Axioms for Modelling Cubical Type Theory in a Topos

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Abstract

The homotopical approach to intensional type theory views proofs of equality as paths. We explore what is required of an interval-like object $\mathtt{I}$ in a topos to give a model of type theory in which elements of identity types are functions with domain $\mathtt{I}$. Cohen, Coquand, Huber and Mörtberg give such a model using a particular category of presheaves. We investigate the extent to which their model construction can be expressed in the internal type theory of any topos and identify a collection of quite weak axioms for this purpose. This clarifies the definition and properties of the notion of uniform Kan filling that lies at the heart of their constructive interpretation of Voevodsky's univalence axiom. Furthermore, since our axioms can be satisfied in a number of different ways, we show that there is a range of topos-theoretic models of homotopy type theory in this style.

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1 Introduction

Cubical type theory [11] provides a constructive justification of Voevodsky's univalence axiom, an axiom that has important consequences for the formalisation of mathematics within Martin-Löf type theory [33]. Working informally in constructive set theory, Cohen *et al* [11] give a model of their type theory using the category $\hat{\mathcal{C}}$ of set-valued contravariant functors on a small category $\mathcal{C}$ which is the Lawvere theory for de Morgan algebra [5, Chapter XI]; see [31]. The representable functor on the generic de Morgan algebra in $\mathcal{C}$ is used as an interval object $\mathtt{I}$ in $\hat{\mathcal{C}}$, with proofs of equality modelled by the corresponding notion of path, that is, by morphisms with domain $\mathtt{I}$. Cohen *et al* call the objects of $\hat{\mathcal{C}}$ *cubical sets*. They have a richer structure compared with previous, synonymous notions [7, 21]. For one thing they allow path types to be modelled simply by exponentials $X^{\mathtt{I}}$, rather than by name abstractions [30, Chapter 4]. More importantly, the de Morgan algebra operations endow $\mathtt{I}$ with structure that considerably simplifies the definition and properties of the constructive notion of Kan filling that lies at the heart of [11]. In particular, the filling operation is obtained from a simple special case that *composes* a filling at one end of the interval to a filling at the other end. Coquand [12] has suggested that this distinctive composition operation can be understood in terms of the properties of partial elements and their extension to total elements, within the internal higher-order logic of toposes [24]. In this paper we show that that is indeed the case and usefully so. In particular, the *uniformity* condition on composition operations [11, Definition 13], which allows one to avoid the non-constructive aspects of the classical notion of Kan filling [6], becomes automatic when the operations

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The interval $\mathtt{I}$ is connected

$$\mathtt{ax}_1 : [\forall(\varphi : \mathtt{I} \to \Omega).\ (\forall(i : \mathtt{I}).\ \varphi\, i \vee \neg\varphi\, i) \Rightarrow (\forall(i : \mathtt{I}).\ \varphi\, i) \vee (\forall(i : \mathtt{I}).\ \neg\varphi\, i)]$$

has distinct end-points $\mathtt{0}, \mathtt{1} : \mathtt{I}$

$$\mathtt{ax}_2 : [\neg(\mathtt{0} = \mathtt{1})]$$

and has a connection algebra structure $\_ \sqcap \_, \_ \sqcup \_ : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$

$$\mathtt{ax}_3 : [\forall(i : \mathtt{I}).\ \mathtt{0} \sqcap x = \mathtt{0} = x \sqcap \mathtt{0} \ \wedge\ \mathtt{1} \sqcap x = x = x \sqcap \mathtt{1}]$$
$$\mathtt{ax}_4 : [\forall(i : \mathtt{I}).\ \mathtt{0} \sqcup x = x = x \sqcup \mathtt{0} \ \wedge\ \mathtt{1} \sqcup x = \mathtt{1} = x \sqcup \mathtt{1}].$$

Cofibrant propositions $\mathtt{Cof} = \{\varphi : \Omega \mid \mathtt{cof}\, \varphi\}$ (where $\mathtt{cof} : \Omega \to \Omega$) include end-point-equality

$$\mathtt{ax}_5 : [\forall(i : \mathtt{I}).\ \mathtt{cof}(i = \mathtt{0}) \ \wedge\ \mathtt{cof}(i = \mathtt{1})]$$

and are closed under binary disjunction

$$\mathtt{ax}_6 : [\forall(\varphi\ \psi : \Omega).\ \mathtt{cof}\, \varphi \Rightarrow \mathtt{cof}\, \psi \Rightarrow \mathtt{cof}(\varphi \vee \psi)]$$

and dependent conjunction

$$\mathtt{ax}_7 : [\forall(\varphi\ \psi : \Omega).\ \mathtt{cof}\, \varphi \Rightarrow (\varphi \Rightarrow \mathtt{cof}\, \psi) \Rightarrow \mathtt{cof}(\varphi \wedge \psi)].$$

Strictness postulate for universe $\mathcal{U}$: any cofibrant-partial type A that is isomorphic to a total type B everywhere that A is defined, can be extended to a total type B' that is isomorphic to B:

$$\mathtt{ax}_8 : \{\varphi : \mathtt{Cof}\}(A : [\varphi] \to \mathcal{U})(B : \mathcal{U})(s : (u : [\varphi]) \to A\, u \cong B) \to (B' : \mathcal{U}) \times \{s' : B' \cong B \mid \forall(u : [\varphi]).\ A\, u = B' \wedge s\, u = s'\}.$$

Figure 1 The axioms

are formulated internally. Our approach has the usual benefit of axiomatics – helping to clarify exactly which properties of a topos are sufficient to carry out each of the various constructions used to model cubical type theory [11]. For example, we show that something weaker than de Morgan algebra on the interval object $\mathtt{I}$ is sufficient and as a result other models emerge, including one based on constructive simplicial sets.

To accomplish all this, we find it helpful to work not in the higher-order predicate logic of toposes, but in an extensional type theory equipped with an impredicative universe of propositions Ω, standing for the subobject classifier of the topos [27]. Working in such a language, our axiomatisation concerns two structures that a topos $\mathcal{E}$ may possess: an object $\mathtt{I}$ that is endowed with some elementary characteristics of the unit interval; and a subobject of propositions $\mathtt{Cof} \rightarrowtail \Omega$ whose elements we call *cofibrant propositions* and which determine the subobjects that are relevant for a Kan-like notion of filling (for example in the case of [11], subobjects generated by unions of faces of hypercubes). Working internally with cofibrant propositions rather than externally with a class of cofibrant subobjects (monomorphisms) leads to an appealingly simple notion of *fibration* (Section 4), with that of Cohen *et al* as an instance when the topos is $\hat{\mathcal{C}}$. These fibrations are type-families equipped with extra structure (*composition* operations) which are supposed to model intensional Martin-Löf type theory, when organised as a Category with Families [14], say. In order that they do so, we make a series of postulates about the interval and cofibrant subobjects that are true of the presheaf model in [11]. For ease of reference these axioms are collected together in Figure 1, written in the language described in Section 2. Axiom $\mathtt{ax}_1$ expresses that the interval $\mathtt{I}$ is internally connected, in the sense that any decidable subset of its elements is either empty or the whole of $\mathtt{I}$. It is used at the end of Section 4.2 to show that the topos natural number object is fibrant and that fibrations are closed under binary coproducts; and it also gets used in proving properties of the glueing construct mentioned below. Axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ give what we call a *path connection algebra* structure on $\mathtt{I}$; they capture some very simple properties of the minimum and maximum operations on the unit interval $[0, 1]$ of real numbers that suffice to ensure contractibility of singleton types (Section 3) and, in combination with $\mathtt{ax}_2$, $\mathtt{ax}_5$ and $\mathtt{ax}_6$, to define path lifting from composition for fibrations (Section 4.2). Indeed only axioms $\mathtt{ax}_2$–$\mathtt{ax}_6$ are needed to show that fibrations provide a model of Π- and Σ-types; and furthermore to show that the path types determined by the interval object $\mathtt{I}$ (Section 3) satisfy the rules for identity types propositionally [13, 34]. Axiom $\mathtt{ax}_7$ is used to get from these propositional identity types to the proper, definitional identity types of Martin-Löf type theory, via a version of Swan's construction [32]; see Section 4.3. In Section 5 we consider univalence [33, Section 2.10] – the correspondence between type-valued paths in a universe and functions that are equivalences modulo path-based equality. To do so we give a non-strict, "up-to-isomorphism" version of the *glueing* construct of Cohen *et al* in the internal type theory of the topos. Axiom $\mathtt{ax}_8$ allows us to regain the strict form of glueing used in [11]. Our development also differs from that of Cohen *et al* by avoiding the need to postulate that cofibrant propositions are closed under $\mathtt{I}$-indexed intersection. However, we do not yet have an internalisation of the modified form of Hofmann-Streicher universe construction [20] used in [11] to obtain a fibrant universe satisfying the univalence axiom; and closure of `Cof` under $\mathtt{I}$-indexed intersection may be needed for that.

In Section 6 we indicate why the model in [11] satisfies our axioms and more generally which other presheaf toposes satisfy them. There is some freedom in choosing the subobject of cofibrant propositions. The path-connection algebra structure we assume for the interval $\mathtt{I}$ (axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$) is much weaker than being a de Morgan algebra. In particular, we can avoid the use of a de Morgan involution operation and as a result the topos of simplicial sets supports a model of the axioms. It remains to be seen whether the univalent universe construction of [11] transfers to this constructive simplicial set model and how that relates to the classical simplicial model of univalent foundations [23]. In Section 7 we conclude by considering some other related work and future directions.

Agda formalisation

The definitions and constructions we carry out in the internal type theory of toposes are sufficiently involved to warrant machine-assisted formalisation. Our tool of choice is Agda [3]. We persuaded it to provide an impredicative universe of mere propositions [33, Section 3.3] using a method due to Escardo [15]. This gives an intensional, proof-relevant version of the subobject classifier Ω and of the type theory described in Section 2. To this we add postulates corresponding to the axioms in Figure 1. We also made modest use of the facility for user-defined rewriting in recent versions of Agda [10], in order to make the connection algebra axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ definitional, rather than just propositional equalities, thereby eliminating a few proofs in favour of computation. Using Agda required us to construct and pass around many proof details that are omitted or elided in the paper version; we found this to be quite bearable and also invaluable for getting the details right. Our development can be found at http://www.cl.cam.ac.uk/~rio22/agda/cubical-topos/root.html.

2 Internal Type Theory of a Topos

We rely on the categorical semantics of dependent type theory in terms of *categories with families* (CwF) [14]. For each topos $\mathcal{E}$ (with subobject classifier $\top : 1 \to \Omega$) one can find a CwF with the same objects, such that the category of families at each object X is equivalent to the slice category $\mathcal{E}/X$. This can be done in a number of different ways; for example [29, Example 6.14], or the more recent references [23, Section 1.3], [26] and [4], which cater for categories more general than a topos (and for contextual/comprehension categories rather than CwFs in the first two cases). Using the objects, families and elements of this CwF as a signature, we get an internal type theory along the lines of those discussed in [27], canonically interpreted in the CwF in the standard fashion [19]. We make definitions and postulates in this internal language for $\mathcal{E}$ using a concrete syntax inspired by Agda [3]. Dependent function types are written as $(x : A) \to B$; their canonical terms are function abstractions, written as $\lambda(x : A) \to t$. Dependent product types are written as $(x : A) \times B$; their canonical terms are pairs, written as (s, t). The subobject classifier Ω becomes an impredicative universe of propositions in the internal type theory with logical connectives, equality and quantifiers $\top, \bot, \neg, \wedge, \vee, \Rightarrow, =, \forall(x : A), \exists(x : A)$. Its universal property gives rise to comprehension subtypes: given $\Gamma, x : A \vdash \varphi(x) : \Omega$, then $\Gamma \vdash \{x : A \mid \varphi(x)\}$ is a type whose terms are those $t : A$ for which $\varphi(t)$ is provable, with the proof being treated irrelevantly.[1] Taking $A = 1$ to be terminal, for each $\varphi : \Omega$ we have a type whose inhabitation corresponds to provability of φ:

$$[\varphi] \triangleq \{\_ : 1 \mid \varphi\} \tag{1}$$

We will make extensive use of these types in connection with the partial elements of a type; see Section 4.1.

Instead of quantifying externally over the objects, families and elements of the CwF associated with $\mathcal{E}$, we will assume $\mathcal{E}$ comes with an internal full subtopos $\mathcal{U}$. In the internal language we use $\mathcal{U}$ as a Russell-style universe (that is, if $A : \mathcal{U}$, then A itself denotes a type) containing Ω and closed under forming products, exponentials and comprehension subtypes.

3 Path Types

The homotopical approach to type theory [33] views elements of identity types as paths between the two elements being equated. We try to take this literally, using paths in a topos $\mathcal{E}$ that are morphisms out of a distinguished object $\mathtt{I}$, called the *interval*. We assume it is equipped with morphisms $\mathtt{0}, \mathtt{1} : 1 \to \mathtt{I}$ and $\_\sqcap\_, \_\sqcup\_ : \mathtt{I} \to \mathtt{I} \to \mathtt{I}$ satisfying axioms $\mathtt{ax}_1$–$\mathtt{ax}_4$ in Figure 1. Axiom $\mathtt{ax}_1$ is an internal connectedness property of the interval that we will not need until Section 6.1. Axiom $\mathtt{ax}_2$ says that the interval is non-trivial. Axioms $\mathtt{ax}_3$ and $\mathtt{ax}_4$ endow it with a form of *path-connection* algebra structure [9] which is used to ensure contractibility of singleton types (see below) and to define path lifting from composition for fibrations (see Section 4.2). In the model of [11] the connection algebra structure is given by the lattice structure of the interval, taking $\_\sqcap\_$ to be binary meet, $\_\sqcup\_$ to be binary join and using the fact that $\mathtt{0}$ and $\mathtt{1}$ are respectively least and greatest elements.

[1] Our Agda development is proof relevant, so that terms of comprehension types contain a proof of membership as a component.

▶ Remark 3.1 (de Morgan involution). In the model of [11] $\mathtt{I}$ is not just a lattice, but also has an involution operation $1 - (\_) : \mathtt{I} \to \mathtt{I}$ (so that $(1 - (1 - i) = i)$ making $\sqcup$ the de Morgan dual of $\sqcap$, in the sense that $i \sqcup j = 1 - ((1 - i) \sqcap (1 - j))$. We have found that although this involution structure is convenient, it is not strictly necessary for the constructions that follow. Instead we can just give a $\mathtt{0}$-version and a $\mathtt{1}$-version of certain concepts; for example, "composing from $\mathtt{1}$ to $\mathtt{0}$" as well as "composing from $\mathtt{0}$ to $\mathtt{1}$" in Section 4.2.

Given $A : \mathcal{U}$, we call terms of type $\mathtt{I} \to A$ *paths* in A. The *path type* associated with A is $\_ \sim \_ : A \to A \to \mathcal{U}$ where

$$a_0 \sim a_1 \triangleq \{p : \mathtt{I} \to A \mid p\,\mathtt{0} = a_0 \wedge p\,\mathtt{1} = a_1\} \tag{2}$$

Can these types be used to model the rules for Martin-Löf identity types? We can certainly interpret the identity introduction rule (reflexivity), since degenerate paths given by constant functions

$$\mathtt{k}\,a\,i \triangleq a \tag{3}$$

satisfy $\mathtt{k} : \{A : \mathcal{U}\}(a : A) \to a \sim a$.[2] However, we need further assumptions to interpret the identity elimination rule, otherwise known as path induction [33, Section 1.12.1]. Coquand has given an alternative (propositionally equivalent) formulation of identity elimination in terms of substitution functions $a_0 \sim a_1 \to P\,a_0 \to P\,a_1$ and contractibility of singleton types $(a_1 : A) \times (a_0 \sim a_1)$; see [7, Figure 2]. The path-connection algebra structure gives the latter, since using $\mathtt{ax}_3$ and $\mathtt{ax}_4$ we have

$$\begin{aligned} &\mathtt{ctr} : \{A : \mathcal{U}\}\{a_0\ a_1 : A\}(p : a_0 \sim a_1) \to (a_0, \mathtt{k}\,a_0) \sim (a_1, p) \\ &\mathtt{ctr}\,p\,i \triangleq (p\,i, \lambda j \to p(i \sqcap j)) \end{aligned} \tag{4}$$

However, to get suitably behaved substitution functions we have to consider families of types endowed with some extra structure; and that structure has to lift through the type-forming operations (products, functions, identity types, etc). This is what the definitions in the next section achieve.

4 Cohen-Coquand-Huber-Mörtberg (CCHM) Fibrations

In this section we show how to generalise the notion of fibration introduced in [11, Definition 13] from the particular presheaf model considered there to any topos with an interval object as in the previous section. To do so we introduce the notion of *cofibrant proposition* and use it to internalise the composition and filling operations described in [11].

4.1 Cofibrant propositions

Kan filling and other cofibrancy conditions on collections of subspaces have to do with lifting maps from a subspace to the whole space. Here we consider subspaces of spaces as subobjects of objects in toposes. Since subobjects are classified by morphisms to Ω, it is possible to consider collections of subobjects that are specified generically by giving a property of propositions, in other words by giving a subobject $\mathtt{Cof} \rightarrowtail \Omega$. A monomorphism $m : A \rightarrowtail B$ is in the corresponding collection if its classifier $\lambda(y : B) \to \exists(x : A).\ m\,x = y : B \to \Omega$ factors

[2] Here and elsewhere we use the Agda convention that braces {} indicate implicit arguments.

through $\texttt{Cof} \rightarrowtail \Omega$. We reduce lifting properties of morphisms out of the domains of such monomorphisms to extension properties of *partial elements* whose domains of definition are in $\texttt{Cof}$. Recall that in intuitionistic logic, partial elelements of a type A are often represented by sub-singletons, that is, by terms $s : A \to \Omega$ satisfying $\forall(x\ x' : A).\ s\,x \wedge s\,x' \Rightarrow x = x'$. However, it will be more convenient to work with an extensionally equivalent representation as dependent pairs $\varphi : \Omega$ and $f : [\varphi] \to A$. The proposition φ is the *extent* of the partial element; in terms of sub-singletons it is equal to $\exists(x : A).\ s\,x$.

▶ **Definition 4.1** (Cofibrant partial elements, $\Box A$). We assume we are given a subobject $\texttt{Cof} \rightarrowtail \Omega$ satisfying axioms $\texttt{ax}_5$–$\texttt{ax}_8$ in Figure 1. We call terms of type $\texttt{Cof}$ *cofibrant propositions.* Given a type $A : \mathcal{U}$, we define the type of *cofibrant partial elements* of A to be

$$\Box A \triangleq (\varphi : \texttt{Cof}) \times ([\varphi] \to A) \tag{5}$$

An *extension* of such a partial element $(\varphi, f) : \Box A$ is an element $a : A$ together with a proof of the following relation:

$$(\varphi, f) \nearrow a \triangleq \forall(u : [\varphi]).\ f\,u = a \tag{6}$$

We postpone discussing axiom $\texttt{ax}_8$ until Section 5. Axioms $\texttt{ax}_5$–$\texttt{ax}_7$ give the simple properties of cofibrant propositions we use to define an internal notion of fibration generalising Definition 13 of [11] and show that it is closed under forming Σ-, Π- and $\texttt{Id}$-types, as well as basic datatypes. In the figure $\texttt{cof} : \Omega \to \Omega$ is the classifying morphism of the subobject $\texttt{Cof} \rightarrowtail \Omega$, so that $\texttt{Cof} = \{\varphi : \Omega \mid \texttt{cof}\ \varphi\}$. The last of these three axioms, $\texttt{ax}_7$, is equivalent to requiring that the collection of cofibrant monomorphisms is closed under composition (proof omitted); we only use this property in order to construct definitional identity types from propositional identity types (see Section 4.3).

Note that axioms $\texttt{ax}_2$ and $\texttt{ax}_5$ together imply that $\texttt{cof}\ \bot$ holds, so that $\emptyset \rightarrowtail A$ is always a cofibrant monomorphism, where $\emptyset$ is the initial object. Axiom $\texttt{ax}_6$ says that the union of two cofibrant subobjects is again cofibrant. This allows us to take the union of compatible cofibrant partial elements, which is used in many of the constructions below.

4.2 Composition and filling structures

Given an interval-indexed family of types $A : \texttt{I} \to \mathcal{U}$, we think of elements of the dependent function type $\Pi_{\texttt{I}} A \triangleq (i : \texttt{I}) \to A\,i$ as dependently typed paths. We call elements of type $\Box(\Pi_{\texttt{I}} A)$ *cofibrant-partial paths.* Given $(\varphi, f) : \Box(\Pi_{\texttt{I}} A)$, we can evaluate it at a point $i : \texttt{I}$ of the interval to get a cofibrant partial element $(\varphi, f) @ i : \Box(A\,i)$:

$$(\varphi, f) @ i \triangleq (\varphi, \lambda(u : [\varphi]) \to f\,u\,i) \tag{7}$$

An operation for *filling from* $\texttt{0}$ in $A : \texttt{I} \to \mathcal{U}$ takes any $(\varphi, f) : \Box(\Pi_{\texttt{I}} A)$ and any $a_0 : A\,\texttt{0}$ with $(\varphi, f) @ \texttt{0} \nearrow a_0$ and extends (φ, f) to a dependently typed path $g : \Pi_{\texttt{I}} A$ with $g\,\texttt{0} = a_0$. This is a form of uniform *Homotopy Extension and Lifting Property* (HELP) [28, Chapter 10, Section 3] stated internally in terms of cofibrant propositions rather than externally in terms of cofibrant monomorphisms, obviating the need to mention uniformity explicitly. Indeed a feature of the present work compared with Cohen *et al* is that the *uniformity* condition on composition/filling operations [11, Definition 13], which allows one to avoid the non-constructive aspects of the classical notion of Kan filling [6], becomes automatic when the operations are formulated in terms of the internal collection $\texttt{Cof}$ of cofibrant propositions. Since we are not assuming any structure on the interval for reversing paths (see Remark 3.1), we also need to consider the symmetric notion of filling from $\texttt{1}$. So we have the following definition.

▶ **Definition 4.2** (Filling structures). Let $\mathtt{\{0,1\}} \triangleq \{i : \mathtt{I} \mid i = \mathtt{0} \vee i = \mathtt{1}\}$; because of axiom $\mathtt{ax}_2$ this is isomorphic to the object of Booleans $(1 + 1)$ and hence there is a function $\overline{} : \mathtt{\{0,1\}} \to \mathtt{\{0,1\}}$ satisfying $\overline{\mathtt{0}} = \mathtt{1}$ and $\overline{\mathtt{1}} = \mathtt{0}$. Then the type of *filling structures* for $\mathtt{I}$-indexed families of types, $\mathtt{Fill} : (e : \mathtt{\{0,1\}})(A : \mathtt{I} \to \mathcal{U}) \to \mathcal{U}$, is defined by:

$$\mathtt{Fill}\, e\, A \triangleq (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A)(a : \{a' : A\, e \mid (\varphi, f) \mathbin{@} e \nearrow a'\}) \to \{g : \Pi_{\mathtt{I}} A \mid (\varphi, f) \nearrow g \ \wedge\ g\, e = a\} \tag{8}$$

A notable feature of [11] compared with preceding work [7] is that such filling structure can be constructed from a simpler *composition* structure that just produces an extension at one end of a cofibrant-partial path from an extension at the other end. We will deduce this using axioms $\mathtt{ax}_3$–$\mathtt{ax}_6$ after defining the main notion of this paper.

▶ **Definition 4.3** (The CwF of CCHM fibrations). A *CCHM fibration* (A, α) over a type $\Gamma : \mathcal{U}$ is a family $A : \Gamma \to \mathcal{U}$ equipped with a fibration structure $\alpha : \mathtt{Fib}\, A$, where $\mathtt{Fib} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U}) \to \mathcal{U}$ is defined by

$$\mathtt{Fib}\, \{\Gamma\}\, A \triangleq (e : \mathtt{\{0,1\}})(p : \mathtt{I} \to \Gamma) \to \mathtt{Comp}\, e\, (A \circ p) \tag{9}$$

Here $\mathtt{Comp} : (e : \mathtt{\{0,1\}})(A : \mathtt{I} \to \mathcal{U}) \to \mathcal{U}$ is the type of *composition structures* for $\mathtt{I}$-indexed families:

$$\mathtt{Comp}\, e\, A \triangleq (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A) \to \{a_0 : A\, e \mid (\varphi, f) \mathbin{@} e \nearrow a_0\} \to \{a_1 : A\, \overline{e} \mid (\varphi, f) \mathbin{@} \overline{e} \nearrow a_1\} \tag{10}$$

Unwinding the definition, if $\alpha : \mathtt{Fib}\, A$ then $\alpha\, \mathtt{0}$ satisfies that for each cofibrant partial path $f : [\varphi] \to \Pi_{\mathtt{I}}(A \circ p)$ over a path $p : \mathtt{I} \to \Gamma$, if $a_0 : A\, \mathtt{0}$ extends the partial element $(\varphi, f) \mathbin{@} \mathtt{0}$, (that is, $\forall (u : [\varphi]).\ f\, u\, \mathtt{0} = a_0$), then $\alpha\, \mathtt{0}\, p\, \varphi\, f\, a_0 : A\, \mathtt{1}$ extends $(\varphi, f) \mathbin{@} \mathtt{1}$, that is $\forall (u : [\varphi]).\ f\, u\, \mathtt{1} = \alpha\, \mathtt{0}\, p\, \varphi\, f\, a_0$; and similarly for $\alpha\, \mathtt{1}$.

CCHM fibrations are closed under re-indexing: given $\gamma : \Delta \to \Gamma$ and $A : \Gamma \to \mathcal{U}$, we get a function $[\_]\gamma : \mathtt{Fib}\, A \to \mathtt{Fib}(A \circ \gamma)$ defined by $[\alpha]\gamma\, e\, p \triangleq \alpha\, e\, (\gamma \circ p)$. It follows that we get the structure of a Category with Families by taking families to be CCHM fibrations (A, α) over each $\Gamma : \mathcal{U}$ and elements of such a family to be dependent functions in $(x : \Gamma) \to A\, x$.

▶ Remark 4.4 (Fibrant objects). We say $A : \mathcal{U}$ is a *fibrant object* if we have a fibration structure for the constant family $\lambda(\_ : 1) \to A$ over the terminal object 1. Note that if $A : \Gamma \to \mathcal{U}$ has a fibration structure, then for each $x : \Gamma$ the type $A\, x : \mathcal{U}$ is fibrant. However the converse is not true: having a family of fibration structures, that is, an element of $(x : \Gamma) \to \mathtt{Fib}(\lambda(\_ : 1) \to A\, x)$, is weaker than having a fibration structure for $A : \Gamma \to \mathcal{U}$. For example, having a fibration structure allows one to transport elements along paths in Γ (see the `subst` functions defined below in (19)), whereas clearly a family of fibrant objects may not possess such transport operations.

If $\alpha : \mathtt{Fill}\, e\, A$, then $\lambda \varphi\, f\, a \to \alpha\, \varphi\, f\, a\, \overline{e} : \mathtt{Comp}\, e\, A$ and so every filling structure gives rise to a composition structure. Conversely, the composition structure of a CCHM fibration gives rise to filling structure:

▶ **Lemma 4.5.** *Given* $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $e : \mathtt{\{0,1\}}$, $\alpha : \mathtt{Fib}\, A$ *and* $p : \mathtt{I} \to \Gamma$, *there is a filling structure* $\mathtt{fill}\, e\, \alpha\, p : \mathtt{Fill}\, e\, (A \circ p)$ *that agrees with* α *at* $\overline{e}$, *that is:*

$$\forall (\varphi : \mathtt{Cof})(f : [\varphi] \to \Pi_{\mathtt{I}} A)(a : A(p\, e)).\ (\varphi, f) \mathbin{@} e \nearrow a \ \Rightarrow\ \mathtt{fill}\, e\, \alpha\, p\, \varphi\, f\, a\, \overline{e} = \alpha\, e\, p\, \varphi\, f\, a \tag{11}$$

Proof. First note that if two partial elements $f : [\varphi] \to A$ and $g : [\psi] \to A$ are *compatible*, that is, if the following relation holds

$$(\varphi, f) \smile (\psi, g) \triangleq \forall(u : [\varphi])(v : [\psi]).\ f\,u = g\,v \tag{12}$$

then by axiom $\mathtt{ax_6}$ their union $f \cup g : [\varphi \vee \psi] \to A$ gives a cofibrant partial element provided that (φ, f) and (ψ, g) are cofibrant partial elements. We use this in a construction of filling from composition that follows [11, Section 4.4], but just using the path-connection algebra structure on $\mathtt{I}$, rather than a de Morgan algebra structure. Suppose $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$, $e : \{\mathtt{0,1}\}$, $\alpha : \mathtt{Fib}\,A$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}(A \circ p)$, $a : A(p\,e)$ with $(\varphi, f)\ @\ e \nearrow a$, and $i : \mathtt{I}$. Then we can define

$$\mathtt{fill}\,e\,\alpha\,p\,\varphi\,f\,a\,i \triangleq \alpha\,e\,(p'\,i)\,(\varphi \vee i = e)\,(f'\,i \cup g\,i)\,a, \text{ where} \tag{13}$$

$p' : \mathtt{I} \to \mathtt{I} \to \Gamma$ is defined by $p'\,i\,j \triangleq p(i \sqcap_e j)$

$f' : (i : \mathtt{I}) \to [\varphi] \to \Pi_{\mathtt{I}}(A \circ (p'\,i))$ is defined by $f'\,i\,u\,j \triangleq f\,u\,(i \sqcap_e j)$

$g : (i : \mathtt{I}) \to \{g' : [i = e] \to \Pi_{\mathtt{I}}(A \circ (p'i)) \mid (\varphi, f'\,i) \smile (i = e, g')\}$ is defined by $g\,i\,v\,j \triangleq a$

and where $\sqcap_e$ is given by $\sqcap_{\mathtt{0}} \triangleq \sqcap$ and $\sqcap_{\mathtt{1}} \triangleq \sqcup$. We omit the proof that the above definition of $\mathtt{fill}$ has the required properties. ◀

Compared with [7], the fact that filling can be defined from composition considerably simplifies the process of lifting fibration structure through the usual type-forming constructs; for example:

▶ **Theorem 4.6** (Fibrant Σ- and Π-types)**.** *There are functions*

$$\mathtt{Fib}_{\Sigma} : \{\Gamma : \mathcal{U}\}\{A_1 : \Gamma \to \mathcal{U}\}\{A_2 : (x : \Gamma) \times A_1\,x \to \mathcal{U}\} \to \mathtt{Fib}\,A_1 \to \mathtt{Fib}\,A_2 \to \mathtt{Fib}(\Sigma\,A_1\,A_2) \tag{14}$$

$$\mathtt{Fib}_{\Pi} : \{\Gamma : \mathcal{U}\}\{A_1 : \Gamma \to \mathcal{U}\}\{A_2 : (x : \Gamma) \times A_1\,x \to \mathcal{U}\} \to \mathtt{Fib}\,A_1 \to \mathtt{Fib}\,A_2 \to \mathtt{Fib}(\Pi\,A_1\,A_2) \tag{15}$$

where $\Sigma\,A_1\,A_2\,x \triangleq (a_1 : A_1\,x) \times A_2(x, a_1)$ *and* $\Pi\,A_1\,A_2\,x \triangleq (a_1 : A_1\,x) \to A_2(x, a_1)$*. These functions are stable under re-indexing. Hence the category with families given by CCHM fibrations has* Σ*- and* Π*-types.*

Proof. The proof uses the above lemma and constructions similar to those in [11, Section 4.5]. We just give the construction of $\mathtt{Fib}_{\Pi}$ here, to show how to avoid the use Cohen *et al* make of de Morgan involution. Given $\Gamma : \mathcal{U}$, $A_1 : \Gamma \to \mathcal{U}$, $A_2 : (x : \Gamma) \times A_1\,x \to \mathcal{U}$, $\alpha_1 : \mathtt{Fib}\,A_1$, $\alpha_2 : \mathtt{Fib}\,A_2$, $e : \{\mathtt{0,1}\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_{\mathtt{I}}((\Pi\,A_1\,A_2) \circ p)$, $g : (\Pi\,A_1\,A_2)(p\,e)$ with $(\varphi, f)\ @\ e \nearrow g$ and $a_1 : A_1(p\,\overline{e})$, using Lemma 4.5 we define

$$\mathtt{Fib}_{\Pi}\,\alpha_1\,\alpha_2\,e\,p\,\varphi\,f\,g\,a_1 \triangleq \alpha_2\,e\,q\,\varphi\,f_2\,a_2, \text{ where} \tag{16}$$

$f_1 : \Pi_{\mathtt{I}}(A_1 \circ p)$

$f_1 \triangleq \mathtt{fill}\,\overline{e}\,\alpha_1\,p\,\bot\ \mathtt{elim}_{\emptyset}\ a_1$

$q : \mathtt{I} \to (x : \Gamma) \times A_1\,x$

$q \triangleq \langle p\ ,\ f_1 \rangle$

$f_2 : [\varphi] \to \Pi_{\mathtt{I}}(A_2 \circ q)$

$f_2\,u\,i \triangleq f\,u\,i\,(f_1\,i)$

$a_2 : \{a_2' : A_2(q\,e) \mid (\varphi, f_2)\ @\ e \nearrow a_2'\}$

$a_2 \triangleq g(f_1\,e)$

where for any $B : \mathcal{U}$, $\mathtt{elim}_\emptyset : [\bot] \to B$ denotes the unique function given by initiality of $[\bot]$, so that $(\bot, \mathtt{elim}_\emptyset) : \Box B$ because $\bot : \mathtt{Cof}$ by $\mathtt{ax}_2$ and $\mathtt{ax}_5$; and furthermore $(\bot, \mathtt{elim}_\emptyset) \nearrow b$ holds for any $b : B$. ◀

The theorem allows us to construct fibration structures for Σ- and Π-types, given fibration structures for their constituent types. But are there any fibration structures to begin with? We answer this question by showing that the natural number object N in the topos is always fibrant. This is proved for the topos of cubical sets $\hat{\mathcal{C}}$ in [7, Section 4.5] by defining a composition structure by primitive recursion. We give a more elementary proof using the fact that the interval object in $\hat{\mathcal{C}}$ satisfies axiom $\mathtt{ax}_1$ (see Theorem 6.1).

▶ **Theorem 4.7** (N is fibrant). *If* N *is an object with decidable equality, then there is a function* $\mathtt{Fib}_\mathtt{N} : \{\Gamma : \mathcal{U}\} \to \mathtt{Fib}(\lambda(\_ : \Gamma) \to \mathtt{N})$. *In particular, if the topos* $\mathcal{E}$ *has a natural number object* $1 \xrightarrow{\mathtt{Z}} \mathtt{N} \xrightarrow{\mathtt{S}} \mathtt{N}$, *then the category with families given by CCHM fibrations has a natural number object.*

Proof. Suppose $\Gamma : \mathcal{U}$, $e : \{0,1\}$, $p : \mathtt{I} \to \Gamma$, $\varphi : \mathtt{Cof}$, $f : [\varphi] \to \Pi_\mathtt{I}(\lambda\_ \to \mathtt{N})$ and $n : \mathtt{N}$ with $(\varphi, f)@e \nearrow n$. By assumption on N, for each $u : [\varphi]$ the property $\lambda(i : \mathtt{I}){\to}(f\,u\,i = n) : \mathtt{I}{\to}\Omega$ is decidable; hence by axiom $\mathtt{ax}_1$ and the fact that $f\,u\,e = n$, we also have $f\,u\,\overline{e} = n$. Therefore we can get $\mathtt{Fib}_\mathtt{N}\,e\,p\,\varphi\,f\,n : \{n' : \mathtt{N} \mid (\varphi, f)\,@\,\overline{e} \nearrow n'\}$ just by defining: $\mathtt{Fib}_\mathtt{N}\,e\,p\,\varphi\,f\,n \triangleq n$. For the last part of the theorem we use the fact that in a topos with natural number object, equality of numbers is decidable. ◀

A similar use of axiom $\mathtt{ax}_1$ suffices to prove:

▶ **Theorem 4.8** (Fibrant coproducts). *Writing* $A_1 \xrightarrow{\mathtt{inl}} A_1 + A_2 \xleftarrow{\mathtt{inr}} A_2$ *for the coproduct of* A_1 *and* A_2 *in* $\mathcal{E}$, *we lift this to families of types,* $\_ \uplus \_ : \{\Gamma : \mathcal{U}\}(A_1\ A_2 : \Gamma \to \mathcal{U}) \to \Gamma \to \mathcal{U}$, *by defining* $(A_1 \uplus A_2)\,x \triangleq A_1\,x + A_2\,x$. *Then there is a function*

$$\mathtt{Fib}_\uplus : \{\Gamma : \mathcal{U}\}\{A_1\ A_2 : \Gamma \to \mathcal{U}\} \to \mathtt{Fib}\,A_1 \to \mathtt{Fib}\,A_2 \to \mathtt{Fib}(A_1 \uplus A_2) \tag{17}$$

and this fibration structure on coproducts is stable under re-indexing. Hence the category with families given by CCHM fibrations has coproducts. ◀

4.3 Identity types

The next result follows from axioms $\mathtt{ax}_2$–$\mathtt{ax}_6$ by a construction like that in [7, Section 4.5].

▶ **Theorem 4.9** (Fibrant path types). *There is a function* $\mathtt{Fib}_\mathtt{Path} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \mathtt{Fib}\,A \to \mathtt{Fib}(\mathtt{Path}\,A)$, *where* $\mathtt{Path}\,A : (x : \Gamma) \times (A\,x \times A\,x) \to \mathcal{U}$ *is given by*

$$\mathtt{Path}\,A\,(x, (a_0, a_1)) \triangleq a_0 \sim a_1 \tag{18}$$

and where $\sim$ *is as in* (2). *This fibration structure on path types is stable under re-indexing.* ◀

These path types in the CwF of CCHM fibrations (Definition 4.3) satisfy the Coquand formulation of identity types with propositional computation properties [7, Figure 2]. Thus in addition to the contractibility of singleton types (4), we get *substitution functions* for transporting elements of a fibration along a path

$$\mathtt{subst} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{\alpha : \mathtt{Fib}\,A\}\{x_0\ x_1 : \Gamma\} \to (x_0 \sim x_1) \to A\,x_0 \to A\,x_1 \tag{19}$$

$$\mathtt{subst}\,p\,a \triangleq \alpha\,0\,p\,\bot\ \mathtt{elim}_\emptyset\ a$$

using the cofibrant partial elements $(\bot, \texttt{elim}_\emptyset)$ mentioned in the proof of Theorem 4.6. By Lemma 4.5 we have that these substitution functions satisfy a propositional computation rule for constant paths (3):

$$\begin{aligned}&\texttt{H} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{\alpha : \texttt{Fib}\, A\}\{x : \Gamma\}(a : A\, x) \to (a \sim \texttt{subst}(\texttt{k}\, x)\, a) \\ &\texttt{H}\, a \triangleq \texttt{fill}\, \texttt{0}\, \alpha\, (\texttt{k}\, x)\, \bot\, \texttt{elim}_\emptyset\, a\end{aligned} \tag{20}$$

To get Martin-Löf identity types with standard definitional, rather than propositional computation properties from these path types, we can use a version of Swan's construction [32] like the one in Section 9.1 of [11], but only using the path-connection algebra structure on $\texttt{I}$, rather than a de Morgan algebra structure. This is the only place that axiom $\texttt{ax}_7$ is used; we need the fact that the universe given by $\texttt{Cof}$ and $[\_] : \texttt{Cof} \to \mathcal{U}$ is closed under dependent products:

▶ **Lemma 4.10.** *The following term of type Ω is provable: $\forall(\varphi : \Omega)(f : [\varphi] \to \Omega).\, \texttt{cof}\, \varphi \Rightarrow (\forall(u : [\varphi]).\ \texttt{cof}(f\, u)) \Rightarrow \texttt{cof}(\exists(u : [\varphi]).\, f\, u)$.*

Proof. Note that if $u : [\varphi]$ then $(\exists(v : [\varphi]).\ f\, v) = f\, u$ and hence $\texttt{cof}(\exists(v : [\varphi]).\ f\, v) = \texttt{cof}(f\, u)$. So $\forall(u : [\varphi]).\ \texttt{cof}(f\, u)$ equals $\varphi \Rightarrow \texttt{cof}(\exists(v : [\varphi]).\ f\, v)$. Therefore from $\texttt{cof}\, \varphi$ and $\forall(u : [\varphi]).\ \texttt{cof}(f\, u)$ by axiom $\texttt{ax}_7$ we get $\texttt{cof}(\varphi \wedge \exists(v : [\varphi]).\ f\, v)$ and hence $\texttt{cof}(\exists(v : [\varphi]).\ f\, v)$, since $(\exists(v : [\varphi]).\ f\, v) \Rightarrow \varphi$. ◀

▶ **Theorem 4.11** (Fibrant identity types)**.** *Define identity types by:*

$$\begin{aligned}&\texttt{Id} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U}) \to (x : \Gamma) \times (A\, x \times A\, x) \to \mathcal{U} \\ &\texttt{Id}\, A\, (x, (a_0, a_1)) \triangleq (p : \texttt{Path}\, A\, (x, (a_0, a_1))) \times \{\varphi : \texttt{Cof} \mid \varphi \Rightarrow \forall(i : \texttt{I}).\, p\, i = a_0\}\end{aligned} \tag{21}$$

Then there is a function $\texttt{Fib}_{\texttt{Id}} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\} \to \texttt{Fib}\, A \to \texttt{Fib}(\texttt{Id}\, A)$ and the fibrations $(\texttt{Id}\, A, \texttt{Fib}_{\texttt{Id}}\, A)$ can be given the structure of Martin-Löf identity types in the CwF of CCHM fibrations.

Proof. Given $\Gamma : \mathcal{U}$, $A : \Gamma \to \mathcal{U}$ and $\alpha : \texttt{Fib}\, A$, using Theorems 4.6 and 4.9 we define $\texttt{Fib}_{\texttt{Id}}\, \alpha \triangleq \texttt{Fib}_\Sigma(\texttt{Fib}_{\texttt{Path}}\, \alpha)\, \beta$, where $\beta : \texttt{Fib}\, \Phi$ with

$$\begin{aligned}&\Phi : (y : (x : \Gamma) \times (A\, x \times A\, x)) \times \texttt{Path}\, A\, y \to \mathcal{U} \\ &\Phi((x, (a_0, a_1)), p) \triangleq \{\varphi : \texttt{Cof} \mid \varphi \Rightarrow \forall(i : \texttt{I}).\, p\, i = a_0\}\end{aligned}$$

and the fibration structure β mapping $e : \{\texttt{0,1}\}$, $p : \texttt{I} \to (y : (x : \Gamma) \times (A\, x \times A\, x)) \times \texttt{Path}\, A\, y$, $\varphi : \texttt{Cof}$, $f : [\varphi] \to \Pi_{\texttt{I}}(\Phi \circ p)$ and $\varphi' : \Phi(p\, e)$ with $(\varphi, f)\, @\, e \nearrow \varphi'$ to the term $\beta\, e\, p\, \varphi\, f\, \varphi' \triangleq \exists(u : [\varphi]).\ f\, u\, \overline{e}$ (using Lemma 4.10 to see that this is well defined). We get the usual introduction, elimination and computation rules for these identity types as follows. Since $\top : \texttt{Cof}$ holds by axiom $\texttt{ax}_5$, identity introduction $\texttt{refl} : \{\Gamma : \mathcal{U}\}\{A : \Gamma \to \mathcal{U}\}\{x : \Gamma\}(a : A\, x) \to \texttt{Id}\, A\, (x, (a, a))$ can be defined by $\texttt{refl}\, a \triangleq (\lambda a\ i \to a, \top)$. Identity elimination

$$\begin{aligned}&\texttt{J} : \{\Gamma : \mathcal{U}\}(A : \Gamma \to \mathcal{U})(x : \Gamma)(a_0 : A\, x)(B : (a : A\, x) \times \texttt{Id}\, A\, (x, (a_0, a)) \to \mathcal{U}) \\ &\quad (\beta : \texttt{Fib}\, B)(a_1 : A\, x)(e : \texttt{Id}\, A\, (x, (a_0, a_1))) \to B(a_0, \texttt{refl}\, a_0) \to B(a_1, e)\end{aligned} \tag{22}$$

is given by $\texttt{J}\, A\, x\, a_0\, B\, \beta\, a_1 (p, \varphi)\, b \triangleq \beta\, \texttt{0}\, \langle p\ ,\ q\rangle\, \varphi\, f\, b$ where $q : (i : \texttt{I}) \to \texttt{Id}\, A\, x\, (a_0, p\, i)$ is $q\, i\, j \triangleq (p(i \sqcap_0 j), \varphi \vee i = \texttt{0})$ and $f : [\varphi] \to \Pi_{\texttt{I}}(B \circ \langle p\ ,\ q\rangle)$ is $f\, u\, i \triangleq b$. ◀

5 Towards Univalence

Voevodsky's *univalence axiom* [33, Section 2.10] for a universe $\mathcal{V}$ in a CwF (with Σ-, Π- and Id-types) states that for every $A, B : \mathcal{V}$ the canonical function from $\mathtt{Id}\,\mathcal{V}\,A\,B$ to $(f : A \to B) \times \mathtt{Equiv}\,f$ is an equivalence. The notion of *equivalence* can be defined in terms of having contractible homotopy fibres [33, Section 4.4]:

$$\mathtt{Contr} : \mathcal{U} \to \mathcal{U} \tag{23}$$
$$\mathtt{Contr}\,A \triangleq (a_0 : A) \times ((a : A) \to a_0 \sim a)$$
$$\mathtt{Equiv} : \{A\ B : \mathcal{U}\}(f : A \to B) \to \mathcal{U} \tag{24}$$
$$\mathtt{Equiv}\,f \triangleq (b : B) \to \mathtt{Contr}((a : A) \times f\,a \sim b)$$

Cohen *et al* construct such a universe in the (CwF associated to the) presheaf topos of cubical sets by adapting the Hofmann-Streicher universe construction for presheaf categories [20]. We currently have no method for expressing this in the internal type theory of a general topos. Nevertheless in this section we present constructions using a *glueing* construction as in [11] that we conjecture suffice to ensure that if a universe of CCHM fibrations exists, then it satisfies most, if not all, of the conditions of the univalence axiom. Specifically we show how to transform equivalences into paths and vice-versa just for fibrant objects, rather than for fibrant families of objects (cf. Remark 4.4). Were there to be a universe of fibrations, then a proof of equality between types A and B in that universe would be a family $P : \mathtt{I} \to \mathcal{U}$ that not only satisfies $P\,\mathtt{0} = A$ and $P\,\mathtt{1} = B$, but also is fibrant. Note that if P has a fibration structure, then A and B are necessarily fibrant objects. We show that given such a family P it is always possible to construct an equivalence $f : A \to B$. Conversely, given an equivalence $f : A \to B$ between fibrant objects, it is always possible to construct such a P, provided cofibrant propositions satisfy a certain strictness property (axiom $\mathtt{ax}_8$ in Figure 1).

We begin by defining a path type for elements of the universe $\mathcal{U}$, in the style of (2). To do this we assume a second universe $\mathcal{U}_1$ with $\mathcal{U} : \mathcal{U}_1$. We sometimes refer to terms of type $\mathcal{U}$ as *small* types and terms of type $\mathcal{U}_1$ as *large* types. Define $\_ \sim_{\mathcal{U}} \_ : \mathcal{U} \to \mathcal{U} \to \mathcal{U}_1$ by

$$A \sim_{\mathcal{U}} B \triangleq \{P : \mathtt{I} \to \mathcal{U} \mid P\,\mathtt{0} = A \wedge P\,\mathtt{1} = B\} \tag{25}$$

▶ **Theorem 5.1.** *There is a function*

$$\mathtt{pathToEquiv} : \{A\ B : \mathcal{U}\}(P : A \sim_{\mathcal{U}} B)(\rho : \mathtt{Fib}\,P) \to (f : A \to B) \times \mathtt{Equiv}\,f \tag{26}$$

Proof. Define maps $f : A \to B$ and $g : B \to A$ as follows:

$$f\,a \triangleq \rho\,\mathtt{0}\,id \perp \mathtt{elim}_\emptyset\ a \qquad\qquad g\,b \triangleq \rho\,\mathtt{1}\,id \perp \mathtt{elim}_\emptyset\ b$$

This definition is well-typed since $P\,\mathtt{0} = A$ and $P\,\mathtt{1} = B$. Since both functions are defined using composition structure, for every $a : A$ and $b : B$ we can use filling (Lemma 4.5) to find dependently typed paths

$$p, q : \Pi_{\mathtt{I}} P \text{ with } p\,\mathtt{0} = a,\ p\,\mathtt{1} = f\,a,\ q\,\mathtt{0} = b \text{ and } q\,\mathtt{1} = g\,b \tag{27}$$

Since $P : \mathtt{I} \to \mathcal{U}$ has a fibration structure, A and B are fibrant objects. Therefore as in Section 4.3, we have path types for them satisfying the properties of identity types propositionally. So it is possible to combine the paths (27) to get $a \sim g\,(f\,a)$ and $b \sim f\,(g\,b)$. Hence f and g are quasi-inverses [33, Section 4.1] and hence in particular f is an equivalence. ◀

We now wish to construct a map going the other way, from equivalences to paths in the universe. To do so we use the notion of cofibrant-partial families of types: given a type $\Gamma : \mathcal{U}$ and a cofibrant property $\Phi : \Gamma \to \mathtt{Cof}$, we define $(\Gamma, \Phi) : \mathcal{U}$ by

$$\Gamma\,,\Phi \triangleq (x : \Gamma) \times [\Phi\, x] \tag{28}$$

and say that a term of type $A : (\Gamma\,, \Phi) \to \mathcal{U}$ is a *cofibrant-partial family of types* over Γ. Next we give a version of the *glueing* construction [11, Section 6], which allows one to extend such a cofibrant-partial family along a function $f : (x : \Gamma)(v : [\Phi\, x]) \to A(x, v) \to B\, x$ to a total family over Γ.

▶ **Definition 5.2** (Glueing). Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \mathtt{Cof}$, $A : \Gamma\,, \Phi \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma)(v : [\Phi\, x]) \to A(x, v) \to B\, x)$, define:

$$\mathtt{Glue}\, \Phi\, A\, B\, f : \Gamma \to \mathcal{U} \tag{29}$$
$$\mathtt{Glue}\, \Phi\, A\, B\, f\, x \triangleq (g : (v : [\Phi\, x]) \to A(x, v)) \times \{b : B\, x \mid \forall (v : [\Phi\, x]).\ f\, x\, v\, (g\, v) = b\}$$
$$\mathtt{glue}\, f : ((x, u) : \Gamma\,, \Phi) \to A(x, u) \to \mathtt{Glue}\, \Phi\, A\, B\, f\, x \tag{30}$$
$$\mathtt{glue}\, f\, (x, u)\, a \triangleq (\lambda(\_ : [\Phi\, x]) \to a,\ f\, x\, u\, a)$$
$$\mathtt{unglue}\, f : (x : \Gamma) \to \mathtt{Glue}\, \Phi\, A\, B\, f\, x \to B\, x \tag{31}$$
$$\mathtt{unglue}\, f\, x \triangleq \mathtt{snd}$$

▶ **Theorem 5.3.** $\mathtt{Glue}\, \Phi\, A\, B\, f$ *has a fibration structure if* A *and* B *have one and if* f *has the structure of an equivalence. In other words there is a function*

$$\begin{aligned}\mathtt{Fib}_{\mathtt{Glue}} : &\{\Gamma : \mathcal{U}\}\{\Phi : \Gamma \to \mathtt{Cof}\}\{A : \Gamma\,, \Phi \to \mathcal{U}\}\{B : \Gamma \to \mathcal{U}\}\\ &(f : (x : \Gamma)(u : [\Phi\, x]) \to A(x, u) \to B\, x) \to\\ &((x : \Gamma)(v : [\Phi\, x]) \to \mathtt{Equiv}(f\, x\, v)) \to \mathtt{Fib}\, A \to \mathtt{Fib}\, B \to \mathtt{Fib}(\mathtt{Glue}\, \Phi\, A\, B\, f)\end{aligned} \tag{32}$$

Proof. In outline, our proof of the theorem is as follows:
- Characterise equivalences in terms of a notion of *extension structure* (cf. Lemma 7 in [11]).
- Show that $\mathtt{unglue}\, f\, x : \mathtt{Glue}\, \Phi\, A\, B\, f\, x \to B\, x$ has such an extension structure when each $f\, x\, v$ does and when B is a CCHM fibration.
- Show that for a family of functions with an extension structure, if the codomain has a fibration structure, then so does the domain. Applying this to $\mathtt{unglue}\, f$, we get that $\mathtt{Glue}\, \Phi\, A\, B\, f$ has a fibration structure.

The details can be found in our Agda development. This proof differs from that in [11] in that it does not need cofibrant propositions to be closed under $\mathtt{I}$-indexed conjunction (cf. the $\forall$ quantifier defined in Section 4.1 of [11]). However, unlike in [11], the construction does not yield an element $\mathtt{Fib}_{\mathtt{Glue}}\, f\, \varepsilon\, \alpha\, \beta$ that restricts to α on Φ – a property that is probably needed for the construction of a univalent universe. ◀

We now have a way to interpret the glueing operation from [11] that meets some of the necessary requirements; see [11, Figure 4]. However, the current construction does not have certain *strictness* properties. In particular cubical type theory requires that, when restricted to a context where Φ holds, glueing should be equal to A "on the nose", that is that for any $x : \Gamma$ with $u : [\Phi\, x]$, we should have $\mathtt{Glue}\ \Phi\, A\, B\, f\, x = A(x, u)$. To satisfy such a requirement we postulate a further axiom $\mathtt{ax}_8$ that allows us to extend a partial type along an *isomorphism*

$$A \cong B \triangleq \{f : A \to B \mid (\exists g : B \to A)\, (g \circ f = id) \land (f \circ g = id)\} \tag{33}$$

to get a total type. Isomorphisms have inverses up to the extensional equality of the internal type theory, in contrast to equivalences which only have inverses up to path equality. Axiom $\mathtt{ax}_8$ says that any partial type A that is isomorphic to a total type B everywhere that A is defined, can be extended to a total type B' that is isomorphic to B.

▶ **Definition 5.4** (Strict Glueing). Given $\Gamma : \mathcal{U}$, $\Phi : \Gamma \to \mathtt{Cof}$, $A : \Gamma\,,\Phi \to \mathcal{U}$, $B : \Gamma \to \mathcal{U}$ and $f : (x : \Gamma)(v : [\Phi\, x]) \to A(x, v) \to B\, x$, define $\mathtt{SGlue}\,\Phi\, A\, B\, f : \Gamma \to \mathcal{U}$ by

$$\mathtt{SGlue}\,\Phi\, A\, B\, f\, x \triangleq \\ \mathtt{fst}(\mathtt{ax}_8\,(\lambda u : [\Phi\, x] \to A(x, u))\,(\mathtt{Glue}\ \Phi\, A\, B\, f\, x)\,(\lambda u : [\Phi\, x] \to \mathtt{glue}\, f\,(x, u))) \quad (34)$$

Note that $\mathtt{SGlue}$ has the desired strictness property: given any $(x, u) : \Gamma\,,\Phi$, by $\mathtt{ax}_8$ we have $A(x, u) = \mathtt{fst}(\mathtt{ax}_8\,(\lambda u : [\Phi\, x] \to A(x, u))\,(\mathtt{Glue}\ \Phi\, A\, B\, f\, x)\,(\lambda u : [\Phi\, x] \to \mathtt{glue}\, f\,(x, u)))$ and hence

$$\forall (x : \Gamma)(u : [\Phi\, x]).\ \mathtt{SGlue}\,\Phi\, A\, B\, f\, x = A(x, u) \quad (35)$$

▶ **Theorem 5.5.** $\mathtt{SGlue}$ *has a fibration structure if A and B have one and f has the structure of an equivalence.*

Proof. It is easy to show that (fibrewise) isomorphisms preserve fibration structures. Hence we obtain a fibration structure on $\mathtt{SGlue}$ by transporting the structure obtained from $\mathtt{Fib}_{\mathtt{Glue}}$ (Theorem 5.3) along the isomorphism from $\mathtt{ax}_8$. ◀

We are now able to construct a map from equivalences to paths in the universe:

▶ **Theorem 5.6.** *There is a function*

$$\mathtt{equivToPath} : \{A\ B : \mathcal{U}\}\{\alpha : \mathtt{Fib}(\lambda\_ : 1 \to A)\}\{\beta : \mathtt{Fib}(\lambda\_ : 1 \to B)\} \\ (f : A \to B) \to (\mathtt{Equiv}\, f) \to (P : A \sim_{\mathcal{U}} B) \times (\mathtt{Fib}\, P) \quad (36)$$

Proof. Define the following:

$$\Phi : \mathtt{I} \to \mathtt{Cof}$$
$$\Phi\, i \triangleq (i = 0) \vee (i = 1)$$
$$C : (\mathtt{I}\,,\Phi) \to \mathcal{U}$$
$$C\,(i, u) \triangleq ((\lambda\_ : [i = 0] \to A) \cup (\lambda\_ : [i = 1] \to B))\, u$$
$$f' : (i : \mathtt{I})(u : [\Phi\, i]) \to C(i, u) \to B$$
$$f'\, i \triangleq (\lambda\_ : [i = 0] \to f) \cup (\lambda\_ : [i = 1] \to id)$$

Now let $P \triangleq \mathtt{SGlue}\ \Phi\, C\,(\lambda\_ : \mathtt{I} \to B)\, f'$ and observe that $P\,\mathtt{0} = A$ and $P\,\mathtt{1} = B$ by the strictness property of $\mathtt{SGlue}$. Further, we can show that f' is an equivalence since f and the identity are both equivalences; and using α, β and $\mathtt{ax}_1$, we can define a fibration structure on C. Hence, by Theorem 5.5, we get a fibration structure on P. ◀

The following theorem shows that for fibrant objects the functions $\mathtt{pathToEquiv}$ and $\mathtt{equivToPath}$ are mutually inverse up to path equality. We omit its proof here.

▶ **Theorem 5.7.** *Given $A, B : \mathcal{U}$, define*

$$\mathtt{pathToPath} : (P : A \sim_{\mathcal{U}} B)(\rho : \mathtt{Fib}\, P) \to (P' : A \sim_{\mathcal{U}} B) \times (\rho' : \mathtt{Fib}\, P') \quad (37)$$
$$\mathtt{pathToPath}\, P\, \rho \triangleq \mathtt{equivToPath}\,(\mathtt{fst}(\mathtt{pathToEquiv}\ P\, \rho))\,(\mathtt{snd}(\mathtt{pathToEquiv}\ P\, \rho))$$
$$\mathtt{equivToEquiv} : (f : A \to B)(e : \mathtt{Equiv}\, f) \to (f' : A \to B) \times (e' : \mathtt{Equiv}\, f') \quad (38)$$
$$\mathtt{equivToEquiv}\, f\, e \triangleq \mathtt{pathToEquiv}\,(\mathtt{fst}(\mathtt{equivToPath}\ f\, e))\,(\mathtt{snd}(\mathtt{equivToPath}\ f\, e))$$

If A and B are fibrant objects, then there are functions

$$\mathtt{pathInv} : (P : A \sim_{\mathcal{U}} B)(\rho : \mathtt{Fib}\, P)(i : \mathtt{I}) \to P\, i \sim_{\mathcal{U}} \mathtt{fst}(\mathtt{pathToPath}\, P\, \rho)\, i \tag{39}$$

$$\mathtt{equivInv} : (f : A \to B)(e : \mathtt{Equiv}\, f) \to f \sim \mathtt{fst}(\mathtt{equivToEquiv}\, f\, e) \tag{40}$$

Since for a function f between fibrant objects $\mathtt{Equiv}\, f$ *is a mere proposition [33, Section 3.3], it follows that there is a function:*

$$\mathtt{equivInv}' : (f : A \to B)(e : \mathtt{Equiv}\, f) \to (f, e) \sim (\mathtt{equivToEquiv}\, f\, e). \tag{41}$$

◀

Note that for any family $A : \Gamma \to \mathcal{U}$, the type $\mathtt{Fib}\, A$ is also a mere proposition. However, without the presence of a univalent universe it is not possible to state the equivalent of $\mathtt{equivInv}'$ for $\mathtt{pathInv}$. We hope to resolve this issue in subsequent work.

6 Satisfying the Axioms

Working informally in a constructive set theory, the authors of [11] give a model of their type theory using the topos $\hat{\mathcal{C}}$ of contravariant set-valued functors on a particular small category $\mathcal{C}$ that they call the *category of cubes*. Its objects are all finite subsets of a fixed, countably infinite set with decidable equality whose elements should be thought of as names of cartesian directions. Given two such subsets X and Y, the $\mathcal{C}$-morphisms $X \to Y$ are all de Morgan algebra [5, Chapter XI] homomorphisms from the free de Morgan algebra on the finite set Y to that on X; composition and identities are as in the (opposite of the) category of sets. Thus $\mathcal{C}$ is in fact a presentation of the algebraic theory of de Morgan algebra as a Lawvere theory [25, 2] and $\mathcal{C}$ is universal among categories with finite products containing an internal de Morgan algebra.

Let us replace $\mathcal{C}$ by an arbitrary small category $\mathbf{C}$ and see what is required of $\mathbf{C}$ for the topos $\hat{\mathbf{C}}$ of presheaves (within Intuitionistic ZF set theory [1, Section 3.2], say) to have an interval object and subobject of cofibrant propositions satisfying the axioms in Figure 1. We do not aim for complete generality, just enough to encompass some examples of independent interest such as $\mathbf{C} = \mathbf{\Delta}$ the category of inhabited finite linearly ordered sets $[0 < 1 < \cdots < n]$, for which $\hat{\mathbf{C}} = \mathbf{sSet}$, the category of *simplicial sets*, widely used in homotopy theory [18].

6.1 The interval object

We take the interval object $\mathtt{I} \in \hat{\mathbf{C}}$ to be the representable functor $\mathrm{y}_{\mathtt{i}} \triangleq \mathbf{C}(\_, \mathtt{i})$ on some object $\mathtt{i} \in \mathbf{C}$. The following theorem gives a useful criterion for such an interval object to satisfy axiom $\mathtt{ax}_1$.

▶ **Theorem 6.1.** *In a presheaf topos $\hat{\mathbf{C}}$, a representable functor $\mathtt{I} = \mathrm{y}_{\mathtt{i}}$ satisfies axiom $\mathtt{ax}_1$ if $\mathbf{C}$ is a* cosifted *category, that is, if finite products in* **Set** *commute with colimits over $\mathbf{C}^{\mathrm{op}}$ [16].*

Proof. $\mathbf{C}$ is cosifted if the colimit functor $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}} : \hat{\mathbf{C}} \to \mathbf{Set}$ preserves finite products. Recall that $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}} : \hat{\mathbf{C}} \to \mathbf{Set}$ is left adjoint to the constant presheaf functor $\Delta : \mathbf{Set} \to \hat{\mathbf{C}}$ and (hence) that for any $c \in \mathbf{C}$ it is the case that $\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_c \cong 1$. So when $\mathbf{C}$ is cosifted we have for any $c \in \mathbf{C}$

$$\hat{\mathbf{C}}(\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}, \Delta\{0,1\}) \cong \mathbf{Set}(\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}), \{0,1\}) \cong \\ \mathbf{Set}(\mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_c \times \mathrm{colim}_{\mathbf{C}^{\mathrm{op}}}\, \mathrm{y}_{\mathtt{i}}, \{0,1\}) \cong \mathbf{Set}(1 \times 1, \{0,1\}) \cong \{0,1\}$$

Since decidable subobjects in $\hat{\mathbf{C}}$ are classified by $1+1=\Delta\{0,1\}$, this means that the only two decidable subobjects of $\mathrm{y}_c \times \mathrm{y}_{\mathtt{i}}$ are the smallest and the greatest subobjects. Since this is so for all $c \in \mathbf{C}$, it follows that $\mathtt{I} = \mathrm{y}_{\mathtt{i}}$ satisfies $\mathtt{ax}_1$. ◀

A more elementary characterisation of cosiftedness is that $\mathbf{C}$ is inhabited and for every pair of objects $c, c' \in \mathbf{C}$ the category of spans $c \leftarrow \cdot \rightarrow c'$ is a connected category [2, Theorem 2.15]. The category $\mathbf{\Delta}$ has this property and hence the natural candidate for an interval in **sSet**, namely $\mathrm{y}_{\mathtt{i}}$ when $\mathtt{i}$ is the 1-simplex $[0<1]$, satisfies $\mathtt{ax}_1$. Any category with finite products trivially has the property. This is the case for $\mathcal{C}$ and thus the interval in the model of [11], where $\mathbf{C} = \mathcal{C}$ and $\mathtt{i}$ is any one-element set (the underlying object of the internal de Morgan algebra), satisfies $\mathtt{ax}_1$.

In addition to $\mathtt{ax}_1$, the other axioms in Figure 1 concerning the interval say that $\mathtt{I}$ is a non-trivial ($\mathtt{ax}_2$) model of the algebraic theory given by $\mathtt{ax}_3$ and $\mathtt{ax}_4$, which we call *path connection algebra*. (See also Definition 1.7 of [17], which considers a similar notion in a more abstract setting.) The 1-simplex in **sSet** is a non-trivial path-connection algebra, the constants being its two end points and the binary operations being induced by the order-preserving binary operations of minimum and maximum on $[0<1]$. The generic de Morgan algebra in $\hat{\mathcal{C}}$ is a non-trivial path-connection algebra: the constants are the least and greatest elements and the binary operations are meet and join. An obvious variation on the theme of [11] would be to replace $\mathcal{C}$ by the Lawvere theory for path-connection algebras.

6.2 Cofibrant propositions and the strictness postulate

In a topos with an interval object, there are many candidates for a subobject $\mathtt{Cof} \rightarrowtail \Omega$ satisfying axioms $\mathtt{ax}_5$–$\mathtt{ax}_7$ in Figure 1. For example, one can take the subobject inductively defined by the requirements that it contains $\{\mathtt{0}\}$ and $\{\mathtt{1}\}$ and is closed under binary union and dependent intersection to obtain a smallest $\mathtt{Cof}$ satisfying $\mathtt{ax}_5$–$\mathtt{ax}_7$. Although it is not described that way, this is the notion of cofibrant proposition in $\hat{\mathcal{C}}$ used for the model in [11].

Next we discuss satisfaction of the strictness axiom $\mathtt{ax}_8$ in a general presheaf topos $\hat{\mathbf{C}}$. We work in the CwF associated with $\hat{\mathbf{C}}$ as in [19, Section 4]. In particular, families over a presheaf $\Gamma \in \hat{\mathbf{C}}$ are given by functors $(\int \Gamma)^{\mathrm{op}} \to \mathbf{Set}$, where $\int \Gamma$ is the usual category of elements of Γ. If $\mathcal{S}$ is a Grothendieck universe in the ambient set theory, then its Hofmann-Streicher lifting [20] to a universe $\mathcal{U}$ in that CwF satisfies that the morphisms $\Gamma \to \mathcal{U}$ in $\hat{\mathbf{C}}$ name the families $(\int \Gamma)^{\mathrm{op}} \to \mathcal{S}$ taking values in $\mathcal{S} \subseteq \mathbf{Set}$.

▶ **Definition 6.2** (Ω_{dec})**.** The subobject classifier Ω in a presheaf topos $\hat{\mathbf{C}}$ maps each $c \in \mathbf{C}$ to the set of *sieves* on c, that is, pre-composition closed subsets $S \subseteq \mathrm{obj}(\mathbf{C}/c)$. Let $\Omega_{\mathrm{dec}} \rightarrowtail \Omega$ be the subpresheaf consisting of those S that are decidable subsets of $\mathrm{obj}(\mathbf{C}/c)$. Of course if the ambient set theory satisfies the Law of Excluded Middle, then $\Omega_{\mathrm{dec}} = \Omega$. In general Ω_{dec} classifies monomorphisms $\alpha : F \rightarrowtail G$ in $\hat{\mathbf{C}}$ for which each injective function $\alpha_c : F(c) \to G(c)$ has decidable image.

▶ **Theorem 6.3.** *Interpreting the universe $\mathcal{U}$ as the Hofmann-Streicher lifting [20] of a Grothendieck universe in* **Set**, *a subobject $\mathtt{Cof} \rightarrowtail \Omega$ in a presheaf topos $\hat{\mathbf{C}}$ satisfies the strictness axiom $\mathtt{ax}_8$ if it is contained in $\Omega_{\mathrm{dec}} \rightarrowtail \Omega$.*

Proof. For each $c \in \mathrm{obj}\,\mathbf{C}$, suppose we are given $S \in \Omega_{\mathrm{dec}}(c)$. Thus S is a sieve on c and for each $c' \in \mathrm{obj}\,\mathbf{C}$ and $\mathbf{C}$-morphism $f : c' \to c$, it is decidable whether or not $f \in S$. We can also regard S as a subpresheaf $S \hookrightarrow \mathrm{y}_c$.

Suppose that we have families $A : (\int S)^{\mathrm{op}} \to \mathcal{S}$, $B : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ and a natural isomorphism s between A and the restriction of B along $S \hookrightarrow \mathrm{y}_c$. For each $\mathbf{C}$-morphism $\cdot \xrightarrow{f} c$, using the decidability of S, we can define bijections $s'(f) : B'(f) \cong B(f)$ given by

$$B'(f) \triangleq \begin{cases} A(f) & \text{if } f \in S \\ B(f) & \text{if } \neg(f \in S) \end{cases} \qquad \text{and} \qquad s'(f) \triangleq \begin{cases} s(f) & \text{if } f \in S \\ f & \text{if } \neg(f \in S) \end{cases}$$

(compare this with Definition 15 in [11]). We make B' into a functor $(\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ by transferring the functorial action of B across these bijections. Having done that, s' becomes a natural isomorphism $B' \cong B$; and by definition its restriction along $S \hookrightarrow \mathrm{y}_c$ is s. ◀

▶ Remark 6.4. As a partial converse of the theorem, we have that if $\mathtt{ax}_8$ is satisfied by the Hofmann-Streicher universe in the CwF associated with $\hat{\mathbf{C}}$, then each cofibrant mono $\alpha : F \rightarrowtail G$ has component functions $\alpha_c : F\,c \to G\,c$ whose images are $\neg\neg$-closed subsets of $G\,c$. To see this we can apply an argument due to Andrew Swan [private communication] that relies upon the fact that in the ambient set theory one has

$$(X = \emptyset) = \forall x \in X.\,\bot = \neg\neg(\forall x \in X.\,\bot) = \neg\neg(X = \emptyset) \tag{42}$$

For suppose given $c \in \mathrm{obj}\,\mathbf{C}$ and $S \in \mathtt{Cof}(c)$. We have to use axiom $\mathtt{ax}_8$ to show that S is a $\neg\neg$-closed subset of $\mathrm{obj}(\mathbf{C}/c)$. Let $A : (\int S)^{\mathrm{op}} \to \mathcal{S}$ be the constant functor mapping each (c', f) to $\{\emptyset\}$; and let $B : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ map each (c', f) to $\{\{\emptyset\}, \{\emptyset \mid f \in S\}\}$ (which does extend to a functor, because S is a sieve). The restriction of B along $S \hookrightarrow \mathrm{y}_c$ is isomorphic to A and so by $\mathtt{ax}_8$ there some $B' : (\int \mathrm{y}_c)^{\mathrm{op}} \to \mathcal{S}$ whose restriction along $S \hookrightarrow \mathrm{y}_c$ is equal to A and some isomorphism $s' : B' \cong B$. For any $(c', f) \in \mathrm{obj}(\int \mathrm{y}_c)$, suppose $X \in B'(c', f)$; then $f \in S \Rightarrow X = \emptyset$, hence $\neg\neg(f \in S) \Rightarrow \neg\neg(X = \emptyset)$ and therefore by (42), $\neg\neg(f \in S) \Rightarrow (X = \emptyset)$. Therefore $\neg\neg(f \in S) \Rightarrow B'(c', f) = \{\emptyset\} \Rightarrow B(c', f) \cong \{\emptyset\} \Rightarrow f \in S$. So S is indeed a $\neg\neg$-closed subset of $\mathrm{obj}(\mathbf{C}/c)$.

Note that this result implies that it is not possible to take $\mathtt{Cof}$ to be the whole of Ω and satisfy $\mathtt{ax}_8$ unless the ambient set theory satisfies the Law of Excluded Middle.

It is not hard to see that Ω_{dec} satisfies $\mathtt{ax}_6$ and $\mathtt{ax}_7$. It also satisfies $\mathtt{ax}_5$ if for example equality of $\mathbf{C}$-morphisms is decidable. Therefore we get a model of our axioms in $\hat{\mathbf{C}}$ by taking $\mathtt{Cof} = \Omega_{\mathrm{dec}}$, for any cosifted $\mathbf{C}$ that has decidable equality of morphisms and a representable with the structure of a path-connection algebra; $\mathcal{C}$ and $\mathbf{\Delta}$ are both examples. In the case of $\hat{\mathcal{C}}$ this is a different choice of cofibrant proposition to the one in [11]. In the case of $\hat{\mathbf{\Delta}} = \mathbf{sSet}$, it remains to be investigated what is the relationship between this constructive model based on CCHM fibrations and Voevodsky's non-constructive model of univalent type theory using classical Kan simplicial sets [23].

7 Related and Future Work

The work reported here was inspired by [11]. We have shown how to express Cohen, Coquand, Huber and Mörtberg's notion of fibration in the internal type theory of a topos (Definition 4.3). We found that quite a simple collection of axioms (Figure 1) suffices for this to model Martin-Löf type theory with path types satisfying a weak form of univalence. The construction in Section 8.2 of [11] of a fibrant universe satisfying the full univalence axiom uses a modified form of Hofmann-Streicher lifting [20] within the ambient constructive set theory. We plan to investigate whether that, or indeed some other universe construction, can be axiomatised within the internal type theory of a topos.

We found that only a simple path-connection algebra, rather than a de Morgan algebra structure, is needed on the interval. Furthermore, the collection of propositions suitable for uniform Kan filling can be chosen in various ways. This allows a model of our axioms in constructive simplicial sets as one example, besides variations on the notion of cubical set. In Section 6 we only considered how presheaf categories can satisfy our axioms. It might be interesting to consider models in sheaf toposes, particularly *gros toposes* such as [22] that allow the interval object to be the usual topological interval.

Our concerns (modelling intensional type theory with path-like identity types, getting cubical and simplicial sets as instances) are similar to those of Gambino and Sattler [17]; although our methods differ, it seems we arrive at the same notion of fibration, although this remains to be investigated. Birkedal *et al* [8] are developing guarded cubical type theory with a semantics based on an axiomatic version of [11] within the internal logic of a presheaf topos. We believe that our approach to modelling cubical type theory is more general than theirs, but that there will be interesting synergies.

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