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Coherent state path integral for linear systems

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We present a computation of the coherent state path integral for a generic linear system using "functional methods" (as opposed to discrete time approaches). The Gaussian phase space path integral is formally given by a determinant built from a first-order differential operator with coherent state boundary conditions. We show how this determinant can be expressed in terms of the symplectic transformation generated by the (in general, time-dependent) quadratic Hamiltonian for the system. We briefly discuss the conditions under which the coherent state path integral for a linear system actually exists. A necessary—but not sufficient—condition for existence of the path integral is that the symplectic transformation generated by the Hamiltonian is (unitarily) implementable on the Fock space for the system.

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I. INTRODUCTION

The coherent state path integral, a variant of the phase space path integral, has long been recognized as a useful tool in quantum mechanics and in quantum field theory (see, e.g., [1–5]). In the quantum mechanical setting this form of the path integral has been studied fairly extensively (see, e.g., [4–8]). In this work the method normally used to define and analyze the coherent state path integral is based upon taking a limit of an approximation based upon discretized paths. In the field theoretic setting the coherent state path integral is mainly used to set up the perturbative evaluation of the S matrix; here "functional methods" (as opposed to discretized path methods) are normally employed. In particular, at the level of the free or semiclassical theory the path integral representation of the vacuum to vacuum transition amplitude, after integrating out the canonical momenta, is expressed in terms of a Fredholm determinant of a linear differential operator (wave operator, Dirac operator, etc.), which features in the approximate quadratic action functional.

In this paper we use functional methods based upon the approaches of [9,10] to compute the coherent state path integral for a generic linear bosonic dynamical system with general coherent state boundary conditions. The number of degrees of freedom can be infinite, so this computation includes field theory. This computation should be relevant for a number of applications, including: any linear quantum mechanical system, linearized/semiclassical approximations, quantum fields in curved spacetime, and parametrized free field theory and various other quantum gravity models [11]. In our computation we do not first integrate out the canonical momenta, nor do we restrict attention to relativistic fields. Therefore, the path integral is formally given in terms of the determinant of a first-order differential operator with coherent state boundary conditions. We show how this determinant can be expressed in terms of the symplectic transformation generated by the (in general, time-dependent) quadratic Hamiltonian for the system. While our computations are somewhat formal, the resulting expression for the coherent state transition amplitude agrees with the (rigorous) result obtained using methods of canonical quantization [12,13].

The computation provided here demonstrates a viable method for evaluating a class of coherent state path integrals using the functional properties of the integral and its concomitant functional determinants. It employs initial/final coherent state boundary conditions throughout. This is in contrast with existing techniques which are largely based upon discretization methods and the use of initial conditions (e.g., the "shooting method") for evaluating functional determinants. For the state of the art in this regard see [5–8]. Of course, the two methods of computation agree where their respective domains of applicability overlap. One can thus view the results presented here as demonstrating an alternative approach within the arsenal of techniques used to compute path integrals.

Our computations are also designed to explore within the path integral formalism subtle quantum field theoretic phenomena that have been uncovered using other methods of quantization (e.g., canonical quantization). In particular, it is well known that for field theories—i.e., systems with an infinite number of degrees of freedom—there exist inequivalent representations of the canonical commutation relations (see, e.g., [14] and references therein). Closely related to this is the fact that many linear canonical transformations—which may include those defining the time evolution of a linear system—cannot be unitarily implemented in the Fock space quantization of a field theory [12,13,15]. This situation is known to occur in a variety of physical settings, e.g., for quantum fields in curved spacetimes [16], for the polarized Gowdy model in general relativity [17], and for parametrized free field theories in dimensions greater than two [18]. It is natural to ask how these important phenomena, which have been understood heretofore using operator techniques, manifest themselves in the path integral formalism. Because we can express the path integral in terms of the symplectic transformations generated by the classical Hamiltonian, the connection with results from canonical quantization on unitary implementability/equivalence becomes immediately accessible. For example, in this paper we shall see explicitly that unitary implementability of dynamical evolution is necessary but not sufficient for the coherent state path integral to exist.

II. PRELIMINARIES

We will be considering the path integral for a linear dynamical system, by which we mean the following. Fix a real Hilbert space, i.e., a real vector space $\mathcal{V}$, complete with respect to a scalar product $(\cdot, \cdot)$. Elements of $\mathcal{V}$ will be denoted $\vec{z}$, $\vec{w}$, etc. The vector space $\mathcal{V}$ is to be the phase space for the system, so we further assume that $\mathcal{V}$ is equipped with a densely defined symplectic form Ω and a Hamiltonian $\mathcal{H}$, which is a densely defined quadratic form on $\mathcal{V}$. As explained e.g., in [19], we require the inner product and symplectic form to satisfy

$$(\vec{z}, \vec{z}) = \frac{1}{4} \underset{\vec{w} \neq 0}{\text{l.u.b.}} \frac{[\Omega(\vec{z}, \vec{w})]^2}{(\vec{w}, \vec{w})}. \tag{2.1}$$

This implies that the symplectic form is bounded on $\mathcal{V}$ and so can be defined with $\mathcal{V}$ as its domain. The scalar product and symplectic form then combine to define a (bounded, skew-adjoint) complex structure J: $\mathcal{V} \to \mathcal{V}$ via

$$\Omega(\vec{z}, \vec{w}) = 2(\vec{z}, J\vec{w}), \tag{2.2}$$

which can be used to define "positive and negative frequency" solutions to the linear field equations defined by $\mathcal{H}$. More precisely, we introduce the complexification $\mathcal{V}^C$ of $\mathcal{V}$ and extend $(\cdot, \cdot)$, Ω and J to $\mathcal{V}^C$ via linearity. We define complex conjugate Hilbert spaces $\mathcal{V}_\pm \subset \mathcal{V}^C$ corresponding to the $\pm i$ eigenspaces of J equipped with the sesquilinear inner products

$$\vec{z}, \vec{w} \in \mathcal{V}_\pm \to 2(\vec{z}^*, \vec{w}). \tag{2.3}$$

Here we use an asterisk for complex conjugation on $\mathcal{V}^C$. For later convenience, we suppose we have fixed a basis: $\{\vec{e}_a\}$ for $\mathcal{V}_+$ and $\{\vec{e}^*_a\}$ for $\mathcal{V}_-$. If $\vec{z} \in \mathcal{V}$, we have[1]

$$\vec{z} = z^a \vec{e}_a + z^{a*} \vec{e}^*_a, \tag{2.4}$$

a relationship we will often denote as either

$$\vec{z} = \begin{pmatrix} z^a \\ z^{a*} \end{pmatrix}, \tag{2.5}$$

or simply

$$\vec{z} = \begin{pmatrix} z \\ z^* \end{pmatrix}. \tag{2.6}$$

If $\vec{z} \in \mathcal{V}^C$ we write

$$\vec{z} = z^a \vec{e}_a + \bar{z}^a \vec{e}^*_a, \tag{2.7}$$

or

$$\vec{z} = \begin{pmatrix} z^a \\ \bar{z}^a \end{pmatrix}, \tag{2.8}$$

or simply

$$\vec{z} = \begin{pmatrix} z \\ \bar{z} \end{pmatrix}, \tag{2.9}$$

so that $\vec{z} = \binom{z^a}{\bar{z}^a}$ is in fact an element of $\mathcal{V}$ if and only if $\bar{z}^a = z^{a*}$. With this notation we have

$$\vec{z} \in \mathcal{V}_+ \leftrightarrow \vec{z} = z^a \vec{e}_a \leftrightarrow \vec{z} = \begin{pmatrix} z^a \\ 0 \end{pmatrix}, \tag{2.10}$$

$$\vec{z} \in \mathcal{V}_- \leftrightarrow \vec{z} = \bar{z}^a \vec{e}^*_a \leftrightarrow \vec{z} = \begin{pmatrix} 0 \\ \bar{z}^a \end{pmatrix}, \tag{2.11}$$

$$\vec{z}^* \in \mathcal{V}_- \leftrightarrow \vec{z} = z^{a*} \vec{e}^*_a \leftrightarrow \vec{z} = \begin{pmatrix} 0 \\ z^{a*} \end{pmatrix}. \tag{2.12}$$

The basis $\{\vec{e}_a, \vec{e}^*_a\}$ is chosen such that

$$(\vec{w}, \vec{z}) = \tfrac{1}{2}(\bar{w}_a z^a + w^a \bar{z}_a), \tag{2.13}$$

$$\Omega(\vec{w}, \vec{z}) = \frac{1}{i}(\bar{z}_a w^a - \bar{w}_a z^a), \tag{2.14}$$

$$J\vec{z} = \begin{pmatrix} iz^a \\ -i\bar{z}^a \end{pmatrix}, \tag{2.15}$$

and

$$\mathcal{H}(\vec{z}) = A_{ab} z^a \bar{z}^b + \tfrac{1}{2} B_{ab} z^a z^b + \tfrac{1}{2} \bar{B}_{ab} \bar{z}^a \bar{z}^b, \tag{2.16}$$

where A is self-adjoint and B, $\bar{B} = B^*$ are symmetric.

As explained, e.g., in [19] the data $\mathcal{V}$, Ω, μ define a "1-particle" Hilbert space and a corresponding Fock space representation of the Heisenberg group. In this representation, the operators corresponding to the classical variables z^b and $\bar{z}^b$ are annihilation and creation operators a^b and $a^{\dagger b}$, respectively.

The path integral we will be studying is over a space of paths $\vec{z} = \vec{z}(t)$ in $\mathcal{V}^C$. The paths will be required to obey coherent state boundary conditions of the form

$$z^a(0) = w^a, \qquad \bar{z}^a(T) = v^a, \tag{2.17}$$

for some given w^a and v^a. Given the Hamiltonian (2.16), symplectic structure (2.14), and boundary conditions (2.17) the action functional on this space of paths is given by

$$Q(\vec{z}) = \int_0^T dt \left\{ \frac{1}{2i}(\dot{\bar{z}}_a z^a - \bar{z}_a \dot{z}^a) - \left(A_{ab} z^a \bar{z}^b + \frac{1}{2} B_{ab} z^a z^b + \frac{1}{2} \bar{B}_{ab} \bar{z}^a \bar{z}^b \right) \right\} + \frac{1}{2i}[\bar{z}_a(T) z^a(T) + \bar{z}_a(0) z^a(0)]. \tag{2.18}$$

[1]The summation convention is in effect and the range of the summation can be infinite and continuous (i.e., integration).

This is just the phase space action functional on $\mathcal{V}$ expressed in a complex coordinate chart and extended to $\mathcal{V}^C$. The boundary terms in (2.18) are there so that Q is differentiable with the boundary conditions (2.17) [2].

In light of (2.17), we fix a path $\vec{\mathbf{z}}(t)$ with these boundary conditions and set

$$\vec{z}(t) = \vec{\mathbf{z}}(t) + \vec{\zeta}(t). \tag{2.19}$$

The variables $\vec{\zeta}(t)$ satisfy the boundary conditions

$$\zeta^a(0) = 0, \qquad \overline{\zeta}^a(T) = 0, \tag{2.20}$$

and are elements of a vector space X with dual X'. The pairing between X and X' is

$$\begin{aligned} \vec{\zeta}(t) &= \begin{pmatrix} \zeta^a \\ \overline{\zeta}^a \end{pmatrix} \in X \\ \vec{\zeta}'(t) &= \begin{pmatrix} \zeta'_a \\ \overline{\zeta}'_a \end{pmatrix} \in X' \\ \langle \vec{\zeta}', \vec{\zeta} \rangle &= \int_0^T dt \frac{1}{2}(\overline{\zeta}'_a(t)\zeta^a(t) + \zeta'_a(t)\overline{\zeta}^a(t)). \end{aligned} \tag{2.21}$$

The action restricted to X is a quadratic form characterized by a symmetric linear operator D: $X \longrightarrow X'$:

$$Q(\vec{\zeta}) = \langle D\vec{\zeta}, \vec{\zeta} \rangle, \tag{2.22}$$

where

$$D\vec{\zeta} = \begin{pmatrix} i\dot{\overline{\zeta}}_a - A_{ba}\zeta^b - \overline{B}_{ab}\overline{\zeta}^b \\ -i\dot{\zeta}_a - A_{ab}\overline{\zeta}^b - B_{ab}\zeta^b \end{pmatrix}. \tag{2.23}$$

We also define the quadratic forms Q_0 and V on X and symmetric operators D_0: $X \longrightarrow X'$ and N: $X \longrightarrow X'$ by

$$Q_0(\vec{\zeta}) = \int_0^T dt \left\{ \frac{1}{2i}(\dot{\overline{\zeta}}_a\zeta^a - \overline{\zeta}_a\dot{\zeta}^a) - A^b_a\overline{\zeta}_b\zeta^a \right\} = \langle D_0\vec{\zeta}, \vec{\zeta} \rangle, \tag{2.24}$$

$$D_0\vec{\zeta} = \begin{pmatrix} i\dot{\overline{\zeta}}_a - A_{ba}\zeta^b \\ -i\dot{\zeta}_a - A_{ab}\overline{\zeta}^b \end{pmatrix}, \tag{2.25}$$

and

$$Q(\vec{\zeta}) = Q_0(\vec{\zeta}) + V(\vec{\zeta}), \tag{2.26}$$

where

$$V(\vec{\zeta}) = \langle N\vec{\zeta}, \vec{\zeta} \rangle, \qquad N\vec{\zeta} = \begin{pmatrix} -\overline{B}_{ab}\overline{\zeta}^b \\ -B_{ab}\zeta^b \end{pmatrix}. \tag{2.27}$$

III. THE PATH INTEGRAL

Our goal is to compute a path integral which is, roughly, of the form

$$\mathcal{I}(v, w) \sim \int [d\vec{z}] e^{iQ(\vec{z})}, \tag{3.1}$$

where the integral is over a space of phase space paths with the boundary conditions (2.17). From the point of view of the Fock space representation of the quantum theory, $\mathcal{I}(v, w)$ should be a coherent state matrix element of the time evolution operator $U(0, T)$ defined by a Hamiltonian corresponding to the classical observable $\mathcal{H}$:

$$\mathcal{I}(v, w) = \langle v|U(0, T)|w \rangle. \tag{3.2}$$

Here the states $|v\rangle$ and $|w\rangle$ are Fock space eigenvectors of the annihilation operator a^b:

$$a^b|v\rangle = v^b|v\rangle, \qquad a^b|w\rangle = w^b|w\rangle. \tag{3.3}$$

An approach to obtaining a rigorous definition of (3.1) can be found in [9,10]. We shall not attempt to define functional integration here; instead we shall proceed formally by postulating a few basic properties, which feature in the rigorous definitions of [9,10], and which any suitable definition of integration should exhibit. We shall assume that integration is a linear operation on a class of functions on the vector space X. The integration operation is normalized relative to the quadratic form Q_0 and is denoted with the symbol $\int_X [d\zeta]_{Q_0}$. The normalization condition is

$$\int_X [d\vec{\zeta}]_{Q_0} e^{iQ_0(\vec{\zeta})} = 1. \tag{3.4}$$

We formally define the path integral (3.1) by

$$\mathcal{I}(v, w) = \int_X [d\vec{\zeta}]_{Q_0} e^{iQ(\vec{\mathbf{z}}+\vec{\zeta})}. \tag{3.5}$$

As we shall see, in the Fock space representation our choice of path integral normalization corresponds to defining the time evolution operator using the normal-ordered Hamiltonian operator associated to the classical expression (2.16), with no additive c-number renormalizations.

To evaluate the path integral we choose

$$\vec{\mathbf{z}}(t) = \begin{pmatrix} \mathbf{z}(t) \\ \overline{\mathbf{z}}(t) \end{pmatrix} \tag{3.6}$$

to be a critical point of the action functional, i.e., $\vec{\mathbf{z}}(t)$ is chosen to be the (unique) path in $\mathcal{V}^C$ satisfying

$$D\vec{\mathbf{z}}(t) = 0, \qquad \mathbf{z}(0) = w, \qquad \overline{\mathbf{z}}(T) = v. \tag{3.7}$$

Using linearity of the integration operation, the path integral now takes the form

$$\mathcal{I}(v, w) = \exp\left\{\frac{1}{2}[v_a\mathbf{z}^a(T) + \overline{\mathbf{z}}_a(0)w^a]\right\} \int_X [d\vec{\zeta}]_{Q_0} e^{iQ(\vec{\zeta})}. \tag{3.8}$$

The remaining path integral is an oscillating Gaussian—or Fresnel—type of integral. It is (at least formally) given in terms of the Fredholm determinant [10][2]:

[2]In principle there could also be a phase factor coming from the index of Q relative to Q_0 [10], but we shall see that this factor is unity.

$$\mathcal{I}(0,0)=\int_X[d\vec{\zeta}]_{Q_0}e^{iQ(\vec{\zeta})}=\det{}^{-1/2}(D_0^{-1}D). \tag{3.9}$$

Hence

$$\mathcal{I}(v,w)=\exp\{\tfrac{1}{2}[v_a\mathbf{z}^a(T)+\overline{\mathbf{z}}_a(0)w^a]\}\det{}^{-1/2}(D_0^{-1}D). \tag{3.10}$$

IV. EVALUATING THE PATH INTEGRAL IN TERMS OF CLASSICAL DYNAMICS

We now show that the result (3.10) can be expressed in terms of the symplectic transformation(s) generated by $\mathcal{H}(t)$ and the boundary conditions (2.17). We begin by giving some relevant features of the symplectic transformation.

We assume the Hamiltonian $\mathcal{H}(t)$, which is allowed to be time-dependent, generates a 1-parameter family of symplectic transformations $S(t)$ on $\mathcal{V}^C$. S is determined by

$$\left(\frac{d}{dt}+H(t)\right)S(t)=0,\qquad H(t)=i\begin{pmatrix}A(t) & \overline{B}(t)\\ -B(t) & -A(t)\end{pmatrix},$$
$$S(0)=id. \tag{4.1}$$

The operators $S(t)$ can be expressed as a Dyson-type of expansion, i.e., the time-ordered exponential of the linear transformations $H(t)$ on $\mathcal{V}^C$ defined by the Hamiltonian $\mathcal{H}(t)$. We shall not need the explicit formula here. Using a block matrix notation paralleling (2.8), we can express the symplectic transformations as

$$S(t)=\begin{pmatrix}\alpha(t) & \beta(t)\\ \overline{\beta}(t) & \overline{\alpha}(t)\end{pmatrix}, \tag{4.2}$$

i.e.,

$$S\vec{\mathbf{z}}=\begin{pmatrix}\alpha^a_b\mathbf{z}^b+\beta^a_b\overline{\mathbf{z}}^b\\ \overline{\alpha}^a_b\overline{\mathbf{z}}^b+\overline{\beta}^a_b\mathbf{z}^b\end{pmatrix}, \tag{4.3}$$

where the Bogoliubov coefficients $\alpha(t)$, $\beta(t)$ satisfy at each t

$$\alpha\colon \mathcal{V}_+\to\mathcal{V}_+,\qquad \beta\colon \mathcal{V}_-\to\mathcal{V}_+, \tag{4.4}$$

$$\overline{\alpha}\colon \mathcal{V}_-\to\mathcal{V}_-,\qquad \overline{\beta}\colon \mathcal{V}_+\to\mathcal{V}_-, \tag{4.5}$$

$$\overline{\alpha}=\alpha^*,\qquad \overline{\beta}=\beta^*, \tag{4.6}$$

$$\alpha\alpha^\dagger-\beta\beta^\dagger=id_{\mathcal{V}_+},\qquad \alpha\beta^T-\beta\alpha^T=0. \tag{4.7}$$

From these equations it follows that $\alpha(t)$ has a bounded inverse and that

$$S^{-1}(t)=\begin{pmatrix}\alpha^\dagger(t) & -\beta^T(t)\\ -\beta^\dagger(t) & \alpha^T(t)\end{pmatrix}. \tag{4.8}$$

Finally, we have that

$$\alpha(0)=id_{\mathcal{V}_+},\qquad \overline{\alpha}(0)=id_{\mathcal{V}_-},$$
$$\beta(0)=0,\qquad \overline{\beta}(0)=0. \tag{4.9}$$

All the dependence of $\mathcal{I}(v,w)$ on the initial and final states is in the exponential of the action sitting in front of the determinant in (3.10). To make this initial/final state dependence explicit, we use the fact that the critical points

$$\vec{\mathbf{z}}(t)=\begin{pmatrix}\mathbf{z}(t)\\ \overline{\mathbf{z}}(t)\end{pmatrix} \tag{4.10}$$

of the action are determined by the symplectic transformations generated by the Hamiltonian (2.16):

$$\vec{\mathbf{z}}(t)=S(t)\vec{\mathbf{z}}(0). \tag{4.11}$$

The boundary conditions (2.17) imply

$$\mathbf{z}(T)=\alpha^{-1\dagger}(T)w+\sigma(T)v \tag{4.12}$$

$$\overline{\mathbf{z}}(0)=\overline{\alpha}^{-1}(T)v-\gamma(T)w, \tag{4.13}$$

where we have defined symmetric operators $\gamma\colon \mathcal{V}_+\to\mathcal{V}_-$ and $\sigma\colon \mathcal{V}_-\to\mathcal{V}_+$ via

$$\gamma=\overline{\alpha}^{-1}\overline{\beta},\qquad \sigma=\beta\overline{\alpha}^{-1}. \tag{4.14}$$

Putting this all together, we get

$$\mathcal{I}(v,w)=\exp\{(\alpha^{-1\dagger})^a_b(T)v_aw^b+\tfrac{1}{2}\sigma^a_b(T)v_av^b$$
$$-\tfrac{1}{2}\gamma^b_a(T)w^aw_b\}\det{}^{-1/2}(D_0^{-1}D). \tag{4.15}$$

The determinant appearing in (4.15) depends upon the Hamiltonian (2.16) via the symplectic transformation S. We will now make this dependence on S explicit. The hypothesis on $M=D_0^{-1}D$ is that $M-1$ is trace-class; this guarantees the determinant is well defined and satisfies the variational identity.

$$\delta\log\det(M)=\mathrm{Tr}(M^{-1}\delta M). \tag{4.16}$$

The trace ("Tr") of an operator $R\colon X\to X$ is defined in terms of an integral kernel $R(t,u)$ with values in the set of operators on $\mathcal{V}^C$:

$$R\vec{z}(t)=\int_0^T du R(t,u)\vec{z}(u). \tag{4.17}$$

We set

$$\mathrm{Tr}(R)=\int_0^T dt\,\mathrm{tr}R(t,t), \tag{4.18}$$

where tr denotes the Hilbert space trace. To use (4.16) we define the differential operator $D_\lambda\colon X\to X'$ by

$$D_\lambda\vec{\zeta}=\begin{pmatrix}i\dot{\zeta}_a-A_{ba}\zeta^b-\lambda\overline{B}_{ab}\overline{\zeta}^b\\ -i\dot{\overline{\zeta}}_a-A_{ab}\overline{\zeta}^b-\lambda B_{ab}\zeta^b\end{pmatrix}, \tag{4.19}$$

(so that $D=D_1$). Setting

$$\mathcal{I}_\lambda = \det^{-1/2}(D_0^{-1} D_\lambda), \tag{4.20}$$

we have

$$\frac{d}{d\lambda} \log \mathcal{I}_\lambda = -\tfrac{1}{2}\mathrm{Tr}\{(D_0^{-1}D_\lambda)^{-1} D_0^{-1} N\} = -\tfrac{1}{2}\mathrm{Tr}(D_\lambda^{-1} N), \tag{4.21}$$

where N was defined in (2.27). Our strategy is to obtain a suitable expression of $\mathrm{Tr}(D_\lambda^{-1}N)$ in terms of symplectic transformations and then solve the differential equation (4.21) in λ with initial condition given by the path integral normalization,

$$\mathcal{I}_0 = 1, \tag{4.22}$$

to find $\mathcal{I}_\lambda$, from which we have $\mathcal{I} = \mathcal{I}_{\lambda=1}$.[3]

To obtain a suitable form of $\mathrm{Tr}(D_\lambda^{-1}N)$ we need to obtain an expression for the Green function $G_\lambda \equiv D_\lambda^{-1}$, which is uniquely determined by the solution of the system

$$D_\lambda \vec{\zeta}(t) = \vec{F}(t), \qquad \zeta^a(0) = 0, \qquad \bar{\zeta}^a(T) = 0, \tag{4.23}$$

so that

$$\vec{\zeta}(t) = G_\lambda \vec{F}(t) = \int_0^T du G_\lambda(t,u)\vec{F}(u). \tag{4.24}$$

The general solution to the differential equation in (4.23) is easily checked to be

$$\vec{\zeta}(t) = \vec{\zeta}_0(t) + \int_0^T du \theta(t-u) S_\lambda(t) S_\lambda^{-1}(u)\Sigma \vec{F}(u), \tag{4.25}$$

where $\vec{\zeta}_0$ is the general solution to $D_\lambda \vec{\zeta}_0 = 0$,

$$\vec{\zeta}_0(t) = S_\lambda(t)\vec{\zeta}_0(0), \tag{4.26}$$

with

$$S_\lambda(t) = \begin{pmatrix} \alpha_\lambda(t) & \beta_\lambda(t) \\ \bar{\beta}_\lambda(t) & \bar{\alpha}_\lambda(t) \end{pmatrix}, \qquad S_\lambda^{-1}(t) = \begin{pmatrix} \alpha_\lambda^\dagger(t) & -\beta_\lambda^T(t) \\ -\beta_\lambda^\dagger(t) & \alpha_\lambda^T(t) \end{pmatrix} \tag{4.27}$$

being the symplectic transformation generated by

$$\mathcal{H}_\lambda(t) = A_{ab}(t) z^a \bar{z}^b + \tfrac{1}{2}\lambda(B_{ab}(t) z^a z^b + \bar{B}_{ab}(t)\bar{z}^a \bar{z}^b). \tag{4.28}$$

$S_\lambda(t)$ satisfies

$$\left(\frac{d}{dt} + H_\lambda(t)\right) S_\lambda(t) = 0, \qquad H_\lambda(t) = i\begin{pmatrix} A(t) & \lambda \bar{B}(t) \\ -\lambda B(t) & -A(t) \end{pmatrix}, \tag{4.29}$$

and, for each value of λ, the obvious generalizations of Eqs. (4.4), (4.5), (4.6), (4.7), and (4.9) hold. In (4.25) θ is the step function

$$\theta(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0, \end{cases} \tag{4.30}$$

and

$$\Sigma = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{4.31}$$

The boundary conditions in (4.23) imply (suppressing indices)

$$\vec{\zeta}_0(t) = \begin{pmatrix} \beta_\lambda(t)\bar{\zeta}_0(0) \\ \bar{\alpha}_\lambda(t)\bar{\zeta}_0(0) \end{pmatrix}, \tag{4.32}$$

where

$$\bar{\zeta}_0(0) = -\bar{\alpha}_\lambda^{-1}(T)\left[\int_0^T du S_\lambda(T) S_\lambda^{-1}(u)\Sigma \vec{F}(u)\right]_{\bar{\zeta}}. \tag{4.33}$$

Here we use a notation for the components of elements of $\mathcal{V}^C$ such that if

$$\vec{v} = \begin{pmatrix} v \\ \bar{v} \end{pmatrix}, \tag{4.34}$$

then

$$(\vec{v})_\zeta = v, \qquad (\vec{v})_{\bar{\zeta}} = \bar{v}. \tag{4.35}$$

Putting all of this together, the solution

$$\vec{\zeta}(t) = \begin{pmatrix} \zeta(t) \\ \bar{\zeta}(t) \end{pmatrix} \tag{4.36}$$

to (4.23) is given by[4]

$$\begin{aligned} \zeta(t) = \int_0^T du\{&(\theta(t-u)\alpha_\lambda(t) - \beta_\lambda(t)\gamma_\lambda(T)) \\ &\times [S_\lambda^{-1}(u)\Sigma\vec{F}(u)]_\zeta - \theta(u-t)\beta_\lambda(t) \\ &\times [S_\lambda^{-1}(u)\Sigma\vec{F}(u)]_{\bar{\zeta}}\} \\ \bar{\zeta}(t) = \int_0^T du\{&(\theta(t-u)\bar{\beta}_\lambda(t) - \bar{\alpha}_\lambda(t)\gamma_\lambda(T)) \\ &\times [S_\lambda^{-1}(u)\Sigma\vec{F}(u)]_\zeta - \theta(u-t)\bar{\alpha}_\lambda(t) \\ &\times [S_\lambda^{-1}(u)\Sigma\vec{F}(u)]_{\bar{\zeta}}\}, \end{aligned} \tag{4.37}$$

[3]This approach will define $\mathcal{I}$ provided D_λ has no zero eigenvalues as λ varies from 0 to 1 [10]. Otherwise there is an additional phase coming from the index of D_λ. We shall see that D_λ has no kernel.

[4]As can be seen from this result, there are no nontrivial solutions to $D_\lambda \vec{\zeta} = 0$ with boundary conditions (2.20). This means that D_λ has no kernel and there are no additional phases to be computed (see [10]). Evidently, this is a simplifying feature of the coherent state path integral.

where

$$[S_\lambda^{-1}\Sigma\vec{F}]_\zeta = -i\alpha_\lambda^\dagger F_\zeta - i\beta_\lambda^T F_{\bar\zeta},$$
$$[S_\lambda^{-1}\Sigma\vec{F}]_{\bar\zeta} = i\beta_\lambda^\dagger F_\zeta + i\alpha_\lambda^T F_{\bar\zeta}. \tag{4.38}$$

The Green function thus takes the form

$$G_\lambda(t,u) = \begin{pmatrix} G_{\zeta\zeta}(t,u) & G_{\zeta\bar\zeta}(t,u) \\ G_{\bar\zeta\zeta}(t,u) & G_{\bar\zeta\bar\zeta}(t,u)\end{pmatrix}, \tag{4.39}$$

where

$$\begin{aligned} G_{\zeta\zeta}(t,u) &= -i\theta(t-u)(\alpha_\lambda(t)\alpha_\lambda^\dagger(u) - \beta_\lambda(t)\beta_\lambda^\dagger(u)) \\ &\quad + i\beta_\lambda(t)(\gamma_\lambda(T)\alpha_\lambda^\dagger(u) - \beta_\lambda^\dagger(u)), \\ G_{\zeta\bar\zeta}(t,u) &= -i\theta(t-u)(\alpha_\lambda(t)\beta_\lambda^T(u) - \beta_\lambda(t)\alpha_\lambda^T(u)) \\ &\quad + i\beta_\lambda(t)(\gamma_\lambda(T)\beta_\lambda^T(u) - \alpha_\lambda^T(u)), \\ G_{\bar\zeta\zeta}(t,u) &= -i\theta(t-u)(\bar\beta_\lambda(t)\alpha_\lambda^\dagger(u) - \bar\alpha_\lambda(t)\beta_\lambda^\dagger(u)) \\ &\quad + i\bar\alpha_\lambda(t)(\gamma_\lambda(T)\alpha_\lambda^\dagger(u) - \beta_\lambda^\dagger(u)), \\ G_{\bar\zeta\bar\zeta}(t,u) &= -i\theta(t-u)(\bar\beta_\lambda(t)\beta_\lambda^T(u) - \bar\alpha_\lambda(t)\alpha_\lambda^T(u)) \\ &\quad + i\bar\alpha_\lambda(t)(\gamma_\lambda(T)\beta_\lambda^T(u) - \alpha_\lambda^T(u)). \end{aligned} \tag{4.40}$$

We then have

$$\begin{aligned} \mathrm{Tr}(G_\lambda N) &= -\int_0^T dt\,\mathrm{tr}\{G_{\zeta\bar\zeta}(t,t)B(t) + G_{\bar\zeta\zeta}(t,t)\bar B(t)\} \\ &= -i\int_0^T dt\,\mathrm{tr}\{\beta_\lambda(t)(\gamma_\lambda(T)\beta_\lambda^T(t) - \alpha_\lambda^T(t))B(t) \\ &\quad + \bar\alpha_\lambda(t)(\gamma_\lambda(T)\alpha_\lambda^\dagger(t) - \beta_\lambda^\dagger(t))\bar B(t)\}. \end{aligned} \tag{4.41}$$

Equation (4.41) can be simplified considerably. In particular, we claim that

$$\mathrm{Tr}\,(G_\lambda N) = \mathrm{tr}\Big(\bar\alpha_\lambda^{-1}(T)\frac{d}{d\lambda}\bar\alpha_\lambda(T)\Big), \tag{4.42}$$

where the trace on the right-hand side of the equation is on $\mathcal{V}_-$. To prove (4.42), we differentiate the equations (4.29) determining S_λ so that

$$\Big(\frac{d}{dt} + H_\lambda\Big)\frac{dS_\lambda}{d\lambda} + \frac{dH_\lambda}{d\lambda}S_\lambda = 0. \tag{4.43}$$

Equation (4.43) can be viewed as an inhomogeneous equation for $dS_\lambda/d\lambda$ with "source" given by $-(dH_\lambda/d\lambda)S_\lambda$. With initial condition

$$\Big(\frac{dS_\lambda}{d\lambda}\Big)_{t=0} = 0, \tag{4.44}$$

the solution is

$$\frac{dS_\lambda(t)}{d\lambda} = -\int_0^t du\,S_\lambda(t)S_\lambda^{-1}(u)\frac{dH_\lambda(u)}{d\lambda}S_\lambda(u). \tag{4.45}$$

Using (4.27) and

$$\frac{dH_\lambda}{d\lambda} = i\begin{pmatrix} 0 & \bar B \\ -B & 0\end{pmatrix}, \tag{4.46}$$

in (4.45) we get

$$\begin{aligned} \frac{d\bar\alpha_\lambda(t)}{d\lambda} &= i\int_0^t du\{[\bar\alpha_\lambda(t)\alpha_\lambda^T(u) - \bar\beta_\lambda(t)\beta_\lambda^T(u)]B(u)\beta_\lambda(u) \\ &\quad + [\bar\alpha_\lambda(t)\beta_\lambda^\dagger(u) - \bar\beta_\lambda(t)\alpha_\lambda^\dagger(u)]\bar B(u)\bar\alpha_\lambda(u)\}. \end{aligned} \tag{4.47}$$

We then have

$$\begin{aligned} \mathrm{tr}\Big(\bar\alpha_\lambda^{-1}(T)\frac{d\bar\alpha_\lambda(T)}{d\lambda}\Big) &= i\int_0^T du\,\mathrm{tr}\{[\alpha_\lambda^T(u) - \gamma_\lambda(T)\beta_\lambda^T(u)] \\ &\quad \times B(u)\beta_\lambda(u) \\ &\quad + [\beta_\lambda^\dagger(u) - \gamma_\lambda(T)\alpha^\dagger(u)]\bar B(u)\bar\alpha_\lambda(u)\} \\ &= \mathrm{Tr}(G_\lambda N). \end{aligned} \tag{4.48}$$

With this result in hand we can obtain the final form for the path integral. Using

$$\mathrm{tr}\Big(\bar\alpha_\lambda^{-1}(T)\frac{d\bar\alpha_\lambda(T)}{d\lambda}\Big) = \frac{d}{d\lambda}[\log\,\det(\bar\alpha_\lambda(T))], \tag{4.49}$$

and (4.21) we have

$$\frac{d}{d\lambda}\{\log\mathcal{I}_\lambda + \tfrac{1}{2}\log\,\det(\bar\alpha_\lambda(T))\} = 0, \tag{4.50}$$

so that, using (4.22), we have

$$\det{}^{-1/2}(D_0^{-1}D) = \mathcal{I}_1 = \det{}^{-1/2}(\bar\alpha_0^{-1}(T)\bar\alpha_1(T)). \tag{4.51}$$

Finally, from (4.15) and $\alpha_1 = \alpha$ we have

$$\begin{aligned} \mathcal{I}(v,w) &= \exp\Big\{(\alpha^{-1\dagger})^a_b(T)v_a w^b + \frac{1}{2}\sigma^a_b(T)v_a v^b \\ &\quad - \frac{1}{2}\gamma^b_a(T)w^a w_b\Big\}\sqrt{\frac{1}{\det(\bar\alpha_0^{-1}(T)\bar\alpha(T))}}. \end{aligned} \tag{4.52}$$

We note that the coherent state path integral result (4.52) for the transition amplitude has been rigorously obtained within the Fock space formalism for a particular class of normal-ordered, time-independent, quadratic Hamiltonians [13].

V. DISCUSSION

Evidently, the path integral as computed in (4.52) makes sense provided the exponential factor and the determinant of the operator $K = \alpha_0^{-1}(T)\alpha(T)$ exist. For the Fredholm determinant of an operator K to be defined it is necessary and sufficient that $J - id$ is a trace-class operator. (In this section "id" denotes the identity operator on $\mathcal{V}_+$.) An important necessary condition for $\det(K)$ to exist is that the Bogoliubov coefficient $\beta(T)$ in $S(T)$ defines a Hilbert-Schmidt operator:

$$\mathrm{tr}\,[\beta(T)\beta^\dagger(T)] < \infty. \tag{5.1}$$

This can be seen by first noting that α_0 is a unitary operator, which follows from the easily established fact that $\beta_0 = 0$. Then we have

$$|\det(K)|^2 = \det(KK^\dagger) = \det(id + \alpha_0^{-1}\beta\beta^\dagger\alpha_0), \quad (5.2)$$

where all operators are evaluated at time T. The operator $\alpha_0\beta\beta^\dagger\alpha_0^\dagger$ is trace-class if and only if $\beta\beta^\dagger$ is, i.e., $\beta(T)$ must be Hilbert-Schmidt. When $\beta(T)$ is Hilbert-Schmidt it follows that the exponential factor in (4.52) exists because each of $\alpha^{-1}(T)$, $\sigma(T)$ and $\gamma(T)$ is bounded. This necessary condition is highlighted here because it is known that the symplectic transformation $S(T)$ is unitarily implementable in the Fock space representation defined by Ω and J if and only if $\beta(T)$ is Hilbert-Schmidt [13,15].[5] Thus the path integral, normalized using Q_0 and interpreted using the Fredholm determinant, fails to exist if the symplectic transformation generated by the classical Hamiltonian fails to be unitarily implementable in the Fock space representation.

It should be emphasized, however, that even if the symplectic transformation corresponding to time evolution from $t = 0$ to $t = T$ *is* unitarily implemented, this does not guarantee that the path integral exists. Implementability means only that the absolute value of the determinant is defined—the phase of the determinant may not be defined.

In the case where the Hamiltonian is time-independent, some sufficient conditions for the Fredholm determinant of K to exist (and hence for the exponential factor to exist as well) can be obtained from the results in [12,13]. For example, a relatively simple sufficient condition is that the quadratic form on $\mathcal{V}_+$ given by B corresponds (via the scalar product on $\mathcal{V}_+$) to a Hilbert-Schmidt operator:

$$\sum_{ab} B_{ab}\overline{B}^{ab} < \infty. \quad (5.3)$$

From the point of view of the Fock space formulation of the quantum system, this implies that the symplectic transformations $S(t)$ form a strongly continuous group and are represented (projectively) as a continuous unitary group generated by a Hamiltonian operator, which is unique up to addition of a multiple of the identity. The Hilbert-Schmidt condition on B is equivalent to requiring that the vacuum state of the Fock representation defined by Ω and J is in the domain of the Hamiltonian.

[5]There are a number of interesting situations where this Hilbert-Schmidt condition fails (see, e.g., [16–20]).

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