PHYSICAL REVIEW B **95**, 155118 (2017)

Topological phases from higher gauge symmetry in 3 + 1 dimensions

Alex Bullivant,[1] Marcos Calçada,[1,2] Zoltán Kádár,[1] Paul Martin,[1] and João Faria Martins[1]

[1]*School of Mathematics, University of Leeds, Leeds, LS2 9JT, United Kingdom*

[2]*Departamento de Matemática e Estatística, Universidade Estadual de Ponta Grossa, Ponta Grossa, PR, Brazil*

(Received 26 September 2016; revised manuscript received 13 February 2017; published 13 April 2017)

We propose an exactly solvable Hamiltonian for topological phases in 3 + 1 dimensions utilizing ideas from higher lattice gauge theory, where the gauge symmetry is given by a finite 2-group. We explicitly show that the model is a Hamiltonian realization of Yetter's homotopy 2-type topological quantum field theory whereby the ground-state projector of the model defined on the manifold M^3 is given by the partition function of the underlying topological quantum field theory for $M^3 \times [0,1]$. We show that this result holds in any dimension and illustrate it by computing the ground state degeneracy for a selection of spatial manifolds and 2-groups. As an application we show that a subset of our model is dual to a class of Abelian Walker-Wang models describing 3 + 1 dimensional topological insulators.

DOI: 10.1103/PhysRevB.95.155118

I. INTRODUCTION

Topological phases of matter have received considerable interest recently due to their practical applications related to various quantum Hall effect phenomena [1–4] and for the realization of topological quantum computation [5,6]. Topological phases of matter have an underlying effective (infrared limit) description given by a topological quantum field theory (TQFT) [4,7–9]. Such theories are independent of the metric structure of space time, so low-energy physical processes are insensitive to local perturbations. Amplitudes of these physical processes are global quantities, topological invariants of the configuration space. In a canonical approach, the Hamiltonian is a sum of mutually commuting constraints, so the ground-state space is their joint eigenspace [5,10,11]. The degeneracy of the ground state and types, fusion, and braiding of possibly exotic excitations fully characterize such a theory [12,13]. An important property of TQFTs is negative corrections to the experimentally observable entanglement entropy (due to global constraints on the correlations) [8,14–16].

In 2 + 1D, all phases of quantum matter with topological order are described by Chern-Simons-Witten/BF theories [17], (twisted) quantum double (QD) models [5,18], and Levin Wen string nets [10]. In 3 + 1D there are few known examples of TQFTs. As such there is limited knowledge of the kinds of observables and quasiexcitations which could be expected to exist in 3 + 1D topological phases of matter. So far in 3 + 1D 1 only the Dijkgraaf-Witten topological gauge theory [19,20] which provides a dual description of symmetry protected topological (SPT) phases when the symmetry is gauged [21], and the Crane-Yetter TQFT [22–24], which describe topological insulators and gauge theories when the input is given by a premodular category, have been studied in the physical literature. Both TQFTs give observables which depend on at most the fundamental group and the signature of the space time. They both support quasiexcitations given by point particles with charge like quantum numbers and fermionic/bosonic mutual statistics and loop excitations which carry both charge and flux like quantum numbers [20,24]. In the search for self correcting quantum error codes, novel Hamiltonian models have been developed in 3 + 1D which exhibit properties of topological phases such as a topologically dependent ground-state degeneracy but do not admit a TQFT description [25]. Models such as Haah's code [25] are given by a lattice gauge theory description [26,27] but are not fixed under renormalization and instead bifurcate into possibly different phases [28].

In this paper we will describe a Hamiltonian formalism for a third type of TQFT, the Yetter homotopy 2-type TQFT [29,30]. Yetter's TQFT uses a *2-group* (equivalently a *crossed module*) to define a TQFT utilizing the ideas of topological higher lattice gauge theory. Unlike the previous theories, such a theory is also sensitive to the homotopy 2-type information of the space time, e.g., the second homotopy group. This feature is likely to be necessary to find nontrivial (meaning neither bosonic nor fermionic) representations of the *loop braid* group[1] [31–33], since the evolution of loop excitations are world sheets.

The aim of this work is to better understand candidate theories for topological phases of matter in 3 + 1D space time. Many questions understood in the 2 + 1D case are still without a full resolution in 3 + 1D. It is known that local[2] (fully extended) n + 1D TQFTs can be defined by appropriate weak n categories [34]. In 2 + 1D this description is captured by fusion categories [35]. In 3 + 1D there is still no suitably general description of the analogous 3-categories, as such the most general formulation of possible 3 + 1D topological phases remains unknown. For known models there are further open questions. For example, what is the relation between the ground-state degeneracy and the number of quasiparticle excitations? In 2 + 1D it is known that there exists a 1-1 correspondence between the ground-state degeneracy on the torus and the number of irreducible quasiparticle excitations, but the analog in 3 + 1D with the ground-state degeneracy on the 3-torus and the number of irreducible excitations is known not to hold [20,36,37]. Another related question is what are the topological quantum numbers needed to classify the ground states and quasiexcitations in 3 + 1D. Such quantum numbers are expected to be related to the generators of the mapping

---

[1] This group is the 3 + 1D analog of the braid group, which governs particle statistics in 2 + 1D.

[2] Here local is the physical requirement that the total space-time partition function can be evaluated in submanifolds of space time and be appropriately "glued" to evaluate the total partition function [74].

class group of the 3-torus $SL(3,\mathbb{Z})$, but a full proof is still lacking. In this paper we will outline a model of topological phases of matter in 3 + 1D with a 2-group gauge symmetry. (In an accompanying paper we explicitly verify the mathematical consistency of such models and also further discuss the topological observables and quantum numbers of the model.)

Defining Hamiltonian formalisms for TQFTs in 2 + 1D and 3 + 1D has been an effective strategy for understanding the spectrum of observables and physical manifestations of topological phases of matter [5,10,18,20,23]. This is the approach taken in this paper, where we will present a Hamiltonian formalism for the Yetter TQFT in 3 + 1D. The structure of the text is as follows. We introduce the basic components of higher lattice gauge theory (HLGT) in Sec. II, including the definition of a crossed module (Sec. II A). This serves as a framework for the remaining structure. We will then outline the Hamiltonian model in Sec. III. Section IV explains why our model is a Hamiltonian formalism of the 4D Yetter TQFT. Finally we describe an inclusion of a class of our Hamiltonian models into the class of Walker-Wang models [23,24] (which form a Hamiltonian presentation of the Crane-Yetter TQFT) in Sec. V.

II. HIGHER LATTICE GAUGE THEORY

In this section we will establish the lattice formulation of higher gauge theories. These are more complicated than ordinary lattice gauge theory. Instead of a group (the gauge group) on lattice edges, here we need two groups, the group of holonomies of ordinary and higher gauge fields. The latter types of holonomies sit on plaquettes and can be thought to arise from surface integral of a non-Abelian 2-connection [38]. Beside the two groups, the physical edge/plaquette geometry induces two maps between them, which satisfy certain compatibility conditions. This collection of data is called a crossed module (crossed modules are equivalent to 2-groups [39]) and replaces the notion of the gauge group in ordinary gauge theories. Just as the structure of the gauge group ensures that gauge-invariant and measurable quantities are independent of the choices made when defining the holonomy and the way holonomies are composed, it is the structure encoded in a crossed module which takes care of the same independence in a higher gauge theory.

For the proof of independence, some algebraic topology is needed. The proof will be published in a companion paper [40]. Here we lean instead on existing work [41] and only sketch the internal consistency of the theory we derive our Hamiltonian model from.

A. Crossed modules

Let G and E be groups, $\partial : E \to G$ a group homomorphism and $\triangleright$ an action of G on E by automorphisms (i.e., the maps $G \times E \to E,(g,e) \mapsto g\triangleright e$ are homomorphisms for both variables). If the Peiffer conditions

$$\partial(g\triangleright e) = g\partial(e)g^{-1} \quad \forall g \in G, \forall e \in E, \tag{1}$$

$$\partial(e)\triangleright f = efe^{-1} \quad \forall e,f \in E \tag{2}$$

are satisfied then the tuple $(G,E,\partial,\triangleright)$ is called a *crossed module*.

An example is $(G,G,\mathbf{id},\triangleright)$ with $g \triangleright h = ghg^{-1}$, the double $\mathcal{D}G$ of the group G. Another example is $(G,\mathrm{AUT}(G),\mathrm{ad},\triangleright)$, where $\mathrm{AUT}(G)$ is the automorphism group of G. Here ad sends a $g \in G$ to conjugation by g, and the action is simply by evaluation of $\mathrm{AUT}(G)$. If V is a representation of a group G then we can build a crossed module $(G,V,\partial,\triangleright)$ where V is a group as a vector space, $\partial(V) = \{\mathbf{1}_G\}$ and where $\triangleright$ is the given action of G on V. Now, we need a lattice encoding the physical space of the theory.

B. Lattices and lattice paths

Given a manifold x we write $bd(x)$ for the boundary and (x) for $x \setminus bd(x)$. Given a set K of subsets of a set we write $|K|_u$ for the union. Given a d-manifold M, a *lattice* L for M is a set of subsets L^i, for each $i = 0,1,2,3$, where $x \in L^i$ is a closed topological i-disk embedded in M, satisfying the following requirements, with $M^i := |\cup_{j=0}^{i} L^j|_u$.

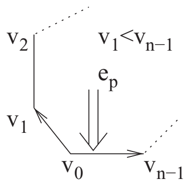
For $i = 1,2,3$ and for $x,y \in L^i$ we have $bd(x) \subset M^{i-1}$; $(x) \cap y = \emptyset$; $(x) \cap M^{i-1} = \emptyset$.

Finally, either $d \leqslant 3$ and $M^3 = M$ or an extension of L exists so that $M^d = M$, with all additional cells in L^4 or above.

(1) An element in L^0 is a point of M, called a *vertex*.
(2) An element in L^1 is called an edge or a *track*.
(3) An element in L^2 is called a face or a *plaquette*.
(4) An element in L^3 is called a *blob*.

A lattice for M is essentially the same as a regular CW-complex decomposition for M (a CW complex is said to be regular if each attaching map is an embedding [42]). Examples are triangulations and cubulations [43].

To describe field configurations succinctly, we need to give extra structure to the lattice. Let us fix a total order on L^0 denoted $<$. We give reference orientation to each element of L^1 such that the source vertex is smaller than the target vertex. (Note that the lattice does not contain 1-gons.) For every element of $p \in L^2$ we distinguish the smallest vertex $v_0(p)$ and fix an orientation for p according to which, for p with $n > 2$ boundary edges, $v_1(p) < v_{n-1}(p)$ for the two neighbors of $v_0(p)$. The default *target* t_p of p is the edge whose source is $v_0(p)$, target is $v_{n-1}(p)$, the default *source* s_p of p is the path with consecutive boundary vertices $v_0(p),v_1(p),v_2(p),\ldots,v_{n-1}(p)$. In figures, if the target path of a face p is an edge then we indicate the target edge by a double arrow, thus the default case is:



For simplicity here we exclude lattices with 2-gons in L^2. Let us call the lattice with chosen total order the *dressed lattice*.

A simple path from vertex v to vertex v' in L is a path in the 1-skeleton M^1 without repeated vertices. Thus a simple path is a 1-disk in M^1 with its boundary decomposed into an ordered pair of vertices (the source and target). Similarly a 2-path is a disk surface P in M^2 with $bd(P)$ decomposed into an ordered pair of simple paths.

Remark. By the time we introduce the Hamiltonian (Sec. III) we will restrict to the subclass of lattices that are triangulations. Before that we will also consider an intermediate restriction on lattices which makes for simple notation. Both of these steps are in the interests of balancing technical utility with clear exposition, rather than matters of necessity.

C. Gauge fields

Given a crossed module $(G,E,\partial,\triangleright)$ and a dressed lattice L, a gauge field configuration is an analog of a conventional one, which is encoded by a map $L^1 \to G$ assigning an element of gauge group G to each edge of L. Beside 1-holonomies associated to directed paths, 2-holonomies are associated to surfaces between two paths with common source and target. Akin to 1-holonomies which describe parallel transport of point particles, 2-homolomies describe the parallel transport of extended 1D objects, with fixed end points over a surface. This is shown diagramatically below where $\gamma,\gamma' \in G$ are two directed paths in L^1 between points v_1 and v_2 and $\Gamma \in E$ is the 2-holonomy associated with the parallel transport of the path γ to the path γ' along a surface in L^2.

$$\begin{tikzcd} v_0 \arrow[r, bend left=50, "\gamma"{name=U}] \arrow[r, bend right=50, "\gamma'"'{name=D}] & v_1 \arrow[Rightarrow, from=U, to=D, "\Gamma"] \end{tikzcd} \tag{3}$$

The 2-holonomy is encoded by functions $L^1 \to G, i \mapsto g_i$ and $L^2 \to E, p \mapsto e_p$. More precisely, we associate an element of G to each oriented edge and an element of E to each face with reference source and target. We call them the 1- and 2-holonomy with given source and target, respectively. Let us assume that a specific oriented edge i is determined by its vertices v,w (with $v < w$). Then we may write $i = vw$, and $g_i = g_{vw}$. We denote a complete 'coloring' of the dressed lattice with such reference-oriented data by L_c and write $\mathcal{G}^L = G^{|L^1|} \times E^{|L^2|}$ for the full set of colorings.

Given $i = vw \in L^1$, the 1-holonomy g_{wv} with $w > v$ is given by $g_{wv} = g_{vw}^{-1}$. The 1-holonomy along a path in the 1-skeleton of the lattice is given by the multiplication of the group elements of subpaths (and hence eventually of g_is, or their inverses, depending on the direction of the edge with respect to that of the path) along the path. Note that 1-holonomy is well defined by (existence of inverses and) associativity of G. Similarly, we can compose 2-holonomies of disk surfaces, where the target of one coincides with the source of the next. By (nonobvious) analogy with 1-holonomy, 2-path 2-holonomy is well defined by the crossed module axioms. Below we explain the analog of inverses and (more briefly) associativity.

First we will establish some notation. In figures, 2-holonomies of faces and other disk-surfaces will be depicted by double arrows, which point toward the reference target edge, just as for the 2-paths themselves. For a face p we will call the 1-holonomy g_{s_p} of the source s_p the source of the 2-holonomy e_p. For example in the next figure we say that the 2-holonomy associated to the triangle has source g_1g_2 and target g_3.

$$\begin{tikzcd} & v_1 \arrow[dr, "g_2"] \arrow[d, Rightarrow, "e"] & & & & \\ v_0 \arrow[ur, "g_1"] \arrow[rr, "g_3"'] & {} & v_2 & = & v_0 \arrow[r, bend left=50, "g_1g_2"{name=U}] \arrow[r, bend right=50, "g_3"'{name=D}] & v_2 \arrow[Rightarrow, from=U, to=D, "e"] \end{tikzcd} \tag{4}$$

Note, that our conventions for composing 1-holonomies throughout the paper is g_1g_2 for consecutive edges 1 and 2. The right hand side of equation (4) demonstrates how each triangle can be viewed as a bigon in the graphical calculus for ease of presentation.

In the following we will adopt a restriction on the lattice: We assume that there can be at most one edge between two vertices and that a face is determined by its boundary vertex set. This is not strictly necessary, but the notation becomes simpler, we will use vw for the unique edge oriented from vertex v to w and vwu for the unique triangle with distinct boundary vertices v,w,u.

The terminology of source and target above comes from the axioms of 2-categories, which is the language used in Pfeiffer [41], for example, to define higher lattice gauge theories. In this paper, we will not define 2-categories but simply write down rules from that formalism which we can use to define our model.

We are now ready to compute the 2-holonomy of an arbitrary 2-path. As already noted, it is intrinsic to the notion of a gauge field that changing the direction of an edge is equivalent to changing the associated group element to its inverse. One can compute the 1-holonomy along a path in the 1-skeleton of the lattice by using these transformations to ensure that the target of the 1-holonomy of an edge in the path coincides with the source of the next.

The 2-holonomy of a 2-path is constructed from the reference 2-holonomies of its plaquettes using a set of rules relating the reference 2-holonomy of each $p \in L^2$ with the 2-holonomy at p with different source and target. This way one multiplies all 2-holonomies of the elements of L^2 that are parts of the surface transformed appropriately so that the target of one is the source of the next.

The fact that the procedure is consistent and independent of the choices made is explained in a companion paper (theorem 5 and lemma 10) [40] (using the language of crossed modules of groupoids equivalent to that of 2-groupoids). Here we only illustrate the composition rules in indicative cases.

For each face $p \in L^2$ we define a 1-holonomy operator

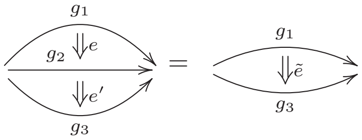
$$H_1(p) \equiv \partial e_p \, g_{s_p} \, g_{t_p}^{-1}. \tag{5}$$

It is also called the *fake curvature* and it corresponds to the curvature 1-form [44] of higher gauge theory.[3] In what follows, we will only consider configurations (unless otherwise stated) where $H_1(p) = \mathbf{1} \in G$ for $p \in L^2$. This is needed for consistency of the lattice formulation of 2d holonomy.

---

[3] In the differential formulation of higher gauge theory, the equations of motion analogous to the vanishing of the field strength in ordinary gauge theory has an additive contribution of the derivative of the map ∂. So whenever the latter is nontrivial the equation $H_1(p) = \mathbf{1}$ is not equivalent to flatness of the 1-connection, hence the adjective "fake."

Let us write down the multiplication convention and the rules of changing source and target of 2-holonomies. We can multiply (compose) the 2-holonomy e with e' if $t_e = s_{e'} (= g_2$ in the figure):



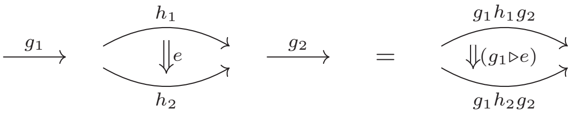
(6)

We use the convention to multiply the group elements from right to left $\tilde{e} = e' \cdot e$. The 'whiskering' rules [41] for changing the source and target of a 2-holonomy are as follows:

(1) We can switch source and target of the 2-holonomy by changing e to e^{-1}.

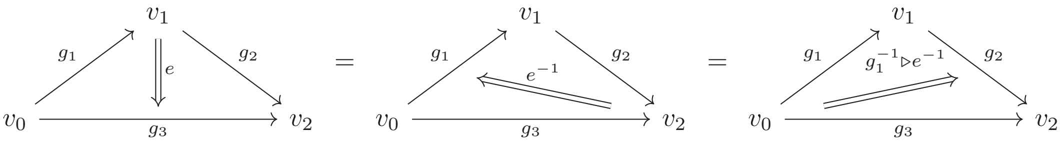
(2) We can change the direction of the source $g_s \in G$ and target $g_t \in G$ of the 2-holonomy simultaneously by changing e to $g_s^{-1} \triangleright e^{-1}$.

(1) We can change the source of both s_e and t_e simultaneously and also the target of both as shown in the figure.



(7)

Here is an example of changing the source s and target t from the reference ones $s = g_1g_2, t = g_3$ of the 2-holonomy e associated to the triangle depicted in the first figure. In the second figure $s = g_3g_2^{-1}, t = g_1$; in the third $s = g_1^{-1}g_3, t = g_2$. The transformation of the value of the 2-holonomy written on the double arrow is computed using the above described rules.

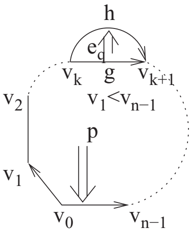


(8)

Note that the *fake flatness* $H_1(p) = \mathbf{1} \in G$ [the lhs is defined by (5)] of the face p is a crucial condition for consistency of the above: Changing the *basepoint* (the vertex corresponding to the source of s_e and t_e) of the 2-holonomy of the face p back to the beginning gives

$$g_n g_{n-1}^{-1} g_{n-2}^{-1} \dots g_1^{-1} \triangleright e_p = \partial e_p \triangleright e_p = e_p e_p e_p^{-1} = e_p$$

by the second Peiffer condition of crossed modules (2). The next example illustrates the composition of 2-holonomies associated with faces with a common boundary edge, where the condition of matching first target and second source is not satisfied.



Using the whiskering rules above, we can change the target and source of the top 2-holonomy e_q to $t(e'_q) = g_{v_0\,v_k} g g_{v_{k+1}\,v_{n-1}}$ and $s(e'_q) = g_{v_0\,v_k} h g_{v_k\,v_{n-1}}$, respectively, by changing e_q to $e'_q \equiv g_{v_0\,v_k} \triangleright e_q^{-1}$, so the holonomy of the big disk $r = p \cup q$ with $s(r) = g_{v_0\,v_k} h g_{v_k\,v_{n-1}}$ and $t(r) = g_{v_0 v_{n-1}}$ is

$$e_r = e_p\big(g_{v_0\,v_k} \triangleright e_q^{-1}\big). \tag{9}$$

Note, that g_{vw} for a nonadjacent vertex pair (v,w) stands for the 1-holonomy along the boundary of the face according to its circular orientation.

We could use the whiskering rule to the direction opposite to the orientation of p, which results in

$$\big(g_{\overline{v_0\,v_k}^o} \triangleright e_q^{-1}\big)e_p, \tag{10}$$

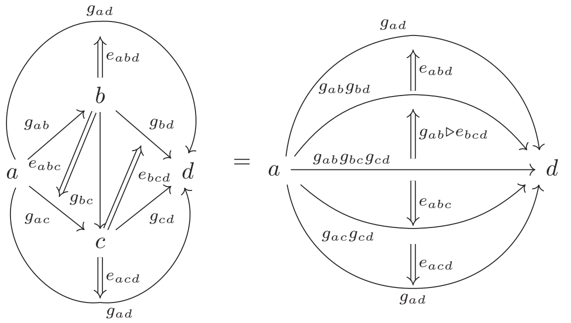
where $g_{\overline{vw}^o}$ denotes the 1-holonomy from v to w along the boundary of p in the direction opposite to its circular orientation ($g_{\overline{v_0\,v_k}^o} = g_{v_0\,v_{n-1}} g_{v_{n-1}\,v_{n-2}} \dots g_{v_{k+1} v_k}$). Due to the fake-flatness condition the expressions (9) and (10) agree:

$$\begin{aligned} \big(g_{\overline{v_0\,v_k}^o} \triangleright e_q^{-1}\big)e_p &= \big(\partial e_p\, g_{v_0\,v_k} \triangleright e_q^{-1}\big)\, e_p \\ &= e_p\big(g_{v_0\,v_k} \triangleright e_q^{-1}\big)e_p^{-1} e_p = e_p\big(g_{v_0\,v_k} \triangleright e_q^{-1}\big). \end{aligned} \tag{11}$$

In the first equality we used fake flatness $\partial e_p = g_{\overline{v_0\,v_k}^o} g_{\overline{v_k\,v_o}^o}$, in the second we used (2).

Finally we give the formula for the 2-holonomy operator $H_2(P) : \mathcal{G}^L \to E$ in the case of a reference tetrahedron $P \in L^2$:

$$P \equiv \{[abcd], s(P) = t(P) = ad, a < b < c < d\}$$



(12)

Using the multiplication convention we read the formula off from the figure

$$H_2(P) = e_{acd} e_{abc} \big(g_{ab} \rhd e_{bcd}^{-1} \big) e_{abd}^{-1}. \tag{13}$$

For the explicit calculation see Appendix A. Note that choosing the base point to be the lowest ordered of the constituting vertices (a), and the direction of the 2-holonomy associated with the triangle acd to be that of the 2-holonomy associated with the tetrahedron, makes the latter unambigously defined. This fact follows from the work [40] (as discussed there, it can be considered as a consequence of the coherence theorem for 2-categories).

III. THE HAMILTONIAN MODEL

Recall that given a crossed module $\mathcal{G} = (G, E, \partial, \rhd)$ and a dressed lattice L we have a set $\mathcal{G}^L$ of gauge field configurations or 'colorings of L.' We write $\mathcal{H}^L$ for the 'large' Hilbert space which has basis $\mathcal{G}^L$ as a $\mathbb{C}$-vector space and has the natural delta-function scalar product. We write $|\bigotimes_{i \in L^1} g_i \bigotimes_{p \in L^2} e_p\rangle$ for a coloring regarded as an element of $\mathcal{H}^L$.

We also have the subset $\mathcal{G}^L_{ff}$ of $\mathcal{G}^L$ of fake-flat colorings. The Hilbert space $\mathcal{H}$ is defined using $\mathcal{G}^L_{ff}$ as the basis. These are the colorings satisfying the constraint at every face $p \in L^2$ that the fake curvature (5) vanishes:

$$\begin{aligned} H_1(p) &= \partial e_p \, g_{s_p} g_{t_p}^{-1} \\ &= \partial e_p g_{v_0 v_1} g_{v_1 v_2} g_{v_2 v_3} \cdots g_{v_{n-2} v_{n-1}} g_{v_0 v_{n-1}}^{-1} = \mathbf{1}_G. \end{aligned}$$

For example for a fake-flat coloring of L a 'triangle' we may choose $e_p \in E$, $g_{v_0 v_1}, g_{v_1 v_2} \in G$ arbitrarily, but then $g_{v_0 v_2} = \partial e_p \, g_{v_0 v_1} g_{v_1 v_2}$ is fixed. Thus $\dim(\mathcal{H}) = |G|^2 |E|$ here. The scalar product for $\mathcal{H}$ is the one induced by that of $\mathcal{H}^L$.

In the following we will often use a simplified notation $|L_c\rangle$ for a basis element of the Hilbert space, with L denoting the dressed lattice and c its coloring. If clear from the context, we will also use this simplified notation for various dimensions and Hilbert spaces, for example for the QD models [5], which corresponds to the finite 'group' crossed module with $E = \{\mathbf{1}\}$ and the restriction to $\mathbb{C}(G^{|L_1|})$ forgetting about the trivial factor corresponding to face coloring. There the lattice L (or rather the underlying manifold) is two dimensional.

A. Gauge transformations

We are now going to define operators in End($\mathcal{H}^L$). We will show in Sec. III B that they restrict to End($\mathcal{H}$). We will show how the 1- and 2-holonomy transform under their action, and hence show that they are gauge transformations. We adopt the latter terminology now.

An intuitive way to think about these operators is as follows. A *vertex* or 1-gauge *transformation* at vertex v is the analog of ordinary G gauge transformation: Edge labels change as they do in ordinary gauge theory. There, the 1-holonomies corresponding to boundaries of faces, also called Wilson loops, transform by conjugation; their traces are observables. In higher gauge theory however, the '1-holonomy' is already different: Compare $g_{ab} g_{bc} g_{ca}$ with $\partial(e_{abc}) g_{ab} g_{bc} g_{ca}$.

Since the 2-holonomy does not physically change under 1-gauge transformations, the face labels are invariant except when the vertex v is the basepoint of the face. An *edge transformation* is a "pure" E 2-gauge transformation: It changes the 2-holonomy associated with each face p adjacent to the edge. For a geometric picture of 2-gauge transformation we refer the reader to section 4.3 of Ref. [40].

The transformation properties of 1-holonomies associated to faces, $H_1(p)$, and 2-holonomies associated to blobs, $H_2(P)$, will be discussed in the next subsection. There the reader can also find figures illustrating the effect of the gauge transformations on a reference triangle. The explicit transformation formulas for the 1-holonomy of a triangle face and the 2-holonomy of a tetrahedron are given in Appendix B.

Let us recall first the definition of left and right multiplication operators for a group G

$$\begin{aligned} L^g &: G \to G, \; h \mapsto gh \\ R^g &: G \to G, \; h \mapsto hg^{-1} \end{aligned}$$

linearly extended to the group algebra $\mathbb{C}G$. In the following we will use the notation L_i^g (R_i^g), $i \in L^1, g \in G$ for the linear operator in End($\mathcal{H}^L$), which acts as left (right) multiplication by g on the tensor factor $\mathbb{C}G$ of $\mathcal{H}^L$ corresponding to the edge i and identity on all other factors—i.e., 'locally' at i. Similarly L_p^e (R_p^e) stands for the same type of local operators in End($\mathcal{H}^L$) acting on the tensor factor $\mathbb{C}E$ corresponding to the face p and identity on all other factors. We will also use $g \rhd_p (\cdot)$ for the operator acting on the tensor factor E of $\mathcal{H}^L$ corresponding to the face p as $e_p \mapsto g \rhd e_p$. The gauge transformation associated with vertex v is defined by

$$A_v^g = \prod_{i \in \star(v)} \mathcal{L}_v^g(i) \prod_{p \in \star'(v)} \mathcal{L}_v^g(p),$$

where $\star(v)$ [$\star'(v)$] is the set of reference edges (faces) adjacent to the vertex v, respectively; the terms in the product are defined as follows:

$$\begin{aligned} \mathcal{L}_v^g(i) &= \begin{cases} L_i^g, & \text{if } v = s(i); \\ R_i^g, & \text{if } v = t(i). \end{cases} \\ \mathcal{L}_v^g(p) &= g \rhd_p (\cdot), \; \text{if } v = v_0(p) \end{aligned}$$

and both families of operators act as identity in all other cases. Note that all factors in the product of the expression of A_v^g act on different tensor factors and their action depends only on the parameter $g \in G$, so they commute: A_v^g is well defined. The gauge transformation associated with edge i is defined by

$$A_i^e = L_i^{\partial e} \prod_{p \in \star(i)} \mathcal{L}_i^e(p),$$

where $\star(i)$ is the set of reference faces adjacent to the edge i and, for face p an n-gon and $k, k+1 \in \{0, \cdots, n\}$,

$$\mathcal{L}_i^e(p) = \begin{cases} R_p^{g_{v_0 v_k} \rhd e}, & \text{if } i = v_k v_{k+1}; \\ L_p^{g_{\overline{v_0 v_{k+1}}^o} \rhd e}, & \text{if } i = v_{k+1} v_k. \end{cases} ,$$

where $g_{\overline{v_0 v_l}^o}$ was defined at (11). Here $g_{v_0 v_0} \equiv \mathbf{1}_G$, $g_{v_0 v_k} \equiv g_{v_0 v_1} g_{v_1 v_2} \cdots g_{v_{k-1} v_k}$, for $k = 1, \cdots, n-1$, $g_{v_0 v_n} \equiv g_{v_0 v_1} g_{v_1 v_2} \cdots g_{v_{n-1} v_0}$, and we use the convention $i = v_n v_{n-1} = v_0 v_{n-1}$. Furthermore, all vertex labels are relative to the face p, but the notation $v_k(p)$ instead of v_k above would be too complicated. Recall that each face p (n-gon) of the lattice has a smallest vertex and a cyclic orientation $v_0, v_1, v_2, \dots, v_{n-1}$, where v_1 is the smaller of the two neighbors of v_0 in p. An earlier remark is also recalled here: If, in the labeling of the face p for $0 < l < n-2$ we have $v_l > v_{l+1}$ in the previous sequence of group elements, then $g_{v_l v_{l+1}} = g^{-1}_{v_{l+1} v_l}$, with $g_{v_{l+1} v_l}$ being the reference 1-holonomy (often called color in the following) of the corresponding edge.

As noted earlier, the crossed module with $E = \{\mathbf{1}\}$ corresponds to the QD model [5] with group G, (but in dimension $d = 3$ here). The set of gauge transformation is the vertex transformations defined above forgetting the trivial factor corresponding to the edge labels.

The operators defined above satisfy the following relations for any vertices v, v', any edges i, i', any elements $g, h \in G$, and $e, f \in E$:

$$A^g_v A^h_v = A^{gh}_v, \tag{14}$$

$$A^g_v A^h_{v'} = A^h_{v'} A^g_v, \quad \text{if } v \neq v' \tag{15}$$

$$A^e_i A^f_i = A^{ef}_i, \tag{16}$$

$$A^e_i A^f_{i'} = A^f_{i'} A^e_i, \quad \text{if } i \neq i' \tag{17}$$

$$A^g_v A^e_i = A^e_i A^g_v, \quad \text{if } v \neq s(i) \tag{18}$$

$$A^g_v A^e_i = A^{g \rhd e}_i A^g_v, \quad \text{if } v = s(i). \tag{19}$$

The proof of these identities are given in Appendix C.

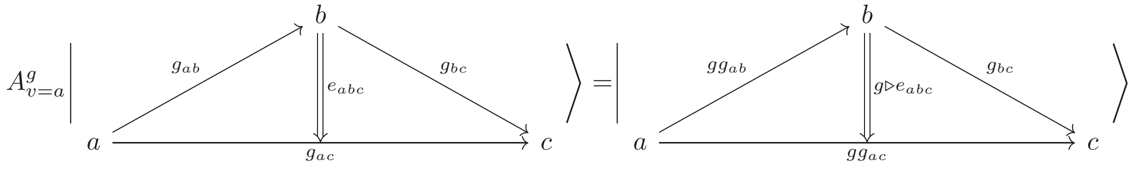
B. Covariance and 2-flatness constraints

We will now define the operators enforcing '2-flatness' of boundaries of blobs $x \in L^3$. At this point we restrict the lattice to be a triangulation. The reason is of technical nature: For triangulations all blobs are tetrahedra and we have an essentially unique expression for their 2-holonomy given by Ref. (13). Even for a cubic lattice, we would need to write a formula with several distinct cases depending on the order of the vertices on the boundary of cubes. From a physical point of view, however, restricting ourselves to triangulations is not severe. (Nevertheless this restriction will be eliminated in our companion paper [40] where we will prove that any lattice can be used.)

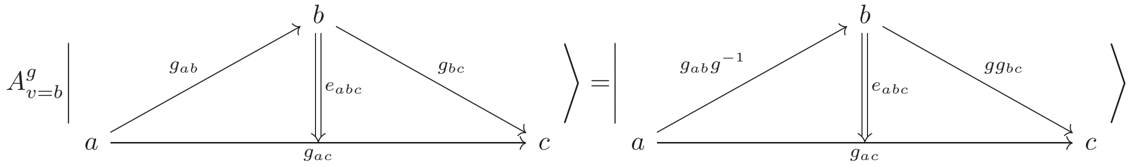
The linear operators enforcing 2-flatness (trivial 2-holonomy) of a tetrahedron $P[= bd(x)$ with $x \in L^3]$ and fake-flatness, regarded as elements of End$(\mathcal{H}^L)$, act on basis elements in $\mathcal{G}^L$ by

$$B_P = \delta_{H_2(P),\mathbf{1}}; \quad B_p = \delta_{H_1(p),\mathbf{1}}. \tag{20}$$

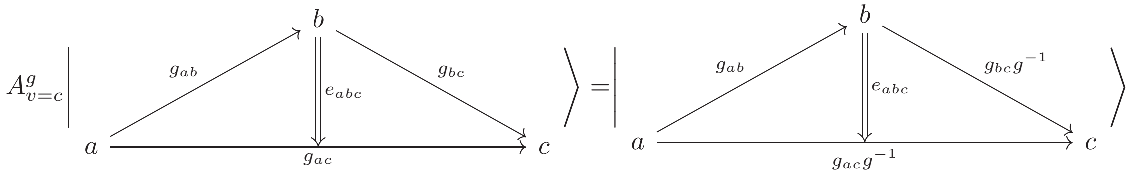
Let us now check how the holonomies $H_1(p) = (\partial e_{abc})\, g_{ab}\, g_{bc}\, g^{-1}_{ac}$ and $H_2(P) = e_{acd}\, e_{abc}\, (g \rhd e^{-1}_{bcd})\, e^{-1}_{abd}$ transform under gauge transformations. The six gauge transformations 'touching' a triangle with vertices $a < b < c$ are depicted below. Caveat: Here and hereafter, we omit to record the changes to the configuration on the rest of the lattice.



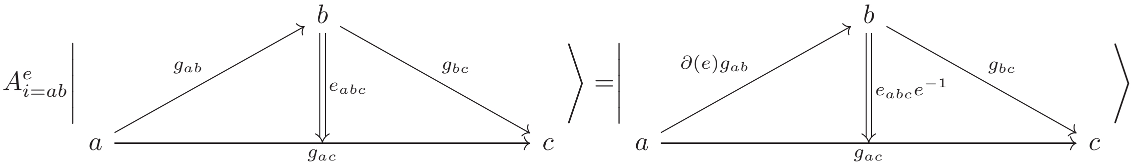
(21)



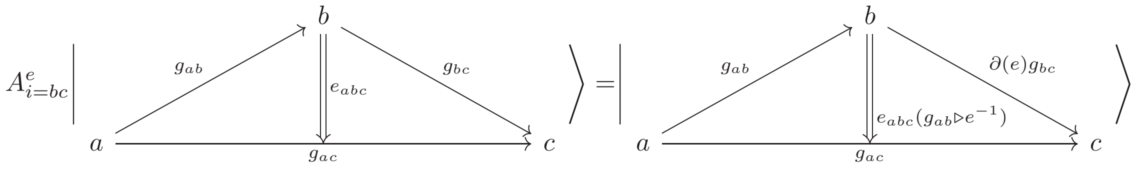
(22)



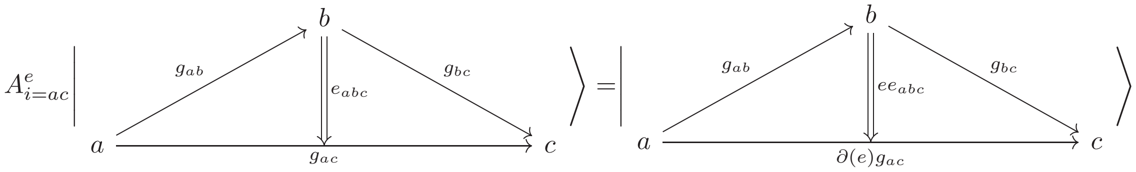
(23)



(24)



(25)



(26)

Now we can compute the transformations of the 1- and 2-holonomy under all possible gauge transformations of the triangle (three vertex gauge transformations and three edge gauge transformations) and the tetrahedron (four vertex gauge transformation and six edge gauge transformations). The formulas are given in Appendix B. An immediate corollary of them is that denoting the transformed quantities by tilde we have

$$H_1(p) = \mathbf{1} \Longleftrightarrow \tilde{H}_1(p) = \mathbf{1}, p \in L^2. \tag{27}$$

Another consequence is

$$\left[B_p, A_v^g\right] = \left[B_p, A_i^e\right] = 0. \tag{28}$$

By (27), operators A_v^g, A_i^e restrict to elements of End($\mathcal{H}$) [preserve $H_1(p) = 1$ with $p \in L^2$]. When restricted we also have

$$H_2(P) = \mathbf{1} \Longleftrightarrow \tilde{H}_2(P) = \mathbf{1}, \quad P = \partial x, x \in L^3. \tag{29}$$

Thus, when restricted to $\mathcal{H}$, they are gauge transformations (transform covariantly).

Hereafter we understand the operators as restricted to $\mathcal{H}$. Consequently the commutation relations

$$\left[B_P, A_v^g\right] = \left[B_P, A_i^e\right] = 0 \tag{30}$$

hold for all possible values of the parameters.

C. The Hamiltonian

Now we define the vertex operators A_v and edge operators A_i as

$$A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g, \quad A_i = \frac{1}{|E|} \sum_{e \in E} A_i^e.$$

Using relation (14) we can verify that A_v is a projector operator, that is $A_v^2 = A_v$:

$$A_v^2 = \frac{1}{|G|^2} \sum_{g,h \in G} A_v^g A_v^h = \frac{1}{|G|^2} \sum_{g,h \in G} A_v^{gh}$$
$$= \frac{1}{|G|^2} \sum_{g',h \in G} A_v^{g'} = \frac{1}{|G|} \sum_{g' \in G} A_v^{g'} = A_v.$$

Similarly, using (16) one obtains $A_i^2 = A_i$. Also, they satisfy the following commutation relations

$$[A_v, A_i] = 0 \tag{31}$$

$$[A_v, A_{v'}] = 0 \tag{32}$$

$$[A_i, A_{i'}] = 0 \tag{33}$$

for any vertices v, v' and edges i, i'. In fact, the relations (32) and (33) follow immediately from relations (15) and (17), respectively. In the same way, (31) follows from (18) if $v \neq s(i)$. If $v = s(i)$, using (19) we obtain

$$\sum_{e \in E, g \in G} A_v^g A_i^e = \sum_{e \in E, g \in G} A_i^{g \triangleright e} A_v^g = \sum_{e' \in E, g \in G} A_i^{e'} A_v^g,$$

where in the last equality we used the fact that the map $g \triangleright (\cdot)$ is a bijection. Then it is clear that (31) holds. Let us now consider the multiplicative operators $B_P = \delta_{H^2(P),\mathbf{1}}$ defined above. It is clear that they mutually commute and they are projections. It is also clear that they commute with the projections A_v and A_i due to (30). Now we can write down a *Hamiltonian of higher lattice gauge theory* in terms of mutually commuting operators in End($\mathcal{H}$)

$$H = -\sum_v A_v - \sum_i A_i - \sum_P B_P, \tag{34}$$

where the summations run over vertices $v \in L^0$, edges $i \in L^1$, and blob boundaries $P = \partial x, x \in L^3$.

Note that one can define a Hamiltonian on $\mathcal{H}^L$ with the same ground-state sector by including the 1-flatness constraint operators:

$$H' = -\sum_v A_v - \sum_i A_i - \sum_P B_P \left(\prod_p B_p \right) - \sum_p B_p, \tag{35}$$

where summation and product over p mean over elements $p \in L^2$. The reason for the modified form of the B_P operators is that without the multiplier that enforces fake flatness, they would not commute with the gauge transformations (as can be seen from the transformation of $H_2(P)$ under A^e_{cd} of the tetrahedron $P = a < b < c < d$), see Appendix B. This way H' is also a sum of mutually commuting projections and the ground states of H and H' agree.

In the restriction to a two-manifold and crossed module $\mathcal{G} = (G,\mathbf{1}_G,\partial,\triangleright)$ we note the model H' reproduces the Kitaev quantum double [5] Hamiltonian for group G. The second term is zero due to the triviality of E and the third term does not enter the equation as there are no blobs in $d = 2$, so we have

$$H = -\sum_v A_v - \sum_p B_p. \tag{36}$$

From this connection, the first term in (34) can be seen naively as the Gauss constraint and the last term as the magnetic constraint. Caveat: Unless fake-flatness constraints are imposed, the 2-holonomy of a blob is ambiguous: The formula (13) depends on the choice of the composition of the 2-holonomies of the boundary faces of the blob [41].

IV. RELATION TO THE 4D YETTER TQFT

A. Ground-state projection as a 4D state sum

In this chapter we will relate our Hamiltonian model in three space dimensions to the Yetter TQFT in four dimensions. In general one can always define a d-dimensional lattice Hamiltonian of mutually commuting projection operators from a $d+1$ dimensional TQFT, see for example Williamson-Wang [45]. For $d = 2$ Kádár *et al.* showed [46] (see also [47]) that the Levin-Wen model [10] is the Hamiltonian version of the Turaev-Viro (TV) TQFT [48,49] in that the ground-state projection of the former defined on the two-dimensional lattice L for M^2 is given by the TV state sum for the $M^2 \times [0,1]$. The rigorous proof was given by Kirillov [50].

A subset of the Levin-Wen Hamiltonian models was shown [51,52] to be the dual lattice description of those corresponding to BF gauge theories: the quantum double models of Kitaev [5]. Here duality means Fourier transformation on the gauge group, as a result of which states are labeled by irreducible representations instead of group elements. Hence, the above statement has to hold in the dual description as well. We will state and prove it in this section and generalizse to the case of our 3D model: Its ground state projection is given by the appropriate 4D Yetter TQFT amplitude.

The general correspondence we will investigate has been known qualitatively. The 2+1 BF-theory action with gauge group $SU(2)$ is equivalent to the Einstein-Hilbert one for Euclidean signature and zero cosmological constant as shown by, e.g., Ooguri and Sasakura [53]. On the other hand, in 1969, Ponzano and Regge derived the gravity action from the asymptotic form of the Wigner-Racah coefficients [54]. These results were the motivation for several works in the 2+1 quantum gravity literature, where the details of the correspondence were well understood for the case of $SU(2)$ [55]: The Hilbert space is the state space of canonical quantum gravity; the TQFT is the corresponding state sum or spin foam model.

We will sketch the derivation of the TQFT state sum from a general BF gauge theory. Then we replace the continuous gauge group with a finite one and work out the correspondence for $d = 2,3$ for lattice and higher lattice gauge theory (LGT, HLGT) and show the pattern, which arises for arbitrary dimension.

B. Ordinary pure lattice gauge theory based on a finite group

Let us first consider the theory defined on an oriented manifold M^D of dimension D by the action with a compact Lie group G and its Lie algebra $\mathfrak{g}$

$$S[B,A] = \int_{M^D} \mathrm{tr}(B \wedge F(A))$$

with $F(A) = dA + [A,A]$ being the ($\mathfrak{g}$-valued) curvature of the connection A, B a locally $\mathfrak{g}$-valued (D-2) form, and tr the Cartan-Killing form of $\mathfrak{g}$. The partition function (a map that associates a scalar to each manifold M^D) is defined formally in terms of the path integral

$$Z_{BF}(M^D) = \int \mathcal{D}B\mathcal{D}A\, e^{iS[B,A]}$$

with $\mathcal{D}A$ and $\mathcal{D}B$ standing for some measure over the space of connections and the B field. To make sense of this formal expression there is a standard discretization procedure, see e.g., Oeckl [56] or Baez [57]. Here we will only sketch the procedure and write down the discrete version of the partition function. Let Δ be a dressed lattice of M^D and $\tilde{\Delta}$ the dual complex (whose $k \leqslant D$ dimensional simplices are in one-to-one correspondence with the $D-k$ simplices of Δ). We dress $\tilde{\Delta}$ similarly to before: we orient edges (which are dual to $(D-1)$ simplices of Δ) and give circular orientation to faces and distinguish basepoints in each face. A gauge configuration is an assignment of a group element to each edge. For a face p we define the exponentiated curvature by $g_p = \prod_{i\in\partial p} g_i$: The multiplication is done in the order along the chosen cyclic orientation starting at $v_0(p)$. The B field, locally being a $D-2$ form, is naturally associated with dual faces. F_p is the curvature variable associated with the dual face p. Now, the integral $\int dB_p e^{iB_pF_p}$ vanishes, unless the curvature vanishes, thus it can be replaced by $\delta_{g_p,\mathbf{1}} = \delta_{g_p,\mathbf{1}}$ in terms of the chosen variables: The vanishing of the curvature in the dual face p is equivalent to the trivial holonomy along the boundary of the face. The expression for the discretized path integral reads

$$Z_{LGT}(M^D,\Delta) = \int \prod_{i\in\Delta^1} dg_i \prod_{p\in\Delta^2} \delta_{g_p,\mathbf{1}}, \tag{37}$$

where Δ^1 (Δ^2) is the set of edges (faces) of $\tilde{\Delta}$ (see Refs. [55–58] for details of the discretization procedure). For compact Lie groups the measure dg_i is the Haar measure on G. The partition function is the sum of all coloring subject to the constraint that the holonomy g_p around each face is trivial: The underlying connection is flat. At this point there is no difference between using Δ or $\tilde{\Delta}$ for edge and face coloring. We will use the former.

Note that we still do not know whether $Z_{LGT} < \infty$ in general, but we are interested here in replacing G with a finite group.[4] For a finite group G, we will choose the measure $\int dg = \sum_{g\in G}$. The partition function needs to be normalized so that it is independent of the lattice used:

$$Z_{LGT}(M^D,\Delta){=}|G|^{-|\Delta^0|}|\{\textit{admissible}\text{ colorings of }\Delta\}|, \tag{38}$$

---

[4] The most studied case of a compact Lie group is $SU(2)$, and finiteness requires gauge fixing [55].

where admissibility means that all faces are flat: $\prod_{p\in\Delta^2}\delta_{g_p,\mathbf{1}} = 1$ and the multiplicative factor ensures independence on the lattice. This formula holds in arbitrary dimension (it is proved in more general settings by Porter [30], Yetter [29], and Faria Martins/Porter [59]).

Generalizing to manifolds with boundary, we can seek to satisfy the (weak) cobordism property of a TQFT (partition functions become partition vectors with composition by dot product—see, e.g., Martin [60, §2.1, §10.2]). For this the above definition has to be generalized. Let the boundary be a closed $d = D - 1$-dimensional manifold with a lattice $\Delta^{(b)} \subset \Delta$. A boundary coloring is denoted by $\Delta_c^{(b)}$ (so c stands for a set $\{g_i \in G, i \in \Delta^{(b)1}\}$). Let us call a coloring of Δ c-admissible if it restricts to $\Delta_c^{(b)}$ and if the flatness constraints for all faces are satisfied: $\prod_{p\in\Delta^2}\delta_{g_p,\mathbf{1}} = 1$. The partition vector is a vector with components indexed by the possible boundary colorings $\Delta_c^{(b)}$, and a component then reads

$$Z_{LGT}\big(M^D,\Delta,\Delta_c^{(b)}\big) = |G|^{-|\Delta^0|+\frac{|\Delta^{(b)0}|}{2}}|\{c\text{-admissible colorings of }\Delta\}|. \quad (39)$$

This way the gluing property of a TQFT holds as follows. Let N^d be a closed submanifold of a manifold M^D, such that N^d separates M^D into two unconnected components. Then write M_1^D and M_2^D for the closures of the two components of $M^D \setminus N^d$. The manifolds M_1^D and M_2^D (with lattices Δ^1,Δ^2, say) have homeomorphic and oppositely oriented boundaries N^d (with lattice $L \subset \Delta_1,\Delta_2$). Then the partition functions satisfy the gluing property of TQFTs. That is, changing the notation as $\int\prod_{i\in L^1} dg_i \to \sum_c$ we can write

$$Z_{LGT}\big(\big(M_1^D \sqcup_{N^d} M_2^D\big),\Delta_1\cup\Delta_2\big) = \sum_c{}' Z_{LGT}\big(M_1^D,\Delta_1,L_c\big)Z_{LGT}\big(M_2^D,\Delta_2,L_c\big), \quad (40)$$

where $M_1 \sqcup_{N^d} M_2$ denotes M^D and the prime in $\sum_c'$ indicates that flatness constraints have to be inserted for all $p \in L^2$. The gluing property states that the partition function of a manifold can be evaluated by evaluating on bounded submanifolds M_1 and M_2 then summing the product of partition functions with boundary labels identified along the intersection N^d.

1. 2 + 1D lattice gauge theory with a finite group

Recall the 2 + 1D Kitaev QD Hamiltonian from (36). The ground state projection reads

$$\mathcal{P}_{gs}^K = \prod_{v\in L^0} A_v \prod_{p\in L^2} B_p.$$

Theorem IV.1. Let M^2 be a 2-manifold with lattice L and let Δ be the 3-dimensional lattice of $M^2 \times [0,1]$ that restricts to $L_0 \simeq L_1 \simeq L$ at the boundaries $M^2 \times \{0\}$ and $M^2 \times \{1\}$. Let the internal edge set be $\{v \times [0,1]\}_{v\in L^0}$. Let L_j^i refer to L_j and L_{jc} the edge colorings of L_j^1. Finally, we consider the QD Hilbert space based on L_j with states $|L_{jc}\rangle$ and identify the Hilbert spaces based on $L_j, j \in 0,1$. Then

$$Z_{LGT}(M^2 \times [0,1],\Delta,L_{0c}\cup L_{1c}) = \langle L_{1c}|\mathcal{P}_{gs}^K|L_{0c}\rangle. \quad (41)$$

The proof is written in Appendix D. Note, that we made a choice for using the edge and face set of Δ for defining the partition function. An alternative approach, more standard in the realm of the Turaev-Viro model [46,57,61], makes use of the edge and face set of the dual lattice $\tilde{\Delta}$. Then, the minimal lattice, which restricts to L^j at the boundary is three translated copies of L connected with vertical edges.[5] That way, no Dirac deltas are needed for boundary faces as those in the middle layer of L. That way, it is not necessary to impose flatness on the boundary faces when defining the partition function for a manifold with boundary. The faces in the middle layer of L enforce flatness for the gauge equivalent boundaries L_{jc}; the proposition is stated identically to the "Fourier dual": the Turaev-Viro partition function on the 3d triangulation matches the Levin-Wen ground state projection on the boundary dual.

C. Higher lattice gauge theory based on a finite crossed module

Fix now a crossed module $(G,E,\partial,\triangleright)$ and a four-manifold M^4. Let us consider the theory given by the $BFCG$ action [44]

$$S[A,B,C,\Sigma] = \int_{M^4}(\mathrm{tr}_{\mathfrak{g}}(B\wedge F_A) + \mathrm{tr}_{\mathfrak{h}}(C\wedge G_\Sigma)),$$

where B is a G-valued 2-form, $F_A = dA + [A,A]$ is the curvature of the connection A, C is an E-valued 1-form, and $G_\Sigma = d\Sigma + A \triangleright \Sigma$ the curvature 3-form of the two-connection Σ corresponding to the gauge group E. We will use the following form of the partition function

$$Z_{BFCG}(M^4) = \int \mathcal{D}A\,\mathcal{D}B\,\mathcal{D}C\,\mathcal{D}G\, e^{iS[A,B,C,G]}$$

whose discretized form defined on the dressed lattice Δ of M^4 is given by

$$Z_{HGT}(M^4) = \int \prod_{i\in L^1} dg_i \prod_{p\in L^2} de_p\,\delta_{H_1(p),\mathbf{1}} \prod_{t\in L^3}\delta_{H_2(t),\mathbf{1}}.$$

For a finite group we can rewrite it analogously to (38) in LGT. Let a coloring be called admissible if all Dirac-delta constraints are satisfied: All faces are fake flat and all blobs are 2-flat. We do the substitutions $\int dg_i \to \sum_{g_i\in G}$ and $\int de_p \to \sum_{e_p\in E}$ and write

$$Z_{HGT}(M^4) = \frac{|E|^{|\Delta^0|-|\Delta^1|}}{|G|^{|\Delta^0|}}|\{\text{admissible colourings of }\Delta\}|,$$

where the multiplicative factor ensures independence on the lattice. This is proved to be the same in arbitrary dimensions [29,30,59]. For manifolds with boundary, we need to modify the above similarly to the LGT case. Let the manifold M^4 with boundary have a lattice decomposition Δ, and let $\Delta^{(b)} \subset \Delta$ denote the boundary lattice. Let $\Delta_c^{(b)}$ be a coloring of $\Delta^{(b)}$: $\{g_i \in G, i \in \Delta^{(b)1}, e_p \in E, p \in \Delta^{(b)2}\}$. We call a coloring of

---

[5] One considers $\tilde{L} \times [0,1]$ and constructs the dual complex of this. It will have a vertex in the middle of each prism connected vertically to the middle points of $\tilde{L}_j$; middle points of neighbor prisms are connected and the duals of $\tilde{L}_j$ are the original graphs L_j at the boundary.

Δ c admissible if it is admissible in Δ and restricts to $\Delta_c^{(b)}$ on $\Delta^{(b)}$. The components of the partition vector read:

$$Z_{HGT}\big(M^4,\Delta,\Delta_c^{(b)}\big) = \frac{|E|^{|\Delta^0|-|\Delta^1|}}{|G|^{|\Delta^0|}} \frac{|G|^{\frac{|\Delta^{(b)0}|}{2}}}{|E|^{\frac{|\Delta^{(b)0}|-|\Delta^{(b)1}|}{2}}} \\ \times |\{c\text{-admissible colourings of } \Delta\}|. \tag{42}$$

This does not depend on the lattice decomposition of M^4 extending a given lattice decomposition of the boundary [29,30].

Theorem IV.2. Let $\Delta_L \equiv \Delta$ be the lattice of $M^3 \times [0,1]$ which restricts to $L_0 \simeq L_1 \simeq L$ at boundaries $M^3 \times \{0\}$ and $M^3 \times \{1\}$ and the internal edge set is $\{v \times [0,1]\}_{v \in L^0}$. Let L_j^i refer to L_j and L_{jc} the colorings of those, where fake flatness is assumed for each face $p \in L_j^2, j = 1,2$. Finally, we consider the Hilbert space $\mathcal{H}$ defined in Sec. III based on L with states $|L_c\rangle$ and identify the Hilbert spaces based on $L_j, j \in 0,1$ with it and consider the projection $\mathcal{P}_{gs}^B$ to the ground state defined by

$$\mathcal{P}_{gs}^B = \prod_{v \in L^0} A_v \prod_{i \in L^1} A_i \prod_{P \in L^3} B_P.$$

Then

$$Z_{HGT}(M^3 \times [0,1], L_{0c}, L_{1c}) = \langle L_{1c}|\mathcal{P}_{gs}^B|L_{0c}\rangle. \tag{43}$$

In words, the ground-state projection of our 3D Hamiltonian model associated with M^3 is given by the Yetter TQFT amplitude on $M^3 \times [0,1]$. The proof is in Appendix E.

D. Hamiltonians corresponding to lattice gauge theories

We will look at the correspondence in arbitrary dimension $d \geqslant 1$ for both ordinary and higher lattice gauge theories. We can observe the following.

(1) The fake-flatness constraints of internal faces and 2-flatness of internal blobs of the $d+1$-dimensional manifold $M^d \times [0,1]$, which we refer to as the *prism lattice*, are equivalent to the bottom and top layer of the prism being connected by gauge transformations. For ordinary gauge theory, the latter is equivalent to the flatness of internal faces.

(2) The 2-flatness (flatness) constraints of internal faces of the boundary lattice are the magnetic operators of the Hamiltonian (in ordinary gauge theory, respectively).

As a consequence, the Hamiltonian in the Hilbert space associated with the d-dimensional lattice L, whose ground-state projections are given by the corresponding $d+1$-dimensional partition function is given by the following table, where the sign −||− means the same formula as on its left for all terms to the right starting from the sign.[6]

<table>
<tr><th>d</th><th>1</th><th>2</th><th>3</th><th></th><th></th></tr>
<tr><td>LGT</td><td>$-\sum_{v\in L^0} A_v$</td><td>$-\sum_{v\in L^0} A_v - \sum_{p\in L^2} B_p$</td><td>−||−</td><td>. . .</td><td></td></tr>
<tr><td>$HLGT$</td><td>$-\sum_{v\in L^0} A_v - \sum_{i\in L^1} A_i$</td><td>$-\sum_{v\in L^0} A_v - \sum_{i\in L^1} A_i$</td><td>$-\sum_{v\in L^0} A_v - \sum_{i\in L^1} A_i - \sum_{P\in L^3} B_P$</td><td>−||−</td><td>. . .</td></tr>
</table>

E. The ground-state degeneracy

We compute here the ground-state degeneracy (GSD) for a few examples. This is given by the trace of the ground-state projection. Using equation (43) and

$$Z_{HGT}(M^d \times S^1) = \sum_{L_{0c}} Z_{HGT}(M^d \times [0,1], L_{0c}, L_{0c}) \tag{44}$$

which follows from equation (40), the GSD can also be computed from the Yetter invariant

$$\begin{aligned}\mathrm{Tr}\big(P_{gs}^B\big) &= Z_{HGT}(M^d \times S^1) \\ &= |G|^{-|L^0|}|E|^{-|L^1|} \\ &\quad \times |\{\text{admissible colouring of } \Delta_L\}|. \end{aligned} \tag{45}$$

Here L is the lattice of M^d, L^i refers to its set of i-dimensional cells as before, and Δ_L is the prism lattice whose top L_1 and bottom L_0 are identified. This also applies to LGT, with obvious modifications.

(1) $d = 1$. Here a minimal lattice of S_1 is the lattice with one edge with its source and target vertex identified. Let us first consider the case of ordinary lattice gauge theory; i.e., E is the trivial group. The GSD is by definition the number of gauge equivalence classes of admissible colorings of S^1. This clearly coincides with the number of conjugacy classes of G. We can also obtain this GSD as $Z_{LGT}(S^1 \times S^1)$, which is $|G|^{-1}$ times the number of colorings of the lattice of the torus; explicitly: $|G|^{-1}\,|\{\phi : \pi_1(T^2) \to G\}| = |\{(g,h) \in G^2 : gh = hg\}$. So in this case, the equality (45) boils down to the well established fact from group theory that the number of conjugacy classes of a finite group equals the order of the group times its commuting fraction, the probability that two elements commute; i.e., $\frac{1}{|G|}\,|\{(g,h) \in G^2 : gh = hg\}| =$ number of conjugacy classes of G.

In the general HGT case and also for S^1, looking at the lhs of (45), the GSD can be expressed as the number of conjugacy classes of $G/\partial(E)$ [noting the image of $\partial(E)$ is a normal subgroup of G]. The rhs of (45) explicitly is:

$$\begin{aligned} &\frac{1}{|G|\,|E|}|\{(g,h,e) \in G \times G \times E : \partial(e) \\ &\quad = [g,h]\}| \\ &\quad = \frac{1}{|G/\partial(E)|}\{(g',h') \in G/\partial(E) \times G/\partial(E) : g'h' = h'g'\} \\ &\quad = \text{number of conjugacy classes of } G/\partial(E) \end{aligned} \tag{46}$$

which follows from the group theoretic fact stated in the above paragraph.

---

[6] The Hamiltonian for $d = 3$ HLGT in the table differs from (34) by an unimportant constant.

(2) $d = 2$. Here, the ordinary gauge theory model is well studied, the dimension of the ground state for the manifold M^2 and gauge group G is $|\{\phi : \pi_1(M^2) \to G\}/\sim|$, where $\sim$ means modulo an overall conjugation $\phi(g) \mapsto h\phi(g)h^{-1}, g \in \pi_1(M^2), h \in G$. That is, the GSD is the number of gauge equivalent classes of flat connections. For $M^2 = T^2$, this is well known to coincide with the number of irreps of the quantum double DG [62]. Computing the GSD from the partition function gives $\frac{1}{|G|}|\{(g_1,g_2,g_3), \in G^3, [g_i,g_j] = \mathbf{1}\}|$. Recall that irreps of DG are given by pairs $(C^A, \pi(Z(C^A)))$. Here C^A is a conjugacy class of G and $\pi(Z(C^A))$ is an irreducible representation of the centralizer subgroup $Z(C^A)$ of a representative of C^A in G. It is straightforward to show the expression and the number of such pairs are in one-to-one correspondence [5,18,19].

For HGT, consider S^2, with a cell decomposition with one single vertex and a unique 2-cells. (This would be a trivial case to consider for ordinary gauge theory.) In this case the GSD can be computed, looking at the lhs of (45) as being the cardinality of the set of orbits of the action of G on $\ker(\partial)$. Computing the GSD from the rhs of (45) yields $\frac{1}{|G|}\sum_{e \in \ker(\partial)} |\{g \in G : g \triangleright e = e\}|$. Elementary tools from group actions tell us that these two coincide.

(3) $d = 3$. Let us consider the T^3 case, using the obvious cell decomposition of the cube, and then identifying sides. Fake flatness of the three distinct faces and 2-flatness of the cube read ($[g_1,g_2] \equiv g_1 g_2 g_1^{-1} g_2^{-1}, g_1, g_2 \in G$):

$$[x,y] = \partial f, \quad [x,z] = \partial e, \quad [y,z] = \partial k,$$
$$f\,(yxy^{-1} \triangleright k^{-1})\,(y \triangleright e)\,(yzy^{-1} \triangleright f^{-1})\,k = e.$$

Let the subset of $G^3 \times E^3$ defined by the joint solutions of the above equations be denoted by S and consider the equivalence relation $\cong$ in S generated by

(with $a \in G$ and $e_x, e_y, e_z \in E$):

$$\begin{aligned}(x,y,z,e,f,k) &\cong (axa^{-1}, aya^{-1}, aza^{-1}, a \triangleright e, a \triangleright f, a \triangleright k),\\ &\cong \left(x, \partial e_y\, y, z, e, (x \triangleright e_y)\, f\, e_y^{-1}, e_y\, k \left(z \triangleright e_y^{-1}\right)\right),\\ &\cong \left(x, y, \partial e_z\, z, (y \triangleright e_z)\, e\, e_z^{-1}, f, (x \triangleright e_z)\, k\, e_z^{-1}\right),\\ &\cong \left(\partial e_x\, x, y, z, e_x\, e \left(z \triangleright e_x^{-1}\right), e_x\, f \left(y \triangleright e_x^{-1}\right) k\right).\end{aligned}$$

The GSD is $|S/\cong|$. For instance consider the crossed module $DG = (G, G, \mathrm{id}, \mathrm{ad})$, (where ad stands for the conjugation action). Here GSD $= 1$, and this is easily computable from the rhs of (45).

Another easily computable example is $(\mathbb{Z}_2, \mathbb{Z}_{2r}, \mathrm{sgn}, \mathrm{id})$, where sgn is the parity ($r \in \mathbb{N}_+$), and id denotes the trivial action. Here GSD $= r^3$. This is easily seen from the rhs of (45): $x,y,z \in G$ are arbitrary, they determine (e,f,k) via the fake-flatness constraints and the 2-flatness of the cube holds by construction. We have three more cubes based on 2- and 3-faces, whose faces are again pairwise identified. The identical argument applies: The new face labels are determined by the fake flatness of the sides, so the four edges labels are arbitrary and all six face labels are determined by the commutators. So we have $|G|^4$ admissible colorings, $|L^0| = 1$ and $|L_1| = 3$. Another way to infer that GSD $= r^3$ is the following. Yetter's state sum $Z_{HGT}(W)$, where W is a closed manifold, depends only on the weak homotopy type of the underlying crossed module [59]. The crossed module $(\mathbb{Z}_2, \mathbb{Z}_{2r}, \mathrm{sgn}, id)$ is weak equivalent to $(\{0\}, \mathbb{Z}_r)$, where we consider the constant map $\partial : \mathbb{Z}_r \to \{0\}$. In general, considering a crossed module of the form $\mathcal{E} = (\mathbf{1}_E, E, \partial, \triangleright)$, where ∂ and $\triangleright$ are trivial maps, we have that $Z_{HGT}(S^1 \times S^1 \times S^1 \times S^1) = |E|^3$. The number of admissible coloring is $|E|^4$ since we can color the 2-cells of the 4-cube with faces identified as we please.

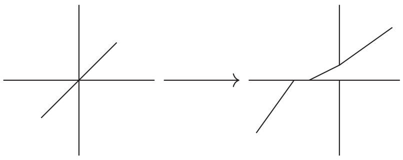


FIG. 1. Resolution of 6-valent vertex to a trivalent vertex.

It can be shown in general that the GSD for ordinary gauge theory can be interpreted as the number of group homomorphisms from the fundamental group of $M^d \times S^1$ to G. In a similar vein it can be shown that for higher gauge theory the GSD can be calculated by counting the number of crossed modules homomorphisms from the fundamental crossed module of $M^d \times S^1$ to the crossed module $\mathcal{G}$ [59].

V. RELATION TO WALKER WANG MODELS

In this section we discuss the relation between the Walker-Wang model [23] and our model. In particular we outline a duality map between our model with the finite crossed module $\mathcal{E} = (\mathbf{1}_E, E, \partial, \triangleright)$, where $\partial : E \to \mathbb{1}_E$ and $\triangleright$ is the identity and the Walker-Wang model based on the symmetric fusion category $\mathcal{M}(\mathcal{E})$, where E is any finite Abelian group.

A. Walker-Wang model

To begin, we briefly outline the Walker-Wang model [23]. The Walker-Wang model is a 3 + 1D model of string-net condensation with ground states proposed to describe time-reversal invariant topological phases of matter in the bulk and chiral anyon theories on the boundary [24]. Such models are believed to be the Hamiltonian realization of the Crane-Yetter-Kauffman TQFT [22] state sum models analogous to the relation between our model and the Yetter's homotopy 2-type TQFT [29].

The Walker-Wang model is specified by two pieces of input data, a unitary braided fusion category (UBFC) $\mathcal{C}$ and a cubulation C of a 3-manifold M^3. In the following we will define the generic model on a trivalent graph Γ defined from the 1-skeleton C^1 of C where vertices are canonically resolved to trivalent vertices, see Fig. 1. We will then restrict the input to a symmetric braided fusion category $\mathcal{E}$ and remove the vertex resolution condition. We will make the assumption that the cubulation of the manifold is simple: Namely all faces have 4-edges and each vertex is 6-valent.[7] Later on we will make

---

[7] Every 3-manifold has a presentation in terms of a cubulation, in other words in terms of a partition into 3-dimensional cubes, which only intersect along a common face. However in some cases the valence of some of the edges of a cubulation may be different than 4, and therefore some vertices may not be six-valent. For some manifolds these features are not avoidable; see Ref. [43].

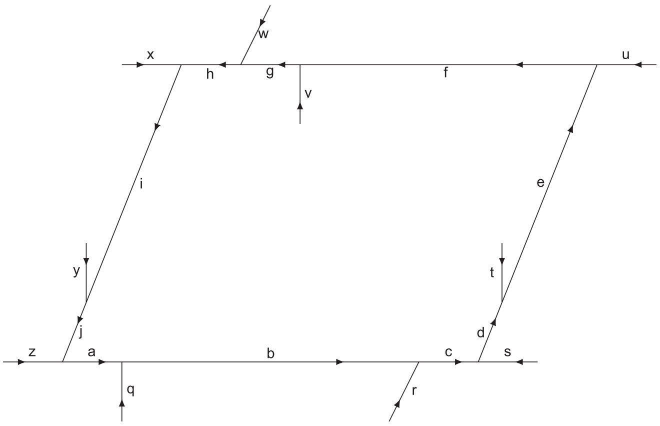


FIG. 2. Trivalent plaquette with oriented edges for Walker-Wang model.

the additional assumption of working with cubulations such that the dual cell decomposition also is a cubulation.

The Walker-Wang model is defined on the trivalent cubic graph Γ (see Fig. 2) with directed edges. The Hilbert space has an orthonormal basis given by colorings of the directed edges of Γ by labels from $\mathcal{L} = \{1,a,b,c,\cdots\}$. For each edge label $a \in \mathcal{L}$ there is a conjugate label $a^* \in \mathcal{L}$ which satisfies the relation $a^{**} = a$. We define the states such that reversing the direction of an edge and conjugating the edge label gives the same state of the Hilbert space as the original configuration. The label set $\mathcal{L}$ has a unique element $1 \in \mathcal{L}$ we call the vacuum which satisfies the relation $1 = 1^*$.

To specify the Hamiltonian we introduce the fusion algebra of the label set [6,12,13,63]. A fusion rule is an associative, commutative product of labels such that for $a,b,c \in \mathcal{L}$, $a \otimes b = \sum_c N^c_{ab} c$. Here $N^c_{ab} \in \mathbb{Z}^+$ is a non-negative integer called the fusion multiplicity. In the following we will restrict to the case of "multiplicity free" which is the restriction $N^c_{ab} \in \{0,1\}$ $\forall a,b,c \in \mathcal{L}$. The fusion multiplicities satisfy the following relations

$$N^c_{ab} = N^c_{ba} \tag{47}$$

$$N^1_{ab} = \delta_{ab^*} \tag{48}$$

$$N^b_{a1} = \delta_{ab} \tag{49}$$

$$\sum_{x\in\mathcal{L}} N^x_{ab} N^d_{xc} = \sum_{x\in\mathcal{L}} N^d_{ax} N^x_{cd}. \tag{50}$$

Given the label set and fusion algebra we define $d : \mathcal{L} \rightarrow \mathbf{R}^+$ such that $\forall a \in \mathcal{L}, d : a \mapsto d_a$ and $d_{a^*} = d_a$. We will refer to d_a as the quantum dimension of the label a. The quantum dimensions are required to satisfy

$$d_a d_b = \sum_c N^c_{ab} d_c. \tag{51}$$

Additionally we define the Frobenius-Shur indicator $\alpha_i = sgn(d_i) \in \{\pm 1\}$ if $i = i^*$ and $\alpha_i = 1$ else, which satisfies

$$\alpha_i \alpha_j \alpha_k = 1 \tag{52}$$

if $N^{k^*}_{ij} = 1$.

Given the fusion algebra and quantum dimensions we define the $6j$ symbols which enforce the associativity of fusion of processes. The $6j$ symbols are a map $F : \mathcal{L}^6 \rightarrow \mathbf{C}$ which satisfy the following relations

$$F^{ijm}_{j^*i^*1} = \frac{v_m}{v_i v_j} N^{m^*}_{ij} \tag{53}$$

$$F^{ijm}_{kln} = F^{klm^*}_{jin^*} = F^{jim}_{lkn^*} = F^{mij}_{nk^*l^*} \frac{v_m v_n}{v_j v_l} = \overline{F^{j^*i^*m^*}_{l^*k^*n}} \tag{54}$$

$$\sum_n F^{mlq}_{kp^*n} F^{jip}_{mns^*} F^{js^*n}_{lkr^*} = F^{jip}_{q^*kr} F^{riq^*}_{mls^*} \tag{55}$$

$$\sum_n F^{mlq}_{kp^*n} F^{l^*m^*i^*}_{pk^*n} = \delta_{iq} \delta_{mlq} \delta_{k^*ip}, \tag{56}$$

where $v_a = \sqrt{d_a}$.

The final piece of data required to define the Walker-Wang model is the braiding relations or R matrices. The R matrices are a map $R : \mathcal{L}^3 \rightarrow \mathbf{C}$ which are required to satisfy the Hexagon equations which ensure the compatibility of braiding and fusion. The Hexagon equations are as follows

$$\sum_g F^{cad^*}_{be^*g} R^e_{gc} F^{abg^*}_{ce^*f} = R^d_{ac} F^{acd^*}_{be^*f} R^f_{bc}$$

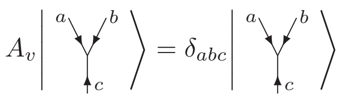
$$\sum_g F^{e^*bd}_{cag} R^e_{ad} F^{e^*ag}_{bcf} = R^d_{ac} F^{e^*bd}_{acf} R^f_{ab}. \tag{57}$$

The data $(\mathcal{L},N,d,F,R)$ forms a UBFC. Examples of solutions to the above data are representations of a finite group or a quantum group.

Using the above data we can write down the Walker-Wang Hamiltonian. The Hamiltonian is of the following form

$$H = -\sum_{v\in\Gamma} A_v - \sum_{p\in\Gamma} B_p, \tag{58}$$

where Γ is the directed, trivalent graph on which the model is defined and the v and p are the vertices and plaquettes of the graph. The plaquettes are defined with reference to the original square faces of C before the vertex resolution. The term A_v is the vertex operator and acts on the 3-edges adjacent to a vertex. We define the action of A_v on states as follows



$$A_v \Big| \quad \Big\rangle = \delta_{abc} \Big| \quad \Big\rangle \tag{59}$$

where $\delta_{abc} = 1$ if $N^{c*}_{ab} \geqslant 1$ and $\delta_{abc} = 0$ else.

The plaquette operator B_p has a slightly more complicated form in terms of the $6j$ symbols and R matrices. Using Fig. 2 as the basis, B_p has the following form

$$\begin{aligned} B^n_p = & \sum_{a',b',c',d',e',f',g',h',i',j'} R^d_{t^*e} \overline{R^{d'}_{t^*e'}} R^{f'}_{v^*g'} \overline{R^f_{v^*g}} \\ & \times F^{qb^*a}_{n^*a'b'^*} F^{rc^*b}_{n^*b'c'^*} F^{sd^*c}_{n^*c'd'^*} F^{te^*d}_{n^*d'e'^*} F^{uf^*e}_{n^*e'f'^*} \\ & \times F^{vg^*f}_{n^*f'g'^*} F^{wh^*g}_{n^*g'h'^*} F^{xi^*h}_{n^*h'i'^*} F^{y^*j^*i}_{n^*i'j'^*} F^{z^*a^*j}_{n^*j'a'^*} \\ & \times |a',b',c',d',e',f',g',h',i',j'\rangle\langle a,b,c,d,e,f,g,h,i,j| \end{aligned} \tag{60}$$

$$B_p = \sum_{n\in\mathcal{L}} \frac{d_n}{D^2} B_p^n. \qquad (61)$$

We define the inner product of such states by

$$\langle a,b,c,\cdots|a',b',c',\cdots\rangle = \delta_{aa'}\delta_{bb'}\delta_{cc'}\cdots. \qquad (62)$$

B. The symmetric braided fusion category $\mathcal{M}(\mathcal{E})$

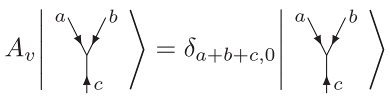
Utilizing the work of Bantay [64] one can define a UBFC for every finite crossed module. Following this construction we will define the symmetric braided fusion category $\mathcal{M}(\mathcal{E})$ induced from the data of the finite crossed module $\mathcal{E} = (\mathbf{1}_E, E, \partial, \triangleright)$, where E is any finite Abelian group and ∂ and $\triangleright$ are trivial.

The label set of $\mathcal{M}(\mathcal{E})$ is given by elements of E, with the vacuum label given by the identity element of E and $a^* = a^{-1}$. The quantum dimension $d_a = 1$ for all $a \in E$ and $D^2 = |E|$. The fusion multiplicities are multiplicity free with $N_{ab}^c = \delta_{a+b,c}$ such that the fusion rules are given by the group composition rules (we use $+$ for the group composition as E is an Abelian group) and $a \otimes b = a + b$ for all $a,b \in E$. We list the data of $\mathcal{M}(\mathcal{E})$ below.

$$\begin{aligned}
\mathcal{L} &= \text{underlying set of } E\\
a\otimes b &= a+b\\
d_a &= 1 \quad \forall a \in \mathcal{L}\\
D^2 &= |E|\\
N_{ab}^c &= \delta_{a+b,c}\\
F_{kln}^{ijm} &= \delta_{i+j,m^{-1}}\delta_{k+l,m}\delta_{l+i,n^{-1}}\delta_{j+k,n}\\
R_{i+j}^k &= \delta_{i+j,k}
\end{aligned} \qquad (63)$$

C. Walker-Wang models for $\mathcal{M}(\mathcal{E})$

Utilizing $\mathcal{M}(\mathcal{E})$ as defined in the previous section as the input data of the Walker-Wang model we may write the terms of the Hamiltonian as follows. The vertex operator acts on basis elements as



(64)

which energetically penalizes configurations of labels around vertices which do not fuse to the identity object.

To define the plaquette operator we first choose an orientation of the plaquette (although the action of B_p is independent of the choice taken). In the following we choose an anticlockwise convention and define $\{e^{+(-)}\} \in p$ as the set of edges with direction parallel (antiparallel) to the choice of orientation. We may then write the plaquette operator for $n \in E$ as follows

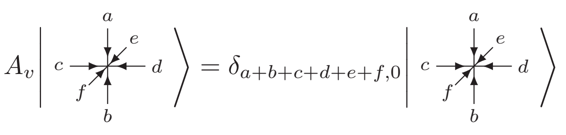
$$B_p^n = \left(\prod_{v\in p} A_v\right) \prod_{e^+\in p} \Sigma_e^n \prod_{e^-\in p} \Sigma_e^{-n}, \qquad (65)$$

where Σ_e^n acts on the label l of edge e such that $\Sigma_e^n : l \mapsto l + n$. The operators Σ_e^n commute for all edges and $\Sigma_e^n\Sigma_e^m = \Sigma_e^{n+m}$. The operator B_p in the Hamiltonian is then defined as

$$B_p = \frac{1}{|E|}\sum_{n\in E} B_p^n. \qquad (66)$$

As such an operator symmetrizes over all group elements the action on basis states is independent of orientation convention for the plaquette.

As the model based on $\mathcal{M}(\mathcal{E})$ does not have any strict dependency on the trivalent lattice we may equally well resolve the trivalent vertices and define the model on a cubic lattice without changing the dynamics of the model. Under such a transformation the vertex operator becomes



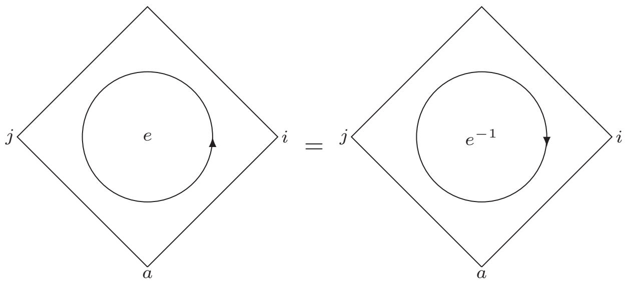
(67)

while the plaquette operator takes the same form with the trivalent vertex operators replaced with the 6-valent counterpart.

D. Yetter model for $\mathcal{E}$ on cubic lattice

As mentioned previously in the text the Yetter model can be equally defined on any cellular decomposition of a 3-manifold. In this section we will outline the model with crossed module of the form of $\mathcal{E}$ on the cubic lattice and show by considering the *dual* of the model that such a model is equivalent to the Walker-Wang model of $\mathcal{M}(\mathcal{E})$ on the cubic lattice.

We begin by defining the Yetter model $\mathcal{E}$ on the cubic lattice following the general procedure outlined in Sec. III. The first step is define an orientation on each square face of the lattice in analogy to the orientation of edges which is inherited from the vertex ordering. Following the previous definitions we choose to orient faces from the lowest ordered vertex on each face which we call the basepoint. We then assign the orientation relative to the two adjacent vertices to the basepoint such that the orientation points to the lowest ordered vertex adjacent to the basepoint. This is demonstrated in the left hand side of equation (68) where the face carries the group element $e \in E$ and vertex a is the basepoint and the orientation is given by the relation $a < i < j$. Reversing the orientation of the face replaces the face label with its inverse as shown in the right hand side of equation (68).



(68)

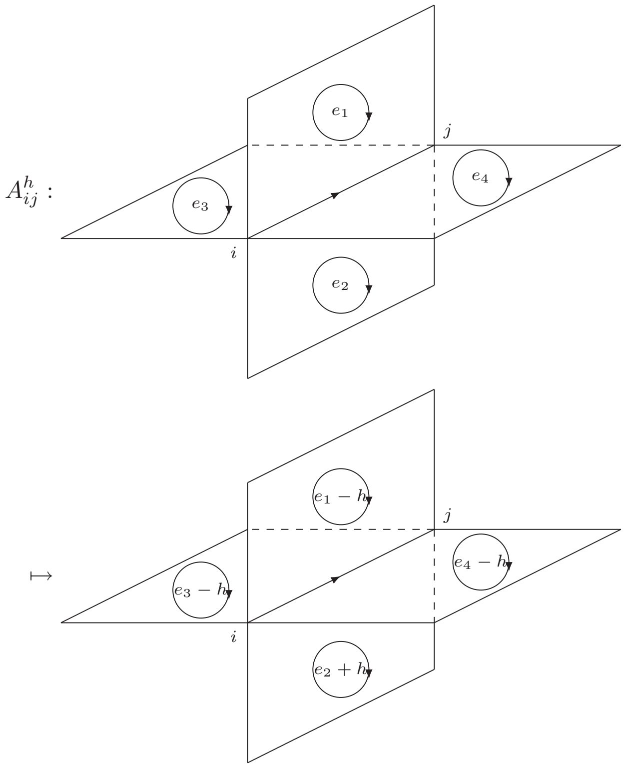
Using the above conventions for the sign of face labels we can now define the 2-flatness condition of a cubic cell. As $\mathcal{E}$ is Abelian and only assigns the identity element to edges, the computation of the 2-holonomy is much simpler than in the general setting. In order to calculate the 2-holonomy of a cubic cell we fix a convention of defining the orientation of faces from either inside or outside of the cubic cell, in the following we choose outside the cell (the flatness condition is independent of such a choice). We then compose the group elements on faces of the cubic with the convention that if the orientation is clockwise we compose the element of the face and if the orientation is anticlockwise we compose the inverse of the face label. We notate this process by introducing the variable $\epsilon \in \{\pm 1\}$ where $\epsilon_f = +1(-1)$ if the face f has clockwise (anticlockwise) orientation such that the 2-holonomy H_2 on the cube can be written as

$$H_2 = \sum_{f \in \text{cube}} e_f^{\epsilon_f} \tag{69}$$

and the 2-flatness condition becomes

$$\sum_{f \in \text{cube}} e_f^{\epsilon_f} = \mathbf{1}_E. \tag{70}$$

We now define the 2-gauge transformation on the cubic lattice. We may neglect the 1-gauge transform as the 1-gauge group is trivial for the crossed module $\mathcal{E}$. The 2-gauge transformation acts on the four faces adjacent to an edge. We notate the 2-gauge transformation as A_{ij}^h on the edge ij where $h \in E$ is the gauge parameter. The gauge transformation has the action of multiplying the faces adjacent to the edge by either h or h^{-1} depending on whether the direction of the edge is parallel or antiparallel to the orientation of the adjacent edges. An example is shown in equation (71).



(71)

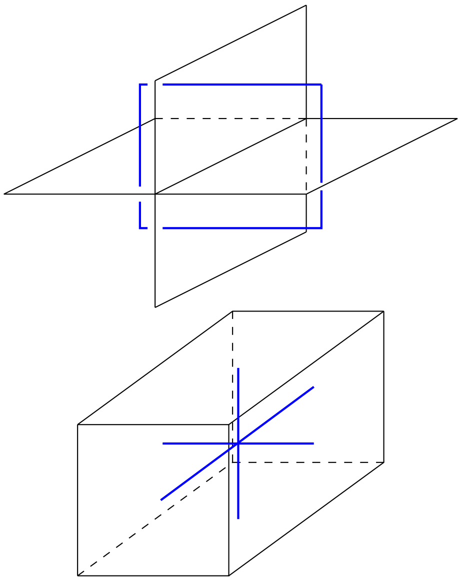
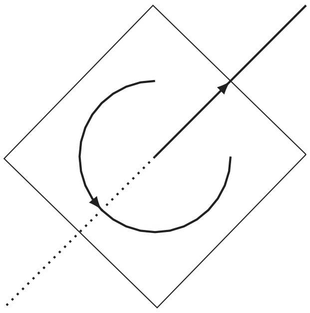


FIG. 3. Examples of the dual of a cubic lattice. The edges of the original lattice are black and the dual edges blue.

1. Model on the dual lattice

After defining the Yetter model for crossed module $\mathcal{E}$ on the cubic lattice Γ, we now define the model on the dual cubulation $\tilde{\Gamma}$. We define dualization by a map which takes the n cells of a cellular decomposition of a d manifold to the (d-n) cells of the dual cellulation. We will make the assumption that we are working with a cubulation such that the dual cell decomposition is also a cubulation; such a restriction is for ease of presentation and the arguments follow straightforwardly outside of such a restriction. In this case the cubes (3-cells) are taken to vertices (0-cells) of the new cellulation, square faces (2-cells) are taken to edges (1-cells), and edges (1-cells) are taken to faces (2-cells). In this way we can canonically map the Yetter model with degrees of freedom on faces to a dual lattice where the face labels are now on edges. Examples are shown in Fig. 3 where black edges are of the original lattice and blue are dual.

Utilizing the duality map discussed previously the direction of dual edges are inherited from the orientation of faces. The direction is defined by the right hand rule, such that if the finger of your right hand points in the direction of the orientation arrow the thumb gives the direction of the dual edge.

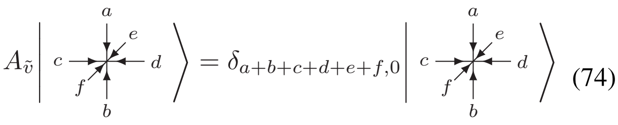


(72)

Using the above convention to define the directed dual lattice, the Hamiltonian form of the Yetter model for the crossed module $\mathcal{E}$ is vastly simplified. Using the definition of the 2-flatness condition discussed in the previous section, on the dual lattice this constraint becomes the condition

$$\sum_{\tilde{e}\in *(\tilde{v})} g_{\tilde{e}}^{\epsilon_{\tilde{e}}} = \mathbf{1}_E, \qquad (73)$$

where $\tilde{e}$ and $\tilde{v}$ are the dual edges and vertices, respectively, $*(\tilde{v})$ is the set of dual edges adjacent to $\tilde{v}$, and $\epsilon_{\tilde{e}} \in \pm 1$ is $+1$ when $\tilde{v}$ is the target of $\tilde{e}$ and -1 when $\tilde{v}$ is the source. Thus we see that the 2-flatness condition on a cubic cell $c \in \Gamma$ becomes a vertex condition on $\tilde{v} \in \tilde{\Gamma}$ the dual lattice. The action of $B_c^{(2)} := A_{\tilde{v}}$ on states of the dual lattice is shown below.



$$A_{\tilde{v}}\Big|\quad\Big\rangle = \delta_{a+b+c+d+e+f,0}\Big|\quad\Big\rangle \qquad (74)$$

We note $B_c^{(2)}$ now has the same action as the vertex operator in the Walker-Wang model of $\mathcal{M}(\mathcal{E})$.

We now consider the edge gauge transformation. On the dual lattice this operator acts on the four edges bounding a plaquette on the dual lattice $\tilde{p}$. As with the plaquette operator B_p in the Walker-Wang model we define the operator $A_{\tilde{p}}^h$ as the gauge transformation on the dual plaquette $\tilde{p}$ by taking an anticlockwise orientation around the plaquette and define $e^{+\tilde{(-)}}$ by whether the dual edge $\tilde{e}$ is parallel (antiparallel) to the orientation convention for the plaquette. We then define $A_e^h := B_{\tilde{p}}^h$ as

$$B_{\tilde{p}}^h = \prod_{\tilde{e}^+\in\tilde{p}} \Sigma_{\tilde{e}}^h \prod_{\tilde{e}^-\in\tilde{p}} \Sigma_{\tilde{e}}^{-h}, \qquad (75)$$

where Σ_e^h is defined as previously and

$$B_{\tilde{p}} = \frac{1}{|E|}\sum_{h\in E} B_{\tilde{p}}^h \qquad (76)$$

such that on the dual lattice $\tilde{\Gamma}$

$$H_{\text{Yetter}}(\mathcal{E}) = -\sum_{\tilde{v}\in\tilde{\Gamma}} A_{\tilde{v}} - \sum_{\tilde{p}\in\tilde{\Gamma}} B_{\tilde{p}}. \qquad (77)$$

E. Comparison of models

Using the discussion outlined in the previous sections we now compare the Yetter model with input $\mathcal{E}$ and the Walker-Wang model with input $\mathcal{M}(\mathcal{E})$. Both models are defined on a cubic lattice Γ with a local Hilbert space defined by $\mathcal{H} = \otimes_{e\in\Gamma}\mathbf{C}^{|E|}$ with edge labels indexed by the group E. The Hamiltonian for the Yetter and Walker-Wang models can, respectively, be written as follows.

$$H_{\text{Yetter}}(\mathcal{E}) = -\sum_{v\in\Gamma} A_v - \sum_{p\in\Gamma}\left(\frac{1}{|E|}\sum_{h\in E}\prod_{e^+\in p}\Sigma_e^h\prod_{e^-\in p}\Sigma_e^{-h}\right)$$

$$H_{WW}(\mathcal{M}(\mathcal{E})) = -\sum_{v\in\Gamma} A_v - \sum_{p\in\Gamma}\left(\frac{1}{|E|}\sum_{h\in E}\prod_{e^+\in p}\Sigma_e^h\prod_{e^-\in p}\Sigma_e^{-h}\right) \times \left(\prod_{v\in p} A_v\right) \qquad (78)$$

Comparing the two equations above the only difference is in the definition of the second term which acts on plaquettes of the lattice. This difference is actually immaterial as the only distinguishing feature of the term $(\prod_{v\in p} A_v)$ is to increase the energy penalty for color configurations which do not satisfy the vertex constraint to twice the energy cost of creating plaquette defects. From such a point of view the two Hamiltonians have the same ground-state configurations and the excitations will have the same measurable properties such as braid statistics but the energy cost will be increased for the creation of vertex violations in the Walker-Wang model in comparison to the energy cost in the Yetter model.

The Yetter model may also be connected to Walker-Wang models in a second manner. Defining the model for the crossed module $G = (G, 1_G, \partial, \triangleright)$ the Hamiltonian is equivalent to the 3 + 1D Kitaev quantum double model describing a conventional discrete group lattice gauge theory. In this case the model is dual to a Walker-Wang model arising from input given by the symmetric braided fusion category defined by representations of G [23,24,65].

VI. DISCUSSION AND OUTLOOK

Here we comment on our main results and discuss the open questions naturally raised by our construction. The main result of our paper is Sec. III. In this section we outline a large class of exactly solvable Hamiltonian models for topological phases in 3 + 1D. Such models utilize the conventions of higher lattice gauge theory to define a topological lattice model on a simplicial triangulation of a closed, compact 3-manifold M. The algebraic data of the model is defined by a crossed module $\mathcal{G} = (G, E, \partial, \triangleright)$ (equivalently a 2-group). Each edge of the triangulation is "colored" by a group element $g \in G$ as in topological gauge theory models [5,18,20,36] while additionally faces of the triangulation are "colored" with an element $e \in E$. In a companion paper [40] we will further describe the mathematical consistency of our model.

In an additional paper[8] we will present results further generalizing the model in order to include crossed module cohomology utilizing the work of Faria Martins and Porter [59]. To extend the model we introduce a 4-cocycle $\omega \in H^4(B\mathcal{G}, U(1))$, where $B\mathcal{G}$ is the classifying space of the crossed module $\mathcal{G}$. Such a cocycle adds a $U(1)$ valued phase to the vertex and gauge transformations while the flatness conditions remain unchanged. The value of ω is determined by considering the gauge transformations as 4-simplices connecting the original lattice coloring to the gauge transformed coloring. ω is then defined by the 4-cocycle of such a complex. Such a phase generalizes the model by allowing for ground states of the model which are not in an equal superposition of basis states as in the current paper.

In Sec. IV we established the relation between our model and the Yetter homotopy 2-type TQFT (Yetter TQFT) [29]. Specifically we showed that the ground-state projector of our model for M^3 is given by the Yetter partition function Z_{Yetter} defined on the 4-manifold $M^3 \times [0,1]$. An intriguing consequence of the proof and the fact that the model can

---

[8] Twisted Higher Symmetry Topological Phases, Bullivant *et al.*

be defined in arbitrary dimension $d > 0$ is that this result also holds for arbitrary dimensions $d > 0$, but for $d < 3$, the Hamiltonian does not contain magnetic operators (there are no blobs in a $d < 2$ dimensional boundary). A direct consequence of this is that the ground-state degeneracy (GSD) can be obtained by the relation $GSD = Z_{\text{Yetter}}(M^d \times S^1)$. We illustrate the formula with several examples in different dimensions for both ordinary and higher gauge theory.

In Sec. V we described a duality between our model with the crossed module $\mathcal{E} = (\mathbf{1}_E, E, \partial, \triangleright)$ and the Walker-Wang model [23] with symmetric-braided fusion category $\mathcal{M}(\mathcal{E})$. The duality is established using the results of Bántay [64] to relate the algebraic data of a crossed module and a braided fusion category. One could expect a further generalization of such results by noting that the crossed module $M = (G, G, \mathbf{1}_G, Ad)$ defines a modular tensor category, the quantum double $D(G)$ of the group G. This observation is seemingly justified by the fact that both models give rise to a unique ground state on all 3-manifolds [24] although further work is needed to establish such a connection.

In two intriguing papers, Kapustin-Thorngren [65,66] investigated the physical role of higher form gauge theories. In their presentation starting from a finite crossed module they define a lattice TQFT partition function [65] using the data defining an equivalence of crossed modules. This data is specified by the quadruple $(\Pi_1, \Pi_2, \alpha, \beta)$ where $\Pi_1 := G/im(\partial)$, $\Pi_2 := ker(\partial)$, $\alpha : \Pi_1 \to Aut(\Pi_2)$ and $\beta \in H^3(B\Pi_1, \Pi_2)$ where $B\Pi_1$ is the classifying space of Π_1. It would be interesting to formally check that the Kapustin-Thorngren partition function is equivalent to the Yetter 2-type TQFT and relate the Hamiltonian model to such a description. This relation would additionally allow for an interpretation of the model in terms of electric and magnetic gauge groups given by Π_1 and Π_2, respectively, and furthermore suggest that such models are physically realized in massive gauge theories where the microscopic gauge group is both partially confined and partially Higgsed [65–67].

More recently Cui presented a generalization of the construction of Crane-Yetter from premodular categories to G-crossed braided fusion categories [45,68,69]. Crossed modules form a subset of G-crossed braided fusion categories and as such our work can be viewed as an example of this construction. These models have yet to be fully explored; it is thought they give rise to manifold invariants which depend on the homotopy 3-type of a manifold and the second Stiefel-whitney class and thus give rise to new, possibly fermionic topological phases in 3 + 1D.

Another avenue for exploration would be to classify the excitation spectrum of the model. There are two complimentary approaches to such a classification. The first is to consider local operators of the model [5]. Using this approach we expect there to be four distinct types of excitations. The four classes of excitations should be point particles and extended-line-like excitations carrying 1-gauge and 2-gauge charges, respectively. Additionally we expect there to be closed-loop-like excitations and membrane type excitations. The second approach is by considering the quantum numbers associated with the ground states of our model. In Sec. IV E we discussed the ground-state degeneracy which is a topological observable of the theory which is independent of local mutations of M which keep the global topology intact, e.g., Pachner moves. In general one can consider other topological observables associated with M which come from global transformations of M which keep the global topology invariant. Such global transformations are indexed by the mapping class group (MCG) of M. The associated observables should give a full classification of quantum numbers in our model. This approach has already been utilized in Refs. [20,36,37,70] for 3 + 1D topological phases using projective representations of $MCG(T^3) = SL(3,\mathbb{Z})$ to understand the quantum numbers for topological gauge theories. $SL(3,\mathbb{Z})$ has two generators given by the S and T matrices. We call eigenstates of the T matrix, $\{|\psi_j\rangle\}$ the quasiparticle basis. The eigenvalues of T in such a basis then give the topological spin of the quasiexcitations associated with the ground state. The exchange statistics of such excitations may be calculated by considering the overlap of each basis state with the S matrix such that the exchange statistics are given by matrix elements $S_{ij} = \langle\psi_j|S|\psi_i\rangle$.

We expect the looplike excitations of our model to form a representation of the loop-braid group. Note that Yetter's TQFT have been shown to give nontrivial invariants of knotted surfaces in 3 + 1D space time [71,72] and furthermore an embedded 1 + 1D TQFT for links in S^3 and their cobordisms [73].

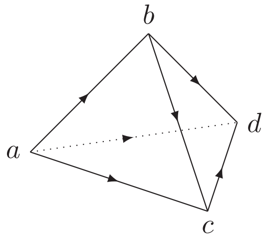
Another possible generalization which should be explored in the future is to consider 3-manifolds with boundary. It is known that BF like theories such as the Walker-Wang model with boundaries reproduce chiral anyon theories on two-dimensional boundaries [24]. Such a relation is suggestive that the boundaries of our model could support nontrivial anyon models.

ACKNOWLEDGMENTS

Z.K. thanks Kirill Krasnov for a discussion on the Ponzano Regge model. A.B. would like to thank Jiannis K. Pachos for helpful discussions and Jamie Vicary for introducing the idea of crossed modules. A.B., Z.K., and P.M.: This work was supported by EPSRC under Grant No. EP/I038683/1. Z.K. also thanks the University of Leeds for support under the Academic Development Fellowship programme. M.C. thanks CNPq for partial financial support via Science Without Borders Programme. J.F.M. was supported by CMA/FCT/UNL, under the project UID/MAT/00297/2013.

APPENDIX A: TETRAHEDRON EXAMPLE

In this section we present the calculation of the 2-holonomy of the boundary of a tetrahedron giving a concrete example of how the vertical composition and whiskering rules of Sec. II C can be applied. Given a tetrahedron with vertices $a < b < c < d$:



(A1)

We can consider splitting the edge *ad* and folding the tetrahedron into the plane as in the following.

$$\begin{tikzcd}
 & b \arrow[dr, "g_{bd}"] \arrow[dd, "g_{bc}"] \arrow[ddl, phantom, "\Downarrow e_{abc}"] & \\
a \arrow[ur, "g_{ab}"] \arrow[dr, "g_{ac}"'] \arrow[rr, bend left=90, "g_{ad}"] \arrow[rr, bend right=90, "g_{ad}"'] \arrow[rr, phantom, bend left=60, "\Uparrow e_{abd}"] \arrow[rr, phantom, bend right=60, "\Downarrow e_{acd}"'] & & d \\
 & c \arrow[ur, "g_{cd}"'] \arrow[u, phantom, shift right=6, "\Uparrow e_{bcd}"] &
\end{tikzcd}\tag{A2}$$

In order to calculate the 2-holonomy associated with this surface it is convenient to *whisker* [see equation (7)] the face labels such that each label has a source and target of an edge from a to d as in equation (A3). Once the face labels are in such a form the 2-holonomy can be calculated straightforwardly through vertical composition [see equation (6)]. This choice is ad-hoc in the sense that if we had chosen to split any edge to flatten the tetrahedron into the plane we would obtain the same 2-holonomy.

$$\begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=90, "g_{ad}"] \arrow[r, bend left=45, "g_{ab}g_{bd}"] \arrow[r, "g_{ab}g_{bc}g_{cd}"] \arrow[r, bend right=45, "g_{ac}g_{cd}"'] \arrow[r, bend right=90, "g_{ad}"'] \arrow[r, phantom, bend left=70, "\Uparrow e_{abd}"] \arrow[r, phantom, bend left=25, "\Uparrow g_{ab}\triangleright e_{bcd}"] \arrow[r, phantom, bend right=25, "\Downarrow e_{abc}"'] \arrow[r, phantom, bend right=70, "\Downarrow e_{acd}"'] & d
\end{tikzcd}\tag{A3}$$

Looking at equation (A2) the labels e_{abd} and e_{acd} already have source and targets given by edges between a and d. As such we only need to apply whiskering to the labels e_{abc} and e_{bcd}. Considering these two labels we can perform the whiskering as follows.

$$\begin{tikzcd}
 & b \arrow[dr, "g_{bd}"] \arrow[dd, "g_{bc}"] \arrow[ddl, phantom, "\Downarrow e_{abc}"] & \\
a \arrow[ur, "g_{ab}"] \arrow[dr, "g_{ac}"'] & & d \\
 & c \arrow[ur, "g_{cd}"'] \arrow[u, phantom, shift right=6, "\Uparrow e_{bcd}"] &
\end{tikzcd}$$

$$= \begin{tikzcd}
a \arrow[rr, bend left=50, "g_{ab}g_{bd}"] \arrow[r, bend left, "g_{ab}g_{bc}"] \arrow[r, bend right, "g_{ac}"'] \arrow[r, phantom, "\Downarrow e_{abc}"] & c \arrow[r, "g_{cd}"'] \arrow[u, phantom, "\Downarrow g_{ab}e_{bcd}"] & d
\end{tikzcd}$$

$$= \begin{tikzcd}[column sep=huge]
a \arrow[r, bend left=45, "g_{ab}g_{bd}"] \arrow[r, "g_{ab}g_{bc}g_{cd}"] \arrow[r, bend right=45, "g_{ac}g_{cd}"'] \arrow[r, phantom, bend left=25, "\Uparrow g_{ab}e_{bcd}"] \arrow[r, phantom, bend right=25, "\Downarrow e_{abc}"'] & d
\end{tikzcd}\tag{A4}$$

Here we made use of the left whiskering identity

$$\begin{tikzcd}
 & & c \arrow[dr, "g_{cd}"] \arrow[d, phantom, "\Downarrow e_{bcd}"] & \\
a \arrow[r, "g_{ab}"'] & b \arrow[ur, "g_{bc}"] \arrow[rr, "g_{bd}"'] & \, & d
\end{tikzcd}$$

$$= \begin{tikzcd}
 & c \arrow[dr, "g_{cd}"] \arrow[d, phantom, "\Downarrow g_{ab}e_{bcd}"] & \\
a \arrow[ur, "g_{ab}g_{bc}"] \arrow[rr, "g_{ab}g_{bd}"'] & \, & d
\end{tikzcd}\tag{A5}$$

between the first and second lines and the right whiskering identity

$$\begin{tikzcd}
 & b \arrow[dr, "g_{bc}"] \arrow[d, phantom, "\Downarrow e_{abc}"] & & \\
a \arrow[ur, "g_{ab}"] \arrow[rr, "g_{ac}"'] & \, & c \arrow[r, "g_{cd}"'] & d
\end{tikzcd}$$

$$= \begin{tikzcd}
 & b \arrow[dr, "g_{bc}g_{cd}"] \arrow[d, phantom, "\Downarrow e_{abc}"] & \\
a \arrow[ur, "g_{ab}"] \arrow[rr, "g_{ac}g_{cd}"'] & \, & d
\end{tikzcd}\tag{A6}$$

between the second and third lines. Additionally we made use of the identity

$$\begin{tikzcd}
 & j \arrow[dr, "g_{jk}"] \arrow[d, phantom, "\Downarrow e_{ijk}"] & \\
i \arrow[ur, "g_{ij}"] \arrow[rr, "g_{ik}"'] & \, & k
\end{tikzcd}
=
\begin{tikzcd}
i \arrow[r, bend left, "g_{ij}g_{jk}"] \arrow[r, bend right, "g_{ik}"'] \arrow[r, phantom, "\Downarrow e_{ijk}"] & k
\end{tikzcd}\tag{A7}$$

which holds for any triangle $i < j < k$.

APPENDIX B: TRANSFORMATION PROPERTIES OF 1- AND 2-HOLONOMIES

In this section we compute the transformation properties of the 1-holonomy of a reference triangle with labels given by the lhs of (21) and a reference tetrahedron with labels given by (12).

$$\begin{aligned}
H_1 &\xrightarrow{A_a^g} \partial(g \triangleright e_{abc}) g g_{ab}\, g_{bc}\, g_{ac}^{-1} g^{-1} \\
&= g(\partial e_{abc}) g_{ab}\, g_{bc}\, g_{ac}^{-1} g^{-1} = g H_1 g^{-1} \\
&\xrightarrow{A_b^g} (\partial e_{abc}) g_{ab} g^{-1}\, g g_{bc}\, g_{ac}^{-1} = H_1 \\
&\xrightarrow{A_c^g} (\partial e_{abc}) g_{ab}\, g_{bc} g^{-1}\, g g_{ac}^{-1} = H_1 \\
&\xrightarrow{A_{ab}^e} \partial(e_{abc}\, e^{-1})(\partial e) g_{ab}\, g_{bc}\, g_{ac}^{-1} = H_1 \\
&\xrightarrow{A_{bc}^e} \partial(e_{abc}\, g_{ab} \triangleright e^{-1}) g_{ab}\, \partial e\, g_{bc}\, g_{ac}^{-1} \\
&= \partial e_{abc}\, g_{ab}\, \partial e^{-1} g_{ab}^{-1}\, g_{ab}\, \partial e\, g_{bc}\, g_{ac}^{-1} = H_1 \\
&\xrightarrow{A_{ac}^e} \partial(e\, e_{abc}) g_{ab}\, g_{bc}\, g_{ac}^{-1} \partial e^{-1} = \partial e H_1 (\partial e)^{-1} \\
H_2 &\xrightarrow{A_a^g} (g \triangleright e_{acd})(g \triangleright e_{abd})\big(g g_{ab} \triangleright e_{bcd}^{-1}\big)(g \triangleright e_{abd}) = g \triangleright H_2 \\
&\xrightarrow{A_b^g} e_{acd}\, e_{abc}\big(g_{ab} g^{-1} \triangleright \big(g \triangleright e_{bcd}^{-1}\big)\big) e_{abd}^{-1} = H_2 \\
&\xrightarrow{A_c^g} H_2
\end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{A^g_d} H_2\\
&\xrightarrow{A^e_{ab}} e_{acd}\, e_{abc}e^{-1}\big((\partial e g_{ab}) \triangleright e^{-1}_{bcd})e_{abd}e^{-1}\big)^{-1}\\
&= e_{acd}\, e_{abc}e^{-1}\big[(\partial e g_{ab}) \triangleright e^{-1}_{bcd}\big]e\, e^{-1}_{abd}\\
&= e_{acd}\, e_{abc}e^{-1}\big(\partial e \triangleright \big(g_{ab} \triangleright e^{-1}_{bcd}\big)\big)ee^{-1}_{abd}\\
&= e_{acd}\, e_{abc}e^{-1}e(g_{ab} \triangleright e^{-1}_{bcd}))e^{-1}ee^{-1}_{abd} = H_2\\
&\xrightarrow{A^e_{bc}} e_{acd}\, e_{abc}(g_{ab} \triangleright e^{-1})\big(g_{ab} \triangleright e\, e^{-1}_{bcd}\big)e^{-1}_{abd}\\
&= e_{acd}\, e_{abc}\big(g_{ab} \triangleright \big(e^{-1}ee^{-1}_{bcd}\big)\big)e^{-1}_{abd} = H_2\\
&\xrightarrow{A^e_{ac}} e_{acd}e^{-1}ee_{abc}\big(g_{ab} \triangleright e^{-1}_{bcd}\big)e^{-1}_{abd} = H_2\\
&\xrightarrow{A^e_{ad}} e\, e_{acd}\, e_{abc}\big(g_{ab} \triangleright e^{-1}_{bcd}\big)\,(ee_{abd})^{-1} = eH_2\, e^{-1}\\
&\xrightarrow{A^e_{bd}} e_{acd}\, e_{abc}(g_{ab} \triangleright (ee_{bcd})^{-1})(e_{abd}(g_{ab} \triangleright e^{-1}))^{-1}\\
&= e_{acd}\, e_{abc}(g_{ab} \triangleright \big(e^{-1}_{bcd}e^{-1}\big)(g_{ab} \triangleright e)e^{-1}_{abd}\\
&= e_{acd}\, e_{abc}(g_{ab} \triangleright \big(e^{-1}_{bcd}e^{-1}e\big)e^{-1}_{abd} = H_2\\
&\xrightarrow{A^e_{cd}} e_{acd}(g_{ac} \triangleright e^{-1})e_{abc}\,[g_{ab} \triangleright (e_{bcd}\big(g_{bc} \triangleright e^{-1}))^{-1}]e^{-1}_{abd}\\
&= e_{acd}(g_{ac} \triangleright e^{-1})e_{abc}\,\big[g_{ab} \triangleright (g_{bc} \triangleright e)e^{-1}_{bcd}\big]e^{-1}_{abd}\\
&= e_{acd}(g_{ac} \triangleright e^{-1})e_{abc}(g_{ab}\, g_{bc} \triangleright e)\big(g_{ab} \triangleright e^{-1}_{bcd}\big)e^{-1}_{abd}\\
&= e_{acd}\big(\big(g_{ac}(g_{ab}\, g_{bc})^{-1}\big) \triangleright e'^{-1}\big)\\
&\quad \times e_{abc}e'\big(g_{ab} \triangleright e^{-1}_{bcd}\big)e^{-1}_{abd}\\
&= e_{acd}\big(\big(H^{abc}_1\big)^{-1}\partial e_{abc}\big) \triangleright e'^{-1}\\
&\quad \times e_{abc}e'\big(g_{ab} \triangleright e^{-1}_{bcd}\big)e^{-1}_{abd}\\
&\approx e_{acd}\, e_{abc}e'^{-1}e^{-1}_{abc}e_{abc}e'\big(g_{ab} \triangleright e^{-1}_{bcd}\big)\big)e^{-1}_{abd} = H_2
\end{aligned}
$$

In the last equation the substitution $e' = g_{ab}\, g_{bc} \triangleright e$ has been made and $\approx$ means equality in the case when $H^{abc}_1 = 1$. Note that the very last relation shows that the 2-holonomy does not transform covariantly if fake flatness of the boundary faces is not imposed.

APPENDIX C: ALGEBRA OF GAUGE TRANSFORMATIONS

In this Appendix the proof of the relations (14) to (19) of gauge transformations is presented. First we remember that the operators R^g_i and L^g_i are representations of the group G, that is, for all $g,h \in G$ we have $R^{gh}_i = R^g_i R^h_i$ and $L^{gh}_i = L^g_i L^h_i$. Then it is clear that $\mathcal{L}^{gh}_v(i) = \mathcal{L}^g_v(i)\mathcal{L}^h_v(i)$. Also, since $g \triangleright (h \triangleright (\cdot)) = gh \triangleright (\cdot)$, one obtains $\mathcal{L}^{gh}_v(p) = \mathcal{L}^g_v(p)\mathcal{L}^h_v(p)$. Using these properties we deduce that

$$
\begin{aligned}
&\prod_{i\in\star(v)} \mathcal{L}^{gh}_v(i) \prod_{p\in\star'(v)} \mathcal{L}^{gh}_v(p)\\
&\quad= \prod_{i\in\star(v)} \mathcal{L}^g_v(i)\mathcal{L}^h_v(i) \prod_{p\in\star'(v)} \mathcal{L}^g_v(p)\mathcal{L}^h_v(p)\\
&\quad= \prod_{i\in\star(v)} \mathcal{L}^g_v(i) \prod_{p\in\star'(v)} \mathcal{L}^g_v(p) \prod_{i\in\star(v)} \mathcal{L}^h_v(i) \prod_{p\in\star'(v)} \mathcal{L}^h_v(p),
\end{aligned}
$$

where in the last equality we used the fact that all the operators in the middle two products commute pairwise, because each one of them acts nontrivially on only a distinct edge or face label. Thus we proved (14).

Now we prove the identity (16). First note that $A^e_i A^f_i$ acts as $g_i \mapsto \partial(e)\partial(f)g_i$ on the edge label of edge i, while A^{ef}_i acts as $g_i \mapsto \partial(ef)g_i$. Both act trivially on the other edge labels and, since $\partial(ef) = (\partial e)(\partial f)$, they have the same action on all edge labels. To prove that they coincide also on face labels, we must consider two cases, (i) $i = v_k v_{k+1}$, and (ii) $i = v_{k+1}v_k$. Also we can suppose that the face p is adjacent to the edge i, otherwise both operators act trivially on the edge label of this face.

(i) If $i = v_k v_{k+1}$, $A^e_i A^f_i$ acts as $e_p \mapsto e_p(g_{v_0 v_k} \triangleright f^{-1})(g_{v_0 v_k} \triangleright e^{-1})$ and A^{ef}_i acts as $e_p \mapsto e_p(g_{v_0 v_k} \triangleright (ef)^{-1})$. As $g \triangleright (\cdot)$ is a homomorphism, we have that the actions coincide.

(ii) If $i = v_{k+1}v_k$, $A^e_i A^f_i$ acts as $e_p \mapsto (g_{\overline{v_0 v_k}} \triangleright e)(g_{\overline{v_0 v_k}} \triangleright f)e_p$ and A^{ef}_i acts as $e_p \mapsto (g_{\overline{v_0 v_k}} \triangleright ef)e_p$. As before, the two sides agree.

So we proved (16).

In order to verify (15) we deduce first that $[\mathcal{L}^g_v(i),\mathcal{L}^h_{v'}(i)] = 0$ and $[\mathcal{L}^g_v(p),\mathcal{L}^h_{v'}(p)] = 0$, if $v \neq v'$. The first relation holds because both operators can act nontrivially only on the edge i and only if one of the vertices is the source and the other is the target of i. However, the operator associated with the source of i is a left multiplication operator and the other associated with the target of i is a right multiplication operator, and these actions are obviously commutative. The second relation follows more easily because at most one of the operators can act nontrivially on a face label. The validity of relation (15) is now a consequence of the fact that all operators in the definitions of A^g_v and $A^h_{v'}$ commute pairwise.

To prove (17) we note that A^e_i and $A^f_{i'}$ commute on edge labels because each one of them acts nontrivially on only one edge label and they are distinct. They also commute on a face label if the associated face is not adjacent to both edges, since in this case at least one of them acts trivially on such a face label. If the face is adjacent to both edges i and i' and they are oppositely oriented, then one is a left action and the other is a right one, so they commute. Assume that $s(i) < t(i)$ and $s(i') < t(i')$. Assume without loss of generality that $s(i') > s(i)$. Then $A^e_i A^f_{i'}$ acts as $e_p \mapsto e_p(g_{v_0 v_{s(i')}} \triangleright f^{-1})(g_{v_0 v_{s(i)}} \triangleright e^{-1})$. The action of $A^f_{i'} A^e_i$ reads:

$$
\begin{aligned}
e_p \mapsto\ & e_p\big(g_{v_0 v_{s(i)}} \triangleright e^{-1}\big)\big(g_{v_0 v_{s(i)}}\partial e\, g_{v_{s(i)}v_{s(i')}} \triangleright f^{-1}\big)\\
=\ & e_p\big(g_{v_0 v_{s(i)}} \triangleright \big(e^{-1}(\partial e g_{v_{s(i)}v_{s(i')}} \triangleright f^{-1})\big)\big)\\
=\ & e_p(g_{v_0 v_{s(i)}} \triangleright \big(e^{-1}e\big(g_{v_{s(i)}v_{s(i')}} \triangleright f^{-1})e^{-1}\big)\big)\\
=\ & e_p(g_{v_0 v_{s(i)}} \triangleright \big(\big(g_{v_{s(i)}v_{s(i')}} \triangleright f^{-1})e^{-1}\big)\big)\\
=\ & e_p\big(g_{v_0 v_{s(i')}} \triangleright f^{-1}\big)\big(g_{v_0 v_{s(i)}} \triangleright e^{-1}\big),
\end{aligned}
$$

where we used the homomorphism property of $\triangleright$ in the first and last equation and the second Peiffer condition in the second. The other case [$s(i) > t(i)$ and $s(i') > t(i')$] is a similar computation.

Let us consider now the proof of the identities (18) and (19). First we note that if i and v are not adjacent to a given face, then

A_i^e and A_v^g commute on all edge labels of edges adjacent to this face and on the face label of this face, because in such cases at least one of the operators is the identity operator. Therefore, for the rest of the proof we consider a face adjacent to both i and v (if it exists). Note that A_i^e and A_v^g commute on edge labels (on face labels) if $v \neq s(i)$ and $v \neq t(i)$ (if $v \neq v_0$), since in this case A_v^g acts trivially on edge labels (on face labels, respectively). Thus we need to verify the identities for the cases: (i) $v = v_0$ and $i = v_0 v_1$, (ii) $v = v_0$ and $i = v_0 v_{n-1}$, (iii) $v = v_0, i \neq v_0 v_1$ and $i \neq v_0 v_{n-1}$, (iv) $v \neq v_0$ and $v = s(i)$, and (v) $v \neq v_0$ and $v = t(i)$. Furthermore, it is enough to verify (iii) only on face labels and (iv),(v) only on edge labels. Now we have

(i) $A_i^{g \rhd e} A_v^g$ acts as $g_i \mapsto \partial(g \rhd e) g g_i$ and $e_p \mapsto (g \rhd e_p)(g \rhd e^{-1})$ and $A_v^g A_i^e$ acts as $g_i \mapsto g\partial(e) g_i$ and $e_p \mapsto g \rhd (e_p e^{-1})$. They agree since $g\partial(e)g_i = g\partial(e)g^{-1}gg_i = \partial(g \rhd e)gg_i$ and $g \rhd (e_p e^{-1}) = (g \rhd e_p)(g \rhd e^{-1})$.

(ii) $A_i^{g \rhd e} A_v^g$ acts as $g_i \mapsto \partial(g \rhd e) g g_i$ and $e_p \mapsto (g \rhd e)(g \rhd e_p)$ and $A_v^g A_i^e$ acts as $g_i \mapsto g\partial(e)g_i$ and $e_p \mapsto g \rhd (e e_p)$ Thus by the previous item the maps agree on edge labels and on face labels by the homomorphisms property of $\rhd$.

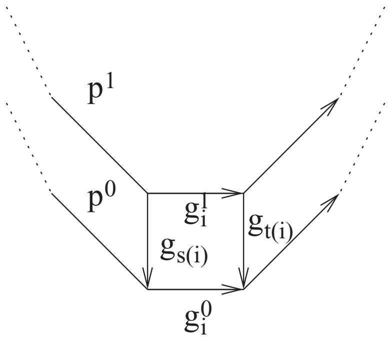
(iii) For $s(i) < t(i)$ the operator $A_i^e A_v^g$ acts as $e_p \mapsto (g \rhd e_p)(g g_{v_0 s(i)} \rhd e^{-1})$ and $A_v^g A_i^e$ acts as $e_p \mapsto g \rhd (e_p (g_{v_0 s(i)} \rhd e^{-1}))$. For $s(i) > t(i)$ the operator $A_i^e A_v^g$ acts as $e_p \mapsto (g g_{\overline{v_0 v_{s(i)}}^o} \rhd e)(g \rhd e_p)$ whereas $A_v^g A_i^e$ acts as $e_p \mapsto g \rhd ((g_{\overline{v_0 v_{s(i)}}} \rhd e) e_p)$. For both cases equality is clear by the homomorphism property of $\rhd$.

(iv) $A_i^{g \rhd e} A_v^g$ acts as $g_i \mapsto \partial(g \rhd e)(g g_i)$ and $A_v^g A_i^e$ acts as $g_i \mapsto g\partial(e)g_i$. As $\partial(g \rhd e)(gg_i) = g\partial e g^{-1} g g_i = g \partial e g_i$, they coincide.

(v) $A_i^e A_v^g$ and $A_v^g A_i^e$ both act as $g_i \mapsto \partial(e) g_i g^{-1}$.
This ends the proof of the relations (14) to (19).

APPENDIX D: PROOF OF THEOREM IV.1

A three cell of Δ is a prism based on a face of L_j.



The figure shows a part of the complex. The lattice Δ consists of L_{0c}, L_{1c} with the coloring given by $\{g_i^0\}_{i \in L_0^1}$, $\{g_i^1\}_{i \in L_1^1}$, $\{g_v\}_{v \in L^0}$, oriented edges in L_0, L_1, and vertical edges assumed to be oriented towards L^0 (downwards in the figure) connecting corresponding vertices of L_j, respectively.

We have a Dirac delta $\delta_{g_p,\mathbf{1}}$ in the partition function for each face. In particular, for an internal face p (this is always a rectangle connecting corresponding edges of L_j) the term $\delta_{g_p,\mathbf{1}}$ enforces $g_i^1 = g_{s(i)} g_i^0 g_{t(i)}^{-1}$. Taking all such faces into account, identifying the coloring L_{0c} with $|L_{0c}\rangle$ we have[9]

$$\prod_{p \in \Delta^{(i)2}} \delta_{g_p,\mathbf{1}} = \langle L_{1c}| \prod_{v \in L^0} A_v^{g_v} |L_{0c}\rangle \tag{D1}$$

and consequently we can write

$$\frac{1}{|G|^{|L^0|}} \prod_{v \in L^0} \sum_{g_v \in G} \prod_{p \in \Delta^{(i)2}} \delta_{g_p,\mathbf{1}} \tag{D2}$$

$$= \langle L_{1c}| \prod_{v \in L^0} \frac{1}{|G|} \sum_{g_v \in G} A_v^{g_v} |L_{0c}\rangle = \langle L_{1c}| \prod_{v \in L^0} A_v |L_{0c}\rangle. \tag{D3}$$

Now, let us compute the prefactor in the definition (39) for the lattice: $|G|^{-\frac{|L_0^0|+|L_1^0|}{2}} = |G|^{-|L_0|}$. Hence the lhs of the formula above almost coincides with the lhs of (41), only $\prod_{p \in \Delta^{(b)2}} \delta_{g_p,\mathbf{1}}$ is missing. This enforces flatness on the faces $p \in L_0^2 \cup L_1^2$. It agrees with the action of the operator $\prod_{p \in L^2} B_p$ on $|L_{0c}\rangle$ times that on $|L_{1c}\rangle$ (note that the lattices L_0 and L_1 are identified, but the states $|L_{0c}\rangle$ and $|L_{1c}\rangle$ are different). However, the operators $B_p, p \in L^2$ and $A_v, v \in L^0$ commute for any pair of labels and are also self-adjoint, so we can simply insert the factor $\prod_{p \in L^2} B_p$ anywhere in the scalar product, using the fact that $B_p^2 = B_p, p \in L^2$, thus inserting B_p only once is sufficient. ■

APPENDIX E: PROOF OF THEOREM IV.2

Consider that $M^3 \times [0,1]$ has the product lattice decomposition Δ. Let us choose a total order on Δ^0 in the following way. The Hilbert space is associated with a dressed L, i.e., L^0 has a total order. Let the total orders in L_i^0 agree with that on L^0 and let any vertex in L_1^0 be smaller than any other in L_0^0. Then denoting the color of the internal edge connecting the vertices in L_0^0 and L_1^0 corresponding to $v \in L^0$ by g_v and the color of the internal face connecting edges in L_0^1 and L_1^1 corresponding to $i \in L^1$ by e_i, the following equality is true.

$$\prod_{P \in \Delta^{3(i)}} \delta_{H_2(P),\mathbf{1}} \prod_{p \in \Delta^{2(i)}} \delta_{H_1(p),\mathbf{1}} = \langle L^{1c}| \prod_{v \in L^0} A_v^{g_v} \prod_{i \in L^1} A_i^{e_i} |L^{0c}\rangle \tag{E1}$$

This is the analog of (D1). To justify it, let us consider the rectangle p depicted in Fig. 4 with boundary edges $g_i^0, g_{s(i)}, g_{t(i)}, g_i^1$. The constraint $\delta_{H_1(p),\mathbf{1}}$ enforces $\partial e_i = g_{s(i)}^{-1} g_i g_{t(i)} (g_i^0)^{-1}$. Equivalently

$$g_i^1 = g_{s(i)} \partial e_i g_i^0 g_{t(i)}^{-1}, \tag{E2}$$

which is precisely the image of g_i^0 under the product of gauge transformations with parameters $g_{s(i)}, g_{t(i)}, e_i$ at vertices $s(i), t(i)$ and edge i, respectively, such that the edge transformation acts first. Note that no other generators in the product act on the edge color g_i. Let us recall that we identify L_{jc} with the vector $|L_{jc}\rangle$ and colors on $\Delta^{(i)}$ are identified with parameters of gauge transformation on the Hilbert space.

Now consider $p^0 \in L_{0c}$ colored with e_p^0 corresponding to the 2-holonomy based at $v_0(p^0)$ and denote the boundary

---

[9] Note, that $|L_{0c}\rangle$ is a basis element, not a generic state in the Hilbert space.

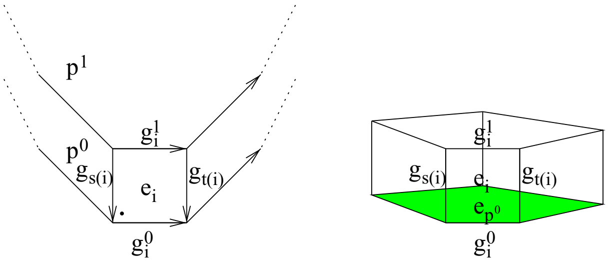


FIG. 4. The left figure shows one of the internal faces connecting corresponding boundary edges of L_j; the top edge label g_i^1 is determined by fake flatness of the rectangle. By our choice of total order on Δ^0, the basepoint of the rectangle is $s(i)$ and the fake-flatness constraint reads as the composition (E2). The right figure shows a blob bounded by the "bucket" consisting of the green pentagon of L_0 with label e_p^0 and the rectangles with labels $e_i \in E, i \in bd(p_0)$. The 2-holonomy of the bucket is $\tilde{e}_p^0$; the "lid pentagon" is colored by e_p^1. By 2-flatness of the blob $\tilde{e}_p^0 = e_p^1$.

edge set by $L^1_{p^0}$. Assume that p^0 is an n-gon. Denote the disk (depicted as a bucket in Fig. 4) by $\tilde{p}^0$, consisting of the rectangles bounded by the edges $\{g_i^0, g_{s(i)}, g_{t(i)}, g_i\}, i \in bd(p^0)$ and p^0. We can compute the 2-holonomy of this disk based at $v_0(p)$, which we denote by $e^0_{\tilde{p}}$. We have to show that this (again via the identification of boundary colorings with basis vectors in the Hilbert space and internal colors as parameters of gauge transformations) agrees with the action of the product $\prod_{v\in L^0} A_v^{g_v} \prod_{i\in L^1} A_i^{e_i}$ on e_p^0 where g_v is the color of the vertical edge pointing toward the vertex v and e_i is the color of the vertical rectangle based at edge i. Introducing the notation i for the ith edge starting from $v_0(p)$ in the circular order the action of the product of edge transformations reads

$$\begin{aligned} e_p^0 \mapsto e_p'^0 \equiv e_{p_0}\big(\partial e_{p_0}^{-1} \triangleright e_n\big)\big(g_{v_0,s(n-1)} \triangleright e_{n-1}^{\pm 1}\big) \\ \times \big(g_{v_0,s(n-2)} \triangleright e_{n-2}^{\pm 1}\big) \dots \big(g_{v_0,s(2)} \triangleright e_2^{\pm 1}\big) e_1^{-1}\ , \end{aligned} \tag{E3}$$

where $e_i^{\pm 1}$ means e_i (e_i^{-1}), if i is oriented opposite (according) to the orientation of $bd(p)$, respectively. Now we apply the vertex transformations. We will show that (i) the rhs of (E3) is unchanged under a vertex gauge transformation $A_v^{g_v}$ with $v \neq v_0$ and (ii) that it changes as $(.) \mapsto g_{v_0} \triangleright (.)$ for $v = v_0$. This is how the face holonomy $e_p'^0$ should change under $\prod_{v\in L^0} A_v^{g_v}$. Consequently, the 2-holonomy $e^0_{\tilde{p}}$ of the disk $\tilde{p}^0$ is given by the image of e_p^0 under gauge transformation.

It is easy to see that $g_{v_0,s(k)}$ changes only if $v = s(k)$ since then $g_{s(k)-1,s(k)} \mapsto g_{s(k)-1,s(k)} g_v^{-1}$, which induces $g_{v_0,s(k)} \mapsto g_{v_0,s(k)} g_v^{-1}$. The label e_k changes also only for $s(k) = v$ as $e_k \mapsto g_v \triangleright e_k$. Altogether

$$g_{v_0,s(k)} \triangleright e_k \mapsto g_{v_0,s(k)} g_v^{-1} \triangleright (g_v \triangleright e_k) = g_{v_0,s(k)} \triangleright e_k$$

so (i) is proved. The action of $A_{v_0}^{g_{v_0}}$ is $e_{p_0} \mapsto g_{v_0} \triangleright e_{p_0}$ and $e_1 \mapsto g_{v_0} \triangleright e_1$ on the two faces with basepoint v_0 and all parenthesis in (E3) transform as $(.) \mapsto g_{v_0} \triangleright (.)$ since $g_{v_0,s(k)} \mapsto g_{v_0} g_{v_0,s(k)}$. Hence, since $(.) \mapsto g \triangleright (.)$ is a homomorphism, we showed (ii), so the claim is proved. Note that the edge labels change as $g_i^0 \mapsto \partial e_i g_i^0 \mapsto g_{s(i)} \partial e_i g_i^0 g_{t(i)}^{-1}$ under $\prod_j A_j^{e_j}$ and $\prod_v A_v^{g_v}$, respectively, in accordance with (E2). Another remark is that the order of the terms on the rhs of (E1) depends on the total order on Δ^0. In particular, if we had chosen $v^0_{s(i)} > v^1_{s(i)}$ the fake-flatness condition would have read $\partial e_i = g_i^1 g_{t(i)} (g_i^0)^{-1} g_{s(i)}^{-1}$ equivalent to the order $A_i^{e_i} A_{s(i)}^{g_{s(i)}}$ of action of gauge transformations. Once the integration is done, this dependence will of course disappear, equivalently the vertex projections defined as averaged gauge transformations mutually commute.

Now, consider the 2-sphere S_{p_0} bounded by the disks $\tilde{p}^0$ and p^1 colored by $e^0_{\tilde{p}}$ and e^1_p, respectively. They have the same boundary $bd(p^1)$, and they are oriented oppositely. Hence, $\delta_{H_2(S_{p_0}),\mathbf{1}} = 1$ iff $e^0_{\tilde{p}} = e_{p^1}$. This completes the proof of (E1).

Let us now determine the multiplicative factors from the definition of the partition vector from (39) for the lattice at hand. We have only boundary vertices $\Delta^0 = L_0^0 \cup L_1^0$ and the edge set is in one-to-one correspondence with $L^0 \cup (L_0^1 \cup L_1^1)$ such that the first factor is for internal edges and the second for external ones, respectively. The multiplicative factor determined from (39) turns out to be $|G|^{-|L^0|}|E|^{-|L^1|}$. This means that we can assign a factor of $|G|^{-1}$ to each term $\sum_{g_v} A_v^{g_v}, v \in L^0$ and a factor of $|E|^{-1}$ to $\sum_{e_i} A_i^{e_i}, i \in L^1$. So, similarly to the ordinary gauge theory case, we found

$$\begin{aligned} &\frac{1}{|G|^{|L^0|}|E|^{|L^1}} \prod_{v\in L^0} \prod_{i\in L_0^1} \sum_{g_v\in G} \sum_{e_i\in E} \prod_{P\in\Delta^{3(i)}} \delta_{H_2(P),\mathbf{1}} \prod_{p\in\Delta^{2(i)}} \delta_{H_1(p),\mathbf{1}} \\ &= \langle L^{c_1}| \prod_{v\in L^0} A_v \prod_{i\in L^1} A_i |L^{c_0}\rangle. \end{aligned}$$

The last step is the identification of the 2-flatness constraints of the blobs in $L_0^3 \cup L_1^3$ with the B_P operators. This goes parallel to the ordinary lattice gauge theory case. ■

---

[1] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
[2] D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[4] E. Fradkin, *Field Theories of Condensed Matter Physics*, 2nd ed. (Cambridge University Press, Cambridge, 2013).
[5] A. Y. Kitaev, Ann. Phys. **303**, 2 (2003).
[6] J. K. Pachos, *Introduction to Topological Quantum Computation* (Cambridge University Press, Cambridge, 2012).
[7] M. F. Atiyah, Publications Mathématiques de l'IHÉS **68**, 175 (1988).
[8] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[9] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[10] M. A. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005).
[11] J. B. Kogut, Rev. Mod. Phys. **51**, 659 (1979).
[12] A. Kitaev, Annals of Physics **321**, 2 (2006).
[13] Z. Wang, *Topological Quantum Computation* (American Mathematical Soc., Providence, Rhode Island, 2010), p. 112.
[14] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[15] T. Grover, A. M. Turner and A. Vishwanath, Phys. Rev. B **84**, 195120 (2011).
[16] A. Bullivant and J. K. Pachos, Phys. Rev. B **93**, 125111 (2016).

[17] A. Lopez and E. Fradkin, *Composite Fermions: A Unified View of the Quantum Hall Regime* (World Scientific, 1998), pp. 195–253.

[18] Y. Hu, Y. Wan, and Y.-S. Wu, Phys. Rev. B **87**, 125114 (2013).

[19] R. Dijkgraaf and E. Witten, Commun. Math. Phys. **129**, 393 (1990).

[20] Y. Wan, J. Wang, and H. He, Phys. Rev. B **92**, 045101 (2015).

[21] C. Wang and M. Levin, Phys. Rev. B **91**, 165119 (2015).

[22] L. Crane and D. Yetter, Quantum Topology **3**, 120 (1993).

[23] K. Walker and Z. Wang, Front. Phys. **7**, 150 (2012).

[24] C. W. von Keyserlingk, F. J. Burnell, and S. H. Simon, Phys. Rev. B **87**, 045107 (2013).

[25] J. Haah, Phys. Rev. A **83**, 042330 (2011).

[26] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B **94**, 235157 (2016).

[27] D. J. Williamson, Phys. Rev. B **94**, 155128 (2016).

[28] J. Haah, Phys. Rev. B **89**, 075119 (2014).

[29] D. N. Yetter, J. Knot Theor. Ramif. **2**, 113 (1993).

[30] T. Porter, J. London Math. Soc. **58**, 723 (1998).

[31] J. C. Baez, A. S. Crans, and D. K. Wise, Adv. Theor. Math. Phys. **11**, 707 (2007).

[32] R. Fenn, R. Rimányi, and C. Rourke, Topology **36**, 123 (1997).

[33] Z. Kadar, P. Martin, E. Rowell, and Z. Wang, Glasg. Math. J. **59**, 359 (2017).

[34] D. Freed, Bull. Am. Math. Soc. **50**, 57 (2013).

[35] B. Bakalov and A. A. Kirillov, *Lectures on Tensor Categories and Modular Functors* (American Mathematical Society Providence, Providence, Rhode Island, 2001), Vol. 21.

[36] H. Moradi and X.-G. Wen, Phys. Rev. B **91**, 075114 (2015).

[37] S. Jiang, A. Mesaros, and Y. Ran, Phys. Rev. X **4**, 031048 (2014).

[38] J. Baez and U. Schreiber, arXiv:hep-th/0412325.

[39] J. C. Baez and A. D. Lauda, Theory Appl. Categ. **12**, 423 (2004).

[40] A. Bullivant, M. Calçada, Z. Kádár, J. F. Martins, and P. Martin, arXiv:1702.00868.

[41] H. Pfeiffer, Ann. Phys. **308**, 447 (2003).

[42] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, 2002).

[43] D. Cooper and W. P. Thurston, Topology **27**, 23 (1988).

[44] F. Girelli, H. Pfeiffer, and E. Popescu, J. Math. Phys. **49**, 032503 (2008).

[45] D. J. Williamson and Z. Wang, Ann. Phys. **377**, 311 (2016).

[46] Z. Kadar, A. Marzuoli, and M. Rasetti, Int. J. Quantum. Inform. **7**, 195 (2009).

[47] R. Koenig, G. Kuperberg, and B. W. Reichardt, Ann. Phys. **325**, 2707 (2010).

[48] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds* (Walter de Gruyter, Berlin, Boston, 1994), Vol. 18.

[49] J. Barrett and B. Westbury, Trans. Amer. Math. Soc. **348**, 3997 (1996).

[50] A. Kirillov, Jr., arXiv:1106.6033.

[51] Z. Kádár, A. Marzuoli, and M. Rasetti, Adv. Math. Phys. **2010**, 671039 (2010).

[52] O. Buerschaper and M. Aguado, Phys. Rev. B **80**, 155136 (2009).

[53] H. Ooguri and N. Sasakura, Mod. Phys. Lett. A **6**, 3591 (1991).

[54] G. Racah, *Spectroscopic and Group Theoretical Methods in Physics: Racah Memorial Volume* (Wiley Interscience Division, Amsterdam, 1968).

[55] L. Freidel and D. Louapre, Class. Quantum Grav. **21**, 5685 (2004).

[56] R. Oeckl, *Discrete Gauge Theory: From Lattices to TQFT* (World Scientific, Imperial College Press, London, 2005).

[57] J. C. Baez, Lect. Notes Phys. **543**, 25 (2000).

[58] L. Freidel and D. Louapre, Nucl. Phys. B **662**, 279 (2003).

[59] J. F. Martins and T. Porter, Theory Appl. Categ **18**, 118 (2007).

[60] P. Martin, *Potts Models and Related Problems in Statistical Mechanics* (World Scientific, Singapore, 1991).

[61] A. Kirillov, Jr. and B. Balsam, arXiv:1004.1533.

[62] R. Dijkgraaf, V. Pasquier, and P. Roche, Nucl. Phys. B, Proc. Suppl. **18**, 60 (1991).

[63] P. H. Bonderson, Non-Abelian anyons and interferometry, Ph.D. thesis, California Institute of Technology (2007).

[64] P. Bantay, Moonshine: The First Quarter Century and Beyond **372**, 1 (2005).

[65] A. Kapustin and R. Thorngren, arXiv:1309.4721.

[66] A. Kapustin and R. Thorngren, Adv. Theor. Math. Phys. **18**, 1233 (2014).

[67] R. Thorngren, arXiv:1309.1489.

[68] S. X. Cui, arXiv:1610.07628.

[69] S. X. Cui, C. Galindo, J. Y. Plavnik, and Z. Wang, Commun. Math. Phys. **348**, 1043 (2016).

[70] H. Moradi and X.-G. Wen, Phys. Rev. Lett. **115**, 036802 (2015).

[71] L. H. Kauffman and J. F. Martins, Compos. Math. **144**, 1046 (2008).

[72] J. Faria Martins, Trans. Amer. Math. Soc. **361**, 4593 (2009).

[73] J. F. Martins, J. Knot Theor. Ramif. **16**, 1181 (2007).

[74] M. Freedman, C. Nayak, K. Walker, and Z. Wang, arXiv:0806.1926.