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Topological properties of some algebraically defined subsets of $\beta\mathbb{N}$

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Abstract

Let S be a discrete semigroup and let the Stone-Čech compactification βS of S have the operation extending that of S which makes βS a right topological semigroup with S contained in its topological center. We show that the closure of the set of multiplicative idempotents in $\beta\mathbb{N}$ does not meet the set of additive idempotents in $\beta\mathbb{N}$. We also show that the following algebraically defined subsets of $\beta\mathbb{N}$ are not Borel: the set of idempotents; the smallest ideal; any semiprincipal right ideal of $\mathbb{N}^*$; the set of idempotents in any left ideal; and $\mathbb{N}^*+\mathbb{N}^*$. We extend these results to βS, where S is an infinite countable semigroup algebraically embeddable in a compact topological group.

Key words: Stone-Čech Compactification, idempotent, Borel
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1. Introduction

Let $(S,\cdot)$ be a discrete semigroup. We take the Stone-Čech compactification βS of S to be the set of ultrafilters on S with the points of S identified with the principal ultrafilters. Given $A \subseteq S$, we let $\overline{A} = \{p \in \beta S : A \in p\}$. The set

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$\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS as well as a basis for the closed sets. The operation on S extends to βS so that the function ρ_p defined by $\rho_p(x) = x \cdot p$ is continuous for each $p \in \beta S$. Furthermore, S is contained in the topological center of βS, meaning that the function λ_y defined by $\lambda_y(x) = y \cdot x$ is continuous for each $y \in S$.

So, for $p, q \in \beta S$, $pq = \lim_{s \to p} \lim_{t \to q} st$, where s and t denote elements of S. For $p, q \in \beta S$ and $A \subseteq S$, $A \in pq$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : xy \in A\}$. Given $p \in \beta S$, $B \in p$, and a sequence $\langle x_s \rangle_{s \in B}$ in a compact Hausdorff topological space, define $p\text{-}\lim_{s \in B} x_s = y$ if and only if for each neighborhood U of y, $\{s \in B : x_s \in U\} \in p$. Then if $p, q \in \beta S$, $B \in p$, and for each $s \in B$, $D_s \in q$, then $pq = p\text{-}\lim_{s \in B}(q\text{-}\lim_{t \in D_s} st)$. If $A \subseteq S$, A^* will denote $c\ell_{\beta S}(A) \setminus A$.

Every compact Hausdorff right topological semigroup T has important algebraic properties, including the fact that it has at least one idempotent. If V is a subset of T, $E(V)$ will denote the set of idempotents in V. T has a smallest two sided ideal, $K(T)$, which is the union of all of the minimal right ideals and the union of all of the minimal left ideals of T. Every right ideal of T contains a minimal right ideal, and every left ideal of T contains a minimal left ideal. The intersection of a minimal right ideal and a minimal left ideal is a group; and all the subgroups of T which arise in this way are algebraically isomorphic and are homeomorphic if they lie in the same minimal right ideal. However, these groups need not be homeomorphic in general. In fact, if S is an infinite cancellative and commutative semigroup, then by [7, Theorem 3] and [6, Lemma 6.40] the maximal groups contained in any minimal left ideal lie in $2^{\mathfrak{c}}$ isomorphism classes. For an elementary introduction to the algebraic structure of compact right topological semigroups, see [6, Part I].

We shall use $\mathbb{N}$ to denote the set of positive integers, ω to denote the set of non-negative integers, $\mathbb{Z}$ to denote the set of all integers and $\mathbb{R}$ to denote the set of real numbers. $\mathbb{H}$ will denote $\bigcap_{n \in \mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. This is a subsemigroup of $\beta\mathbb{N}$ which contains all the idempotents. In $\beta\mathbb{N}$, we shall use $+$ to denote the semigroup operation which extends addition in $\mathbb{N}$, although this operation is very far from being commutative. We shall simply use juxtaposition to denote the semigroup operation which extends multiplication in $\mathbb{N}$.

The idempotents in $K(\beta S)$ have been particularly important in combinatorics and topological dynamics. These are called minimal idempotents. If p is a minimal idempotent in βS, then $p\beta Sp$ is the group $(p\beta S) \cap (\beta Sp)$.

The relation between the two operations on $\beta\mathbb{N}$ has had important applications in combinatorics. It was shown in [4, Lemma 2.5] that $E(\beta\mathbb{N}, \cdot) \cap c\ell E(\beta\mathbb{N}, +) \neq \emptyset$ and as a consequence, given any finite coloring of $\mathbb{N}$, there is an infinite finite sum set S and an infinite finite product set P for which $S \cup P$ is monochromatic. It was not until fourteen years later that an elementary proof of this fact was found in [2]. It was shown in [5, Theorem 7.6] that $(\mathbb{N}^* + \mathbb{N}^*) \cap K(\beta\mathbb{N}, \cdot) = \emptyset$, so in particular $K(\beta\mathbb{N}, +) \cap K(\beta\mathbb{N}, \cdot) = \emptyset$. We remark that it remains an open question whether $(\mathbb{N}^* + \mathbb{N}^*) \cap (\mathbb{N}^* \cdot \mathbb{N}^*) = \emptyset$.

In [1, Theorem 5.4] it was shown that $E\big(K(\beta\mathbb{N},\cdot)\big)\cap c\ell E\big(K(\beta\mathbb{N},+)\big)\neq\emptyset$. In [8, Corollary 2.3], it was shown that $(\mathbb{N}^*+\mathbb{N}^*)\cap c\ell K(\beta\mathbb{N},\cdot)=\emptyset$, so in particular $K(\beta\mathbb{N},+)\cap c\ell K(\beta\mathbb{N},\cdot)=\emptyset$. In Section 2 of the present paper, we show that $E(\beta\mathbb{N},+)\cap c\ell E(\beta\mathbb{N},\cdot)=\emptyset$.

Anyone who has worked with $\beta\mathbb{N}$, will not be surprised to learn that some of the algebraically defined subsets of $\beta\mathbb{N}$ are not topologically simple, even though they are very simple to define algebraically. In Section 3, we prove that the following subsets of $\beta\mathbb{N}$ are not Borel: the set of idempotents; any semiprincipal right ideal of $\mathbb{N}^*$; the smallest ideal of $\beta\mathbb{N}$; the set of idempotents in any left ideal of $\beta\mathbb{N}$. In Section 4, we extend these results to infinite countable semigroups which can be algebraically embedded in compact Hausdorff topological groups.

2. $E(\beta\mathbb{N},+)$ does not meet the closure of $E(\beta\mathbb{N},\cdot)$

We begin by introducing the *uniform compactification* of $\mathbb{R}$.

Theorem 2.1. *Let $\mathbb{R}$ have the usual topology. There is a topological compactification $\mu\mathbb{R}$ of $\mathbb{R}$ such that*

(1) the operation $+$ on $\mathbb{R}$ extends to $\mu\mathbb{R}$ making $(\mu\mathbb{R},+)$ a right topological semigroup with $\mathbb{R}$ contained in its topological center and

(2) if Y is a compact topological group and $h:\mathbb{R}\to Y$ is a (uniformly) continuous homomorphism, then there is an extension $\bar{h}:\mu\mathbb{R}\to Y$ such that $\bar{h}$ is a continuous homomorphism.

Proof. [6, Theorems 21.41, 21.43, and 21.45] □

Lemma 2.2. *Let $L:\beta\mathbb{N}\to\mu\mathbb{R}$ be the continuous extension of* $\log:\mathbb{N}\to\mathbb{R}$.

(1) For all $u\in\beta\mathbb{N}$ and all $v\in\mathbb{N}^*$, $L(u+v)=L(v)$.

(2) For all $u,v\in\beta\mathbb{N}$, $L(uv)=L(u)+L(v)$.

Proof. [8, Lemma 2.1] □

We regard the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, and $\pi:\mathbb{R}\to\mathbb{T}$ will denote the canonical homomorphism.

Definition 2.3. Let $D=\{x\in\mu\mathbb{R}:$ for every continuous homomorphism $\varphi:\mu\mathbb{R}\to\mathbb{T}\,,\ \varphi(x)=0\}$.

Note that D is a compact subsemigroup of $\mu\mathbb{R}$ which contains all the idempotents of $\mu\mathbb{R}$.

Lemma 2.4. *If $s,t\in\mathbb{R}$ and $0<s<t$, then $(s+D)\cap(t+D)=\emptyset$.*

Proof. Suppose $x,y\in D$ and $s+x=t+y$. Pick $n>t$ and define $h:\mathbb{R}\to\mathbb{T}$ by $h(z)=\pi(\frac{z}{n})$. Then h is a uniformly continuous homomorphism, so by Theorem 2.1 (2) there is a continuous homomorphism $\bar{h}:\mu\mathbb{R}\to\mathbb{T}$ extending h. Then $h(s)=h(s)+\bar{h}(x)=\bar{h}(s+x)=\bar{h}(t+y)=h(t)+\bar{h}(y)=h(t)$, a contradiction. □

Definition 2.5. $C = \{q \in \mathbb{H} : qq = q\}$.

Lemma 2.6. *Let q be an idempotent in $(\beta\mathbb{N}, \cdot)$.*

(1) If $2\mathbb{N} \in q$, then $q \in \mathbb{H}$.

(2) If $2\mathbb{N} \notin q$, then either $q = 1$ or there is some $r \in \mathbb{H}$ such that $q = r + 1$.

Proof. Note that if $q \neq 1$, then $q \in \mathbb{N}^*$. Assume first that $2\mathbb{N} \in q$. Let $n \in \mathbb{N}$ and let $h : \beta\mathbb{Z} \to \mathbb{Z}_{2^n}$ denote the continuous extension of the natural homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_{2^n}$. By [6, Corollary 4.23], h is a homomorphism. Let $h(q) = m \in \{0, 1, 2, \ldots, 2^n - 1\}$ and note that m is even since $2\mathbb{N} \in q$. Then $m \equiv m^2 \pmod{2^n}$. If $m \neq 0$, we can write $m = 2^r s$, where $r, s \in \mathbb{N}$, $r < n$ and s is odd. We then have $2^r s = 2^{2r} s^2 + t2^n$ for some $t \in \mathbb{Z}$. We obtain a contradiction by dividing this equation by 2^r. So $m = 0$ and thus $q \in \mathbb{H}$.

Now assume that $2\mathbb{N} \notin q$ and $q \neq 1$. It suffices to show that for all $k \in \mathbb{N}$, $2^k\mathbb{N} + 1 \in q$, for then we may let $r = \{-1 + A : A \in q\}$. So suppose that for some $k \in \mathbb{N}$, $2^k\mathbb{N} + 1 \notin q$ and pick the least such k, noting that $k > 1$. Then since $2^{k-1}\mathbb{N} + 1 \in q$, we have that $2^k\mathbb{N} + 2^{k-1} + 1 \in q$. By [6, Theorem 5.8], we may pick x and y in $2^k\mathbb{N} + 2^{k-1} + 1$ such that $xy \in 2^k\mathbb{N} + 2^{k-1} + 1$. This is easily seen to be impossible. □

Lemma 2.7. *If $p = p + p$ and $p \in c\ell\{q \in \beta\mathbb{N} : qq = q\}$, then $p \in c\ell C$.*

Proof. Let $A \in p$. We will show that there exists $q \in \mathbb{H} \cap \overline{A}$ such that $qq = q$. Since $p = p + p$, $2\mathbb{N} \in p$. Pick $q \in \beta\mathbb{N} \cap \overline{2\mathbb{N} \cap A}$ such that $qq = q$. By Lemma 2.6 (1), $q \in \mathbb{H}$. □

Lemma 2.8. *Let $p \in \beta\mathbb{N}$. There is at most one $n \in \mathbb{N}$ such that $(\beta\mathbb{N} + p) \cap c\ell(nC) \neq \emptyset$.*

Proof. By Lemma 2.2 (2), $L[C] \subseteq D$, because $L[C]$ is contained in the set of idempotents of $\mu\mathbb{R}$. Also by Lemma 2.2 (2), if $n \in \mathbb{N}$, then $L[nC] = L(n) + L[C] \subseteq L(n) + D$ and so $L[c\ell(nC)] = c\ell L[nC] \subseteq L(n) + D$. By Lemma 2.2 (1), $L[\beta\mathbb{N} + p] = \{L(p)\}$. If $(\beta\mathbb{N} + p) \cap c\ell(nC) \neq \emptyset$, then $L(p) \in L(n) + D$. It follows from Lemma 2.4 that there is at most one value of n for which this can hold. □

Lemma 2.9. *Let A and B denote σ-compact subsets of $\beta\mathbb{N}$. If $\overline{A} \cap \overline{B} \neq \emptyset$, then $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.*

Proof. [6, Theorem 3.40] □

Theorem 2.10. *The closure of C does not meet $\mathbb{N}^* + \mathbb{N}^*$.*

Proof. Suppose that $p + q \in c\ell(C)$, where $p, q \in \mathbb{N}^*$. We observe that there is at most one value of m in $\mathbb{N}$ for which $m + q \in \mathbb{H}$. By Lemma 2.8 there is at most one value of n in $\mathbb{N}$ for which $(\beta\mathbb{N} + q) \cap c\ell(nC) \neq \emptyset$. Let $M = \{n \in \mathbb{N} : n + q \notin \mathbb{H}\}$ and let $K = \{n \in \mathbb{N} : (\beta\mathbb{N} + q) \cap c\ell(nC) = \emptyset\}$. Then $p + q \in c\ell(M + q)$ and $C \subseteq c\ell\big(\bigcup_{n \in K} c\ell(nC)\big)$ so $c\ell(C) \subseteq c\ell\big(\bigcup_{n \in K} c\ell(nC)\big)$.

It follows from Lemma 2.9 that $p' + q \in c\ell(nC)$ for some $p' \in \beta\mathbb{N}$ and some $n \in K$, or else $m + q \in c\ell\big(\bigcup_{n\in K} c\ell(nC)\big)$ for some $m \in M$. The first possibility contradicts the definition of K. The second possibility contradicts the definition of M, because $\bigcup_{n\in N} c\ell(nC) \subseteq \mathbb{H}$. □

Corollary 2.11. *There is no additve idempotent of $\beta\mathbb{N}$ in the closure of the multiplicative idempotents of $\beta\mathbb{N}$.*

Proof. Lemma 2.7 and Theorem 2.10. □

Observe that we did not prove that $(\mathbb{N}^* + \mathbb{N}^*) \cap c\ell E(\beta\mathbb{N}, \cdot) = \emptyset$.

Question 2.12. *Is $(\mathbb{N}^* + \mathbb{N}^*) \cap c\ell E(\beta\mathbb{N}, \cdot) = \emptyset$?*

Notice that by Lemma 2.6, this question is equivalent to asking whether $(\mathbb{N}^* + \mathbb{N}^*) \cap c\ell\{q \in \mathbb{H} + 1 : qq = q\} = \emptyset$.

3. Subsets of $\beta\mathbb{N}$ which are not Borel

In this section, we shall show that the following subsets of $(\beta\mathbb{N}, +)$ are not Borel: the set of idempotents in $\beta\mathbb{N}$, the smallest ideal of $\beta\mathbb{N}$; any semiprincipal right ideal of $\mathbb{N}^*$; the set of idempotents in any minimal left ideal of $\beta\mathbb{N}$.

Lemma 3.1. *Every Borel subset of $\beta\mathbb{N}$ is the union of a family of compact subsets of $\beta\mathbb{N}$ of cardinality at most $\mathfrak{c}$.*

Proof. We remind the reader that a family of subsets of a topological space contains all the Borel subsets of the space if it contains all the open sets and all the closed sets, and is closed under countable unions and countable intersections. This follows from the fact that the Borel sets can be defined inductively by starting from the family of open sets and their complements, and carrying out the process of forming countable unions and countable intersections ω_1 times.

Let $\mathcal{F}$ denote the family of subsets of $\beta\mathbb{N}$ which are the union of $\mathfrak{c}$ or fewer compact subsets of $\beta\mathbb{N}$. $\mathcal{F}$ contains the open subsets of $\beta\mathbb{N}$, because $\beta\mathbb{N}$ has a basis of $\mathfrak{c}$ clopen sets, and $\mathcal{F}$ obviously contains the closed subsets of $\beta\mathbb{N}$. It is also obvious that $\mathcal{F}$ is closed under countable unions. To see that $\mathcal{F}$ is closed under countable intersections let $\langle A_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{F}$ and for each $n \in \mathbb{N}$, pick a set $\mathcal{D}_n$ of at most $\mathfrak{c}$ compact subsets of $\beta\mathbb{N}$ such that $A_n = \bigcup \mathcal{D}_n$. Then $\bigcap_{n=1}^{\infty} A_n = \bigcup\{\bigcap_{n=1}^{\infty} F(n) : F \in \times_{n=1}^{\infty}\mathcal{D}_n\}$ and $|\times_{n=1}^{\infty}\mathcal{D}_n| \le \mathfrak{c}^{\omega} = \mathfrak{c}$. □

Definition 3.2. (a) For $n \in \mathbb{N}$ we define $\mathrm{supp}(n) \subseteq \omega$ by $n = \sum_{t\in\mathrm{supp}(n)} 2^t$.

(b) For $n \in \mathbb{N}$, $\theta(n) = \min\big(\mathrm{supp}(n)\big)$ and $\phi(n) = \max\big(\mathrm{supp}(n)\big)$.

The functions θ and ϕ extend to continuous functions from $\beta\mathbb{N}$ to $\beta\omega$ which we denote by the same symbols.

Lemma 3.3. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{H}$ on which ϕ is injective. If x is a point of accumulation of the sequence, then $x \notin \mathbb{N}^* + \mathbb{N}^*$.*

Proof. Suppose that $x = y + z$, where $y, z \in \mathbb{N}^*$. We observe that there are at most two values of n for which $\phi(x_n) \in \{\phi(z), \phi(z)+1\}$ and at most one value of k in $\mathbb{N}$ for which $k + z \in \mathbb{H}$. Put $M = \big\{n \in \mathbb{N} : \phi(x_n) \notin \{\phi(z), \phi(z)+1\}\big\}$ and $K = \{k \in \mathbb{N} : k + z \notin \mathbb{H}\}$. Since $x \in cl\{x_n : n \in M\}$ and $y + z \in cl(K + z)$, it follows from Lemma 2.9 that (i) $x' = k + z$ for some $x' \in cl\{x_n : n \in M\}$ and some $k \in K$, or else (ii) $x_n = u + z$ for some $n \in M$ and some $u \in \beta\mathbb{N}$. We shall refute both these possibilities.

If (i) holds, $k + z \in \mathbb{H}$, contradicting the definition of K.

So assume that (ii) holds. For $s \in \mathbb{N}$, pick $i_s \in \{0, 1\}$ such that

$$B_s = \{t \in \mathbb{N} : \phi(s + t) = \phi(t) + i_s\} \in z\,.$$

Pick $j \in \{0, 1\}$ such that $D = \{s \in \mathbb{N} : i_s = j\} \in u$. Since $u + z = u\text{-}\lim_{s \in D}(z\text{-}\lim_{t \in B_s} s + t)$, we have that $\phi(x_n) = \phi(u + z) = \phi(z) + j$ so that $n \notin M$, a contradiction. □

Definition 3.4. We put $P = \{2^n : n \in \mathbb{N}\}$.

Since $\phi(2^n) = \theta(2^n) = n$ for every $n \in \mathbb{N}$, $\phi[P^*] = \theta[P^*] = \mathbb{N}^*$. So $|\phi[P^*]| = |\theta[P^*]| = 2^{\mathfrak{c}}$. We observe that $P^* \subseteq \mathbb{H}$. So, for any $x \in \beta\mathbb{N}$ and any $p \in P^*$, $\phi(x + p) = \phi(p)$, and for any $y \in \mathbb{H}$, $\theta(p + y) = \theta(p)$ by [6, Lemma 6.8].

Theorem 3.5. *$\mathbb{N}^* + \mathbb{N}^*$ and $\mathbb{H} + \mathbb{H}$ are not Borel subsets of $\beta\mathbb{N}$.*

Proof. By Lemma 3.3, ϕ takes on only finitely many values on any compact subset of $\mathbb{N}^* + \mathbb{N}^*$. As we just observed, $|\phi[P^*]| = 2^{\mathfrak{c}}$ and so given any $p \in \mathbb{H}$, $|\phi[p + P^*]| = 2^{\mathfrak{c}}$ and therefore ϕ takes on $2^{\mathfrak{c}}$ values on $\mathbb{H} + \mathbb{H}$. Thus Theorem 3.1 applies. □

Theorem 3.6. *The set of idempotents in $\beta\mathbb{N}$ is not a Borel subset of $\beta\mathbb{N}$.*

Proof. Let E be the set of idempotents in $\beta\mathbb{N}$. By Lemma 3.3 any subset of E on which ϕ assumes infinitely many values, has limit points in $\beta\mathbb{N} \setminus E$. So, for every compact subset C of E, $\phi[C]$ is finite. If E were Borel, it would follow from Lemma 3.1 that $|\phi[E]| \le \mathfrak{c}$. However, for every $p \in P^*$, there exists an idempotent q in the left ideal $\beta\mathbb{N} + p$ of $\beta\mathbb{N}$. Since $\phi(q) = \phi(p)$, $|\phi[E]| = 2^{\mathfrak{c}}$. □

The proofs of the next three theorems are similar.

Theorem 3.7. *$K(\beta\mathbb{N})$ is not a Borel subset of $\beta\mathbb{N}$.*

Proof. It is sufficient to show that $K(\beta\mathbb{N}) \cap \mathbb{H}$ is not a Borel subset of $\beta\mathbb{N}$. We assume the contrary. Suppose that C is a compact subset of $K(\beta\mathbb{N}) \cap \mathbb{H}$. It follows from Lemma 3.3 that $\phi[C]$ is finite, because any subset of $\mathbb{H}$ on which ϕ assumes infinitely many values has limit points which are in $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ and are therefore in $\mathbb{N}^* \setminus K(\beta\mathbb{N})$. So, by Lemma 3.1, ϕ assumes at most $\mathfrak{c}$ distinct values on $K(\beta\mathbb{N}) \cap \mathbb{H}$.

If q is a minimal idempotent in $\beta\mathbb{N}$ and $p \in P^*$, then $\phi(q + p) = \phi(p)$. Since $q + p \in K(\beta\mathbb{N}) \cap \mathbb{H}$, it follows that $\phi[K(\beta\mathbb{N}) \cap \mathbb{H}] = 2^{\mathfrak{c}}$. This contradiction establishes that $K(\beta\mathbb{N}) \cap \mathbb{H}$ is not Borel. □

The next theorem is a corollary to Theorem 4.4, but has a simpler proof, so we present it separately.

Theorem 3.8. *Let $q \in \mathbb{N}^*$ and let $R = q + \mathbb{N}^*$. Then R is not a Borel subset of $\beta\mathbb{N}$.*

Proof. We shall show that $R \cap \mathbb{H}$ is not Borel. We assume the contrary. It follows from Lemma 3.3 that $\phi[C]$ is finite if C is a compact subset of $R \cap \mathbb{H}$. So, by Lemma 3.1, $|\phi[R \cap \mathbb{H}]| \leq \mathfrak{c}$. Since R is a right ideal in $\mathbb{N}^*$, R contains an idempotent u. For every $p \in P^*$, $u + p \in R \cap \mathbb{H}$ and $\phi(u + p) = \phi(p)$. We have observed that $|\phi[P^*]| = 2^{\mathfrak{c}}$, and so $\phi[R \cap \mathbb{H}] = 2^{\mathfrak{c}}$. This contradiction establishes that $R \cap \mathbb{H}$ is not Borel. □

Notice that if $q \in \mathbb{N}^*$, then $q + \mathbb{N}^*$ and $q + \beta\mathbb{N}$ differ by a countable set, so it is also true that $q + \beta\mathbb{N}$ is not Borel.

Theorem 3.9. *Let L be a left ideal of $\beta\mathbb{N}$. Then $E(L)$ is not a Borel subset of $\beta\mathbb{N}$.*

Proof. Let A be a subset of $E(L)$ and assume that $\theta[A]$ is infinite. Choose a sequence $\langle q_n \rangle_{n=1}^{\infty}$ in A such that $\theta(q_n) \neq \theta(q_m)$ when $m \neq n$ and $\{\theta(q_n) : n \in \mathbb{N}\}$ is discrete. By [6, Theorem 6.15.1], if p is any cluster point of $\langle q_n \rangle_{n=1}^{\infty}$, then $p + p \notin c\ell E(\beta\mathbb{N})$ and so $p \in L \setminus E(L)$. Hence, if C is a compact subset of $E(L)$, $\theta[C]$ is finite. So, if $E(L)$ were Borel, it would follow from Lemma 3.1 that $|\theta[E(L)]| \leq \mathfrak{c}$. However, for every $p \in P^*$, there exists an idempotent q in $L \cap (p + \beta\mathbb{N})$. Then $q = p + x$ for some $x \in \beta\mathbb{N}$. This equation implies that $x \in \mathbb{H}$ and hence that $\theta(q) = \theta(p)$. So $\theta[E(L)] = 2^{\mathfrak{c}}$. □

Question 3.10. *Are any of the maximal groups in $K(\beta\mathbb{N})$ Borel?*

We conjecture that the answer to this question is "no".

4. Subsets of βS which are not Borel

In this section, we extend the results of Section 3 to countable semigroups which can be algebraically embedded in compact Hausdorff topological groups. We observe that this includes all countable commutative semigroups, as well as free semigroups and free groups with countably many generators.

Throughout this section, S will denote a countably infinite semigroup which can be algebraically embedded in a compact Hausdorff topological group. By Lemma 4.1 below, we may suppose that S is contained in a compact metrizable topological group C with identity e. We assume that G is a countable subgroup of C which contains S. We regard S and G as subsets of C_d, the group C with the discrete topology, and we regard βS and βG as subsets of βC_d. We use $h : \beta C_d \to C$ to denote the continuous homomorphism which extends the identity map from C_d onto C. This maps an ultrafilter on C to the point of C to which it converges.

Let $V = G^* \cap h^{-1}[\{e\}]$. By [6, Theorem 7.28] there exists a function $\psi : \omega \xrightarrow[\text{onto}]{\text{1-1}} G$, whose continuous extension to a homeomorphism from $\beta\omega$ onto βG maps $\mathbb{H}$ isomorphically onto V. We shall use the same symbol ψ to denote this continuous extension. Since e has a countable base of neighbourhoods in C, we note that V is a compact G_δ-subsemigroup of G^*, which contains all the idempotents of G^*.

The following lemma is well known. We thank Jan van Mill for providing the simple argument presented here.

Lemma 4.1. *Let G be a countable group which is embeddable in a compact Hausdorff topological group. Then G is embeddable in a compact metrizable topological group.*

Proof. Assume that G is contained in a compact Hausdorff topological group H with identity e. By [3, Theorem 8.7], there is a closed normal G_δ sugroup N of H such that $N \cap G = \{e\}$ and H/N is metrizable. □

Lemma 4.2. *Let M be a nonempty compact G_δ subset of $\mathbb{H}$. Then $|\phi[M]| = |\theta[M]| = 2^{\mathfrak{c}}$.*

Proof. Note that if U is open in $\beta\mathbb{N}$, then $U = \bigcup\{\overline{A} : A \subseteq U \cap \mathbb{N}\}$ and for $A \subseteq \mathbb{N}$, $\phi[\overline{A}] = \overline{\phi[A]}$, so $\phi : \beta\mathbb{N} \to \beta\omega$ is an open map. Since $\phi[M] \subseteq \mathbb{N}^*$, $\phi[M]$ contains a nonempty G_δ subset of $\mathbb{N}^*$. Therefore by [6, Theorem 3.36], $\phi[M]$ has nonempty interior in $\mathbb{N}^*$ and thus $|\phi[M]| = 2^{\mathfrak{c}}$.

By the same argument, $|\theta[M]| = 2^{\mathfrak{c}}$. □

Theorem 4.3. *$E(S^*)$ is not Borel in βS, $K(\beta S)$ is not Borel in βS, and if L is a left ideal of βS, then $E(L)$ is not Borel in βS.*

Proof. Let $M = \psi^{-1}[V \cap S^*]$, a compact G_δ subsemigroup of $\mathbb{H}$. By Lemma 3.3, ϕ can only assume a finite number of values on any compact subset of $\mathbb{H} + \mathbb{H}$. By Lemma 4.2, $|\phi[M]| = 2^{\mathfrak{c}}$. For every $x \in M$, there exists an idempotent p in the left ideal $M + x$ of M. Since $\phi(p) = \phi(x)$ by [6, Lemma 6.8], it follows that $|\phi[E(M)]| = 2^{\mathfrak{c}}$. So, by Lemma 3.1, $E(M)$ is not a Borel subset of $\beta\omega$. Since ψ is a homeomorhism, $\psi[E(M)] = E(S^*)$ is not a Borel subset of βG. This implies that $E(S^*)$ is not a Borel subset of βS, because βS is a clopen subset of βG.

Similarly, for every $x \in M$, the left ideal $M + x$ of M meets $K(M)$. So $\phi[K(M)] = 2^{\mathfrak{c}}$. It follows from Lemmas 3.3 and 3.1 that $K(M)$ is not a Borel subset of $\beta\omega$, and hence that $\psi[K(M)]$ is not a Borel subset of βG. Now $\psi[K(M)] = K(V \cap S^*)$. Since $V \cap S^*$ contains all the idempotents of S^*, $V \cap S^*$ meets $K(S^*)$ and so, by [6, Theorem 1.65], $K(V \cap S^*) = K(\beta S) \cap V$. So $K(\beta S)$ is not a Borel subset of βG, and therefore it is not a Borel subset of βS.

Now let L be a left ideal in S^*. It follows from [6, Theorem 6.15.1] that θ can assume only a finite number of values on any compact subset of $\psi^{-1}[E(L)] = E(\psi^{-1}[L])$. We observe that $\psi^{-1}[L \cap V]$ is a left ideal of M. If $x \in M$, there is an idempotent p in the intersection of $\psi^{-1}[L \cap V]$ and the right ideal $x + M$ of M. Then $p = x + y$ for some $y \in M$ so $\theta(p) = \theta(x)$ by [6, Lemma 6.8]. By

Lemma 4.2, $|\theta[M]| = 2^{\mathfrak{c}}$ and so $|\psi^{-1}[E(L)]| = 2^{\mathfrak{c}}$. It follows from Lemma 3.1 that $\psi^{-1}[E(L)]$ is not Borel in $\beta\omega$, and hence that $E(L)$ is not Borel in βG. So $E(L)$ is not Borel in βS. □

Theorem 4.4. *If $p \in S^*$, pS^* is not a Borel subset of βS.*

Proof. Let $c = h(p)$. We may assume that $c \in G$.

Let $q = c^{-1}p \in \beta G$. Then $q \in V$ because h is a homomorphism. Let $x = \psi^{-1}(q) \in \mathbb{H}$, let $W = S^* \cap V$ and let $M = \psi^{-1}[W]$. Since W is a compact G_δ subsemigroup of V, M is a compact G_δ subsemigroup of $\mathbb{H}$.

Note that $qS^* \cap V = qW$. (If $r \in S^*$ and $qr \in V$, then $h(r) = e$, so $r \in W$. The other inclusion is immmediate.) Now $\psi^{-1}[qW] = x + M$. By [6, Lemma 6.8], $\phi[x + M] = \phi[M]$ and so by Lemma 4.2, $|\phi[x + M]| = 2^{\mathfrak{c}}$. By Lemma 3.3, ϕ takes only finitely many values on any compact subset of $\mathbb{H} + \mathbb{H}$. So by Lemma 3.1, $x + M$ is not Borel in $\mathbb{H}$ and hence it is not Borel in $\beta\omega$. Therefore $qS^* \cap V = qW = \psi[x + M]$ is not Borel in βG, and so qS^* is not Borel in βG. Since λ_c is a homeomorphism of βG, $cqS^* = pS^*$ is not Borel in βG. This implies that pS^* is not Borel in βS, because βS is a clopen subset of βG. □

References

[1] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.

[2] V. Bergelson and N. Hindman, *Additive and multiplicative Ramsey Theorems in $\mathbb{N}$ – some elementary results*, Comb. Prob. and Comp. **2** (1993), 221-241.

[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis, I*, Springer-Verlag, Berlin, 1963.

[4] N. Hindman, *Partitions and sums and products of integers*, Trans. Amer. Math. Soc. **247** (1979), 227-245.

[5] N. Hindman, *Sums equal to products in $\beta\mathbb{N}$*, Semigroup Forum **21** (1980), 221-255.

[6] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.

[7] N. Hindman and D. Strauss, *Algebraic and topological equivalences in the Stone-Čech compactification of a discrete semigroup*, Topology and its Applications **66** (1995), 185-198.

[8] D. Strauss, *The smallest ideals of $\beta\mathbb{N}$ under addition and multiplication*, Topology and its Applications **149** (2005), 289-292.