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Rookie to Expert: Implementing Rook Theory in Strategic Games

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Imagine a chess board–but forget the rules. The board is just a flat square with smaller colored squares on it and the pieces are just objects. Now choose eight of the pawns and place them on the diagonal. You just made a non-attacking rook arrangement. Now make another arrangement where the pawns can only be on white squares and cannot share a row or column. This is another rook arrangement in which the black cells were restricted. How many ways do you think the pieces could be arranged on the white cells without sharing a row or column? That is the question to which rook polynomials can give an answer.

Rook polynomials provide a method of counting permutations with restricted positions. The theory was first developed by Kaplansky and Riordan in [Kaplansky-Riordan 1946], but problems that can be modeled by permutations with restrictions have been investigated since the eighteenth century when Pierre de Montmort worked on the *probleme des rencontres*. This is a special type of problem that asks the probability of not drawing numbered balls on their numbered draw. For example, not drawing ball number three on the third draw. This simple problem can be solved using rook polynomials and has a number of interesting applications. In this paper, we summarize the methods of finding rook polynomials in two and three dimensions that were discussed in [Kaplansky-Riordan 1946, Alayont-Krzywonos 2013, Michaels-Rosen 1991]. Finally, we present a strategy game we developed based on rook polynomial theory which includes an explanation of the game, game testing, reviews, and possible adjustments.

Rook Theory in Two Dimensions

A rook polynomial is a generating polynomial which counts all possible ways of placing nonattacking rooks on the available cells of a board. Nonattacking means no two rooks can occupy the same row or column. The number of rooks that can be placed on a board with m rows and n columns cannot be greater than either m or n. For example, our 8 x 8 chess board could have at most 8 rooks. A board is like a chess board of varying dimensions that, in addition to having available cells, could also have restricted cells where a rook cannot be placed. In the introductory example of the chess board, the white cells were available while the black cells were restricted. More formally, a board is composed of cells written in the form (i, j) where $1 \leq i \leq m$ and $1 \leq j \leq n$. These cell coordinates form a subset of the full, unrestricted board $[m] \times [n]$.

The rook polynomial of a board B shows the number of ways to place k rooks on a board B where $k \geq 0$. It is written as $R_B(x) = r_0(B) + r_1(B)x + ... + r_k(B)x^k + ...$ where r_0 is the number of ways to place 0 rooks, r_1 is the number of ways to place 1 rook, etc. For all boards, $r_0 = 1$, and r_1 equals the number of available cells. For a rectangular board with no restrictions, the theorem for finding all terms of the polynomial is as follows:

Theorem 1. *The number of ways of placing k nonattacking rooks, with $0 \leq k \leq min\{m, n\}$, on an $m \times n$ board with no restrictions is equal to* $\binom{m}{k}\binom{n}{k}k!$.

Note the theorem accounts for square boards where $m = n$ and for rectangular boards where $m \neq n$. So although most of our examples discuss square boards for simplicity, rectangular boards can also be represented by rook polynomials.

For a simple board like this 2 x 2 with no restrictions, we could simply count the number of ways to place rooks using brute force.

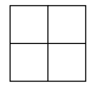


Figure 1. 2×2 board

In this case, we can place up to two rooks and there are two ways to place them (on either diagonal). There are four available cells so there are four ways to place one rook. Finally there is one way to place zero rooks, as always. The rook polynomial found using Theorem 1 confirms this.

$$\binom{2}{0}\binom{2}{0}0! + \binom{2}{1}\binom{2}{1}1!x + \binom{2}{2}\binom{2}{2}2!x^2 = 1 + 4x + 2x^2$$

In rook polynomials, the exponent of the x terms gives the number of rooks which are being placed. The respective coefficient tells the number of ways to place that number of rooks. Since the coefficient of the x term is four, we can see that there are four ways to put one rook on the board. Likewise, there are two ways to place the maximum two rooks on the board, which agrees with our brute force method of counting.

Unlike an unrestricted 2×2 board, some boards are too large to count by hand. We can use three theorems of rook polynomials to make counting the number of rook arrangements easier. The first of these theorems is Disjoint Board Decomposition theorem which calculates the polynomial of the original board from the polynomials of its disjoint subboards. Disjoint subboards are smaller sections of the original board which have no rows or columns in common. Figure 2 gives an example of an original board being separated into two disjoint subboards. If a board can be broken into these disjoint subboards, we can treat these subboards as new boards and give them their own rook polynomial. The theorem then says to multiply these separate rook polynomials to find the rook polynomial of the original board.

Theorem 2 (Disjoint Board Decomposition)**.** *Let A and B be boards that share no rows or columns. Then the rook polynomial of the board $A \sqcup B$ consisting of the union of the cells in A and B is $R_{A\sqcup B}(x) = R_A(x) \times R_B(x)$.*

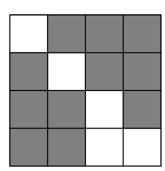


Figure 2. Original 4×4 board

(a) Subboard made from rows and columns 1 & 2

(b) Subboard made from rows and columns 3 & 4

The 4×4 board above is an example of when the Disjoint Board Decomposition theorem could be used because the larger board is the union of the two smaller disjoint subboards. Since no rook can be placed on the excess restricted cells, they do not affect the rook polynomial and can be removed from the smaller disjoint subboards. Although it is simple to see that there is only one way to place four rooks on the original board, we will verify the theorem by finding the polynomials for the subboards and multiplying them in accordance with Theorem 2.

$$\begin{aligned} R_A(x) &= 1 + 2x + 1x^2 \\ R_B(x) &= 1 + 3x + 1x^2 \\ R_{A \cup B}(x) &= (1 + 2x + x^2)(1 + 3x + x^2) \\ R_{A \cup B}(x) &= 1 + 5x + 8x^2 + 5x^3 + x^4 \end{aligned}$$

This verifies that there is one way to place four rooks on this board. It also simplifies counting the ways to place less than four rooks on the board, which makes it especially useful for larger boards with more available cells. It is simple to see why there are five ways to place three rooks by examining the cases of splitting three rooks among the two subboards. Either we place one rook on subboard A and two on subboard B, or we place two rooks on A and one rook on B. In the first case with one on A, there are two ways to place one on A and one way to place two rooks on B. That means there are two ways to place the three rooks between the subboards. We can see this in the from $2x \cdot x^2 = 2x^3$. In the second case where two rooks are on subboard A, there is only one way to place two rooks on A and three ways to place one rook on subboard B. Again we see this in the rook polynomial from $x^2 \cdot 3x = 3x^2$. These two cases comprise the five ways to place three rooks. The proof of this theorem applies this logic to all k rooks on a general board $A \sqcup B$.

The second theorem, called Complementary Board theorem, is useful when restricted cells are outnumbered by available cells. It uses the same technique of forming the rook polynomial as in Theorem 1, but does so for the forbidden cells instead of the available cells. For example, we could use complementary board theorem on a 3×3 board with three forbidden cells as in the board below.

(A) Original board B (B) Complementary board $\overline{B}$

In this case, instead of looking at the available cells of the original board, it is easier to take the complement which turns the available cells into restricted cells and the restricted cells into available cells. In the above figure, taking the complement turns board (A) into board (B) which only has three available cells. This makes finding the rook polynomial much simpler. Since we still have a 3×3 board, we can place up to three rooks. There is one way to place zero rooks, three ways to place one rook, three ways to place two rooks and one way to place three rooks. Therefore, the rook polynomial of the complementary board is $R_{\overline{B}}(x) = 1 + 3x + 3x^2 + x^3$. However, this is not the original board's rook polynomial. To find the number of ways to place the rooks on the original board, we can use the inclusion-exclusion principle. This is done by excluding the union of all the sets from the universal set as illustrated in the following theorem.

Theorem 3 (Complementary Board Theorem)**.** *Let $\overline{B}$ be the complement of B inside $[m] \times [n]$ and $R_B(x) = \Sigma r_i(B) x^i$ the rook polynomial of B. Then the number of ways to place k nonattacking* *rooks on* $\overline{B}$ *is*

$$r_k(\overline{B}) = \sum_{i=0}^{k} (-1)^i \binom{m-i}{k-i} \binom{n-i}{k-i} (k-i)! r_i(B).$$

To find the number of ways to place three rooks, m, k, and n would all equal three, which would make the binomial coefficients all equal 1 in the above formula. Thus, the formula simplifies to

$$3! - r_1 \cdot 2! + r_2 \cdot 1! - r_3 \cdot 0! \;=\; 3! - 3 \cdot 2! + 3 \cdot 1! - 1 \cdot 0! \;=\; 2$$

Since this answer is so small we can verify it by simply counting the number of ways to place three rooks on the original board. This, however, is not always possible, which makes the complementary board theorem very useful for more complicated boards.

The last theory of simplifying a large board is the Cell Decomposition theorem. This is centered on the possibility of two cases, either placing a rook on a certain cell or not placing it there. We consider both cases and make appropriate rook polynomials, then combine the polynomials to make a rook polynomial for the original board, as illustrated in the following theorem.

Theorem 4 (Cell Decomposition). *Let* B *be a board,* B' *be the board obtained by removing the row and column corresponding to a cell from* B*, and* B'' *be the board obtained by deleting the same cell from* B*. Then* $R_B(x) = xR_{B'} + R_{B''}(x)$*.*

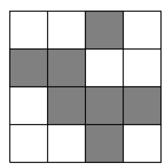


FIGURE 5. Original board B

(A) Board B' excluding row 4 and column 1

(B) Board B'' excluding the cell $(4,1)$

Considering the example board shown in Figure 5 we will show how to use the Cell Decomposition theorem where $(4,1)$ is the chosen cell. First, we will look at the case in which we place a rook on the cell $(4,1)$. With this placement, we then create a new board called B' in which we eliminate the fourth column and first row, since we cannot place another rook in those positions. From this new board, we make its rook polynomial using Theorem 1: $R_{B'} = 1 + 4x + 3x^2$. For the second case where we do not place a rook on $(4,1)$, we create a new board in which it is forbidden. This constitutes the new board B'' whose rook polynomial is $R_{B''} = 1+8x+18x^2+13x^3+2x^4$. Finally we combine them to find the polynomial for the original board B, by doing $R_B = xR_{B'} + R_{B''}$ as stated in Theorem 4. We multiply the rook polynomial of B' by x because we have already chosen where one rook is. The rook polynomial of board B'' does not get multiplied by x because we did not place any rooks on that board. For our example we have,

$$\begin{aligned} R_B &= x(1 + 4x + 3x^2) + (1 + 8x + 18x^2 + 13x^3 + 2x^4) \\ &= 1 + 9x + 22x^2 + 16x^3 + 2x^4. \end{aligned}$$

Therefore, our rook polynomial tells us there are two ways to place four rooks on the original board. While we happened to choose the cell $(4, 1)$ for our original rook placement, we could have chosen any cell. The goal is to make the two new boards as simple as possible, so some cells will be more advantageous than others. Choosing $(4, 1)$ made B' fairly simple by eliminating four other available cells, however, any initial cell would result in the same final rook polynomial.

Rook Theory in Three and Higher Dimensions

The theory of rook polynomials in two dimensions can be generalized to boards of three and higher dimensions. The application of rook theory to three dimensions was first introduced in [Zindle 2007] and further developed in [Alayont-Krzywonos 2013]. Whereas in two dimensions our boards were size $m \times n$, in three dimensions they are $m \times n \times p$, where p is the number of layers. This can be generalized to higher dimensions d such that the board is a subset of $[m_1] \times [m_2] \times ... \times [m_d]$. A cell in three dimensions is now a cube with coordinates (i, j, k) where $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq p$. Instead of referring to rows and columns, boards in three dimensions have slabs, walls and layers. Cells with the same i coordinate lie in the same slab, and cells with the same j coordinate lie in the same wall. As mentioned before, p refers to the layer, so all cells with the same j coordinate are in the same layer.

Just like in two dimensions, a rook in three and higher dimensions will attack any other rook that shares one of its coordinates. This means no two rooks can share a slab, wall or layer in a three dimensional board. Theorem 5 finds the rook polynomial for a three dimensional board by treating the three dimensional board as a two dimensional board that is extended up a number of layers.

Theorem 5. *Let A be an $m \times n$ board and B be a three-dimensional extension of A with p layers. Then, for k rooks where $0 \leq k \leq min\{m, n, p\}$, $r_k(B) = \frac{p!}{(p-k)!} r_k(A)$*

Theorem 5 works well for three dimensions, but not for higher dimensions. Instead we can use Theorem 6 as a general formula for three and higher dimensions.

Theorem 6. *There are $\binom{m_1}{k}\binom{m_2}{k}...\binom{m_d}{k}(k!)^{d-1}$ ways to place k nonattacking rooks, with $0 \leq k \leq min_i\{m_i\}$, on a full $m_1 \times m_2 \times ... \times m_d$ board in d dimensions.*

Looking at a simple $2 \times 2 \times 2$ board, we can use the above theorems to find the rook polynomial.

(A) First Layer (B) Second Layer

Since it is only two layers, we could use either Theorem 5 or 6, but we will use Theorem 6 which finds the coefficients of each term.

$$\binom{2}{0}\binom{2}{0}\binom{2}{0}(0!)^2x^0 + \binom{2}{1}\binom{2}{1}\binom{2}{1}(1!)^2x^1 + \binom{2}{2}\binom{2}{2}\binom{2}{2}(2!)^2x^2 \quad = \quad 1 + 8x + 4x^2$$

This polynomial means that there is one way to place zero rooks, 8 ways to place one (since there are four options for each of two layers) and four ways to place two rooks.

The calculation of the Disjoint Board Decomposition and Cell Decomposition theorems are about the same as in two dimensions, except the new boards used in the calculations are defined a bit differently. In three dimensions two boards are disjoint if they do not share any walls, slabs or layers. In higher dimensions disjoint boards do not share any layers. The theorems for disjoint subboards and cell decomposition are as follows.

Theorem 7 (Disjoint Board Decomposition)**.** *Let A and B be two boards in three or higher dimensions that share no layers. Then the rook polynomial of the board $A \cup B$ consisting of the union of the cells in A and B is $R_{A \cup B} = R_A(x) \times R_B(x)$.*

Theorem 8 (Cell Decomposition)**.** *Let B be a board, B' be the board obtained by removing the layers that correspond to a cell from B, and B" be the board obtained by removing the same cell from B. Then $R_B(x) = xR_{B'}(x) + R_{B"}(x)$.*

The theorem for complementary boards in three and higher dimensions is also similar to its version in two dimensions.

Theorem 9 (Complementary Board Theorem)**.** *Let B be the complement of B inside $[m_1] \times [m_2] \times ... \times [m_d]$ and let $R_B(x) = \sum r_i(B)x^i$ be the rook polynomial of B. Then the number of ways to place k nonattacking rooks on B is*

$$r_k(B) = \sum_{i=0}^{k} (-1)^i \binom{m_1 - i}{k - i} \binom{m_2 - i}{k - i} ... \binom{m_d - i}{k - i} (k - i)!^{d-1} r_i(B)$$

We refer the reader to [Alayont-Krzywonos 2013] for proofs and more in depth discussion of the above higher dimensional rook theory results.

Games Based on Rook Polynomials

Using rook theory, we developed a strategic game with the goal of making math more approachable to younger players. It is a puzzle type of game, meant to develop logical thinking in players around the ages of 3rd to 8th grade. There are four levels that describe different scenarios in which rook theory could be applied. This is a noncompetitive game and does not require scoring. Each level takes approximately $5 - 10$ minutes depending on difficulty and the ability of the player. Players can work alone or in pairs to make it easier.

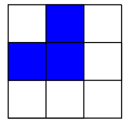
The levels increase in difficulty and should be played in the order that they appear in this paper. For the first three levels, the materials needed are the premade game boards and game pieces that represent the rooks (such as beads, pebbles, or brand name board game pieces). For the last level, the materials needed are a $3 \times 3 \times 3$ cube with monkey cutouts which represent the rooks and bushes which restrict specific cells. The shell of my cube is made from 18 inch wooden rods and the smaller cells are formed from green yarn. The monkeys and bushes are attached to pipe cleaners that hang on the yarn. Pictures of the materials are provided in the Appendix.

The object of the game is to have players place the game pieces as if they are nonattacking rooks. That is, no two game pieces should share a row or column, and in the three dimensional level, the monkey pieces should not share a wall, slab or layer. The level is complete when the player(s) make the specified number of correct rook arrangements on the board. There are two strategies that can be used to complete the arrangements. One strategy is to think of the board as decreasing by a row and column after each rook placement, and then placing another rook on this "smaller" board. The second strategy is to place a rook in each row and column until a correct arrangement has been made. Below you will find short descriptions of each game followed by reviews from players and finally adjustments and extensions.

Windmills on the River

This board was my first level, meant to be very simple to introduce the player to the concept of placing the rooks on the unrestricted cells without putting them in the same row or column. To represent the windmills, we used yellow game pieces from the board game Sorry$^{\text{TM}}$. The story we told was:

> Here is a field with a river (blue cells) going through it. We have three windmills to place on the land. Their wind flows left and right and up and down in straight lines. This means they cannot be in the same row or column, otherwise they would blow each other over. Where should they be put? Can they all be on the same side of the river? Why or why not?

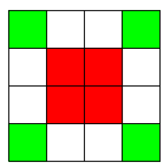


There is no way to put all three rooks on the same side. This board has a rook polynomial of $1 + 6x + 7x^2 + 1x^3$ so there is only one way to place all three rooks on the board.

Scenic Picnic Table

The board for the second level can be seen below. We used the game pieces from the game Clue™ which are small people. We had the players choose the four people that they wanted to use (out of six). Although each piece is a different person, we treated each piece as the same. That is, switching the individual people does not make a new rook arrangement. The story goes,

> Four people are about to sit around a table (the red cells) to enjoy dinner. There are four shrubs on the corners (the green cells). They want to enjoy the view, so they don't want to sit across from each other. They also need room to eat so only one person should be on each side of the table. Find all the ways that they can sit around the table.

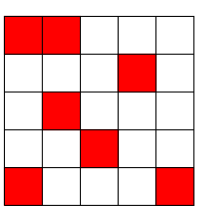


This board's rook polynomial is $1 + 8x + 20x^2 + 16x^3 + 4x^4$ which means there are four ways to place four rooks on this board. Since we asked the players to find all four ways, we told them if they had repeated an arrangement.

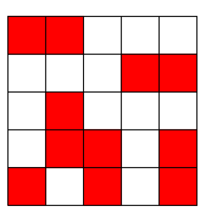
Fire in the Field

For this level we used the white pebbles as the rooks. The players are instructed to find one arrangement per sublevel. The story for this game was:

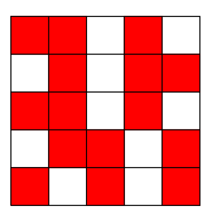
> This field is on fire! Firemen need to stand in the field to put out the fire, but you need to decide where they go. They don't want to be in the same row or column because they don't want to spray each other with their hoses. Can you find a way to place them so they can put the fire out? For second and third sublevels: The fire has spread! Where should the firemen go now?



(A) Sublevel 1



(B) Sublevel 2



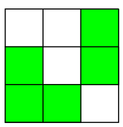
(C) Sublevel 3

For this game there were three sublevels where the "fire was spreading." Once the players completed one sublevel we put more red pebbles on the board to represent the fire spreading in the design of the next sublevel. The players should not move the red pebbles. The rook polynomial of the first sublevel is $1 + 18x + 105x^2 + 231x + 3 + 170x^4 + 25x^2$. The rook polynomial of the second sublevel is $1 + 14x + 63x^2 + 107x^3 + 61x^4 + 7x^3$. The last sublevel's rook polynomial is $1 + 10x + 34x^2 + 46x^3 + 22x^4 + 2x^5$.

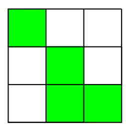
Monkeys in the Jungle

The final level used the three dimensional cube with the monkeys representing the rooks and the bushes restricting certain cells. This level was the most challenging. The story is,

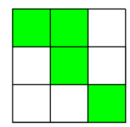
> There are three monkeys in the jungle that love to swing around. They need space to play so no two monkeys can be in the same direction. Can you find a way to place the monkeys in the open spaces so that they have enough room to swing around?



(A) Bottom Layer



(B) Middle Layer



(C) Top Layer

After telling the story we showed the players what we meant by not being in the same direction by giving non-examples where the monkeys were in the same wall, slab and layer. This board has a rook polynomial of $1 + 14x + 28x^2 + 6x^3$. The players are asked to only find one arrangement, but they are welcome to find more.

Observations and Reviews

My first game testers were two female sixth graders. They worked together to get correct arrangements. It was not too hard that they lost motivation, but it was still challenging enough that they never found the solution on the first try. We also tested my game on elementary and middle school aged players at a school math event. Players' grades ranged from 3rd to 8th grade. Some players worked with partners while others worked alone.

Some players got stuck on the picnic table game when finding multiple ways because they could not remember what arrangements they had already made. They liked using the people game pieces in this game as opposed to the beads or Sorry$^{\text{TM}}$ game pieces because it helped them visualize the story. At the school math event, most players were faster at finding arrangements for this board instead of the Windmills on the River board. Perhaps the shape of the rooks around the square was easier to visualize than the abnormal triangle shape of the solution for Windmills on the River.

For the Fire in the Field level players liked that we put out 8+ beads that represented the firefighters and asked how many would they need if the firefighters cannot share a row or column. They thought that was a challenge and did not realize until we told them after that there should be five people because there are five rows and columns. For the additional sublevels we had spread the fire using the red pebbles, but it was suggested that we draw more fire cells onto a new piece of paper and tape them to the plastic sheath on top of the board instead. At times there was confusion about whether the red pebbles could be moved so drawing more fire cells or explicitly telling the players to not move the red pebbles could prevent this confusion. At the school math event, some players were able to recognize that the second column in the third sublevel only has one available cell in which a rook must be placed. This type of logical observation is exactly what we would like the players to make.

It took longer for all players to find a solution for the Monkeys in the Jungle, as intended. Many players would find a solution that satisfied two dimensions, without checking the third dimension. With a couple verbal tips, they eventually found a solution. When players had an opportunity to find a second arrangement it took far less time, which was encouraging. Players liked that the monkeys were attached to the pipe cleaners because they were easy to move, but it was necessary to re-glue the monkeys to the pipe cleaners a few times. Laminating the monkeys would be beneficial to prevent accidental ripping. Some players said the bushes were nice because they did not block the monkeys too much, but they could have been a little bigger. It seemed that there was a disconnect between the two dimensional boards and the three dimensional board. Perhaps a smaller three dimensional board before the Monkeys in the Jungle level, or modifying the instructions to make the goal more clear would help.

Overall everyone enjoyed the game and said it was really fun. They were all excited about moving to the next level. Most players appreciated the order of the games and said the three dimensional level would be too hard to play right away. (Contrarily, one boy at school math event started with the Monkeys in the Jungle level first, then came back later to play the other levels.) After we had played all the games we asked some players if they realized the games were related to math. They said they had no idea and one player said she was not good at math. We explained that the math application finds out how many ways the objects can be placed on the board. A couple players said they didn't realize that there was only one way to put the objects on the River Bank board, so they suggested revealing how many ways there were to place the pieces before they started. One way we checked if the player understood the game was if they could self-assess and catch their mistakes. This did not happen often though. This was especially an issue in the Monkeys in the Jungle level. We found while some players were trying to find a solution they would move the rooks into a correct arrangement and without recognizing that solution, keep moving the rooks into an incorrect position. Perhaps if everyone had played with a partner, that partner could have pointed out any violating rows or columns.

Adjustments

As my testers played the games there were a few things that we noticed could be fixed. Since players had an easier time with the Scenic Picnic Table level, perhaps that should be first and Windmills on the River should be second. Players seemed to be having difficulty generalizing the strategies of the two dimensional boards to the three dimensional board. Perhaps a $2 \times 2 \times 2$ board could be added to simplify the transition. Originally we did not make a board of this size because we thought it would be too simple since the only solutions would be on the diagonals. However, maybe a simple three dimensional board is what the players need to make the connections.

As one girl played she took her hand and physically blocked the row and column that a rook was in already. It would have been helpful to have a dry erase marker that they could write with on the plastic cover of the boards to better visualize and remember which cells were already blocked. Since a few players struggled to remember which arrangements they had already made in the Scenic Picnic table game, it would probably be helpful to provide a small blank version of the board that they could mark off their previous arrangements on. This would only be relevant for the levels in which the players are asked to find more than one arrangement. However, if a teacher wanted to extend this to an entire class s/he could print off the simplified boards located in the instructions above and have players simply draw a dot or x in a cell that a rook/game piece would occupy.

There are various adjustments that could be made to make this game adaptable for different age levels. As previously mentioned to make the game more challenging players could work alone instead of in pairs. For Fire in the Field and Monkeys in the Jungle, the player could find multiple arrangements instead of just one. The player could also be asked the minimum amount of cells that when blocked would not allow the maximum amount of rooks to be placed. For instance in Fire in the Field sublevel 2, blocking the two cells in the bottom row would inhibit five rooks being placed so its minimum amount of additionally blocked cells would be two. It could also be asked where else could those two extra blocks be placed so that they are not in the same row or column and still inhibit the maximum number of rooks being placed. These are interesting questions because they probe more at the player's logic and give them a chance to modify the boards.

If a player is struggling, there are a number of helpful hints that could be given. The instructor could give verbal hints by telling the players if they are close to a solution or whether a certain rook needs to be moved. For example, several players would place a rook on (5,4) in the third sublevel of the Fire in the Field game and not recognize that it needed to be moved. After a while we would take off the rooks that were in the wrong spot. For Monkeys in the Jungle we would turn the board so that the players could see which direction violated the rules.

To get a player started on an arrangement, the instructor could place a rook in a helpful position. For Windmills on the River (1,1) is an easy starting place. Any initial placement in Scenic Picnic Table is equally advantageous as the next, so the instructor can place a second rook based on a rook that a player has already placed. In order to find all four it would be helpful if the instructor kept track of which cells they have already included in an arrangement and try to point them in a new direction.

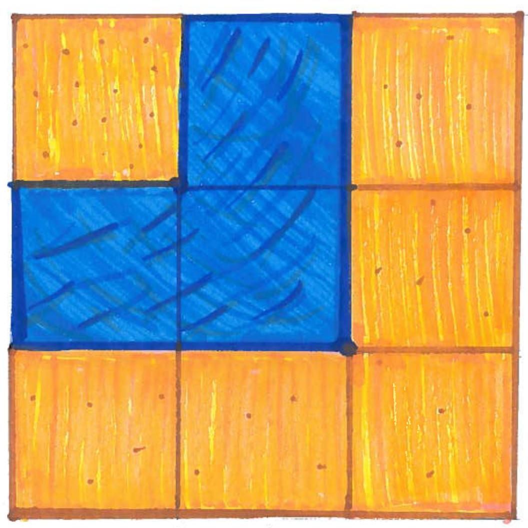
For the first sublevel of Fire in the Field, there are many good starting points. The cells (1,4) and (2,5) could lead to a player seeing a cut diagonal arrangement that includes (3,1), (4,2), (5,3). The second sublevel could start in the (2,3) position for the arrangement including (1,5), (3,4), (4,1), (5,2). This is just one suggestion of many for the first two sublevels. It is advised that the instructor try to direct a player based on what they already have. A more difficult hint for the third sublevel would be (2,1). An easier hint would be (3,3). The instructor should keep in mind that ideally the player would be finding a different arrangement for each sublevel. For Monkeys in the Jungle, two arrangements are possible from starting on the bottom layer in (1,2,1), so that might be a good place for players to start. Encouraging the players to look in every direction is essential for completion of this level.

This game is designed to have players casually engage in mathematics while practicing logical thinking. The instructor is encouraged to ask players questions as they are working to get the players to express their thought processes. The question can be as simple as "why did you place a game piece there?" Asking players why a certain arrangement did not work could also be helpful to get at the logic of placing one rook in each column and row. In Windmills on the River or the third sublevel of Fire in the Field there are columns and rows where there is only one available cell. Asking a player "If you were placing your first rook and you wanted to be sure it was in the final arrangement, where would you place it?" might get them to think about the concept that the only available cell in a row or column must be included in the arrangement. We would encourage instructors to have players self assess whether a certain arrangement is correct or not, instead of always pointing out what the violation is. We believe self assessing would be beneficial to see if the players are actually understanding the rules or if they are just putting the rooks on the board randomly.

Appendix

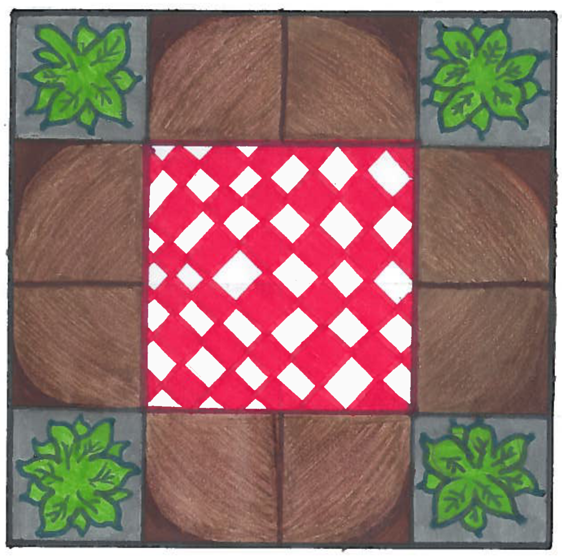
Rooks: Three yellow game pieces from the board game Sorry™

Story: Here is a field with a river (blue cells) going through it. We have three windmills to place on the land. Their wind flows left and right and up and down in straight lines. This means they cannot be in the same row or column, otherwise they would blow each other over. Where should they be put? Can they all be on the same side of the river? Why or why not?



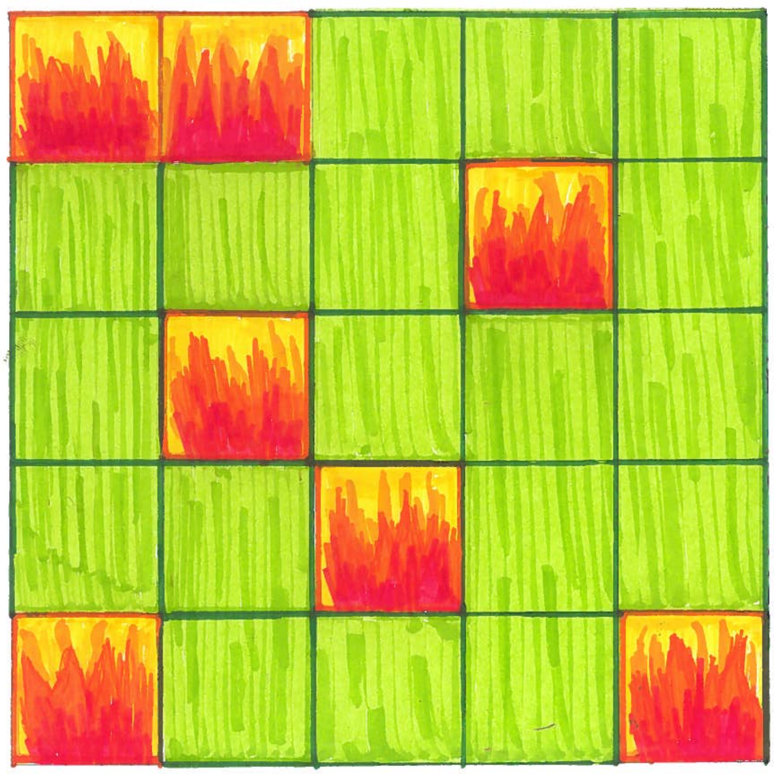
Rooks: Four Clue™ game pieces which are small people.

Story: Four people are about to sit around a table (the red cells) to enjoy dinner. There are four shrubs on the corners (the green cells). They want to enjoy the view, so they don't want to sit across from each other. They also need room to eat so only one person should be on each side of the table. Find all the ways that they can sit around the table.



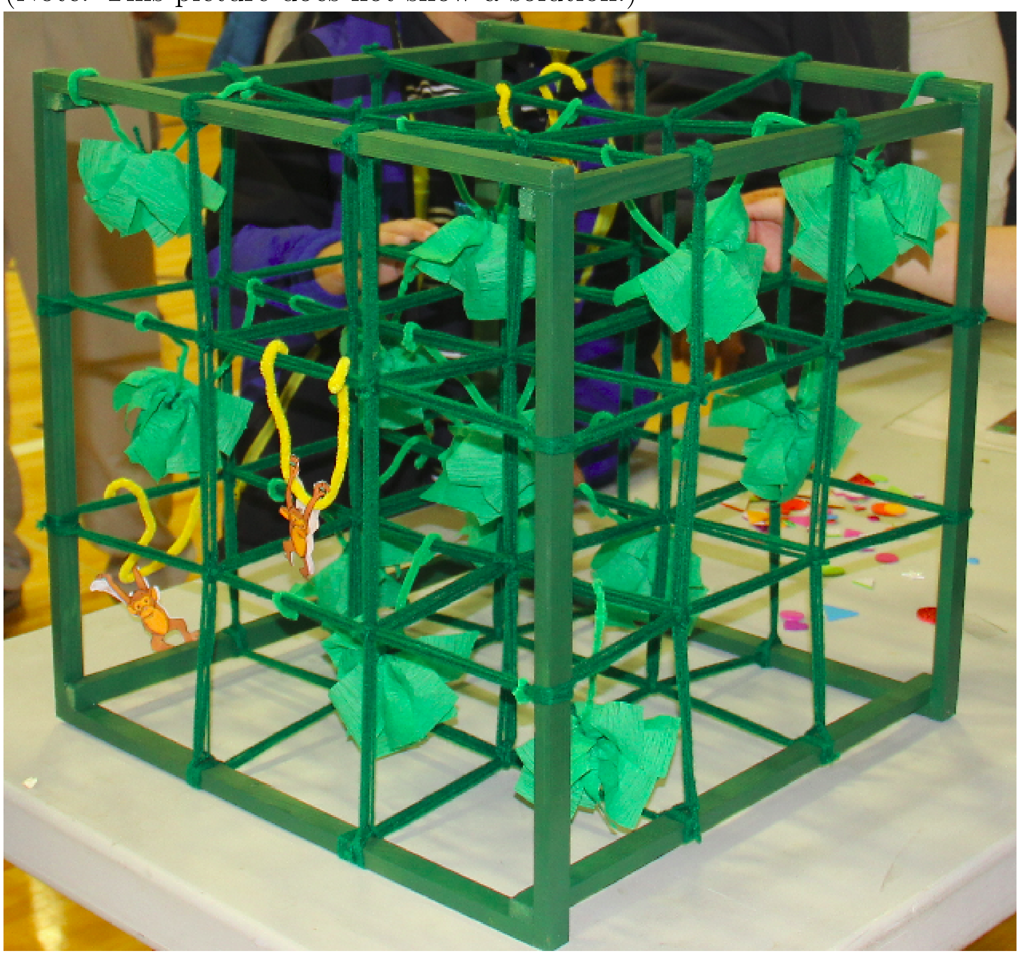
Rooks: Five white pebbles

Story: This field is on fire! Firemen need to stand in the field to put out the fire, but you need to decide where they go. They don't want to be in the same row or column because they don't want to spray each other with their hoses. Can you find a way to place them so they can put the fire out? For second and third sublevels: The fire has spread! Where should the firemen go now?



Rooks: Three monkeys. Bushes were used to restrict cells.

Story: There are three monkeys in the jungle that love to swing around. They need space to play so no two monkeys can be in the same direction. Can you find a way to place the monkeys in the open spaces so that they have enough room to swing around?
(Note: This picture does not show a solution.)



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