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A numerical solution for the shape of fabric-formed concrete structures

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Abstract

This paper details a new numerical method for determining the form of a section of flexible, impermeable and inextensible hanging fabric subject to the hydrostatic load imposed by wet concrete. A closed form solution is already known to exist in the form of incomplete elliptic integrals, but this can be difficult to implement in practice. The numerical method presented here was developed by the authors, and the method has been subsequently implemented by researchers investigating the behaviour of fabric-formed concrete beams. It is thought that this method may be of use to other designers and investigators interested in the form of flexible formworks and also of wider interest for the design of flexible scoops or other hydrostatically loaded structures. The method is shown to be applicable to full, part-full and overfull, i.e. surcharged, containers. The method's accuracy is demonstrated by comparison with the predictions of the closed form solution and with reference to the conclusions of previous empirical investigations. Comparison is also made on a graphical basis to a number of reported hydrostatically determined forms, with good agreement being shown. This indicates that the approach has direct application as a form-finding procedure for fabric-formed concrete structures.

Keywords: concrete, fabric formwork, form-finding, hydrostatic load

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1. Introduction

Wet concrete is effectively liquid. In order to form useful structural elements, liquid concrete must be contained by formwork until it is suitably cured. If the container is rigid, the determination of the final form of the concrete is simple as the concrete takes the shape of the container into which it is poured. If the container is soft, such that it undergoes significant out of plane deformations when filled, the determination of the final form of the concrete is more difficult. This paper provides a numerical solution to the two-dimensional case of this problem. The method was originally developed by the authors for the determination of the sectional profile of a fabric-formed concrete beam. Both this specific application and the more general numerical solution are discussed.

2. Fabric-formed concrete elements

While rigid, impermeable, rectilinear containers constructed from earth, timber, steel or plastic are the principal methods used for the forming of concrete elements, there are alternatives. A design for the use of draped flexible fabrics as formwork for variable section concrete spanning elements was patented by Gustav Lilienthal (1897). Fletcher (1917) explicitly observed in relation to his own textile formwork designs that, despite the obvious potential advantages of variable section concrete elements in achieving optimised profiles, they are "not generally put into practice because of the prohibitive cost of constructing a form in which to mold such a shape". This continued to be the case for much of the last century, prompting Nervi (1956) to note that:

> "although reinforced concrete has been used for over a hundred years and with
>
> increasing interest during the last few decades, few of its properties and
>
> potentialities have been fully exploited so far. Apart from the unconquerable
>
> inertia of our minds, which do not seem able to adopt freely any new ideas, the

main cause of this delay is a trivial technicality: the need to prepare wooden

forms."

In the latter half of the twentieth century, a number of innovators including Miguel Fisac and

Kenzo Unno explored the architectural possibilities of quasi-structural concrete elements and

panels formed in fabrics. Exploratory work by Mark West, has led to resurgence in interest in

fabric formed concrete as an architectural language and as a means for achieving optimised

structural sections. Approaches to the optimisation of concrete elements using variable

section profiles, often with fabric formwork, is a growing research area with analytical and

experimental work carried out in recent years by Bailliss (2006), Garbett (2008), Veenendaal

(2008), Foster (2010), Lee (2011), Hashemian (2012), Orr (2012). While the sometimes

bulbous profile of fabric formed concrete elements is a natural consequence of the flexibility

of the concrete filled fabric, various approaches may be adopted to limit formwork

deformations for the purposes of optimisation, including tensioning of the fabric, stitching or

pinching of the web, and the use of rigid web formers. A range of woven materials have been

used as fabric formwork including hessian, geotextiles and polyesters (Orr et al. 2011).
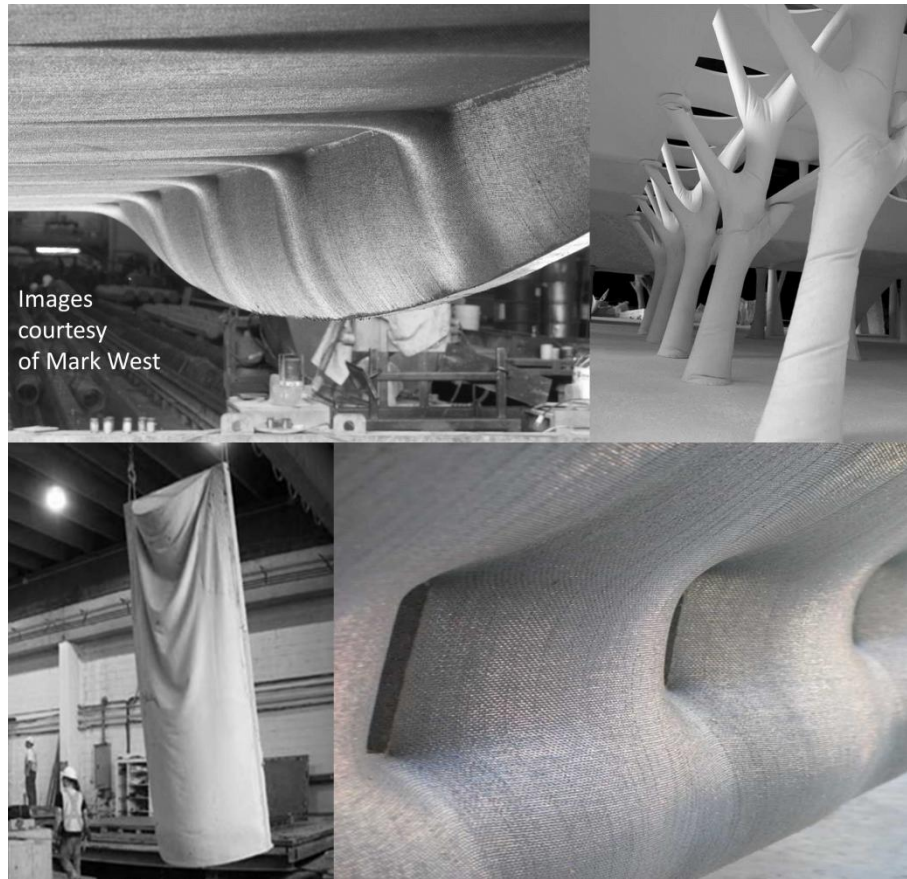
Figure 1. Architectural possibilities afforded by fabric-formed concrete. Images courtesy of Mark West.

Aside from the architectural possibilities opened up by the curvilinear forms that result from the use of fabric formwork (Figure 1), there are a number of structural, economic and environmental benefits:

1. Structurally efficient profiles may be obtained without the need for elaborate formwork, leading to reduced concrete volumes compared with 'equivalent' rectangular elements (Orr et al. 2011). This material saving has a direct benefit in terms of reduction in embodied carbon and cost.

2. A significant proportion of the load carried by typical concrete structures is due to the self-weight of the structural elements themselves. Profile optimisation thus has a

further beneficial effect in terms of reducing element self-weight. This reduction affects not only the design of the element itself, but also the design of the supporting elements. Lighter spanning elements can lead to smaller column sizes and reduced foundations. Smaller columns also lead to greater net internal areas, thereby further increasing the value proposition.

3. Shallower portions of such beams, typically in regions of lower bending moment, can provide natural paths for services (Figure 2). This may allow ceiling voids to be reduced and commensurate increases in floor to ceiling height, or reductions in overall storey height, to be achieved. Furthermore, the aesthetic quality of the curvilinear beams that result may encourage architectural designers to forego suspended ceilings altogether, enabling the benefits of the exposed concrete thermal mass to be more readily exploited.

4. Improved surface concrete quality, both mechanical and aesthetic, has been observed as a result of the reduced water to cement ratio of the surface concrete due to the permeability of the formwork (Ghaib & Gorski 2001). This has particularly important implications in situations where concrete grade is governed by durability requirements, with enhanced surface quality potentially allowing a reduction in concrete grade (Orr et al. 2011). The higher quality of the surface concrete further enhances the aesthetic case for the foregoing of suspended ceilings and the exposure of the structural mass.

5. Suitable fabric selection can allow for a reusable, compact, lightweight formwork system that reduces transportation costs and construction waste.

Despite the potential benefits, designing for flexible formwork presents a number of challenges – not least of which has been the historical difficulty of determining the final

shape of the fabric container when filled with concrete. This was the challenge that led the

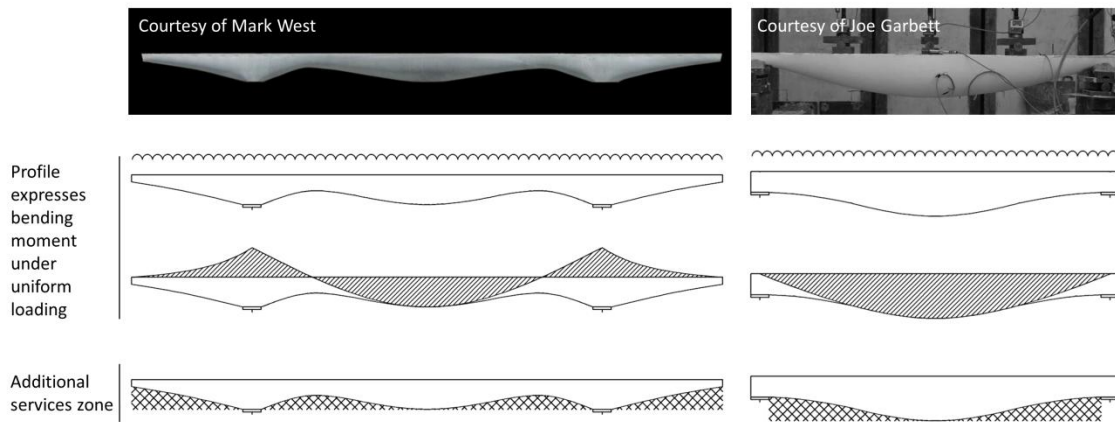authors to develop the numerical solution presented here.

Figure 2: Indicative relationships between profile, bending moment and resulting additional service zone for typical fabric-formed flexural members. Photographic images courtesy of Mark West, and Joe Garbett (Garbett 2008).

3. Form finding

Flexible, inextensible elements may take a variety of forms dependent on loading condition:

- the *parabola* – uniform vertical pressure by plan length,

- the *catenary* – uniform vertical pressure by arc length,

- the *lintearia* – hydrostatic normal pressure,

- the *valeria* – uniform normal pressure.

These forms have long been the subject of enquiry and their solutions have been closely

related to the development of elastic bending theory (Levein 2008). A comparison of these

forms is shown in Figure 3 for profiles of equal sag and varying span. While the forms of the

parabola, catenary and valeria are relatively easy to describe mathematically, the form of the

lintearia or hydrostatically loaded profile is not. This is problematic because precise

description of the form taken by a flexible fabric or container subject to hydrostatic load is

thought to be of potential interest in a wide variety of engineering applications. A rigorous

closed form solution is provided by Iosilevskii (2010) in the form of incomplete elliptic

integrals, but the mathematics involved is likely to be unfamiliar to the non-specialist and

may be difficult to implement. In the context of fabric-formed concrete elements,

experimental investigations have determined some approximate empirical relationships

between characteristic dimensions (Baillis 2006), but have been unable to provide

comprehensive descriptions of the fabric profile. The numerical approach presented here was

developed in order to provide such a description (Foster 2010) and thereby open up

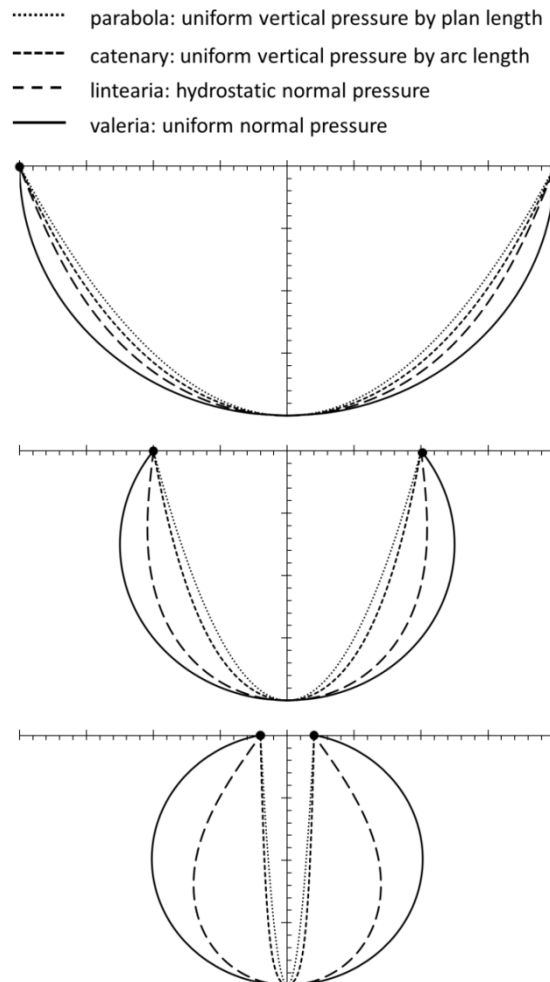possibilities for the design of fabric-formed concrete elements.

Figure 3: Comparison of parabola, catenary, lintearia and valeria profiles of equal sag and varying span

4. Formulation

Consider a flexible, inextensible fabric having negligible out of plane stiffness and self-weight. The fabric is hung between two nodes positioned such that a line connecting them is horizontal. The fabric supports a fluid of density ρ. The nodes are separated by an opening breadth b; the fabric is of length p and forms the hanging perimeter of a section of maximum depth d. It is taken as axiomatic that $p \geq b$. A coordinate system of mutually orthogonal axes can be defined such that the origin is located at the lowest point of the hanging fabric (Figure 4). The x-axis is horizontal and the y-axis is vertical. Thus the coordinates of the hanging nodes are $(\pm b/2, d)$ and the coordinates of the lowest point of the hanging perimeter are $(0, 0)$.
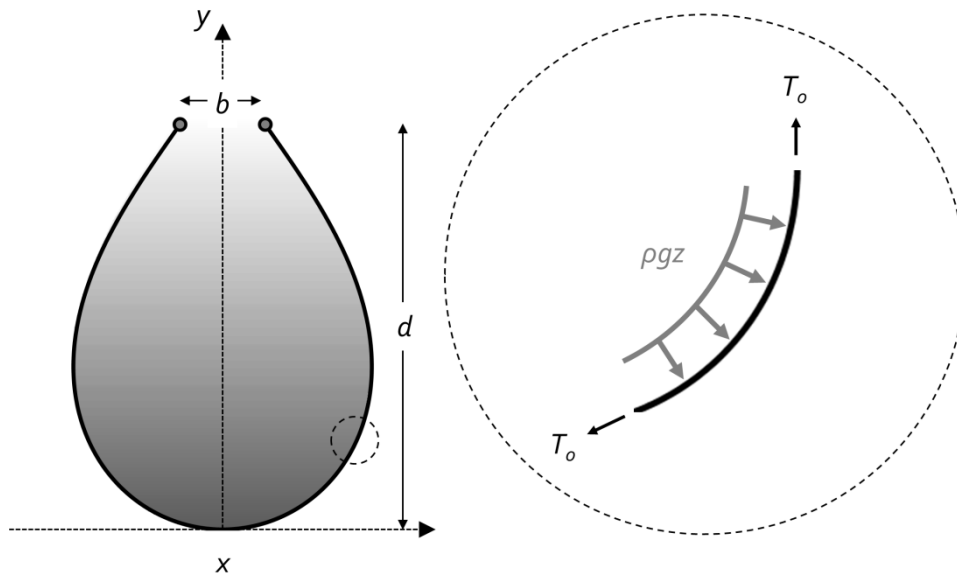

Figure 4: Coordinate system and forces on free element of fabric

The fabric will freely deform under any applied out-of-plane forces, and static equilibrium can only be maintained by resisting forces carried entirely in-plane. By definition, all hydrostatic forces must act perpendicular to the fabric and the fabric will freely deform to carry the resisting forces as a tension in plane. It follows that the tension T_0 in the fabric is constant around the perimeter of any given hydrostatically determined profile (Iosilevskii

2010). Thus, the hanging perimeter can be thought of as forming a two dimensional surface minimizing surface tension. An alternative approach to the same problem might be to think in terms of minimizing the height in *y* of the centre of mass of the enclosed fluid.

Since the only applied forces acting on the fabric are the hydrostatic forces due to weight of the fluid, it can be deduced that the curvature of the profile at any given point is solely determined by the hydrostatic pressure at that point. Given that hydrostatic pressure ρg varies linearly with hydraulic height *z* in *y*, a simple model for the variation in curvature κ and thus the hydrostatically determined profile of the perimeter can be defined:

$$T_0 = \frac{\rho g z}{\kappa}$$
                                                                    Equation 1

Rather than proceeding towards a closed-form analytical characterisation *cf.* Iosilevskii (2010), the hydrostatically determined form can be investigated by starting from the point (0, 0) and 'walking out' a perimeter in accordance with the relation between the curvature and *z*. The starting point of (0, 0) was chosen here for the simplification available by symmetry and because section depth is often a determining structural and architectural characteristic. It should be noted that 'walking out' the perimeter from (0, 0) simply generates a curve that conforms to a chosen relation between curvature and depth. Altering this relation by varying a coefficient *k* allows a family of curves to be generated. These are, in effect, the curves of the *elastica*.
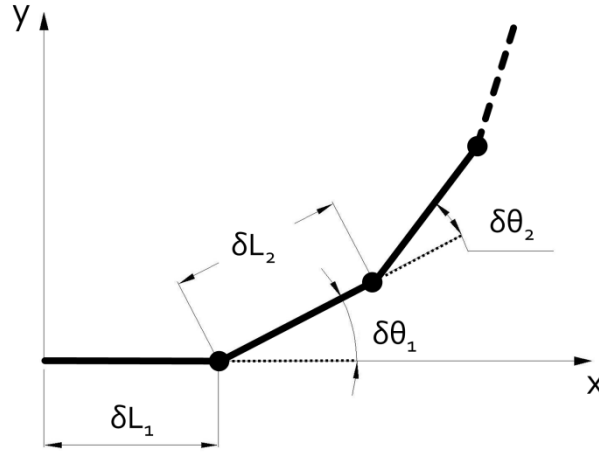
5. Procedure

Figure 5: Characterisation of the 'walking out' procedure, beginning at the lowest level of the hanging fabric profile. The perimeter is thus described by a finite number of discrete steps.

'Walk out' from (0, 0) by small increment δL_1 (Figure 5), arriving at coordinates:

$$(x_1, y_1) = (\delta L_1, 0)$$ **Equation 2**

Where y_n is the y coordinate of the nth increment, define z_n by:

$$z_n = z - y_n$$ **Equation 3**

Define a small angle $\delta\theta_n$ by:

$$\delta\theta_n = kz_n$$ **Equation 4**

Proceed a further small increment δL_2, at $\delta\theta_1$ to δL_1, arriving at coordinates:

$$(x_2, y_2) = (\delta L_1 + \delta L_2 \cos(\delta\theta_1), \delta L_2 \sin(\delta\theta_1))$$ **Equation 5**

Proceed a further small increment δL_3, at $\delta\theta_2$ to δL_2, arriving at coordinates:

$$(x_3, y_3) = \begin{pmatrix} \delta L_1 + \delta L_2 \cos(\delta\theta_1) + \delta L_3 \cos(\delta\theta_1 + \delta\theta_2), \\ \delta L_2 \sin(\delta\theta_1) + \delta L_3 \sin(\delta\theta_1 + \delta\theta_2) \end{pmatrix}$$ **Equation 6**

Thus, for the nth small increment δL_n, at $\delta\theta_{n-1}$ to δL_{n-1}, arrive at coordinates:

$$(x_n, y_n) = \left(\sum_{i=1}^{n} \delta L_i \cos\left(\sum_{j=1}^{i-1} \delta\theta_j \right), \sum_{i=2}^{n} \delta L_i \sin\left(\sum_{j=1}^{i-1} \delta\theta_j \right) \right)$$ **Equation 7**

Choosing a constant value for δL, the expression can be further simplified to:

$$(x_n, y_n) = \left(\delta L \sum_{i=1}^{n} \cos\left(\sum_{j=1}^{i-1} \delta\theta_j \right), \delta L \sum_{i=2}^{n} \sin\left(\sum_{j=1}^{i-1} \delta\theta_j \right) \right)$$ **Equation 8**

The summations to determine these coordinates are easily tabulated or programmed, and an iterative 'solver' or similar algorithm can then be applied to converge on a value of k such that a set of coordinates $(x_n, y_n) \approx (b/2, d)$ is obtained to a desired accuracy.

Small angle theory is not used for determining coordinates as a global coordinate system is adopted and coordinates at each step determined directly in reference to this. The progressive summation of small angles would quickly render assumptions about the behaviour of small angles inappropriate. Alternatively, a local coordinate system might have been adopted that 'walks out' with the fabric. For the purposes of the original research it was considered more convenient to refer directly to the global coordinate system.

Previous work (Bailiss 2006; Iosilevskii 2010) considered a fabric container where the free surface of the fluid is level with the opening i.e., $z = d$. An advantage of the procedure detailed above is that it applies for the over-full case where the fluid is placed under an additional surcharge pressure expressed in head, i.e., $z > d$. This is a useful result because this is the condition that occurs in the bottom flange of a fabric-formed I or T-beam, where stiffened web formers are used to create a more efficient web-profile.

Further, the procedure is also readily adapted to the profile of a partially-full container i.e., $z < d$. It is only necessary to set curvatures to zero once $y_n \geq z$ i.e., for $y_n \geq z$: $\delta\theta_n = 0$. In terms of modelling, the profile is allowed to "walk out" in a straight line above the level of the

fluid. This is understood physically by the absence of further out of plane applied loading above the level of the fluid. This is also a potentially useful result for the design of supporting falsework because the partially-full case is the condition that occurs during the casting process.
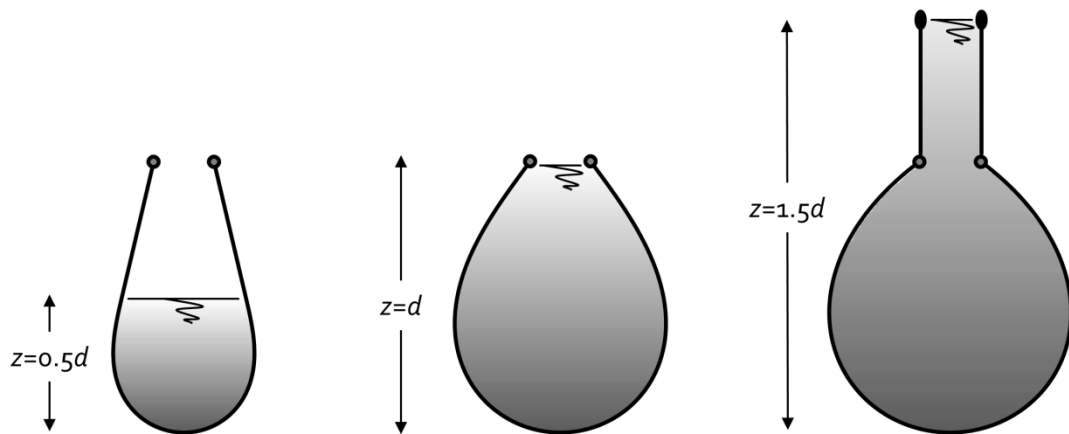

Figure 6 - Part-full; full; and overfull (surcharged) fabric profiles for constant *b* and *d*

6. Analysis

Various non-dimensional values for the principal parameters for the *b*, *d*, *z* and *p* can be calculated and their values plotted. The resulting family of curves define a notional surface on which any simple hydraulic sectional profile can be located (Figure 7).

The surface provides a scheme in which to relate the key dimensions of the possible simple hydrostatic geometries. The surface also illustrates the relation between some analytically determined boundary conditions and the numerically obtained curves.

Notably:

- Regardless of the relation *z*/*d*:

  - When $d/p \to 0$: $b/p \to 1$ because the fabric is flat.

- As $z/d \rightarrow 0$:

  - When $b/p \rightarrow 0$: $d/p \rightarrow 1/2$ because the fabric is folded in half vertically.

- As $z/d \rightarrow \infty$:

  - The form of the *lintearia* tends to that of the *valeria* (*c.f.* Figure 3)

  - When $b/p \rightarrow 0$: $d/p \rightarrow 1/\pi$ because the bulb is circular in section.
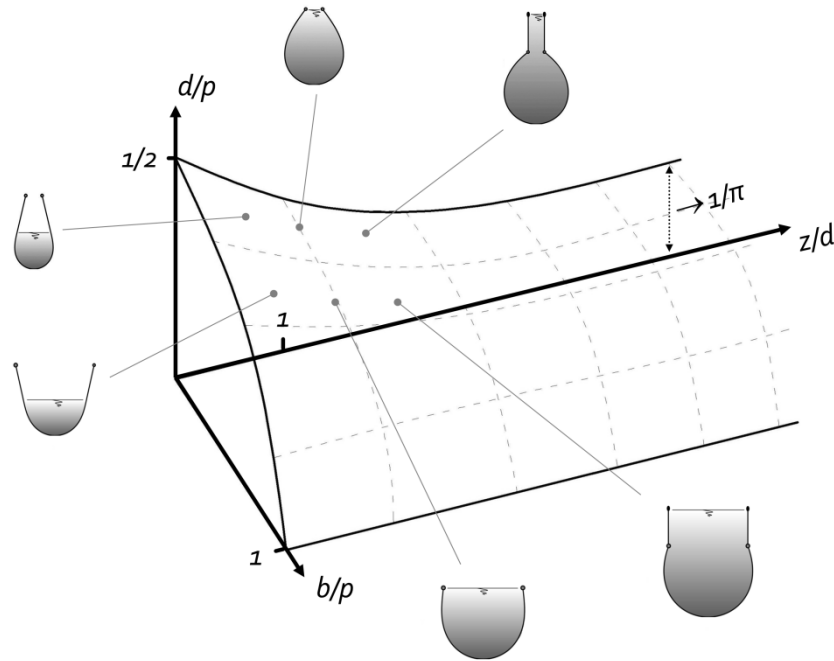

Figure 7: Sketch of non-dimensional surface and some indicative physical interpretations.

7. Validation

Figure 8 compares the results of the numerical procedure described above with Bailiss'

(2006) empirical equation for some normalised characteristic dimensions of an arbitrary

section; and with Iosilevskii's (2010) analytical results obtained by the evaluation of elliptic

integrals. It is apparent that there is good agreement with Bailiss' physical models and very

good agreement with the analytical results. Bailiss' (2006) best-fit approximations can thus

be revisited and their accuracy improved. However, this approach has little further value in

the sense that detailed profiling rather than estimations of characteristic dimensions is

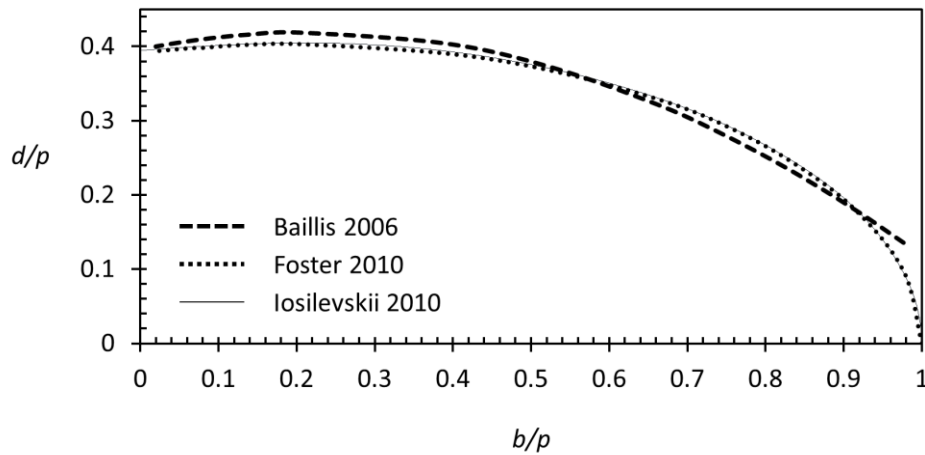required for further progress in optimised design (Garbett 2008).

Figure 8: Comparison of Bailiss' (2006) non-dimensional relation obtained experimentally, that obtained by 'walking out' [6] and Iosilevskii's (2010) plot of opening breadth against depth normalised by perimeter.

Empirical validation is more challenging, as records of as-built versus as-designed fabric-

formed sections have proved difficult to obtain. However, it has been possible to graphically

compare the profiles against the hydrostatically loaded profiles obtained experimentally by

Bailiss (2006). Bailiss physical model consisted of a fabric strip, restrained in plane but free

to deform out of plane. The strip was then hydrostatically loaded with a viscous liquid.

Various opening breadths of opening were modelled. Figure compares the profiles of Bailiss'

(2006) physical models with those predicted by the numerical method based on opening

breadth and hanging depth as indicated by the images. Although the initial information is less

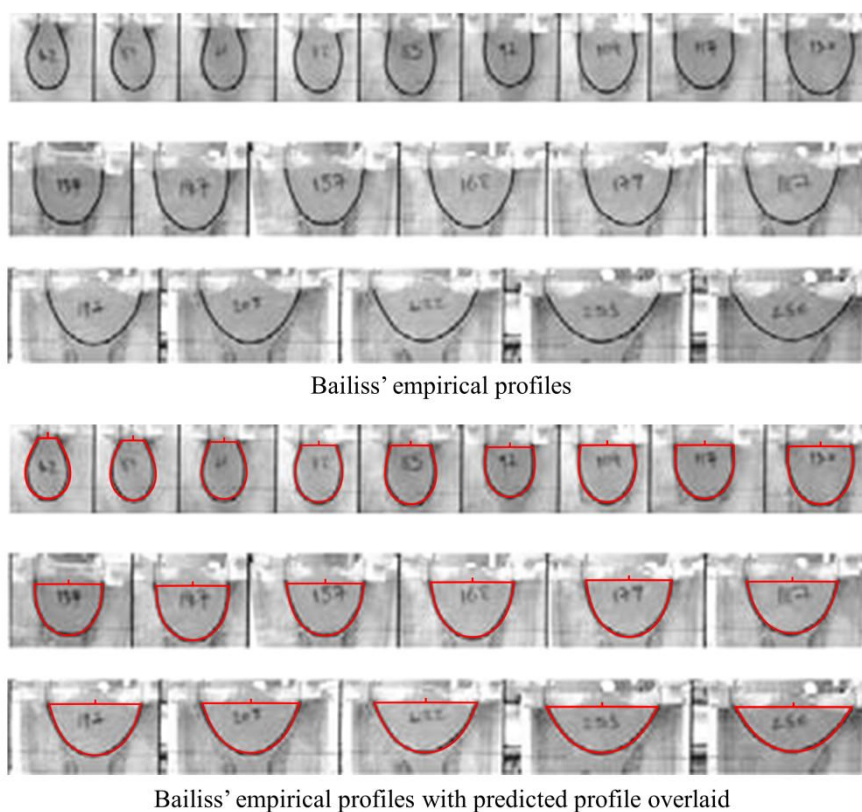than ideal, the predicted profiles are a good fit.

Bailiss' empirical profiles

Bailiss' empirical profiles with predicted profile overlaid

Figure 9 - Comparison with the profiles of Bailiss' physical models (data from Bailiss 2006). Predicted profile in red.

8. Application to fabric-formed concrete

The numerical procedure proposed can be applied to generate a range of example profiles, enabling the production of simple design charts. The particular advantage of such charts is that they are accessible to non-engineering professionals and might therefore facilitate conceptual design and initial form generation within interdisciplinary design teams. Charts can be generated to suit a variety of design situations. A useful example, expressing breadth and fabric perimeter in relation to section depth is shown in Figure 10.
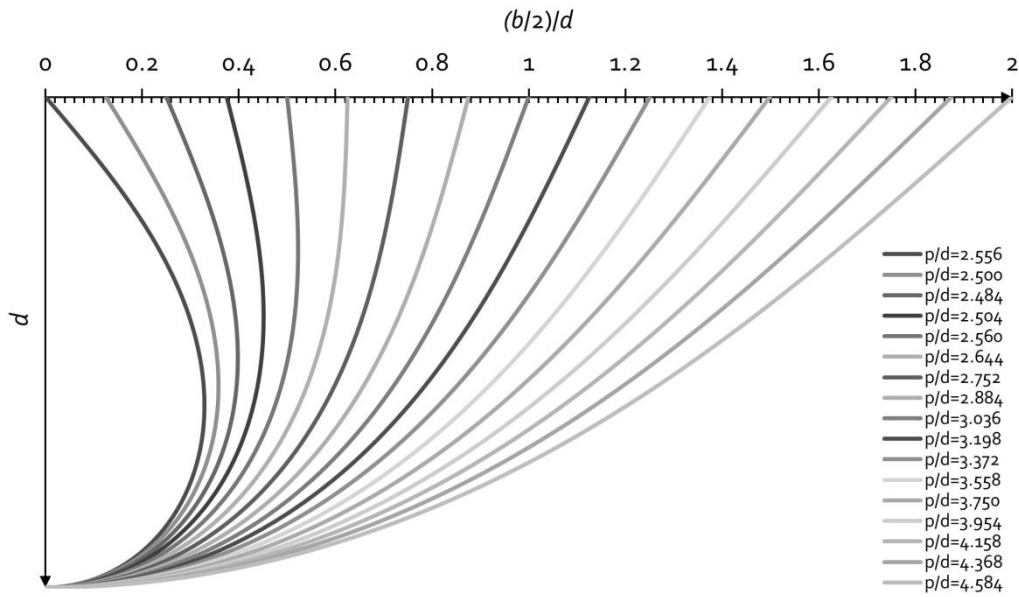
Figure 10: Design chart for hydrostatic profiles expressing *b* and *p* in relation to *d* for the typical case *z=d*

The approach detailed above may also be applied to non-prismatic elements by discretizing the longitudinal profile into a series of sectional profiles. As an example, Figure 11 illustrates the generation of sectional profiles for non-prismatic beams fabricated by Garbett (2008), the first using free hanging fabric, the second using stiff web formers to force a more slender web geometry through the mid-span. While Garbett (2008) was only able to obtain indicative profiles based on Bailiss' (2006) equations for characteristic dimensions, detailed sectional profiles have been obtained here by the numerical method. Note that the profile of the bulb at the base of the T-section is obtained as for the over-full case.
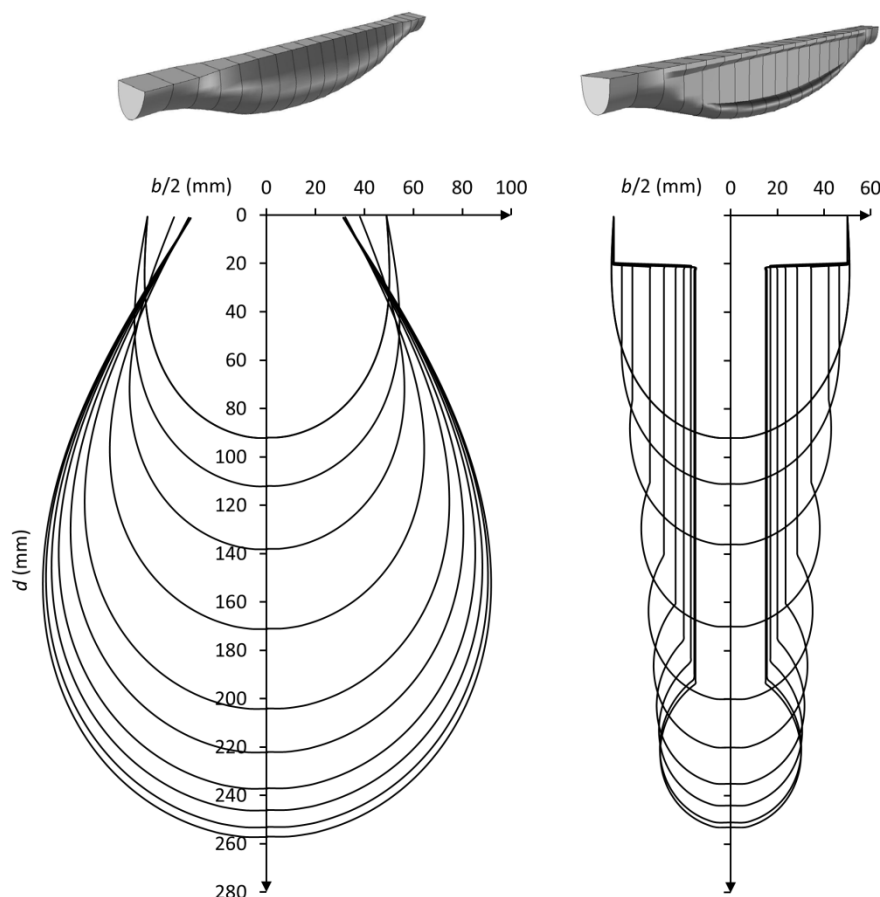
Figure 11 – Detailed sectional profiles generated by the method presented here, for beams fabricated by Garbett (2008). Original isometric approximations (top) courtesy of Joe Garbett.

The number of sections required to suitably model a non-prismatic element will depend on the longitudinal curvatures and the fitness of any interpolation functions used to join the sections. However, longitudinal curvatures in a fabric-formed beam are typically small compared with the transverse curvatures meaning that the number of sections need not be great. Orr (2012) has successfully applied this sectional method to the generation of optimized non-prismatic fabric-formed concrete beams. More complex three-dimensional analysis for fabric formed concrete has been carried out combining dynamic relaxation with finite element analysis by Veenendaal (2008).

A further practical benefit of the approach is that if 'real' values of ρg are used, then the tension T_0 in the fabric at a section is known, along with the profile of the fabric. This allows the vertical and horizontal forces F_x and F_y exerted by the hanging fabric formwork to be determined (Figure 12), allowing the supporting structure to be designed appropriately.

$$F_x = \frac{T_0(x_n - x_{n-1})}{\delta L}$$ **Equation 9**

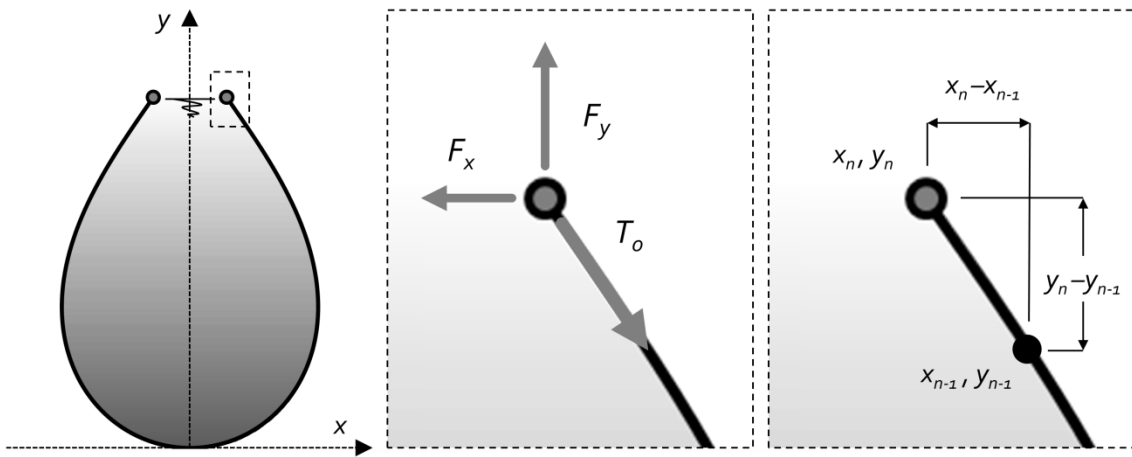$$F_y = \frac{T_0(y_n - y_{n-1})}{\delta L}$$ **Equation 10**

Figure 12 Forces F_x and F_y at fabric supports can be determined from T_0

A limitation of the presented method is the assumption of inextensibility of the fabric formwork. Real fabrics are not inextensible and axial deformation of the fabric will increase the length of the hanging perimeter p and hence the cross-sectional profile will deviate from that predicted. However, for the sizes of structure normally considered, the extension of the geotextile fabrics typically used can in most cases be treated as negligible. The expected extension of the fabric can be verified approximately since T_0 for an inextensible fabric is in any case known. The assumption of inextensibility also limits the application of this method to tensioned fabrics.

9. Conclusion

A numerical method has been presented for determining the form of flexible, impermeable and inextensible hanging fabric subject to hydrostatic load. This method is shown to predict the profiles of part-filled, filled and over-filled. i.e. surcharged, and flat-bottomed fabrics, such as those used for moulding fabric-formed concrete beams. The method shows good agreement against analytical and empirical results reported in the literature, thereby allowing engineers and architects to accurately predict the geometry of fabric-formed concrete structures.

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