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Study of Minimum Void Ratio for Soils with a Range of Grain-Size Distributions

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STUDY OF MINIMUM VOID RATIO FOR SOILS WITH A RANGE OF GRAIN-SIZE DISTRIBUTIONS

A Masters Project Presented

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A Master of Science Project

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ABSTRACT

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Directed by Dr. C. S. Chang

Minimum void ratio or maximum packing density is an important soil property in geotechnical engineering. It apply to volume change tendency control, fluid conductivity control and particles movement.

Previous researchers have attempted to predict maximum packing density by empirical/graphic method, rock correction method, alpha method. Based on the concepts of F. de Larrard in concrete mixture research, we have developed a mathematic model that can predict the minimum void ratio for soils with a wide range of particle size.

Probability density function-lognormal distribution was tested and used to provide a reasonable representation for soils with a range of grain-size distribution. We incorporate the lognormal distribution in the mathematical model, and predict the minimum void ratio for various types of soil gradations. The validity of the model is evaluated.

The evaluation of the model is also performed on several sets of data in the literature, which include binary packing system of steel balls, ternary packing system of spheres, mixtures of round and crushed aggregates, and soils containing gravelly sand with silt. Comparison of the results will be discussed.

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1. INTRODUCTION

Void ratio or solid volume fraction of aggregates is essential factor to describe physical properties of soils. Insufficient fulfillment of requirements of void ratio or compaction would lead magnificent loss. Compactness of granular material is of importance for construction and transportation structures like pavements (Fragaszy and Sneider, 1991). Packing of particles is important to many represented industries, including concrete mixture, soil geology and ceramics. Study of making voids filled was meager published around 1930’s. Research interest of high-density packing of ceramics and metal particles was renewed around 1954, for the reason of impetus of atomic energy and space research. However, those were mainly considering packing of uranium oxide and optimum particle size distribution (PSD) for maximum packing density.(McGeary, 1961). In 1957, a method of prediction of maximum dry density was proposed by Humphres using empirical and graphical method. It is shown in Figure-1.1 a).

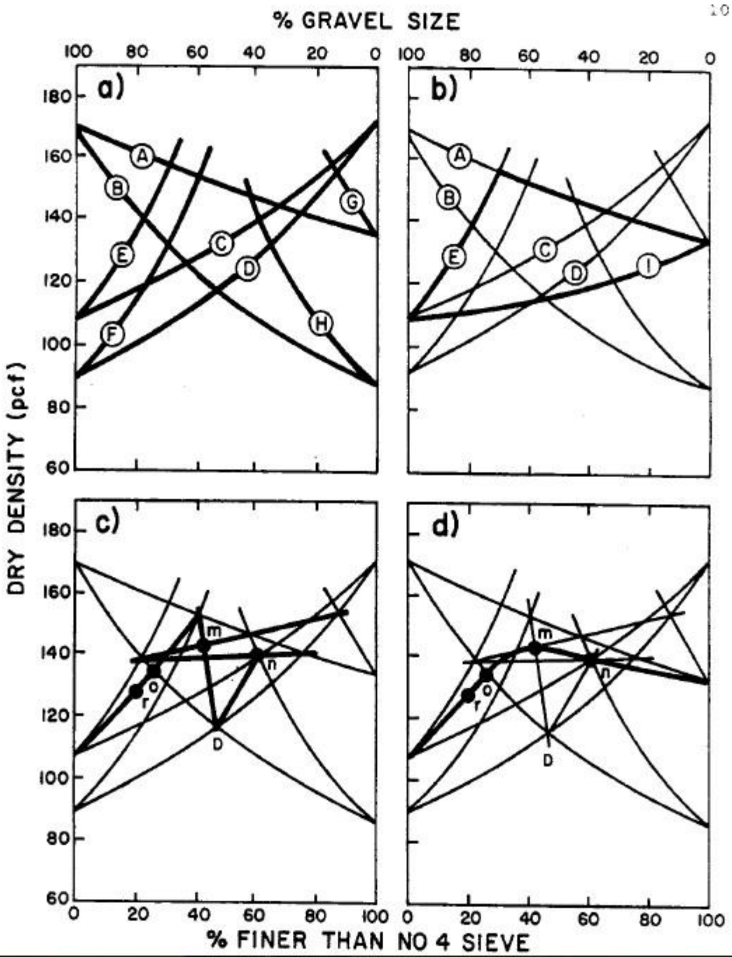


Figure-1.1 a) Derivation of Humphres Maximum Density Curve (Humphres, 1957), cited at (Fragaszy and Sneider, 1991)

Even this method did not considered effect of soil-interaction, Humphres’ math nature is the sound foundation of computer-based modeling. Around 1986, AASHTO T 224-86 specifications for compaction includes a rock correction factor that can be used when the percent of gravel size particles is less than or equal to 70%. This is also a way to describe or limit soil-interaction. Similar to it, Alpha method proposed by

Fragaszy et al (1989) defined a factor, α, to consider soil interaction. This is illustrated in Figure-1.1 b).

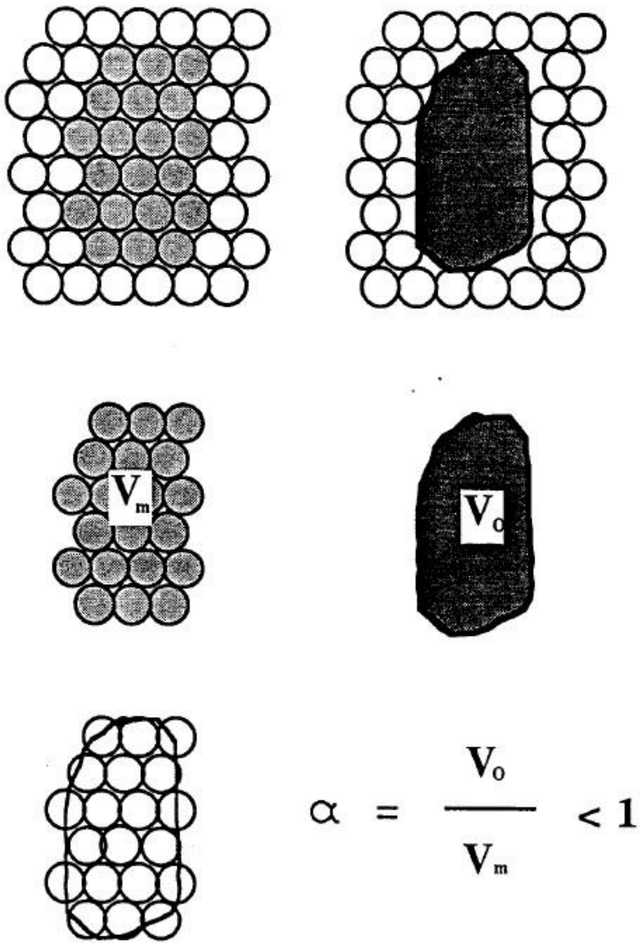


Figure-1.1 b) Illustration showing Derivation of Alpha Term (Fragaszy et al, 1989), cited at (Fragaszy and Sneider, 1991)

In 1991, based on experiments, "Humphres method", "AASTO rock correction" and "Alpha method" were compared together by Fragaszy and Sneider. In 1999, F. de Larrard published a detailed book about concrete mixture proportioning (F. de Larrard, 1999). Larrard detailed analyzed virtual packing density and actual packing density. Using the concept of dominant particle size, a mathematical model is developed for predicting the minimum void ratio of a packing containing a range of particle sizes.

This paper is organized as follows: Chapter 2 introduces the analytical method for predicting minimum void ratio, including by general packing, binary packing and ternary packing. Chapter 3 then explains how to use lognormal distribution to model soils with a range of particle size distributions. Chapter 4 and 5 test the model with literature data of spherical particles and soils or aggregate. Finally, Appendix presents parametric study of coefficient of interaction and model.

2. ANALYTICAL METHOD

A mathematical model for predicting the minimum void ratio for sand-silt mixture has been proposed by Chang and Yin (2011) and Chang and Meidani (2012). The model was applicable for gap-graded soils, which have two particle size-groups, and the two sizes are very different in magnitudes. Based on the concept used by Francois de Larrard(1999) for concrete mixture, the model by Chang and Meidani (2012) is extended to a be applicable for soils with an arbitrary gradation (or particle size distribution). The extended model is briefly summarized below.

2.1 General Packing

Multiple particle-sizes of system are $d_1 > d_2 > d_3 > .... > d_n,\ n \geq 2$ .The solid volume fraction of each class is denoted by

$$y_i = \frac{v_i}{\sum_{j=1}^{n} v_j}, \quad \sum_{j=1}^{n} y_j = 1 \tag{1}$$

For the i[th] class of particles with size d_i, its minimum void ratio is e_i. Assuming we know the values of e_i for each individual class. For a mixture with given y_i for each class, our objective is to estimate the value of minimum void ratio *e* of the mixture (or the maximum density).

2.1.1Without Interaction

If there is no interaction between particles, minimum void ratio *e* can be expressed as

$$\hat{e}_i = e_i - \left[e_i \sum_{j=1}^{i-1} y_j + (1+e_i) \sum_{j=i+1}^{n} y_j \right] \tag{2}$$

$$e = \underset{1 \leq i \leq n}{Max}\ \hat{e}_i \tag{3}$$

2.1.2 Full Interaction

Full interaction happens on condition that particle sizes are same, expression is

$$\bar{e} = \sum_{j=1}^{n} e_j y_j \tag{4}$$

2.1.3 Partial Interaction

When partial interaction are existing between particles, the true void ratio is affected by "wall" effect b_{ij} due to larger size particles, and "loosening" effect a_{ij} due to small size particles. The interaction coefficients, a_{ij} and b_{ij} were introduced in the following equation,

$$a_{ij} = \left(1 - \left(1 - \frac{d_j}{d_i}\right)^p\right)^q \quad (5)$$

$$b_{ij} = \left(1 - \left(1 - \frac{d_i}{d_j}\right)^s\right)^t \quad (6)$$

where *p, q, s,*and *t* are both positive numbers decided by inner properties of mixture. Two physical effects can be interpreted in binary system: $d_1 > d_2$ and class 2 grain is inserted in the porosity of coarse-size packing (class 1grains dominant), if void is no longer capable to fit, there is a decrease of volume of class 1 grains (loosening effect); when some isolated coarse grains are overwhelmed in fine grains (class 2 grains dominant), there is a further amount of voids in the packing of class 2 grains located in the interface vicinity (wall effect).(de Larrard, 1999)

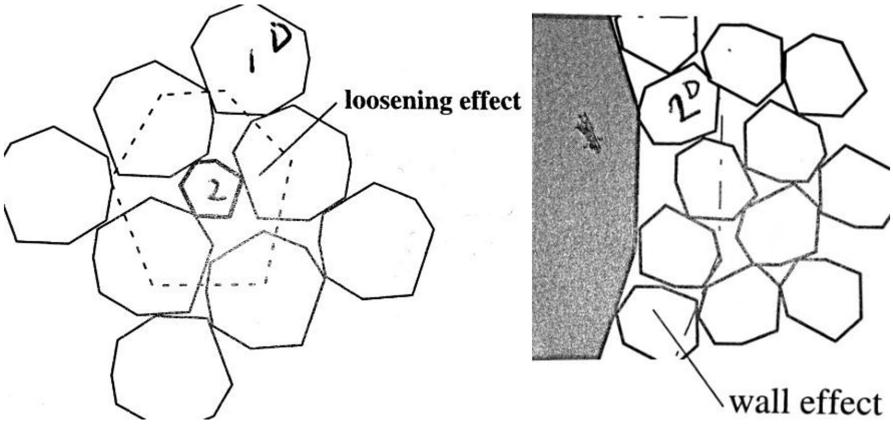


Figure-2.1 a) Loosening effect exerted by a fine grain in a coarse grain packing; wall effect exerted by a coarse grain on a fine grain packing (de Larrard, 1999)

The equation expressing the void ratio under the circumstance that *i* class dominants should be written as

$$\hat{e}_i = e_i - \left[e_i \sum_{j=1}^{i-1} y_j - b_{ij} \sum_{j=1}^{i-1} e_j y_j\right] - \left[(1+e_i) \sum_{j=i+1}^{n} y_j - a_{ij} \sum_{j=i+1}^{n} (1+e_j) y_j\right] \quad (7)$$

$$e = \underset{1 \le i \le n}{Max}\ \hat{e}_i \quad (8)$$

In this report, equation (7) plays an important role of prediction of void ratio in mixture. And it follows $0 \le a_{ij} \le 1$ and $0 \le b_{ij} \le 1$; 0 represents no-interaction and 1 represents full interaction.

To predict minimum void ratio in general packing system at different mixture, we need data of d_1 to d_n, e_1 to e_n, *p, q, s* and *t* for interaction coefficients.

2.2 Binary System

If a mixture system consist of only two grain sizes (uniform), $d_1 > d_2$, n=2, we have following equations,

$$\hat{e}_1 = e_1 - \left[(1+e_1) y_2 - a_{12} (1+e_2) y_2 \right], \quad (9)$$

$$\hat{e}_2 = e_2 - [e_2 y_1 - b_{21} e_1 y_1], \quad (10)$$

$$e = Max(\hat{e}_1, \hat{e}_2) \quad (11)$$

where parameters a_{12} and b_{21} can be determined from experiments (*e* vs. y_2 curve) by equation (12) and (13),

$$\frac{\partial e}{\partial y_2} = -(1+e_1) + a_{12} (1+e_2), \quad (12)$$

$$\frac{\partial e}{\partial y_1} = -e_2 + b_{21} e_1, \quad (13)$$

In curve of *e* vs. y_2, y_2 represent fine grain proportion. Equation (9) means that size of *i=1* dominant the system. Therefore, e_1 would decide the minimum void ratio and it represents coarse soil. Thus, the partial derivative of void ratio for y_2 is rate of change for coarse soil. As for y_1, y_1=1-y_2, $\frac{\partial e}{\partial y_1} = -\frac{\partial e}{\partial y_2}$, then $\frac{\partial e}{\partial y_2}$ represents the rate of change for fine soil. By this way, a_{12} and b_{21} can then be determined.

To predict minimum void ratio in binary system at different mixture, we need data of d_1, d_2, e_1, e_2, *p*, *q*, *s* and *t* for interaction coefficients.

2.3 Ternary System

It will be established with the help of equations (5) (6) (7) that ternary model will be acquired as n=3. The physical relation and mutual interaction between particles were expressed by a_{ij} and b_{ij}. Therefore, the equations describing ternary packing are

$$\hat{e}_1 = e_1 - \left[(1+e_1)(y_2 + y_3) - a_{12} (1+e_2) y_2 - a_{13} (1+e_3) y_3 \right], \quad (14)$$

$$\hat{e}_2 = e_2 - [e_2 y_1 - b_{21} e_1 y_1] - [(1+e_2) y_3 - a_{23} (1+e_3) y_3], \quad (15)$$

$$\hat{e}_3 = e_3 - [e_3 (y_1 + y_2) - b_{31} e_1 y_1 - b_{32} e_2 y_2], \quad (16)$$

$$e = Max(\hat{e}_1, \hat{e}_2, \hat{e}_3) \quad (17)$$

To predict minimum void ratio in ternary system at different mixture, we need data of d_1, d_2, d_3, e_1, e_2, e_3, *p*, *q*, *s* and *t* for interaction coefficients.

The values of a_{ij} and b_{ij} will be obtained by Eqs (5, 6) in which the parameters *p*, *q*, *s* and *t* for coefficient of interaction needs to be determined. We will describe how to determine the interaction coefficients later in this report.

3. PROBABILITY DENSITY FUNCTION FOR PARTICLE SIZE DISTRIBUTION

3.1 Log-normal distribution

In probability density function (PDF) theory, the log-normal distribution (with two parameters) may be defined as the distribution of a random variable whose logarithm is normally distributed (Crow and Kunio, 1988). If the random variable X is log-normally distributed, then Y=log(X) has a normal distribution. It is used in this paper to approximate the particle size distribution of packing system of spherical particles, aggregates and soils. The probability density function of a log-normal distribution is:

$$f_X\left(x;\mu;\sigma\right)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}\ ,\ x>0 \tag{18}$$

where μ and σ means the mean value and standard deviationof the variable's natural logarithm. Then we wrote them as $\mu_{\ln x}$ and $\sigma_{\ln x}$ in this report.

Assuming in binary packing system (fine and coarse soil), each size has a size range in form of log-normal distribution. A log-normal distribution with mean μ_x and variance v_x has parameters (Mood et al, 1974)

$$\mu_{\ln x}=\ln\left(\frac{\mu_x^{\ 2}}{\sqrt{v_x+\mu_x^{\ 2}}}\right)\ ,\ \sigma_{\ln x}=\sqrt{\ln\left(1+\frac{v_x}{\mu_x^{\ 2}}\right)}\ . \tag{19}$$

Based on $\mu_{\ln x}$ and $\sigma_{\ln x}$ calculated above, for easy calculation, five particle sizes (selected by writer) can be generated by dividing log-normal distribution into five histograms whose summation is close to 99.73%. First, it is assumed that normal distribution incorporates 99.7% confidence belt with 3-sigma rule. It is divided into five equal distances (base) within six standard deviations. Then, five probability histograms can be constructed with same base. The centers of them are normal distributed five particle sizes. Last, with the help of relation between log-normal distribution and normal distribution, natural logarithms of them are used to represent grain size distribution in this paper. Five particle sizes will then be produced from one particle size with a range as following Table-3.1 a),

Table-3.1 a) Five particle sizes in log-normal distribution

$d_5=e^{\mu_{\ln x}-2\times\frac{6\sigma_{\ln x}}{5}}$	$d_4=e^{\mu_{\ln x}-\frac{6\sigma_{\ln x}}{5}}$	$d_3=e^{\mu_{\ln x}}$	$d_2=e^{\mu_{\ln x}+\frac{6\sigma_{\ln x}}{5}}$	$d_1=e^{\mu_{\ln x}+2\times\frac{6\sigma_{\ln x}}{5}}$

Assuming void ratios of these five particles remain the same (Adjustment coefficient will be introduced in Chapter 5.1). Each area of histogram (percentage) in log-normal distribution is also constant value, which can be calculated in following Table-3.1 b),

Table-3.1 b) Five area of histograms (percentage) in log-normal distribution

a_5= 0.03458	a_4= 0.23832	a_3= 0.45149	a_2= 0.23832	a_1= 0.03458

3.2 Probability Density Function determined from soil gradation

Fragaszy and Sneider (1991) studied three methods of prediction of maximum density, including humphres' graphical method, AASHTO Rock correction method (1986) and alpha method Fragaszy et al proposed at 1989. Experiments data were acquired by using the Washington Test method No.606 (WTM 606), whose method manipulates vibration under static load to make soil dense (Fragaszy and Sneider, 1991). Fragaszy separated soil into a gravel portion (plus #4 sieve) and a fine portion (minus #4 sieve), then two parts were mixed at different ratios to test the actual maximum dry density by means of WTM 606. Since each component passed through one sieve and was completely retained on the following smaller sieve, assuming that amount of retained percentage is equal to the average magnitude of upper and under sieve. Table-3.2 presents the soil description and approximate calculation of mean *m* and variance *v*.

Table 8.2- Soil description of No.1 to No.8 (Fragaszy and Sneider, 1991)

	Soil No.	No.1	No.2	No.3	No.4	No.5	No.6	No.7	No.8
coarse (mm)	>19	5	6	2	5	5	5	1	10
	19-9.5	45	54	53	65	55	50	34	65
	9.5-4.75	50	40	45	30	40	45	65	25
	μ_x (mm)	10.95	11.72	11.15	12.38	11.66	11.31	9.67	12.99
	v_x (mm)	16.03	15.69	13.92	13.22	15.14	15.71	12.43	14.01
Fine (mm)	4.75-2.36	30	15	23	24	21	14	20	25
	2.36-1.7	13	7	12	6	10	9	15	12
	1.7-0.6	29	18	22	23	25	29	36	23
	0.6-0.4	9	15	7	18	10	10	9	5
	0.4-0.3	4	15	12	11	9	6	5	3
	0.3-0.419	5	16	11	13	12	12	7	7
	0.149-0.074	1	7	7	3	6	5	2	5
	<0.074	9	7	6	2	7	15	6	20
	μ_x (mm)	1.74	1.06	1.43	1.40	1.36	1.13	1.51	1.47
	v_x (mm)	1.73	1.38	1.71	1.70	1.61	1.30	1.38	1.84

After calculation of mean μ_x and variance v_x of each set of grain size distribution, grain size distribution of soils can be represented roughly by log-normal distribution built by them. Comparisons of cumulative distribution are shown in the following Figures-3.2 a),

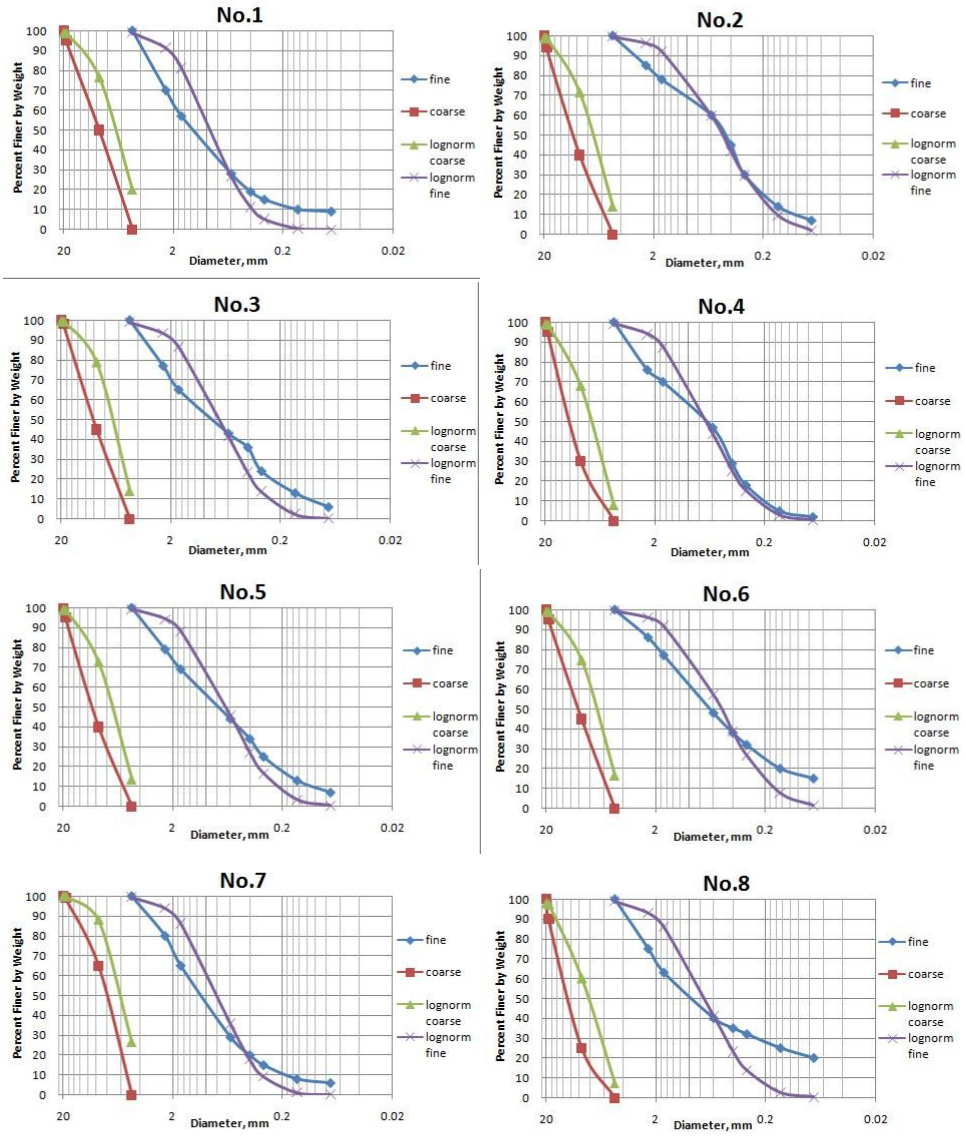


Figure-3.2 a) Comparisons of cumulative distribution between experiments grain size distribution and log-normal distribution built by μ_x and v_x. (The gravel soils are poorly graded and are classified as GP; the minus #4 fraction of four soils is classified SW-SM, two are classified SM, and the remaining two are classified SP and SM(Fragaszy and Sneider, 1991).)

Since Fragaszy separated soil into two portions: a gravel portion (plus #4 sieve) and a fine portion (minus #4 sieve). Range of particle size is from 4.75mm to 20mm for coarse soils, and 4.75mm to 0 for fine soils. We separated each of them into 10 portions. Bins of coarse soils are: 20-18.475, 18.475-16.95, 16.95-15.425, 15.425-13.9, 13.9-12.375, 12.375-10.85, 10.85-9.325, 9.325-7.8, 7.8-6.275, 6.275-4.75mm; their center-size are: 19.2375, 17.7125, 16.1875, 14.6625, 13.1375, 11.6125, 10.0875, 8.5625, 7.0375, 5.5125 mm. Bins of fine soils are: 4.75-4.275, 4.275-3.8, 3.8-3.325, 3.325-2.85, 2.85-2.375, 2.375-1.9, 1.9-1.425, 1.425-0.95, 0.95-0.475, 0.475-0mm; their center-size are: 4.5125, 4.0375, 3.5625, 3.0875, 2.6125, 2.1375, 1.6625, 1.1875, 0.7125, 0.2375mm. Thus, based on gradation of coarse and fine soils, frequency histogram vs. center-sizes can then be drawn for Soils No.1 to No.8 as in Figure-3.2 b) to Figure-3.2 q)

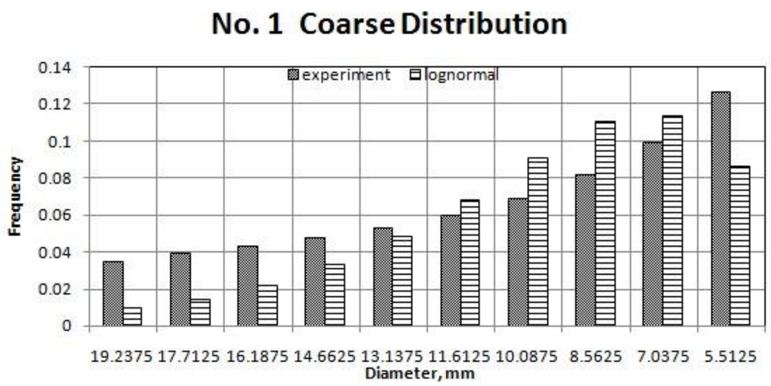


Figure-3.2 b) No.1 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

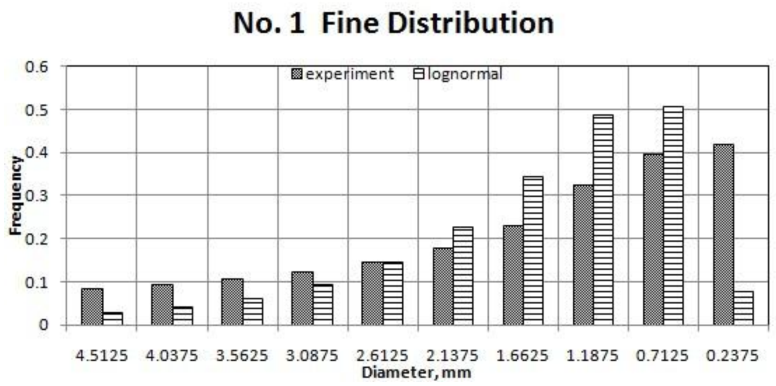


Figure-3.2 c) No.1 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

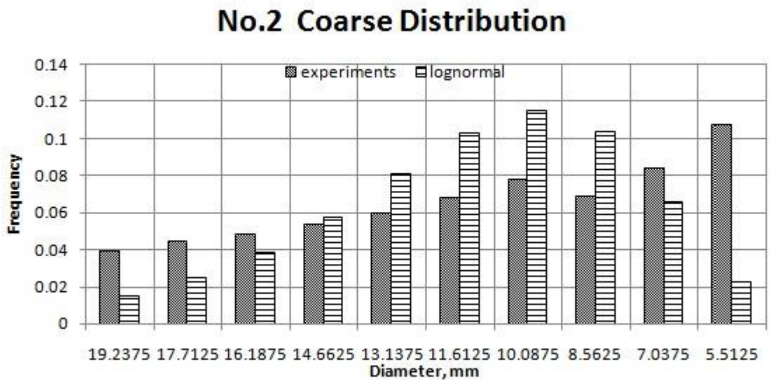


Figure-3.2 d) No.2 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

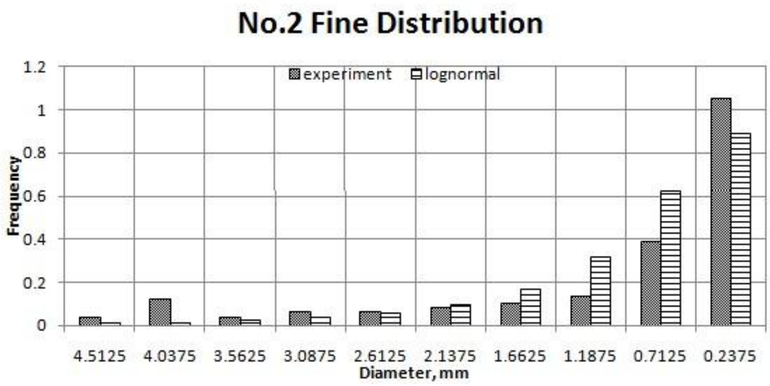


Figure-3.2 e) No.2 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

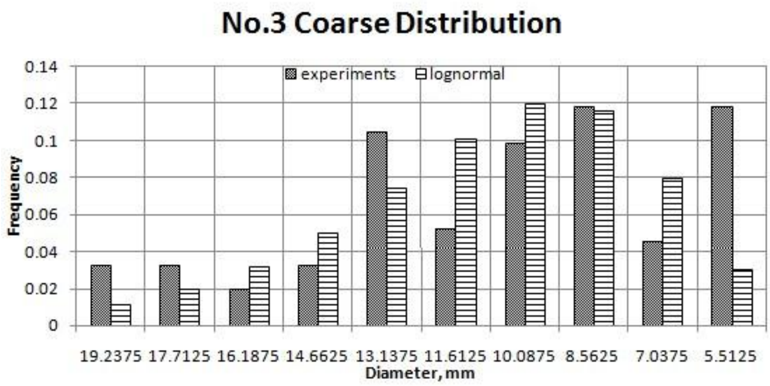


Figure-3.2 f) No.3 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

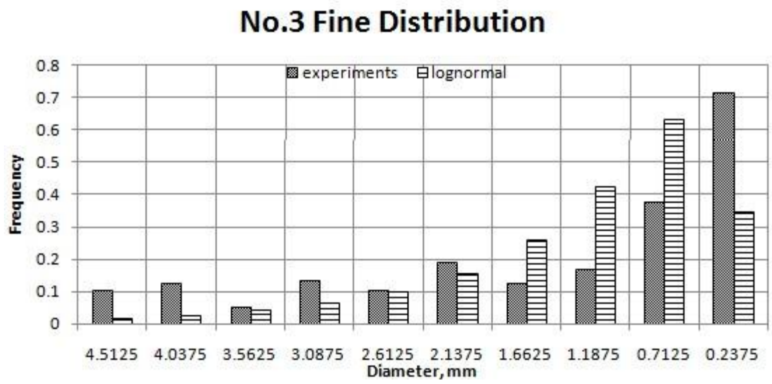


Figure-3.2 g) No.3 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

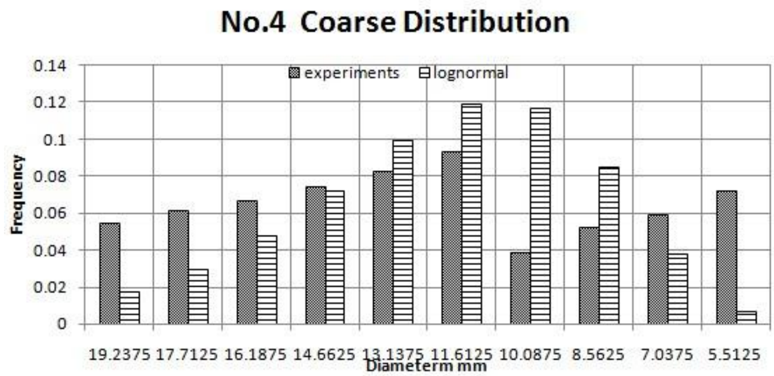


Figure-3.2 h) No.4 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

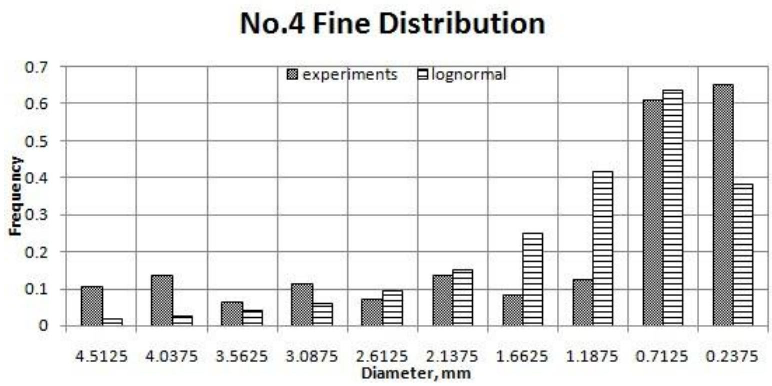


Figure-3.2 i) No.4 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

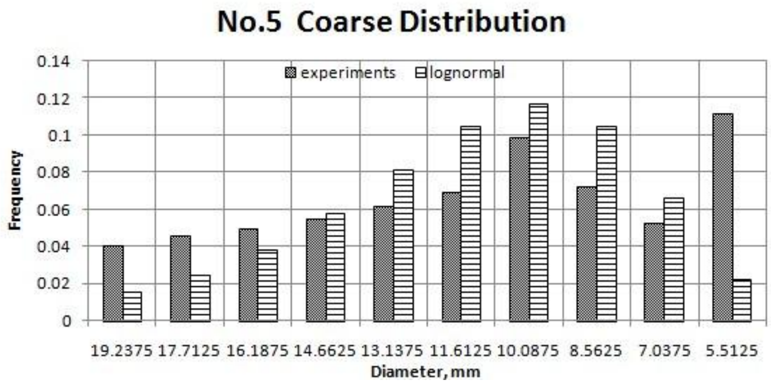


Figure-3.2 j) No.5 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

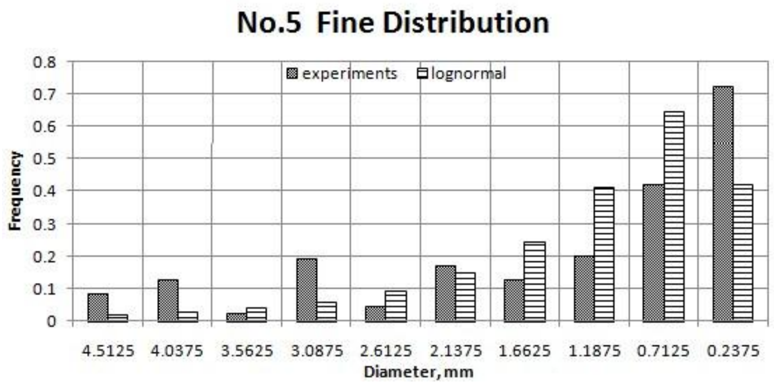


Figure-3.2 k) No.5 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

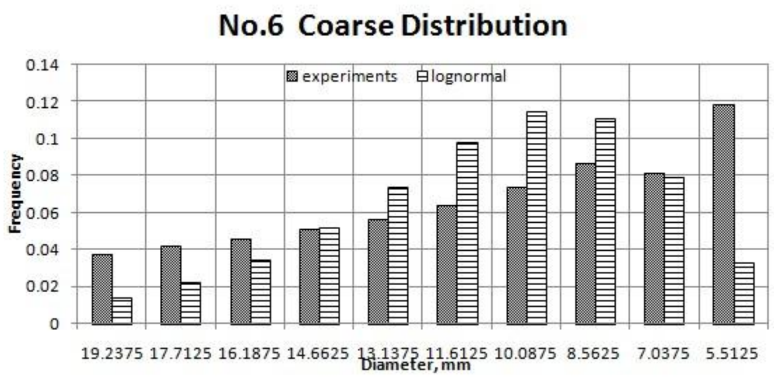


Figure-3.2 l) No.6 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

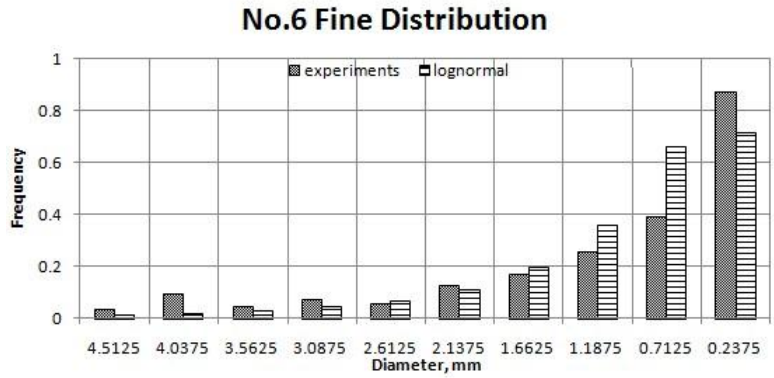


Figure-3.2 m) No.6 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

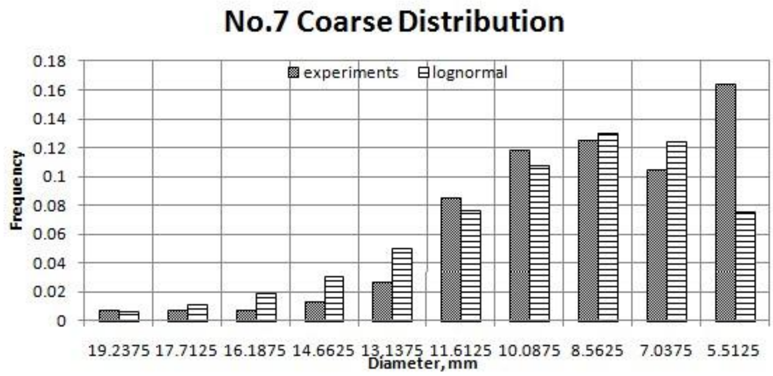


Figure-3.2 n) No.7 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

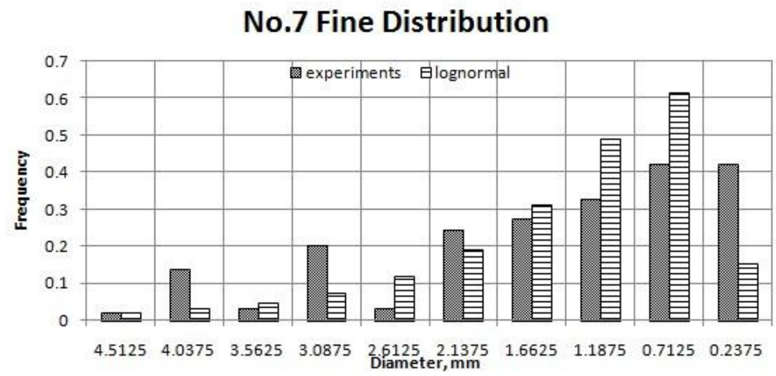


Figure-3.2 o) No.7 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

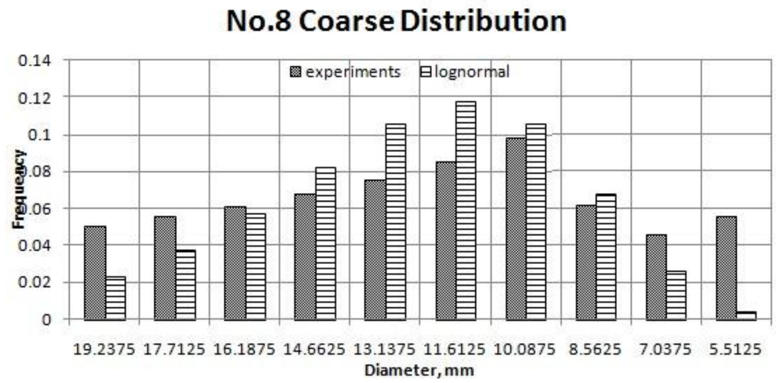


Figure-3.2 p) No.8 Coarse Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

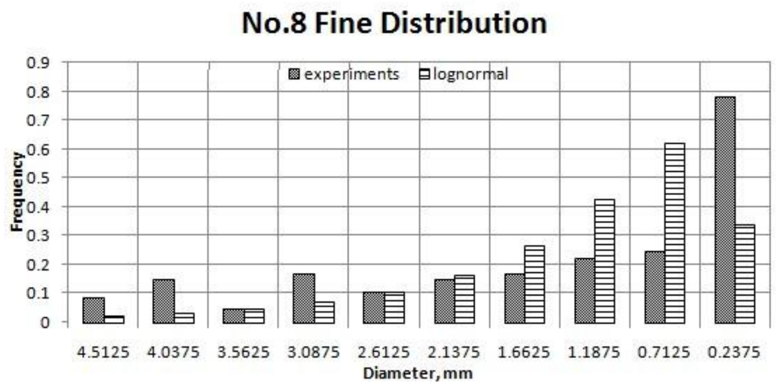


Figure-3.2 q) No.8 Fine Distribution: Comparison between experiments grain size distribution and log-normal distribution built by *m* and *v*. (Fragaszy and Sneider, 1991).)

3.3 Probability Density Function determined from given particle size range

If there is no available soil grain size distribution in literature, method to acquire variance is to subtract minimum diameter $d_{\min}$ from maximum diameter $d_{\max}$. First, mean μ_x is simply the average value of $d_{\min}$ and $d_{\max}$. Then, we can skip the process of determination of variance instead of directly calculate σ_{lnx} by dividing subtraction of $\ln(d_{\min})$ and $\ln(d_{\max})$ into six standard deviations σ_{lnx} (six is arbitrary). In study of concrete mixture, two families of aggregate were selected: rounded aggregate from the Loire (Decize quarry), with nearly spherical shapes; crushed angular aggregate from the Pont de Colonne quarry at Arnay le Duc (F. de Larrard, 1999). Parameters of μ_{lnx} and σ_{lnx} can be calculated in the following equations, and Table-3.3 a) and Table 3.3 b) represent results of two sets of data of aggregates.

$$\mu_{\ln x} = \ln\left(\frac{d_{\max} + d_{\min}}{2}\right) \tag{20}$$

$$\sigma_{\ln x} = \frac{\ln(d_{\max}) - \ln(d_{\min})}{6} \tag{21}$$

Table-3.3 a) Soil description of Rounded aggregate (F. de Larrard, 1999)

Rounded aggregate					
Names	d_{min}(mm)	d_{max}(mm)	particle density	μ_{lnx} (mm)	σ_{lnx} (mm)
R<05	0.08	0.5	0.593	-1.2378744	0.3054302
R05	0.5	0.63	0.592	-0.5709295	0.0385186
R1	1	1.25	0.609	0.117783	0.0371906
R2	2	2.5	0.616	0.8109302	0.0371906
R4	4	5	0.6195	1.5040774	0.0371906
R8	8	10	0.628	2.1972246	0.0371906

Table-3.3 b) Soil description of Crushed aggregate (F. de Larrard, 1999)

Crushed angular aggregate					
Names	d_{min}(mm)	d_{max}(mm)	packing density	μ_{lnx} (mm)	σ_{lnx} (mm)
C<05	0.08	0.5	0.63	-1.237874	0.3054302
C05	0.5	0.63	0.516	-0.57093	0.0385186
C1	1	1.25	0.507	0.117783	0.0371906
C2	2	2.5	0.529	0.8109302	0.0371906
C4	4	5	0.537	1.5040774	0.0371906
C8	8	10	0.572	2.1972246	0.0371906

3.4 Probability Density Function determined from mean particle size

Except for above two situations, mean value or median value of particle sizes could be assumed as μ_{lnx}. It can be easily deduced from relation between log-normal distribution and normal distribution: if X is random variable of log-normal distribution, ln(X) will be random variable of normal distribution and reverse analysis is also feasible. Based on equation (19) (20) (21), five particle sizes can then be determined. In the following two chapters, detailed examples will be demonstrated.

4. PACKING SYSTEM OF SPHERICAL PARTICLES

In this chapter, two detailed prediction examples of spherical particles were shown clearly. One is predicted in binary system with steel shot, and another ternary system with DEM simulation data is interpreted based on binary method to calculate coefficient of interactions of three sizes.

4.1 Spherical Metal Shot In Binary System

Literature data for packing of spherical metal shot is fromR. K. McGeary,1961. Note that McGeary used "pure" components consisting of a single mesh size in research of mechanical packing of spherical particles. Besides, the calculated average sphere diameter was used rather than size of the sieve where spherical retained in each passing through. Therefore, assuming that variance *v* of data is 0.

Relation between solid volume fraction γ and void ratio are

$$e = \frac{v_v}{v_s} \ , \ \gamma = \frac{v_s}{v_T} \ , \ e = \frac{1}{\gamma} - 1 \qquad (22)$$

Where v_V is the volume of void-space; v_s is the volume of solids, and v_T is the total or bulk volume. Packing-density diagram plotted by McGeary is shown in Figure-4.1 a), where "theoretical density" in plot can be transferred to void ratio based on equation (22).

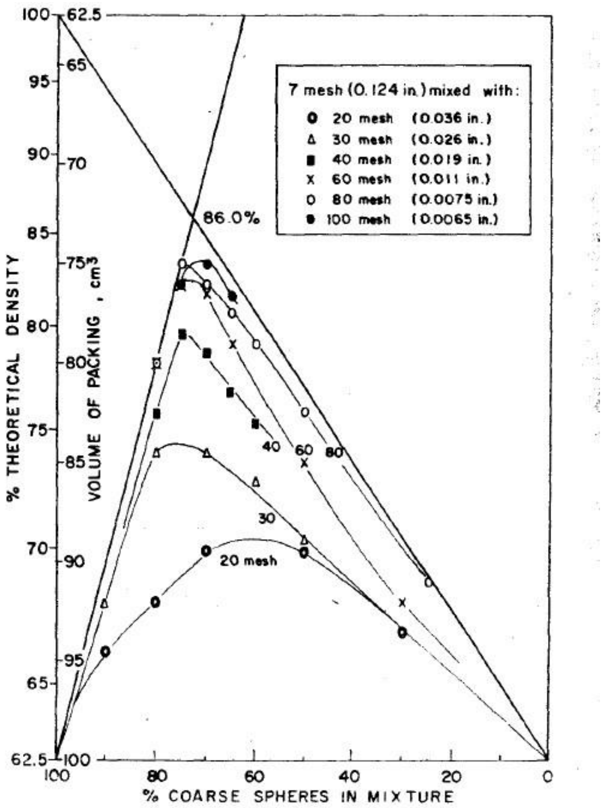


Figure-4.1 a) Binary mechanical packing of coarse steel shot with 6 other fine sizes (R. K. McGeary, 1961)

Binary model needs ten particle sizes: d_1 to d_{10}; *p*, *q*, *s* and *t* for each coefficient of interactions: a_{12} and b_{21}; void ratios: $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. It is already assumed that for coarse particles of d_1 to d_5, void ratios of them stay $\mathbf{e_1}$; for fine particles of d_6 to d_{10}, void ratio of them keep $\mathbf{e_2}$. Void ratio under 100%coarse is $\mathbf{e_1}$, void ratio under 0%coarse is $\mathbf{e_2}$.

As method described in Chapter2.2, to acquire coefficient of interactions: a_{12} and b_{21}, the experiments data shown in Figure-4.1 a) were redrawn in Figure-4.1 b) with rate of change lines for the two sides.

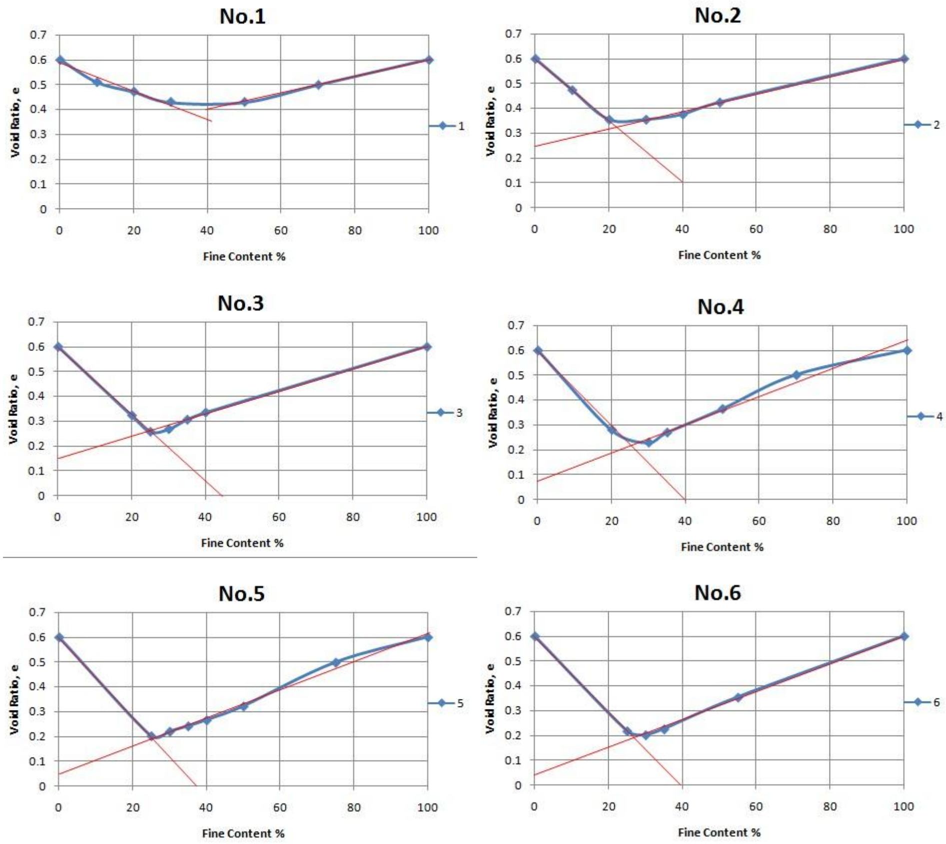


Figure-4.1 b) Six sets of rate of change lines describing slopes of coarse and fine (R. K. McGeary, 1961)

In Equatioin (12), $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_1}$ side (coarse), a_{12} can be determined; since it is known in Chapter2.2 that $\frac{\partial e}{\partial y_1} = -\frac{\partial e}{\partial y_2}$, $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_2}$ side (fine), b_{21} can be determined. Then six sets of coefficient of interactions a_{12} and b_{21} can be calculated.

Results are represented in Table-4.1 a).

Table-4.1 a) Coefficient of interactions of No.1 to No.6 (R. K. McGeary, 1961)

Item No.	μ_{x1}(inch)	μ_{x2}(inch)	μ_{x1}(mm)	μ_{x2}(mm)	ratio	a_{12}	b_{21}
No.1	0.124	0.036	3.1496	0.9144	0.29032	0.60938	0.44444
No.2	0.124	0.026	3.1496	0.6604	0.20968	0.21875	0.41667
No.3	0.124	0.019	3.1496	0.4826	0.15323	0.16667	0.25000
No.4	0.124	0.011	3.1496	0.2794	0.08871	0.06250	0.05000
No.5	0.124	0.0075	3.1496	0.1905	0.06048	0.02344	0.06667
No.6	0.124	0.0065	3.1496	0.1651	0.05242	0.06250	0.06667

where μ_x is mean value of particle sizes: $\mu_{x1}=d_3$, $\mu_{x2}=d_8$; $ratio=\mu_{x1}/\mu_{x2}$

Then parameters p, q, s and t can be estimated through equations of coefficient of interactions (5) (6) by given six sets of a_{12} and $b21$ in Table-4.1 a). Two fitting graphs were shown in Figure-4.1 b). Study of fitting p, q, s and t to get a_{ij} or b_{ij} can be found in Appendix.

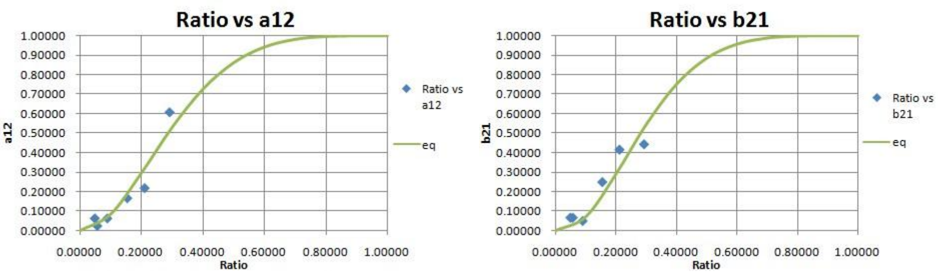


Figure-4.1 c) Fitting graphs of coefficient of interactions for six sets of steel shot (R. K. McGeary, 1961)

After fitting, we get $p=4$, $q=2.3$ for a_{12}, and $s=4.5$, $t=2.7$ for b_{21}. Take No.1 mixture for example.

Since particle size range is 0 (assuming variance is 0), $d_1=d_2=d_3=d_4=d_5=3.1496$, $d_6=d_7=d_8=d_9=d_{10}=0.9144$; $e_1=e_2=e_3=e_4=e_5=0.6(\mathbf{e_1})$, $e_6=e_7=e_8=e_9=e_{10}=0.6(\mathbf{e_2})$. And a_i is known that in Table-3.1 b) that $a_1=a_6=0.03458$, $a_2=a_7=0.23832$, $a_3=a_8=0.45149$, $a_4=a_9=0.23832$, $a_5=a_{10}=0.03458$.

For fines (i = 6, 7, 8, 9, 10), fc, is percent of fine content in the mixture system. y_i can be obtained in following equations, where

$$y_i = fc \times a_i \tag{23}$$

For coarse (i= 1, 2, 3, 4, 5), *cc*, is percent of coarse content in the mixture system. y_i can be obtained in following equations,

$$y_i = cc \times a_i \tag{24}$$

Put all data above into Binary Equation (9, 10, 11). Prediction of minimum void ratio under different ratio of fine and coarse content can then be obtained. Figure-4.1 c) illustrates six sets of prediction results.

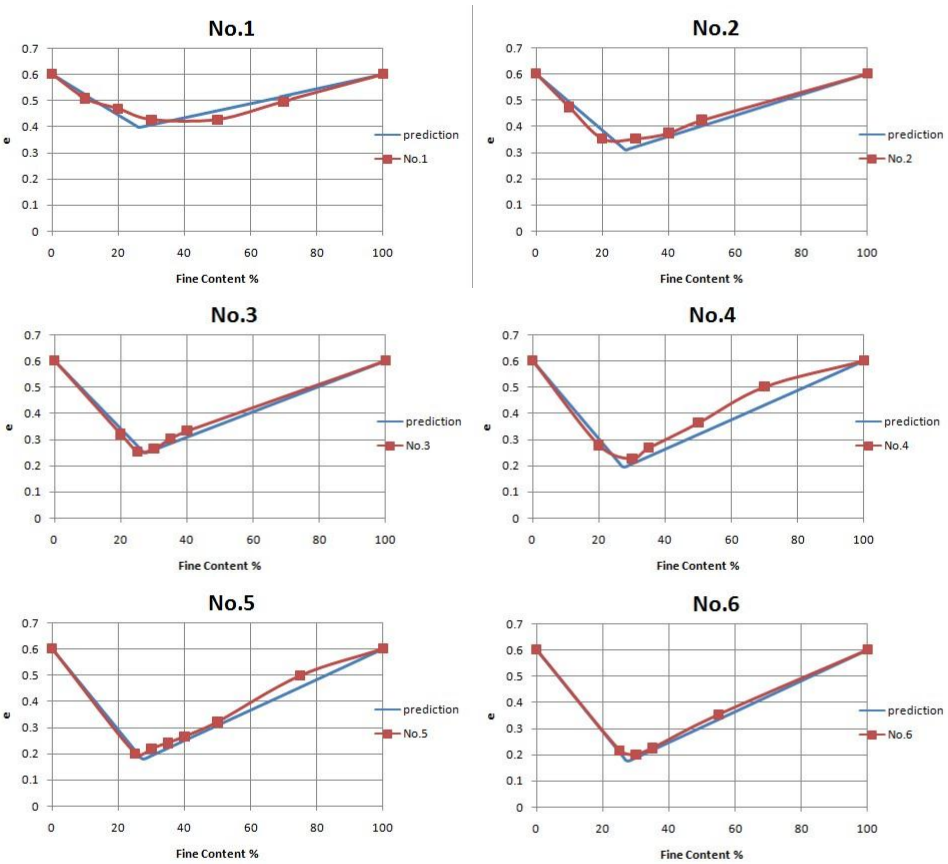


Figure-4.1 d) Comparison between binary system prediction and experiments (Steel shot experiments made byMcGeary, 1961)

4.2 Spheres In Ternary System

Literature data for packing spheres in ternary system were from research of Yi et al (2012), who took advantage of discrete element method (DEM). 28 sets of DEM experiments data showed that particle sizes consist of large (L), medium (M) and small (S) class, are 24.4 mm, 11.6 mm, 6.4 mm, respectively. Raw data are shown in Table-4.2 a)

Table-4.2 a) Packing density and void ratio in ternary system (Yi et al, 2012)

cases	1	2	3	4	5	6	7	8	9	10	11	12	13	14
XL	0	0	0	0	0	0	0	0.17	0.17	0.17	0.25	0.25	0.28	0.28
XM	0	0.28	0.33	0.5	0.67	0.72	1	0.17	0.415	0.66	0.25	0.5	0	0.72
XS	1	0.72	0.67	0.5	0.33	0.28	0	0.66	0.415	0.17	0.5	0.25	0.72	0
PD	0.62	0.64	0.64	0.65	0.65	0.65	0.63	0.66	0.67	0.66	0.67	0.67	0.67	0.65
e	0.613	0.563	0.563	0.538	0.538	0.538	0.587	0.515	0.493	0.515	0.493	0.493	0.493	0.538
cases	15	16	17	18	19	20	21	22	23	24	25	26	27	28
XL	0.33	0.33	0.33	0.415	0.415	0.5	0.5	0.5	0.66	0.67	0.67	0.72	0.72	1
XM	0	0.34	0.67	0.17	0.415	0	0.25	0.5	0.17	0	0.33	0	0.28	0
XS	0.67	0.33	0	0.415	0.17	0.5	0.25	0	0.17	0.33	0	0.28	0	0
PD	0.68	0.69	0.65	0.69	0.68	0.7	0.7	0.66	0.7	0.72	0.66	0.73	0.66	0.63
e	0.471	0.449	0.538	0.449	0.471	0.429	0.429	0.515	0.429	0.389	0.515	0.370	0.515	0.587

XL+XM+XS=1

Ternary model needs fifteen particle sizes: d_1 to d_{15}; *p*, *q*, *s* and *t* for each coefficient of interactions: a_{ij} and b_{ij}; void ratios: $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ ***and*** $\boldsymbol{e_3}$. It is already assumed that for large particles (L) of d_1 to d_5, void ratios of them stay $\mathbf{e_1}$; for medium particles (M) of d_6 to d_{10}, void ratio of them keep $\mathbf{e_2}$; for small particles (S) of d_{11} to d_{15}, void ratio of them keep $\mathbf{e_3}$.

Method of for dealing with ternary system is to establish coefficient of interactions a_{ij} and b_{ij} based on known binary mixture in ternary system. Among 28 sets of assemblies, 15 of them are binary mixture with 3 different particle sizes.It is set that larger particle mean size is named by μ_{x1}, and smaller one's is named by μ_{x2}. Then binary experiments data (M-S, L-S, L-M) were drawn with fitting line describing slopes of coarse and fine. They were drawn in Figure-4.2 a)

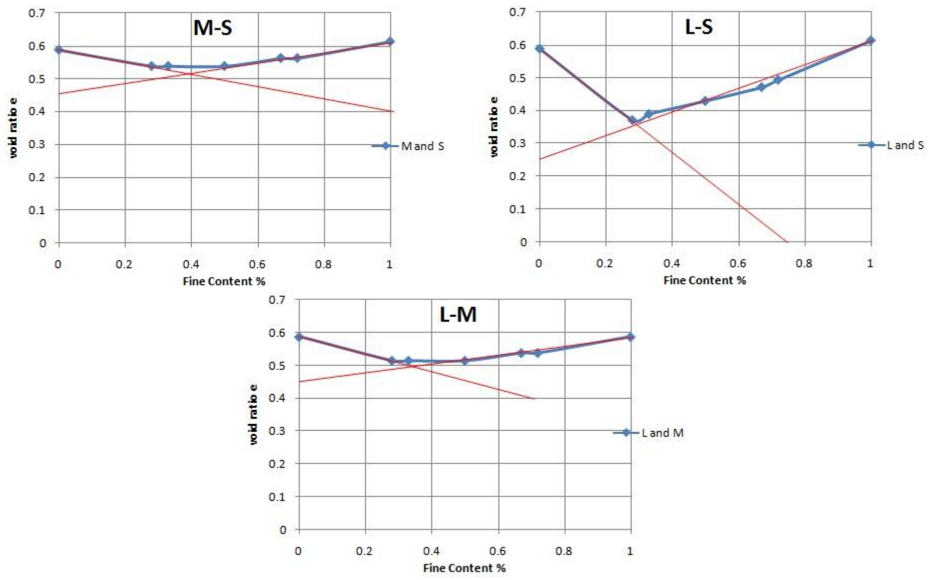


Figure-4.2 a) Three sets of rates of change line describing slopes of coarse and fine (Yi et al, 2012)

Same method as Chapter 4.1, a_{ij} and b_{ij} can be determined by slopes of relative coarse and fine by equation (12) (13). Table-4.2 b) lists their volume fractions, ratio and coefficient of interactions.

Table-4.2 b) Coefficient of interactions obtained by binary mixture of data (Yi et al, 2012)

Item No.	μ_{x2} (mm)	μ_{x1} (mm)	ratio	aij	bij
No.1	6.4	11.6	0.5517	0.8680	0.7662
No.2	6.4	24.4	0.2623	0.4921	0.4257
No.3	11.6	24.4	0.4754	0.8314	0.7662

where *ratio*$=\mu_{x2}/\mu_{x1}$

Then parameters *p*, *q*, *s* and *t* can be estimated through equations of coefficient of interactions (5) (6). Two fitting graphs were shown in Figure-4.2 b).

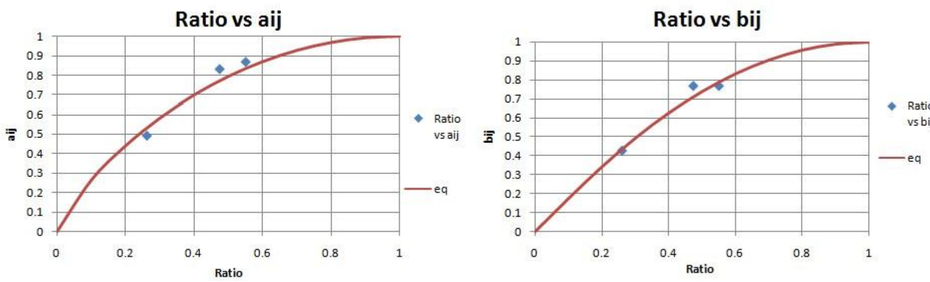


Figure-4.2 b) Fitting graphs of coefficient of interactions for three sets of spheres in DEM (Yi et al, 2012)

Therefore, we get *p=2*, *q=0.8* for a_{ij}, and *s=2*, *t=1.05* for b_{ij}.

Assuming variance of range (v_x) is 1 for each particle size distribution (Results from $v_x=0.01$ is also shown). Based on equation (19), μ_{lnx} and σ_{lnx} values can be determined by μ_x and v_x. ***e₁***, ***e₂*** and ***e₃*** are also known data from experiments.

Table-4.2 c) lists some parameters:

Table-4.2 c) Parameters can used in the model of ternary system (Yi et al, 2012)

μ_{x3}	μ_{x2}	μ_{x1}	**e3**	**e2**	**e1**	p	q
6.40000	11.60000	24.40000	0.61290	0.58730	0.58730	2.00000	0.80000
σ_{lnx3}	σ_{lnx2}	σ_{lnx1}	μ_{lnx3}	μ_{lnx2}	μ_{lnx1}	s	t
0.15531	0.08605	0.04097	1.84424	2.44730	3.19374	2.00000	1.05000

By means of Chapter 3.1 and Table-3.1 a), fifteen particle sizes, fifteen void ratios and corresponding log-normal areas (percentage) were produced in Table-4.2 d).

Table-4.2 d) Fifteen particle sizes, log-normal areas, void ratios of ternary system (Yi et al, 2012)

d_{15}	d_{14}	d_{13}	d_{12}	d_{11}	d_{10}	d_9	d_8	d_7	d_6	d_5	d_4	d_3	d_2	d_1
4.35575	5.24811	6.32328	7.61872	9.17955	9.40075	10.42333	11.55714	12.81428	14.20816	22.09662	23.21003	24.37953	25.60797	26.89831
a_{15}	a_{14}	a_{13}	a_{12}	a_{11}	a_{10}	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1
0.03458	0.23832	0.45149	0.23832	0.03458	0.03458	0.23832	0.45149	0.23832	0.03458	0.03458	0.23832	0.45149	0.23832	0.03458
e_{15}	e_{14}	e_{13}	e_{12}	e_{11}	e_{10}	e_9	e_8	e_7	e_6	e_5	e_4	e_3	e_2	e_1
0.61290	0.61290	0.61290	0.61290	0.61290	0.58730	0.58730	0.58730	0.58730	0.58730	0.58730	0.58730	0.58730	0.58730	0.58730

Put all data above into Equation (14, 15, 16, 17). Prediction of minimum void ratio under different mixture of large, medium and small content can then be obtained. Figure 4.2.c) - Figure 4.2 e) will show comparison between the ternary prediction results and experiments.

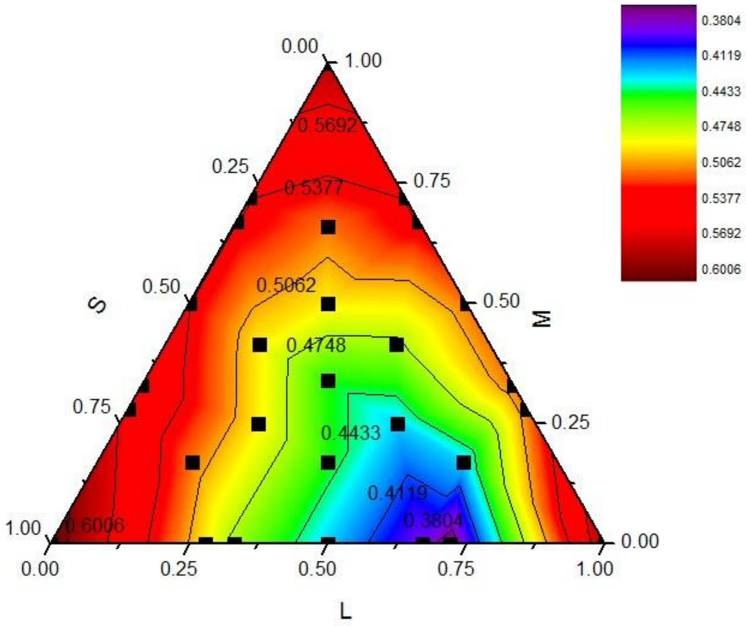


Figure-4.2 c) Void ratio- ternary contour of experiments (Yi et al, 2012)

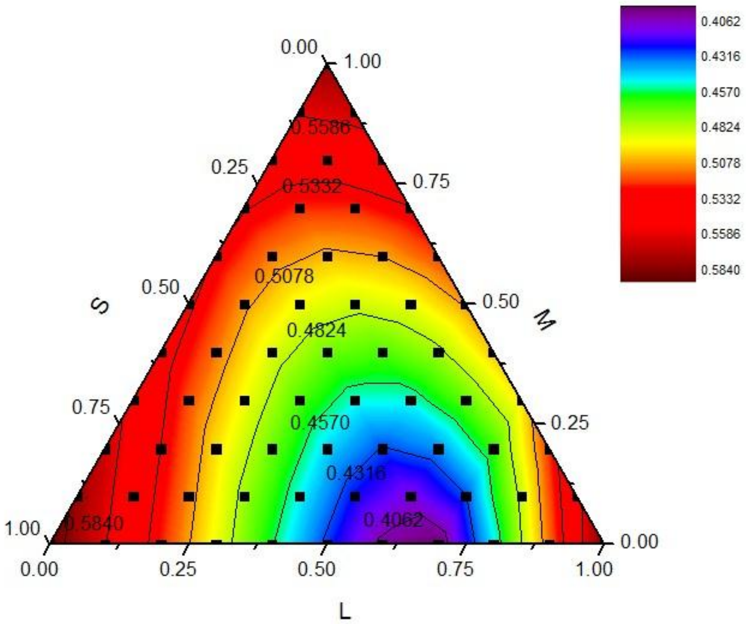


Figure-4.2 d) Void ratio- ternary contour of model with $v_x=1$ (Yi et al, 2012)

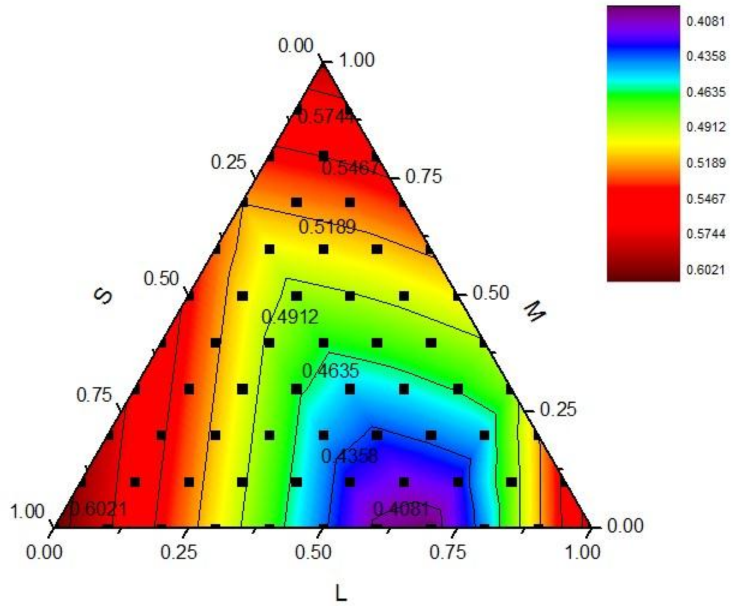


Figure-4.2 e) Void ratio- ternary contour of model with v_x*=0.01* (Yi et al, 2012)

If three particle sizes range were called *d1*, *d2*, *d3* with variance of 1 and 0.01.Log-normal distributions used in prediction model were drawn from Figure -4.2 f) – Figure-4.2 h) for both conditions: v_x=1 and v_x=0.01.

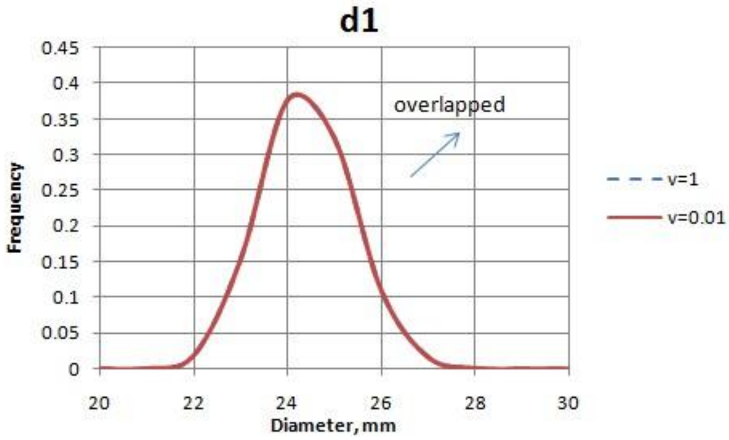


Figure-4.2 f) Log-normal distribution describing size range with v=1 and v=0.01, large sizes d1 (Yi et al, 2012)

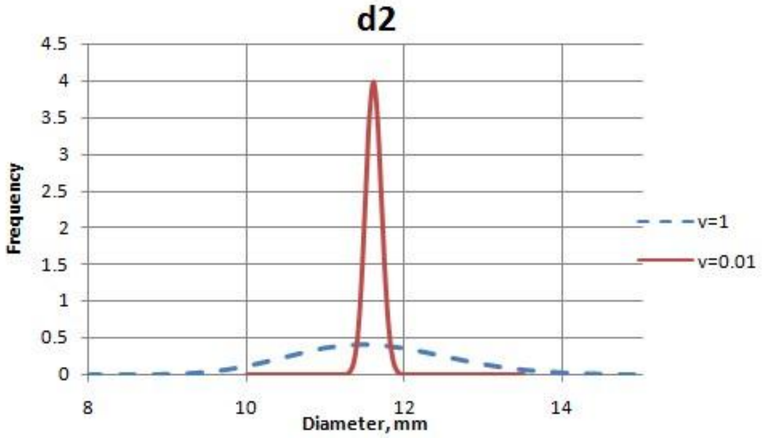


Figure-4.2 g) Log-normal distribution describing size range with v=1 and v=0.01, medium sizes d2 (Yi et al, 2012)

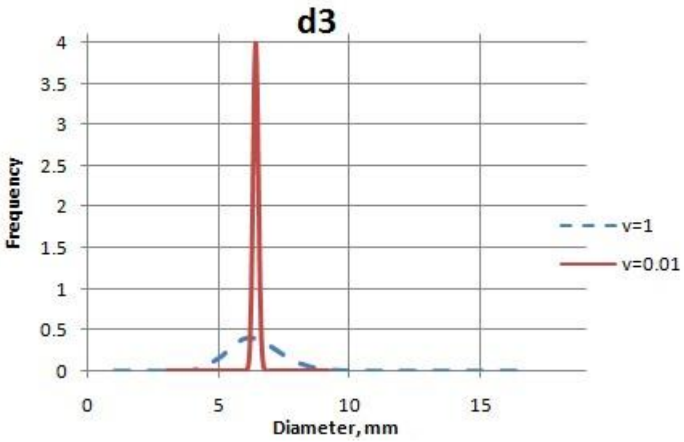


Figure-4.2 h) Log-normal distribution describing size range with v=1 and v=0.01, small sizes d3 (Yi et al, 2012)

5. PACKING SYSTEM OF AGGREGATE OR SOIL

When it comes to soils or concrete aggregate, method used in the paper also works. However, coefficient of interaction would be different than spherical particles. There will be three parts to give detailed examples to compare experiments and the proposed method.

5.1 Gravelly Sand with Silt

Literature data for packing of gravelly sand with silt is from Fragaszy and Sneider, 1991. Densities and void ratios of packing experiments were list in Table 5.1 a) - Table-5.1 h)

Table-5.1 a) Data of packing density and void ratio- No.1 (Fragaszy and Sneider, 1991)

No.1						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	120	0.720	0.388	gravel	fine
40	60	128.2	0.766	0.305	2.7	2.67
70	30	124.8	0.743	0.346		
85	15	116	0.690	0.450		
100	0	111	0.659	0.518		

Table-5.1 b) Data of packing density and void ratio- No.2 (Fragaszy and Sneider, 1991)

No.2						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	121.5	0.718	0.392	gravel	fine
20	80	127.3	0.753	0.328	2.71	2.71
40	60	132.3	0.782	0.278		
70	30	136.6	0.808	0.238		
85	15	125.4	0.742	0.349		
100	0	114.5	0.677	0.477		

Table-5.1 c) Data of packing density and void ratio- No.3 (Fragaszy and Sneider, 1991)

No.3						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	126	0.737	0.357	gravel	fine
20	80	131	0.767	0.304	2.73	2.74
40	60	133.9	0.784	0.275		
70	30	131.6	0.772	0.296		
85	15	114.4	0.671	0.490		
100	0	106	0.622	0.607		

Table-5.1 d) Data of packing density and void ratio- No.4 (Fragaszy and Sneider, 1991)

No.4						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	119	0.704	0.421	gravel	fine
40	60	132.7	0.782	0.278	2.73	2.71
70	30	138.5	0.815	0.227		
85	15	128.7	0.756	0.322		
100	0	119	0.699	0.432		

Table-5.1e) Data of packing density and void ratio- No.5 (Fragaszy and Sneider, 1991)

No.5						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	122	0.746	0.340	gravel	fine
20	80	127	0.772	0.295	2.7	2.62
40	60	130.5	0.789	0.268		
70	30	126.2	0.756	0.323		
85	15	116.2	0.693	0.443		
100	0	106	0.629	0.589		

Table-5.1f) Data of packing density and void ratio- No.6 (Fragaszy and Sneider, 1991)

No.6						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	135	0.816	0.225	gravel	fine
20	80	136.5	0.822	0.216	2.7	2.65
50	50	137.2	0.822	0.217		
70	30	129.6	0.774	0.293		
85	15	124.2	0.739	0.353		
100	0	109	0.647	0.546		

Table-5.1g) Data of packing density and void ratio- No.7 (Fragaszy and Sneider, 1991)

No.7						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	128.5	0.771	0.297	gravel	fine
20	80	129.6	0.777	0.287	2.68	2.67
50	50	130.7	0.783	0.277		
70	30	124.6	0.746	0.341		
85	15	111	0.664	0.506		
100	0	104	0.622	0.608		

Table-5.1h) Data of packing density and void ratio- No.8 (Fragaszy and Sneider, 1991)

No.8						
%gravel	%fine	density,pcf	density	e	specific gravity	
0	100	139.8	0.830	0.205	gravel	fine
20	80	140.7	0.834	0.198	2.71	2.7
40	60	137.6	0.816	0.226		
50	50	135.4	0.802	0.247		
70	30	125.3	0.742	0.348		
85	15	111.4	0.659	0.517		
100	0	100.5	0.594	0.683		

Binary model needs ten particle sizes: d_1 to d_{10}; *p*, *q*, *s* and *t* for each coefficient of interactions: a_{12} and b_{21}; void ratios: $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. It is already assumed that for coarse particles of d_1 to d_5, void ratios of them stay $\mathbf{e_1}$; for fine particles of d_6 to d_{10}, void ratio of them keep $\mathbf{e_2}$. Void ratio under 100% coarse is $\mathbf{e_1}$, void ratio under 0%coarse is $\mathbf{e_2}$.

As method described in Chapter2.2, to acquire coefficient of interactions: a_{12} and b_{21}, the experiments data shown in Table-5.1 a) - Table-5.1h) were redrawn in Figure-5.1 a) with rate of change lines for the two sides.

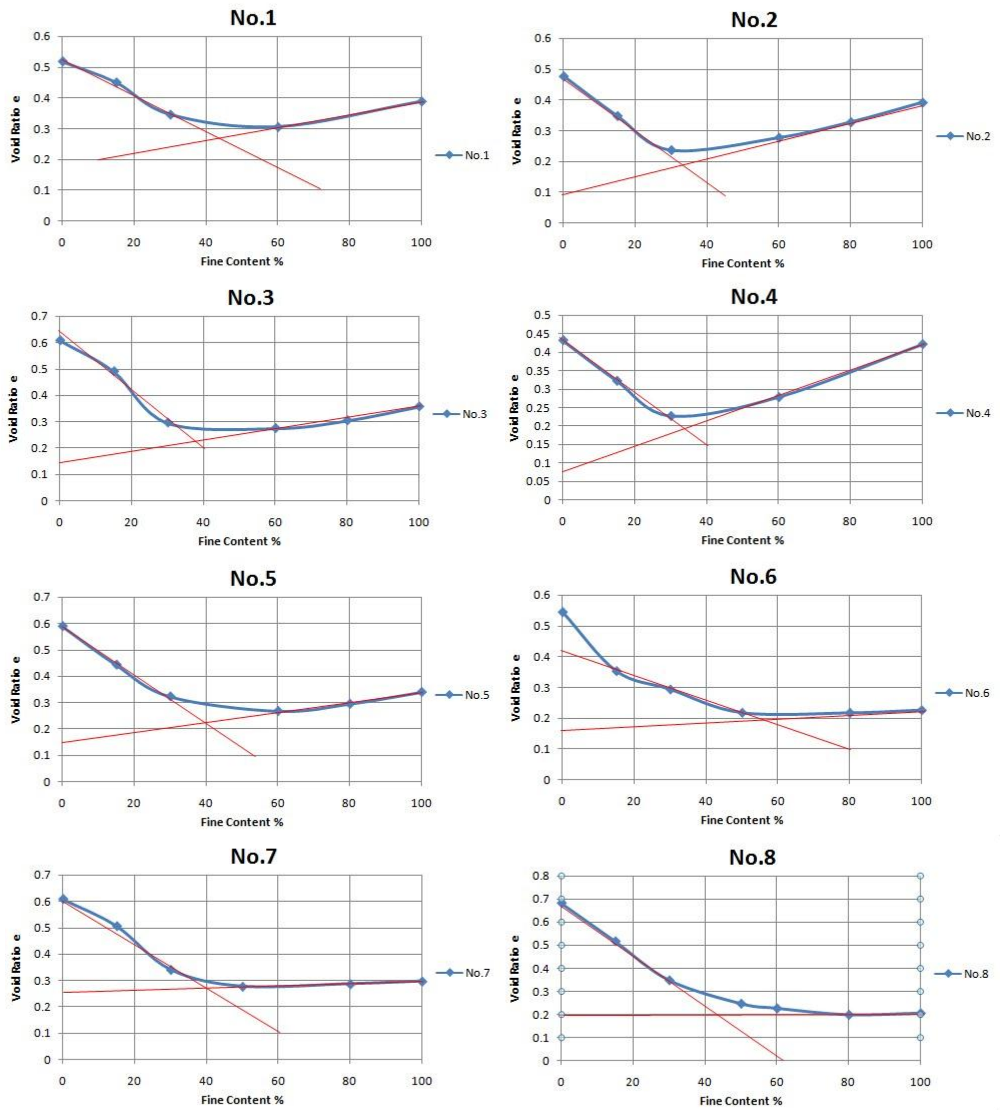


Figure-5.1 a) Eight sets of rate of change line describing slopes of coarse and fine (Fragaszy and Sneider, 1991)

In Equatioin (12), $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_1}$ side (coarse), a_{12} can be determined; since it is known in Chapter 2.2 that $\frac{\partial e}{\partial y_1} = -\frac{\partial e}{\partial y_2}$, $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_2}$ side (fine), b_{21} can be determined. Then six sets of coefficient of interactions a_{12} and b_{21} can be calculated.

Results are represented in Table-5.1 i).

Table-5.1 i) Particle size ratio and coefficient of interactions of No.1 to No.8 (Fragaszy and Sneider, 1991)

Soil No.	ratio	a_{ij}	b_{ij}
No.1	0.16	0.66330	0.34579
No.2	0.09	0.45937	0.20969
No.3	0.13	0.43432	0.24707
No.4	0.11	0.51209	0.17380
No.5	0.12	0.50973	0.25448
No.6	0.10	0.80707	0.31153
No.7	0.16	0.58719	0.41118
No.8	0.11	0.48260	0.29298

Ratio in this table was obtained by: $ratio = \frac{\mu_{x2}}{\mu_{x1}}$, where μ_{x2} and μ_{x1} are mean value from fine and coarse in Table 9.2- Soil description of No.1 to No.8 (Fragaszy and Sneider, 1991)

Then parameters p, q, s and t can be estimated through equations of coefficient of interactions (5) (6) by given eight sets of a_{12} and $b21$ in Table-4.1 i). Two fitting graphs were shown in Figure-5.1 b). Study of fitting p, q, s and t to get a_{ij} or b_{ij} can be found in Appendix

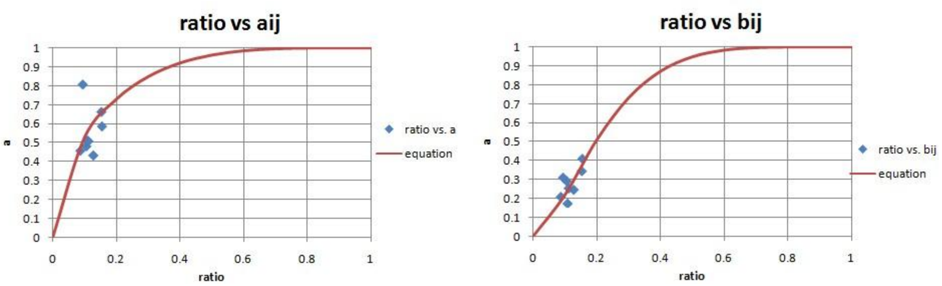


Figure-5.1b) Fitting graphs of coefficient of interactions for eight sets of gravelly sand with silt (Fragaszy and Sneider, 1991)

Therefore, we get *p=4, q=0.6*for a_{ij}, and *s=5, t=1.7* for b_{ij}. Take No.1 mixture for example.

Since particle size variance can seen from Table-3.2, we can get d_1=24.06552, d_2=15.73185, d_3=10.28406, d_4=6.96107, d_5=6.72278, d_6=4.39474, d_7=3.11080, d_8=1.39018, d_9=0.62125, d_{10}=0.27763;

e_1=e_2=e_3=e_4=e_5=0.51784(**e_1**), e_6=e_7=e_8=e_9=e_{10}=0.3884(**e_2**). And a_i is known that in Table-3.1 b) that a_1=a_6=0.03458, a_2=a_7=0.23832, a_3=a_8=0.45149, a_4=a_9=0.23832, a_5=a_{10}=0.03458.

For fines (i= 6, 7, 8, 9, 10), *fc*, is percent of fine content in the mixture system. y_i can be obtained in following equations, where

$$y_i = fc \times a_i \tag{23}$$

For coarse (i= 1, 2, 3, 4, 5), *cc*, is percent of coarse content in the mixture system. y_i can be obtained in following equations,

$$y_i = cc \times a_i \tag{24}$$

Put all data above into Binary Equation (9, 10, 11). Prediction of minimum void ratio under different ratio of fine and coarse content can then be obtained. Figure-5.1 c) illustrates eight sets of prediction results.

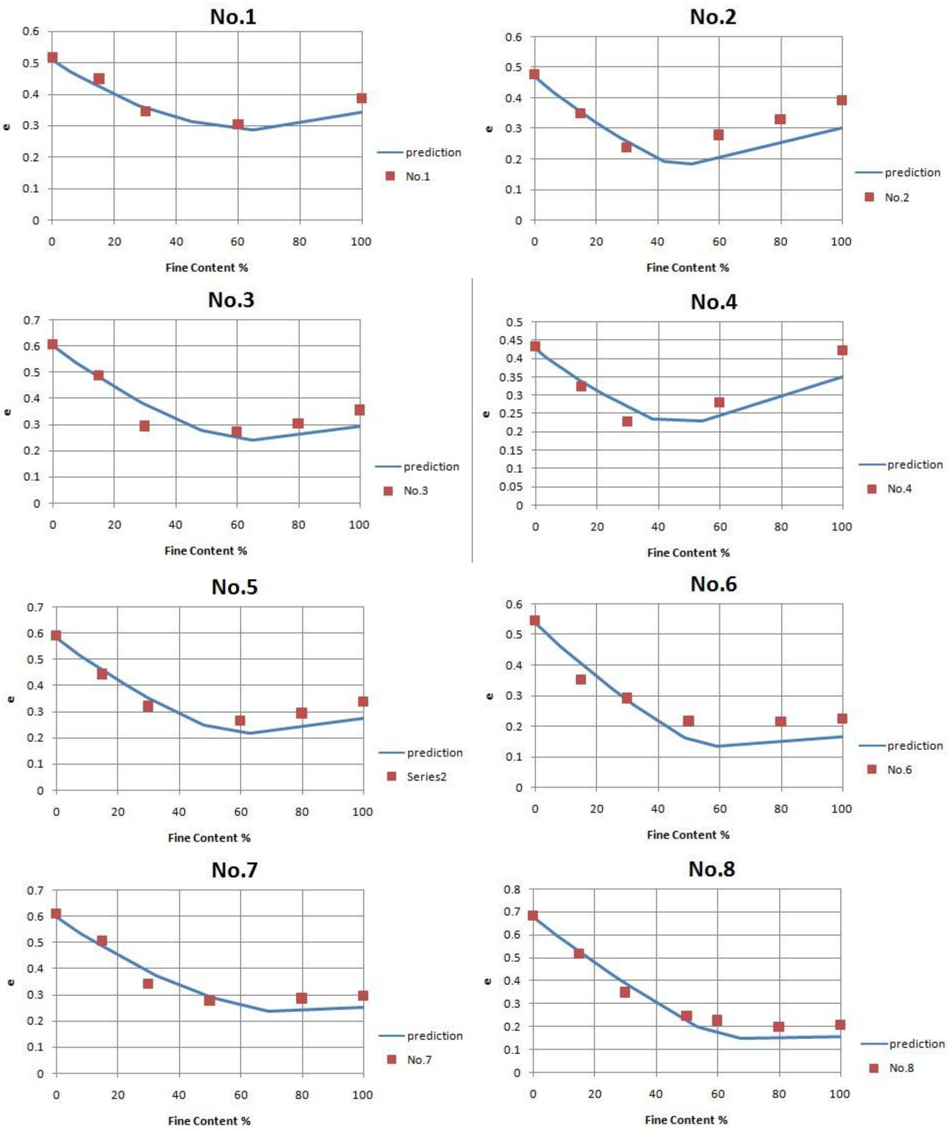


Figure-5.1 c)- Comparison of void ratios between experiments and model (Fragaszy and Sneider, 1991)

Since void ratio e_1 and e_2 used in model cannot represent real condition, an adjustment coefficient, "α" , is then proposed. The input value of void ratio of fines, e_2, was putted into model as a process of iterating until prediction curve fits experiment data. Adjustment coefficients for e_2 from soils No.1 to No.8 are: 1.1, 1.3, 1.15, 1.15, 1.15, 1.35, 1.15 and 1.25. Thus, after each original e_2 times α, the adjusted prediction results are shown in Figure-5.1 d).

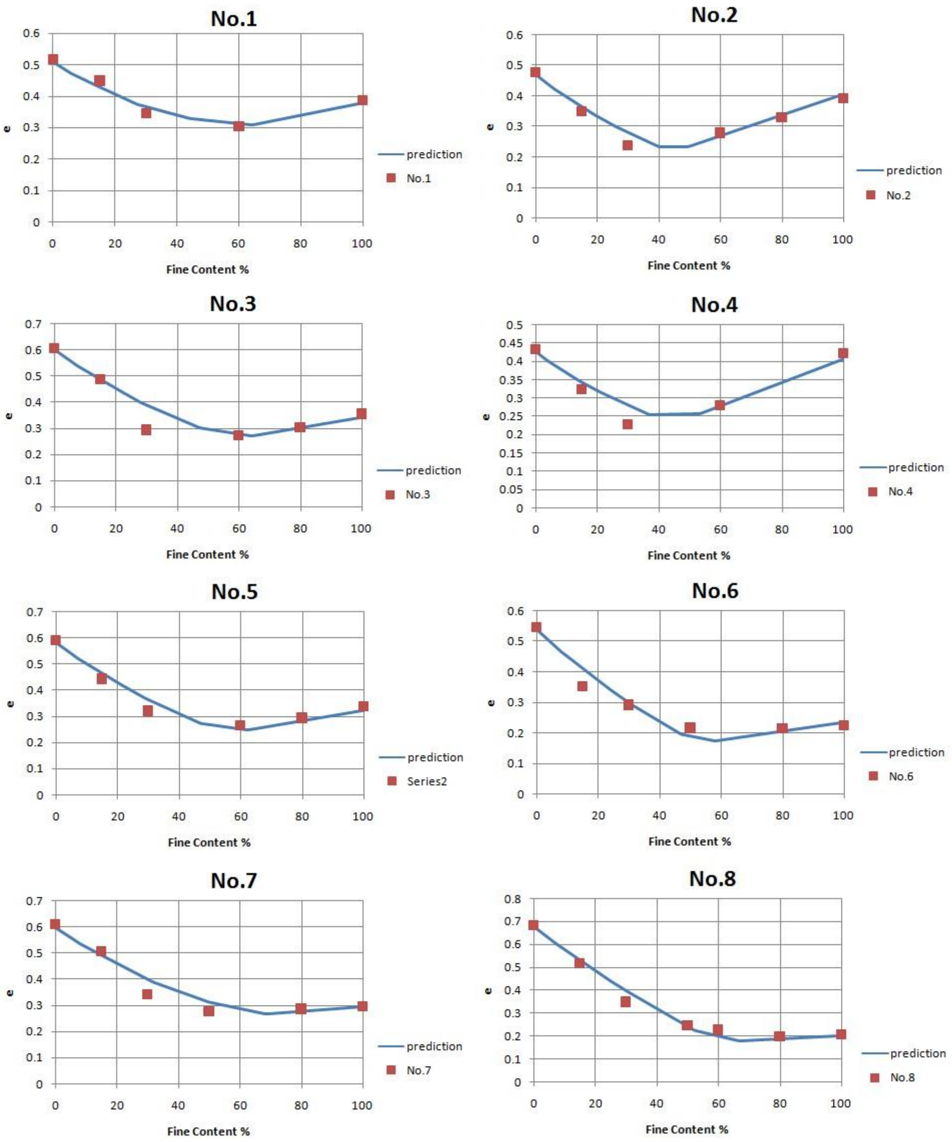


Figure-5.1 d)- Comparison of void ratios between experiments and model with adjustment coefficient, α (Fragaszy and Sneider, 1991)

Besides, real particle size distributions were also put into model to investigate the validities of modeling log-normal distribution. Take Soil No.1 into account, as it is mentioned in Chapter 3, bins of coarse soils are: 20-18.475, 18.475-16.95, 16.95-15.425, 15.425-13.9, 13.9-12.375, 12.375-10.85, 10.85-9.325, 9.325-7.8, 7.8-6.275, 6.275-4.75mm; their center-size are: 19.2375, 17.7125, 16.1875, 14.6625, 13.1375, 11.6125, 10.0875, 8.5625, 7.0375, 5.5125 mm. Average value of every two of them was used to represent d_1 to d_5 in model: 18.475, 15.425, 12.375, 9.325 and 6.275 mm; probability of each size is sum of area of two adjacent frequency columns what we talked in Chapter 3. Thus, a_1 to a_5 are: 0.11375, 0.13829, 0.17268, 0.23007 and 0.34520. Bins of fine soils are: 4.75-4.275, 4.275-3.8, 3.8-3.325, 3.325-2.85, 2.85-2.375, 2.375-1.9, 1.9-1.425, 1.425-0.95, 0.95-0.475, 0.475-0mm; their center-size are: 4.5125, 4.0375, 3.5625, 3.0875, 2.6125, 2.1375, 1.6625, 1.1875, 0.7125, 0.2375mm. Average value of every two of them was used to represent d_6 to d_{10} in model: 4.275, 3.325, 2.375, 1.425 and 0.475mm; probability of each size is sum of area of two adjacent frequency columns what we talked in Chapter 3. Thus a_6 to a_{10} are: 0.08507, 0.10934, 0.15412, 0.26346 and 0.38800.

Then, adjustment coefficients were back fit the experiments data: 1.15 for $\boldsymbol{e_2}$ and 1.1 for $\boldsymbol{e_1}$. Prediction of No.1 soils which used real size distribution was shown in Figure-5.1 e), compared to prediction by using log-normal distribution.

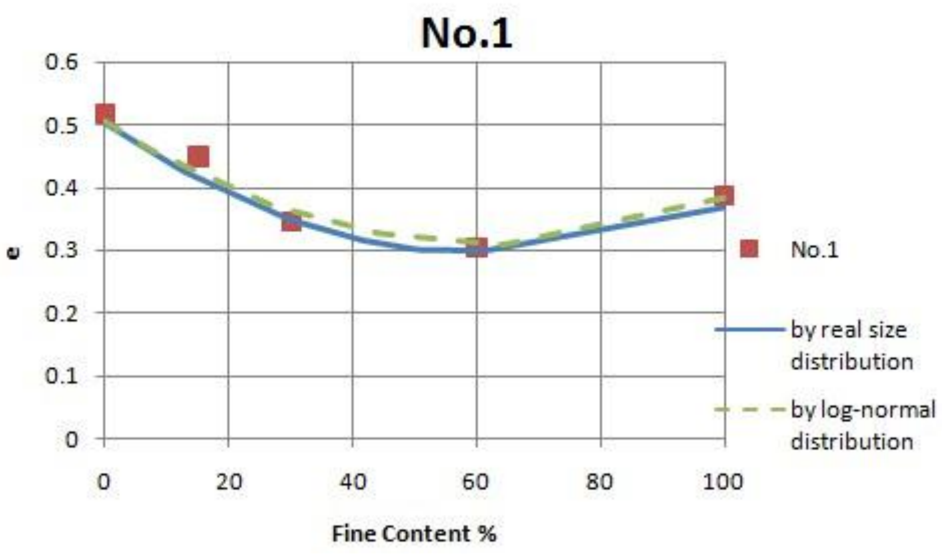


Figure-5.1 e)- Comparison results of void ratios between by real size distribution and by log-normal distribution for Soil No.1 (Fragaszy and Sneider, 1991)

5.2 Rounded Aggregate

Literature data for packing of rounded aggregate is from F. de Larrard, 1999. Densities and void ratios of packing experiments were list in Table 5.2 a) –Table 5.2 b)

Table-10.2 a) Data of packing density and void ratio- ra part 1 (F. de Larrard, 1999)

R8R05			R8R1			R4R05			R8R2		
%fine	PD	e	%fine	PD	e	%fine	PD	e	%fine	PD	e
0	0.628	0.592	0	0.628	0.592	0	0.6195	0.614	0	0.628	0.592
5	0.657	0.522	5	0.6545	0.528	5	0.645	0.550	5	0.653	0.531
10	0.6865	0.457	10	0.6795	0.472	10	0.6715	0.489	10	0.682	0.466
15	0.71	0.408	15	0.707	0.414	15	0.689	0.451	15	0.697	0.435
20	0.729	0.372	20	0.724	0.381	20	0.706	0.416	20	0.714	0.401
25	0.754	0.326	25	0.742	0.348	25	0.7265	0.376	25	0.7235	0.382
30	0.758	0.319	30	0.748	0.337	30	0.7485	0.336	30	0.728	0.374
40	0.753	0.328	40	0.7285	0.373	40	0.736	0.359	40	0.723	0.383
50	0.7385	0.354	50	0.7095	0.409	50	0.725	0.379	50	0.705	0.418
60	0.7165	0.396	60	0.6965	0.436	60	0.7	0.429	60	0.689	0.451
70	0.68	0.471	70	0.677	0.477	70	0.6745	0.483	70	0.671	0.490
80	0.652	0.534	80	0.6585	0.519	80	0.648	0.543	80	0.646	0.548
90	0.6195	0.614	90	0.635	0.575	90	0.614	0.629	90	0.632	0.582
100	0.592	0.689	100	0.609	0.642	100	0.592	0.689	100	0.616	0.623

Table-11.2 b) Data of packing density and void ratio- ra part 2 (F. de Larrard, 1999)

R2R05			R8R4			R1R05		
%fine	PD	e	%fine	PD	e	%fine	PD	e
0	0.616	0.623	0	0.628	0.592	0	0.609	0.642
5	0.635	0.575	5	0.6375	0.569	5	0.624	0.603
10	0.663	0.508	10	0.643	0.555	10	0.633	0.580
15	0.678	0.475	15	0.654	0.529	15	0.64	0.563
20	0.692	0.445	20	0.66	0.515	20	0.656	0.524
25	0.708	0.412	25	0.663	0.508	25	0.666	0.502
30	0.718	0.393	30	0.6595	0.516	30	0.6705	0.491
40	0.708	0.412	40	0.6565	0.523	40	0.6635	0.507
50	0.693	0.443	50	0.6535	0.530	50	0.6545	0.528
60	0.67	0.493	60	0.649	0.541	60	0.644	0.553
70	0.656	0.524	70	0.6445	0.552	70	0.636	0.572
80	0.633	0.580	80	0.638	0.567	80	0.6215	0.609
90	0.613	0.631	90	0.629	0.590	90	0.61	0.639
100	0.592	0.689	100	0.6195	0.614	100	0.592	0.689

Binary model needs ten particle sizes: d_1 to d_{10}; *p*, *q*, *s* and *t* for each coefficient of interactions: a_{12} and b_{21}; void ratios: $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. It is already assumed that for coarse particles of d_1 to d_5, void ratios of them stay $\mathbf{e_1}$; for fine particles of d_6 to d_{10}, void ratio of them keep $\mathbf{e_2}$. Void ratio under 100% coarse is $\mathbf{e_1}$, void ratio under 0%coarse is $\mathbf{e_2}$.

As method described in Chapter2.2, to acquire coefficient of interactions: a_{12} and b_{21}, the experiments data shown in Table-5.2 a) - Table-5.2 b) were redrawn in Figure-5.2 a) with rate of change lines for the two sides.

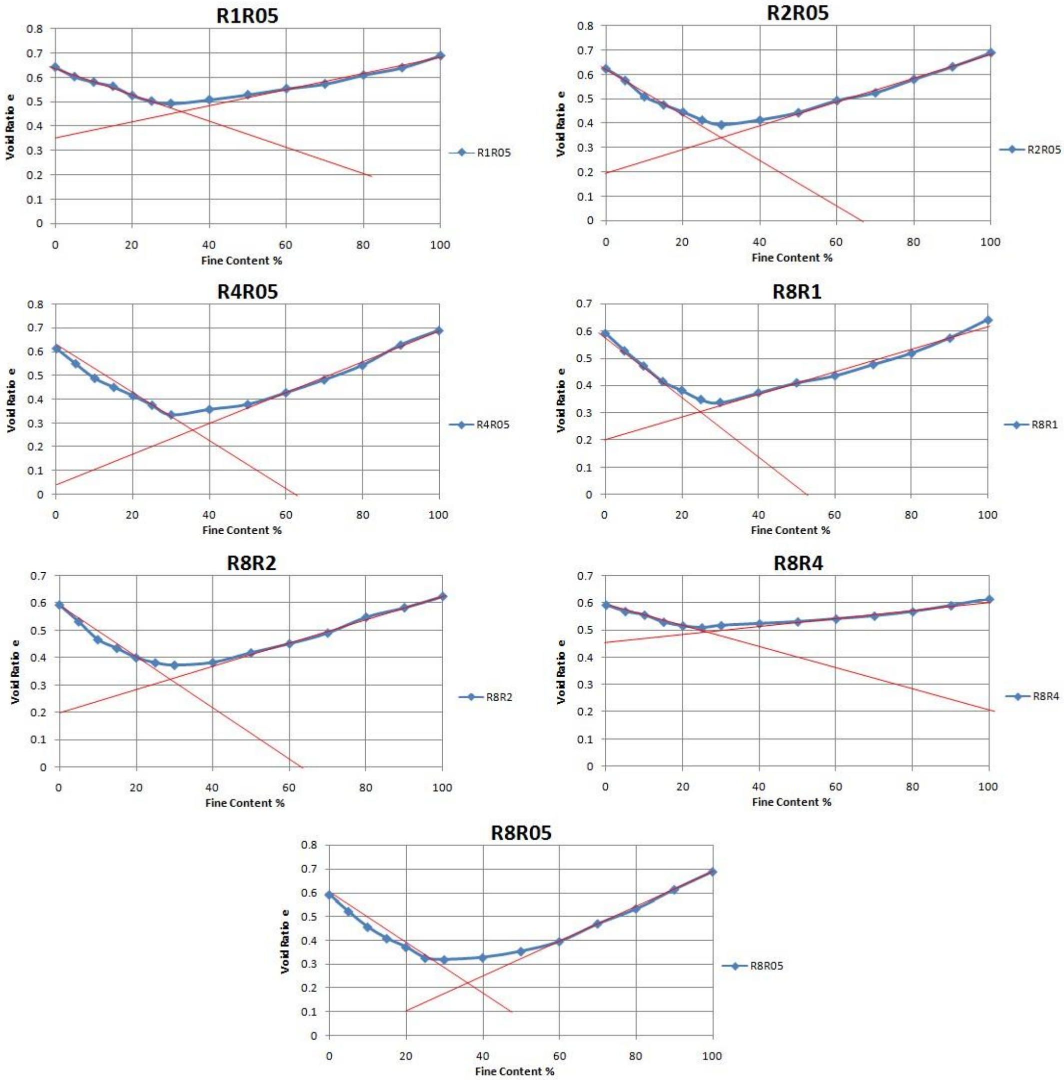


Figure-5.2 a) Seven sets of fitting line describing slopes of coarse and fine-round aggregate(F. de Larrard)

In Equatioin (12), $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_1}$ side (coarse), a_{12} can be determined; since it is known in Chapter 2.2 that $\frac{\partial e}{\partial y_1} = -\frac{\partial e}{\partial y_2}$, $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_2}$ side (fine), b_{21} can be determined. Then seven sets of coefficient of interactions a_{12} and b_{21} can be calculated.

Results are represented in Table-5.2 c).

Table-12.2 c) Seven sets of coefficient of interactions (F. de Larrad)

ratio	0.0625	0.125	0.125	0.25	0.25	0.5	0.5
R	R8R05	R8R1	R4R05	R8R2	R2R05	R8R4	R1R05
a_{12}	0.2949	0.2483	0.3496	0.3727	0.3933	0.7434	0.6450
b_{21}	-0.0798	0.3917	0.0814	0.3376	0.3208	0.7837	0.5451

Where ratio it can also be calculated from Table-3.3 a) Soil description of Rounded aggregate (F. de Larrard, 1999) that ratio of different combinations of mean value of dmin and dmax

Then parameters *p*, *q*, *s* and *t* can be estimated through equations of coefficient of interactions (5) (6) by given eight sets of a_{12} and *b*21 in Table-5.2 c). Two fitting graphs were shown in Figure-5.2 b). Study of fitting *p*, *q*, *s* and *t* to get a_{ij} or b_{ij} can be found in Appendix.

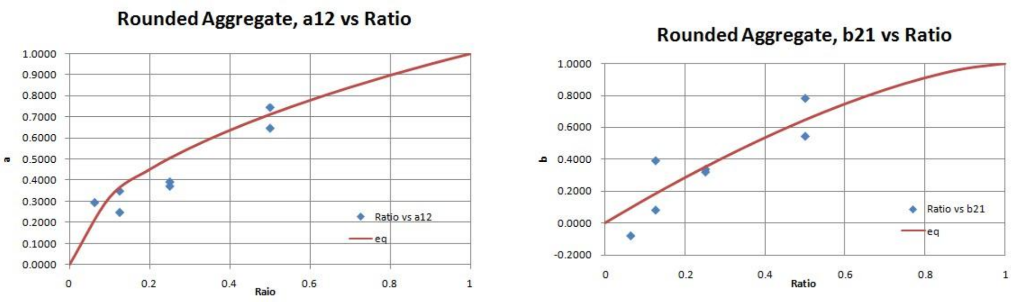


Figure-5.2 b) Fitting graphs of coefficient of interactions for seven sets of rounded aggregate (Fragaszy and Sneider, 1991)

Therefore, we get *p=1.02, q=0.0.5* for a_{ij}, and *s=1.5, t=1* for b_{ij}. Take R1R05 mixture for example.

Since particle size variance can seen from Table-3.3 a) Soil description of Rounded aggregate (F. de Larrard, 1999), we can get d_1=1.23003, d_2=1.17634, d_3=1.125, d_4=1.07590, d_5=1.028936, d_6=0.61972, d_7=0.59173, d_8=0.565, d_9=0.53948, d_{10}=0.51511;

$e_1=e_2=e_3=e_4=e_5$=0.64204($\mathbf{e_1}$), $e_6=e_7=e_8=e_9=e_{10}$=0.68919($\mathbf{e_2}$). And a_i is known that in Table-3.1 b) that $a_1=a_6$=0.03458, $a_2=a_7$=0.23832, $a_3=a_8$=0.45149, $a_4=a_9$=0.23832, $a_5=a_{10}$=0.03458.

For fines (i= 6, 7, 8, 9, 10), *fc*, is percent of fine content in the mixture system. y_i can be obtained in following equations, where

$$y_i = fc \times a_i \tag{23}$$

For coarse (i= 1, 2, 3, 4, 5), *cc*, is percent of coarse content in the mixture system. y_i can be obtained in following equations,

$$y_i = cc \times a_i \tag{24}$$

Put all data above into Binary Equation (9, 10, 11). Prediction of minimum void ratio under different ratio of fine and coarse content can then be obtained. Figure-5.2 c) illustrates seven sets of prediction results.

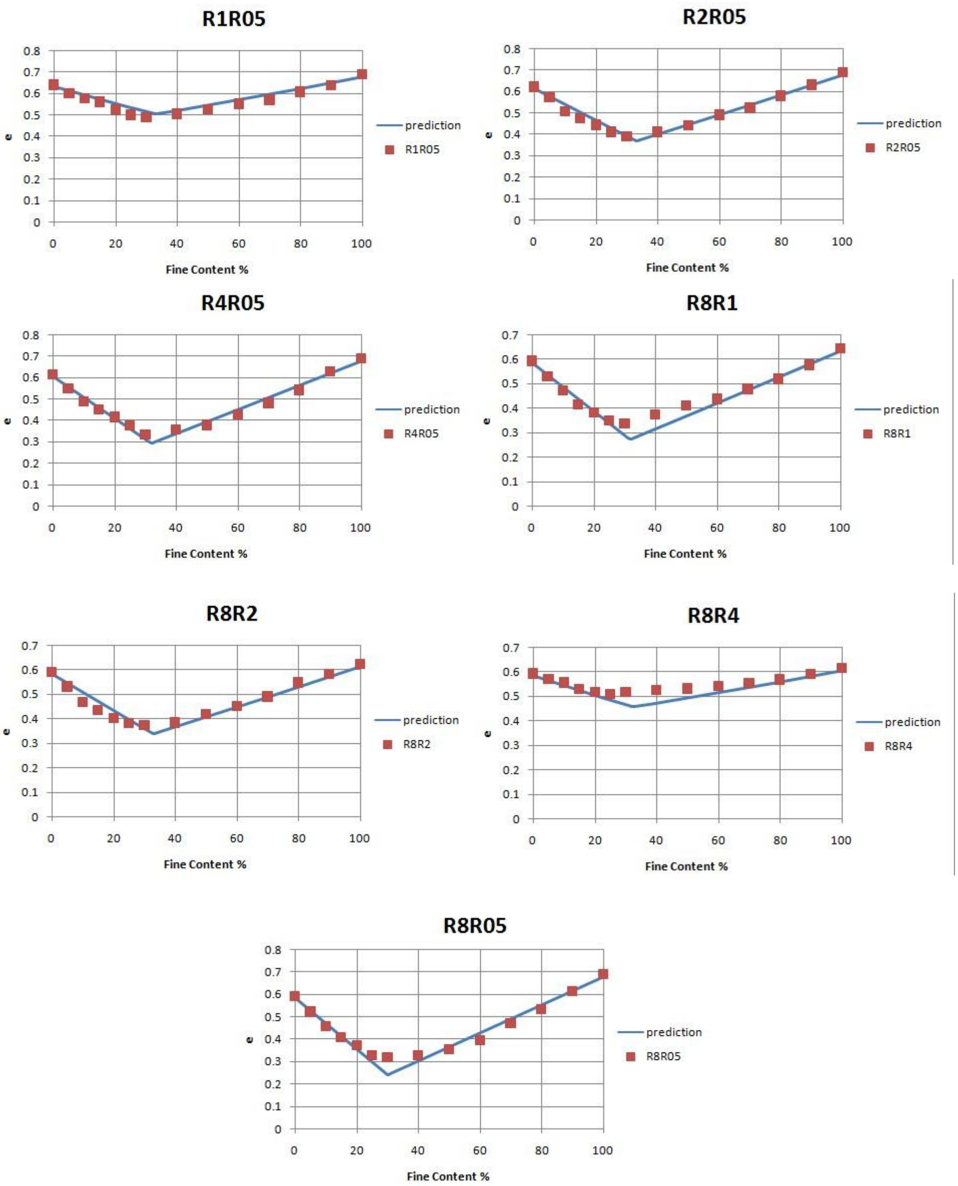


Figure-5.2 c) Comparison of void ratios between experiments and model, rounded aggregate (F. de Larrard, 1999)

5.3 Crushed Aggregate

Literature data for packing of crushed aggregate is from F. de Larrard, 1999. Densities and void ratios of packing experiments were list in Table-5.3 a) – Table-5.3 b)

Table-13.3 a) Data of packing density and void ratio- ca part 1 (F. de Larrard, 1999)

C8C05			C8C1			C4C05			C8C2		
%fine	PD	e	%fine	PD	e	%fine	PD	e	%fine	PD	e
0	0.572	0.748	0	0.572	0.748	0	0.537	0.862	0	0.572	0.748
5	0.62	0.613	5	0.613	0.631	5	0.591	0.692	5	0.597	0.675
10	0.642	0.558	10	0.646	0.548	10	0.6185	0.617	10	0.611	0.637
15	0.676	0.479	15	0.6755	0.480	15	0.638	0.567	15	0.625	0.600
20	0.705	0.418	20	0.699	0.431	20	0.669	0.495	20	0.634	0.577
25	0.731	0.368	25	0.7215	0.386	25	0.693	0.443	25	0.643	0.555
30	0.7365	0.358	30	0.7245	0.380	30	0.711	0.406	30	0.651	0.536
40	0.723	0.383	40	0.7025	0.423	40	0.691	0.447	40	0.643	0.555
50	0.6941	0.441	50	0.6705	0.491	50	0.667	0.499	50	0.6335	0.579
60	0.6585	0.519	60	0.638	0.567	60	0.64	0.563	60	0.6245	0.601
70	0.616	0.623	70	0.611	0.637	70	0.603	0.658	70	0.5975	0.674
80	0.583	0.715	80	0.5965	0.676	80	0.571	0.751	80	0.5695	0.756
90	0.5655	0.768	90	0.5435	0.840	90	0.545	0.835	90	0.5435	0.840
100	0.516	0.938	100	0.507	0.972	100	0.516	0.938	100	0.529	0.890

Table-14.3 b) Data of packing density and void ratio- ca part 2 (F. de Larrard, 1999)

C2C05			C8C4			C1C05		
%fine	PD	e	%fine	PD	e	%fine	PD	e
0	0.529	0.890	0	0.572	0.748	0	0.507	0.972
5	0.54	0.852	5	0.5825	0.717	5	0.527	0.898
10	0.552	0.812	10	0.5875	0.702	10	0.532	0.880
15	0.5515	0.813	15	0.588	0.701	15	0.545	0.835
20	0.566	0.767	20	0.592	0.689	20	0.552	0.812
25	0.573	0.745	25	0.5955	0.679	25	0.5485	0.823
30	0.594	0.684	30	0.594	0.684	30	0.555	0.802
40	0.588	0.701	40	0.5875	0.702	40	0.556	0.799
50	0.582	0.718	50	0.587	0.704	50	0.549	0.821
60	0.579	0.727	60	0.587	0.704	60	0.546	0.832
70	0.568	0.761	70	0.572	0.748	70	0.5425	0.843
80	0.5555	0.800	80	0.564	0.773	80	0.537	0.862
90	0.534	0.873	90	0.553	0.808	90	0.53	0.887
100	0.516	0.938	100	0.537	0.862	100	0.516	0.938

Binary model needs ten particle sizes: d_1 to d_{10}; *p*, *q*, *s* and *t* for each coefficient of interactions: a_{12} and b_{21}; void ratios: $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. It is already assumed that for coarse particles of d_1 to d_5, void ratios of them stay $\mathbf{e_1}$; for fine particles of d_6 to d_{10}, void ratio of them keep $\mathbf{e_2}$. Void ratio under 100% coarse is $\mathbf{e_1}$, void ratio under 0%coarse is $\mathbf{e_2}$.

As method described in Chapter 2.2, to acquire interaction coefficients: a_{12} and b_{21}, the experiments data shown in Table-5.3 a) - Table-5.3 b) were redrawn in Figure-5.3 a) with rate of change lines for the two sides.

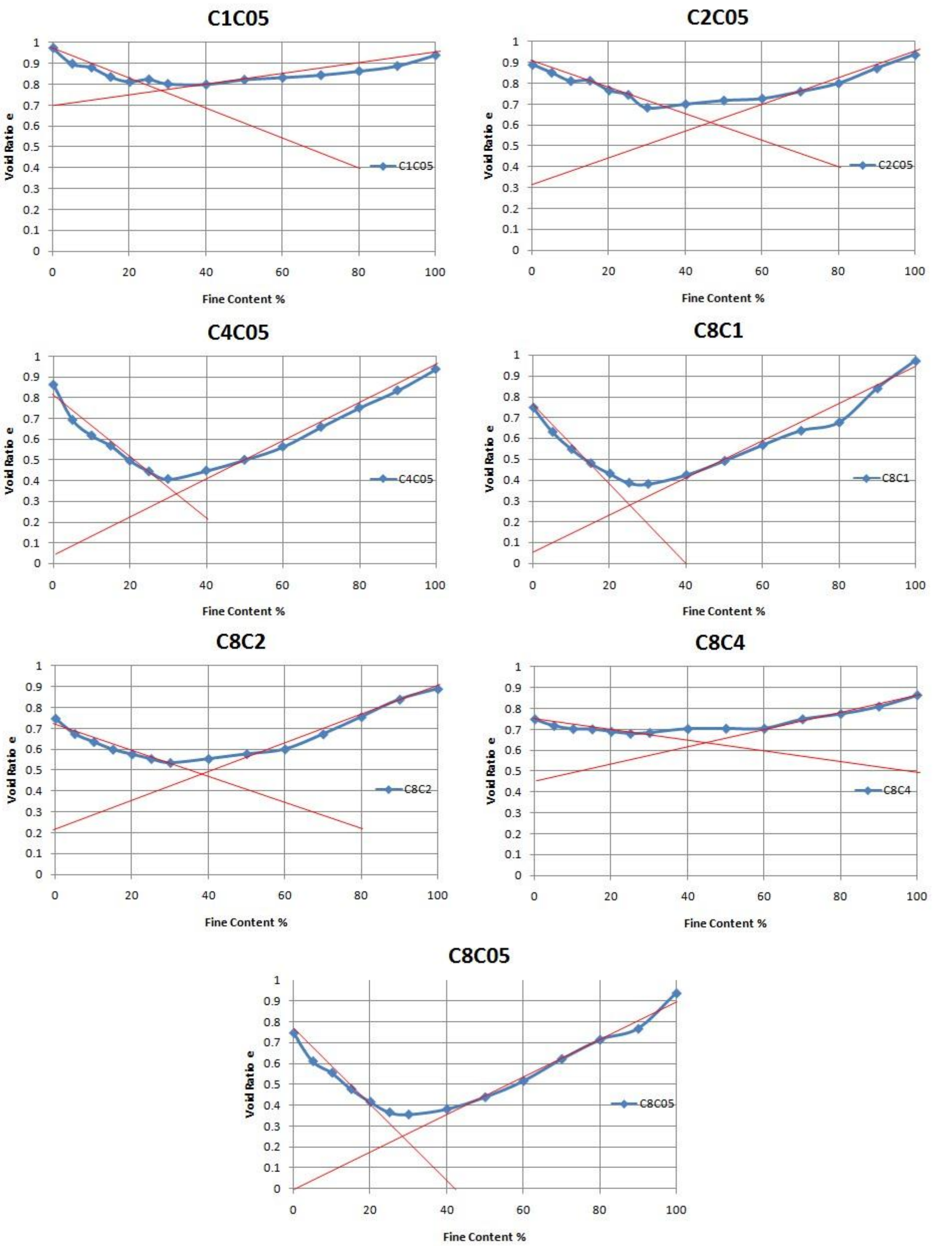


Figure-5.3 a) Seven sets of fitting line describing slopes of coarse and fine- crushed aggregate (F. de Larrard)

In Equatioin (12), $\frac{\partial e}{\partial y_2}$ equals slope of **e₁** side (coarse), a_{12} can be determined; since it is known in Chapter 2.2 that $\frac{\partial e}{\partial y_1} = -\frac{\partial e}{\partial y_2}$, $\frac{\partial e}{\partial y_2}$ equals slope of $\mathbf{e_2}$ side (fine), b_{21} can be determined. Then seven sets of coefficient of interactions a_{12} and b_{21} can be calculated.

Results are represented in Table-5.3 c).

Table-15.3 c) Seven sets of coefficient of interactions (F. de Larrad)

Ratio	0.0625	0.125	0.125	0.25	0.25	0.5	0.5
C	C8C05	C8C1	C4C05	C8C2	C2C05	C8C4	C1C05
a_{12}	-0.017	-0.062	0.187	0.594	0.659	0.806	0.649
b_{21}	0.064	0.137	0.046	0.267	0.337	0.601	0.720

Where ratio it can also be calculated from Table-3.3 b) Soil description of Crushed aggregate (F. de Larrard, 1999) that ratio of different combinations of mean value of dmin and dmax

Then parameters *p*, *q*, *s* and *t* can be estimated through equations of coefficient of interactions (5) (6) by given eight sets of a_{12} and *b*21 in Table-5.3 c). Two fitting graphs were shown in Figure-5.2 b). Study of fitting *p*, *q*, *s* and *t* to get a_{ij} or b_{ij} can be found in Appendix.

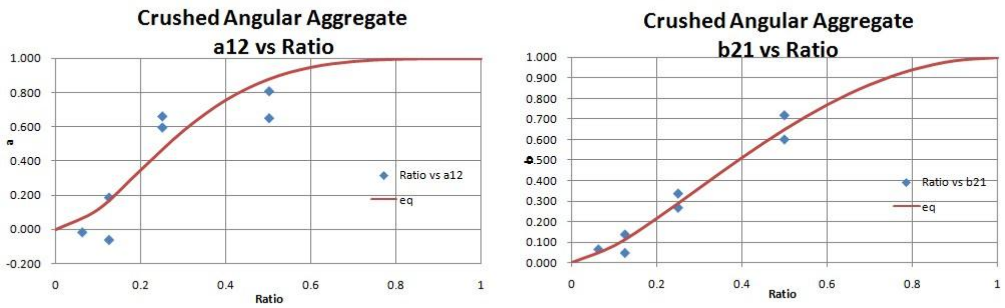


Figure-5.3 b) Fitting graphs of coefficient of interactions for seven sets of crushed aggregate (Fragaszy and Sneider, 1991)

Therefore, we get $p=4$, $q=2$ for a_{12}, and $s=2$, $t=1.5$ for b_{21}. Take C8C05 mixture for example.

Since particle size variance can seen from Table-3.3 b) Soil description of Crushed aggregate (F. de Larrard, 1999), we can get d_1=9.84026, d_2=9.41076, d_3=9, d_4=8.60717, d_5=8.23149, d_6=0.61972, d_7=0.59173, d_8=0.565, d_9=0.53948, d_{10}=0.51511;

$e_1=e_2=e_3=e_4=e_5$=0.74825(**e_1**), $e_6=e_7=e_8=e_9=e_{10}$=0.93798(**e_2**). And a_i is known that in Table-3.1 b) that $a_1=a_6$=0.03458, $a_2=a_7$=0.23832, $a_3=a_8$=0.45149, $a_4=a_9$=0.23832, $a_5=a_{10}$=0.03458.

For fines (i= 6, 7, 8, 9, 10), *fc*, is percent of fine content in the mixture system. y_i can be obtained in following equations, where

$$y_i = fc \times a_i \tag{23}$$

For coarse (i= 1, 2, 3, 4, 5), *cc*, is percent of coarse content in the mixture system. y_i can be obtained in following equations,

$$y_i = cc \times a_i \tag{24}$$

Put all data above into Binary Equation (9, 10, 11). Prediction of minimum void ratio under different ratio of fine and coarse content can then be obtained. Figure-5.3 c) illustrates seven sets of prediction results.

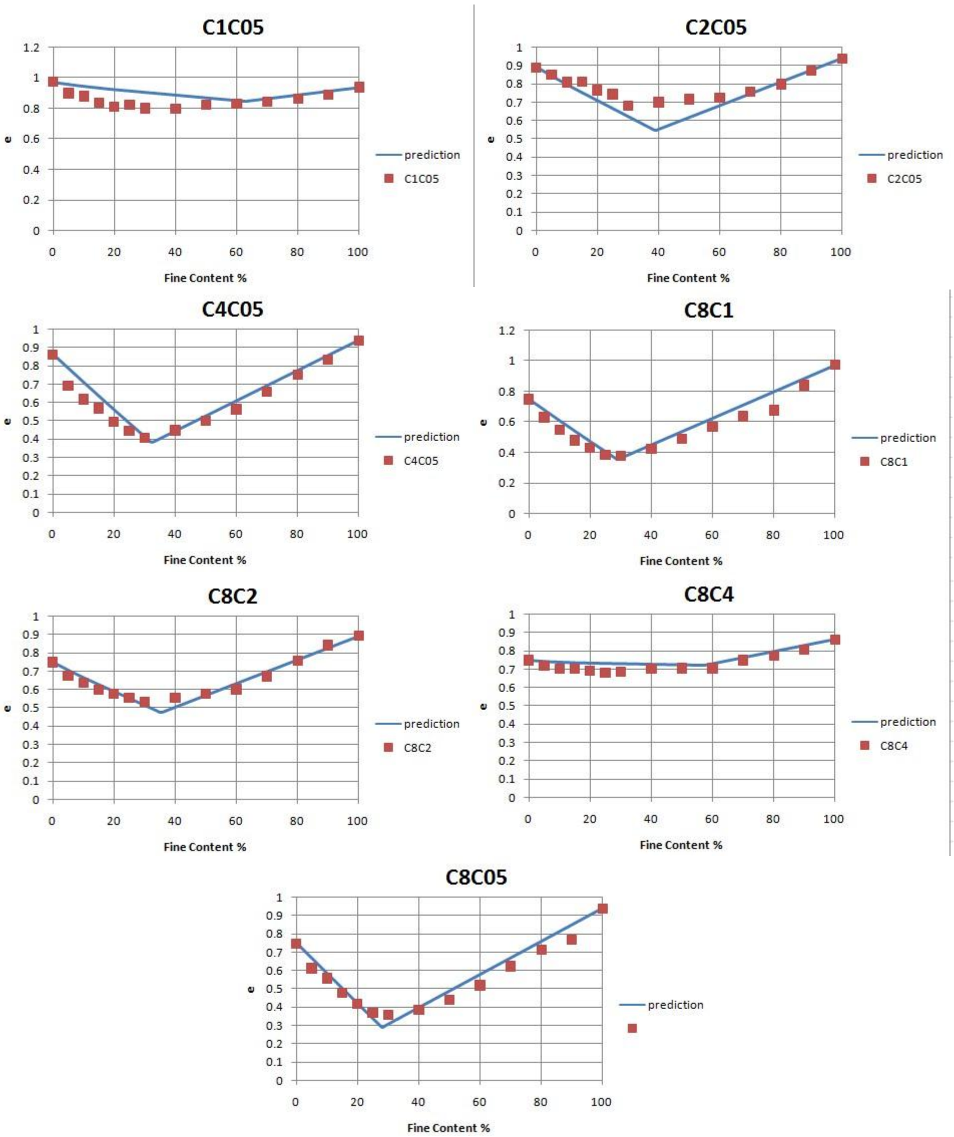


Figure-5.3 c) Comparison of void ratios between experiments and model, crushed aggregate (F. de Larrard, 1999)

6. SUMMARY

1. The aim of this model is to predict minimum void ratios curve based on two known minimum void ratios of fine and coarse grains with given mixture.
2. Author manipulated VBA language to make the modeling of equation (7) and (8).
3. The model can correctly predict minimum void ratio at different mixture of fine and coarse particles, referring to spherical balls, gravelly sand with sand and rounded/crushed aggregate.
4. Log-normal distribution can be used to approximate particle size distribution for granular soils
5. For small given particle size range, the predicted results value of minimum void ratio are same when using two particle classes and 10 particle classes to put into the model.

7. APPENDIX

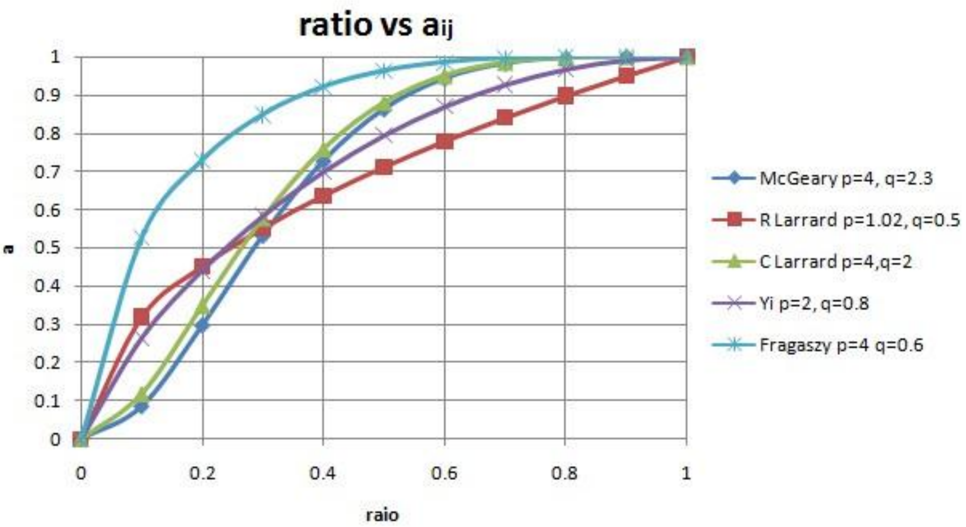
7.1 Coefficient of Interaction

Equations describing coefficient of interactions in this method are determined by "wall effect" and "loosening effect" parameters: *p*, *q*, *s*, *t*. Concepts were explained in Chapter 2.1.3.

$$a_{ij} = \left(1 - \left(1 - \frac{d_j}{d_i}\right)^p\right)^q \qquad (5)$$

$$b_{ij} = \left(1 - \left(1 - \frac{d_i}{d_j}\right)^s\right)^t \qquad (6)$$

A summary of coefficient of interactions used from literature were shown in the Figure-7.1 a).



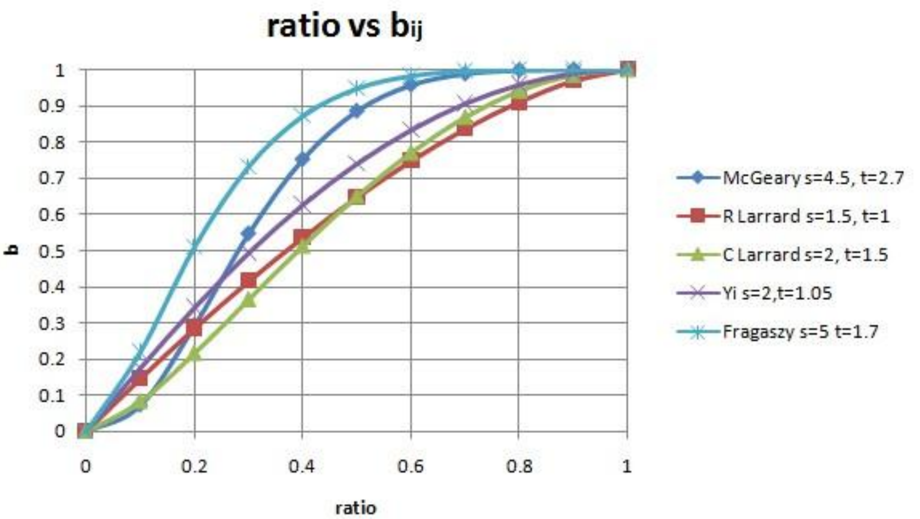


Figure-7.1 a) Summary five sets of coefficient of interaction

Since equation forms for a_{ij} and b_{ij} are same, scale of equations are shown in Figure-7.1 b).

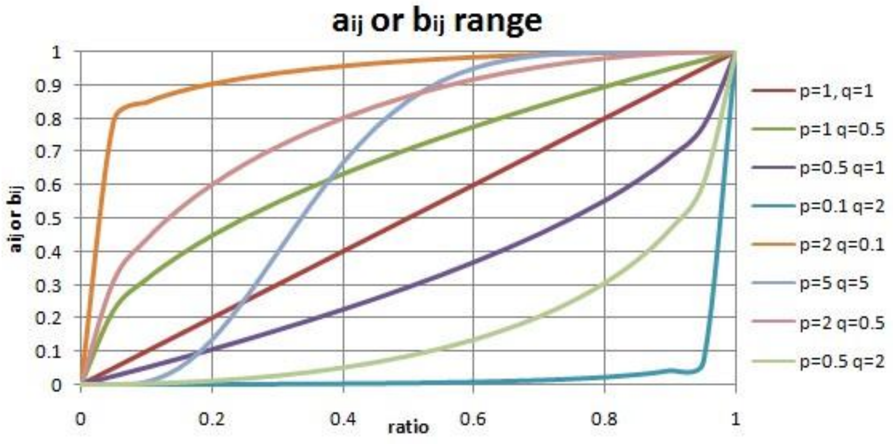


Figure-7.1 b) Scale of equation-coefficient of interaction

Effect of increasing q for a_{ij} is shown in Figure

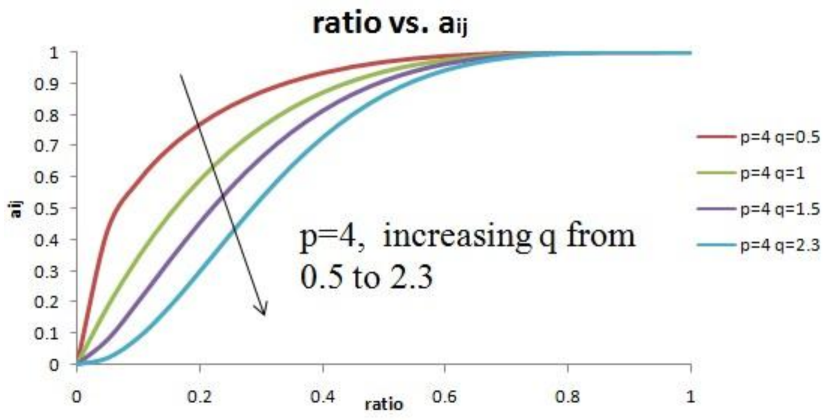


Figure-7.1 c) p=4, increasing q from 0.5 to 2.3

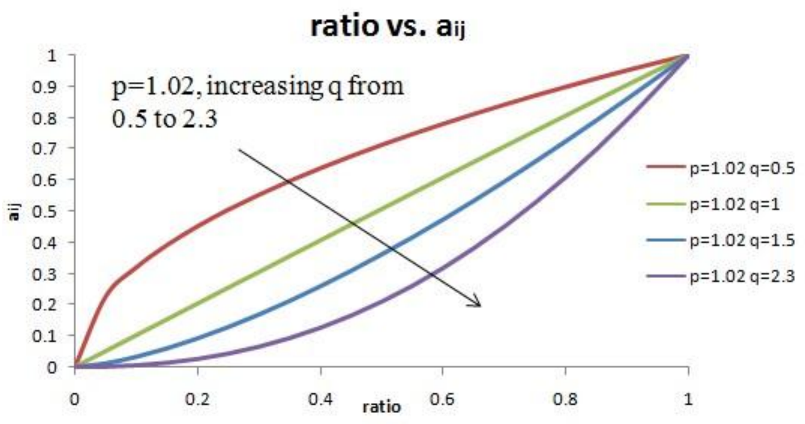


Figure-7.1 d) p=1.02, increasing q from 0.5 to 2.3

Effect of increasing q for a_{ij} is shown in Figure-7.1 e)

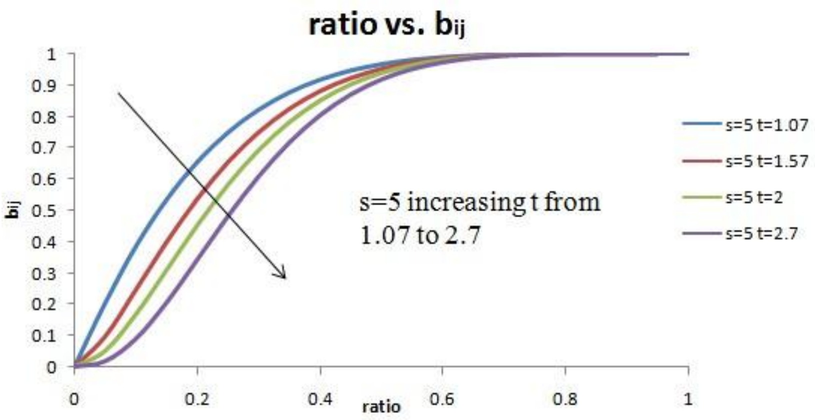


Figure-7.1 e) s=5, increasing t from 1.07 to 2.7

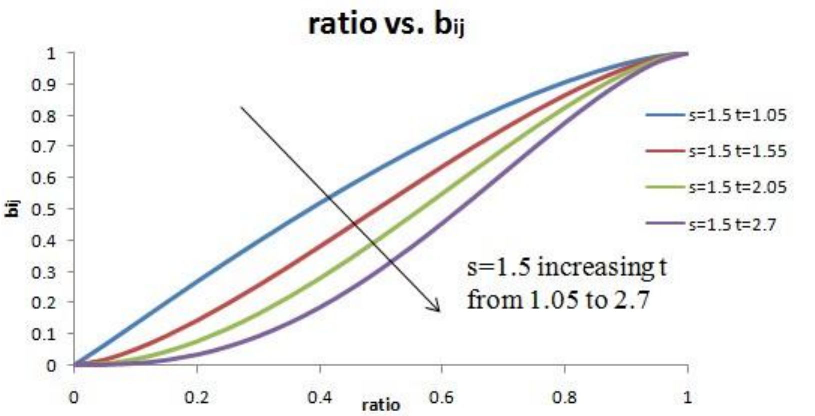


Figure-7.1 f) s=1.02, increasing t from 1.05 to 2.7

7.2 Parametric Analysis

In binary system, if there are ten active parameters influencing trend or direction of minimum void ratios: e_1, e_2, $m_1(\mu_{x1})$, $m_2(\mu_{x2})$, p, q, s, t, v_1, v_2. Their influences on results are shown separately in Figure-7.2 a) –Figure-7.2 j).

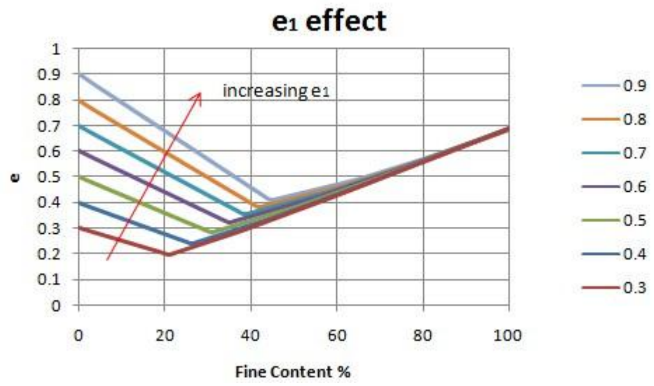


Figure-7.2 a) Overall influence from increasing e_1

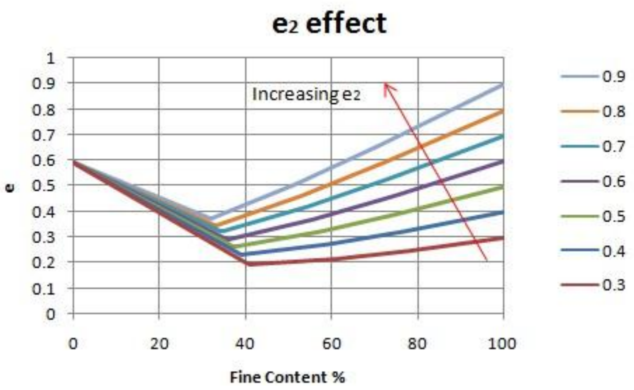


Figure-7.2 b) Overall influence from increasing e_2

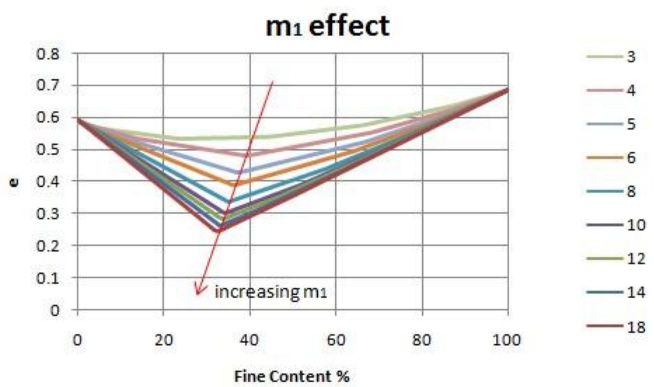


Figure-7.2 c) Overall influence from increasing m_1

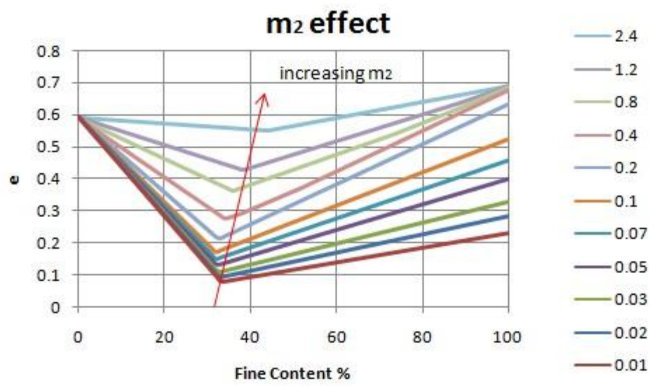


Figure-7.2 d) Overall influence from increasing m_2

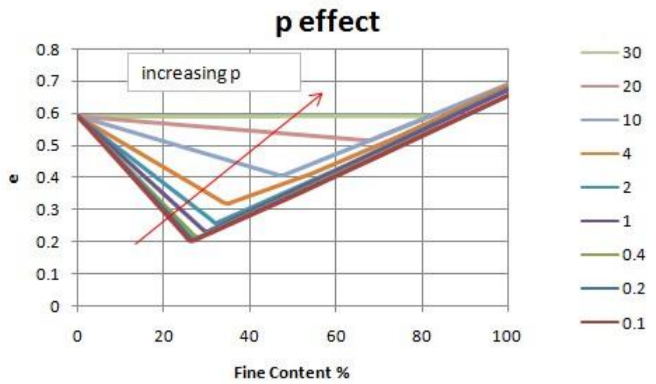


Figure-7.2 e) Overall influence from increasing p

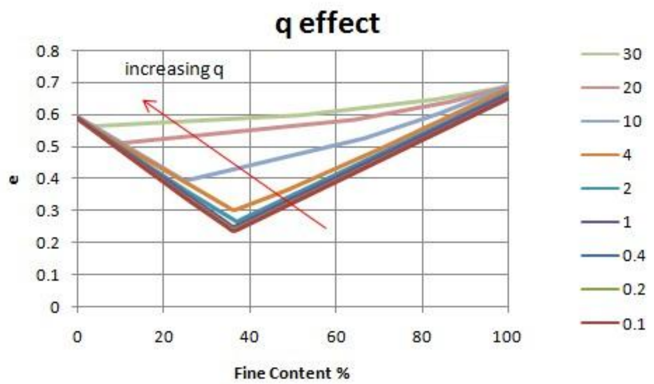


Figure-7.2 f) Overall influence from increasing q

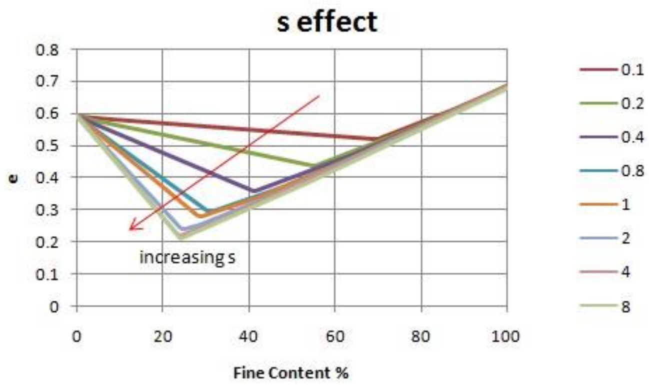


Figure-7.2 g) Overall influence from increasing s

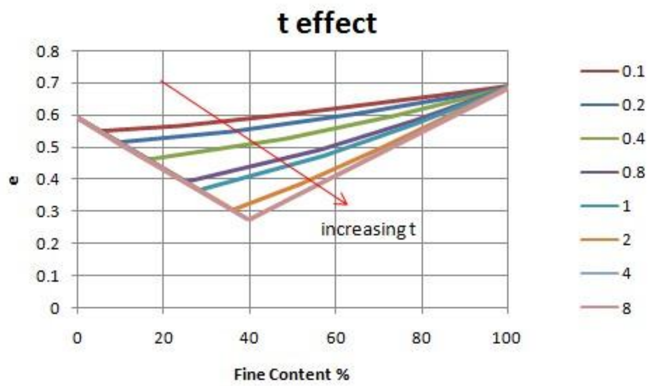


Figure-7.2 h) Overall influence from increasing t

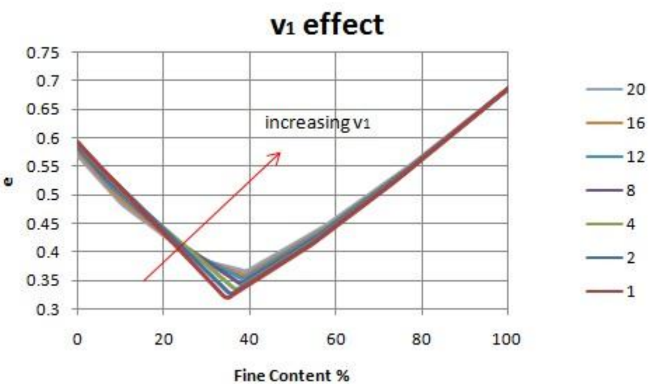


Figure-7.2 i) Overall influence from increasing v_1

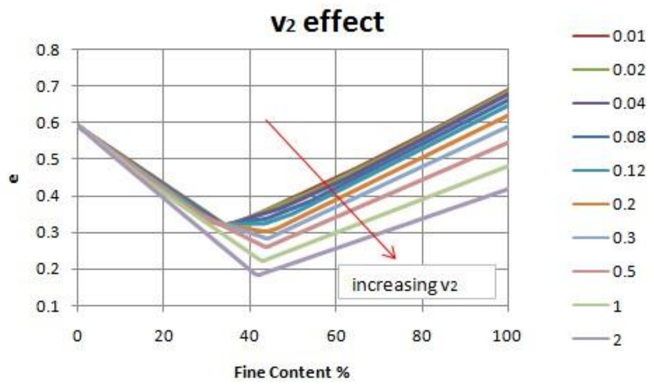


Figure-7.2 j) Overall influence from increasing v_2

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