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Methods for Estimating the Critical Shear Stress of Individual Fractions in Mixed-Size Sediment

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Two methods are commonly used to estimate the critical shear stress of individual fractions in mixed-size sediment, one using the largest grain displaced, the other using the shear stress that produces a small value of transport rate for each fraction. The initial-motion results produced by the two methods are typically different: largest-grain critical shear stresses vary with roughly the square root of grain size, and reference transport critical shear stresses show little variation with grain size. Comparison of the two methods is seldom possible because both methods can rarely be applied to the same data. The one case known for which both methods can be used suggests that the typical differences in initial-motion results reflect more methodological influence than real differences in the initial motion of different sediments. Although the two classes of methods may not be directly compared, a general definition of initial-motion in mixed-size sediment is presented that allows the characteristic differences between the results to be explained in terms of sampling and scaling considerations inherent in the mixed-size initial-motion problem. The initial-motion criterion defined also provides a rational basis for collecting comparable and reproducible data using the two classes of method.

INTRODUCTION

Methodological problems have always haunted the study of incipient motion of sediments. Even in the relatively simple case of sediments that are nearly uniform in size, it has long been realized that different methods, or even variations of the same method, give different values of the critical shear stress for initiation of grain motion [e.g., *Neill*, 1968; *Miller et al.*, 1977]. These methodological problems are carried over to the case of sediments containing a mixture of sizes and are combined with additional problems unique to mixed-size sediment that are not well-understood.

Recent work is, in general, agreement that relatively smaller sizes in a mixture show greater resistance to movement and a higher critical shear stress than sediment in a unisize bed and that relatively larger sizes show less resistance to movement and a lower critical shear stress in a mixture. There is considerable discrepancy in the published results, however. These discrepancies are not random, but fall into two groups that may be associated with two broad classes of methods for determining initial motion. The first method involves estimating the critical shear stress for the individual fractions τ_{ci} as the bed shear stress that produces a small transport rate for each fraction. Transport rates of individual fractions are measured for a number of flows and the shear stress that corresponds to a small reference transport rate is determined from a fitted relation between shear stress and transport rate for each fraction [*Parker et al.*, 1982; *Day*, 1980]. The second method involves determining the largest clast in a sediment mixture that is moved by a given bed shear stress. This quantity may be measured directly using the largest clast found in a transport sample [*Andrews*, 1983; *Carling*, 1983] or measured visually by observing the largest grain moving over an area of the bed [*Hammond et al.*, 1984]. The largest mobile grain is assumed to represent initial-motion conditions if coarser grains are available in the bed. The two general classes of initial-motion methods will be termed here the reference transport method and the largest-grain method. When τ_{ci} is measured with the reference transport method, it is found to have little, if any, dependence on the relative size of the fraction, expressed, for example, as the ratio of the fraction size to the median size of the mixture, D_i/D_{50} [*Parker et al.*, 1982; *Wilcock and Southard*, this issue]. The largest-grain method typically produces a variation of τ_{ci} with roughly the square root of D_i/D_{50} (for review see *Komar* [1987]), although *Andrews* [1983] found only a weak dependence of τ_{ci} on D_i/D_{50} using the largest grain found in transport samples.

Two problems prevent direct comparison of initial-motion results determined by the two different classes of initial-motion methodology. The two initial-motion methods are typically applied to very different portions of a transported sample. A largest-grain τ_{ci} estimate uses only a single grain per sample, typically one in the coarse part of the grain size distribution. A reference shear stress τ_{ci} estimate incorporates the transport rates of all the grains in a sample, as well as the transport rates of other samples that are used to fit an interpolating function between dimensionless forms of the transport rate and bed shear stress for each fraction. The second problem is that in most cases, only one of the two methods can be generally applied to a particular set of transport data. The largest-grain method requires that grains coarser than the largest in the transport sample be available in the bed, a condition often not met by transport samples. On the other hand, a collection of single transport samples from which largest-grain estimates may be made may not be sufficient to adequately define the relation between transport rate and shear stress that is necessary for the reference transport method. Both the mutually exclusive nature typical of the appropriate data sets, and the different portions of the transport size distribution sampled, leave open the question of whether differences in initial-motion results between different sediments and flow conditions are real or merely reflect the different methods used to determine the initial motion. One goal of this paper is to present a definition of initial motion of individual fractions in mixed-size sediment that provides a rational basis for comparing data from different sediments. Although direct com-

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parisons between the two methods cannot be made because of the reasons stated above, a consistent definition of mixed-size initial motion can be used to demonstrate the manner in which the different methods can give very different results. These differences are of a nature that suggest that the reported differences in $\tau_{ci} - D_i/D_{50}$ relations reflect a methodological influence rather than a true variation in initial-motion relations for different sediments and flows. This supposition is supported by the one case known to the author in which both classes of initial-motion method may be applied to the same data set.

In addition to problems of comparability among different data sets, there are general questions concerning the measurement of initial motion of mixed-size sediments that have yet to be answered. Can either of the existing, practical methods be compared with the shear stress at which the first movement of a given fraction is directly observed (if such an observation can be made)? Is one method more appropriate than another for comparing the critical shear stress of grains of a different size in the same mixture? Which method can be compared with critical shear stresses estimated from theoretical models? The rational definition of incipient motion below provides a basis for answering these questions.

DEFINING INITIAL MOTION FOR MIXED-SIZE SEDIMENTS

It has long been recognized that a basic problem encountered when determining the critical shear stress is that it can be estimated only with data from flows with some grain motion, for which the bed shear stress already exceeds critical. A second and more fundamental problem is that the bed shear stress is a fluctuating quantity, and one cannot precisely define a value below which there is no motion. Both problems lead naturally to a definition of τ_c in terms of a small but finite number of grains in motion. But the number of grains displaced depends on the area of the bed examined and the length of time over which grain displacements may occur. An initial-motion criterion must therefore be defined so that the critical shear stress determined for different sediments, or for different fractions in a sediment mixture, are comparable, so that empirical data on critical shear stress can be combined into a general model or compared to theoretical results.

Neill and Yalin [1969] defined initial motion for unisize sediments in terms of an equal number of grains displaced from geometrically similar areas in kinematically similar time periods. They argued that geometrically similar bed areas for two different grain sizes D_1 and D_2 are related as

$$A_1/D_1^2 = A_2/D_2^2 \tag{1}$$

and kinematically similar sample periods are related as

$$t_1 u_{*1}/D_1 = t_2 u_{*2}/D_2 \tag{2}$$

where u_* is the bed shear velocity $u_* = [\tau_0/\rho]^{1/2}$, τ_0 is the bed shear stress, and ρ is water density. Equation (1) provides a scaling so that the same number of grains are displaced from a bed population containing equal numbers of grains (a constant percentage of the grains available are displaced). Equation (2) provides a time scale that may be considered to represent the number of turbulent fluctuations in bed shear stress a grain experiences [*Yalin*, 1977]. This is true if the mean period of near-bed turbulent eddies scales with the ratio k_s/u_*, where k_s is the bed roughness (which itself may be scaled by D). Thus a longer sample period is necessary for coarser unisize beds.

Defining n as the number of grains displaced per unit bed area and unit time, an equal number of grains displaced requires

$$n_1 A_1 t_1 = n_2 A_2 t_2 \tag{3}$$

Inserting the similarity ratios into this requirement produces the initial-motion criterion

$$nD^3/u_* = \text{const} \tag{4}$$

Neill and Yalin [1969] suggest a value of 10^{-6} for the constant in (4), because it is close to the lower limit that can be practically observed in open-channel flow. Equation (4) is not the only unisize initial-motion criterion that has been suggested. *Yalin* [1977] introduced another criterion

$$\frac{nD^3}{((s-1)gD)^{1/2}} = \text{const} \tag{5}$$

Although derived only in part using similarity arguments, (5) can be presented in terms of the scaling arguments that lead to (4). In this case, if the similarity ratio for the bed area is the same, kinematic similarity of sample periods would be defined as

$$t_1(g/D_1)^{1/2} = t_2(g/D_2)^{1/2} \tag{6}$$

Still other similarity ratios could be used, and these would produce other initial-motion criteria. Alternative ratios are more reasonably defined for the sample period than for the sample area, for which an intuitive argument can be made that the size of the bed area observed must be increased for the larger grains in the mixture. The two sample period similarity ratios presented above are the simplest possible using a dimensional analysis of the relevant physical parameters.

The initial-motion criteria of (4) or (5) cannot be directly applied to mixed-size sediment for two reasons. First, the sampling period in both is scaled with grain size. There are no obvious time scales that vary with grain size in a mixed-size bed; the time scale of the turbulent fluctuations should be a function of the overall bed roughness, but should not vary with size within the mixture. For individual fractions in a mixture, then, initial motion should be defined in terms of m, the number of grains displaced per unit bed area. Second, because the grains of each fraction do not cover the entire bed surface, similar bed areas must be scaled with the proportion f_{ai} of each fraction present on the bed surface as well as grain size. Geometrically similar bed areas for individual fractions in mixed-size sediments are defined as

$$f_{a1}\frac{A_1}{D_1^2} = f_{a2}\frac{A_2}{D_2^2} \tag{7}$$

The requirement that the same number of grains are detached from a bed area containing equal numbers of grains is

$$m_1 A_1 = m_2 A_2 \tag{8}$$

Inserting (7) into (8) yields an initial-motion criterion for each fraction in a mixture

$$m_i D_i^2/f_{ai} = \text{const} \tag{9}$$

In practice, an initial-motion criterion such as (4) or (5) would be used for the sediment mixture as a whole in conjunction with (9) applied to the individual fractions in the mixture. Equations (4) or (5) provide a means of standardizing initial-motion observations between different mixed-size sediments, if a characteristic roughness length for the sediment can be defined. The most commonly used roughness length is D_{65}, suggested on the basis of experimental work by *Einstein* [1950]. The author has found D_{65} to provide a reasonable approximation of the roughness length for flume experiments near incipient motion with three mixed-size sediments all with the same mean size and phi standard deviations of 0.2, 0.5, and 1.0 [*Wilcock*, 1987]. These results support the choice of D_{65} as an estimate of the bed roughness near incipient motion and also suggest that a characteristic roughness length may be independent of mixture sorting, which, after some measure of the mean grain size, should be the dominant mixture parameter determining roughness length. With D_{65} and u_* known, and using a constant value of 10^{-6} in (4), n is completely determined. Because $m = nt$, and D_i and f_{ai} are known for each fraction, only t or the constant in (9) remain to be determined. It seems most natural and practical to use a convenient sampling period appropriate to (4) or (5). With the sample period known, m is also determined for the roughness size (e.g., D_{65}), and the value of the constant in (9) is computed from m, D_{65}, and f_{a65}. Because (9) standardizes the initial-motion observations only within each mixture (not from mixture to mixture), a predetermined value of the constant in (9) is not needed.

If the bed surface grain size distribution is described in terms of the volumetric proportion of each fraction by weight, a conversion to an areal proportion by weight must be made, which is approximately

$$\frac{f_{a1}}{f_{a2}} = \frac{f_{m1}}{f_{m2}}\left(\frac{D_2}{D_1}\right) \tag{10}$$

and (9) becomes

$$m_i D_i^{\,3}/f_{mi} = \text{const} \tag{11}$$

The practical implications of (9) or (11) are fairly severe. An example helps to illustrate. Consider a sediment mixture that contains the relatively modest range of sizes such that the coarsest fraction is 16 times the size of the finest and that each of these extreme fractions contains the same areal percentage of the bed by weight. If one is measuring grain displacements over a fixed area of the bed for a fixed length of time, (9) states that at incipient motion the number of grains displaced must vary with the square of grain size, or 256 of the fine grains must be displaced for every one of the coarsest grains. If the two fractions with a 16-fold difference in grain size contain an equal volumetric percentage of the mixture by weight, and a constant observation area and period are used, (11) requires that 4096 of the fine grains be displaced for each of the coarse grains. Or, for a constant observation period and the same number of grains observed in motion (say, one grain in each fraction), the observation area for the coarse grains would have to be 4096 times the size of the area for the fine grains to produce a constant value of the initial-motion criterion in (11). Beyond the scaling problems, serious operational problems are involved in simultaneously determining the bed surface size distribution and f_{ai} along with the number of grains of each size being detached. Finally, it must be recognized that the proportion f_{ai} is both a scaling parameter and a dependent variable of the problem. Because the proportion of a given fraction on the bed can change during the sampling period, or from one sample period to the next, or with different bed shear stresses that correspond to τ_{ci} for different fractions, it is difficult to imagine how a truly representative initial-motion sample could be obtained.

Comparison of Initial Motion Methods

Because the two initial-motion methods involve measurements of entirely different quantities, it is not clear a priori that results obtained from the different methods can be directly compared. Because initial-motion data from both methods are commonly combined without distinction, and because it is not clear which, if either, of the methods should be used to calibrate semianalytical initial-motion models, it is worth pursuing as direct a comparison of the two methods as possible. We will make a comparison in a general, abstract fashion, as well as in terms of the only data set that allows computation of τ_{ci} from both methods. The implications of the scaling problems embodied in (4), (5), and (9) will be also examined in the context of both initial-motion methods.

In principle the two initial-motion methods should give identical results in all but one special case. The ideal conditions for identical results include a sediment bed with a smooth, continuous grain size distribution and a very large number of transport samples from a broad range of flows with a very small increment in bed shear stress from one to the next. Assuming that scaling requirements have been met, complete comparability between the two methods requires several conditions: (1) the flows must not exceed the shear stress necessary to move the coarsest grain in the bed; (2) the samples must be of sufficient duration to ensure that all sizes have experienced the full range of fluctuating bed shear stress; (3) initial-motion conditions for individual fractions in a mixture must increase continuously with grain size; and (4) the reference transport rate must be chosen so that the reference shear stress is close to that at the true initial motion of the various grain sizes in the bed.

In practice, the two methods appear to give very different results. The problems associated with the two methods are entirely different and are considered in order.

Largest-Grain Method

The most common departures of the largest-grain method from the ideal involve sample areas not large enough to give coarse grains an equal opportunity to be sampled, and samples that do not contain the coarser sizes in the bed even though they are taken from flows that exceed the shear stress necessary to move the coarsest grain in the bed. The effect of these sampling and scaling problems can be illustrated with the help of schematic diagrams in Figures 1 and 2 showing a set of "true" initial-motion relations and the related largest-grain results. The figures represent a variety of relations between the critical shear stress τ_{ci} and D_i/D_{50} that are chosen to include the natural range of relations between these two variables. These range from $\tau_{ci} \propto D$ (no relative size effect on the initial motion of individual grains) to $\tau_{ci} \propto$ const (relative size effects exactly balance the difference in mass between different fractions, the equal mobility hypothesis of *Parker et al.* [1982]).

Figures 1 and 2 expressly show the result of sampling flows whose bed shear stress exceeds that necessary to move all sizes in the sediment bed. Figure 1 expresses the initial-motion relations in terms of

$$\frac{\tau_{ci}}{\tau_{c50}} = \left[\frac{D_i}{D_{50}}\right]^a \tag{12}$$

The subscript c in (12) represents the critical shear stress for the fraction i. The subscript lg in Figure 1 represents the largest-grain estimate of τ_{ci}. The true initial-motion relations, shown in Figure 1*a*, are simple straight lines with a log slope of a in (12). The lines for the largest-grain method, shown in Figure 1*b*, each consist of two parts: one is identical to its counterpart in Figure 1*a* and the other follows a vertical trend at D_{max} if the bed shear stress exceeds that necessary to move the largest grain on the bed. One line in Figure 1*b* is exceptional: the case of equal mobility, for which the exponent a in

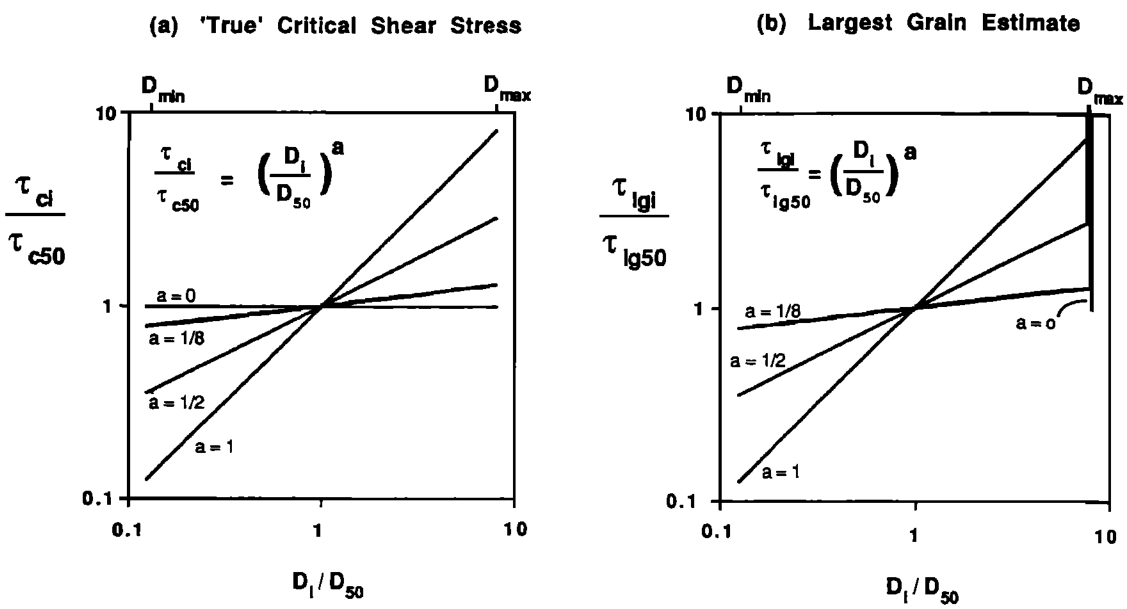

Fig. 1. Schematic initial-motion relations for the largest-grain estimate of the critical shear stress. (*a*) Four different values of the exponent a in (12). (*b*) The largest-grain estimate of the equivalent relations Figure 1*a* and includes the case where the bed shear stress exceeds that necessary to move the coarsest grain in the mixture.

(12) is zero. In this case the largest grain on the bed should be found in all samples when the bed shear stress is large enough to move any sediment, and the appropriate line in Figure 1*b* is a vertical line segment beginning at the point $(D_{max}/D_{50}, 1)$. In this special case the largest-grain method gives results opposite to the true relation.

Figure 2 illustrates the dimensionless version of Figure 1, using

$$\frac{\tau_{ci}^*}{\tau_{c50}^*} \equiv \frac{(\tau_{ci})/((s-1)\rho g D_i)}{(\tau_{c50})/((s-1)\rho g D_{50})} = \frac{\tau_{ci}}{\tau_{c50}}\frac{D_{50}}{D_i} = \left[\frac{D_i}{D_{50}}\right]^b \qquad (13)$$

From (12) and (13), $b = a - 1$, except when $\tau_0 > \tau_{cmax}$, for which D_{lg} is invariant with τ. Figure 2*a* shows the true initial-motion relations and Figure 2*b* shows the largest-grain estimates of these relations. As in Figure 1*b*, the largest-grain lines in Figure 2*b* have two segments: one is identical to those in Figure 2*a*, and the other is a vertical line segment at $D = D_{max}$ for which the bed shear stress exceeds that needed to move the coarsest grain. The special case of equal mobility corresponds to an exponent $b = -1$ in (13) and is represented on Figure 2*b* by a vertical line segment beginning at $(D_{max}/D_{50}, D_{50}/D_{max})$. Again this line contrasts strongly with that determined using the reference method in Figure 2*a*.

The second sampling problem associated with the largest-grain method, that large, scarce grains have a diminished chance of being observed in motion, is only part of the more fundamental problem of scale, for which (9) or (11) can serve as a guide. This problem may be illustrated best with an ex-

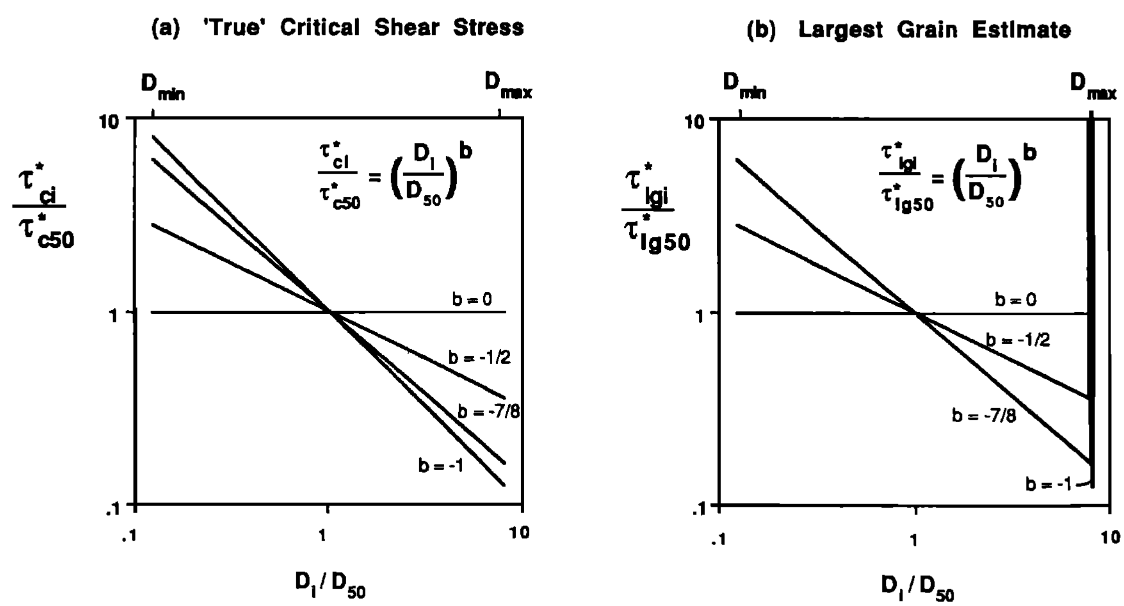

Fig. 2. Schematic initial-motion relations for the largest-grain estimate of the critical Shields parameter. Figure 2 is identical to Figure 1 except that the critical shear stress is expressed as a Shields parameter.

ample. *Hammond et al.* [1984] used the largest-grain method with video observations of initial motion of gravel by a tidal current. Although aware of the scaling principles behind (5), they were unable to alter the viewing area used. Hence the value of m in (5) was essentially constant, because the number of grains observed moving and the bed area were not varied from fraction to fraction. The coarsest particle they observed moving was 4.82 cm in size, compared with a mean size observed moving of 1.7 cm and a minimum size of 0.3 cm. Approximately 4.5% by weight of the coarsest size was present in grab samples of the surface material, compared with 10.5% of the mean size and 1% of the finest size. The scaling provided by (11) requires that for similarity and for the constant viewing area used by Hammond et al., the critical shear stress for each fraction would correspond to the flows that over a constant sample period, displaced 922 of the finest grains and 53 of the mean-sized grains for each of the coarsest grains observed moving.

Similar scaling problems exist when the critical shear stress is estimated from the largest grain found in a transport sample. In this case, however, the problem may be crudely addressed by examining transport samples of the same size at each flow strength (although this would require very long sampling periods at the lower transport rates). This approach at least provides an equal chance of a grain of a given fraction (say, a very large one) to be found in any sample, although it obviously does not provide equal chances for grains of different sizes to be sampled.

If truly similar observation areas or sample sizes are not achieved, how do deviations from similarity in the observation technique affect initial-motion results? The degree to which initial-motion results would vary between similar and nonsimilar samples cannot be exactly determined, although the different trends of the results are fairly clear. For a given bed shear stress and a nonsimilar sampling technique, the size of the largest grain sampled will, in general, be smaller than that obtained with a sampling technique that meets a similarity criterion such as (9) or (11). (For example, the probability of observing the displacement of a coarser grain will increase if the viewing area and/or sample period are increased.) On Figure 1*b*, nonsimilar sampling techniques will result in a point falling above the true initial-motion line. Because grains larger than the mean size are predominantly observed using the largest-grain method and because the scaling problems become particularly extreme for rare, very large grains, one can expect that a nonsimilar technique would produce points that fall above the right-hand part of the true line on Figure 1*b* and that these points would follow a trend that is steeper than the true initial-motion line. This tendency would be reinforced by any samples that were taken for values of bed shear stress that exceeded that necessary to move all fractions in the mixture.

Both sampling and scaling problems in the largest-grain method can produce trends in Figure 1 that are steeper than the true initial-motion line. This is exactly the direction in which the published largest-grain initial-motion results ($\tau_{lgi} \approx \alpha[D_i/D_{50}]^{1/2}$) differ from results determined with the reference transport method ($\tau_{ri} \neq f[D_i/D_{50}]$).

Reference Transport Method

Although the reference transport method does not measure directly the true instant of initial motion for each fraction in a mixture, it has the distinct advantage that it may be used far more reliably than the largest-grain method. The basic requirement is that accurate transport samples be obtained for enough flows (perhaps on the order of 5–10) that the relation between shear stress and transport rate is well defined for all fractions in the vicinity of the reference transport rate. The problems of sample duration and sample area important to the largest-grain method are not as restrictive for determining the fractional transport rate, which is naturally expressed as a percentage of the entire transport and scaled by the proportion of each fraction present in the bed. Thus the absence of a rare, very large grain in the sample will not strongly affect the results. If a sufficiently large transport sample is taken, the proper scaling for determining the reference shear stress may be directly incorporated when computing the fractional transport rates.

A number of different reference transport criteria have been suggested. *Parker et al.* [1982] (hereafter referred to as PKM) define the reference transport rate in terms of a constant value of the fractional transport parameter W_i^*

$$W_i^* = \frac{(s-1)gq_{bi}}{f_i(u_*)^3} \tag{14}$$

where q_{bi} is the transport rate of fraction i in terms of volume transport per unit width and time, and f_i is the proportion of fraction i in the bed sediment. The PKM reference transport rate ($W_i^* = 0.002$) is shown in Figure 3(top), along with, for illustration, fractional transport data for six fractions from a sediment mixture with a mean size of 1.8 mm, a sorting of 1ϕ, and a flow depth of 11 cm in a laboratory flume [*Wilcock*, 1987].

Day [1980] defines the reference transport rate using the *Ackers and White* [1973] (hereafter referred to as AW) transport model, in which the transport parameter *Ggr* can be defined as

$$Ggr = \frac{q_b}{VD}\left[\frac{u_*}{V}\right]^n \tag{15}$$

where V is mean flow velocity, D is a representative grain size, and the exponent n varies with a dimensionless measure of grain size from 1.0 for silt-sized grains to 0.0 for grains coarser than about 2.0–3.0 mm. The AW reference transport rate is $Ggr = 10^4$. The AW reference-transport criterion may be expressed in terms of W_i^* and τ_i^* as

$$W_i^* = \frac{10^{-4}}{\tau_{*i}}\left[\frac{V}{u_*}\right]^{n+1} \tag{16}$$

An exact comparison between the PKM and AW transport parameters is possible only for specific combinations of grain size, fluid viscosity, flow velocity, and bed shear velocity. However, a general comparison can be made by using the flow resistance formula contained in the AW model to substitute for V/u^* in (16)

$$\frac{V}{u_*} = (32)^{1/2}\log_{10}\left[\frac{10d}{D}\right] \tag{17}$$

Because flow depth d is virtually constant for the experimental data in Figure 3(top), a value of 11 cm can be used in (17) to produce W^*, τ^* reference transport criteria for the six fractions shown.

It may be seen in Figure 3(bottom) that the different reference transport criteria can give different reference shear stresses. The reference shear stresses corresponding to the AW criterion are some 20% larger than those computed using the PKM criterion. Because the reference transport criterion are at an angle to each other, and because the fractional transport data tend to follow a steep, concave-downward trend in this

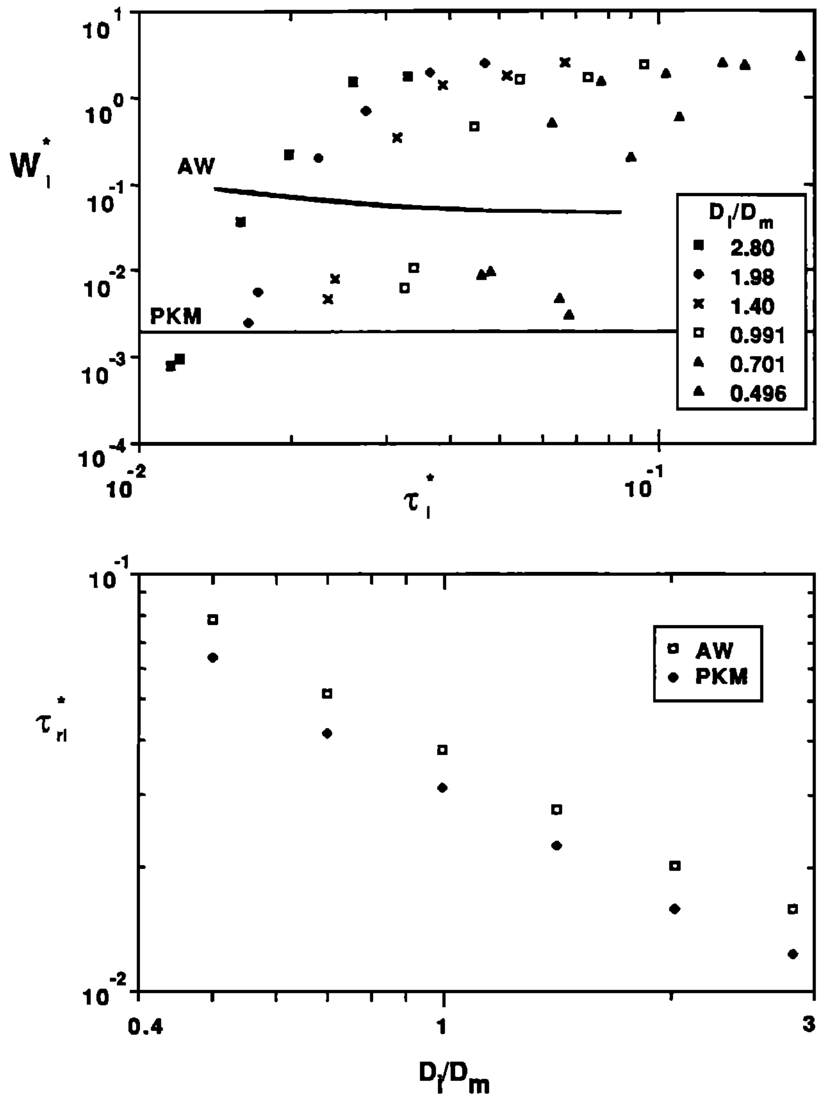

Fig. 3. (Top) Reference transport criteria. PKM, *Parker et al.* [1982]; AW, after *Ackers and White* [1973]. Fractional transport rates of six fractions from a poorly sorted mixture are shown for illustration. (Bottom) Reference shear stresses estimated using the PKM and AW criteria for the data illustrated in Figure 3(top).

graph, the reference shear stresses differ not only in magnitude but also in the variation of τ_{ri}^* with relative grain size (Figure 3(bottom)). The AW criterion gives a relation between τ_{ri}^* and D_i/D_{50} that is slightly curved and gentler in slope than the equivalent PKM results. Equation (13) may be fitted to these data; the value of the exponent b is 0.94 for the PKM criterion and 0.91 for the AW criterion. Larger differences between the methods are evident in other fractional transport rate data [*Wilcock and Southard*, this issue].

The values chosen for the PKM and AW reference transport criteria are, within a range of low transport rates, essentially arbitrary. Because each is given in terms of a constant value of a particular dimensionless transport parameter, they share the advantages and disadvantages of that parameter and are appropriately applied as reference values for transport rates described using that parameter, which is the use for which they were originally derived. For example, PKM used their reference transport criterion (which does not include D_i) to provide a collapse of the transport rate data for individual fractions that was undistorted with respect to grain size. The scaling arguments presented earlier suggest, however, that for the purpose of defining a general initial-motion criterion, and by extension a reference transport rate, grain size should be included in the initial-motion criterion. Because initial-motion relations have significance beyond serving as a reference level for analysis of transport rate data, it is worth considering whether the initial-motion definition given earlier can be used to provide a rational basis for defining a reference transport criterion.

The relation between the number of grains of a fraction moving per unit bed area m_i and the fraction's transport rate is

$$m_i \forall_{gi} u_{gi} = q_{bi} \tag{18}$$

where $\forall_{gi}$ is the volume and u_{gi} is the mean velocity of grains in fraction i. $\forall_{gi}$ can be described without serious error as $\forall_{gi} = \alpha D_i^3$, where α is taken to be a constant for each fraction. Using this expression for the grain volume and combining constant terms, substitution of m_i from (18) into (9) yields

$$q_{bi}/f_{ai} = (\text{const}) D_i u_{gi} \tag{19}$$

This may be expressed in terms of W_i^* and τ_i^*

$$W_i^* = \frac{\text{const}}{\tau_i^*}\frac{u_{gi}}{u_*} \tag{20}$$

It may be seen from (20) that the mixed-size initial-motion criterion may be expressed as a simple function of W^* and τ^* only if u_{gi} is a simple function of u_* (and not of D_i/D_{50}). Conversely, if a constant, low value of a transport rate, expressed in terms of any combination of W_i^* and τ_i^* (including the Einstein transport parameter $q_{bi}^* = W_i^*/\tau_i^{*3/2}$) is to be used as a reference value for estimating τ_{ci}, it will be consistent with (9) only if $u_{gi} = cu_*$, where c is a constant.

A mixed-size reference transport criterion based on (9), if it could be defined, has the advantage that it satisfies similarity arguments that provide each fraction on the bed a representative chance of being observed or sampled. Unfortunately, the transformation of (9) into a transport rate cannot be done at present and depends on an unknown relation between u_{gi}, bed shear stress, and relative grain size. *Fernandez Luque and Van Beek* [1976] found from direct film measurements of grain motion that the grain velocity of unisize sediment is well described by a simple function of u_*. It is entirely unknown at present whether such a consistent relation may be defined for individual fractions in a size mixture or whether such a relation might change with grain size within the mixture.

Comparison Using Oak Creek Data

The Oak Creek data collected by *Milhous* [1973] are particularly valuable here because they are the only data we know that include both extensive transport measurements and largest-grain measurements for conditions where grains coarser than those measured were available in the sediment bed. Thus identical initial-motion results should be obtainable from these data if an appropriate reference transport rate is selected and if scaling problems are not serious. Strikingly, different initial-motion relations, however, have been found. *Komar* [1987] found $\tau_{lgi} \propto D_i^{0.57}$ using the largest-grain method, whereas PKM found essentially no dependence of τ_{ri} on D_i using the reference transport method. These two relations are based on different subsets of the Oak Creek data: Komar examined all 66 samples from the best data set of Milhous whereas PKM examined a subset of these data using the 22 samples with the highest transport rate. (It is worth noting that the lowest transport rate used by PKM, 0.28 g m^{-1} s^{-1}, is fairly small.) To allow a direct comparison of the two initial-motion methods, Figure 4 presents the reference transport and largest-grain results for only the uppermost 22 points. The largest-grain data are identical to the appropriate data subset in Komar, and the reference transport data are those fitted by PKM. Note that the scales of log axes in Figure 4 are unequal in order to show the data points clearly.

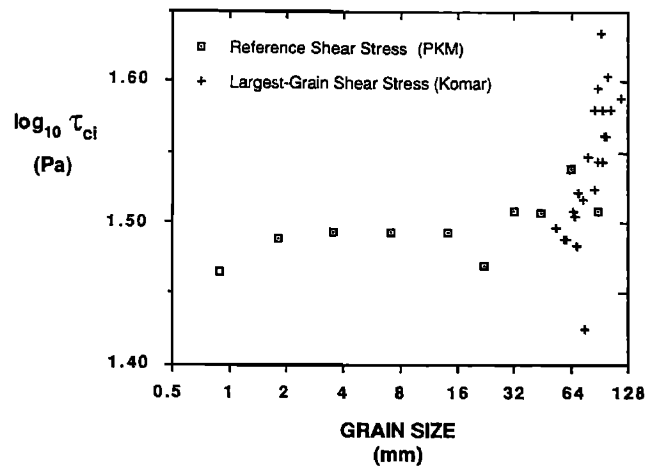

Fig. 4. Initial-motion results for the Oak Creek data of *Milhous* [1973]. Data labeled Komar are estimated using the largest-grain method [*Komar*, 1987]; data labeled PKM are estimated using the PKM reference transport method [*Parker et al.*, 1982].

The most important point about Figure 4 is that even for the same transport samples, the initial-motion results found with different methods are strikingly different. The largest-grain data show τ_{lgi} to vary with D_i/D_{50} to the power 0.405; PKM found τ_{ri} to vary with D_i/D_{50} to the power 0.018. An even larger exponent for the largest-grain method would be obtained if the line were fitted only to the points farthest to the right on Figure 4, a procedure that would appear reasonable given the sampling problems inherent in this method. An additional obvious difference between the initial-motion results is that the largest-grain data represent only the coarsest part of the bed mixture. The smallest grain measured is coarser than more than 80% of the bed mixture. Both the preponderance of coarse grains and the difference in the τ_{ci}-D_i/D_{50} relations produced by the two methods are about what one would expect if the true initial-motion relation for this sediment is close to equal mobility ($a = 0$ in equation (3)), particularly given the sampling problems inherent in the largest-grain determinations. Such problems appear to be important with the Oak Creek data. The availability of coarser grains in the Oak Creek sediment bed may have been strongly limited. Although grains coarser than those sampled are found in the bed, their percentage of the total bed material (all of which is assumed available for motion for the subset of the Milhous data examined by PKM) is vanishingly small (reported as 0% for $D_i > 102$ mm in PKM). In addition, there is a strong positive correlation between the size of the coarsest grain trapped and the total quantity of sediment collected. Hence there is a relatively lower probability for a coarse grain to be found in the smaller transport samples, which were taken at the lower values of bed shear stress, than in the larger samples at higher flows and greater transport rates. Because the total shear stress and sample size are highly correlated, it is difficult to assess the relative importance of each in determining the variation with bed shear stress of the largest grain sampled. If all samples had been equal in size to the largest, some of the lower-shear stress samples would have included coarser grains, although it is not clear whether this would have produced a vertical τ_{ci}-D relation that would be in agreement with the reference transport results.

The Oak Creek data illustrate that strikingly different results are obtained from the same data by using different initial-motion methods. Clearly, the results produced by one method cannot be used to confirm or deny those produced by the other. More important, the difference in the results may be explained in terms of the sampling and scaling problems associated with the largest-grain method. However, even if the largest-grain estimate of the critical shear stress could be scaled to match a criterion such as (9), it is not clear that the two methods would give similar results because, even under ideal conditions, the two methods give fundamentally different results when the special case of equal mobility is approached.

OTHER POSSIBLE SOURCES OF DISCREPANCY IN CRITICAL SHEAR STRESS RESULTS

Although the Oak Creek data demonstrate that methodological differences alone are sufficient to produce the observed discrepancies in critical shear stress, it is worth considering if the differences observed have other causes. All the initial-motion measurements that show an approximate square-root dependence of τ_{ci} on D_i have been made using the largest-grain method with field data [*Carling*, 1983; *Hammond et al.*, 1984; *Komar*, 1987]. The initial-motion results showing a near size independence of τ_{ci} have been made using the reference transport method in both the laboratory [*Day*, 1980*b*; *Wilcock and Southard*, this issue] and in the field [*Parker et al.*, 1982] and using the largest-grain method in the field [*Andrews*, 1983]. Because all laboratory measurements have produced near size independence of τ_{ci}, while many of the field measurements have produced a size dependence, the possibility exists that these differences result from basic differences in transport conditions between field and flume, in addition to differences in the measuring techniques.

One difference between field and flume that might produce the observed discrepancies among initial-motion results concerns the presence of equilibrium transport conditions. All the flume experiments discussed in this paper have been carried to an equilibrium transport condition, wherein the transport rate and size distribution vary about a stable mean under steady flow conditions. In contrast, many of the field studies correspond to transport conditions where the bed and local transport are adjusting to changes in the flow and sediment discharge. If the grain size distribution of the bed surface, on which the size distribution of the displaced sediment must depend, lags behind the flow conditions at the time of sampling, the sediment sampled (whether visually or directly) will be different to some extent from the sediment that would be sampled if the flow and sediment discharge developed to equilibrium conditions. It is difficult to say exactly how disequilibrium transport conditions would influence initial-motion measurements. The magnitude of the effect would depend on, among other things, the magnitude, rate of change, and direction of change of the flow strength, sediment transport rate, and transport size distribution input to the sampling site. Also important are the distribution of sediment at different elevations within the transport system (providing a possibly varying population of sediment available for transport) and the rate with which the sediment bed may adjust its surface size distribution and bed configuration in response to the changing flow and transport conditions. In order for the field results to be consistently different from the flume results, however, it would be necessary to argue that disequilibrium field conditions differ from equilibrium laboratory conditions in a consistent manner. For example, if initial-motion estimates from disequilibrium conditions consistently show a greater dependence on grain size, it would follow that the disequilibrium bed surface would have to be consistently finer grained than the corresponding equilibrium surface for the same flow and sediment (as the bed surface coarsened, the largest grain sampled would tend to increase, thus producing a smaller size variation in the sampled largest grains and a lesser variation of τ_{lgi} with grain size). No such pattern is mentioned or probable, however, in the timing of the field largest-grain samples.

A second difference between field and flume that might produce the different initial-motion results concerns the magnitude of sampling errors that might be expected in field and flume data. The accuracy of both initial-motion methods depends on the accuracy of the transport samples used in the analysis. For example, if coarser sediment were systematically underrepresented in hand samples of transport in the field, both initial-motion methods would produce a critical shear stress for coarse fractions that was larger than the correct value. The transport samples used to produce the initial-motion results discussed in this paper do not support the notion that systematic differences in sampling error can explain the differences in critical shear stress observed between the largest-grain and reference transport methods. Two of the field studies [*Carling*, 1983; *Milhous*, 1973] used a bed load sampling slot that extended the full width of the stream and provided sampling accuracy equivalent to that in a flume. Another field study used data from both a slot sampler and hand-held Helley-Smith samplers [*Andrews*, 1983], but produced largest-grain initial-motion results close to equal mobility, which is typical of the flume results.

The Oak Creek data make it clear that methodological differences alone are sufficient to produce the variation typically observed between largest-grain and reference transport results. Although real variations in initial-motion results may be expected under disequilibrium flow conditions, and sampling errors should have a noticeable impact on critical shear stress estimates, these factors apparently do not contribute to explaining the differences observed in initial-motion results between the largest-grain and reference transport methods.

Application of Initial Motion Methods

Without an accurate conversion between the initial-motion results determined by the different methods available, care must be taken to compare initial-motion data only with previous results determined with the same method and with the same type of data. In situations where the number of transport samples is insufficient to define transport relations for each fraction, only the largest-grain method may be used and only then if there is evidence that coarser, immobile grains are available for transport. Unfortunately, if many natural rivers are found to operate at conditions near equal mobility, it is not clear how useful the largest-grain method would be for making any critical shear stress estimate, because most, if not all, flows producing sediment transport would involve the transport of all grains (including the coarsest). The reference transport method is appropriate for scaling the transport rates of individual fractions, provided that the same transport parameter is used for both the transport analysis and the reference shear stress calculations.

Unless properly scaled, neither initial-motion method will give results that may be interpreted as representing some true, mean shear stress at which individual fractions in a mixture begin moving. A truly general and properly scaled initial-motion methodology is needed to allow future work to focus on the physics of the initial-motion problem in mixed-size sediments and to permit verification of theoretical models based on experimental data on the pivoting angle of individual sizes in mixed-size sediment [*Wiberg and Smith*, 1987]. Equation (9) (or equation (11)) provides the basis for a properly scaled initial-motion criterion; (20) provides a practical means of applying this criterion if an accurate method can be found to relate the mean transport velocity of each fraction to its relative grain size and the bed shear stress.

Conclusions

There are two general methods for determining the critical shear stress for individual fractions in mixed-size sediment. One associates the critical shear stress with the largest grain in the mixture that can be moved by a given flow. The other approximates the critical shear stress as that shear stress that produces a small reference transport rate of a given fraction.

In principle, the two classes of methods should give identical results except for the special case in which all fractions begin moving at the same bed shear stress. In practice, the two methods do not give the same results: the largest-grain critical shear stresses typically vary with roughly the square root of D_i/D_{50}, whereas estimates of the critical shear stress made with the reference transport method show little dependence on grain size. Because the same data can only seldom be analyzed with both initial-motion methods, there is little opportunity to determine whether these observed differences are real, or merely the artifact of different methodologies. The one case known for which the same data may be analyzed with both methods (Oak Creek data of *Milhous* [1973]) suggests that the differences observed may be largely methodological. In the Oak Creek case, the initial-motion results differ in magnitude and direction in exactly the same way as the results typically reported when the two methods are applied to different, mutually exclusive data sets.

The different results produced by the two initial-motion methods may be explained in terms of the difficult scaling problems associated with estimating true initial-motion conditions for individual fractions in mixed-size sediment. These scaling problems are illustrated by the initial-motion criterion for individual fractions in mixed-size sediment developed in this paper. Scaling problems are more serious for the largest-grain method than for the reference transport method, because fractional transport rates may be naturally defined in a fashion that incorporates the necessary scaling considerations. An understanding of the methodological influence on mixed-size initial-motion results is important because it suggests that there is little basis for concluding that initial-motion relations determined for one sediment with one method reflect a truly different physical situation when compared with initial-motion relations determined for a different sediment with the other method. The scaling and sampling problems that may be identified with the use of a mixed-size initial-motion criterion, and the results of the Oak Creek data, suggest that the natural variation in the critical shear stress of individual fractions in mixed-size sediments may not be as large as previously thought.

The practical considerations involved in determining initial motion for mixed-size sediment suggest strongly that the reference transport method is preferable to the largest-grain method. Unfortunately, a generally accepted and properly scaled reference transport criterion is not now available. Such a criterion is definable but requires an unknown relationship between the transport velocity of individual fractions, their relative grain size, and the bed shear stress, so that an initial-motion criterion can be converted to a transport rate. Existing reference transport criteria are necessarily arbitrary and tied to a particular dimensionless transport variable. We have demonstrated that different reference transport criteria produce different initial-motion results, with the clear implication that models of mixed-size sediment transport that use a particular dimensionless transport rate parameter must also use a reference value of the same transport rate parameter in determining the reference shear stress for each fraction.

Even if equivalent reference transport and largest-grain criteria could be derived from the same general initial-motion criterion and properly scaled measurements were made with each method, the initial-motion results would still be considerably different for cases where all sizes begin moving at nearly the same bed shear stress. In these cases, the largest grain in the mixture would be present in most, if not all, of the largest-grain samples, and would give a vertical $\tau_{ci}^* - D_i/D_{50}$ relation that would be perpendicular to the reference transport relation. Transport data from western United States rivers [*Milhous*, 1973, *Andrews*, 1983; *Andrews and Erman*, 1986] and flume experiments [*Day*, 1980*b*; *Wilcock and Southard*, this issue] show transport conditions to be close to equal mobility and suggest that these conditions may be relatively common.

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