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# Skyrmions, Rational Maps & Scaling Identities

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## Abstract

Starting from approximate Skymion solutions obtained using the rational map ansatz, improved approximate Skymions are constructed using scaling arguments. Although the energy improvement is small, the change of shape clarifies whether the true Skymions are more oblate or prolate.

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# 1 Skyrmions and Rational Maps

The Skyrme model [1] is an effective field theory used to describe nuclei. For static fields, the classical solutions of the Skyrme model are stationary points (either minima or saddle points) of the energy functional

$$E = \frac{1}{12\pi^2} \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \text{tr} (R_i^2) - \frac{1}{16} \text{tr} ([R_i, R_j]^2) + m^2 \text{tr}(\mathbb{I} - U) \right\} d^3\mathbf{x}, \quad (1)$$

where  $\mathbf{x} = (x_1, x_2, x_3)$  are the Cartesian coordinates, and  $U(\mathbf{x})$  is an  $SU(2)$ -valued scalar field, known as the Skyrme field, which describes pions nonlinearly. The gradient of the Skyrme field is captured by the current components  $R_i = \partial_i U U^{-1}$  ( $i = 1, 2, 3$ ). In (1), the energy and length units have been scaled away, leaving only the pion mass parameter  $m$ , which is the physical pion mass in scaled units. [In many past studies the pion mass term  $m^2 \text{tr}(\mathbb{I} - U)$  was omitted.] In this study, when we consider massive pions, we assume that  $m = 1$ , since only for  $m$  at or close to this value can the proton and delta masses be reproduced (for more details, see [2] and References therein). Also, when  $m = 1$ , nuclei are represented by Skyrmions that have a fairly uniform density, rather than a hollow, roughly spherical structure [3, 4].

Finiteness of the energy requires that  $U$  approaches the identity matrix  $\mathbb{I}$  at spatial infinity (and this boundary condition is also imposed if  $m = 0$ ). Hence,  $\mathbb{R}^3$  is topologically compactified to a large 3-sphere.  $U$  is then a mapping from  $S^3 \mapsto SU(2)$ , and is classified by the integer-valued degree (winding number)

$$B = -\frac{1}{4\pi^2} \int_{\mathbb{R}^3} \text{tr} (R_1 R_2 R_3) d^3\mathbf{x}, \quad (2)$$

which is a topological invariant. In particular,  $B$  classifies the solitonic sectors of the model and in applications to nuclei,  $B$  is identified with the baryon number of the Skyrme field configuration. A Skyrmion is a static soliton of minimal energy for given  $B$ , although sometimes, more loosely, a local minimum or saddle point of the energy functional is also called a Skyrmion.

In [5], minimal energy Skyrmions with massless pions were approximated by an ansatz involving a rational map  $R$  between Riemann spheres. Here, a point in  $\mathbb{R}^3$  is labelled by

coordinates  $(r, z)$  where  $r$  is the radial distance from the origin and  $z = \tan(\theta/2) e^{i\varphi}$  specifies the direction from the origin ( $\theta, \varphi$  are the usual spherical polar coordinates). Let  $R$  be a degree  $N$  rational map of the form  $R(z) = \frac{p(z)}{q(z)}$ , where  $p$  and  $q$  are polynomials in  $z$  with no common factors, such that  $\max[\deg(p), \deg(q)] = N$ . Then the ansatz for the Skyrme field is

$$U(r, z) = \exp \left[ \frac{if(r)}{1 + |R(z)|^2} \begin{pmatrix} 1 - |R(z)|^2 & 2\overline{R(z)} \\ 2R(z) & |R(z)|^2 - 1 \end{pmatrix} \right], \quad (3)$$

where  $f(r)$  is a real profile function satisfying the boundary conditions  $f(0) = k\pi$  (for some integer  $k$ ) and  $f(\infty) = 0$ . It is straightforward to verify that the baryon number of this field is  $B = kN$ . In the remainder of this paper we only consider  $k = 1$ , thus  $B = N$ . An attractive feature of the rational map ansatz (3) is that it leads to a simple energy expression which is minimized by first minimizing with respect to the parameters in the rational map  $R(z)$ , and then determining the profile function  $f(r)$  by solving an ordinary differential equation. This approach gives close approximations to true Skyrmions. Clearly, this procedure encounters difficulties when there are two or more Skyrmion solutions, either saddle points or genuine minima, with very similar energies. The ansatz might lead to the wrong energy ordering.

For massless pions the minimal energy Skyrmions up to  $B = 22$  are all well-approximated by the rational map ansatz [6]. However, the inclusion of the pion mass term in the energy functional makes the Skyrme model more realistic in describing real nuclei. The solutions change dramatically when  $m = 1$ , especially for large baryon number ( $B \geq 8$ ). Some Skyrmions are known to be prolate (cigarlike) or oblate (pancakelike), for example, those with baryon numbers  $B = 8$  and  $B = 12$ , respectively [4]. Spherical shell-like configurations given by the rational map ansatz are still useful starting points in the search for true solutions, but they are unstable [3], because their hollow interiors have substantial potential energy when  $m = 1$ , and the fields relax to more compact structures, some being rather flat. This way the Skyrmion reduces its volume to surface area ratio and thus reduces its energy. Moreover, the Skyrmions with massive pions are smaller in size and exponentially localized, compared to Skyrmions with massless pions which are algebraically localized. This can be seen in Figure 1.

## 2 Independent Length Rescalings

Recently, Manton [7] studied the effect of independent length rescalings in the three Cartesian directions in the Skyrme model. The simple Derrick scaling identity [8] and further novel identities were derived, relating contributions to the total energy of a Skyrmion. For an exact Skyrmion solution, the identities are satisfied exactly.

In what follows we will study the effect of scaling manipulations on the energy functional for various Skyrme field configurations which are not exact solutions, for both massless and massive pions. In particular, for an approximate Skyrmion described by the rational map ansatz, we can lower the energy by a rescaling of the form

$$x_1 \rightarrow \lambda_1 x_1, \quad x_2 \rightarrow \lambda_2 x_2, \quad x_3 \rightarrow \lambda_3 x_3, \quad (4)$$

with the parameters  $\lambda_i$  not all the same. For a general Skyrme field,  $U(x_1, x_2, x_3)$  is replaced under such a rescaling by  $\tilde{U}(x_1, x_2, x_3) = U(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$  and the energy  $\tilde{E}$  of the rescaled field  $\tilde{U}(\mathbf{x})$  is the modified version of the original energy (1),

$$\begin{aligned} \tilde{E} = \frac{1}{12\pi^2} \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \frac{\lambda_1}{\lambda_2 \lambda_3} \text{tr}(R_1^2) - \frac{1}{2} \frac{\lambda_2}{\lambda_3 \lambda_1} \text{tr}(R_2^2) - \frac{1}{2} \frac{\lambda_3}{\lambda_1 \lambda_2} \text{tr}(R_3^2) - \frac{1}{8} \frac{\lambda_1 \lambda_2}{\lambda_3} \text{tr}([R_1, R_2]^2) \right. \\ \left. - \frac{1}{8} \frac{\lambda_2 \lambda_3}{\lambda_1} \text{tr}([R_2, R_3]^2) - \frac{1}{8} \frac{\lambda_3 \lambda_1}{\lambda_2} \text{tr}([R_3, R_1]^2) + \frac{m^2}{\lambda_1 \lambda_2 \lambda_3} \text{tr}(\mathbb{I} - U) \right\} d^3 \mathbf{x}, \quad (5) \end{aligned}$$

where the current components  $R_i$  are evaluated for the original field  $U(\mathbf{x})$ .

For an exact Skyrmion solution  $U(\mathbf{x})$ ,  $\tilde{E}$  is stationary with respect to  $\lambda_i$  at  $\lambda_i = 1$  (since the energy is stationary with respect to any smooth change of the field that preserves the boundary condition). As in [7], we may restrict to special rescalings which together span all possibilities. In particular, a rescaling in the  $(x_2, x_3)$  plane with no rescaling of  $x_1$  can be achieved by setting  $\lambda_2 = \lambda_3 = \lambda$  and  $\lambda_1 = 1$  in (5). Then the derivative of  $\tilde{E}$  with respect to  $\lambda$  vanishes at  $\lambda = 1$ . This gives identity (6). Similarly, by permutation, the following identities are obtained:

$$\mathcal{I}_1 : \quad \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \text{tr}(R_1^2) + \frac{1}{8} \text{tr}([R_2, R_3]^2) + m^2 \text{tr}(\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0, \quad (6)$$

$$\mathcal{I}_2 : \quad \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \text{tr}(R_2^2) + \frac{1}{8} \text{tr}([R_3, R_1]^2) + m^2 \text{tr}(\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0, \quad (7)$$

$$\mathcal{I}_3 : \quad \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \text{tr}(R_3^2) + \frac{1}{8} \text{tr}([R_1, R_2]^2) + m^2 \text{tr}(\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0. \quad (8)$$

We refer to the left hand sides of these identities as the scaling integrals  $\{\mathcal{I}_i : i = 1, 2, 3\}$ . The sum of the above three identities is Derrick's identity for Skyrmions, which is also obtained by considering the uniform rescaling  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ .

In [6], it has been shown that for  $m = 0$  and  $B \geq 7$ , there are increasingly many Skyrmions, with different symmetries and different shapes, whose energies are very close to the minimal value. The additional configurations are local minima or saddle points of the Skyrme energy, whose energies are difficult to distinguish numerically. In particular for  $B = 9$  there are two Skyrmion possibilities with  $D_{4d}$  and  $T_d$  symmetries; the first is probably the global minimum and the second a saddle point. For  $B = 10$  there are at least four solutions close to minimal energy, but for  $B = 11$  the minimal energy Skyrmion appears isolated. It can be difficult to ensure numerically that the fields have fully relaxed to a solution. One good way to test if they have would be to check whether the scaling identities (6)-(8) are satisfied.

The rational map ansatz gives good *approximations* to exact Skyrmion solutions, which generally do not satisfy all these scaling identities. An optimal rescaling should deform the rational map configurations closer to the exact Skyrmions. Moreover, after rescaling, the scaling identities will all be satisfied. The argument is as follows. We start with the approximate solution and evaluate the seven contributions to the energy that occur in (1), and with modified coefficients in (5). We then find the parameters  $\lambda_i$  that minimize (5). This tells us how to rescale the initial field so as to obtain an improved approximate solution. The rescaled field satisfies the identities (6)-(8), as it has minimal energy with respect to further rescalings. An alternative formulation of this, more convenient computationally, is to say that after rescaling, the identities

$$\tilde{\mathcal{I}}_1 : \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \frac{\lambda_1}{\lambda_2 \lambda_3} \text{tr} (R_1^2) + \frac{1}{8} \frac{\lambda_2 \lambda_3}{\lambda_1} \text{tr} ([R_2, R_3]^2) + \frac{m^2}{\lambda_1 \lambda_2 \lambda_3} \text{tr} (\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0, \quad (9)$$

$$\tilde{\mathcal{I}}_2 : \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \frac{\lambda_2}{\lambda_3 \lambda_1} \text{tr} (R_2^2) + \frac{1}{8} \frac{\lambda_3 \lambda_1}{\lambda_2} \text{tr} ([R_3, R_1]^2) + \frac{m^2}{\lambda_1 \lambda_2 \lambda_3} \text{tr} (\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0, \quad (10)$$

$$\tilde{\mathcal{I}}_3 : \int_{\mathbb{R}^3} \left\{ -\frac{1}{2} \frac{\lambda_3}{\lambda_1 \lambda_2} \text{tr} (R_3^2) + \frac{1}{8} \frac{\lambda_1 \lambda_2}{\lambda_3} \text{tr} ([R_1, R_2]^2) + \frac{m^2}{\lambda_1 \lambda_2 \lambda_3} \text{tr} (\mathbb{I} - U) \right\} d^3 \mathbf{x} = 0 \quad (11)$$

are satisfied, where  $\lambda_1, \lambda_2, \lambda_3$  are the parameters we find, and  $U, R_1, R_2$  and  $R_3$  are the original field and current components before rescaling.

The energy is lowered by this rescaling, so one gets closer to an exact solution. More interesting, perhaps, than the small reduction of energy is the small change in shape. We learn from the calculation whether the true Skyrmion is more prolate or oblate than the approximate Skyrmion (which itself is rather round), or triaxial.

Note that in all cases, the product of the rescaling parameters is very close to unity, i.e.  $\lambda_1\lambda_2\lambda_3 = 1$ , if the starting point is the optimised rational map ansatz with the profile function worked out numerically. This is because the starting point satisfies Derrick's identity.

After rescaling a Skyrme field, the baryon density becomes

$$\tilde{B}(\mathbf{x}) = -\frac{\lambda_1\lambda_2\lambda_3}{4\pi^2} \text{tr}(R_1R_2R_3) , \quad (12)$$

where the right hand side denotes  $\lambda_1\lambda_2\lambda_3$  times the original baryon density evaluated at  $\tilde{\mathbf{x}} = (\lambda_1x_1, \lambda_2x_2, \lambda_3x_3)$ . Its integral is of course unchanged. In particular, if  $\lambda_1\lambda_2\lambda_3 = 1$ , the baryon density transforms as a scalar quantity, so that the new density at  $\mathbf{x}$  is the old density at  $\tilde{\mathbf{x}}$ .

In order to make some practical use of the scaling identities (6)-(8) it would be best to work with Skyrmions that are known to be, or are expected to be, very far from spherically symmetric. Examples are the Skyrmions with  $m = 1$  and baryon number a multiple of four, composed of  $B = 4$  subunits [4] as in the  $\alpha$ -particle model of nuclei. These Skyrmions sometimes look like part of the infinite Skyrme crystal.

We will calculate here the optimal rescaling of approximate Skyrmions given by the rational map ansatz, for  $B = 6, 9, 10$  and  $11$ , and for both  $m = 0$  and  $m = 1$ . In some of these cases the exact solutions are known, so we do not learn much new. For  $B = 9$  and  $B = 11$  the exact solutions for  $m = 1$  are not known, so here we get some insight into the shapes of the true Skyrmions. Since the rescalings are all quite close to unity (within a few percent), it is important that the calculations of the parameters  $\lambda_i$  are not dominated by numerical errors in the energy integrals. In fact, the numerical errors in the rational map approach are smaller than 1%.

We now describe our procedure for finding these configurations, and rescaling them. Initially, a rational map is introduced with a given topological charge  $B$  and assumed symmetry. Then, the energy is minimized (using a simulated annealing process) with respect to the pa-

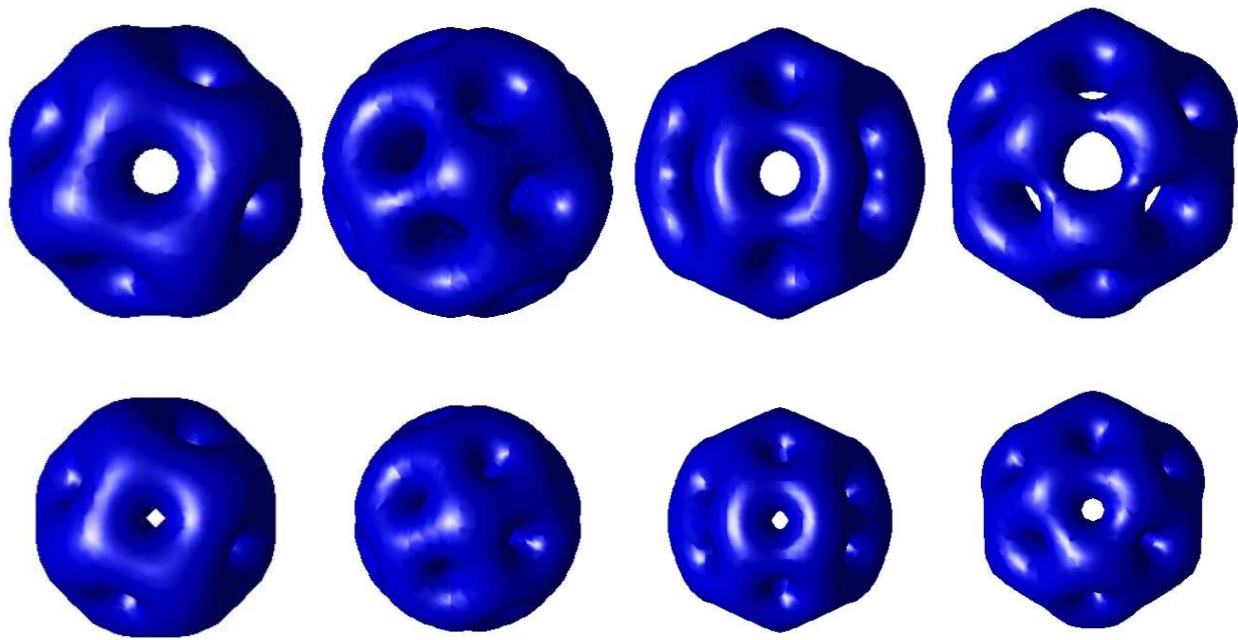


Figure 1: Baryon density isosurfaces of the rescaled (approximate) Skyrmions for  $B = 6, 9, 10$  and  $11$  Skyrmions with  $m = 0$  (first row) and  $m = 1$  (second row). Each corresponds to a value of  $\tilde{\mathcal{B}} = 0.035$ .



rameters appearing in the rational map. In practice, this means minimizing an integral  $\mathcal{I}$ , given in [5], which depends only on the rational map  $R$ . Using this familiar approach, we have re-derived the results obtained in [5], [6] and [9]. The rational maps we obtain are the following:

$$B = 6$$

The rational map has  $D_{4d}$  symmetry, with the  $D_4$  generated by  $z \mapsto iz$  and  $z \mapsto 1/z$ , and has the functional form

$$R(z) = \frac{z^4 + i\alpha}{z^2 (i\alpha z^4 + 1)}. \quad (13)$$

The energy is minimized when  $\alpha = 0.15853$ . The reflection symmetry arises because the parameter  $\alpha$  is real.

$$B = 9$$

The energy minimizing rational map in this case has  $D_{4d}$  symmetry and has the form

$$R(z) = \frac{z(\alpha + i\beta z^4 + z^8)}{1 + i\beta z^4 + \alpha z^8}, \quad (14)$$

where  $\alpha = -3.37764$  and  $\beta = 11.20311$ .

$$B = 10$$

The minimal energy Skyrmion for  $m = 1$  is found to have  $D_{2h}$  symmetry [3]. This symmetry is consistent with the  $\alpha$ -particle model, where the field configuration consists schematically of a pair of cubic  $B = 4$  Skyrmions separated by two  $B = 1$  Skyrmions. The optimal rational map with this symmetry is of the form

$$R(z) = \frac{\alpha + \beta z^2 + \gamma z^4 + \delta z^6 + \epsilon z^8 + z^{10}}{1 + \epsilon z^2 + \delta z^4 + \gamma z^6 + \beta z^8 + \alpha z^{10}}, \quad (15)$$

with  $\alpha = 0.2772$ ,  $\beta = -9.3594$ ,  $\gamma = 14.81$ ,  $\delta = 4.977$  and  $\epsilon = 3.015$ .

$$B = 11$$

The rational map here is  $D_{3h}$ -symmetric, and of the form

$$R(z) = \frac{z^9 + \alpha z^6 + \beta z^3 + \gamma}{z^2 (\gamma z^9 + \beta z^6 + \alpha z^3 + 1)} \quad (16)$$

where  $\alpha = -2.4719$ ,  $\beta = -0.8364$  and  $\gamma = -0.1264$ .

For each of these optimised rational maps, the profile function  $f(r)$  is obtained by solving the radial equation

$$f'' \left( 1 + \frac{2B \sin^2 f}{r^2} \right) + \frac{2f'}{r} + \frac{\sin 2f}{r^2} \left( B (f'^2 - 1) - \frac{\mathcal{I} \sin^2 f}{r^2} \right) - m^2 \sin f = 0, \quad (17)$$

using a *shooting method*.  $B$  is the baryon number and  $\mathcal{I}$  is the integral mentioned above, evaluated on the optimised rational map. To avoid singularities at the origin, the so-called local analysis (see, for details, Reference [10]) has been applied. Near  $r = 0$ , the profile function can be approximated by the Frobenius series  $f(r) = \pi - ar^\sigma$  where the indicial exponent  $\sigma$  is determined by taking the positive root of the quadratic equation  $\sigma^2 + \sigma - 2B = 0$ , and  $a$  is the shooting parameter. The other boundary condition is  $f(\infty) = 0$ .

The Skyrme field is then transformed back from Riemann to spherical coordinates, i.e.  $(r, z, \bar{z}) \rightarrow (r, \theta, \varphi)$ , by inverting the equation  $z = \tan(\theta/2) e^{i\varphi}$ , and the current components  $R_i$  are calculated. Numerical integrations, for evaluating the contributions to the energies  $E$  and  $\tilde{E}$ , and to the scaling integrals  $\mathcal{I}_i$  and  $\tilde{\mathcal{I}}_i$ , are performed by the *Gauss-Kronrod* method.

Finally the values of the rescaling parameters  $\lambda_i$  are obtained by minimizing the energy (5) using (once more) a simulated annealing process. They are presented in Tables 1 and 2. The values are slightly further from 1 for  $m = 1$  than for  $m = 0$ .

$m = 0$	$B = 6$	$B = 9$	$B = 10$	$B = 11$
$\lambda_1$	0.98954	1.01272	1.02050	1.00441
$\lambda_2$	0.98954	1.01272	0.98811	1.00441
$\lambda_3$	1.02118	0.97524	0.99149	0.99103

Table 1: Values of the scaling parameters  $\lambda_i$  for Skyrmions with massless pions.

The energies of the approximate Skyrmions, before and after rescaling, are presented (up to five decimal places) in Table 3, while the values of the scaling integrals are presented in Tables 4 and 5 and prove that after rescaling, the scaling identities are almost exactly satisfied. This is a check on our numerics. Note also that  $\lambda_1 \lambda_2 \lambda_3$  is very close to 1, as anticipated. The changes in the energies are rather small, and much smaller than the 1%

$m = 1$	$B = 6$	$B = 9$	$B = 10$	$B = 11$
$\lambda_1$	0.98921	1.01427	1.02224	1.00455
$\lambda_2$	0.98921	1.01427	0.98785	1.00455
$\lambda_3$	1.02226	0.97268	0.99052	0.99095

Table 2: Values of the scaling parameters  $\lambda_i$  for Skyrmions with massive ( $m = 1$ ) pions.

$B$	$G$	$E(m = 0)/B$	$\tilde{E}(m = 0)/B$	$E(m = 1)/B$	$\tilde{E}(m = 1)/B$
6	$D_{4d}$	1.13726	1.13675	1.33151	1.33092
9	$D_{4d}$	1.11681	1.11610	1.31788	1.31696
10	$D_{2h}$	1.11218	1.11172	1.31561	1.31504
11	$D_{3h}$	1.10976	1.10968	1.31632	1.31622

Table 3: Values of energy per baryon before rescaling (1) and after rescaling (5), for  $m = 0$  and  $m = 1$ .  $G$  is the symmetry group of the Skyrmion.

$m = 0$	$B = 6$	$\tilde{B} = 6$	$B = 9$	$\tilde{B} = 9$	$B = 10$	$\tilde{B} = 10$	$B = 11$	$\tilde{B} = 11$
$\mathcal{I}_1$	0.00793	$-1.33 \cdot 10^{-7}$	-0.00934	$1.59 \cdot 10^{-9}$	0.00882	$1.29 \cdot 10^{-9}$	-0.00332	$5.11 \cdot 10^{-8}$
$\mathcal{I}_2$	0.00793	$-1.05 \cdot 10^{-7}$	-0.00934	$-3.90 \cdot 10^{-12}$	-0.01509	$3.74 \cdot 10^{-9}$	-0.00332	$-1.11 \cdot 10^{-10}$
$\mathcal{I}_3$	-0.01593	$1.21 \cdot 10^{-9}$	0.01866	$1.04 \cdot 10^{-9}$	0.00624	$8.44 \cdot 10^{-10}$	0.00662	$-3.27 \cdot 10^{-11}$

Table 4: Values of the scaling integrals (6)-(8) before rescaling and after rescaling, for  $m = 0$ .

$m = 1$	$B = 6$	$\tilde{B} = 6$	$B = 9$	$\tilde{B} = 9$	$B = 10$	$\tilde{B} = 10$	$B = 11$	$\tilde{B} = 11$
$\mathcal{I}_1$	0.00882	$-8.15 \cdot 10^{-7}$	-0.01094	$-6.73 \cdot 10^{-9}$	0.00975	$-2.61 \cdot 10^{-9}$	-0.00357	$6.74 \cdot 10^{-9}$
$\mathcal{I}_2$	0.00882	$-7.80 \cdot 10^{-7}$	-0.01094	$-1.45 \cdot 10^{-9}$	-0.01734	$7.01 \cdot 10^{-10}$	-0.00357	$-2.06 \cdot 10^{-8}$
$\mathcal{I}_3$	-0.01765	$-2.27 \cdot 10^{-9}$	0.02188	$-9.31 \cdot 10^{-10}$	0.00759	$4.16 \cdot 10^{-10}$	0.00715	$-6.14 \cdot 10^{-9}$

Table 5: Values of the scaling integrals (6)-(8) before rescaling and after rescaling, for  $m = 1$ .

- 2% differences in energy between the approximate Skyrmions before rescaling and the exact Skyrmion solutions. That implies that rescaling accounts for only a small part of the required change of the field needed to reach the exact solutions. Note also that nuclei with even baryon numbers have larger binding energies than those with odd ones. Encouragingly, Skyrmions with even baryon number generally have lower energy per baryon than Skyrmions with odd baryon number. This is seen in our results for  $B = 9, 10$  and  $11$  presented in Table 3.

Finally, baryon density isosurfaces are used to visualize the rescaled Skyrmion configurations and these are presented in Figure 1. The shapes of the rescaled configurations are not very different from the unrescaled ones, since the values of the scaling parameters are close to 1. This can be seen in Figure 2, where plots of the unrescaled and rescaled baryon density isosurfaces of the  $B = 11$  Skyrmion with massless pions are presented. In order to illustrate more clearly the deformation of the Skyrmions under rescaling we show in Figure 3 the  $B = 6$  massless Skyrmion with the true rescaling parameters changed to the values  $\lambda_1 = \lambda_2 = 0.9$  and  $\lambda_3 = 1/\lambda_1\lambda_2$ , and the  $B = 9$  massless Skyrmion with the rescaling parameters changed to  $\lambda_1 = \lambda_2 = 1.1$  and  $\lambda_3 = 1/\lambda_1\lambda_2$ .

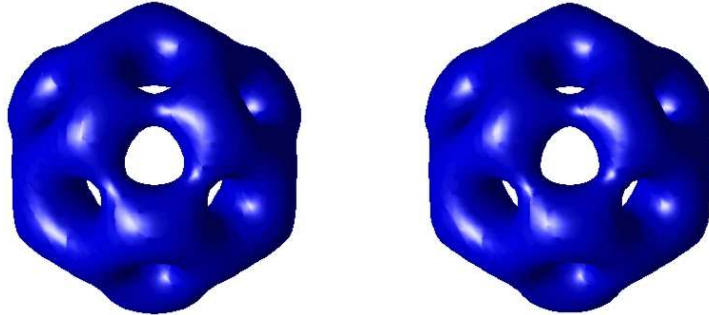


Figure 2: Isosurfaces of the unrescaled (left) and rescaled (right) baryon density for the  $B = 11$  Skyrmion with  $m = 0$ . Each corresponds to a value of  $\tilde{\mathcal{B}} = 0.035$ .

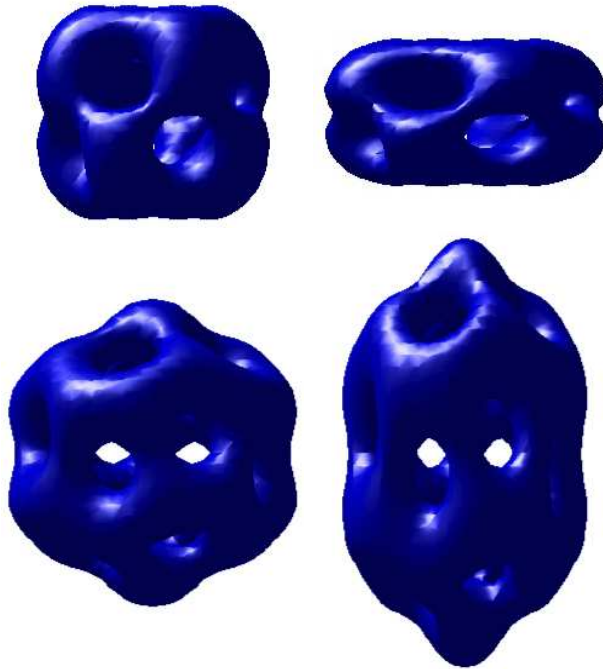


Figure 3: Isosurfaces of the *true* rescaled (left) and *exaggerated* rescaled (right) baryon density at a value of the baryon density which is one quarter of the maximum (i.e.  $0.25 \tilde{\mathcal{B}}_{\max}$ ), for the  $B = 6$  and the  $B = 9$  Skyrmion with  $m = 0$ .

### 3 Conclusions

We have started with a Skyrme field configuration of low but not minimal energy (1), for selected values of  $B$ . This is constructed using the rational map ansatz. We have then rescaled the field, obtaining the energy (5), and have minimized this numerically, thereby finding a lower energy field configuration with the same symmetry. The rescaled field satisfies the scaling identities (6)-(8), or equivalently (9)-(11), and is therefore closer to a true Skyrmion solution. In particular, we learn from the values of the rescaling parameters whether the true Skyrmion is more oblate or prolate than the rational map approximation suggests. This change of shape affects (slightly) the moments of inertia, which in turn will affect estimates of the energy spectrum of the quantized rotating Skyrmion, modelling real nuclei.

The  $B = 6$  configuration shrinks along the  $x_3$ -axis and stretches in the  $(x_1, x_2)$ -plane. It has been observed previously that the rational map approximation to the  $B = 6$  Skyrmion is slightly prolate, and rather too much so, in the sense that its classical electric quadrupole moment is positive, and rather too large to match the measured quadrupole moment of Lithium-6 [11]. The rescaling we have found makes the approximate Skyrmion less prolate, and this is better for fitting the quadrupole moment (although we have not recalculated it). Unfortunately this conclusion is not totally convincing, because if one compares the moments of inertia of the approximate  $B = 6$  Skyrmion obtained using the rational map ansatz [11] with the moments of inertia of the exact solution [9], then it appears that the exact solution is more prolate.

The  $B = 9$  configuration stretches along the  $x_3$ -axis and shrinks in the  $(x_1, x_2)$ -plane, so it becomes more prolate. Similarly the  $B = 11$  configuration stretches along the  $x_3$ -axis and shrinks in the  $(x_1, x_2)$ -plane, becoming prolate under rescaling. These are novel insights, and suggest in particular that the  $B = 11$  solution is not closely related in shape to the known  $B = 12$  solution with  $D_{3h}$  symmetry [9], which is significantly oblate. For  $B = 10$  the three scaling parameters are independent, because the field configuration is triaxial, which agrees with the structure of the exact  $B = 10$  Skyrmion (with  $m = 1$ ) found numerically.

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