plane} + \beta(u) du \otimes du, \tag{22}$$

then the field equations (21) are

$$a[\alpha g_{plane} + \beta \, du \otimes du] + b[\frac{\alpha''}{\alpha} - \frac{3}{2}\left(\frac{\alpha'}{\alpha}\right)^2] du \otimes du = 0. \tag{23}$$

From (23) it is easy to see that the Ricci tensor of a metric with plane wave symmetry is proportional to $du \otimes du$ (this is the case $j = 2$, $\rho = 0$ mentioned earlier), so (21) has no solutions unless $a = 0$, *i.e.*, the cosmological constant must vanish.

Corollary. Any constraints that can be placed by generally covariant field equations upon a metric with plane wave symmetry are equivalent to the vacuum Einstein equations with vanishing cosmological constant.

5. Remarks

(1) The ability of the plane wave symmetric spacetimes to satisfy so many field equations is reminiscent of the idea of "critical solutions" [9].   These are field configurations which (i) are invariant under some symmetry group G, and (ii) are critical points for any G-invariant action functional. The existence of critical solutions can, in many instances, be viewed as an infinite-dimensional analog of Michel's theorem [10], which states that a point is a critical point for all G-invariant functions if and only if it is "isolated in its stratum". It is not clear if one can view gravitational plane waves as critical solutions of this type, if only because the metrics with plane wave symmetry are *not* critical points of the Einstein-Hilbert action. On the other hand, by working in four dimensions and by restricting the form of the wave profile, it is possible to enlarge the plane wave symmetry group such that the resulting group invariant metrics *are* automatically Ricci-flat. For example, let κ be a constant, let

$$f_{ij} = \kappa \begin{pmatrix} \cos(2\kappa u) & \sin(2\kappa u) \\ \sin(2\kappa u) & -\cos(2\kappa u) \end{pmatrix},$$

and adjoin to the 5 generators in (3) the vector field

$$Z = \frac{\partial}{\partial u} - \kappa(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}).$$

The commutator of Z with $\frac{\partial}{\partial v}$ vanishes, and it is a straightforward exercise to check that the commutator of Z with the 4 independent vector fields defined by Y is a linear combination of those vector fields. The enlarged group is thus six-dimensional with four dimensional orbits; the resulting group invariant metrics define homogeneous spacetimes with plane wave symmetry (see the article by Ehlers and Kundt [3] for another example). Using the enlarged symmetry group, the general form of the group-invariant metric is now

$$\begin{aligned} g = a\{ &- 2du \otimes dv + dx \otimes dx + dy \otimes dy \\ &+ [b + \kappa \cos(2\kappa u)(x^2 - y^2) + 2\kappa \sin(2\kappa u)xy)]du \otimes du\}, \end{aligned} \tag{24}$$

with a and b constants. This metric is Ricci-flat for all values of a, b and κ. Therefore, these spacetimes will solve all generally covariant vacuum field equations by virtue of their symmetry, and perhaps can be understood via Michel's theorem. In any case, it is worth noting in this regard that gravitational plane waves satisfy *all* generally covariant vacuum field equations; the vast majority of these equations are not derivable from a local variational principle.

(2) The same sort of arguments as used in §4 can be used to investigate the behavior of generally covariant tensor fields of other types. For example, it is not hard to establish the following.

Theorem 2.　Let g be a metric with plane wave symmetry. **(i)** All generally covariant scalar fields are constant when evaluated on g. Here "constant" means as a function on M and as a functional of metrics with plane wave symmetry. **(ii)** All generally covariant 1-forms, 2-forms, and 3-forms vanish when evaluated on g. **(iii)** All generally covariant 4-forms are constant multiples of the volume form of g.

(3) It is possible to consider a "mini-superspace" description of spacetimes admitting the plane wave symmetry group. The symmetry group is defined by a choice of wave profile as discussed in §2. The mini-superspace $\mathcal{S}$ is defined in terms of the space of metrics of the form (5), so that points in $\mathcal{S}$ are specified by the values of α and β; the mini-superspace is two-dimensional. As we have seen, the only field equations that can be imposed are the vacuum Einstein equations, which take the form

$$\alpha'' - \frac{3}{2}\frac{1}{\alpha}\alpha'^2 = 0. \tag{25}$$

Evidently, the variable β is "pure gauge" and completely drops out of the field equations. This reflects the fact that the value of β can be varied at will by making a coordinate transformation of the form

$$v \to v + \Lambda(u). \tag{26}$$

We might as well drop β from the mini-superspace. Defining q via

$$\alpha = \frac{1}{q^2}, \tag{27}$$

the field equations take the elementary form

$$q'' = 0. \tag{28}$$

Note that the reduced equations of motion, (25) or (28), are not invariant with respect to reparametrizations of the "time" u, contrary to what might be expected. This is due to the fact that the symmetry generators (3) depend explicitly upon the u coordinate, so that the symmetry group defining the reduced equations of motion is sensitive to reparametrizations of u.

Because the scalar curvature vanishes when evaluated on metrics with plane wave symmetry, one cannot simply insert the general metric with plane wave symmetry into the Einstein-Hilbert Lagrangian to obtain a reduced Lagrangian describing the dynamics on the plane wave mini-superspace.‡ Nevertheless, the equation of motion (28) obviously admits a Lagrangian, so one can view a gravitational plane wave (with a given wave profile) as an autonomous Hamiltonian system with one degree of freedom.

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‡ For necessary and sufficient conditions on a symmetry group G such that one *can* make the symmetry reduction at the level of the Lagrangian, see [11] and references therein.

Acknowledgments

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