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SPHERICAL TROPICALIZATION

A Dissertation Presented

by

TASSOS VOGIANNOU

Submitted to the Graduate School of the
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DOCTOR OF PHILOSOPHY

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Department of Mathematics and Statistics

SPHERICAL TROPICALIZATION

A Dissertation Presented

by

TASSOS VOGIANNOU

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DEDICATION

For my brother

ACKNOWLEDGEMENTS

ABSTRACT

SPHERICAL TROPICALIZATION

SEPTEMBER 2016

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In this thesis, I extend tropicalization of subvarieties of algebraic tori over a trivially valued algebraically closed field to subvarieties of spherical homogeneous spaces. I show the existence of tropical compactifications in a general setting. Given a tropical compactification of a closed subvariety of a spherical homogeneous space, I show that the support of the colored fan of the ambient spherical variety agrees with the tropicalization of the closed subvariety. I provide examples of tropicalization of subvarieties of GL_n, SL_n, and PGL_n.

PREFACE

Algebraic geometry is concerned with algebraic varieties, which are geometric objects that locally arise as solution sets of polynomial equations over some algebraically closed field k, for instance $\mathbb{C}$. Varieties can be rather complicated, and their study and classification are quite challenging. For certain classes of varieties, seemingly difficult questions of algebro-geometric nature have easy combinatorial description. In particular, given the algebraic torus $\mathbb{T} = (k^\times)^n$, one can ask what are the possible ways to embed $\mathbb{T}$ in a normal variety X as a dense open subset such that the action of $\mathbb{T}$ on itself extends to all of X. Such varieties are called toric, and are in an on-to-one correspondence with fans (collections of cones) in a lattice which is isomorphic to $\mathbb{Z}^n$ (the lattice of 1-parameter subgroups of $\mathbb{T}$). Moreover, the properties of a toric variety are reflected in the combinatorial structure of its fan.

The study of valuations of fields (maps from the multiplicative subgroup of a field to $\mathbb{Q}$) and of logarithmic maps on complex algebraic varieties leads to the observation that a variety (of a certain kind) gives rise to a convex object in $\mathbb{R}^n$, called the tropicalization of the variety (see [BG], [EKL]). From these ideas and the relevant work of many people, including I. Itenberg, D. Maclagan, G. Mikhalkin, B. Sturmfels, a new branch of geometry, called tropical geometry, emerged, and it has many applications within and outside mathematics. Tropical geometry can be described as a piece-wise linear version of algebraic geometry. The correspondence of toric varieties with fans in $\mathbb{Z}^n$ may appear to be unrelated to the tropicalization of varieties inside a torus. However, J. Tevelev showed in [Te] that this is not the case, and that there is a connection between the two via certain compactifications of a subvariety of $\mathbb{T}$ inside toric varieties, called tropical compactifica-

tions. This demonstrates that tropicalization is not an artificial construction, but rather has a deep connection with the geometry of the subvariety in hand.

Toric varieties form a subcategory of a bigger class of varieties called spherical varieties. Along with toric varieties, many interesting varieties are spherical: GL_n, SL_n, SO_n, symmetric varieties, and flag varieties, to name a few. Because of this, they have been studied extensively by many researchers, such as M. Brion, F. Knop, D. Luna, F. Pauer, T. Vust. In a complete analogy with toric varieties, spherical varieties are classified by colored fans (fans with additional information on them, called "colors") in (a certain subset of) a lattice $\mathbb{Z}^n$ [LV].

The main goal of my thesis is to extend the ideas of tropical geometry to the category of spherical varieties. I extend tropicalization and tropical compactifications from the toric to the spherical case. The relation with tropical compactifications shows that tropicalization is a natural operation.

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INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic, $K = k((t))$ the field of Laurent series over k, and $\overline{K} = \bigcup_n k((t^{1/n}))$ the field of Puiseux series over k (which is the algebraic closure of K in characteristic zero). Consider the discrete valuation

$$\nu : K^\times \to \mathbb{Z}, \quad \sum_n c_n t^n \mapsto \min\{n : c_n \neq 0\}$$

(k is trivially valued). We denote by ν its extension to a valuation $\overline{K}^\times \to \mathbb{Q}$ defined similarly.

Let $\mathbb{T}^n = (k^\times)^n$ be the algebraic torus of dimension n over k, $\Lambda = \operatorname{Hom}(\mathbb{T}^n, k^\times)$ its character group, and $Q = \operatorname{Hom}(\Lambda, k^\times) \cong \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. The valuation ν induces a surjective map:

$$val : T(\overline{K}) \to Q, \tag{1.1}$$

that sends $(x_1(t), \ldots, x_n(t))$ in $T(\overline{K}) \cong (\overline{K}^\times)^n$ to $(\nu(x_1(t)), \ldots, \nu(x_n(t)))$ in $Q \cong \mathbb{Q}^n$. Given a closed subvariety $Y \subseteq \mathbb{T}^n$, the tropicalization of Y, denoted $\operatorname{Trop} Y$, is the image $val(Y(\overline{K})) \subseteq Q$. It is a piece-wise linear set in $Q \cong \mathbb{Q}^n$ which is the support of a fan. The combinatorial structure of the tropicalization of Y carries information about the original variety, and is usually easier to work with. More about tropicalizations and their use can be found in, among many others, [MB], [IMS], [G], [EKL].

Given a closed subvariety $Y \subseteq T$, a tropical compactification of Y is a compactification $\overline{Y} \subseteq X$, i.e. $\overline{Y}$ is a complete variety, in a toric variety X associated to a fan $\mathcal{F}$ in Q, such

that the multiplication map of $\overline{Y}$:

$$\mu_{\overline{Y}} : T \times \overline{Y} \to X, \quad (g,x) \mapsto gx \tag{1.2}$$

is faithfully flat. Tropical compactifications possess some nice properties. For instance, if X is smooth, then the boundary of $\overline{Y}$ is divisorial and has combinatorial normal crossings. If $\overline{Y} \subseteq X$ is a tropical compactification, then $\operatorname{Supp} \mathcal{F} = \operatorname{Trop} Y$, which suggests a way to construct such compactifications. Tropical compactifications were introduced, and their existence and relation to tropicalizations were shown, in [Te].

The applications of tropicalization and tropical compactifications have motivated their extension to more general settings than subvarieties of tori. In particular, tropical compactifications for the case k is not trivially valued (non-constant coefficient case) were introduced in [LQ]. Tropicalization of subvarieties of toric varieties is treated in [P]. Tropicalizations and tropical compactifications of log-regular varieties were introduced in [U]. Our goal is to extend tropicalization and tropical compactifications to subvarieties of spherical homogeneous spaces for an arbitrary connected reductive group. Spherical means that the action of a Borel subgroup on the homogeneous space has an open orbit. The reason such generalization is possible is that, as in the toric case, the equivariant open dense embeddings of a spherical homogeneous space are in a bijection with combinatorial data (colored fans) in a lattice. This correspondence, introduced in [LV], is described briefly in §3.1.

Let G be a connected reductive group over k, $B \subseteq G$ a Borel subgroup, and let G/H be a spherical homogeneous space for some closed algebraic subgroup $H \subseteq G$. In §3.2 we define a map analogous to (1.1):

$$val : G/H(\overline{K}) \to \mathrm{Q}, \tag{1.3}$$

where $\mathrm{Q} = \operatorname{Hom}(\Lambda, \mathbb{Q}) \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, and Λ is the subgroup of characters of B that are weights of B-semi-invariant functions on G/H. The image of this map is the valuation cone $\mathcal{V}$ (defined in §3.1). Then the tropicalization of a closed subvariety $Y \subseteq G/H$ is

defined to be $\operatorname{Trop} Y = val(Y(\overline{K}))$. We will see (Rem. 4.2) that we can work over K instead of $\overline{K}$, i.e. find the image $val(Y(K))$ instead of $val(Y(\overline{K}))$, and then multiply by scalars in $\mathbb{Q}_{\geq 0}$ to get the rest of $\operatorname{Trop} Y$ (here $G/H(K)$ is viewed as a subset of $G/H(\overline{K})$ via the morphism $\operatorname{Spec} \overline{K} \to \operatorname{Spec} K$ induced by the inclusion $K \hookrightarrow \overline{K}$).

Consider open dense G-embeddings $G/H \to X$ on normal varieties. Such a variety X is called spherical. Given a spherical variety X, one can take the closure $\overline{Y} \subseteq X$ of a closed subvariety $Y \subseteq G/H$. Write

$$\mu_{\overline{Y}} : G \times \overline{Y} \to X, \quad (g, x) \mapsto gx, \tag{1.4}$$

for the multiplication map of $\overline{Y}$.

Definition 1.1. The closure $\overline{Y} \subseteq X$ is called a *tropical compactification* of Y if $\overline{Y}$ is complete and the multiplication map $\mu_{\overline{Y}}$ is faithfully flat.

In §4 we obtain the result:

Theorem 1.2. Let Y be a closed subvariety of a spherical homogeneous space G/H. Then:

   (i) Tropical compactifications of Y in toroidal spherical varieties exist.

  (ii) If $\overline{Y} \subseteq X$ is a tropical compactification, where X is a spherical variety associated to a colored fan $\mathcal{F}$, then $\operatorname{Supp} \mathcal{F} = \operatorname{Trop} Y$.

The term toroidal is explained in Definition 3.6, and the support of a colored fan is defined in §3.1. In part (ii), the tropical compactification $\overline{Y} \subseteq X$ is not assumed to be in a toroidal spherical variety. A direct consequence of this theorem is that the tropicalization of any closed subvariety of G/H is a piece-wise linear object in $\mathbf{Q}$ that is the support of a fan.

We show the existence part of the theorem in §2 in a vastly more general setting (Thm. 2.31), where G is replaced by a surjective smooth (relatively) affine group scheme with connected fibers over a normal noetherian scheme S, G/H by a homogeneous G-scheme U (Def. 2.28) that admits an equivariant compactification (Def. 2.1), and Y by

a closed subscheme, flat over S, such that $\mu(G \times_S Y) = U$, where $\mu : G \times X \to X$ is the multiplication map of X. The proof of (i) follows almost immediately from this. The second result of the theorem is based on a spherical version of *Tevelev's lemma* (Lem. 4.5).

In §5 we work on some examples of spherical tropicalization. First we show that tropicalization of subvarieties of a torus $\mathbb{T}^n$, when viewed as a spherical homogeneous space G/H for $G = B = \mathbb{T}^n$ and H the trivial subgroup, is the same as the usual toric tropicalization. Thus spherical tropicalization is indeed an extension of the toric one.

The linear algebraic group GL_n is a spherical homogeneous space when $G = \mathrm{GL}_n \times \mathrm{GL}_n$ is acting on it by left and right multiplication. Recall that if $x = (x_{ij}(t))$ is an invertible matrix with entries in K, there are matrices $g = (g_{ij})$ and $h = (h_{ij})$ with entries in $k[[t]]$, such that gxh is in (inverse) Smith normal form, i.e.

$$
gxh = \begin{pmatrix}
t^{\alpha_1} & 0 & \dots & 0 \\
0 & t^{\alpha_2} & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & t^{\alpha_n}
\end{pmatrix},
$$

for some integers $\alpha_1 \geq \dots \geq \alpha_n$. The integers $\alpha_1, \dots, \alpha_n$ are called the *invariant factors* of x. An invertible matrix x with entries in $\overline{K}$ can be viewed as a matrix with entries in $k((t^{1/m}))$ for some m. Thus there are some matrices g, h with entries in $k[[t^{1/m}]]$, such that gxh is diagonal with entries $(t^{1/m})^{\alpha_1}, \dots, (t^{1/m})^{\alpha_n}$ along the diagonal, for some integers $\alpha_1 \geq \dots \geq \alpha_n$. We call $\alpha_1/m, \dots, \alpha_n/m$ the *invariant factors* of x. We show that for a certain choice of a Borel group and basis of Λ, which give rise to a dual basis on Q, the tropicalization of a closed subvariety of GL_n is a set in $Q \cong \mathbb{Q}^n$ that can be calculated as in the following theorem.

Theorem 1.3. Let Y be a closed subvariety of GL_n, defined by some ideal $I \subseteq k[\mathrm{GL}_n]$. Then $\mathrm{Trop}\, Y$ consists of the n-tuples $(\alpha_1, \dots, \alpha_n)$ of invariant factors (in decreasing order) of invertible matrices with entries in $\overline{K}$, that satisfy the equations of I.

Figure 1: Tropicalization of $Y = V(x_{11} - x_{22}, x_{12}^3 - x_{21})$
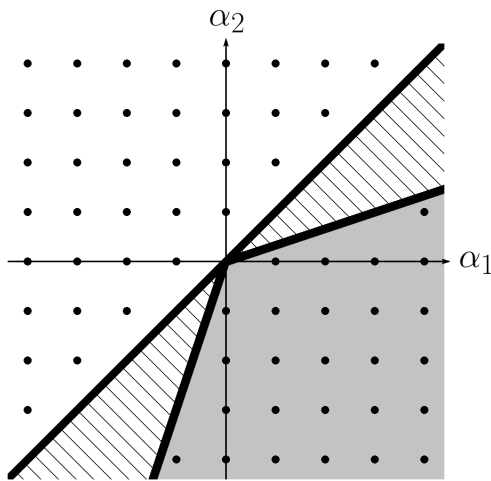
As mentioned earlier, we may work over K when calculating the tropicalization of a subvariety.

If the closed subvariety of GL_n admits a parametrization, then $\mathrm{Trop}\, Y$ can be calculated in a straightforward and elementary way. For instance, to find the tropicalization of the variety $V(x_{11} - x_{22}, x_{12}^3 - x_{21}) \subset \mathrm{GL}_2$, where x_{ij} are coordinates for GL_2, one can write an invertible matrix with entries in K that satisfies the equations $x_{11} = x_{22}$ and $x_{12}^3 = x_{21}$, which is of the form

$$\begin{pmatrix} y(t) & z(t) \\ z(t)^3 & y(t) \end{pmatrix}, \quad y(t), z(t) \in K,$$

and determine what are the possible invariant factors of this matrix. The tropicalization of this variety is drawn in Figure 1. The lightly shaded area is the rest of the valuation cone.

If the closed subvariety $Y = V(I)$ does not admit a parametrization, one has to take the valuations of equations that Y satisfies to impose restrictions on the possible invariant factors of invertible matrices with entries in K, and then find which numbers bounded by these restrictions appear as invariant factors of such matrices. For instance, if $I = (x_{11}^2 x_{12} - x_{22}^5 + x_{11}x_{21}^3 - 1)$, then any matrix $(x_{ij}(t))$ with entries in K that satisfies

Figure 2: Tropicalization of the SL_2-representation variety of $\pi_1(\mathbb{S}_{0,3})$
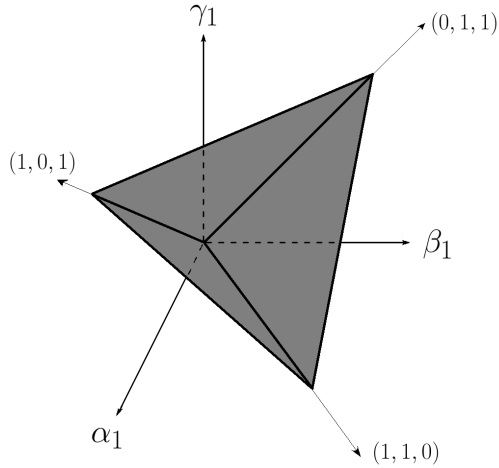
the equations of I must also satisfy:

$$\min\left\{2\nu(x_{11}(t)) + \nu(x_{12}(t)), -5\nu(x_{22}(t)), \nu(x_{11}(t)) + 3\nu(x_{21}(t))\right\} = 0.$$

One has to be careful with this approach. Given a set of generators $f_1, \ldots, f_n$ for the ideal I, the tropicalization of Y is not the intersection of the $\mathrm{Trop}\, V(f_i)$ (see Example 5.4), which is also the case in toric tropicalization.

If the equations defining the ideal I come from matrix products, Horn's inequalities (Eq. (5.1) and (5.2)) may be helpful for determining $\mathrm{Trop}\, Y$. Given three matrices A, B, C with entries in K such that $AB = C$, the integers that appear as invariant factors of them are described by Horn's inequalities (see §5.5). We demonstrate this in Example 5.5.

The cases of subvarieties of SL_n and PGL_n are similar and are treated in §5.3 and §5.4. In §5.5 we find the tropicalization of the G-representation variety of the fundamental group of the sphere with 3 punctures, where G is GL_n or SL_n. The points in $\mathrm{Trop}\, Y$ are precisely the ones that satisfy Horn's inequalities. For the case $G = \mathrm{SL}_2$, the tropicalization is given in Figure 2.

In the toric case, one can construct a tropical compactification for a closed subvariety Y of a torus by embedding the torus in any ambient complete space, and modifying it with

blow-ups until the multiplication map of the closed subvariety becomes flat. This amounts to refining the fan of the ambient space. Then the cones that lie outside $\mathrm{Trop}\,Y$ can be removed, so that the multiplication map becomes surjective, hence faithfully flat. The closure of Y in the toric variety defined by the resulting fan is a tropical compactification of Y. The same idea works in the spherical case. We demonstrate this in §5.6.

In summary, spherical tropicalization appears as a natural extension of the toric one. The latter fits in this context as a special case in which tropicalization carries the "most possible" information. On the other end of the spectrum are generalized flag varieties, that is spherical homogeneous spaces G/H for which H is a parabolic subgroup, i.e. it contains some Borel subgroup, in which case $\mathcal{V}$ is a point and tropicalization is trivial.

This thesis is organized as follows. Chapter §2 is devoted to the proof of the existence of tropical compactifications in a general setting, and can be skipped in first reading. Chapter §3.1 is an introduction to spherical varieties. In §3.2 tropicalization of subvarieties of a spherical homogeneous space is introduced. In §4 we prove *Tevelev's Lemma* for the spherical case, and then Theorem 1.2. These three sections are mostly devoid of examples; there is only an easy one to explain the Luna-Vust theory of spherical varieties, and to show how the tropicalization can be calculated in this easy case. All substantial examples are presented in §5.

In this thesis, a variety is a reduced separated scheme of finite type over an algebraically closed field. Given an algebraic group G, a homogeneous space is an irreducible G-variety (hence integral), such that the action of G is transitive, i.e. there is a unique orbit. The terms affine, projective, and quasi-projective are relative, over a scheme S. The only exception is when we pick an affine open set in a scheme, in which case we write is as the spectrum of a ring.

C H A P T E R   2

TROPICAL COMPACTIFICATIONS

The purpose of this chapter is to show the existence of tropical compactifications in a general setting (Thm. 2.31). The existence of tropical compactifications of subvarieties of spherical homogeneous spaces will then follow as a special case (Thm. 1.2). This chapter is rather technical; the reader who is willing to take in faith the existence of tropical compactifications can skip it.

Definition 2.1. Let G be a group scheme. An *equivariant compactification* of a G-scheme X is a proper G-scheme X' with an open dense G-embedding $X \hookrightarrow X'$. Given an equivariant compactification $X \hookrightarrow X'$, we can view X as an open dense G-stable subset of X'.

Let S be a scheme, G a group scheme over S, U a G-scheme over S, and $Y \subseteq U$ a closed subscheme. The main idea for showing that Y admits a tropical compactification is to find an equivariant compactification $U \hookrightarrow X$, take the closure $\overline{Y} \subseteq X$, which is proper, and then find an equivariant projective birational modification of X that fixes U and makes the multiplication map of the "modified" $\overline{Y}$ flat. The basic problem is the existence of such a modification of X. We proceed by showing that a coherent G-sheaf $\mathcal{M}$ on some G-scheme $\mathcal{X}$ with a projective G-morphism $\mathcal{X} \to \mathcal{Y}$ can be "flattened" in an equivariant way by some modification $\mathcal{Y}' \to \mathcal{Y}$, and then we specialize to flattening of coherent G-sheaves on $G \times_S X$ with respect to the multiplication map. If $\mathcal{M}$ is the structure sheaf $\mathcal{O}_{G \times_S X} / \mathcal{I}_{G \times_S Y}$ of $G \times_S Y$, then this is equivalent to flattening of the multiplication map of Y.

In §2.1 we review flattening of a coherent sheaf $\mathcal{M}$ on a scheme $\mathcal{X}$ for a projective morphism $f : \mathcal{X} \to \mathcal{Y}$, which is due to M. Gruson and L. Raynaud (see [RG] or [R, Chap. 4]). In [RG] more general cases are treated, i.e. when f is not projective, but for the purpose of this thesis this one is sufficient. We define the pure transform and flattening of a coherent sheaf, and then we state the existence of flattenings. Then we define the pure transform and flattening of a closed subscheme of $\mathcal{Y}$, and show their existence.

In §2.2 we extend the results of §2.1 to an equivariant setting. In particular, we show that if all schemes and sheaves considered have a G-structure and morphisms are equivariant, then equivariant flattenings, of sheaves or of closed subschemes, exist. In §2.3 we specialize the results of §2.2 to the case f is the multiplication map $\mu : G \times_S X \to X$, and $\mathcal{M}$ a sheaf on $G \times_S X$ which is the pullback of a (not necessarily equivariant) sheaf on X by the second projection. The multiplication map is not, in general, projective, but we can overcome this problem with an equivariant compactification $G \hookrightarrow G'$ in a projective scheme G' (under certain conditions on G).

Finally, in §2.4 we introduce homogeneous schemes, which is a generalization of homogeneous spaces, and show that tropical compactifications of a closed subscheme of a homogeneous scheme that admits an equivariant compactification exist. We show that for a homogeneous scheme over a field, there are tropical compactifications inside normal schemes.

2.1 The Pure Transform and Flattening

Let X be a scheme, Y a noetherian scheme, $f : X \to Y$ a morphism of finite type, and $\mathcal{M}$ a coherent sheaf on X. If U is an open set in Y, we write $\mathcal{M}|_U$ for the restriction $\mathcal{M}|_{f^{-1}(U)}$. Assume that $U \subseteq Y$ is an open dense set such that $\mathcal{M}|_U$ is flat (over U). When Y is reduced, the existence of such open dense set is guaranteed by Grothendieck's generic flatness [EGAIV, Th. 6.9.1].

Let $u : \widetilde{Y} \to Y$ be a projective birational morphism that restricts to an isomorphism on open dense sets $\widetilde{U} \xrightarrow{\sim} U$, and $\widetilde{X} = X \times_Y \widetilde{Y}$:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\;\tilde{u}\;} & X \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\widetilde{Y} & \xrightarrow{\;u\;} & Y
\end{array}
$$

Consider the subsheaf $\widetilde{\mathcal{N}}$ of the pullback $\widetilde{\mathcal{M}} = \tilde{u}^*\mathcal{M}$ of sections supported on $\tilde{f}^{-1}(\widetilde{Y} - \widetilde{U})$: for any open set $\widetilde{V} \subseteq \widetilde{X}$,

$$
\Gamma(\widetilde{V}, \widetilde{\mathcal{N}}) = \left\{ s \in \Gamma(\widetilde{V}, \widetilde{\mathcal{M}}) : s_P = 0 \text{ for all } P \in \widetilde{X} \text{ with } \tilde{f}(P) \in \widetilde{U} \right\}.
$$

Definition 2.2. The quotient sheaf $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ is called the *pure transform* of $\mathcal{M}$ with respect to (u, U).

When the open set U is clear from the context, we may call $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ the *pure transform* of $\mathcal{M}$ with respect to f, and if both u and U are clear, we may call $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ the *pure transform* of $\mathcal{M}$.

Proposition 2.3. [R, Chap. 4, §1] A coherent sheaf $\mathcal{P}$ on $\widetilde{X}$ is the pure transform of $\mathcal{M}$ if and only if there is a coherent subsheaf $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ such that the following are satisfied:

(i) $\mathcal{P}$ is the quotient $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$,

(ii) $\widetilde{\mathcal{N}}$ vanishes on $\tilde{f}^{-1}(\widetilde{U})$, and

(iii) $\mathrm{Ass}(\mathcal{P}) \subseteq \tilde{f}^{-1}(\widetilde{U})$.

Proposition 2.4. If $\mathcal{M}$ is flat over Y and $\mathrm{Ass}(\widetilde{Y}) \subseteq \widetilde{U}$ (e.g. if $\widetilde{Y}$ is integral), then the pure transform of $\mathcal{M}$ with respect to U is $\widetilde{\mathcal{M}}$, i.e. $\widetilde{\mathcal{N}} = 0$.

Proof. From Proposition 2.3 it suffices to show $\mathrm{Ass}(\widetilde{\mathcal{M}}) \subseteq \tilde{f}^{-1}(\widetilde{U})$. Due to flatness of $\widetilde{\mathcal{M}}$, the associated points of $\widetilde{\mathcal{M}}$ map to associated points in $\widetilde{Y}$ [EGAIV, Th. 3.3.1], hence $\mathrm{Ass}(\widetilde{\mathcal{M}}) \subseteq \tilde{f}^{-1}(\widetilde{U})$. $\qquad\square$

If $\widetilde{\mathcal{N}} \neq 0$, then $\widetilde{\mathcal{M}}$ is certainly not flat over $\widetilde{Y}$. Therefore, instead of asking whether $\widetilde{\mathcal{M}}$ is flat over $\widetilde{Y}$, it is more natural to ask if the pure transform $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ is flat over $\widetilde{Y}$.

Definition 2.5. The projective birational morphism $u : \widetilde{Y} \to Y$, that restricts to an isomorphism on open dense sets $\widetilde{U} \xrightarrow{\sim} U$, is called a *flattening* of $\mathcal{M}$ with respect to (f, U), if the pure transform of $\mathcal{M}$ with respect to it is flat over $\widetilde{Y}$.

Theorem 2.6. [R, Chap. 4, §1, Thm. 1] For any quintuplet $(X, Y, f, \mathcal{M}, U)$, where

$\quad X \quad$ is a scheme,

$\quad Y \quad$ is a noetherian scheme,

$\quad f \quad$ is a projective morphism $X \to Y$ of finite type,

$\quad \mathcal{M} \quad$ is a coherent sheaf on X, and

$\quad U \quad$ is an open dense set in Y such that $\mathcal{M}|_U$ is flat,

there is a flattening $\widetilde{Y} \to Y$ of $\mathcal{M}$ with respect to f. If Y is integral, there is a flattening $\widetilde{Y} \to Y$ of $\mathcal{M}$ with $\widetilde{Y}$ integral.

Let $Z \subseteq X$ be a closed subscheme, and $\mathcal{I}_Z$ the associated sheaf of ideals on X. In this context, U will be an open dense set in Y such that the restriction $f|_Z : Z \to Y$ is flat over U.

Definition 2.7. The *pure transform* of Z with respect to (u, U) is the scheme-theoretic closure $\overline{\widetilde{u}^{-1}(Z \cap f^{-1}(U))} \subseteq \widetilde{X}$.

Write $\widetilde{Z}$ for the pure transform of Z, and $\mathcal{I}_{\widetilde{Z}}$ for the associated sheaf of ideals on $\widetilde{X}$.

Definition 2.8. The projective birational morphism $u : \widetilde{Y} \to Y$, that restricts to an isomorphism on open dense sets $\widetilde{U} \xrightarrow{\sim} U$, is called a *flattening* of Z with respect to (f, U), if $f|_{\widetilde{Y}}$ is flat over $\widetilde{Y}$.

Lemma 2.9. The pure transform of the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Z$ on X is the quotient sheaf $\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}$ on $\widetilde{X}$.

Proof. Consider the coherent sheaves $\mathcal{M} = \mathcal{O}_X/\mathcal{I}_Z$ on X and $\widetilde{\mathcal{M}} = \tilde{u}^*\mathcal{M} = \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\tilde{u}^{-1}(Z)}$ on $\widetilde{X}$. The pure transform of $\mathcal{M}$ is then the quotient $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}} = (\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\tilde{u}^{-1}(Z)})/\widetilde{\mathcal{N}}$, where $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ is the subsheaf of sections with support outside $\tilde{f}^{-1}(\widetilde{U})$. It is of the form $\mathcal{I}_{\widetilde{Z}'}/\mathcal{I}_{\tilde{u}^{-1}(Z)}$ for a sheaf of ideals $\mathcal{I}_{\widetilde{Z}'}$ on $\widetilde{X}$ containing $\mathcal{I}_{\tilde{u}^{-1}(Z)}$, that determines a closed subscheme $\widetilde{Z}' \subseteq \widetilde{X}$ contained in $\tilde{u}^{-1}(Z)$. Then clearly $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}} = \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}'}$. We claim that $\widetilde{Z}' = \widetilde{Z}$.

From the definition of the pure transform of $\mathcal{M}$, $\widetilde{\mathcal{N}} = \mathcal{I}_{\widetilde{Z}'}/\mathcal{I}_{\tilde{u}^{-1}(Z)}$ vanishes on $\tilde{f}^{-1}(\widetilde{U})$, hence

$$\mathcal{I}_{\widetilde{Z}'}|_{\tilde{f}^{-1}(\widetilde{U})} = \mathcal{I}_{\tilde{u}^{-1}(Z)}|_{\tilde{f}^{-1}(\widetilde{U})},$$

which implies

$$\widetilde{Z}' \cap \tilde{f}^{-1}(\widetilde{U}) = \tilde{u}^{-1}(Z) \cap \tilde{f}^{-1}(\widetilde{U}) = \tilde{u}^{-1}(Z \cap f^{-1}(U)).$$

From the definition of the pure transform of Z, $\widetilde{Z} \subseteq \widetilde{Z}'$. Furthermore, $\widetilde{Z}' \cap \tilde{f}^{-1}(\widetilde{U}) = \widetilde{Z} \cap \tilde{f}^{-1}(\widetilde{U})$, and hence

$$\left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}'}\right)|_{\tilde{f}^{-1}(\widetilde{U})} = \left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}\right)|_{\tilde{f}^{-1}(\widetilde{U})}.$$

Assume that the inclusion $\widetilde{Z} \subseteq \widetilde{Z}'$ is strict. Let $P \in \widetilde{Z}' - \widetilde{Z}$. Pick an affine open set $V = \operatorname{Spec} A$ in $\widetilde{X}$ containing P. Write $\widetilde{Z}' \cap V = V(\mathfrak{a})$, $\widetilde{Z} \cap V = V(\mathfrak{b})$ with $\mathfrak{a} \subset \mathfrak{b}$ ideals of A (strict inclusion). Let $a \in A$ be such that $a \in \mathfrak{b}$ but $a \notin \mathfrak{a}$, so that a is zero in $A/\mathfrak{b}$, but non-zero in $A/\mathfrak{a}$. Let $\mathfrak{p}$ be in $f^{-1}(\widetilde{U}) \cap V$. If $\mathfrak{p} \notin \widetilde{Z}'$ then clearly $\left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}'}\right)_{\mathfrak{p}} = 0$. If $\mathfrak{p}$ is in $\widetilde{Z}' \cap \tilde{f}^{-1}(\widetilde{U}) \cap V = \widetilde{Z} \cap \tilde{f}^{-1}(\widetilde{U}) \cap V$, then since $\left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}'}\right)|_{\tilde{f}^{-1}(\widetilde{U})} = \left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}\right)|_{\tilde{f}^{-1}(\widetilde{U})}$,

$$a = 0 \text{ in } \left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}'}\right)_{\mathfrak{p}} = \left(\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}\right)_{\mathfrak{p}} = (A/\mathfrak{b})_{\mathfrak{p}}.$$

Thus a is a non-zero local section supported outside $\tilde{f}^{-1}(\widetilde{U})$. This contradicts the definition of the pure transform of $\mathcal{M}$, hence $\widetilde{Z}' = \widetilde{Z}$. $\qquad\square$

Corollary 2.10. If μ_Z is flat and $\operatorname{Ass}(\widetilde{Y}) \subseteq \widetilde{U}$ (e.g. if $\widetilde{Y}$ is integral), then the pure transform of Z is $\tilde{u}^{-1}(Z)$.

Proof. Flatness of μ_Z is equivalent to flatness of $\mathcal{O}_X/\mathfrak{I}_Z$. The pure transform of $\mathcal{O}_X/\mathfrak{I}_Z$ with respect to U is then $\tilde{u}^*(\mathcal{O}_X/\mathfrak{I}_Z) = \mathcal{O}_{\widetilde{X}}/\mathfrak{I}_{\tilde{u}^{-1}(Z)}$ (Prop. 2.4), and also $\mathcal{O}_{\widetilde{X}}/\mathfrak{I}_{\widetilde{Z}}$. It follows that $\widetilde{Z} = \tilde{u}^{-1}(Z)$. $\qquad\square$

Theorem 2.11. For any quintuplet (X, Y, f, Z, U), where

$\quad X \quad$ is a scheme,

$\quad Y \quad$ is a noetherian scheme,

$\quad f \quad$ is a projective morphism $X \to Y$ of finite type,

$\quad Z \quad$ is a closed subscheme of X, and

$\quad U \quad$ is an open set in Y such that $f|_Z$ is flat over U,

there is a flattening $\widetilde{Y} \to Y$ of Z with respect to f. If Y is integral, there is a flattening $\widetilde{Y} \to Y$ of Z with $\widetilde{Y}$ integral.

Proof. Flatness of $f|_Z$ over U is equivalent to flatness of the coherent sheaf $\mathcal{O}_X/\mathfrak{I}_Z$ over U. Apply Theorem 2.6 to get a flattening $u : \widetilde{Y} \to Y$ of this sheaf:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \tilde{u}\ } & X \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\widetilde{Y} & \xrightarrow{\ u\ } & Y
\end{array}
$$

If Y is integral, we may assume that $\widetilde{Y}$ is as well. From Lemma 2.9, the pure transform of $\mathcal{O}_X/\mathfrak{I}_Z$ is $\mathcal{O}_{\widetilde{X}}/\mathfrak{I}_{\widetilde{Z}}$. Flatness of this sheaf is equivalent to flatness of the restriction $\tilde{f}|_{\widetilde{Z}}$, and we are through. $\qquad\square$

2.2 Equivariant Flattening

In this section we extend the results of §2.1 to an equivariant setting. Our main goal is to prove an equivariant version of Theorem 2.6. We follow the same steps as in [R, Chap. 4, §1, Thm. 1], carrying equivariance along the way. This proof is based on the existence of the Quot scheme, so our first goal is to define a group action on it and show that all relevant morphisms are equivariant.

Let S be a scheme and G a group scheme over S. All schemes and morphisms considered are over S. Let X be a G-scheme, Y a noetherian G-scheme, $f : X \to Y$ a G-morphism of finite type, and $\mathcal{M}$ a coherent G-sheaf on X. Write

$$\mu : G \times_S X \to X, \quad (g, x) \mapsto gx$$

for the multiplication map of X. Assume that $U \subseteq Y$ is a G-stable open dense set such that $\mathcal{M}|_U$ is flat. If Y is reduced and G is flat and locally of finite type, then such an open dense set exists. Indeed, there is an open set U' (not necessarily G-stable) such that $\mathcal{M}|_{U'}$ is flat. The image $U = \mu(G \times_S U')$ is then a G-stable open set such that $\mathcal{M}|_U$ is flat.

Let $u : \widetilde{Y} \to Y$ be a projective birational G-morphism that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$, and $\widetilde{X} = X \times_Y \widetilde{Y}$:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \tilde{u}\ } & X \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\
\widetilde{Y} & \xrightarrow{\ u\ } & Y
\end{array}
$$

Since f and u are G-morphisms, $\widetilde{X}$ has a natural structure as a G-scheme, with which $\tilde{f}$ and $\tilde{u}$ are G-morphisms, and $\tilde{f}^{-1}(\widetilde{U})$ is a G-stable open set.

Proposition 2.12. If G is flat and of finite type, then the pure transform of $\mathcal{M}$ (Def. 2.2) is a G-sheaf on $\widetilde{X}$.

Proof. Write $\widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ for the pure transform of $\mathcal{M}$, where $\widetilde{\mathcal{M}} = \tilde{u}^*\mathcal{M}$, and $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ is the subsheaf of sections supported outside $\tilde{f}^{-1}(\widetilde{U})$. Since $\tilde{u}$ is a G-morphism, $\widetilde{\mathcal{M}}$ is a G-sheaf. It suffices to show that $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ is a G-subsheaf. Write

$$\tilde{\mu} : G \times_S \widetilde{X} \to \widetilde{X} \quad \text{and} \quad \widetilde{\mathrm{pr}}_2 : G \times_S \widetilde{X} \to \widetilde{X}$$

for the multiplication map of $\widetilde{X}$ and the second projection of $G \times_S \widetilde{X}$, respectively, and

$$\alpha : \tilde{\mu}^*\widetilde{\mathcal{M}} \to \widetilde{\mathrm{pr}}_2^*\widetilde{\mathcal{M}}$$

for the isomorphism of $\mathcal{O}_{G \times_S \widetilde{X}}$-modules that defines the G-structure on $\widetilde{\mathcal{M}}$. We want to show that $\alpha(\tilde{\mu}^* \widetilde{\mathcal{N}}) \subseteq \widetilde{\text{pr}}_2^* \widetilde{\mathcal{N}}$. Since $\tilde{\mu}$ is flat and $\widetilde{\mathcal{N}} \subseteq \widetilde{\mathcal{M}}$ is the subsheaf of sections supported outside $\tilde{f}^{-1}(\widetilde{U})$, $\tilde{\mu}^* \widetilde{\mathcal{N}}$ is the subsheaf of sections of $\tilde{\mu}^* \widetilde{\mathcal{M}}$ supported outside $\tilde{\mu}^{-1}(\tilde{f}^{-1}(\widetilde{U}))$, and similarly for $\widetilde{\text{pr}}_2^* \mathcal{N}$ [H, II, Ex. 1.20]. Note that

$$\tilde{\mu}^{-1}(\tilde{f}^{-1}(\widetilde{U})) = G \times_S \tilde{f}^{-1}(\widetilde{U}) = \widetilde{\text{pr}}_2^{-1}(\tilde{f}^{-1}(\widetilde{U})),$$

as $\tilde{f}^{-1}(\widetilde{U})$ is G-stable. The isomorphism α sends a section supported outside of $\tilde{\mu}^{-1}(\tilde{f}^{-1}(\widetilde{U}))$ to a section supported outside of $\tilde{\mu}^{-1}(\tilde{f}^{-1}(\widetilde{U}))$, and so outside of $\widetilde{\text{pr}}_2^{-1}(\tilde{f}^{-1}(\widetilde{U}))$, therefore $\alpha(\tilde{\mu}^* \widetilde{\mathcal{N}}) \subseteq \widetilde{\text{pr}}_2^* \widetilde{\mathcal{N}}$ as required. This completes the proof. $\qquad\square$

Definition 2.13. The projective birational G-morphism $u : \widetilde{Y} \to Y$, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$, is called an *equivariant flattening* (or a *G-flattening*) of $\mathcal{M}$ with respect to (f, U), if the pure transform of $\mathcal{M}$ with respect to it is a G-sheaf that is flat over $\widetilde{Y}$.

From now on we assume that f is projective. Let $\mathfrak{Quot}_{\mathcal{M}/X/Y}$ be the Quot functor, i.e. the contravariant functor $\mathsf{Sch}_Y \to \mathsf{Set}$ such that

$$\mathfrak{Quot}_{\mathcal{M}/X/Y}(T) = \left\{ \begin{array}{c} \text{Coherent quotients of the pullback of} \\ \mathcal{M} \text{ on } T \times_Y X \text{ that are flat over } T \end{array} \right\}$$

for any Y-scheme T. This functor is represented by the Quot scheme Q (when Y is noetherian, f is projective, and $\mathcal{M}$ coherent). It is a disjoint union $\coprod_i \mathrm{Q}_i$ of projective schemes Q_i over Y (see [TDTE] from *Fondements de la Géometrie Algébrique* or [N]). Write $\pi : \mathrm{Q} \to Y$ for the structure morphism. We can view Q as a scheme over S via the composition of π with $Y \to S$, in which case π is a morphism over S.

Lemma 2.14. Q has a natural structure of a G-scheme, with which π is a G-morphism.

Proof. Given a scheme T, we define an action of $G_S(T)$ on $\mathrm{Q}_S(T)$, functorial on T, as follows. Let $g \in G_S(T)$ and $s \in \mathrm{Q}_S(T)$; we want to define an element $gs \in \mathrm{Q}_S(T)$. We

view T as a scheme over Y via $y = \pi \circ s$, in which case s is a morphism over Y, and y a morphism over S:

$$
\begin{array}{ccc}
T & \xrightarrow{\ s\ } & Q \\
& y \searrow\ \ \pi \swarrow & \\
& Y & \\
& \downarrow & \\
& S &
\end{array}
$$

Then $s \in Q_Y(T) = \mathfrak{Quot}_{\mathcal{M}/X/Y}(T)$. In particular, s corresponds to a coherent quotient $\mathcal{N}$ of $\tilde{y}^*\mathcal{M}$ that is flat over T:

$$
\begin{array}{ccc}
T \times_Y X & \xrightarrow{\ \tilde{y}\ } & X \\
\downarrow & & \downarrow f \\
T & \xrightarrow{\ y\ } & Y
\end{array}
$$

The morphism $T \times_Y X \to T$ induces a map $G_S(T) \to G_S(T \times_Y X)$. Let $\tilde{g}$ be the image of g under this map. Note that $\tilde{y} \in X_S(T \times_Y X)$, so that $\tilde{g}\tilde{y} = \widetilde{g\tilde{y}}$ is also an element in $X_S(T \times_Y X)$, where $\widetilde{g\tilde{y}}$ is given by the cartesian diagram

$$
\begin{array}{ccc}
T \times_Y X & \xrightarrow{\ \widetilde{gy}\ } & X \\
\downarrow & & \downarrow f \\
T & \xrightarrow{\ gy\ } & Y
\end{array}
$$

(here $T \times_Y X$ and $T \times_Y X \to T$ are as in the above cartesian diagram). Since $\mathcal{M}$ is a G-sheaf, there is an isomorphism of sheaves on $T \times_Y X$:

$$
\phi : \tilde{y}^*\mathcal{M} \to \widetilde{gy}^*\mathcal{M}
$$

The quotient sheaf $\mathcal{N}$ is identified via ϕ with a coherent quotient sheaf of $\widetilde{gy}^*\mathcal{M}$ that is flat over T. This gives a point in $Q_Y(T) \subseteq Q_S(T)$, where T is a scheme over Y via gy. We define gs to be this point. Showing the properties of a group action and functoriality on T is easy and is omitted.

Now we show that π is a G-morphism. Let T be a scheme. Let $\pi_T : Q_S(T) \to Y_S(T)$ be the map induced by π on T-points, and let $g \in G_S(T)$, $s \in Q_S(T)$. Let y be the

image of s in $Y_S(T)$. From the definition of gs, $\pi_T(gs) = gy = g\pi_T(s)$, and so π is a G-morphism. $\qquad\square$

Let R be a noetherian scheme and $y : R \to Y$ a morphism such that the coherent sheaf $\tilde{y}^*\mathcal{M}$ is flat over R:

$$
\begin{array}{ccc}
X \times_Y R & \xrightarrow{\ \tilde{y}\ } & X \\
\downarrow & & \downarrow{f} \\
R & \xrightarrow{\ y\ } & Y
\end{array}
$$

This gives a point in $\mathfrak{Quot}_{\mathcal{M}/X/Y}(R)$, hence a morphism $s : R \to Q$ over Y.

Lemma 2.15. If R is a G-scheme and y a G-morphism, then s is also a G-morphism.

Proof. Let T be a scheme, and write $s_T : R_S(T) \to Q_S(T)$ for the induced map on T-points. Given $g \in G_S(T)$ and $r \in R_S(T)$, we want to show that $s_T(gr) = gs_T(r)$. The image $s_T(r)$ is the point in $Q_S(T) = \mathfrak{Quot}_{\mathcal{M}/X/Y}(T)$ associated to the sheaf $\tilde{r}^*\tilde{y}^*\mathcal{M} = (\tilde{y} \circ \tilde{r})^*\mathcal{M}$ on $X \times_Y T$ (which is the coherent quotient of $(\tilde{y} \circ \tilde{r})^*\mathcal{M}$ by the zero sheaf, and is flat over T):

$$
\begin{array}{ccccc}
T \times_Y X & \xrightarrow{\ \tilde{r}\ } & X \times_Y R & \xrightarrow{\ \tilde{y}\ } & X \\
\downarrow & & \downarrow & & \downarrow{f} \\
T & \xrightarrow{\ r\ } & R & \xrightarrow{\ y\ } & Y
\end{array}
$$

Let $\tilde{g} \in G_S(T \times_Y X)$ be the image of g under the map $G_S(T) \to G_S(T \times_Y X)$ induced by $T \times_Y X \to T$. Note that $\tilde{y} \circ \tilde{r} \in X_S(T \times_Y X)$ and, as in the proof of Lemma 2.14, $\tilde{g}(\tilde{y} \circ \tilde{r}) = (g(y \circ r))^{\sim} = (y \circ gr)^{\sim}$:

$$
\begin{array}{ccc}
T \times_Y X & \xrightarrow{\ \widetilde{y \circ gr}\ } & X \\
\downarrow & & \downarrow{f} \\
T & \xrightarrow{\ y \circ gr\ } & Y
\end{array}
$$

The equality $g(y \circ r) = y \circ gr$ follows from the equivariance of y. There is an isomorphism of sheaves on $T \times_Y X$:

$$
\phi : (\tilde{y} \circ \tilde{r})^* \mathcal{M} \to \widetilde{y \circ gr}^* \mathcal{M}
$$

The sheaf $(\tilde{y} \circ \tilde{r})^* \mathcal{M}$ (as a quotient of $(\tilde{y} \circ \tilde{r})^* \mathcal{M}$ by the zero sheaf) is identified with the sheaf $((y \circ gr)^\sim)^* \mathcal{M}$. This is the coherent sheaf, flat over T, that determines the point $gs_T(r)$ in $\mathrm{Q}_S(T)$.

The image $s_T(gr)$ is the point in $\mathrm{Q}_S(T)$ associated to the sheaf $((y \circ gr)^\sim)^* \mathcal{M}$ on $X \times_Y T$ (which is the coherent quotient of $((y \circ gr)^\sim)^* \mathcal{M}$ by the zero sheaf, and is flat over T). This is precisely $gs_T(r)$, thus s is a G-morphism. $\qquad\square$

Theorem 2.16. Let S be a scheme and G a flat group scheme over S of finite type. For any quintuplet $(X, Y, f, \mathcal{M}, U)$, where

$\quad X \quad$ is a G-scheme,

$\quad Y \quad$ is a noetherian G-scheme,

$\quad f \quad$ is a projective G-morphism $X \to Y$ of finite type,

$\quad \mathcal{M} \quad$ is a coherent G-sheaf on X, and

$\quad U \quad$ is a G-stable open set in Y such that $\mathcal{M}|_U$ is flat,

there is a G-flattening $\widetilde{Y} \to Y$ of $\mathcal{M}$ with respect to f. If Y is integral, there is a G-flattening $\widetilde{Y} \to Y$ of $\mathcal{M}$ with $\widetilde{Y}$ integral.

Proof. The sheaf $\mathcal{M}|_U$ is the pullback of $\mathcal{M}$ on $f^{-1}(U) = U \times_Y X$ by the G-embedding $U \to Y$, and so by Lemma 2.15 it induces a G-morphism $v : U \to \mathrm{Q}$ over Y. Let $\widetilde{Y}$ be the scheme-theoretic image of v, which is a G-stable closed subscheme of Q, and let $w : U \to \widetilde{Y}$ be the induced G-morphism, and $s : \widetilde{Y} \hookrightarrow \mathrm{Q}$ the associated closed G-embedding. Write $u : \widetilde{Y} \to Y$ for the structure morphism.

We claim that u is a G-flattening of $\mathcal{M}$. Since Y is noetherian, U has finitely many irreducible components, and the same holds for its image $\widetilde{Y}$. Therefore $\widetilde{Y}$ lies in finitely many Q_i in the decomposition of Q, so that $u : \widetilde{Y} \to Y$ is projective. If in addition Y

is integral, there is only one irreducible component, and so $\widetilde{Y}$ is integral. Furthermore, u is a G-morphism since it is the composition $\pi \circ s$. The composition $u \circ w$ is the open G-embedding $U \hookrightarrow Y$, hence w is also an open G-embedding. Let $\widetilde{U} = w(U)$, which is a G-stable open set in $\widetilde{Y}$. The structure morphism $u : \widetilde{Y} \to Y$ is birational, restricting to an isomorphism $\widetilde{U} \xrightarrow{\sim} U$. In summary, u is a projective birational G-morphism, and it restricts to an isomorphism on G-stable open sets $\widetilde{U} \xrightarrow{\sim} U$. If Y is integral, we may further assume that $\widetilde{Y}$ is integral.

We show that the pure transform of $\mathcal{M}$ is a G-sheaf that is flat over $\widetilde{Y}$. It is a G-sheaf from Proposition 2.12. The morphism $s : \widetilde{Y} \hookrightarrow Q$ is an element of $Q_Y(\widetilde{Y}) = \mathfrak{Quot}_{\mathcal{M},X,Y}(\widetilde{Y})$ and it corresponds to a quotient sheaf $\mathcal{P} = \widetilde{\mathcal{M}}/\widetilde{\mathcal{N}}$ on $\widetilde{X} = X \times_Y \widetilde{Y}$, where $\widetilde{\mathcal{M}} = \tilde{u}^*\mathcal{M}$, that is coherent and flat over $\widetilde{Y}$:

$$
\begin{array}{ccc}
\widetilde{X} & \xrightarrow{\ \tilde{u}\ } & X \\
{\scriptstyle \tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
\widetilde{Y} & \xrightarrow{\ u\ } & Y
\end{array}
$$

We show that $\mathcal{P}$ is the pure transform of $\mathcal{M}$.

The morphism $w : U \hookrightarrow \widetilde{Y}$ (over Y) induces a map $Q_Y(\widetilde{Y}) \to Q_Y(U)$. In terms of elements of the set $\mathfrak{Quot}_{\mathcal{M},X,Y}(\widetilde{Y})$, this map sends a coherent quotient of the pullback of $\mathcal{M}$ on $X \times_Y \widetilde{Y}$ that is flat over $\widetilde{Y}$ to its pullback on $f^{-1}(U)$, which is a coherent quotient of $\mathcal{M}|_U$ that is flat over U:

$$
\begin{array}{ccccc}
f^{-1}(U) & \xhookrightarrow{\ \tilde{w}\ } & X \times_Y \widetilde{Y} & \longrightarrow & X \\
{\scriptstyle f|_{f^{-1}(U)}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle f} \\
U & \xhookrightarrow{\ w\ } & \widetilde{Y} & \longrightarrow & Y
\end{array}
$$

Thus the image of s in $Q_Y(U)$, which is $w \circ s = v$, corresponds to $\tilde{w}^*\mathcal{P}$ (which is a coherent quotient of $\mathcal{M}|_U$ flat over U). It follows that $\tilde{w}^*\mathcal{P} = \mathcal{M}|_U$. As an open immersion w is flat, hence

$$
\tilde{w}^*\mathcal{P} = \tilde{w}^*\widetilde{\mathcal{M}}/\tilde{w}^*\widetilde{\mathcal{N}} = (\tilde{w}^*\tilde{u}^*\mathcal{M})/\tilde{w}^*\widetilde{\mathcal{N}} = \mathcal{M}|_U/\tilde{w}^*\widetilde{\mathcal{N}}.
$$

Thus $\mathcal{M}|_U = \mathcal{M}|_U/\tilde{w}^*\widetilde{\mathcal{N}}$ and we deduce that $\tilde{w}^*\widetilde{\mathcal{N}}$ is the zero sheaf. Taking the pullback of $\tilde{w}^*\widetilde{\mathcal{N}}$ by the isomorphism $\tilde{u}|_{\tilde{f}^{-1}(\widetilde{U})} : \tilde{f}^{-1}(\widetilde{U}) \xrightarrow{\sim} f^{-1}(U)$, we see that $\widetilde{\mathcal{N}}|_{\widetilde{U}} = 0$, i.e. $\widetilde{\mathcal{N}}$ vanishes on $\tilde{f}^{-1}(\widetilde{U})$:

$$
\begin{array}{ccccc}
\tilde{f}^{-1}(\widetilde{U}) & \xrightarrow{\tilde{u}|_{\tilde{f}^{-1}(\widetilde{U})}} & f^{-1}(U) & \xhookrightarrow{\tilde{w}} & \widetilde{X} \\
\tilde{f}|_{\tilde{f}^{-1}(\widetilde{U})}\downarrow & & \downarrow f|_{f^{-1}(U)} & & \downarrow \tilde{f} \\
\widetilde{U} & \xrightarrow{u|_{\widetilde{U}}} & U & \xhookrightarrow{w} & \widetilde{Y}
\end{array}
$$

The associated points of $\widetilde{Y}$ are contained in $\widetilde{U}$. Indeed, assume there is $P \in \mathrm{Ass}(\widetilde{Y})$ with $P \notin \widetilde{U}$. Pick some affine open set containing P, say $\widetilde{V} = \mathrm{Spec}\, A$, and let $\mathfrak{p} \subset A$ be the prime ideal associated to P. It is an associated prime of A, that is $\mathfrak{p} = \mathrm{Ann}(a)$ for some (non-zero) $a \in A$. The support of a is the closure of $\mathfrak{p}$ (in $\widetilde{V}$), which is $V(\mathfrak{p})$:

$$\mathfrak{q} \in \mathrm{Supp}\, a \Leftrightarrow a \neq 0 \text{ in } A_{\mathfrak{q}} \Leftrightarrow \mathfrak{q} \supseteq \mathrm{Ann}\,(a) \Leftrightarrow \mathfrak{q} \in V(\mathfrak{p}).$$

Since $P \in \widetilde{Y} - \widetilde{U}$, $V(\mathfrak{p})$ is contained in $\widetilde{Y} - \widetilde{U}$, and so $a \in \Gamma(\widetilde{V}, \mathcal{O}_{\widetilde{Y}})$ is a non-zero section supported outside of $\widetilde{U}$. In particular, the subsheaf of $\mathcal{O}_{\widetilde{Y}}$ consisting of sections with support on $\widetilde{Y} - \widetilde{U}$ is not empty, and it corresponds to a closed subscheme $\widetilde{Y}' \subset \widetilde{Y} \subseteq Q$ (strict inclusion) containing U. This violates the minimality of the scheme-theoretic image $\widetilde{Y}$.

Due to the flatness of $\mathcal{P}$, the associated points of $\mathcal{P}$ map to associated points of $\widetilde{Y}$ [EGAIV, Th. 3.3.1]. We deduce $\mathrm{Ass}(\mathcal{P}) \subseteq \tilde{f}^{-1}(\widetilde{U})$, and Lemma 2.3 implies that $\mathcal{P}$, which is flat over $\widetilde{Y}$, is the pure transform of $\mathcal{M}$. This completes the proof. $\qquad\square$

From now on we assume that G is flat and of finite type. Let $Z \subseteq X$ be a G-stable closed subscheme, and $\mathcal{I}_Z$ the associated G-sheaf of ideals on X. In this context, U will be a G-stable open set in Y such that the restriction $f|_Y : Z \to Y$, which is a G-morphism, is flat over U. Write $\widetilde{Z}$ for the pure transform of Z, and $\mathcal{I}_{\widetilde{Z}}$ for the associated sheaf of ideals on $\widetilde{X}$.

Proposition 2.17. The pure transform of Z (Def. 2.7) is a G-stable closed subscheme of $\widetilde{X}$.

Proof. The quotient $\mathcal{O}_X/\mathcal{I}_Z$ is a G-sheaf, and by Proposition 2.12 so is its pure transform, which is $\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}$ (Lem. 2.9). Since $\mathcal{O}_{\widetilde{X}}/\mathcal{I}_{\widetilde{Z}}$ is a G-sheaf, $\widetilde{Z}$ is a G-stable closed subscheme of $\widetilde{X}$. $\square$

Definition 2.18. The projective birational G-morphism $u : \widetilde{Y} \to Y$, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$, is called an *equivariant flattening* (or a *G-flattening*) of Z with respect to (f, U), if $\widetilde{Z}$ is G-stable and $\tilde{f}|_{\widetilde{Z}}$ is flat over $\widetilde{Y}$.

Theorem 2.19. Let S be a scheme and G a flat group scheme over S of finite type. For any quintuplet (X, Y, f, Z, U), where

X is a G-scheme,

Y is a noetherian G-scheme,

f is a projective G-morphism $X \to Y$ of finite type,

Z is a G-stable closed subscheme of X, and

U is a G-stable open set in Y such that $f|_Y$ is flat over U,

there is a G-flattening $\widetilde{Y} \to Y$ of Z with respect to f. If Y is integral, there is a G-flattening $\widetilde{Y} \to Y$ of Z with $\widetilde{Y}$ integral.

Proof. The proof is the same with the one for the non-equivariant case; use Theorem 2.16 instead of 2.6, and Proposition 2.17 to get that $\widetilde{Z}$ is G-stable. $\square$

2.3   Flattening of the Multiplication Map

In this section we specialize the result of §2.2 to the case f is the multiplication map $G \times_S X \to X$ of a G-scheme X, and $\mathcal{M}$ is the pullback of a (not necessarily equivariant) coherent sheaf on X by the second projection of $G \times_S X$.

Let S be a scheme and G a surjective flat group scheme over S of finite type. All schemes and morphisms considered are over S. Let X be a noetherian G-scheme of finite type, and write

$$\mu : G \times_S X \to X, \quad (g, x) \mapsto gx$$

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for the multiplication map of X. Let G act on $G \times_S X$ by multiplication on the first factor, i.e. $g(h, x) = (gh, x)$. Then the multiplication map μ is a G-morphism.

Let $\mathcal{M}$ be a coherent sheaf on X. Then the pullback $\mathcal{N} = \mathrm{pr}_2^* \mathcal{M}$ is a coherent G-sheaf on $G \times_S X$ (even if $\mathcal{M}$ is not a G-sheaf). Let U be a G-stable open set such that $\mathcal{M}|_U$ is flat, which is equivalent to $\mathcal{N}|_U$ being flat (since pr_2 is faithfully flat).

Let $f : \widetilde{X} \to X$ be projective birational G-morphism, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$:

$$
\begin{array}{ccc}
G \times_S \widetilde{X} & \xrightarrow{\ \tilde{f}\ } & G \times_S X \\
\Big\downarrow{\scriptstyle \tilde{\mu}} & & \Big\downarrow{\scriptstyle \mu} \\
\widetilde{X} & \xrightarrow{\ f\ } & X
\end{array}
$$

The morphism $\tilde{\mu}$ is the multiplication map of $\widetilde{X}$, and $\tilde{f} = 1_G \times f$.

Definition 2.20. The projective birational G-morphism $f : \widetilde{X} \to X$, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$, is called a *flattening* of $\mathcal{M}$ with respect to U, if it is a G-flattening of $\mathcal{N}$ with respect to (μ, U).

Theorem 2.21. Let S be a normal noetherian scheme and G a surjective smooth affine group scheme over S with connected fibers. Then for any triplet $(X, \mathcal{M}, U)$, where

$\quad X \quad$ is a noetherian G-scheme,

$\quad \mathcal{M} \quad$ is a coherent G-sheaf on X, and

$\quad U \quad$ is a G-stable open set in X such that $\mathcal{M}|_U$ is flat,

there is a G-flattening $\widetilde{X} \to X$ of $\mathcal{M}$. If X is integral, there is a G-flattening $\widetilde{X} \to X$ of $\mathcal{M}$ with $\widetilde{X}$ integral.

Proof. Consider the G-isomorphism given by

$$
\phi : G \times_S X \xrightarrow{\sim} G \times_S X, \quad (g, x) \mapsto (g, g^{-1}x).
$$

where G acts on the left copy of $G \times_S X$ by multiplication on both factors, i.e. $g(h, x) = (gh, gx)$, and on the right copy of $G \times_S X$ by multiplication on the left factor. A morphism

$f : \widetilde{X} \to X$ is a G-flattening of $\mathcal{N}$ with respect to μ if and only if it is a G-flattening of $\phi^*\mathcal{N}$ with respect to $\mathrm{pr}_2 = \mu \circ \phi$. Note that when G acts on both factors of $G \times_S X$, pr_2 is a G-morphism.

Let G act on itself by left multiplication. From [Su2, Thm. 4.9] there is an equivariant compactification $G \hookrightarrow G'$ of G (Def. 2.1) in a projective scheme G'. Then $G \times_S X$ is a G-stable open subset of $G' \times_S X$. The G-sheaf $\phi^*\mathcal{N}$ extends to a coherent G-sheaf $\mathcal{P}$ on $G' \times_S X$ such that $\mathcal{P}|_{G \times_S X} = \phi^*\mathcal{N}$: the pushforward of $\phi^*\mathcal{N}$ on $G' \times_S X$ is a quasi-coherent G-sheaf and can be written as the union (i.e. the direct limit) of its coherent G-subsheaves that restrict to $\phi^*\mathcal{N}$ on $G \times_S X$ (see [Th, Cor. 2.4], [B, Lem. 1], or [LM, Cor. 15.5]).

The second projection $\mathrm{pr}'_2 : G' \times_S X \to X$ is projective as a base change of $G' \to S$. Since $\mathcal{M}|_U$ is flat, $(\phi^*\mathcal{N})|_U$ is also flat (with respect to the second projection $G \times_S U \to U$), and so is $\mathcal{P}|_U$. From Theorem 2.16 there is a G-flattening $f : \widetilde{X} \to X$ of $\mathcal{P}$:

$$
\begin{array}{ccc}
G' \times_S \widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & G' \times_S X \\
{\scriptstyle \widetilde{\mathrm{pr}}'_2}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pr}'_2} \\
\widetilde{X} & \xrightarrow{\ f\ } & X
\end{array}
$$

If X is integral, we may assume that $\widetilde{X}$ is also integral. Restricting to the G-stable open set $G \times_S X$, we see that f is a G-flattening of $\mathcal{P}|_{G \times_S X} = \phi^*\mathcal{N}$ with respect to pr_2, hence a G-flattening of $\mathcal{N}$ with respect to μ. $\qquad\square$

Remark 2.22. If $S = \mathrm{Spec}\, k$ for an algebraically closed field k, and G is a linear algebraic group, we may drop the requirement that G has connected fibers, i.e. is connected. We use [Su1, Thm. 3] instead of [Su2, Thm. 4.9].

Let Y be a closed subscheme of X (not necessarily G-stable). Write $\mu_Y : G \times_S Y \to X$ for the restriction of the multiplication map μ on $G \times_S Y$. We call it the *multiplication map* of Y. In this context, U will be a G-stable open set in X such that the morphism μ_Y is flat over U. Note that $G \times_S Y$ is a G-stable closed subscheme of $G \times_S X$, for the given G-structure on $G \times_S X$. Moreover, any G-stable closed subscheme of $G \times_S X$ is of

this form.

Definition 2.23. The *pure transform* of Y with respect to (f, U) is the scheme-theoretic closure $\overline{f^{-1}(Y \cap U)} \subseteq \widetilde{X}$.

Write $\widetilde{Y}$ for the pure transform of Y.

Proposition 2.24. The pure transform of $G \times_S Y$ with respect to (f, U) is $G \times_S \widetilde{Y}$.

Proof. The pure transform of $G \times_S Y$ is a G-stable closed subscheme of $G \times_S \widetilde{X}$ (Prop. 2.17), and so of the form $G \times_S \widetilde{Y}'$ for a closed subscheme $\widetilde{Y}' \subseteq \widetilde{X}$. It is also the closure of the following set in $G \times_S \widetilde{X}$:

$$\tilde{f}^{-1}\left((G \times_S Y) \cap (G \times_S U)\right) = \tilde{f}^{-1}\left(G \times_S (Y \cap U)\right) = G \times_S f^{-1}(Y \cap U),$$

which is contained in $G \times_S \widetilde{Y}$ by the definition of $\widetilde{Y}$ (since U is G-stable, $\mu^{-1}(U) = G \times_S U$). Therefore, there are inclusions:

$$G \times_S f^{-1}(Y \cap U) \subseteq G \times_S \widetilde{Y}' \subseteq G \times_S \widetilde{Y}.$$

Take the images under the second projection $G \times_S \widetilde{X} \to \widetilde{X}$ and then the closures in $\widetilde{X}$, to get $\widetilde{Y}' = \widetilde{Y}$. $\qquad\square$

Definition 2.25. The projective birational G-morphism $f : \widetilde{X} \to X$, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \hookrightarrow U$, is called an *equivariant flattening* (or a *G-flattening*) of Y with respect to U, if it is a G-flattening of $G \times_S Z$ with respect to (μ, U).

Theorem 2.26. Let S be a normal noetherian scheme and G a surjective smooth affine group scheme over S with connected fibers. For any triplet (X, Y, U), where

$\quad X \quad$ is a noetherian G-scheme,

$\quad Y \quad$ is a closed subscheme of X, and

$\quad U \quad$ is a G-stable open set in X such that μ_Y is flat over U,

there is a G-flattening $\widetilde{X} \to X$ of Y. If X is integral, there is a G-flattening $\widetilde{X} \to X$ of Y with $\widetilde{X}$ integral.

Proof. Consider the coherent G-sheaf $\mathcal{O}_{G\times_S X}/\mathcal{I}_{G\times_S Y} = \mathrm{pr}_2^*(\mathcal{O}_X/\mathcal{I}_Y)$. Flatness of μ_Y over U is equivalent to flatness of $(\mathcal{O}_{G\times_S X}/\mathcal{I}_{G\times_S Y})|_U$, and so of $(\mathcal{O}_X/\mathcal{I}_Y)|_U$. Apply Theorem 2.26 to get a flattening $f : \widetilde{X} \to X$ of $\mathcal{O}_X/\mathcal{I}_Y$:

$$
\begin{array}{ccc}
G\times_S \widetilde{X} & \xrightarrow{\ \widetilde{f}\ } & G\times_S X \\
{\scriptstyle \tilde{\mu}}\downarrow & & \downarrow{\scriptstyle \mu} \\
\widetilde{X} & \xrightarrow{\ f\ } & X
\end{array}
$$

If X is integral, we may assume that $\widetilde{X}$ is also integral. By definition, this is a G-flattening of $\mathcal{O}_{G\times_S X}/\mathcal{I}_{G\times_S Y}$ with respect to μ. Flatness of the pure transform of $\mathcal{O}_{G\times_S X}/\mathcal{I}_{G\times_S Y}$, which is $\mathcal{O}_{G\times_S \widetilde{X}}/\mathcal{I}_{G\times_S \widetilde{Y}}$ by Lemma 2.9 and Proposition 2.24, is equivalent to flatness of $\tilde{\mu}_{\widetilde{Y}} : G\times_S \widetilde{Y} \to \widetilde{X}$, where $\widetilde{Y}$ is the pure transform of Y, and $G\times_S \widetilde{Y}$ the one of $G\times_S Y$. Thus f is a G-flattening of $G\times_S Y$ with respect to (μ, U), that is a flattening of Y. $\qquad\square$

2.4 Tropical Compactifications

Let S be a scheme and G a flat group scheme over S of finite type. All schemes and morphisms considered are over S. Let U be a noetherian G-scheme. We consider open dense G-embeddings $U \hookrightarrow X$ for a variable noetherian G-scheme X. For such an embedding, U is viewed as a G-stable open dense subset of X. Write $\mu : G\times_S X \to X$ for the multiplication map of X. Let $Y \subseteq U$ be a closed subscheme, typically not G-stable.

Definition 2.27. The scheme-theoretic closure $\overline{Y} \subseteq X$ is called a *tropical compactification* of Y if $\overline{Y}$ is proper, and the multiplication map

$$
\mu_{\overline{Y}} : G\times_S \overline{Y} \to X, \quad (g,y) \mapsto gy
$$

is faithfully flat.

We say that U is *geometrically homogeneous* if for any algebraically closed field k, and any morphism $\mathrm{Spec}\, k \to S$, the geometric fiber $U_k = U \times_S \mathrm{Spec}\, k$ is a homogeneous space, i.e. the action of $G_k = G \times_S \mathrm{Spec}\, k$ on it is transitive.

Definition 2.28. We say that the scheme U is *homogeneous* if:

(i) it is geometrically homogeneous,

(ii) it is flat and of finite type, and

(iii) the fibers of the morphism

$$\psi : G \times_S U \to U \times_S U, \quad (g, u) \mapsto (gu, u)$$

are reduced, of the same dimension.

Remark 2.29. If $S = \operatorname{Spec} k$ for some algebraically closed field k, and U is a homogeneous space, then U is a homogeneous scheme over k. Indeed, it is certainly geometrically homogeneous, flat and of finite type over k. The fibers of the morphism ψ are the stabilizers of the action of G, which are all isomorphic to the subvariety $H \subseteq G$, hence reduced and of the same dimension.

Lemma 2.30. If S is normal and noetherian, G is smooth, and U is homogeneous, then the morphism ψ in Definition 2.28 is flat.

Proof. The scheme U is smooth over S. Indeed, it is flat and of finite type, and geometric fibers are homogeneous spaces, hence smooth. The first projection $G \times_S U \to G$ is smooth as a base change of $U \to S$, so that its composition with the structure morphism $G \to S$, namely the structure morphism $G \times_S U \to S$, is also smooth. In particular, it is a normal morphism (in the sense of [EGAIV, Def. 6.8.1]). Since S is normal, so is $G \times_S U$ (see [EGAIV, Prop. 6.14.1]). Therefore $G \times_S U$ is a disjoint union of integral schemes. For a similar reason, $U \times_S U \to S$ is also smooth and $U \times_S U$ normal. Applying [HKT, Lemma 10.12] on each component of $G \times_S U$ shows that ψ is flat. $\square$

Theorem 2.31. Let S be a normal noetherian scheme and G a surjective smooth affine group scheme over S with connected fibers. For any pair (Y, U), where

U is a homogeneous scheme that admits an equivariant compactification in a noetherian scheme (Def. 2.1), and

Y is a closed subscheme of U that is flat over S, and is such that

$$\mu(G \times_S Y) = U,$$

there is a tropical compactification $\overline{Y} \subseteq X$. If U admits an equivariant compactification in an integral noetherian scheme, there is a tropical compactification $\overline{Y} \subseteq X$ in some integral noetherian scheme X.

Proof. The multiplication map $\mu_Y : G \times_S Y \to U$ is flat. Indeed, since $Y \to S$ and $G \times_S U \to U \times_S U$ are flat (Lem. 2.30), so are the base changes

$$
\begin{array}{ccc}
G \times_S Y \hookrightarrow G \times_S U & \qquad U \times_S Y \longrightarrow Y \\
\downarrow \qquad\qquad \downarrow & \qquad \downarrow \qquad\qquad \downarrow \\
U \times_S Y \hookrightarrow U \times_S U & \qquad U \longrightarrow S
\end{array}
$$

The map $G \times_S Y \to U \times_S Y$ is given by $(g, y) \mapsto (gy, y)$, while $U \times_S Y \to U$ is $(u, y) \mapsto u$. Their composition, which is flat, is μ_Y.

Let $U \hookrightarrow X$ be a an equivariant compactification of U with X noetherian, and let $\overline{Y} \subseteq X$ be the closure of Y. Applying Proposition 2.26 on $(X, \overline{Y}, U)$, we get a flattening of $\overline{Y}$, that is a projective birational G-morphism $f : \widetilde{X} \to X$, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$, such that the multiplication map $\tilde{\mu}_{\widetilde{Y}} : G \times_S \widetilde{Y} \to \widetilde{X}$ is flat, where $\widetilde{Y}$ is the pure transform of $\overline{Y}$, i.e. the closure of $f^{-1}(\overline{Y} \cap U) = f^{-1}(Y)$ in $\widetilde{X}$. If X is integral, we may assume that $\widetilde{X}$ is also integral. Since f is projective and X proper over S, $\widetilde{X}$ is also proper, and so is $\widetilde{Y}$. We identify $\widetilde{U} \cong U$ via f, and we view $\widetilde{X}$ as a noetherian (integral) G-scheme containing U as a G-stable open dense set, and $\widetilde{Y}$ as the closure of Y in $\widetilde{X}$.

A morphism is faithfully flat if and only if it is flat and surjective. Since $\tilde{\mu}_{\widetilde{Y}} : G \times_S \widetilde{Y} \to \widetilde{X}$ is flat, the image $\tilde{\mu}_{\widetilde{Y}}(G \times_S \widetilde{Y})$ is open in $\widetilde{X}$. If $\tilde{\mu}_{\widetilde{Y}}$ is not surjective, replace $\widetilde{X}$ by $\tilde{\mu}(G \times_S \widetilde{Y})$, which contains $\tilde{\mu}(G \times_S Y) = \mu(G \times_S Y) = U$. Note that, after this change, $\widetilde{X}$ is not necessarily proper. Then $\widetilde{Y} \subseteq \widetilde{X}$ is a tropical compactification of Y. If X is

integral, so is $\widetilde{X}$. □

Corollary 2.32. Let $S = \operatorname{Spec} k$ for an algebraically closed field, and let G be a linear algebraic group over k. For any pair (Y, U), where

$\quad U \quad$ is a homogeneous space, and

$\quad Y \quad$ is a closed subvariety of U,

there is a tropical compactification $\overline{Y} \subseteq X$.

Proof. From Remark 2.29 we know that U is a homogeneous scheme, and it is a smooth variety, hence normal, so it admits an equivariant compactification (see [Su1, Thm. 3]). Clearly $\mu(G \times Y) = U$. Repeat the proof of Theorem 2.31 using Remark 2.22 when applying Proposition 2.26, to ignore the condition on G regarding connectedness. □

The existence of an equivariant compactification of U is not a strong condition. Indeed:

Proposition 2.33. [Su2, Thm. 4.13] Let S be a normal noetherian scheme and G a surjective smooth affine group scheme over S with connected fibers. If U is a G-scheme that satisfies the following:

(i) U is flat and of finite type,

(ii) for any closed point $P \in S$, the fiber U_P is geometrically normal, and

(iii) for any point $P \in S$ of codimension 1, i.e. such that its closure is a subscheme of codimension 1, the fiber U_P is geometrically integral,

then U admits an equivariant compactification.

Our discussion so far guarantees the existence of a tropical compactification for a closed subscheme Y of a homogeneous scheme U, under certain conditions on $S, G, U,$ and Y. Having constructed one, we can get more tropical compactifications by appropriate birational modifications (on integral noetherian ambient G-schemes X), as shown in the

following proposition. Therefore, we can partially order tropical compactifications under the relation:

$$\widetilde{Y} \succeq \overline{Y} \quad \text{if there is a proper birational } G\text{-morphism } \widetilde{X} \to X$$

of integral noetherian G-schemes containing U as

an open subset, that restricts to the identity on U.

This is an analog of Proposition 2.5 in [Te] for the toric case. In the toric case, this ordering has a combinatorial meaning: we can get more tropical compactifications by refining the fan of the corresponding toric variety. We will see that the same holds for tropical compactifications in spherical varieties.

Proposition 2.34. Let S be a scheme, G a surjective flat group scheme over S of finite type, and U a noetherian G-scheme. Let $\overline{Y} \subseteq X$ be a tropical compactification of Y, and $f : \widetilde{X} \to X$ a proper birational G-morphism, with $\widetilde{X}$ an integral G-scheme, that restricts to an isomorphism on G-stable open dense sets $\widetilde{U} \xrightarrow{\sim} U$. Let $\widetilde{Y}$ be the pure transform of $\overline{Y}$. If we identify $\widetilde{U} \cong U$ via f, $\widetilde{Y}$ is a tropical compactification of Y in $\widetilde{X}$, and is equal to $f^{-1}(\overline{Y})$.

Proof. Let $\mu : G \times_S X \to X$ be the multiplication map of X. Consider the coherent sheaves $\mathcal{M} = \mathcal{O}_X / \mathcal{I}_{\overline{Y}}$ on X and $\mathcal{N} = \mathrm{pr}_2^* \mathcal{M} = \mathcal{O}_{G \times_S X} / \mathcal{I}_{G \times_S \overline{Y}}$ on $G \times_S X$. Faithful flatness of the multiplication map $\mu_{\overline{Y}}$ is equivalent to faithful flatness of $\mathcal{N}$ with respect to μ. The pure transform of $\mathcal{N}$ is $\tilde{f}^* \mathcal{N} = \mathcal{O}_{G \times_S \widetilde{X}} / \mathcal{I}_{G \times_S \widetilde{Y}}$ (Prop. 2.4, 2.24, and Lem. 2.9), which is faithfully flat. Faithful flatness of this sheaf is equivalent to faithful flatness of $\tilde{\mu} : G \times_S \widetilde{Y} \to \widetilde{X}$. Furthermore, $\widetilde{Y} = f^{-1}(\overline{Y})$ (Cor. 2.10), and so it is proper since f is. Thus $\widetilde{Y} \subseteq \widetilde{X}$ is a tropical compactification. $\square$

Proposition 2.35. Let S be a scheme and G a group scheme normal over S (in the sense of [EGAIV, Def. 6.8.1]). Let $\phi : \hat{X} \to X$ be the normalization of X. Then $\hat{X}$ has a natural structure of a G-scheme, with which ϕ is a G-morphism.

Proof. The fiber product $G \times_S \hat{X}$ is normal (see [EGAIV, Prop. 6.14.1]). The composition $\mu \circ (1_G \times \phi)$ is a dominant morphism $G \times_S \hat{X} \to X$, and by the universal property of

normalization, there is a unique morphism $\hat{\mu} : G \times_S \hat{X} \to \hat{X}$ that makes the following diagram commute:

$$
\begin{array}{ccc}
\hat{X} & & \\
\uparrow{\scriptstyle \hat{\mu}} & \searrow{\scriptstyle \phi} & \\
G \times_S \hat{X} & \xrightarrow{\ \mu \circ (1_G \times \phi)\ } & X
\end{array}
$$

Showing that $\hat{\mu}$ satisfies the properties of a multiplication map is easy and omitted. Commutativity of the latter diagram is equivalent to ϕ being a G-morphism. $\qquad \square$

Corollary 2.36. Let k be an algebraically closed field, G be a linear algebraic group over k, and U a homogeneous space. Then for any closed subvariety $Y \subseteq U$, a tropical compactification $\overline{Y} \subseteq X$, with X a normal variety, exists.

Proof. The statement follows from Corollary 2.32, Proposition 2.34, and Lemma 2.35. The normalization of any variety is a projective birational morphism. Since U is a homogeneous space, it is normal and so the normalization of an equivariant compactification X of U restricts to an isomorphism on U. $\qquad \square$

CHAPTER 3

SPHERICAL TROPICALIZATION

In this chapter we introduce tropicalization for subvarieties of spherical homogeneous spaces. In §3.1 we review some results on spherical varieties regarding their classification. All of the results are due to D. Luna and T. Vust, when working over an algebraically closed field of characteristic zero. The extension to positive characteristic is due to F. Knop. More details and proofs of the statements can be found in [LV], or in any survey on spherical varieties, for instance [Ti], or the more elementary [K], which also contains the case of positive characteristic. In §3.2 we introduce tropicalization for subvarieties of spherical homogeneous spaces.

Let k, K, $\overline{K}$, and ν be as in §1. Let G be a connected reductive group over k. Fix a Borel subgroup $B \subseteq G$. Let G/H be a spherical homogeneous space for some subgroup $H \subseteq G$. Recall that spherical means the action of B on G/H has an open (dense) orbit.

Definition 3.1. A *spherical embedding* $G/H \hookrightarrow X$ is an open G-embedding of G/H in a normal variety X. The G-variety X is called a *spherical variety*.

3.1 Spherical Varieties

In this section we explain the classification of spherical varieties for a homogeneous space G/H in terms of colored fans. First we introduce the lattice where these fans live. Then we explain how valuations correspond to points in the lattice. We define colored fans, and finally we explain their correspondence with spherical varieties.

Let $\mathfrak{X} = \mathrm{Hom}(B, k^\times)$ be the group of characters of B. Let $k(G/H)^{(B)}$ be the multiplicative group of B-semi-invariant rational functions on G/H:

$$\left\{ f \in k(G/H)^\times : \text{there is } \chi \in \mathfrak{X} \text{ such that } gf = \chi(g)f \text{ for all } g \in B \right\},$$

where G and B act on $k(G/H)$ by left translations, i.e. if $g \in G$ and $f \in k(G/H)$, $gf(x) = f(g^{-1}x)$ for all x such that $g^{-1}x$ is in the domain of f. There is a homomorphism

$$k(G/H)^{(B)} \to \mathfrak{X}, \quad f \mapsto \chi_f$$

where $\chi_f : B \to k^\times$ is the character associated to f. The kernel of this map is the set of constant (non-zero) functions, hence $k(G/H)^{(B)}/k^\times$ injects in $\mathfrak{X}$. Denote by Λ its image. It is a finitely generated free abelian subgroup of $\mathfrak{X}$. Let $\mathrm{Q} = \mathrm{Hom}(\Lambda, \mathbb{Q})$, which is isomorphic to $\Lambda^\vee \otimes_\mathbb{Z} \mathbb{Q}$, where $\Lambda^\vee$ is the dual lattice of Λ.

Any $\mathbb{Q}$-valuation of $k(G/H)$ (trivial on $k^\times$) can be restricted to B-semi-invariant functions, and then induce a homomorphism $k(G/H)^{(B)}/k^\times \to \mathbb{Q}$, i.e. an element of Q. Thus there is a map

$$\varrho : \{\mathbb{Q}\text{-valuations of } k(G/H)\} \to \mathrm{Q}.$$

Denote by $\mathcal{V}$ the set of G-invariant valuations of $k(G/H)$, i.e. valuations $v : k(G/H)^\times \to \mathbb{Q}$ such that $v(gf) = v(f)$ for all $g \in G$. Then ϱ restricts to an injection on $\mathcal{V}$. We identify $\mathcal{V}$ with its image in Q, so that a G-invariant valuation can be viewed as an element in Q. As a subset of Q, $\mathcal{V}$ is a convex cone (but not necessarily strictly convex), called the *valuation cone*. We will also view ϱ as a map from the set of prime divisors of G/H to Q, sending a prime divisor D to $\varrho(v_D)$, where v_D is the valuation induced by D.

Let $\mathcal{D}$ be the set of B-stable prime divisors of G/H. It is a finite set, since B has an open dense orbit in G/H. The elements of $\mathcal{D}$ are called *colors*.

Definition 3.2. A (*strictly convex*) *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C} \subseteq \mathrm{Q}$ and $\mathcal{F} \subseteq \mathcal{D}$, that satisfy the following:

(i) $\mathcal{C}$ is a strictly convex cone.

(ii) $\mathcal{C}$ is generated by $\varrho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$.

(iii) The relative interior of $\mathcal{C}$ intersects $\mathcal{V}$ non-trivially.

(iv) The set $\varrho(\mathcal{F})$ does not contain 0.

A *face* of a colored cone $(\mathcal{C}, \mathcal{F})$ is a pair $(\mathcal{C}_0, \mathcal{F}_0)$, where $\mathcal{C}_0$ is a face of $\mathcal{C}$ that intersects $\mathcal{V}$ non-trivially, and $\mathcal{F}_0 = \mathcal{F} \cap \varrho^{-1}(\mathcal{C}_0)$.

Definition 3.3. A *colored fan* $\mathcal{F}$ is a (non-empty) collection of colored cones $(\mathcal{C}, \mathcal{F})$ such that:

(i) Every face of a cone in $\mathcal{F}$ is also in $\mathcal{F}$.

(ii) Any element $v \in \mathcal{V}$ lies in the interior of at most one cone.

A spherical variety is called *simple* if it contains a unique closed G-orbit. Any spherical variety admits a covering by finitely many simple spherical open subvarieties.

Theorem 3.4. There is a bijection:

$$\left\{ \begin{array}{c} \text{Spherical embeddings} \\ G/H \to X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Colored} \\ \text{fans in Q} \end{array} \right\}$$

that restricts to:

$$\left\{ \begin{array}{c} \text{Simple spherical} \\ \text{embeddings } G/H \to X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Colored} \\ \text{cones in Q} \end{array} \right\}$$

We describe the association between simple spherical varieties and colored cones; the extension to arbitrary spherical varieties and colored fans is straightforward. Let X be a simple spherical variety for the spherical homogeneous space G/H, with unique closed G-orbit Y. Let $\mathcal{B} \subseteq \mathcal{V}$ be the set of G-stable prime divisors containing the closed orbit Y, and let $\mathcal{F}$ be the set of B-stable prime divisors containing Y that are not G-stable (equivalently, the ones that intersect G/H). We identify any $D \in \mathcal{F}$ with the intersection $D \cap G/H$, which is a B-stable prime divisor of G/H, i.e. a color. Thus we can view $\mathcal{F}$

as a subset of $\mathcal{D}$. Let $\mathcal{C}$ be the cone in Q generated by $\mathcal{B}$ and $\varrho(\mathcal{F})$. Then $(\mathcal{C}, \mathcal{F})$ is the colored cone associated to X.

The *support* of a colored fan $\mathcal{F}$ is

$$\operatorname{Supp} \mathcal{F} = \left(\bigcup_{(\mathcal{C}, \mathcal{F}) \in \mathcal{F}} \mathcal{C} \right) \cap \mathcal{V}.$$

A spherical variety is complete if and only if the support of the associated fan is all of $\mathcal{V}$.

Definition 3.5. An *equivariant compactification* of a spherical variety X is a complete spherical variety X' (for the same homogeneous space G/H) with an open dense G-embedding $X \hookrightarrow X'$. Given an equivariant compactification $X \hookrightarrow X'$, we view X as a G-stable open subset of X'.

Definition 3.6. A spherical variety X is called *toroidal* if the associated colored fan has no colors, i.e. if $(\mathcal{C}, \mathcal{F})$ is a cone of the fan, then $\mathcal{F} = \varnothing$.

Given a spherical variety, one can always find an equivariant compactification of it by completing the colored fan, as is done in the case of toric varieties. If the given variety is toroidal, one may assume that the equivariant compactification occurs in a toroidal spherical variety. Any spherical variety X is dominated by a toroidal one, i.e. there is a surjective proper birational G-morphism $X' \twoheadrightarrow X$, that restricts to the identity on G/H, with X' a toroidal spherical variety. If X is a toroidal spherical variety associated to a fan $\mathcal{F}$, we will write $\mathcal{C}$ instead of $(\mathcal{C}, \varnothing)$ for a colored cone in $\mathcal{F}$.

Example 3.7. Let $G = \mathrm{SL}_2$ with Borel subgroup B consisting of the upper triangular matrices, and consider the homogeneous space $G/H = \mathbb{A}^2 - \{0\}$, where 0 is the origin, and the action is given by left multiplication (the elements of $\mathbb{A}^2 - \{0\}$ are viewed as column vectors). Here H is the subgroup

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2 : a \in k \right\}.$$

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Let g_{ij} be coordinates for SL_2, and x, y coordinates for $\mathbb{A}^2 - \{0\}$. There are two B-orbits, namely

$$O = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{A}^2 \,|\, y \neq 0 \right\}, \quad D = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{A}^2 \,|\, x \neq 0 \right\}.$$

The orbit O is open, while D is closed. In particular $\mathbb{A}^2 - \{0\}$ is spherical. Also, D is a B-stable prime divisor, given by the equation $y = 0$.

The group of characters of the Borel subgroup $\mathcal{X}$ is isomorphic to $\mathbb{Z}$, where $n \in \mathbb{Z}$ is identified with

$$\chi_n : B \to k^\times, \quad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto a^n.$$

The field of rational functions is $k(x, y)$. The B-semi-invariant rational functions of $\mathbb{A}^2 - \{0\}$, up to multiplication by scalars, are y^n for $n \in \mathbb{Z}$. The character associated to y^n is χ_n. Therefore $\Lambda = \mathcal{X}$, generated by y (or $\chi = \chi_1$), and $Q = \mathrm{Hom}(\Lambda, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$, spanned by

$$\chi^* : \Lambda \to \mathbb{Q}, \quad \chi^*(y) = 1.$$

Consider the valuation $v : k(\mathbb{A}^2 - \{0\})^\times \to \mathbb{Q}$ that gives the order of vanishing of a function $f \in k(\mathbb{A}^2 - \{0\})^\times$ at the origin, i.e. if $f = y^n f_1(x, y)/y^m f_2(x, y)$ with y not dividing f_1 or f_2, then

$$v(f) = n - m.$$

This is a G-invariant valuation of $k(\mathbb{A}^2 - \{0\})$. The valuation $-v : k(\mathbb{A}^2 - \{0\})^\times \to \mathbb{Q}$ that sends $f \in k(\mathbb{A}^2 - \{0\})^\times$ to the negative of its total degree, i.e. if $f = f_1(x, y)/f_2(x, y)$ for polynomials f_1, f_2, then

$$-v(f) = \deg f_2 - \deg f_1$$

(equivalently, $-v$ gives the order of vanishing at the origin), is also G-invariant. Note that it is not equal to the negative of v in general, but only for B-semi-invariant functions. Therefore $\mathcal{V} = Q$. Since $v(y) = 1$, $v = \chi^*$ (as elements in Q) for the choice of basis on Q we have made. There is only one color, the B-orbit D, so that $\mathcal{D} = \{D\}$. It measures the

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Table 1: Spherical varieties for the homogeneous space $\mathbb{A}^2 - \{0\}$

Variety	G-orbits	Colored cones	Colored fan
$\mathbb{A}^2 - \{0\}$	$\mathbb{A}^2 - \{0\}$	$(0, \varnothing)$	
$\mathbb{A}^2$	0	$(\mathcal{R}, \mathcal{D})$	
$\mathrm{Bl}_0\, \mathbb{A}^2$	E	$(\mathcal{R}, \varnothing)$	
$\mathbb{P}^2 - \{0\}$	$W = 0$	$(-\mathcal{R}, \varnothing)$	
$\mathbb{P}^2$	$W = 0,\ 0$	$(\mathcal{R}, \mathcal{D}),\ (-\mathcal{R}, \varnothing)$	
$\mathrm{Bl}_0\, \mathbb{P}^2$	$W = 0,\ E$	$(\mathcal{R}, \varnothing),\ (-\mathcal{R}, \varnothing)$	

order of vanishing of a function on D, which is $y = 0$. If v_D is the associated valuation on $k(\mathbb{A}^2 - \{0\})$, then clearly $v_D(y) = 1$, hence $\varrho(D) = \chi^*$.

Let $\mathcal{R}$ denote the cone in Q generated by χ^*, and $-\mathcal{R}$ the one generated by $-\chi^*$. There are three distinct non-trivial colored cones in Q, and six colored fans. These fans are listed in Table 1, along with their maximal cones, the corresponding spherical varieties, and their closed G-orbits. One can see that the cone $\mathcal{R}$ adds to $\mathbb{A}^2 - \{0\}$ "limit points at the origin," while $-\mathcal{R}$ adds "limit points at infinity." The colored cone $(\mathcal{R}, \mathcal{D})$ adds a point at the origin, while $(\mathcal{R}, \varnothing)$ adds the exceptional divisor of the blowup of the plane at the origin, denoted E. Note that even though the dimension of any non-trivial colored fan is 1, the spherical varieties associated to fans with colors have a closed G-orbit of codimension 2, which is not the case for toroidal spherical embeddings. Also, the complete spherical varieties, namely $\mathbb{P}^2$ and $\mathrm{Bl}_0\, \mathbb{P}^2$, are supported on all of $\mathcal{V}$.

Now we demonstrate how to find the fan of a spherical variety. Consider the projective space $\mathbb{P}^2$ with homogeneous coordinates W, X, Y. Let $\mathbb{A}^2 - \{0\} \hookrightarrow \mathbb{P}^2$ be the embedding of $\mathbb{A}^2 - \{0\}$ in the affine plane $W \neq 0$ inside $\mathbb{P}^2$, so that $x = X/W$ and $y = Y/W$. The action of SL_2 extends naturally to an action on all of $\mathbb{P}^2$:

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} (W : X : Y) = (W : g_{11}X + g_{12}Y : g_{21}X + g_{22}Y),$$

for any $(g_{ij}) \in \mathrm{SL}_2$, $(W : X : Y) \in \mathbb{P}^2$. Thus $\mathbb{P}^2$ is a spherical variety.

There are two closed G-orbits, the boundary of the affine plane on $\mathbb{P}^2$ and the origin

0:

$$Y_1 = \left\{ (0 : X : Y) \in \mathbb{P}^2 : X, Y \in k \right\}, \quad Y_2 = \left\{ (1 : 0 : 0) \right\}.$$

The orbit Y_1 contains "limit points of $\mathbb{A}^2 - \{0\}$ at infinity," while Y_2 contains a "limit point at the origin." Each of them will give a cone in Q. There is a unique B-stable prime divisor containing Y_1, namely $W = 0$, which is G-stable. Let v_1 be the valuation associated to it. The rational function y on $\mathbb{A}^2 - \{0\}$ can be written as Y/W in $k(\mathbb{P}^2) = k(\mathbb{A}^2 - \{0\})$, so that $v_1(y) = -1$, hence $v_1 = -\chi^*$ in Q. The colored cone associated to Y_1 is then $(-\mathcal{R}, 0)$. There is a unique B-stable prime divisor containing Y_2, which is $Y = 0$, and is not G-stable. Its intersection with $\mathbb{A}^2 - \{0\}$ is the prime divisor $y = 0$, i.e. the color D. Thus, the colored cone associated to Y_2 is $(\mathcal{R}, \mathcal{D})$. The fan of $\mathbb{P}^2$ is then $\mathcal{F} = \{(0, \varnothing), (\mathcal{R}, \mathcal{D}), (-\mathcal{R}, \varnothing)\}$.

3.2 Tropicalization of Subvarieties of G/H

In this section we define a tropicalization map from the $\overline{K}$-points of G/H to the valuation cone $\mathcal{V}$. The tropicalization of a subvariety $Y \subseteq G/H$ will then be the image of $Y(\overline{K}) \subseteq G/H(\overline{K})$.

Any K-point of G/H defines a G-invariant discrete valuation, and moreover, any G-invariant valuation of G/H is a scalar multiple of a valuation defined by a K-point [LV, Sect. 4]. Roughly speaking, each K-point defines a "formal curve" in G/H, with a limit point in a G-stable divisor of some spherical variety for the homogeneous space G/H. The valuation induced by this K-point measures the order of vanishing of a rational function along that curve at the limit point. We explain this association.

Let $\gamma : \operatorname{Spec} K \to G/H$ be an element in $G/H(K)$. We want to define a discrete valuation $v_\gamma : k(G/H)^\times \to \mathbb{Z}$. Let $f \in k(G/H)^\times$. The domain of f may not contain the image of γ, but due to homogeneity, for most $g \in G$, i.e. for g from an open (dense) set of G, the one of gf does. Then one can take the pullback $\gamma^*(gf) = (gf) \circ \gamma$, which

is an element of K, i.e. a Laurent series in t. The value $\nu(\gamma^*(gf))$ may depend on the choice of g, but there is an open set of G for which it is constant, and it increases on the complement. An element $g \in G$ from this open set is referred to as a *sufficiently general element of G*. The image $v_\gamma(f)$ is then defined to be $\nu(\gamma^*(gf))$ for a sufficiently general $g \in G$. For an arbitrary $g \in G$, for which the domain of gf contains $\operatorname{im}\gamma$, $\nu(\gamma^*(gf)) \geq v_\gamma(f)$. Thus there is a map

$$val : G/H(K) \to \mathcal{V}, \quad \gamma \mapsto v_\gamma.$$

Any $\gamma : \operatorname{Spec}\overline{K} \to G/H$ factors through $\operatorname{Spec}k((t^{1/n}))$ for some $n > 0$. Indeed, if $U = \operatorname{Spec}A$ is an affine open set in G/H containing the image of γ, then the restriction of γ to a morphism $\operatorname{Spec}\overline{K} \to U$ is induced by a k-algebra homomorphism $\gamma^* : A \to \overline{K}$. If $x_1, \ldots, x_m$ is a set of generators for A (as a k-algebra), the images $\gamma^*(x_i)$ are Puiseux series and they all lie in some $k((1/n))$ for some $n \geq 0$. It follows that γ^* factors through $k((t^{1/n}))$, so that $\operatorname{Spec}\overline{K} \to U$ factors through $\operatorname{Spec}k((t^{1/n}))$, and the same holds for γ. If $\tilde{\gamma} : \operatorname{Spec}k((\tilde{t})) \to G/H$ is the induced morphism, where $\tilde{t} = t^{1/n}$, we define $v_\gamma = v_{\tilde{\gamma}}/n$. Thus we can extend the map *val* to a surjection

$$val : G/H(\overline{K}) \twoheadrightarrow \mathcal{V}, \quad \gamma \mapsto v_\gamma,$$

Remark 3.8. If $v_\gamma \in \operatorname{Trop}Y$ for some $\gamma \in G/H(K)$, then from the construction of the extension of *val* on $G/H(\overline{K})$ it is immediate that there is some $v_{\tilde{\gamma}} \in \operatorname{Trop}Y$ for $\tilde{\gamma} \in G/H(K)$ that lies in the same ray in $\mathcal{V}$ as v_γ, and $\operatorname{im}\tilde{\gamma} = \operatorname{im}\gamma$.

An alternative way to calculate $v_\gamma(f)$, given $\gamma : \operatorname{Spec}K \to G/H$ and $f \in k(G/H)^\times$, is the following. Let $k(G)$ be the field of rational functions on G, and let $L = k(G)((t))$ be the field of Laurent series over $k(G)$. Consider the valuation

$$\nu : L^\times \to \mathbb{Z}, \quad \sum_n c_n t^n \mapsto \min\{n : c_n \neq 0\}$$

where the coefficients c_n are in $k(G)$. Let $\psi_\gamma = \mu \circ \phi_\gamma$ be the morphism $\operatorname{Spec}L \to G/H$,

with ϕ_γ induced as in the diagram:

$$
\begin{array}{ccc}
\operatorname{Spec} k(G) & \longrightarrow & G \\
 & \nearrow & \quad p_1 \nearrow \\
\operatorname{Spec} L \dashrightarrow[\phi_\gamma] G \times G/H & \xrightarrow{\mu} & G/H \\
 & \searrow & \quad p_2 \searrow \\
\operatorname{Spec} K & \xrightarrow{\gamma} & G/H
\end{array}
$$

where $\operatorname{Spec} L \to \operatorname{Spec} k(G)$ and $\operatorname{Spec} L \to \operatorname{Spec} K$ are the morphism induced by the inclusions of fields $k(G) \to L$ and $K \to L$, respectively, $\operatorname{Spec} k(G) \to G$ is the morphism that sends the unique point of $k(G)$ to the generic point of G, and $\mu : G \times G/H \to G/H$ is the multiplication map. The pullback $\psi_\gamma^*(f)$ is an element in L, i.e. a Laurent series with coefficients in $k((G))$. Then $v_\gamma(f) = \nu(\psi_\gamma^*(f))$. Roughly speaking, the pullback $\psi_\gamma^*(f)$ is the function f along the curve defined by γ, permuted by an arbitrary element of G, which appears as parameters in the coefficients of the series. The extension of this to $\overline{K}$-points of G/H in straightforward.

Now let $Y \subseteq G/H$ be a closed subvariety. The set of $\overline{K}$-points of Y is a subset of $G/H(\overline{K})$.

Definition 3.9. The *tropicalization* of Y is $\operatorname{Trop} Y = val(Y(\overline{K}))$.

We will see later that it is enough to find the set $val(Y(K))$. Multiplying this set by scalars in $\mathbb{Q}_{\geq 0}$ gives the rest of $\operatorname{Trop} Y$.

Example 3.10. Let $G = \operatorname{SL}_2$ and $G/H = \mathbb{A}^2 - \{0\}$ be as in Example 3.7, and pick the same basis for Q. We describe the map *val* in this case and then find all possible tropicalizations of curves in $\mathbb{A}^2 - \{0\}$.

The $\overline{K}$-points of $\mathbb{A}^2 - \{0\}$ correspond to homomorphisms of k-algebras $k[x, y] \to \overline{K}$ such that not both x, y map to zero. Given such $\gamma : \operatorname{Spec} \overline{K} \to \mathbb{A}^2 - \{0\}$, write x_γ and y_γ for the images of $x, y \in k[x, y]$ under the map $\gamma^* : k[x, y] \to \overline{K}$. They are Puiseux series

in t. Given $(g_{ij}) \in \text{SL}_2$, we have

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g_{22}x - g_{12}y \\ -g_{21}x + g_{11}y \end{pmatrix}$$

so that the rational function gy is the sum of x and y with some (constant) coefficients. If $g = (g_{ij})$ is sufficiently general, the terms of $-g_{21}x$ and $g_{11}y$ with the lowest exponents in t do not cancel, and therefore

$$v_\gamma(y) = \nu(\gamma^*(gy)) = \min\{\nu(x_\gamma), \nu(y_\gamma)\}.$$

Thus $val(\gamma) = c\chi^*$, where $c = \min\{\nu(x_\gamma), \nu(y_\gamma)\}$.

Alternatively, $k(G) = k(g_{ij})$, $L = \bigcup_n k(g_{ij})((t^{1/n}))$, and the morphism $\psi_\gamma : \text{Spec } L \to \mathbb{A}^2 - \{0\}$ corresponds to the homomorphism of k-algebras:

$$\psi_\gamma^* : k[x, y] \to L, \quad f(x, y) \mapsto f(g \cdot (x_\gamma, y_\gamma)),$$

where

$$g \cdot (x_\gamma, y_\gamma) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x_\gamma \\ y_\gamma \end{pmatrix} = \begin{pmatrix} g_{11}x_\gamma + g_{12}y_\gamma \\ g_{21}x_\gamma + g_{22}y_\gamma \end{pmatrix}.$$

The pullback $\psi_\gamma^*(y)$ is then $g_{21}x_\gamma + g_{22}y_\gamma$. In this expression, no term from $g_{21}x_\gamma$ cancels with a term from $g_{22}y_\gamma$, since they have distinct coefficients in $k(G)$. Therefore

$$v_\gamma(y) = \nu(\psi_\gamma^*(y)) = \min\{\nu(x_\gamma), \nu(y_\gamma)\},$$

and as before, $val(\gamma) = c\chi^*$, where $c = \min\{\nu(x_\gamma), \nu(y_\gamma)\}$.

Let C be a curve in $\mathbb{A}^2 - \{0\}$ given by an equation

$$f(x, y) = \sum_{n,m} c_{n,m} x^n y^m = 0.$$

A $\overline{K}$-point $\gamma : \text{Spec } \overline{K} \to \mathbb{A}^2 - \{0\}$ factors through C precisely when the kernel of $\gamma^* : k[x, y] \to \overline{K}$ contains $f(x, y)$, i.e. if $f(x_\gamma, y_\gamma) = 0$, where $x_\gamma = \gamma^*(x)$ and $y_\gamma = \gamma^*(y)$, as before. Write $f(x, y) = f_0(x, y) + c$, where $c = c_{0,0}$ is the constant term. If $c \neq 0$, then $f(x_\gamma, y_\gamma) = 0$ implies

$$\min_{(n,m)} \{n\nu(x_\gamma) + m\nu(y_\gamma)\} \leq 0,$$

where (n, m) are the pairs of non-negative integers, not both of which are zero, such that $c_{n,m} \neq 0$. It is clear that one of $\nu(x_\gamma)$ and $\nu(y_\gamma)$ has to be non-positive, hence $v_\gamma(y) \leq 0$ and $val(\gamma) = c\chi^*$ with $c \leq 0$. It follows that $\mathrm{Trop}\, C = val(C(\overline{K}))$ is the ray $-\mathcal{R}$, i.e. the ray generated by $-\chi^*$ in Q:

$$\makebox[3cm]{\hrulefill}\!\!\circ$$

In case $c = 0$, there is no restriction on $v_\gamma(f)$, and $\mathrm{Trop}\, C$ is all of $\mathcal{V} = $ Q:

$$\makebox[2cm]{\hrulefill}\!\!\circ\!\!\makebox[2cm]{\hrulefill}$$
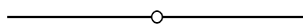
In other words, the tropicalization of a curve "passing through the origin" is all of $\mathcal{V}$, while the tropicalization of a curve not passing through it is the ray $-\mathcal{R}$.

C H A P T E R 4

TROPICAL COMPACTIFICATIONS IN SPHERICAL

VARIETIES

Let k be an algebraically closed field, G a connected reductive group over k, and G/H a spherical homogeneous space for some closed subgroup $H \subseteq G$. We use standard notation on spherical varieties, which was introduced in §3. Let $Y \subseteq G/H$ be a closed subvariety. If $G/H \hookrightarrow X$ is an open (dense) G-embedding and X is a normal variety, then X is a spherical variety. Hence any tropical compactification of Y in a normal variety occurs in a spherical variety. Our goal is to prove Theorem 1.2.

The main tool for the proof of Theorem 1.2 is Proposition 4.5, which is an extension of Lemma 2.2 (*Tevelev's Lemma*) in [Te] from toric to spherical varieties. To prove this, we first need to show that if $v \in \operatorname{Trop} Y$, then the whole ray of v is in $\operatorname{Trop} Y$.

Lemma 4.1. If $v \in \operatorname{Trop} Y$, then $cv \in \operatorname{Trop} Y$ for any $c \in \mathbb{Q}_{\geq 0}$.

Proof. Let $v \in \operatorname{Trop} Y$, say $v = v_\gamma$ for some $\gamma \in Y(\overline{K})$, and pick any $c \in \mathbb{Q}_{\geq 0}$. Consider the endomorphism of $\overline{K}$ (as a k-algebra):

$$\phi^* : \overline{K} \to \overline{K}, \quad f(t) \mapsto f(t^c).$$

This induces a morphism $\phi : \operatorname{Spec} \overline{K} \to \operatorname{Spec} \overline{K}$ of schemes over k. The composition $\tilde{\gamma} = \gamma \circ \phi$ is a $\overline{K}$-point of Y. We claim that $v_{\tilde{\gamma}}$, which is in $\operatorname{Trop} Y$, is equal to cv.

Let $f \in k(G/H)^\times$, and let $g \in G$ be such that the domain of gf contains $\operatorname{im} \gamma = \operatorname{im} \tilde{\gamma}$. Then $\tilde{\gamma}^*(gf)(t) = \gamma^*(gf)(t^c)$, and hence

$$\nu(\tilde{\gamma}^*(gf)) = c\,\nu(\gamma^*(gf))$$

(here we use that $c \geq 0$). Since this is true for any $g \in G$ with gf defined on $\operatorname{im}\gamma$, it follows that $v_{\tilde{\gamma}}(f) = cv_\gamma(f)$, and hence $v_{\tilde{\gamma}} = cv$. $\qquad\square$

Remark 4.2. From this lemma and Remark 3.8 follows that to find $\operatorname{Trop} Y$ it suffices to find $val(G/H(K))$ and then multiply by scalars in $\mathbb{Q}_{\geq 0}$.

Let $R = k[[t]]$ be the ring of power series over k. It is a discrete valuation ring with field of fractions K. If γ is a K-point of G/H, and $G/H \hookrightarrow X$ a spherical embedding, then due to separatedness there is at most one morphism $\theta : \operatorname{Spec} R \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\operatorname{Spec} K & \xrightarrow{\;\gamma_X\;} & X \\
\downarrow & \nearrow\raisebox{-1ex}{θ} & \\
\operatorname{Spec} R & &
\end{array}
$$

where γ_X is the composition of γ with $G/H \hookrightarrow X$. If such a morphism θ exist, write x and ξ for the images of the closed and the generic point of $\operatorname{Spec} R$, respectively. The point ξ is the image of γ_X, and is in G/H. The point x is called the *limit point* of γ in X, denoted $\lim \gamma$. It lies in the closure of ξ. We say that $\lim \gamma$ exists in X if the morphism θ exist. If $X \to X'$ is a G-morphism of spherical varieties, that fixes G/H, then the image of the limit point of γ in X (if it exists) is the limit point of γ in X'. This can be extended to $\overline{K}$-points of G/H, since any morphism $\operatorname{Spec} \overline{K} \to G/H$ factors through $\operatorname{Spec} k((t^{1/n}))$ for some $n \in \mathbb{Z}_{\geq 0}$.

The following lemma, in a more general form, is in [LV, Sect. 4.8], but we include the proof for completeness. Given a cone $\mathcal{C} \subseteq \mathcal{V}$, denote by $\mathcal{C}^\circ$ its relative interior.

Lemma 4.3. If $\mathcal{R} \subseteq \mathcal{V}$ is a ray, X the associated toroidal simple spherical variety with closed G-orbit O, and $v_\gamma \in \mathcal{R}^\circ$ for some $\gamma \in G/H(K)$, then $\lim \gamma$ exists in X and lies in the closed orbit of X.

Proof. The ray $\mathcal{R}$ is generated by some G-invariant discrete valuation v_D, associated to a G-stable prime divisor $D \subset X$ containing O. Since X is toroidal and $\dim \mathcal{C} = 1$, O is

of codimension 1, hence $D = O$. Write $v_\gamma = cv_D$ for some $c \in \mathbb{Q}_{\geq 0}$.

Let $X \hookrightarrow X'$ be an equivariant compactification (Def. 3.5), with X' a toroidal spherical variety. Due to properness of X', there is a (unique) morphism $\theta : \operatorname{Spec} R \to X'$ such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\ \gamma_{X'}\ } & X' \\ \downarrow & {\Large\nearrow}\ \theta & \\ \operatorname{Spec} R & & \end{array}$$

Write $\tilde{x}$ and $\tilde{\xi}$ for the closed and the generic point of $\operatorname{Spec} R$, respectively, and let $x = \theta(\tilde{x})$ and $\xi = \theta(\tilde{\xi})$ be their images in X'.

Consider the induced map on stalks:

$$\theta_x^* : \mathcal{O}_{X',x} \to \mathcal{O}_{\operatorname{Spec} R, \tilde{x}}.$$

We can view any $f \in \mathcal{O}_{X',x}$ as a rational function on X', i.e. an element in $k(X') = k(G/H)$, that is defined on $\operatorname{im} \gamma = \operatorname{im} \gamma_{X'}$, in which case

$$\gamma^*(f) = \gamma_{X'}^*(f) = \theta_x^*(f).$$

Let $U = \operatorname{Spec} A$ be an affine open set in X' that contains x. Let $\mathfrak{p} \subset A$ be the prime ideal associated to x. We view θ_x^* as a map $A_\mathfrak{p} \to R$ via the natural identifications. It is a local homomorphism: if $f \in A_\mathfrak{p}$, then $\theta_x^*(f)$ is a unit, i.e. a series in $R = k[[t]]$ with a non-zero constant term, precisely when f is a unit, hence $\nu(\gamma^*(f)) = 0$ if f does not vanish at x, and $\nu(\gamma^*(f)) > 0$ otherwise.

We show that $x \in O$. Assume the opposite is true. Write O' for the closure of O in X'. We consider three different cases: (i) $x \in G/H$, (ii) $x \notin G/H$ and $x \notin O'$, and (iii) $x \in O' - O$.

(i) Pick an affine open set in X' that contains x and intersects O', say $U = \operatorname{Spec} A$. Such an affine open set always exists: if $U_0 \subseteq X'$ is an affine open set containing a point of O', then for an appropriate $g \in G$, $U = gU_0$ is an affine open set that contains x and still

intersects D (also because x and D are both in the simple spherical variety X, which is quasi-projective). Let $\mathfrak{p} \subset A$ be the prime ideal associated to x, and write $O' \cap U = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$.

Since $x \notin O'$, $\mathfrak{p} \not\supseteq \mathfrak{a}$, say $f \in \mathfrak{a}$ but $f \notin \mathfrak{p}$. We view $f \in A$ as a rational function on X'. It is a unit in $A_\mathfrak{p}$, hence $\nu(\gamma^*(f)) = 0$, which implies $v_\gamma(f) \leq 0$. On the other hand, since $f \in \mathfrak{a}$, f vanishes on $O' \cap U$ and so $cv_D(f) > 0$, as v_D measures the order of vanishing on $O' \cap U$.

(ii) Since x is in the boundary $X' - G/H$ but not in O', it lies in a G-stable prime divisor D' which is distinct from D. The associated G-invariant non-zero valuation $v_{D'}$ is different from $v_D = cv_\gamma$ (and not a positive multiple of it). Thus for some rational function $f \in k(G/H)^\times$, $v_{D'}(f) > 0$ but $v_\gamma(f) = 0$, if v_D and $v_{D'}$ are not collinear in $\mathcal{V}$, or $v_\gamma(f) < 0$, if they are.

Pick an affine open set $U = \operatorname{Spec} A$ containing x, and hence intersecting D', and let $\mathfrak{p}$ be the prime ideal associated to x. Write $D' \cap U = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. Since $x \in D'$, $\mathfrak{p} \supseteq \mathfrak{a}$. If $f \in A$ is such that $v_{D'}(f) > 0$, then f vanishes on $D' \cap U$, i.e. $f \in \mathfrak{a}$, and hence $f \in \mathfrak{p}$. Then f is not a unit in $A_\mathfrak{p}$, so that $\nu(\gamma^*(f)) > 0$. For any g from an open set of G, gf is a rational function on X' vanishing at x. Indeed, if $O_x \subseteq D'$ is the G-orbit where x is in, then $O_x \cap U$ is a non-empty open set, since it contains x, and f vanishes on it. Then, for g from an open set of G, the intersection of the domain of gf with O_x is an open set containing x, and gf vanishes at it. Thus gf is not a unit in $\mathcal{O}_{X',x}$, and as before $\nu(\gamma^*(gf)) > 0$. It follows that $v_\gamma(f) > 0$. Since this holds for any regular function around x, there is no rational function f on X' such that $v_{D'}(f) > 0$ but $v_\gamma(f) \leq 0$.

(iii) If $x \in O' - O$, then in particular $O' - O$ is non-empty and O' strictly contains O. Also, O is open dense in O', hence $O' - O$ is a closed set in X', which is G-stable. Let $U = \operatorname{Spec} A$ be an affine open set containing x, and so intersecting O' and O, and let $\mathfrak{p} \subset A$ be the prime ideal associated to x. Write $O' \cap U = V(\mathfrak{a})$ and $(O' - O) \cap U = V(\mathfrak{b})$

(with $\mathfrak{a}, \mathfrak{b}$ radical). Since $O' - O \subset O'$, $\mathfrak{a} \subset \mathfrak{b}$ (strict inclusion). Also, since $x \in O' - O$, $\mathfrak{p} \supseteq \mathfrak{b}$.

Let $f \in A$ be such that $f \in \mathfrak{b}$ but $f \notin \mathfrak{a}$, and so $f \in \mathfrak{p}$. We view f as a rational function on X'. In particular f does not vanish on $O' \cap U$, hence $cv_D(f) = 0$. On the other hand, f is not a unit in $A_\mathfrak{p}$, hence $\nu(\gamma^* f) > 0$. Since $O' - O$ is G-stable, f vanishes on $(O' - O) \cap U$, and $x \in O' - O$, like in (ii), for any g from an open set of G, gf is a rational function on X' such that $\nu(\gamma^*(gf)) > 0$. We deduce that $v_\gamma(f) > 0$, which is a contradiction.

Since $x \in O$, in particular $x \in X$ and hence $\operatorname{im} \theta \subseteq X$. Thus θ factors through X; abusing notation, write $\theta : \operatorname{Spec} R \to X$. We have a commutative diagram:

$$
\begin{array}{ccc}
\operatorname{Spec} K & \xrightarrow{\gamma_X} & X \\
\downarrow & \nearrow{\scriptstyle \theta} & \\
\operatorname{Spec} R & &
\end{array}
$$

It follows that $\lim \gamma$ exists in X and is equal to x, which is in O. $\qquad \square$

Lemma 4.4. If $\mathcal{C} \subseteq \mathcal{V}$ is a (non-trivial) cone, X the associated toroidal simple spherical variety with closed G-orbit O, and $x \in O$, then there is a $\gamma \in G/H(K)$ such that $\lim \gamma = x$ in X and $v_\gamma \in \mathcal{C}°$. Moreover, if $Y \subseteq G/H$ is a closed subvariety, $\overline{Y} \subseteq X$ its closure, and the point $x \in O$ lies in $\overline{Y}$, we may assume that $\gamma \in Y(K)$.

Proof. Note that the first statement is a special case of the second for $Y = G/H$, so we only need to prove the second one. Pick an affine open set $U = \operatorname{Spec} A$ containing x, and let $\mathfrak{p} \subset A$ be the prime ideal associated to x. Write $\overline{Y} \cap U = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subseteq A$. Then the local ring $\mathcal{O}_{\overline{Y}, x} = (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}$.

Let B be a discrete valuation ring that dominates $(A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}$, i.e. $(A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}$ is contained in B and $\mathfrak{m}_B \cap (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} = \mathfrak{m}_{(A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}}$, where $\mathfrak{m}_{(A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}}}$ and $\mathfrak{m}_B$ are the maximal ideals. The completion $\widehat{B}$ of B is isomorphic to $R = k[[t]]$, and we identify it with this ring (this follows from *Cohen's Structure Theorem*, see [E, Prop. 10.16]). The maps $\phi^* : A/\mathfrak{a} \hookrightarrow K$

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and $\psi^* : A/\mathfrak{a} \hookrightarrow \widehat{B}$ that come from compositions of the following inclusions

$$A/\mathfrak{a} \hookrightarrow (A/\mathfrak{a})_{\mathfrak{p}/\mathfrak{a}} \hookrightarrow B \hookrightarrow \widehat{B} \hookrightarrow K$$

give rise to morphisms $\phi : \operatorname{Spec} K \to \overline{Y}$ and $\psi : \operatorname{Spec} \widehat{B} \to \overline{Y}$ of schemes over k, such that ϕ is the composition of the natural morphism $\operatorname{Spec} K \to \operatorname{Spec} \widehat{B}$ with ψ. The image of ϕ is actually in Y: since ϕ^* is an inclusion, the preimage of the zero ideal in K is the zero ideal in $A/\mathfrak{a}$, hence a generic point, which must be in the open set $Y \cap U \subseteq \overline{Y} \cap U$. Thus ϕ factors through Y and we have a K-point of Y, say $\gamma \in Y(K)$:

Furthermore, from the construction of B, $(\psi^*)^{-1}(\mathfrak{m}_{\widehat{B}}) = \mathfrak{p}/\mathfrak{a}$, so that the closed point of $\operatorname{Spec} \widehat{B}$ maps to x in $\overline{Y}$ via ψ.

The composition of ϕ with $\overline{Y} \hookrightarrow X$ is the same as γ_X. Let θ be the composition of ψ with $\overline{Y} \hookrightarrow X$. We have a commutative diagram:

It follows that $\lim \gamma$ exists and is equal to x.

Now we show that $v_\gamma \in \mathcal{C}^\circ$. Let $\mathcal{C}_0$ be the ray generated by v_γ in $\mathcal{V}$, and let X_0 be the associated toroidal simple spherical variety, with closed G-orbit O_0. Write $Y_0 \subseteq X_0$ for the closure of Y. If $\mathcal{C}_0$ is in $\mathcal{C}^\circ$ then we are done. Assume not. We consider two cases, (i) $\mathcal{C}_0$ is not contained in $\mathcal{C}$, and (ii) $\mathcal{C}_0$ is in $\mathcal{C} - \mathcal{C}^\circ$.

(i) Let $\mathcal{F}$ be a fan (without colors) that contains both $\mathcal{C}_0$ and $\mathcal{C}$, and let X' be the associated toroidal spherical variety. There are open G-embeddings $X_0 \hookrightarrow X'$ and $X \hookrightarrow X'$ that fix G/H. We treat X_0 and X as G-stable open subsets of X'. Since the cones $\mathcal{C}_0$ and $\mathcal{C}$ don't intersect (except at the origin), the orbit O_0 is disjoint from the

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orbit O in X'. Since $v_\gamma \in \mathcal{C}_0^\circ$, from Lemma 4.3 we know that $\lim \gamma$ exists in X_0 and is in O_0. Then clearly $\lim \gamma$ exists in X' and is the same point in O_0. But this cannot be true, because $\lim \gamma = x$ as shown above, which is in O.

(ii) There is a birational G-morphism $X_0 \to X$ that fixes G/H. Since $\mathcal{C}_0$ is not contained in $\mathcal{C}^\circ$, the closed orbit O_0 of X_0 does not map to the closed orbit O of X, but to an orbit of smaller codimension. From Lemma 4.3, $\lim \gamma$ exists in X_0 and is in O_0. The image of $\lim \gamma$ under $X_0 \to X$, which is the limit point of γ in X, is a point in the boundary of X that is not in O. But by construction, the limit point of γ in X lies in the orbit O. $\qquad\square$

Proposition 4.5. Let X be a simple toroidal spherical variety with closed G-orbit O, and let $\mathcal{C}$ be the associated cone in Q. Then $\mathrm{Trop}\, Y$ intersects the relative interior of $\mathcal{C}$ if and only if the closure $\overline{Y} \subseteq X$ intersects the closed orbit O.

Proof. First assume that $\mathrm{Trop}\, Y \cap \mathcal{C}^\circ \neq \varnothing$, and let $v \in \mathrm{Trop}\, Y \cap \mathcal{C}^\circ$. From Remark 3.8 we may assume that $v = v_\gamma$ with $\gamma \in Y(K)$. Let X_0 be the toroidal simple spherical variety associated to $\mathcal{C}_0$, O_0 the closed G-orbit of X_0, and $Y_0 \subseteq X_0$ the closure of Y. Since $\mathcal{C}_0$ is in $\mathcal{C}^\circ$, there is a G-morphism $f : X_0 \to X$ that fixes G/H and sends O_0 to O. Also, f maps Y_0 to $\overline{Y}$. Therefore, if x is a point in $Y_0 \cap O_0$, then $f(x) \in \overline{Y} \cap O$, hence it suffices to show that $Y_0 \cap O_0$ is non-empty. From Lemma 4.3, the limit point of γ is in O_0:

$$
\begin{array}{ccc}
\mathrm{Spec}\, K & \xrightarrow{\ \gamma_{X_0}\ } & X_0 \\
\downarrow & \nearrow{\scriptstyle \theta} & \\
\mathrm{Spec}\, R & &
\end{array}
$$

Since γ_{X_0} factors through Y, the image of the generic point of $\mathrm{Spec}\, R$ under θ, say ξ, is in Y. As $\lim \gamma$ is in the closure of ξ, it is also in the closure Y_0. Thus $\lim \gamma \in Y_0 \cap O_0$ and we are through.

Now assume that $\overline{Y} \cap O \neq \varnothing$. Pick $x \in \overline{Y} \cap O$. From Lemma 4.4 there is a K-point $\gamma \in Y(K)$ such that $v_\gamma \in \mathcal{C}^\circ$. Clearly $v_\gamma \in \mathrm{Trop}\, Y$, and so $\mathrm{Trop}\, Y \cap \mathcal{C}^\circ \neq \varnothing$. This completes the proof. $\qquad\square$

The proofs of the following propositions are the same with the ones of Propositions 2.3 and 2.5 in [Te] for the toric case, but they are short and we include them for the sake of completeness, with the appropriate modifications.

Proposition 4.6. Let X be a toroidal spherical variety, and let $\mathcal{F}$ be the associated fan. Then $\overline{Y}$ is complete if and only if $\operatorname{Trop} Y \subseteq \operatorname{Supp} \mathcal{F}$.

Proof. First assume that $\overline{Y}$ is complete but $\operatorname{Trop} Y$ is not contained in $\operatorname{Supp} \mathcal{F}$. Let $X \hookrightarrow X'$ be an equivariant compactification of X in some toroidal spherical variety X', associated to a fan $\mathcal{F}'$ containing $\mathcal{F}$. Since X' is complete, $\operatorname{Supp} \mathcal{F}' = \mathcal{V}$, hence there is a cone $\mathcal{C}$ of $\mathcal{F}'$ whose interior does not intersect $\mathcal{F}$ and contains a point of $\operatorname{Trop} Y$. Let Y' be the closure of Y (or of $\overline{Y}$) in X'. Since $\overline{Y}$ is complete, $Y' = \overline{Y}$. Thus Y' does not intersect the boundary $X' - X$. This boundary contains the closed G-orbit corresponding to $\mathcal{C}$, and this contradicts Proposition 4.5.

Now assume that $\operatorname{Trop} Y \subseteq \operatorname{Supp} \mathcal{F}$, but $\overline{Y}$ is not complete. Let $X \hookrightarrow X'$, $\mathcal{F}'$, and Y' be as above. Since $\overline{Y}$ is not complete but Y' is, as a closed subvariety of a complete variety, the inclusion $\overline{Y} \subset Y'$ is strict. In particular, Y' intersects some G-orbit in $X' - X$, which corresponds to a cone $\mathcal{C}$ of $\mathcal{F}'$ whose interior does not intersect $\mathcal{F}$. By Proposition 4.5, $\mathcal{C}^\circ$ intersects $\operatorname{Trop} Y$, but this is not the case as the latter is contained in $\operatorname{Supp} \mathcal{F}$. $\square$

Proposition 4.7. If $\overline{Y}$ is a tropical compactification of Y in a toroidal spherical variety X associated to a fan $\mathcal{F}$, then $\operatorname{Supp} \mathcal{F} = \operatorname{Trop} Y$.

Proof. Assume that the support of $\mathcal{F}$ is not $\operatorname{Trop} Y$. From Proposition 4.6, $\operatorname{Supp} \mathcal{F}$ contains $\operatorname{Trop} Y$. Let $v \in \operatorname{Supp} \mathcal{F}$ be an element not in $\operatorname{Trop} Y$. Then the entire ray generated by v is not in $\operatorname{Trop} Y$ (Lem. 4.1). Let $\mathcal{F}'$ be a refinement of $\mathcal{F}$ that contains a cone $\mathcal{C}$ that does not intersect $\operatorname{Trop} Y$ (for instance, the ray of v), and let X' be the toroidal spherical variety defined by it.

There is a proper birational G-morphism $f : X' \to X$ that fixes G/H. The closure $Y' \subseteq X'$ of Y, which is the pure transform of $\overline{Y}$ with respect to f, is a tropical compact-

ification of Y (Prop. 2.34). The multiplication morphism $\mu'_{Y'} : G \times Y' \to X'$ is faithfully flat, hence surjective, and so Y' intersects every G-orbit of X'. But from Proposition 4.5 this is not the case for the closed G-orbit associated to the cone $\mathcal{C}$. $\square$

Proof of Theorem 1.2. From Corollary 2.36 tropical compactifications of Y exist. Let $\overline{Y} \subseteq X$ be a tropical compactification, and let $\mathcal{F}$ be the fan associated to X. Let $\mathcal{F}'$ be the fan that results after removing all colors from $\mathcal{F}$, i.e. $\mathcal{F}'$ consists of all cones $\mathcal{C} \cap \mathcal{V}$ for $(\mathcal{C}, \mathcal{F}) \in \mathcal{F}$, and let X' be the associated toroidal spherical variety. In particular, $\operatorname{Supp} \mathcal{F}' = \operatorname{Supp} \mathcal{F}$. There is a proper birational G-morphism $f : X' \to X$ restricting to the identity on G/H. The closure $Y' \subseteq X'$ of Y is a tropical compactification of Y in a toroidal spherical variety (Prop. 2.34).

For the second statement, given a tropical compactification $\overline{Y} \subseteq X$, let $\mathcal{F}$, $\mathcal{F}'$, X', and $f : X' \to X$ be as before. From Proposition 4.7,

$$\operatorname{Supp} \mathcal{F} = \operatorname{Supp} \mathcal{F}' = \operatorname{Trop} Y.$$

This completes the proof. $\square$

CHAPTER 5

EXAMPLES OF SPHERICAL TROPICALIZATION

In this chapter we list a series of examples of tropicalization of subvarieties of various spherical homogeneous spaces. We use notation as in §3. First we treat the case G/H is a torus, and show that spherical tropicalization agrees with toric tropicalization, and so it is indeed an extension of the latter. Particular examples of tropicalization of subvarieties of tori are readily available in the literature, and we do not provide any. In §5.2 we find all possible tropicalizations of subvarieties of the puncture n-space $\mathbb{A}^n - \{0\}$, which completes Examples 3.7 and 3.10.

In §5.3 and §5.4 we consider GL_n, SL_n, and PGL_n, viewed as spherical homogeneous spaces under the action of $\mathrm{GL}_n \times \mathrm{GL}_n$, $\mathrm{SL}_n \times \mathrm{SL}_n$, and $\mathrm{PGL}_n \times \mathrm{PGL}_n$, respectively, by multiplication on the left and on the right, and we prove Theorem 1.3, and the equivalent ones for SL_n and PGL_n, Theorems 5.2 and 5.6, and provide some short examples of tropicalization of subvarieties of them. In the last two sections we treat two special cases of subvarieties of GL_n and SL_n, and products of them. In §5.5 we consider the G-representation variety of the fundamental group of the sphere with 3 punctures, for $G = \mathrm{GL}_n$ or SL_n, which is a subvariety of $G \times G \times G$. We find that the tropicalization of this variety is given by the Horn's inequalities. In §5.6 we demonstrate how to construct a tropical compactification with blow-ups.

We frequently make use of Remark 3.8 without mentioning so, i.e. when we calculate a tropicalization we use K-points instead of $\overline{K}$-points, and then we multiply with scalars in $\mathbb{Q}_{\geq 0}$.

5.1 Subvarieties of Tori

Let G be the n-torus $\mathbb{T}^n$ with Borel subgroup $B = G$, and consider the spherical homogeneous space $G/H = \mathbb{T}^n$, i.e. H is the trivial subgroup. Let $x_1, \ldots, x_n$ be coordinates for $\mathbb{T}^n$.

The group of characters $\mathcal{X}$ of $\mathbb{T}^n$ is isomorphic to $\mathbb{Z}^n$, where $\mathbf{a} = (a_1, \ldots, a_n)$ is identified with

$$\chi_{\mathbf{a}} : \mathbb{T}^n \to k^\times, \quad (x_1, \ldots, x_n) \mapsto x_1^{-a_1} \cdots x_n^{-a_n},$$

The lattice Λ is generated by $x_1, \ldots, x_n$. The character associated to x_i is $\chi_i = \chi_{\mathbf{e}_i}$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 in the i-th position), so that $\Lambda = \mathcal{X}$. Then $Q = \mathrm{Hom}(\Lambda, \mathbb{Q})$ is isomorphic to $\mathbb{Q}^n$, spanned by $\chi_1^*, \ldots, \chi_n^*$, where

$$\chi_i^* : \Lambda \to \mathbb{Q}, \quad \chi_i^*(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

Clearly there are no colors.

Let $\gamma \in \mathbb{T}^n(\overline{K})$, and write $x_{i,\gamma}$ for the image of x_i under the map $\gamma^* : k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \overline{K}$, which is non-zero. For any i and any $g = (g_1, \ldots, g_n)$ in $\mathbb{T}^n$, $gx_i = g_i^{-1}x_i$. Thus acting on x_i by some $g \in G$ only scales it by a constant in $k^\times$, hence $\nu(\gamma^*(gx_i)) = \nu(\gamma^*(x_i)) = \nu(x_{i,\gamma})$ and so $v_\gamma(x_i) = \nu(x_{i,\gamma})$. Then $val : \mathbb{T}^n(\overline{K}) \to Q$ sends γ to

$$val(\gamma) = \nu(x_{1,\gamma})\chi_1^* + \cdots + \nu(x_{n,\gamma})\chi_n^*,$$

or $(\nu(x_{1,\gamma}), \ldots, \nu(x_{n,\gamma}))$, in Q. This map is the same with the one of the toric tropicalization.

5.2 Subvarieties of the Punctured Affine n-space

Let $G = \mathrm{SL}_n$, with Borel subgroup B the set of upper triangular matrices, act on the punctured affine n-space $G/H = \mathbb{A}^n - \{0\}$ by left multiplication; the elements of $\mathbb{A}^n$ are

viewed as column vectors. Let g_{ij} be coordinates for G, and let $x_1, \ldots, x_n$ be coordinates for $\mathbb{A}^n - \{0\}$. This homogeneous space is spherical with open B-orbit:

$$O = \{(x_1, \ldots, x_n) \in \mathbb{A}^n : x_n \neq 0\}.$$

We have worked the case $n = 2$ in Examples 3.7 and 3.10. The situation is similar for any n. The group of characters of the Borel subgroup $\mathfrak{X}$ is isomorphic to $\mathbb{Z}^{n-1}$, where $\mathbf{m} = (m_1, \ldots, m_{n-1})$ is identified with

$$\chi_{\mathbf{m}} : B \to k^\times, \quad \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \prod_{i=1}^{n-1} a_{ii}^{-1} \end{pmatrix} \mapsto \prod_{i=1}^{n-1} a_{ii}^{m_i}.$$

The lattice Λ is generated by x_{nn}. The character associated to it is $\chi = \chi_{\mathbf{u}}$, where $\mathbf{u} = (1, 1, \ldots, 1)$. Therefore Λ is a lattice of dimension 1 inside $\mathfrak{X}$, and $Q \cong \mathbb{Q}$, spanned by χ^*, where

$$\chi^* : \Lambda \to k^\times, \quad \chi^*(x_n) = 1.$$

The valuation cone $\mathcal{V}$ is all of Q. The set of colors $\mathcal{D}$ consists of only one element, namely the B-stable divisor

$$D = \{(x_1, \ldots, x_n) \in \mathbb{A}^n - \{0\} : x_n = 0\}$$

We see that the lattice is of dimension 1. The spherical embeddings are similar to the ones of the case $n = 2$: in Table 1 replace the number "2" with the number "n". Also, the geometric description is the same. The cone $(-\mathcal{R}, \varnothing)$ adds the boundary of $\mathbb{A}^n$ in $\mathbb{P}^n$, the cone $(\mathcal{R}, \mathcal{D})$ adds a point at the origin, while $(\mathcal{R}, \varnothing)$ adds the exceptional divisor of the blow up of the affine n-space at the origin.

In Example 3.10, for the case $n = 2$, we have seen that if C is a curve, Trop C is either all of $\mathcal{V}$, if the curve passes through the origin, or the ray pointing to the left, namely $-\mathcal{R}$, if it does not. In particular, if Trop C is $-\mathcal{R}$, then the unique tropical compactification of

C occurs in $\mathbb{P}^2 - \{0\}$. Roughly speaking, this compactification adds the limit points of C at infinity. On the other hand, if $\operatorname{Trop} C$ is $\mathcal{V}$, then the unique tropical compactification of C in a toroidal spherical variety occurs in $\operatorname{Bl}_0 \mathbb{P}^2$. This compactification adds the limit points of C at infinity, as well as the limit points at the origin.

The situation is similar for $\mathbb{A}^n - \{0\}$. Let $Y \subseteq \mathbb{A}^n - \{0\}$ be a closed subvariety. Then $\operatorname{Trop} Y$ is either all of $\mathcal{V}$, if Y "contains the origin," or $-\mathcal{R}$, if it does not:

Y does not contain the origin Y contains the origin

If $\operatorname{Trop} Y$ is $-\mathcal{R}$, the unique tropical compactification of Y occurs in $\mathbb{P}^n - \{0\}$, where the limit points of Y at infinity are added. If Y contains the origin, the unique tropical compactification of C in a toroidal spherical variety occurs in $\operatorname{Bl}_0 \mathbb{P}^n$, where in addition to the limit points at infinity, the ones at the origin are also added.

Remark 5.1. Given a curve $C \subseteq \mathbb{A}^2 - \{0\}$, let $\overline{C} \subseteq \mathbb{X}$ be a tropical compactification with $\mathbb{X}$ toroidal, in which case $\mathbb{X}$ is $\mathbb{P}^2 - \{0\}$ or $\operatorname{Bl}_0 \mathbb{P}^2$. In particular $\mathbb{X}$ contains the boundary $(W = 0)$ of $\mathbb{A}^2 - \{0\}$ in $\mathbb{P}^2 - \{0\}$ (with homogeneous coordinates W, X, Y), and the intersection number $\overline{C} \cdot (W = 0)$ is the degree of the curve C. On the other hand, if $\mathbb{X} = \operatorname{Bl}_0 \mathbb{P}^2$, the intersection number $\overline{C} \cdot E$, where E is the exceptional divisor, is the "order of vanishing of C at the origin," which is the term of smallest degree of the equation defining C. It is, in general, smaller than $\deg C$. We see that if we define multiplicities as in the toric case [ST], the balancing condition does not hold.

5.3   Subvarieties of GL_n and SL_n

In this section we prove Theorem 1.3, along with its analog for the case of SL_n:

Theorem 5.2. Let Y be a closed subvariety of SL_n, defined by some ideal $I \subseteq k[\operatorname{SL}_n]$. Then $\operatorname{Trop} Y$ consists of the $(n-1)$-tuples $(\alpha_1, \ldots, \alpha_{n-1})$ of the $n-1$ greatest invariant

factors (in decreasing order) of matrices of determinant 1 with entries in $\overline{K}$, that satisfy the equations of I.

We treat the cases of GL_n and SL_n at the same time, since they are similar. Let $G = \mathrm{GL}_n$ or SL_n. Consider the group $G \times G$ with Borel subgroup B consisting of pairs of an upper and a lower triangular matrix. Then $(G \times G)/H = G$ is a homogeneous space, where the action is given by left and right multiplication, i.e.

$$(g, h) \cdot x = gxh^{-1} \quad (g, h) \in G \times G, \ x \in G.$$

Here $H = \{(g, g) \in G \times G : g \in G\}$. It is spherical with open B-orbit

$$O = \{x \in G \,|\, x_{nn} \neq 0\}.$$

Let g_{ij}, h_{ij} be coordinates for the group $G \times G$, and x_{ij} for the homogeneous space G.

If $G = \mathrm{GL}_n$, the group of characters of the Borel subgroup $\mathfrak{X}$ is isomorphic to $\mathbb{Z}^{2n}$, where $(\mathbf{l}, \mathbf{m}) = (l_1, \ldots, l_n, m_1, \ldots, m_n) \in \mathbb{Z}^{2n}$ is identified with

$$\chi_{(\mathbf{l},\mathbf{m})} : B \to k^\times, \quad ((a_{ij}), (b_{ij})) \mapsto \prod_{i=1}^{n} a_{ii}^{-l_i} b_{ii}^{m_i}.$$

If $G = \mathrm{SL}_n$, then $\mathfrak{X} \cong \mathbb{Z}^{2(n-1)}$, where $(\mathbf{l}, \mathbf{m}) = (l_1, \ldots, l_{n-1}, m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{2(n-1)}$ is identified with

$$\chi_{(\mathbf{l},\mathbf{m})} : B \to k^\times, \quad ((a_{ij}), (b_{ij})) \mapsto \prod_{i=1}^{n-1} a_{ii}^{-l_i} b_{ii}^{m_i}.$$

If $G = \mathrm{GL}_n$, the lattice Λ is generated by the (classes of) B-semi-invariant functions

$$f_i' = \det \begin{pmatrix} x_{i,i} & x_{i,i+1} & \cdots & x_{i,n} \\ x_{i+1,i} & x_{i+1,i+1} & \cdots & x_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,i} & x_{n,i+1} & \cdots & x_{n,n} \end{pmatrix} \quad \text{for } i = 1, \ldots, n$$

In particular $f_1' = \det x$ and $f_n' = x_{nn}$. The character associated to f_i' is $\chi_i' = \chi_{(\mathbf{m}_i, \mathbf{m}_i)}$, where $\mathbf{m}_i = (0, \ldots, 0, 1, \ldots, 1)$ (the first non-zero entry is the i-th one). A better set of generators is $f_1, \ldots, f_n$, where $f_i = f_i'/f_{i+1}'$ for $i < n$, and $f_n = f_n'$. The character

associated to f_i is $\chi_i = \chi'_i/\chi'_{i+1} = \chi_{(\mathbf{e}_i,\mathbf{e}_i)}$ for $i < n$, and the one to f_n is $\chi_n = \chi_{(\mathbf{e}_n,\mathbf{e}_n)}$, where $\mathbf{e}_i = (0,\ldots,0,1,0,\ldots,0)$ (the 1 in the i-th entry). The vector space Q is n-dimensional, spanned by the dual basis $\chi_1^*,\ldots,\chi_n^*$:

$$\chi_i^* : \Lambda \to \mathbb{Q}, \quad \chi_i^*(f_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

There are $n-1$ colors, which are the B-stable prime divisors $D_2,\ldots,D_n$ given by the functions $f'_2,\ldots,f'_n$, and $\varrho(D_i) = (\chi'_i)^*$ in Q.

The situation when $G = \mathrm{SL}_n$ is similar: define $f'_2,\ldots,f'_n$ as for $G = \mathrm{GL}_n$, i.e. ignore the determinant function, and then let $f_1 = 1/f'_2$, and $f_i = f'_i/f'_{i+1}$ for $i = 2,\ldots,n-1$. The character χ'_i associated to f'_i is $\chi_{(\mathbf{l},\mathbf{l})}$, where $\mathbf{l} = (-1,\ldots,-1,0,\ldots,0)$ (the first zero in the i-th position). The character χ_i associated to f_i is as in the case of $G = \mathrm{GL}_n$. The vector space Q is $(n-1)$-dimensional, spanned by $\chi_1^*,\ldots,\chi_{n-1}^*$, defined as in the case of GL_n.

We now construct the tropicalization map $val : G(\overline{K}) \to$ Q. Let $\gamma \in G(\overline{K})$, and write $\gamma^* : k[G] \to \overline{K}$ for the associated homomorphism of k-algebras, where $k[G] = k[x_{ij}]_{\det x}$ if $G = \mathrm{GL}_n$, and $k[G] = k[x_{ij}]/(1 - \det x)$ when $G = \mathrm{SL}_n$. Let $x_{ij,\gamma} = \gamma^*(x_{ij})$ for any i, j, and write x_γ for the matrix $(x_{ij,\gamma})$. Let $\alpha_1,\ldots,\alpha_n$ be the invariant factors of x_γ, in decreasing order.

If $L = \bigcup_m k(G \times G)((t^{1/m}))$, the morphism $\psi_\gamma : \mathrm{Spec}\, L \to G$ (see §3.2) is induced by the map

$$\psi_\gamma^* : k[G] \to L, \quad f(x) \mapsto f(gx_\gamma h^{-1}).$$

Then $v_\gamma(f'_i)$ is the smallest value of the valuations of all $i \times i$ minors of the matrix x_γ, and $v_\gamma(f_i) = v(f'_i) - v(f'_{i+1})$, with the special cases $v(f_n) = v(f'_n)$ if $G = \mathrm{GL}_n$, or $v(f_1) = -v(f'_2)$ if $G = \mathrm{SL}_n$. This is a well-known method for calculating the invariant factors of a matrix, hence $v(f_i) = \alpha_i$. Therefore, if $G = \mathrm{GL}_n$,

$$val(\gamma) = \alpha_1 \chi_1^* + \cdots + \alpha_n \chi_n^* \quad \text{in Q},$$

Figure 3: Valuation cone of GL_2
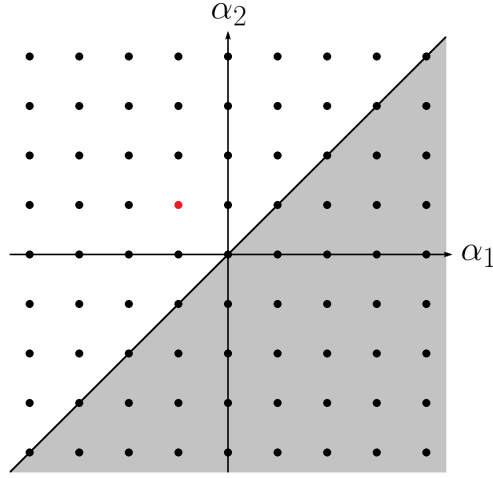
i.e. $val(\gamma)$ is the vector $(\alpha_1, \ldots, \alpha_n)$ with respect to the given basis. Similarly, if $G = \mathrm{SL}_n$, $val(\gamma) = (\alpha_1, \ldots, \alpha_{n-1})$.

Proofs of Theorems 1.3 and 5.2. Let $Y \subseteq G$ be a closed subvariety given by an ideal $I \subseteq k[G]$. Given $\gamma \in G(\overline{K})$, let $\gamma^* : k[G] \to \overline{K}$ and x_γ be as before. The $\overline{K}$-point γ factors through Y when $I \subseteq \ker \gamma^*$, i.e. when $h(x_\gamma) = 0$ for all $h \in I$, and the statements of the theorems follow. $\square$

If $G = \mathrm{GL}_n$, the valuation cone, which is the image of val, is the set

$$\mathcal{V} = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q} : \alpha_1 \geq \cdots \geq \alpha_n\}.$$

while if $G = \mathrm{SL}_n$, it is the set

$$\mathcal{V} = \left\{(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Q} : \alpha_1 \geq \cdots \geq \alpha_{n-1} \text{ and } \sum_{i=1}^{n-1} \alpha_i + \alpha_{n-1} \geq 0\right\},$$
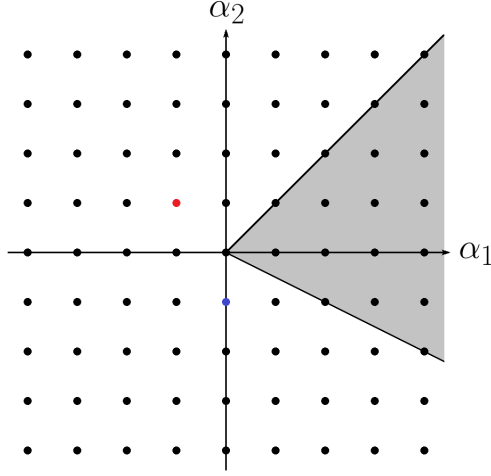
since the sum of the greatest $n-1$ invariant factors of a matrix of determinant 1 is equal to the negative of the smallest invariant factor, which must be greater or equal to $-\alpha_{n-1}$. The valuation cones of GL_2 and SL_3 are the lightly shaded areas in Figures 3 and 4.

Example 5.3. Let C be the line in GL_2 defined by the ideal

$$I = (x_{11} - x_{12} - 1,\ x_{12} - x_{21},\ x_{22}).$$

Figure 4: Valuation cone of SL_3

A matrix with entries in K that satisfies the equations $x_{11} = x_{12} + 1$, $x_{12} = x_{21}$, and $x_{22} = 0$ is of the form

$$\begin{pmatrix} z(t) + 1 & z(t) \\ z(t) & 0 \end{pmatrix}, \quad z(t) \in K.$$

The determinant of this matrix is $-z(t)^2$. If $\nu(z(t)) \geq 0$, then the smallest invariant factor is $\alpha_2 = 0$, and $\alpha_1 = \nu(-z(t)^2)$, so that α_1 can be any positive integer. This gives the ray consisting of the positive α_1-axis in Q, say $\mathcal{R}_1$. If $\nu(z(t)) < 0$, then the smallest invariant factor is $\alpha_2 = \nu(z(t))$, and $\alpha_1 + \alpha_2 = \nu(-z(t)^2) = 2\alpha_2$, so that $\alpha_1 = \alpha_2$. This corresponds to the ray along the line $\alpha_1 = \alpha_2$ in Q on the third quadrant, call it $\mathcal{R}_2$. In Figure 5 we draw $\mathrm{Trop}\, C$, which is the union of the two rays $\mathcal{R}_1$ and $\mathcal{R}_2$.

Since $\mathrm{Trop}\, C$ is of dimension 1, it completely determines the toroidal spherical variety in which the tropical compactification occurs (in this example there are no non-toroidal spherical varieties supproted on $\mathrm{Trop}\, C$). We describe this spherical variety.

We view GL_2 as a quasi-affine variety in $\mathbb{A}^4$ (with coordinates x_{ij}). Consider the projective space $\mathbb{P}^4$ with homogeneous coordinates

$$(X_0, X) = \left(X_0, \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \right).$$

We identify $\mathbb{A}^4$ with the affine space ($X_0 \neq 0$) in $\mathbb{P}^4$. The action of $GL_2 \times GL_2$ on GL_2

Figure 5: Trop $V(x_{11} - x_{12} - 1, x_{12} - x_{21}, x_{22})$
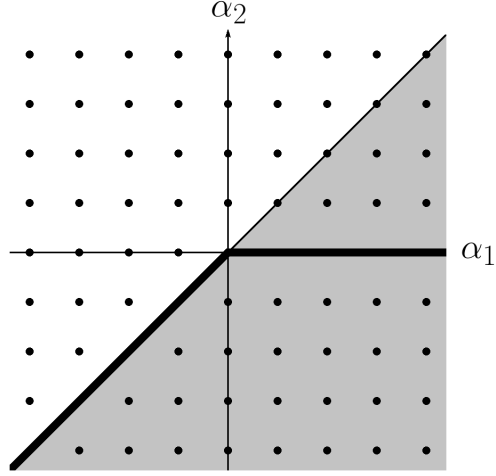
extends to an action on all of $\mathbb{P}^4$:

$$(g, h) \cdot (X_0, X) = (X_0, gXh^{-1}), \quad (g, h) \in \mathrm{GL}_2 \times \mathrm{GL}_2, \ (X, X_0) \in \mathbb{P}^4,$$

and so $\mathrm{GL}_2 \hookrightarrow \mathbb{P}^4$ is a spherical embedding. Its colored fan is given in Figure 11. The rays $\mathcal{R}_1$ and $\mathcal{R}_2$ are cones of this fan, and so the spherical varieties associated to them are GL_2-stable open subvarieties of $\mathbb{P}^4$.

The ray $\mathcal{R}_1$ corresponds to the embedding of GL_2 in the punctured affine space $\mathbb{X}_1 = \mathbb{A}^4 - \{0\}$, i.e. it adds the rank 1 matrices in $\mathbb{A}^4$, and $\mathcal{R}_2$ corresponds to the embedding of GL_2 in

$$\mathbb{X}_2 = \big\{ (X_0, X) \in \mathbb{P}^4 : \det X \neq 0 \big\}$$

i.e. it adds invertible matrices at infinity. The tropical compactification of C occurs in the variety $\mathbb{X} \subset \mathbb{P}^4$ that results when $\mathbb{X}_1$ and $\mathbb{X}_2$ are glued along GL_2, i.e. their union inside $\mathbb{P}^4$. The closure $\overline{Y} \subset \mathbb{P}$ contains two points in the boundary, namely

$$\left(1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \quad \text{and} \quad \left(0, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Example 5.4. Let Y_1 be the hyperplane $(x_{11} = 1)$ in GL_2. An invertible matrix $(x_{ij}(t))$ with entries in K that satisfies the equation $x_{11} = 1$ must satisfy $\nu(x_{11}(t)) = 0$, hence

the smallest invariant factor of such matrix, say α_2, is always non-positive. There is no restriction on the biggest invariant factor. Indeed, if (α_1, α_2) is a pair of integers with $\alpha_1 \geq \alpha_2$ and $\alpha_2 \leq 0$, then the following matrix

$$\begin{pmatrix} 1 & t^{\alpha_1} \\ t^{\alpha_2} & 0 \end{pmatrix}$$

is an invertible matrix that satisfies the equation $x_{11} = 1$ and has invariant factors (α_1, α_2). Therefore the tropicalization of Y_1 is the one of Figure 6.

If $Y_2 = V(x_{21} - x_{12}^2)$, then $\text{Trop}\, Y_2$ is all of the valuation cone. Indeed, for any pair of integers (α_1, α_2) with $\alpha_1 \geq \alpha_2$, the matrix

$$\begin{pmatrix} t^{\alpha_1} & 0 \\ 0 & t^{\alpha_2} \end{pmatrix}$$

satisfies the equation $x_{21} = x_{12}^2$ and has invariant factors (α_1, α_2).

Now consider the subvariety $Y = V(x_{11} - 1, x_{21} - x_{12}^2)$ of GL_2. An invertible matrix with entries in K that satisfies the equations $x_{11} = 1$ and $x_{21} = x_{12}^2$ is of the form

$$\begin{pmatrix} 1 & y(t) \\ y^2(t) & z(t) \end{pmatrix}, \quad y(t), z(t) \in K.$$

The determinant of this matrix is $z(t) - y^3(t)$. There are four cases:

(i) If $\nu(y(t)), \nu(z(t)) \geq 0$, then $\alpha_2 = 0$ and α_1 can be any positive number, which gives the positive α_1-axis.

(ii) If $\nu(y(t)) \leq 0, \nu(z(t)) \geq 0$, then $\alpha_2 = 2\nu(y(t))$ and $\alpha_1 = \nu(y(t))$. This is the ray along the line $\alpha_2 = -\alpha_1/2$, on the third quadrant.

(iii) If $\nu(y(t)) \geq 0, \nu(z(t)) \leq 0$, then $\alpha_2 = \nu(z(t))$ and $\alpha_1 = 0$, which is the negative α_2-axis.

(iv) If $\nu(y(t)), \nu(z(t)) \leq 0$, then there are three subcases. If $\nu(z(t))$ is more than $2\nu(y(t))$ or less than $3\nu(y(t))$, then we get back the ray along $\alpha_2 = -\alpha_1/2$ or the negative
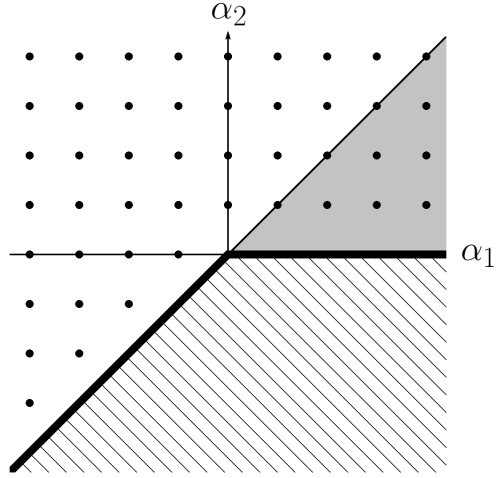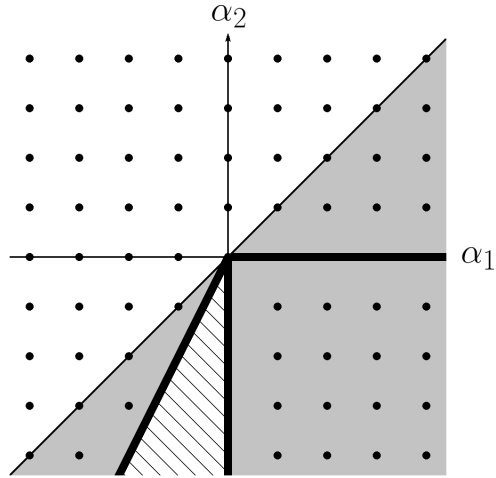
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Figure 6: Trop $V(x_{11} - 1)$

Figure 7: Trop $V(x_{11} - 1, x_{21} - x_{12}^2)$

α_2-axis, respectively. If $3\nu(y(t)) \leq \nu(z(t)) \leq 2\nu(y(t))$, then $\alpha_2 = \nu(z(t))$ and $\alpha_2/2 \leq \alpha_1 \leq 0$, and we get the cone between $\alpha_2 = -\alpha_1/2$ and the negative α_2-axis.

The tropicalization of Y_2 is given in Figure 7. Note that even though $Y = Y_1 \cap Y_2$, the tropicalization of Y is strictly smaller than the intersection $\mathrm{Trop}\, Y_1 \cap \mathrm{Trop}\, Y_2$.

Example 5.5. Assume that $\overline{K}$ is algebraically closed, which holds if char $k = 0$. Consider the special orthogonal group SO_4 as a subvariety of SL_4:

$$\mathrm{SO}_4 = \left\{ x \in \mathrm{SL}_4 : x^t x = e \right\},$$

where e is the identity matrix. Let $x(t)$ be a matrix of determinant 1 with entries in K that satisfies $x(t)^t x(t) = e$. The invariant factors of $x(t)^t$ are the same with the ones of $x(t)$, while the ones of e are all zero. Then, the invariant factors $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of $x(t)$ must satisfy the following Horn's inequalities (see §5.5):

$$\alpha_1 + \alpha_4 \geq 0 \quad \text{and} \quad \alpha_2 + \alpha_3 \geq 0.$$

Since $x(t)$ has determinant 1, $\alpha_4 = -\alpha_1 - \alpha_2 - \alpha_3$, and the first inequality becomes $\alpha_2 + \alpha_3 \leq 0$, hence $\alpha_3 = -\alpha_2$. This forces $\alpha_4 = -\alpha_1$. We show that any quadruple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ that satisfies these two conditions is in $\mathrm{Trop}\,SO_4$.

Pick $(\alpha_1, \alpha_2, -\alpha_2, -\alpha_1)$ with $\alpha_1 \geq \alpha_2 \geq 0$. The matrix with entries in $\overline{K}$:

$$\begin{pmatrix} t^{-\alpha_1} & \sqrt{1 - t^{-2\alpha_1}} & 0 & 0 \\ -\sqrt{1 - t^{-2\alpha_1}} & t^{-\alpha_1} & 0 & 0 \\ 0 & 0 & t^{-\alpha_2} & \sqrt{1 - t^{-2\alpha_2}} \\ 0 & 0 & -\sqrt{1 - t^{-2\alpha_2}} & t^{-\alpha_2} \end{pmatrix}$$

is orthogonal, of determinant 1, and has invariant factors $(\alpha_1, \alpha_2, -\alpha_2, -\alpha_1)$. It follows that

$$\mathrm{Trop}\,SO_4 = \left\{ (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{V} : \alpha_3 = -\alpha_2 \right\}.$$

It is the cone of dimension two with extremal rays the α_1-axis and the ray which is the intersection of the planes $\alpha_1 = \alpha_2$ and $\alpha_3 = -\alpha_2$, for $\alpha_2 \geq 0$. We draw $\mathrm{Trop}\,SO_4$ in Figure 8. It is the dark gray area; the lightly shaded area is the plane $\alpha_1 = \alpha_2$ (for $\alpha_1, \alpha_2 \geq 0$).

5.4   Subvarieties of PGL_n

Here we describe tropicalization of subvarieties of PGL_n. The situation is similar to the one of GL_n and SL_n. For a homogeneous matrix with entries in K it does not make sense to ask for its invariant factors. For instance, if $\alpha \in \mathbb{Z}$, I and $t^\alpha I$ refer to the same

Figure 8: Tropicalization of $SO_4 \subset SL_4$
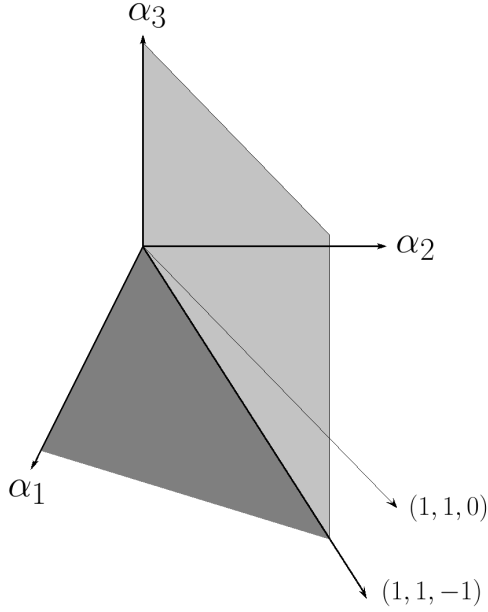
homogeneous matrix with entries in K, but the invariant factors of I are all 0, while the ones of $t^\alpha I$ are all α. Given any homogeneous matrix, there is a representation of it for which the smallest invariant factor is 0, and moreover, any such representation has the same invariant factors. We call these invariant factors, without the last one which is zero, the *relative invariant factors* of the homogeneous matrix. This definition extends naturally to invariant factors of a homogeneous matrix with entries in $\overline{K}$.

Theorem 5.6. Let Y be a closed subvariety of PGL_n, defined by some homogeneous ideal $I \subseteq k[PGL_n]$. Then $\text{Trop}\, Y$ consists of the $(n-1)$-tuples $(\alpha_1, \ldots, \alpha_{n-1})$ of the relative invariant factors (in decreasing order) of invertible homogeneous matrices with entries in $\overline{K}$, that satisfy the homogeneous equations of I.

Let $G = PGL_n \times PGL_n$, with Borel subgroup B consisting of pairs of an upper and a lower triangular homogeneous matrix, and consider the spherical homogeneous space $G/H = PGL_n$, where the action is given by left and right multiplication, i.e.

$$(g, h) \cdot X = gXh^{-1} \quad (g, h) \in PGL_n \times PGL_n, \ X \in PGL_n.$$

The group H and the open B-orbit are as in the case of GL_n and SL_n. Let g_{ij}, h_{ij} be homogeneous coordinates for $\mathrm{PGL}_n \times \mathrm{PGL}_n$, and X_{ij} homogeneous coordinates for the homogeneous space PGL_n.

The group of characters of the Borel subgroup $\mathfrak{X}$ is isomorphic to $\mathbb{Z}^{2(n-1)}$, where $(\mathbf{l}, \mathbf{m}) = (l_1, \ldots, l_{n-1}, m_1, \ldots, m_{n-1}) \in \mathbb{Z}^{2(n-1)}$ is identified with

$$\chi_{(\mathbf{l},\mathbf{m})} : B \to k^\times, \quad ((A_{ij}), (B_{ij})) \mapsto \prod_{i=1}^{n-1} \left(\frac{A_{ii}}{A_{nn}} \right)^{-l_i} \left(\frac{B_{ii}}{B_{nn}} \right)^{m_i}.$$

The lattice Λ is generated by the (classes of) B-semi-invariant functions:

$$f_i = \frac{f'_i}{X_{nn} f'_{i+1}}, \quad \text{for } i = 1, \ldots, n-1,$$

where the f'_i are given by

$$f'_i = \det \begin{pmatrix} X_{i,i} & X_{i,i+1} & \ldots & X_{i,n} \\ X_{i+1,i} & X_{i+1,i+1} & \ldots & X_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n,i} & X_{n,i+1} & \ldots & X_{n,n} \end{pmatrix} \quad \text{for } i = 1, \ldots, n$$

The character associated to f_i is $\chi_i = \chi_{(\mathbf{e}_i, \mathbf{e}_i)}$, where $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 in the i-th entry). The vector space Q is $(n-1)$-dimensional, spanned by the dual basis $\chi_1^*, \ldots, \chi_{n-1}^*$:

$$\chi_i^* : \Lambda \to \mathbb{Q}, \quad \chi_i^*(f_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

There are $n-1$ colors, which are the B-stable prime divisors D_i given by the zero sets of the homogeneous polynomials $f'_2, f'_3, \ldots, f'_n$.

We construct the tropicalization map $val : \mathrm{PGL}_n(\overline{K}) \to \mathrm{Q}$. Let $\gamma \in \mathrm{PGL}_n(\overline{K})$, and write $\gamma^* : k[X_{ij}]_{\det X}^{(k^\times)} \to \overline{K}$ for the associated homomorphism of k-algebras. There is some pair (k, l), such that not all functions

$$X_{ij,\gamma} = \gamma^* \left(\frac{X_{ij} X_{kl}^{n-1}}{\det X} \right)$$

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are zero. For such (k, l), consider the homogeneous matrix $X_\gamma = (X_{ij,\gamma})$. Write $\alpha_1, \ldots, \alpha_{n-1}$ for the relative invariant factors of x_γ, in decreasing order.

If $L = \bigcup_m k(\mathrm{PGL}_n \times \mathrm{PGL}_n)((t^{1/m}))$, the morphism $\psi_\gamma : \mathrm{Spec}\, L \to \mathrm{PGL}_n$ (see §3.2) is induced by the map

$$\psi_\gamma^* : k[X_{ij}]_{\det X}^{(k^\times)} \to L, \quad f(X) \mapsto f(gX_\gamma h^{-1}).$$

Since $f_i = (f_i'/f_{i+1}')/X_{nn}$, $\nu(f_i(X_\gamma))$ is the i-th invariant factor of X_γ (for a fixed representation) minus the smallest one (see the case of GL_n and SL_n in §5.3), i.e. the i-th relative invariant factor. Therefore

$$val(\gamma) = \alpha_1 \chi_1^* + \cdots + \alpha_{n-1} \chi_{n-1}^* \quad \text{in } \mathbb{Q},$$

i.e. $val(\gamma)$ is the vector $(\alpha_1, \ldots, \alpha_{n-1})$ with respect to the given basis.

Proof of Theorem 5.6. The proof is the same as the one of Theorems 1.3 and 5.2 for GL_n and SL_n. $\qquad\square$

The valuation cone, which is the image of val, is the set

$$\mathcal{V} = \{(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Q} : \alpha_1 \geq \cdots \geq \alpha_{n-1} \geq 0\}.$$

We draw the valuation cone for the case $n = 3$ in Figure 9.

Example 5.7. Consider the subvariety $Y \subset \mathrm{PGL}_3$ given by the homogeneous ideal $I = (X_{11} - X_{13}, X_{12}, X_{21}, X_{22}^2 - X_{33}, X_{23}, X_{31}, X_{32})$. An invertible homogeneous matrix with entries in K that satisfies the equations $X_{11} = X_{13}$, $X_{12} = X_{21} = X_{23} = X_{31} = X_{32} = 0$, and $X_{22}^2 = X_{33}$ is of the form

$$\begin{pmatrix} Y(t) & 0 & Y(t) \\ 0 & Z(t) & 0 \\ 0 & 0 & Z(t)^2 \end{pmatrix}, \quad Y(T), Z(T) \in K.$$

We consider two cases, (i) $\nu(Y(T)) \leq \nu(Z(T))$, and (ii) $\nu(Z(T)) \leq \nu(Y(T))$.

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Figure 9: Valuation cone of PGL$_3$
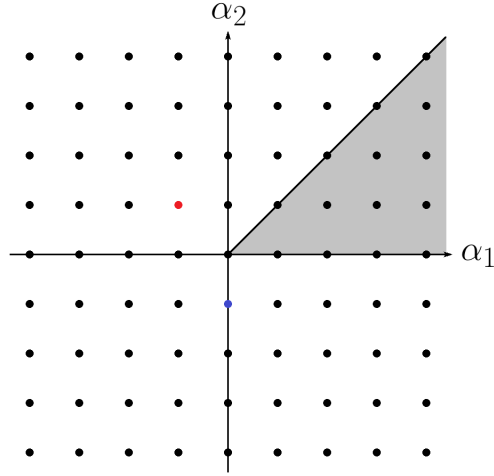
(i) Rescale the matrix by $Y(T)^{-1}$, so that its smallest invariant factor is zero:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & Z(t) & 0 \\ 0 & 0 & Z(t)^2 \end{pmatrix}$$

The second greatest invariant factor is $\alpha_2 = \nu(Z(t))$, while the greatest one is $\alpha_1 = 2\nu(Z(t)) = 2\alpha_2$. This gives the ray along the line $\alpha_1 = 2\alpha_2$, in the first quadrant.
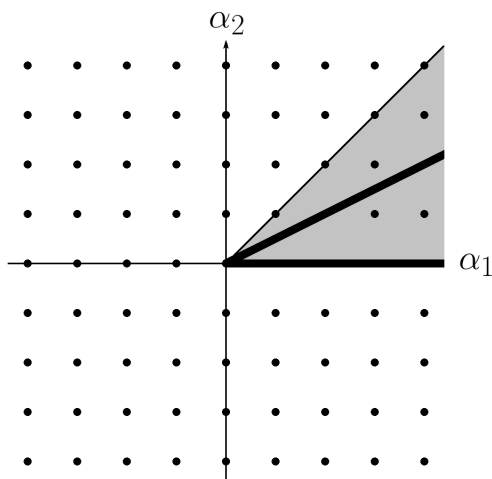
(ii) Rescale the matrix by $Z(T)^{-2}$, so that it becomes

$$\begin{pmatrix} Y(t) & 0 & Y(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The second greatest invariant factor is $\alpha_2 = 0$, while the greatest one is $\alpha_1 = \nu(Y(t))$, which can be any positive integer. This is the positive α_1-axis.

We draw Trop Y in Figure 10. It consists of two rays, the α_1-axis, and the ray along the line $\alpha_1 = 2\alpha_2$.

Figure 10: Trop $V(X_{11} - X_{13}, X_{12}, X_{21}, X_{22}^2 - X_{33}, X_{23}, X_{31}, X_{32})$

5.5 Tropicalization of the Representation Variety of $\pi_1(\mathbb{S}_{0,3})$

Let $\mathbb{S}_{0,3}$ denote the Riemann sphere with 3-punctures. The fundamental group is given by the following generators and relations:

$$\pi_1(\mathbb{S}_{0,3}) = \langle a, b, c : abc = 1 \rangle = \langle a, b, c : ab = c^{-1} \rangle,$$

where a, b, c are loops around the first, second, and third puncture, respectively. This is of course isomorphic to the free group in 2 generators, but this representation is more natural for the problem.

Let G be GL_n or SL_n. Then the G-representation variety of $\pi_1(\mathbb{S}_{0,3})$ is

$$\mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3})) = \mathrm{Hom}(\pi_1(\mathbb{S}_{0,3}), G) = \left\{ (x, y, z) \in G^3 : xy = z^{-1} \right\}.$$

We view G^3 as a homogeneous space via the action of $G^6 = (G \times G)^3$ by multiplication on the left and on the right. The lattice Q has dimension $3n$ if $G = \mathrm{GL}_n$, and $3(n-1)$ if $G = \mathrm{SL}_n$. Consider the standard basis for Q, which is an extension of the one given in §5.3 to the product of three copies of G.

If $G = \mathrm{GL}_n$, the set $\mathrm{Trop}\,\mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))$ consists of (positive scalar multiples of) $(3n)$-tuples of integers $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_n)$ that appear as invariant factors of matrices x, y, z with entries in K, such that $xy = z^{-1}$. We write $(\gamma_1', \ldots, \gamma_n')$ for

the invariant factors of the matrix z^{-1}, which are $\gamma_1' = -\gamma_n$, $\gamma_2' = -\gamma_{n-2}$, etc. It suffices to find the $(3n)$-tuplets of integers $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1', \ldots, \gamma_n')$ that appear as invariant factors of matrices x, y, z' with entries in K, such that $xy = z'$. The case $G = \mathrm{SL}_n$ is similar.

The solution to this problem is given by the Horn's inequalities (it is equivalent to Horn's problem, see [F, Thm. 7 & 17]). In particular, the elements of the $(3n)$-tuple $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \gamma_1', \ldots, \gamma_n')$ appear as invariant factors of matrices x, y, z' such that $xy = z'$ if and only if

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \gamma_i', \tag{5.1}$$

and

$$\sum_{k \in K} \gamma_i' \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_i \quad \text{for all } (I, J, K) \in T_r^n, \tag{5.2}$$

where I, J, K are subsets of $\{1, \ldots, n\}$ of the same cardinality, and T_r^n are defined inductively as

$$T_r^n = \left\{ (I, J, K) \in U_r^n : \begin{array}{c} \text{for every } p < r \text{ and } (F, G, H) \in T_p^r, \\ \displaystyle\sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + p(p+1)/2 \end{array} \right\}$$

where U_r^n are the sets of triplets (I, J, K) given by:

$$U_r^n = \left\{ (I, J, K) : \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + r(r+1)/2 \right\}.$$

The only case for which the tropicalization can be drawn is when $G = \mathrm{SL}_2$. In this case, $\mathrm{Trop}\, \mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))$ is given by the inequalities:

$$\alpha_1 \leq \beta_1 + \gamma_1, \quad \beta_1 \leq \gamma_1 + \alpha_1, \quad \gamma_1 \leq \alpha_1 + \beta_1.$$

We draw the tropicalization of $\mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))$ in Figure 2. The valuation cone is the first quadrant.

It is not yet known to the author what is a tropical compactification of $\mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))$, and which fan is associated to the ambient space. Oftentimes, compactifications of the

representation variety are used to derive a compactification of the corresponding character variety, see [Ko] and [M]. Recall that the character variety is the quotient of the representation variety by the action of G by conjugation. In our case,

$$\mathrm{Char}_G(\pi_1(\mathbb{S}_{0,3})) = \mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))/\!\!/G.$$

An interesting question to ask is whether a tropical compactification of $\mathrm{Rep}_G(\pi_1(\mathbb{S}_{0,3}))$ will produce a meaningful compactification of $\mathrm{Char}_G(\pi_1(\mathbb{S}_{0,3}))$, e.g. one with combinatorial normal crossings.

5.6 Tropical Compactification of the Maximal Torus of GL_2

Consider the maximal torus

$$T = \{x \in \mathrm{GL}_2 : x_{12} = x_{21} = 0\}.$$

We want to find a tropical compactification of T. The idea is to find the tropicalization of T, and begin with a "naive" compactification $\overline{T} \subset \mathbb{X}$ such that the colored fan of $\mathbb{X}$ is supported on $\mathrm{Trop}\,T$. Then exhibit successive blow-ups of $\mathbb{X}$ at the locus of "problematic" points until the multiplication map of the pure transform becomes flat. This amounts to refining the fan of $\mathbb{X}$ and removing colors.

The tropicalization of T is all of the valuation cone. Indeed, given a pair of integers (α_1, α_2) with $\alpha_1 \geq \alpha_2$, the invertible matrix

$$\begin{pmatrix} t^{\alpha_1} & 0 \\ 0 & t^{\alpha_2} \end{pmatrix}$$

satisfies the equations that define T and has invariant factors α_1, α_2. Thus we should begin by compactifying T in a spherical variety supported on $\mathcal{V}$, i.e. a complete spherical variety.

We view GL_2 as an open subset of $\mathbb{A}^4$, with coordinates x_{ij}, which in turn is embedded in $\mathbb{P}^4$, with homogeneous coordinates $(X_0, X) = (X_0, (X_{ij}))$, and is identified with $(X_0 \neq$

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0) (see Example 5.3). The spherical variety $\mathbb{P}^4$ has two closed GL_2-orbits: the origin of $\mathbb{A}^4$, say 0, which is the zero matrix, and the set of rank 1 matrices at infinity. There are three other orbits, the set of invertible matrices at infinity, the matrices of rank 1 in $\mathbb{A}^4$, and the open orbit GL_2. The colored fan of $\mathbb{P}^4$ is shown in Figure 11.

Let $T' \subset \mathbb{P}^4$ be the closure of T. We claim that the multiplication map $\mu_{T'} : \mathrm{GL}_2 \times \mathrm{GL}_2 \times T' \to \mathbb{P}^4$ is flat everywhere but $\mu_{T'}^{-1}(0) = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \{0\}$. We first show that all fibers of $\mu_{T'}$ but the one over $0 \in \mathbb{P}^4$ have the same dimension.

Proposition 5.8. Let G be an algebraic group over k, X a G-variety, and $Y \subseteq X$ a closed subvariety. The non-empty fibers of the multiplication map of Y:

$$\mu_Y : G \times Y \to X, \quad (g, y) \mapsto gy$$

over points in an orbit O have dimension $\dim G + \dim(Y \cap O) - \dim O$.

Proof. First we show that all fibers over points in O have the same dimension. Let $x, y \in O$, say $y = hx$ for some $h \in G$. Consider the isomorphism of varieties:

$$\phi : G \times Y \xrightarrow{\sim} G \times Y, \quad (g, z) \mapsto (gh, z).$$

The fiber of $\mu_Y \circ \phi$ over x is the same as the fiber of μ over y, and is also isomorphic to the fiber of μ_Y over x. Therefore the fibers of μ_Y over x and over y are isomorphic, hence of the same dimension.

Assume that $Y \cap O$ is non-empty. The multiplication map μ_Y restricts to a surjective morphism $G \times (Y \cap O) \to O$. The fibers over points from an open set of O have dimension:

$$\dim(G \times (Y \cap O)) - \dim O = \dim G + \dim(Y \cap O) - \dim O.$$

From the above all fibers over O have the same dimension, which must be $\dim G + \dim(Y \cap O) - \dim O$, and we are through. $\square$

The dimension of $\mathrm{GL}_2 \times \mathrm{GL}_2$ is 8, while the one of $\mathbb{P}^4$ is 4. We use Proposition 5.8 on each orbit of $\mathbb{P}^4$ to show that the dimension of all fibers but the one over $0 \in \mathbb{P}^4$ is 6. For each orbit O, we need to show that $\dim(T' \cap O) - \dim O = -2$.

(i) If $O = \mathrm{GL}_2$, then $T' \cap O = T$ is of dimension 2, while $\dim O = 4$.

(ii) The orbit O of rank 1 matrices in $\mathbb{A}^4$, i.e. in $(X_0 \neq 0)$, is the divisor $(\det x = 0) \subset \mathbb{A}^4$, without the origin 0, hence of dimension 3. It intersects T' at the following points in $\mathbb{A}^4$:

$$\begin{pmatrix} x_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & x_{22} \end{pmatrix}, \quad x_{11}, x_{22} \in k^\times.$$

The set of matrices of the first of the above two forms is the line $V(x_{12}, x_{21}, x_{22})$, which is of dimension 1, without 0. Similarly for the set of matrices of the second form. Therefore the intersection $T' \cap O$ is a union of two lines minus a point, hence of dimension 1.

(iii) Let O be the orbit of invertible matrices at infinity. It is an open set in the hyperplane $(X_0 = 0)$, hence of dimension 3. Its intersection with T' consists of diagonal matrices at infinity:

$$\left(0, \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix} \right), \quad X_{11}, X_{22} \in k^\times, \ X_{11}X_{22} \neq 0.$$

It is isomorphic to a projective line in $\mathbb{P}^3$, minus two points, and so of dimension 1.

(iv) Let O be the orbit of rank 1 matrices at infinity:

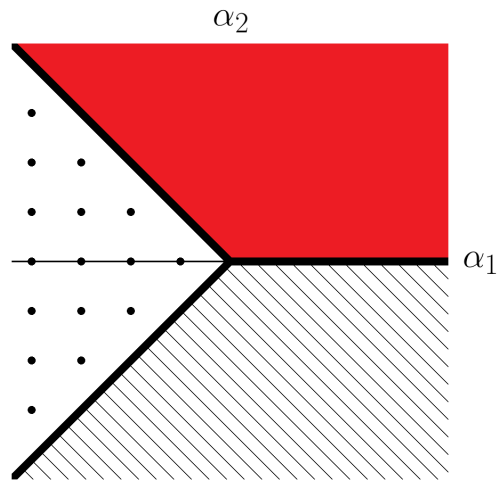$$O = \left\{ (0, X) \in \mathbb{P}^4 : \det X = 0 \right\}.$$

It is a divisor on the hyperplane $(X_0 = 0) \cong \mathbb{P}^3$, hence of dimension 2. It intersects T' at:

$$\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \quad \text{and} \quad \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

This is a set if two points, hence $\dim(T' \cap O) = 0$.

The fiber over 0 is $\mathrm{GL}_2 \times \mathrm{GL}_2 \times \{0\}$, which is of dimension 8. We see that $\mu_{T'}$ is equidimensional everywhere but at the origin. Flatness of $\mu_{T'}$ over $\mathbb{P}^4 - \{0\}$ follows from

Figure 11: The fan of $\mathbb{P}^4$

the following proposition, which is a direct consequence of [EGAIV, Prop. 6.1.5]. The closed set T' is Cohen-Macaulay as a complete intersection in $\mathbb{P}^4$, and GL_2 is an open set in $\mathbb{A}^4$, thus $\mathrm{GL}_2 \times \mathrm{GL}_2 \times T'$ is Cohen-Macaulay.

Proposition 5.9. Let $\phi : X \to Y$ be a morphism of varieties. Suppose that:

   (i) Y is nonsingular,

  (ii) X is Cohen-Macaulay, and

 (iii) for all $y \in f(X)$, $\dim X = \dim Y + \dim \phi^{-1}(y)$.

Then ϕ is flat.

Consider the blow-up $\mathrm{Bl}_0 \, \mathbb{P}^4$, and write $\pi : \mathrm{Bl}_0 \, \mathbb{P}^4 \to \mathbb{P}^4$ for the natural proper birational morphism that restricts to an isomorphism $\mathrm{Bl}_0 \, \mathbb{P}^4 - E \xrightarrow{\sim} \mathbb{P}^4 - \{0\}$. The exceptional divisor E is isomorphic to $\mathbb{P}^3$. We view its elements as 2×2 homogeneous matrices; write Y_{ij} for the associated homogeneous coordinates. The action of $\mathrm{GL}_2 \times \mathrm{GL}_2$ on $\mathbb{P}^4 - \{0\}$ extends to an action on $\mathrm{Bl}_0 \, \mathbb{P}^4$ by left and right multiplication on the homogeneous matrices of the exceptional divisor. Under this action π is a GL_2-morphism. Thus $\mathrm{Bl}_0 \, \mathbb{P}^4$ is a spherical variety for the homogeneous space GL_2. The closed GL_2-orbits are the set of

Figure 12: The fan of $\mathrm{Bl}_0\, \mathbb{P}^4$
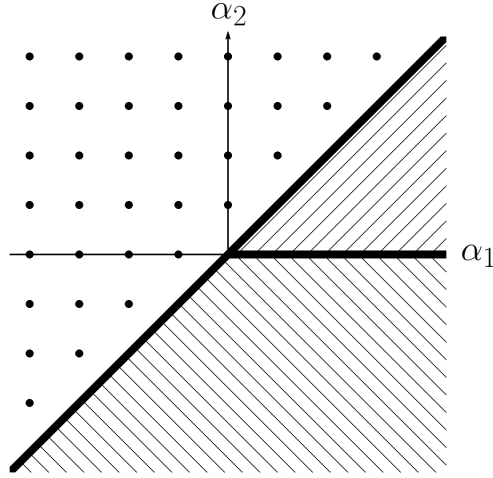
matrices of rank 1 at infinity, and the matrices of rank 1 in the exceptional divisor. The fan associated to $\mathrm{Bl}_0\, \mathbb{P}^4$ is given in Figure 12. In particular $\mathrm{Bl}_0\, \mathbb{P}^4$ is toroidal.

We claim that the closure $\overline{T} \subset \mathrm{Bl}_0\, \mathbb{P}^4$ is a tropical compactification of T. Completeness of $\overline{T}$ follows from completeness of $\mathrm{Bl}_0\, \mathbb{P}^4$, or from the fact $\mathrm{Trop}\, T = \mathrm{Supp}\, \mathcal{F}$, where $\mathcal{F}$ is the fan associated to $\mathrm{Bl}_0\, \mathbb{P}^4$ (Prop. 4.6). Also, $\mathrm{Trop}\, T = \mathrm{Supp}\, \mathcal{F}$ implies that $\overline{T}$ intersects all orbits of $\mathrm{Bl}_0\, \mathbb{P}^4$ (Prop. 4.5), so that the multiplication map $\mu_{\overline{T}} : \mathrm{GL}_2 \times \mathrm{GL}_2 \times \overline{T} \to \mathrm{Bl}_0\, \mathbb{P}^4$ is surjective. We show that it is also flat.

Since π is a GL_2-morphism that restricts to an isomorphism $\mathrm{Bl}_0\, \mathbb{P}^4 - E \xrightarrow{\sim} \mathbb{P}^4 - \{0\}$, the multiplication maps of $\overline{T}$ and T' agree away from the exceptional divisor and the origin 0:

$$\mu_{\overline{T}}\big|_{\mathrm{GL}_2 \times \mathrm{GL}_2 \times (\overline{T} - E)} = \mu_{T'}\big|_{\mathrm{GL}_2 \times \mathrm{GL}_2 \times (T' - \{0\})}$$

(as morphisms to $\mathrm{Bl}_0\, \mathbb{P}^4 - E \cong \mathbb{P}^4 - \{0\}$). The intersection $\overline{Y} \cap E$ consists of the diagonal homogeneous matrices of E. One can check using Proposition 5.8 that all fibers of $\mu_{\overline{T}}$ over E are of dimension 6; this case is identical with the case of fibers over the hyperplane $(X_0 = 0)$. Therefore all fibers of $\mu_{\overline{T}}$ are of the same dimension.

The closed set $\overline{T}$, which is the pure transform of T', is Cohen-Macaulay as a complete intersection; this can be easily checked on the standard charts $U_{ij} \cong \mathbb{P}^4$ of $\mathrm{Bl}_0\, \mathbb{P}^4 \subset \mathbb{P}^4 \times \mathbb{P}^3$.

Applying Proposition 5.9 we get that $\mu_{\overline{T}}$ is flat, and since it is surjective, faithfully flat.

We deduce $\overline{Y} \subset \mathrm{Bl}_0 \mathbb{P}^4$ is a tropical compactification.

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