Delay-independent incremental stability in time-varying monotone systems satisfying a generalized condition of two-sided scalability

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Abstract

Monotone systems generated by delay differential equations with explicit time-variation are of importance in the modeling of a number of significant practical problems, including the analysis of communications systems, population dynamics, and consensus protocols. In such problems, it is often of importance to be able to guarantee delay-independent incremental asymptotic stability, whereby all solutions converge toward each other asymptotically, thus allowing the asymptotic properties of all trajectories of the system to be determined by simply studying those of some particular convenient solution. It is known that the classical notion of quasimonotonicity renders time-delayed systems monotone. However, this is not sufficient alone to obtain such guarantees. In this work we show that by combining quasimonotonicity with a condition of scalability motivated by wireless networks, it is possible to guarantee incremental asymptotic stability for a general class of systems that includes a variety of interesting examples. Furthermore, we obtain as a corollary a result of guaranteed convergence of all solutions to a quantifiable invariant set, enabling time-invariant asymptotic bounds to be obtained for the trajectories even if the precise values of time-varying parameters are unknown.

Key words: Monotone systems; asymptotic stability; time-delay; time-varying systems; nonlinear systems

1 Introduction

Monotone systems represent an important class of dynamical systems that are of interest both for their applicability to a number of practical problems and for their rich mathematical structure. The order-preserving structure of these systems allows strong results about their stability properties to be obtained. In the celebrated work [17], Hirsch established results of generic convergence, guaranteeing convergence of almost every bounded solution of any time-invariant system for which the monotonicity property holds strongly to the equilibrium set, provided this set is nonempty. In systems of differential equations that are not autonomous, however, the equilibrium set of even a monotone system will frequently be empty, so the generic convergence results are not directly applicable. Instead, a property that is often of interest is the concept of incremental asymptotic stability. The system is said to be incrementally asymptotically stable, in some specified set of initial conditions, if all solutions starting within this set converge to each other uniformly, and so this idea is of particular use in problems of system tracking or prediction. This concept, proposed in [3] and often referred to simply as incremental stability, was seen in [29] to be closely linked to the notions of convergence and contraction studied in [7] and [24] respectively, and has been investigated in numerous interesting works such as [13, 14, 4, 26, 20, 11]. Two recent papers in which incremental asymptotic stability in general nonlinear systems with time-delays has been studied are [27, 6]. [27] presented a Lyapunov–Krasovskii framework for verifying incremental asymptotic stability in delayed systems, while [6] proposed an incremental formulation of Lyapunov–Razumikhin approaches and used this to formulate sufficient algebraic conditions for incremental asymptotic stability in Lur'e systems.

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Monotone systems have been extensively studied in the context of a variety of applications, including the motivating problem of population modeling [16], the analysis of biological systems [19], and the notion of antagonistic consensus [2]. One particular area in which they have received significant attention is in the control of antenna uplink powers in wireless networks. This follows from the seminal paper [32], in which it was shown that a general class of such power control algorithms can be modeled in discrete-time by a general monotone system satisfying a condition of scalability. Stability of a continuous-time version of this framework with time-varying delays was considered in [23]. Stability issues in general classes of delayed autonomous monotone systems have been addressed in the literature, e.g. in [30, 5, 10]. It was shown in [9] that when explicit time-variation is incorporated within the class of systems considered in [23], versions of the monotonicity and scalability properties introduced in [32] can be sufficient to guarantee that all trajectories converge to one another, independent of arbitrary bounded time-varying delays. However, the notion of monotonicity used in the analysis is significantly stronger than the classical property of quasimonotonicity that is needed to specify a general monotone system of delay differential equations, thus restricting its applicability to wider classes of systems. As such, it is desirable to investigate whether these approaches can be generalized to yield conclusions guaranteeing delay-independent incremental asymptotic stability in general monotone systems under a condition inspired by the property of scalability.

Within this paper we will demonstrate that this is possible, proving results guaranteeing incremental asymptotic stability for a broad class of time-varying nonlinear systems encompassing a range of interesting practical examples. Moreover, we shall in fact see that, similarly to the extensions provided to the wireless network analysis in [31], this framework can be further generalized by combining the quasimonotonicity and scalability conditions into a single, weaker property of uniform two-sided quasiscalability, thereby yielding conclusions of delay-independent incremental asymptotic stability even for classes of systems that may not be monotone. As a corollary, we also show that monotonicity properties can be exploited in this context to deduce convergence of all solutions to a bounded invariant set, thus allowing time-invariant asymptotic bounds for the trajectories to be obtained.

The paper is structured as follows. We begin in Section 2 by reviewing some of the theory concerning stability and convergence of systems of delay differential equations. We present in Section 3.1 the general framework of strictly positive monotone systems of delay differential equations within which our analysis will take place. In Section 3.2, we formulate an assumption of uniform two-sided quasiscalability, and we show that this condition alone can be used to deduce incremental asymptotic stability whenever the system admits a solution satisfying a particular boundedness condition. Section 3.3 then presents a constraint on the system's explicit time-variation that allows us to guarantee the existence of a bounded invariant set. These results are then combined, in Section 3.4, to yield our main result of guaranteed delay-independent incremental asymptotic stability for systems for which both of the foregoing assumptions are satisfied. Various applications are given in Section 4, and finally conclusions are drawn in Section 5.

2 Preliminaries

2.1 Notation

Within the paper, we will use $\mathbb{R}_+$ to denote the set of nonnegative real numbers $\{s \in \mathbb{R} \colon s \geq 0\}$ and $\mathring{\mathbb{R}}_+$ to denote the set of positive reals $\{s \in \mathbb{R} \colon s > 0\}$. In the preliminaries, the notation $\|\cdot\|$ can represent any norm on $\mathbb{R}^n$, however, within our analysis we will work mainly with the infinity norm, denoted by $\|\cdot\|_\infty$. Inequalities in $\mathbb{R}^n$ are defined as follows: $x \geq y$ means $x_i \geq y_i$ for all i, $x > y$ means $x_i \geq y_i$ for all i and $x \neq y$, and $x \gg y$ means $x_i > y_i$ for all i. We will use $\mathcal{C}([a,b],\Omega)$ to denote the Banach space of continuous functions mapping $[a,b] \subseteq \mathbb{R}$ into $\Omega \subseteq \mathbb{R}^n$, with elemental norm $\|\phi\|_{\mathcal{C}} = \sup_{a\leq\theta\leq b} \|\phi(\theta)\|$. Inequalities in $\mathcal{C}([a,b],\Omega)$ are treated pointwise, e.g. $\phi \geq \psi$ means $\phi(\theta) \geq \psi(\theta)$ for all $\theta \in [a,b]$.

2.2 Background theory

We wish to investigate the long-term behavior of solutions of a general class of nonautonomous systems of differential equations with arbitrary bounded time-delays. A detailed study of the theory of such systems can be found in [15]. Following this framework, if $x \in \mathcal{C}([t_0 - r, t_0 + A], \Omega)$ for a given $t_0 \geq 0$ and $A > 0$, we define the segment $x_t \in \mathcal{C}([-r,0],\Omega)$ of x as $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-r, 0]$, for any $t \in [t_0, t_0 + A]$. The general delay differential equation can then be written as

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $f : \mathbb{R}_+ \times \mathcal{C}([-r,0],\Omega) \to \mathbb{R}^n$ is assumed to be continuous in its first argument and to satisfy a local Lipschitz property, uniformly in t, in its second argument [1], which ensure existence and uniqueness of solutions and their continuous dependence on the initial data ([15], Theorems 2.2.1, 2.2.2, and 2.2.3). The function x is then said to be a solution of (1) through (t_0, ϕ) for the initial condition $\phi \in \mathcal{C}([-r,0],\Omega)$ if $x(t)$ satisfies (1) for all $t \geq t_0$ and $x_{t_0} = \phi$ on $[-r, 0]$. When explicit consideration of the initial conditions is required, this solution can be denoted by $x(t, t_0, \phi)$, with the corresponding delayed segment written as $x_t(t_0, \phi)$.

[1] By this we mean that for any compact subset $\Phi \subseteq \Omega$, there exists a constant $L \geq 0$ such that $\|f(t,\phi) - f(t,\psi)\| \leq L\|\phi - \psi\|_{\mathcal{C}}$ for all $\phi, \psi \in \mathcal{C}([-r,0],\Phi)$ and all $t \geq 0$.

In this paper, we wish to investigate when it is possible to guarantee certain properties of stability and convergence for the system (1). We now define a standard class of comparison functions in terms of which the relevant notions can be formulated. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if

- For each fixed $\tilde{s} \geq 0$, $\beta(0, \tilde{s}) = 0$ and $\beta(s, \tilde{s})$ is strictly increasing in s,
- For each fixed $s > 0$, $\beta(s, \tilde{s})$ is decreasing in $\tilde{s}$ and $\lim_{\tilde{s}\to\infty} \beta(s, \tilde{s}) = 0$.

We first introduce a notion of uniform asymptotic stability with respect to a particular trajectory of (1).

Definition 1 The solution $X(t)$ of (1) is uniformly asymptotically stable in a set of initial conditions $S \subseteq \mathcal{C}([-r, 0], \Omega)$ if there exists a function $\beta \in \mathcal{KL}$ such that for all $t_0 \geq 0$, all $\phi \in S$, and all $t \geq t_0$,

$$\|x_t(t_0, \phi) - X_t\|_{\mathcal{C}} \leq \beta\left(\|\phi - X_{t_0}\|_{\mathcal{C}}, t - t_0\right).$$

It is globally uniformly asymptotically stable if the set S can be chosen as the entire space $\mathcal{C}([-r, 0], \Omega)$.

Our second convergence definition represents a uniform asymptotic stability property for the increment between arbitrary solutions of (1) and is generally a stronger requirement than Definition 1.

Definition 2 The system (1) is incrementally asymptotically stable in a set of initial conditions $S \subseteq \mathcal{C}([-r, 0], \Omega)$ if there exists a function $\beta \in \mathcal{KL}$ such that for all $t_0 \geq 0$, all $\phi, \psi \in S$, and all $t \geq t_0$,

$$\|x_t(t_0, \phi) - x_t(t_0, \psi)\|_{\mathcal{C}} \leq \beta\left(\|\phi - \psi\|_{\mathcal{C}}, t - t_0\right).$$

Definitions 1 and 2 encode two distinct stability notions for the system (1): whereas Definition 1 requires uniform convergence of any arbitrary solution of (1) to the particular trajectory $X(t)$, Definition 2 requires uniform convergence to zero of the increment between any arbitrary pair of solutions. A similar comparison was discussed in detail in [29]. The advantage of considering these uniform notions of asymptotic stability is that they will allow us to obtain conclusions that both guarantee convergence and specify that this convergence occurs at a uniform rate with transients that can be uniformly bounded across the allowed space of initial conditions. In the context of applications, the motivation behind considering incremental asymptotic stability is that it allows uniform convergence of all solutions to *any* choice of trajectory that is convenient to analyze to be guaranteed without needing to specify this particular trajectory a priori.

To analyze stability properties of functional differential equations of the form (1), [28] considered the functional $\bar{V}(t, x_t) = \sup_{-r\leq\theta\leq 0} V(t+\theta, x(t+\theta))$ and derived conditions on the function V such that $\bar{V}$ is decreasing along system trajectories. This involves the time-derivative along the system trajectory $x(t+h, t, \phi)$, defined as $\dot{V}(t, \phi(0)) = \limsup_{h\to 0+} \frac{1}{h}\{V(t+h, x(t+h, t, \phi)) - V(t, \phi(0))\}$.

Theorem 3 (Razumikhin Theorem) *Let $x = 0$ be a solution of (1) with $\Omega = \mathbb{R}^n$. Suppose that $f : \mathbb{R}_+ \times \mathcal{C}([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ in (1) takes $\mathbb{R}_+\times$ bounded sets in $\mathcal{C}([-r, 0], \mathbb{R}^n)$ into bounded sets in $\mathbb{R}^n$, and q, u, v, $w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, nondecreasing functions with $q(s) > s$ and $u(s)$, $v(s)$, $w(s) > 0$ for all $s > 0$, $u(0) = v(0) = 0$, $u(s) \to \infty$ as $s \to \infty$, and v strictly increasing. Suppose further that there exists a continuous function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ such that:*

(i) $u(\|x\|) \leq V(t, x) \leq v(\|x\|)$, $\forall\, t \in \mathbb{R}_+$, $\forall\, x \in \mathbb{R}^n$,
(ii) $\dot{V}(t, x(t)) \leq -w(\|x(t)\|)$ *if* $V(t+\theta, x(t+\theta)) \leq q(V(t, x(t)))$ *for all* $\theta \in [-r, 0]$, *where* $x(t)$ *is any trajectory of* (1).

Then $x = 0$ is globally uniformly asymptotically stable.

This theorem will enable us to guarantee global uniform asymptotic stability of the origin for a transformed system, upon which inversion of the coordinate transformation can yield stability results for the original system.

3 Stability and convergence analysis

3.1 Problem formulation

We wish to investigate in this section the stability properties of a general class of delay differential equation system of the form (1). As discussed in Section 1, it is of interest and relevance to consider systems that satisfy notions of monotonicity and positivity, which we now define and assume to hold for the system (1).

We will assume that the nonlinearity within the system (1) satisfies the property of quasimonotonicity ([21, 25, 18]), which states that

$$f_i(t, \phi) \geq f_i(t, \psi) \text{ whenever } \phi \geq \psi \text{ and } \phi_i(0) = \psi_i(0) \quad (2)$$

for all $i \in \{1, \ldots, n\}$, all $t \geq 0$, and all ϕ, $\psi \in \mathcal{C}([-r, 0], \Omega)$. Under this condition, it holds that $x_{t_0} \geq \tilde{x}_{t_0}$ implies $x(t) \geq \tilde{x}(t)$ for all $t \geq t_0$ for arbitrary solutions $x(t)$ and $\tilde{x}(t)$, meaning that (1) defines a monotone system.

Our second assumption is that (1) defines a strictly positive system, in the sense that the open convex cone $\mathring{\mathbb{R}}^n_+ = \{x \in \mathbb{R} \colon x \gg 0\}$ is positively invariant with respect to (1). We can then restrict our attention to the domain $\Omega = \mathring{\mathbb{R}}^n_+$. Due to the quasimonotonicity property (2), strict positivity will immediately follow for any system if $f(t, 0) \gg 0$ for all $t \geq 0$, which gives an easy check for verifying [2] the positivity requirement.

[2] The condition $f(t, 0) \gg 0$ is only sufficient for strict positivity, so failure of this condition to hold does not preclude the system from being strictly positive.

There exists an extensive theory detailing stability and convergence results for monotone systems that are time-invariant, as described in [30, 18]. Using these results, we can immediately see that in cases where (1) has no explicit time-variation, almost all bounded solutions are guaranteed to converge to the set of equilibria of the system. However, in the more general setting in which the system (1) depends explicitly upon time, the analysis becomes more involved and these results fail to hold. In particular, boundedness of trajectories can be violated, so it will be important to establish properties that can guarantee the existence of bounded trajectories. Furthermore, the limiting behavior will now in general not simply consist of equilibrium points, but rather can have a more general time-varying form. Consequently, the convergence definitions introduced in Definitions 1 and 2 will be of use in the stability analysis in the more general time-varying setting. Our aim within this section is to introduce conditions motivated by practical problems studied in [32, 31] and to establish how these conditions can be used to deduce useful results about the convergence properties of the time-varying system (1).

3.2 Incremental asymptotic stability

In order to guarantee incremental asymptotic stability, it is necessary to endow the system with a property that will ensure contractiveness between trajectories. In the context of discrete-time wireless network power control algorithms, [32] ensured such contractiveness by considering a class of interference functions I satisfying the scalability property $I(\alpha\phi) \ll \alpha I(\phi)$ for all $\alpha > 1$ in addition to the discrete-time monotonicity property $I(\phi) \leq I(\psi)$ for all $\phi \leq \psi$. The analysis in [31] showed that these properties can be combined into the single, weaker condition of two-sided scalability

$$\tfrac{1}{\alpha} I(\phi) \ll I(\psi) \ll \alpha I(\phi) \text{ whenever } \tfrac{1}{\alpha}\phi \leq \psi \leq \alpha\phi. \quad (3)$$

However, when these frameworks are generalized to continuous-time, as in [23, 9], the resulting systems satisfy monotonicity properties that are overly restrictive. Instead, in order to investigate incremental asymptotic stability in general continuous-time monotone systems, we aim here to relax these monotonicity properties to require only the classical condition of quasimonotonicity as stated in (2). To achieve this, for the time-invariant case let us suppose that the nonlinearity in (1) satisfies the scalability property $f(\alpha\phi) \ll \alpha f(\phi)$ for all $\alpha > 1$ in addition to quasimonotonicity (2). Motivated by the form of (3), we then suppose that $\frac{1}{\alpha}\phi \leq \psi \leq \alpha\phi$, whence applying the quasimonotonicity and scalability conditions in each case gives:

- If $\psi_i(0) = \frac{1}{\alpha}\phi_i(0)$, then $f_i(\phi) < \alpha f_i(\psi)$,
- If $\psi_i(0) = \alpha\phi_i(0)$, then $f_i(\psi) < \alpha f_i(\phi)$.

Consequently, we obtain a condition in which the inequality in (3) is split based upon the endpoint of the interval $[\frac{1}{\alpha}\phi_i(0), \alpha\phi_i(0)]$ at which the value $\psi_i(0)$ is situated. Finally, in order to incorporate the time-variation present within our system model (1), we include uniformity within each of the inequalities. Thus, we arrive at our main assumption about the nonlinearity f in system (1), which as stated holds for all $i \in \{1, \dots, n\}$ and all $t \geq 0$, and in terms of some given $\phi \in \mathcal{C}([-r, 0], \mathring{\mathbb{R}}^n_+)$:

(A) *Uniform two-sided quasiscalability:* For any $\alpha > 1$, if $\frac{1}{\alpha}\phi \leq \psi \leq \alpha\phi$, then
  (a) $f_i(t, \phi) \leq \alpha f_i(t, \psi) - \delta(\alpha)$ whenever $\frac{1}{\alpha}\phi_i(0) \leq \psi_i(0) \leq \frac{1}{\alpha - \eta(\alpha)}\phi_i(0)$,
  (b) $f_i(t, \psi) \leq \alpha f_i(t, \phi) - \delta(\alpha)$ whenever $(\alpha - \eta(\alpha))\phi_i(0) \leq \psi_i(0) \leq \alpha\phi_i(0)$,
  where $\delta, \eta : (1, \infty) \to \mathring{\mathbb{R}}_+$ are both continuous and nondecreasing.

Remark 4 It follows from how we arrived at this formulation that assumption (A) represents a generalization of the combination of quasimonotonicity with a uniform scalability condition, meaning in particular that systems of the form (1) that satisfy these properties fit within the framework considered here. Condition (A) also includes the more restrictive assumptions considered in our preliminary conference paper [8] as a special case. It is important to note that quasimonotonicity, uniform scalability, and two-sided quasiscalability are all pointwise conditions on the time-varying nonlinearity f that can be checked without requiring any knowledge of the solutions of the system. As such, they can be each be verified a priori by considering properties of the nonlinearity. Furthermore, they are all distributed conditions, meaning that they can be verified independently for each component. As it will be seen in Section 4, in many practical applications these conditions arise naturally from modeling assumptions, even when precise values of the system parameters are unknown.

Remark 5 The scalability property used in the formulation of assumption (A) can be thought of as a strict version of subhomogeneity. Without this strictness in the inequalities, the condition would not in general be able to preclude the existence of multiple limiting behaviors in time-varying systems of the form (1), so incremental asymptotic stability would not necessarily follow.

Remark 6 If the system's continuity properties are strengthened by requiring that the nonlinearity f be uniformly continuous, then the uniformization function η in condition (A) is not required. However, the assumption of uniform continuity can be restrictive, and so we choose to state the condition in the form given in (A) in order to derive our results in a more general setting. Indeed, Example 2 in Section 4 illustrates a situation in which the system nonlinearity is not uniformly continuous and thus this more general setup is necessary to prove incremental asymptotic stability.

Using only the condition (A), we are now able to prove, by means of a logarithmic coordinate change and Lyapunov–Razumikhin analysis, a result of guaranteed incremental asymptotic stability for (1) assuming the existence of some bounded solution.

Theorem 7 *Suppose that* (1) *admits some solution* $X(t)$ *for which there exist scalar constants* $a, A, M > 0$ *such that*

- $a \leq X(t) \leq A$,
- $-M \leq f(t, X_t) \leq M$,
- f *satisfies assumption* (A) *with* $\phi = X_t$,

all hold for all $t \geq 0$. *Then* (1) *is incrementally asymptotically stable in the set of initial conditions* $S = \mathcal{C}([-r, 0], \Xi)$ *for any compact set* $\Xi \subset \mathring{\mathbb{R}}^n_+$.

Proof. Due to the strict positivity of (1), we know that given any $x_{t_0} \gg 0$, we must have $x(t)$, $X(t) \gg 0$ for all $t \geq t_0$. Consequently, we may apply to (1) the transformation $y_i = \log(\frac{x_i}{X_i})$, giving the transformed system

$$\frac{dy_i}{dt} = \frac{1}{X_i e^{y_i}}\Big[f_i(t, \mathrm{diag}(e^{(y_t)_j})X_t) - e^{y_i} f_i(t, X_t)\Big] \qquad (4)$$

with state space $\mathbb{R}^n$, where $\mathrm{diag}(e^{\phi_j})$ is used to denote the function taking as value the $n \times n$ diagonal matrix with diagonal entries $e^{\phi_1(\theta)}, e^{\phi_2(\theta)}, \ldots, e^{\phi_n(\theta)}$ at each $\theta \in [-r, 0]$. Observe that the boundedness of $f(t, X_t)$ and the fact that the local Lipschitz property for $f(t, \cdot)$ holds uniformly with resepct to t imply that right-hand side in (4) maps $\mathbb{R}_+ \times$ bounded sets in $\mathcal{C}([-r, 0], \mathbb{R}^n)$ into bounded sets in $\mathbb{R}^n$. Furthermore, the particular solution $X(t)$ of (1) is transformed to the equilibrium $y = 0$ of (4). Thus, this coordinate change will allow us to investigate the stability properties of the solution $X(t)$ for (1) by using Theorem 3 to analyze those of the origin for (4).

The proof now proceeds in three steps:

Step 1: We first use Theorem 3 to establish global uniform asymptotic stability of $y = 0$ for (4).

Let us define, for the system (4), the candidate Lyapunov–Razumikhin function $V(y) = e^{\max_i |y_i|} - 1$, which is continuous and satisfies condition (i) of Theorem 3 with $u(s) = v(s) = e^s - 1$ by virtue of the fact that $\max_i |y_i| = \|y\|_\infty$. Let I denote any component choice such that $|y_I| = \max_i |y_i|$.

In order to satisfy condition (ii) of Theorem 3, let us suppose that $q(V(y(t))) \geq \sup_{-r \leq \theta \leq 0} V(y(t + \theta))$ for q defined as $q(s) = e^{\tilde{q}(\log(1+s))} - 1$, in terms of some $\tilde{q}$ which is defined on $\mathbb{R}_+$. This implies that, for all j,

$$\tilde{q}(|y_{I(t)}(t)|) \geq \sup_{-r \leq \theta \leq 0} |y_j(t + \theta)|\,. \qquad (5)$$

We will show that it is possible to choose $\tilde{q}$ such that all of the properties required in Theorem 3 are satisfied.

For the value of the time-derivative of V along the system trajectories, there are three significant cases [3] , namely

$$\dot{V}(y(t)) = \begin{cases} -e^{-y_{I(t)}(t)}\dot{y}_{I(t)}(t) & \text{if } y_{I(t)}(t) < 0, \\ e^{y_{I(t)}(t)}\dot{y}_{I(t)}(t) & \text{if } y_{I(t)}(t) > 0, \\ 0 \quad \text{if } y_{I(t)}(t + \theta) = 0,\ \forall\, \theta\ \in [-r, 0], \end{cases} \qquad (6)$$

The third case is trivial, leaving two important cases:

(i) $y_I < 0$, whence $e^{\tilde{q}(-y_I)} > 1$ and (5) gives $e^{-\tilde{q}(-y_I)}X_t \leq \mathrm{diag}(e^{(y_t)_j})X_t \leq e^{\tilde{q}(-y_I)}X_t$. Therefore, the uniform two-sided quasiscalability assumption (Aa) with $\alpha = e^{\tilde{q}(-y_I)}$ implies that

$$f_I(t, \mathrm{diag}(e^{(y_t)_j})X_t) \geq e^{-\tilde{q}(-y_I)} f(t, X_t) + e^{-\tilde{q}(-y_I)}\delta(e^{\tilde{q}(-y_I)}), \qquad (7)$$

whenever $e^{-\tilde{q}(-y_I)} \leq e^{y_I} \leq \frac{1}{e^{\tilde{q}(-y_I)} - \eta(e^{\tilde{q}(-y_I)})}$. Due to (5) and the property that η is nondecreasing, this necessarily holds in particular whenever

$$e^{-y_I} \leq e^{\tilde{q}(-y_I)} \leq e^{-y_I} + \eta(e^{-y_I}). \qquad (8)$$

Substituting (7) into (4), rearranging, and invoking (5) and the nondecreasing nature of δ then gives

$$\frac{dy_I}{dt} \geq \frac{1}{X_I}\big[(e^{-\tilde{q}(-y_I) - y_I} - 1) f_I(t, X_t) + e^{-\tilde{q}(-y_I) - y_I}\delta(e^{-y_I})\big]. \qquad (9)$$

The above shows that, whenever it holds that

$$e^{-y_I} \leq e^{\tilde{q}(-y_I)} \leq e^{-y_I} + \frac{1}{2M}\delta(e^{-y_I}) \qquad (10)$$

and

$$e^{-y_I} \leq e^{\tilde{q}(-y_I)} \leq e^{-2y_I}, \qquad (11)$$

then (9) yields $\frac{dy_I}{dt} \geq \frac{e^{y_I}\delta(e^{-y_I})}{2A}$, whence by (6) we have

$$\dot{V}(y(t)) \leq -\frac{\delta(e^{-y_I})}{2A}. \qquad (12)$$

(ii) $y_I > 0$, whence $e^{\tilde{q}(y_I)} > 1$ and (5) gives $e^{-\tilde{q}(y_I)}X_t \leq \mathrm{diag}(e^{(y_t)_j})X_t \leq e^{\tilde{q}(y_I)}X_t$. Therefore, invoking (Ab) with $\alpha = e^{\tilde{q}(y_I)}$ gives

$$f_I(t, \mathrm{diag}(e^{(y_t)_j})X_t) \leq e^{\tilde{q}(y_I)} f(t, X_t) - \delta(e^{\tilde{q}(y_I)}), \qquad (13)$$

[3] The remaining case in which $y_I(t) = 0$ and $y_I(t+\theta) \neq 0$ for some $\theta \in [-r, 0)$ is not needed in order to apply Theorem 3, since such a possibility can never occur under assumption (5).

whenever $e^{\tilde{q}(y_I)} - \eta(e^{\tilde{q}(y_I)}) \le e^{y_I} \le e^{\tilde{q}(y_I)}$. As in case (i), inequality (5) and the property that η is nondecreasing mean that this necessarily holds in particular whenever

$$e^{y_I} \le e^{\tilde{q}(y_I)} \le e^{y_I} + \eta(e^{y_I}). \tag{14}$$

Using (13) and again applying (5) together with the nondecreasing nature of δ, (4) then becomes

$$\frac{dy_I}{dt} \le \frac{1}{X_I}\left[(e^{\tilde{q}(y_I)-y_I} - 1)f_I(t, X_t) - e^{-y_I}\delta(e^{y_I})\right]. \tag{15}$$

Analogously to case (i), whenever it holds that

$$e^{y_I} \le e^{\tilde{q}(y_I)} \le e^{y_I} + \frac{1}{2M}\delta(e^{y_I}), \tag{16}$$

we then see that (15) implies $\frac{dy_I}{dt} \le -\frac{e^{-y_I}\delta(e^{y_I})}{2A}$, whence by (6) we have

$$\dot{V}(y(t)) \le -\frac{\delta(e^{y_I})}{2A}. \tag{17}$$

Based on the foregoing calculations, let us define $\tilde{q}(0) = 0$ and $\tilde{q}(s) = \min\{\log(e^s + \eta(e^s)), \log(e^s + \frac{1}{2M}\delta(e^s)), 2s\}$ for all $s > 0$. Recalling that $q(s) = e^{\tilde{q}(\log(1+s))} - 1$, this gives $q(0) = 0$ and $q(s) = s + \min\{\eta(1+s), \delta(1+s), s(1+s)\}$ for all $s > 0$, which clearly, since δ and η are positive, continuous, and nondecreasing, defines a choice of q satisfying all of the properties required in Theorem 3. Moreover, by construction all of (8), (10), (11), (14), and (16) are satisfied, so $\tilde{q}$ explicitly defines a function satisfying all of the properties required within the foregoing analysis. Therefore, if we also define $w(0) = 0$ and $w(s) = \frac{\delta(e^s)}{2A}$ for all $s > 0$, then w also satisfies all of the properties required in Theorem 3, and inequalities (12) and (17) yield $\dot{V}(y(t)) \le -w(\|y\|_\infty)$ whenever (5) holds. This is precisely condition (ii) in Theorem 3, so $y = 0$ is a globally uniformly asymptotically stable solution of (4).

Step 2: Next, we show that global uniform asymptotic stability of $y = 0$ for (4) implies uniform asymptotic stability of $X(t)$ for (1).

As a result of Step 1, it follows from Definition 1 that there exists $\beta \in \mathcal{KL}$ such that, for all $t_0 \ge 0$ and all solutions $y(t)$ of (4), $\|y_t\|_\mathcal{C} \le \beta(\|y_{t_0}\|_\mathcal{C}, t - t_0)$ for all $t \ge t_0$. By the coordinate change, this implies that, for all $t_0 \ge 0$, all solutions $x(t)$ of (1), and all $t \ge t_0$,

$$\|\log x_t - \log X_t\|_\mathcal{C} \le \beta(\|\log x_{t_0} - \log X_{t_0}\|_\mathcal{C}, t - t_0). \tag{18}$$

Now let Ξ be any compact set in $\mathring{\mathbb{R}}^n_+$ and suppose that $x(t)$ denotes the solution $x(t, t_0, x_{t_0})$ of (1) through an arbitrary $x_{t_0} \in S = \mathcal{C}([-r, 0], \Xi)$ at some $t_0 \ge 0$. It follows from compactness that there exist scalar constants $0 < b \le a$ and $B \ge A$ such that $S \subseteq \mathcal{C}([-r, 0], [b, B])$, whence we immediately have $b \le x_{t_0}, X_{t_0} \le B$ (using also the given boundedness properties of the solution $X(t)$). Therefore, the logarithmic geometry implies that $\|\log x_{t_0} - \log X_{t_0}\|_\mathcal{C} \le \frac{1}{b}\|x_{t_0} - X_{t_0}\|_\mathcal{C}$, so (18) gives

$$\|\log x_t - \log X_t\|_\mathcal{C} \le \beta\left(\tfrac{1}{b}\|x_{t_0} - X_{t_0}\|_\mathcal{C}, t - t_0\right). \tag{19}$$

Additionally, it follows from (18) and the boundedness of x_{t_0}, X_{t_0} that

$$\begin{aligned} \|\log x_t\|_\mathcal{C} &\le \|\log X_t\|_\mathcal{C} + \beta(\|\log x_{t_0} - \log X_{t_0}\|_\mathcal{C}, t - t_0) \\ &\le \|\log X_t\|_\mathcal{C} + \beta(\|\log x_{t_0}\|_\mathcal{C} + \|\log X_{t_0}\|_\mathcal{C}, t - t_0) \\ &\le C + \beta(2C, 0) \end{aligned} \tag{20}$$

by the definition of a $\mathcal{KL}$ function, where $C = \max\{|\log b|, |\log B|\}$. In particular, (20) implies that $x_t, X_t \le D$ for all $t \ge t_0$, where $D = \max\{B, e^{C+\beta(2C,0)}\}$ is a constant independent of both t_0 and t. Therefore, the logarithmic geometry again yields $\|x_t - X_t\|_\mathcal{C} \le D\|\log x_t - \log X_t\|_C$, whence (19) gives

$$\|x_t - X_t\|_\mathcal{C} \le \hat{\beta}(\|x_{t_0} - X_{t_0}\|_\mathcal{C}, t - t_0) \tag{21}$$

for all $t \ge t_0$, in terms of the $\mathcal{KL}$ function $\hat{\beta}(s, \tilde{s}) = D\beta(\frac{1}{b}s, \tilde{s})$. Since (21) holds for all $t_0 \ge 0$ and all initial conditions $x_{t_0} \in S$, $X(t)$ is uniformly asymptotically stable in the set of initial conditions S by Definition 1.

Step 3: Finally, we show that (21) can be used to infer incremental asymptotic stability. To see this, we follow the method of the proof of [29, Theorem 8] [4].

Choose any $t_0 \ge 0$ and let $x(t)$ and $\tilde{x}(t)$ represent solutions of (1) through arbitrary $x_{t_0}, \tilde{x}_{t_0} \in S$. It follows from (20) that all solutions with initial conditions in S remain always within the closed and bounded set $\mathcal{C}([-r, 0], [e^{-C-\beta(2C,0)}, e^{C+\beta(2C,0)}])$. Hence by the local Lipschitz property for $f(t, \cdot)$ and the fact that it holds uniformly with respect to t, there exists a constant $\gamma \ge 0$ such that, for all $t \ge t_0$,

$$\|f(t, x_t) - f(t, \tilde{x}_t)\| \le \gamma\|x_t - \tilde{x}_t\|_\mathcal{C}. \tag{22}$$

Integrating (1) and applying (22) then gives $\|x(t) - \tilde{x}(t)\| \le \|x(t_0) - \tilde{x}(t_0)\| + \gamma\int_{t_0}^t \|x_s - \tilde{x}_s\|_\mathcal{C}\, ds$. Thus, as $\gamma \ge 0$ and $\|x_s - \tilde{x}_s\|_\mathcal{C} \ge 0$ for all s, we get $\|x_t - \tilde{x}_t\|_\mathcal{C} = \sup_{\theta\in[-r,0]} \|x(t+\theta) - \tilde{x}(t+\theta)\| \le \|x_{t_0} - \tilde{x}_{t_0}\|_\mathcal{C} + \gamma\int_{t_0}^t \|x_s - \tilde{x}_s\|_\mathcal{C}\, ds$ for all $t \ge t_0$. Then Gronwall's Lemma gives

$$\|x_t - \tilde{x}_t\|_\mathcal{C} \le \|x_{t_0} - \tilde{x}_{t_0}\|_\mathcal{C} e^{\gamma(t-t_0)} \tag{23}$$

[4] A method similar to the proof of [29, Theorem 8] was also used in [6, Proposition 1] to deduce a result connecting the properties of uniform convergence and incremental asymptotic stability in delayed systems when all solutions converge to a compact positively invariant set.

for all $t \geq t_0$. Now note also that (21) applies with respect to each solution $x(t)$ and $\tilde{x}(t)$, so the triangle inequality and the fact that $b \leq x_{t_0}, \tilde{x}_{t_0}, X_{t_0} \leq B$ give

$$\begin{aligned} \|x_t - \tilde{x}_t\|_{\mathcal{C}} &\leq \hat{\beta}\left(\|x_{t_0} - X_{t_0}\|_{\mathcal{C}}, t - t_0\right) \\ &\quad + \hat{\beta}\left(\|\tilde{x}_{t_0} - X_{t_0}\|_{\mathcal{C}}, t - t_0\right) \\ &\leq 2\hat{\beta}\left(B - b, t - t_0\right) \end{aligned} \tag{24}$$

for all $t \geq t_0$. Combining (23) and (24), we finally get

$$\begin{aligned} \|x_t - \tilde{x}_t\|_{\mathcal{C}} \leq \min\{&\|x_{t_0} - \tilde{x}_{t_0}\|_{\mathcal{C}} e^{\gamma(t-t_0)}, \\ &2\hat{\beta}\left(B - b, t - t_0\right)\}. \end{aligned} \tag{25}$$

That there exists a function $\check{\beta} \in \mathcal{KL}$ which is an upper bound for the right-hand side in (25) can then be verified either by directly considering the behavior of the terms in the minimum here or by an easy application of Lemma 4.1 in [1], giving the result that

$$\|x_t - \tilde{x}_t\|_{\mathcal{C}} \leq \check{\beta}\left(\|x_{t_0} - \tilde{x}_{t_0}\|_{\mathcal{C}}, t - t_0\right)$$

for all $t \geq t_0$. Since this holds for all $t_0 \geq 0$ and all choices of $x_{t_0}, \tilde{x}_{t_0} \in S$, it follows from Definition 2 that the system (1) is incrementally asymptotically stable in the set of initial conditions S. □

Remark 8 Note that the explicit use of quasimonotonicity (2) is not required within the proof of Theorem 7. Instead, the required monotonicity properties are encoded in the two-sided quasiscalability assumption (A), which can permit weaker monotonicity notions.

Remark 9 The Lyapunov–Razumikhin function used in the proof of Theorem 7 can be viewed as a function of the logarithmic difference $\log x - \log X$ between solutions. Furthermore, in (12) and (17), bounds on its derivative are established in terms of this logarithmic difference. This contrasts with [6], in which the derivatives of the incremental Lyapunov–Razumikhin functions developed were constrained in terms of the additive difference $x - X$. It is necessary to follow this distinct approach in order to be able to exploit the inherent multiplicative nature of the two-sided quasiscalability condition (A).

Remark 10 The fact that uniformity in the asymptotic stability properties can be established only with respect to initial conditions taking values in arbitrarily large compact sets, as opposed to in the whole of $\mathring{\mathbb{R}}^n_+$, is a consequence of the fact that the two-sided quasisaclability property (A) ensures contraction only in the ratio between solutions rather than directly in their difference. A uniform correspondence between these two notions, as established in Step 2 of the proof of Theorem 7, is only feasible when the permitted initial conditions are bounded both above and away from the coordinate axes. We will see in Example 2 that nonuniformity in the convergence when the initial conditions take values in the whole positive orthant $\mathring{\mathbb{R}}^n_+$ is indeed possible under the assumptions in Theorem 7.

3.3 *Existence of an invariant set*

One way in which the boundedness requirements within Theorem 7 can be ensured is by guaranteeing the existence of a compact positively invariant set for (1). We now show that this can be achieved through an additional assumption stipulating the existence of points at which the nonlinearity f has fixed sign, and moreover that the time-variation on the interval between these points is bounded. To be precise, we assume:

(B) *Bounded time-variation:* there exist points $z^{\min} \leq z^{\max}$ in $\mathring{\mathbb{R}}^n_+$ such that $f(t, z^{\min}) \geq 0$ and $f(t, z^{\max}) \leq 0$ for all $t \geq 0$. Moreover, there exists a constant K such that $-K \leq f(t, \phi) \leq K$ for all $\phi \in \mathcal{C}([-r, 0], [z^{\min}, z^{\max}])$ and all $t \geq 0$.

Remark 11 In systems in which the explicit time-variation satisfies known bounds, the technical condition (B) frequently arises naturally by calculating the points $z_i^{\min}$ and $z_i^{\max}$ as fixed points of the functions $f_i^{\min}(\phi) = \inf_{t \geq 0} f_i(t, \phi)$ and $f_i^{\max}(\phi) = \sup_{t \geq 0} f_i(t, \phi)$. The boundedness requirement can often then follow from the continuity of $f^{\min}$ and $f^{\max}$ and the compactness of the set $[z^{\min}, z^{\max}]$. Examples of this will be seen in the applications in Section 4.

Remark 12 In the case where $z^{\min} = z^{\max}$, these points correspond to an equilibrium of (1). Therefore, our framework can also be used to recover stability properties of monotone systems with equilibria.

By recalling ideas from the theory of monotone systems presented in [30], we can use the property (B) together with quasimonotonicity (2) to construct a positively invariant set for (1).

Lemma 13 *Suppose that assumption* (B) *holds. Then the interval* $\mathcal{J} = \mathcal{C}([-r, 0], [z^{\min}, z^{\max}]) = \{\phi\colon z^{\min} \leq \phi \leq z^{\max}\}$ *is a non-empty positively invariant set for* (1).

Proof. The interval $\mathcal{J}$ is non-empty because $z^{\min} \leq z^{\max}$. Positive invariance follows immediately from Theorem 5.2.1 and Remark 5.2.1 in [30], which establish through use of quasimonotonicity (2) that the regions $[z^{\min}, \infty) \cap \mathring{\mathbb{R}}^n_+$ and $(-\infty, z^{\max}] \cap \mathring{\mathbb{R}}^n_+$ are both positively invariant. □

3.4 *Incremental asymptotic stability and convergence of all trajectories*

We proved in Section 3.2 that, assuming certain boundedness properties, assumption (A) implies an incremental stability result given by Theorem 7. In Section 3.3 we then established that assumption (B) implies a boundedness result given by Lemma 13. Therefore, it is natural to combine Theorem 7 and Lemma 13 in order to prove delay-independent incremental asymptotic stability for the system (1) under both assumptions (A) and (B).

Theorem 14 *Suppose that f satisfies assumption* (B) *and also satisfies assumption* (A) *for all* $\phi \in \mathcal{C}([-r, 0], [z^{\min}, z^{\max}])$. *Then the system* (1) *is incrementally* *asymptotically stable in the set of initial conditions* $S = \mathcal{C}([-r,0],\Xi)$ *for any compact set* $\Xi \subset \mathring{\mathbb{R}}^n_+$.

Proof. If f satisfies assumption (B), then Lemma 13 guarantees that $\mathcal{J} = \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$ defines a non-empty, bounded, positively invariant set for (1). Thus, if we specify any $X_0 \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$, then necessarily $X(t) \in [z^{\min}, z^{\max}]$ for all $t \geq 0$. Moreover, the property (B) also then ensures that $-K \leq f(t, X_t) \leq K$ for all $t \geq 0$. Furthermore, since $X_t \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$ for all $t \geq 0$ and assumption (A) holds for all $\phi \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$, it follows that (1) satisfies assumption (A) with $\phi = X_t$ for all $t \geq 0$. We therefore see that all of the assumptions of Theorem 7 are satisfied. Consequently, from Theorem 7 we conclude that (1) is incrementally asymptotically stable in the set of initial conditions $S = \mathcal{C}([-r,0],\Xi)$ for any compact $\Xi \subset \mathring{\mathbb{R}}^n_+$. □

The fact that Theorem 14 proves incremental asymptotic stability in arbitrarily large sets of initial conditions implies the following result guaranteeing convergence of all trajectories.

Corollary 15 *Suppose that* f *satisfies assumption* (B) *and also satisfies assumption* (A) *for all* $\phi \in \mathcal{C}([-r,0], [z^{\min}, z^{\max}])$. *Then for any two solutions* $x(t)$ *and* $\tilde{x}(t)$ *of* (1), $x(t) - \tilde{x}(t) \to 0$ *as* $t \to \infty$.

Proof. Choose S such that $x_{t_0}, \tilde{x}_{t_0} \in S$. Then incremental asymptotic stability in S implies that the increment $x(t) - \tilde{x}(t) \to 0$ as $t \to \infty$. □

Another consequence of the arguments used to prove Theorem 14 is the following corollary, which quantifies bounds on the asymptotic behavior of trajectories directly from the constraints on the nonlinearity in (B).

Corollary 16 *Suppose that* f *satisfies assumption* (B) *and also satisfies assumption* (A) *for all* $\phi \in \mathcal{C}([-r,0], [z^{\min}, z^{\max}])$. *Then every solution of* (1) *satisfies* $x(t) \to [z^{\min}, z^{\max}]$ *as* $t \to \infty$.

Proof. $X(t) \in [z^{\min}, z^{\max}]$ for all $t \geq -r$ and $x(t) - X(t) \to 0$ as $t \to \infty$ by Corollary 15, so $x(t) \to [z^{\min}, z^{\max}]$ as $t \to \infty$. □

The foregoing analysis demonstrates that the existence of a region with respect to which the two-sided quasiscalability property (A) holds, in conjunction with appropriate conditions such as (B) that lead to the existence of a bounded invariant set, yield some strong global performance guarantees. In particular, Theorem 14 guarantees that the asymptotic behavior of all solutions will be identical for arbitrary bounded time-varying delays, with uniform convergence rates and uniformly bounded transients within any closed and bounded set of initial conditions. Moreover, all solutions are guaranteed to be bounded, prohibiting pathological divergent behavior. Furthermore, Corollary 16 allows explicit bounds on the system's asymptotic behavior to be determined independently of both the initial states and the delays present in the system.

4 Applications

Example 1 As a first application, consider the behavior of the general class of time-varying wireless network uplink power control algorithms studied in [9] motivated by the setting in [32]. Letting x_i denote the power of the signal transmitted by user i and $I_i(t,x)$ denote the generalized interference nonlinearity measured at i, a general class of time-delayed uplink power control schemes can be modeled by the equation $\frac{dx_i}{dt} = k_i(-x_i + I_i(t, x^{d_i}))$, where $x^{d_i}(t) = (x_1(t+\theta_{i1}(t)), \ldots, x_n(t+\theta_{in}(t)))^T$ for continuous $\theta_{ij} : \mathbb{R}_+ \to [-r, 0]$. The functions I are assumed to fit within a uniform continuous-time version of the framework [5] of [32]:

(i) if $x \geq \tilde{x}$, then $I(t,x) \geq I(t,\tilde{x})$,

(ii) $I_i(t,x) - \frac{1}{\alpha} I_i(t, \alpha x) \geq \hat{\delta}(x, \alpha)$ for all $\alpha > 1$ for some $\hat{\delta} : \mathbb{R}^n_+ \times (1,\infty) \to \mathring{\mathbb{R}}_+$ that is continuous and nondecreasing in all variables and satisfies $\tilde{\alpha}\hat{\delta}(x,\alpha) \geq \hat{\delta}(\tilde{\alpha}x, \alpha)$ for all $\tilde{\alpha} > 1$.

In the framework studied above, then, we have $f_i(t,\phi) = k_i\{-\phi_i(0) + I_i[t, \mathrm{diag}(\phi_j(\theta_{ij}(t)))]\}$ and it can be seen that property (i) implies the quasimonotonicity condition (2). Moreover, it can be shown that $f(t,0) \gg 0$ for all $t \geq 0$, whence the strict positivity property follows immediately. Thus, this example fits within the framework analyzed in Section 3. To verify the two-sided quasiscalability property (A) consider arbitrary $\alpha > 1$ and $\phi \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$ for any potential choices of $z^{\min}$ and $z^{\max}$, and let $\frac{1}{\alpha}\phi \leq \psi \leq \alpha\phi$. We then see that:

(i) If $\frac{1}{\alpha}\phi_i(0) \leq \psi_i(0) \leq \frac{1}{\alpha - \eta(\alpha)}\phi_i(0)$, these properties give $f_i(t,\phi) - \alpha f_i(t,\psi) \leq \frac{\eta(\alpha)}{\alpha-\eta(\alpha)} \max_j z_j^{\max} - \hat{\delta}(z^{\min}, \alpha)$.

(ii) Analogously, if $(\alpha - \eta(\alpha))\phi_i(0) \leq \psi_i(0) \leq \alpha\phi_i(0)$, then $f_i(t,\psi) - \alpha f_i(t,\phi) \leq \eta(\alpha) \max_j z_j^{\max} - \hat{\delta}(z^{\min}, \alpha)$.

From cases (i) and (ii) we thus see that the required conditions (Aa) and (Ab) are both satisfied if we make the choice $\delta(\alpha) = \frac{1}{2}\hat{\delta}(z^{\min}, \alpha)$ and $\eta(\alpha) = \min\{\alpha - 1, \frac{\hat{\delta}(z^{\min},\alpha)}{2\max_j z_j^{\max}}\}$. Both of these definitions are, as required, positive, continuous, and nondecreasing. Hence, assumption (A) holds with respect to all $\phi \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$. Therefore, we are assured by Theorem 7 that any system having this form that admits a bounded solution will necessarily be incrementally asymptotically stable [6] in all closed and bounded sets

[5] This framework could also be relaxed to the formulation in [31], and our analysis would still be applicable to deduce incremental asymptotic stability through application of Theorem 7. Note, however, that by using the framework in [32], we are additionally able to deduce asymptotic bounds on the trajectories by application of Corollary 16, which would not be feasible with the relaxed monotonicity properties in [31].

[6] Note that the notion of incremental stability is stronger than the results established in [9], where the uniformity of convergence was not explicitly addressed.

of initial conditions. Furthermore, Theorem 14 guarantees incremental asymptotic stability provided that the system admits time-invariant bounds of the form (B). A particular example here is the prototypical Foschini–Miljanic [12] interference function with time-variation in the link gains G_{ij}, corresponding to relative motion of the network users, $I_i(t,x) = \frac{1}{G_{ii}(t)}(\sum_{j\neq i} G_{ij}(t)x_j + \nu_i)$. Assuming time-variation with minimum and maximum saturation at $G_{ij}^{\min}$ and $G_{ij}^{\max}$, taking the link gains $G_{ii} = G_{ii}^{\max}$, $G_{ij} = G_{ij}^{\min}$, and $G_{ii} = G_{ii}^{\min}$, $G_{ij} = G_{ij}^{\max}$ respectively yields time-independent $J^{\min}$ and $J^{\max}$ for which $J^{\min}(x) \leq I(t,x) \leq J^{\max}(x)$ for all $t \geq 0$ and all $x \in \mathring{\mathbb{R}}_+^n$. Whenever these bounding nonlinearities admit positive fixed points such that $J^{\min}(z^{\min}) = z^{\min}$ and $J^{\max}(z^{\max}) = z^{\max}$, the Brouwer fixed point theorem guarantees that $z^{\min} \leq z^{\max}$. It thus follows as required that $f(t, z^{\min}) \geq 0$, $f(t, z^{\max}) \leq 0$, and $-z^{\max} + J^{\min}(z^{\min}) \leq f(t,\phi) \leq -z^{\min} + J^{\max}(z^{\max})$ for all $\phi \in \mathcal{C}([-r,0],[z^{\min}, z^{\max}])$, whence assumption (B) is satisfied. Therefore, by calculating these fixed points we can formulate a limiting set $[z^{\min}, z^{\max}]$ to which all trajectories are guaranteed to converge by Corollary 16.

Example 2 A second example is the scalar Gompertz equation discussed in [22]

$$\dot{x} = x(a(t) - \log x), \tag{26}$$

where $a(t) \in [\underline{a}, \overline{a}]$ for some nonnegative constants $\underline{a}$, $\overline{a}$. For this system, it can be shown that quasimonotonicity, strict positivity, and assumptions (A) and (B) all hold when delays are included within the x-dependence of the term $a(t)x$. Then Theorem 14 and Corollary 16 guarantee that the system is incrementally asymptotically stable in all closed and bounded sets of initial conditions and that all solutions approach the invariant set $[e^{\underline{a}}, e^{\overline{a}}]$ under arbitrary bounded delays. However, when the delays are instead incorporated within the linear factor in the second term $-x\log x$, property (A) is seen to be violated. As such our results do not apply in this case, and we expect that the incremental asymptotic stability property may fail. Indeed, this is seen to be the case for the simple example with $a(t) \equiv 1$ in the plots shown in Fig. 1. Whereas in the first of these plots, we observe delay-independent incremental asymptotic stability for the system trajectories when the delays appear only within the first term, when the delays are included instead within the second term, we see in the second plot that incremental asymptotic stability breaks down for larger delay values and the convergence of trajectories in this case becomes delay-dependent. Note that in all cases the system remains strictly positive and admits a bounded trajectory, in the form of the equilibrium at e, along which the nonlinearity is bounded. Consequently, the failure of Theorem 7 to be applicable is due to the violation of assumption (A), demonstrating that the convergence breakdown occurs here exclusively as a result of this violation. In this way, this example demonstrates

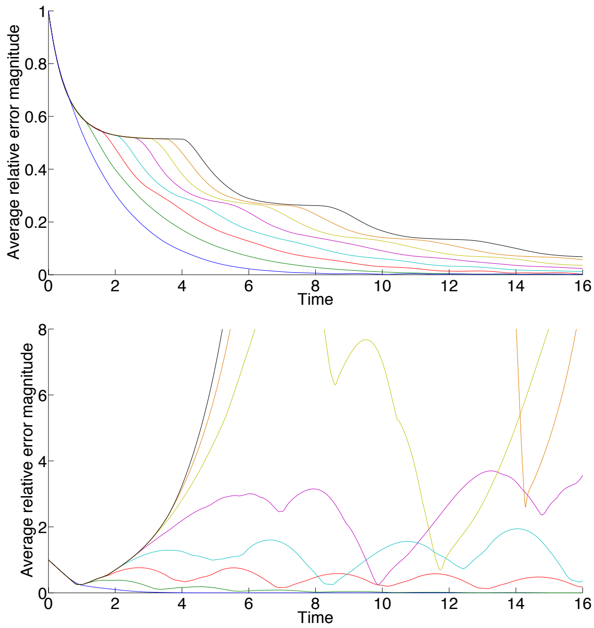

Fig. 1. Evolution of the relative error for trajectories of the Gompertz equation, averaged over 16 random initial conditions. Delays are included in the first term (first plot) and the second term (second plot) on the right-hand side of (26), for 8 constant time-delay values, ranging from 0.5 to 4. The first plot shows delay-independent stability while the second plot exhibits convergence breakdown when the delays reach 2.

the importance and necessity of the uniform two-sided quasiscalability assumption (A) in order for incremental asymptotic stability to hold.

This example can also offer a simple illustration of why the stability conclusions in Theorems 7 and 14 require compact sets of initial conditions. To see this, we consider the simple undelayed case of (26) with $a(t) \equiv 1$, for which the analytical solution

$$x(t) = \exp(1 - [1 - \log x(0)]e^{-t}) \tag{27}$$

can easily be derived. Let us now suppose that (26) were globally incrementally asymptotically stable. Then, in particular, the solution $x = e$ must be globally uniformly asymptotically stable, whence by Definition 1 there exists $\beta \in \mathcal{KL}$ such that

$$|x(t) - e| \leq \beta(|x(0) - e|, t) \tag{28}$$

holds for all $x(0) \in \mathring{\mathbb{R}}_+$ and all $t \geq 0$. Let us now consider the initial conditions $x^k(0) = e^{-k}$ for $k \in \mathbb{N}$. Then, substituting the corresponding solutions $x^k(t)$ from (27) into (28) and noting that $\beta(|x^k(0) - e|, t) \leq \beta(e,t)$ gives the inequality

$$\begin{aligned} &e - \exp(1 - (1+k)e^{-t}) \leq \beta(e,t) \\ \Rightarrow\ & k \leq (1 - \log(e - \beta(e,t)))e^t - 1, \end{aligned} \tag{29}$$

which must be satisfied by all $k \in \mathbb{N}$ and all $t \geq 0$. But by definition there exists some T for which $\beta(e,T) \leq e - 1$, whence (29) becomes $k \leq e^T - 1$, which is clearly violated for all sufficiently large choices of k. Thus, the contradiction implies that (26) cannot [7] be globally incrementally asymptotically stable on the whole positive orthant $\mathring{\mathbb{R}}_+$.

Example 3 We finally provide a simple example that illustrates how Theorem 14 can be combined with comparison results from classical monotone systems theory to yield insights into the behavior of systems that do not directly fit within our framework. Consider a simplified consensus protocol including arbitrary bounded noise and a uniform time-varying delay

$$\dot{x}_i = -x_i + \sum_{j \neq i} a_{ij}(t) x_j^d + \nu_i(t), \tag{30}$$

with $x^d = x(t + \theta(t))$ for a continuous map $\theta : \mathbb{R}_+ \to [-r, 0]$ and each $\nu_i(t) \in (-N, N)$. The matrix $A = (a_{ij}(t))_{i,j}$ is row-stochastic and, for simplicity of the analysis, has each $a_{ii} = 0$ and satisfies the uniform two-hop connectivity property that there exists $\rho > 0$ such that, given any pair (i, j) there exists some $k \neq i, j$ with $a_{ik}(t), a_{jk}(t) \geq \rho$ for all $t \geq 0$. Due to the possibly negative noise, both the strict positivity requirement and the two-sided quasiscalability assumption (A) may be violated. However, if $z_i = x_i - x_m$ where $x_m = \min_j x_j$, then

$$\dot{z}_i < -z_i + \sum_{j \neq i,k} a_{ij}(t) z_j^d + (a_{ik}(t) - \rho) z_k^d + 2N. \tag{31}$$

Based on this, we introduce the bounding system

$$\dot{Z}_i = -Z_i + \sum_{j \neq i,k} a_{ij}(t) Z_j^d + (a_{ik}(t) - \rho) Z_k^d + 2N. \tag{32}$$

Applying the classical comparison result [18, Theorem 4.1] to (31) and (32) shows that if $z_{t_0} = Z_{t_0}$, then $z(t) \leq Z(t)$, and so the solutions of (31) are bounded above by those of (32). Moreover, (32) defines a strictly positive monotone system of the form (1) for which assumption (A) is satisfied by similar arguments as in Example 1. Furthermore, the system nonlinearity satisfies $f(t, \frac{2N}{\rho}) = 0$, so as discussed in Remark 12, the bounded time-variation property (B) is satisfied with $z^{\min} = z^{\max} = \frac{2N}{\rho}$. Thus, Theorem 14, and in particular Corollary 16, apply for (32), guaranteeing incremental asymptotic stability and furthermore the result that $Z(t) \to \frac{2N}{\rho}$ for all solutions of (32). Thus, by choosing $Z_{t_0} = z_{t_0}$, the comparison result gives $z(t) \to [0, \frac{2N}{\rho}]$ as $t \to \infty$ for any z_{t_0}. Therefore, for all initial conditions $x_{t_0} \in \mathcal{C}([-r, 0], \mathbb{R}^n)$, the solutions of (30) satisfy $\max_i x_i(t) - \min_i x_i(t) \to [0, \frac{2N}{\rho}]$, quantifying a delay-independent bound on the maximum asymptotic deviation from global consensus that can occur.

5 Conclusions

We have considered the behavior of the solutions of a class of time-varying, strictly positive monotone systems of delay differential equations. Motivated by its utility in the analysis of a class of power control schemes in wireless networks, we first considered a scalability condition and combined this with quasimonotonicity to formulate a weaker combined condition of two-sided quasiscalability. Using this condition, we were able to prove, whenever certain boundedness properties hold, a result guaranteeing incremental asymptotic stability. We then discussed that, by imposing a condition constraining the time-variation of the system nonlinearity, it is possible to guarantee the existence of a non-empty, bounded, positively invariant set. This guarantees the requisite boundedness properties, meaning that by combining our results we were able to prove that the existence of a bounded region on which the system's time-variation is suitably constrained and with respect to which the two-sided quasiscalability assumption is satisfied can be sufficient to guarantee that the system is incrementally asymptotically stable in arbitrarily large sets of initial conditions, independent of arbitrary time-varying delays. Consequently, all solutions are guaranteed to converge to one another asymptotically, at a uniform rate and with uniformly bounded transients within any set of initial conditions taking values in any compact subset of $\mathring{\mathbb{R}}^n_+$. The significance of this result lies in the fact that it allows the behavior of all trajectories to be robustly determined by simply studying the evolution of one particular convenient choice of solution. Furthermore, it was seen that the specified constraints on the time-variation of the nonlinearity, which can often be determined directly from the system model, quantify a limiting region to which all solutions converge, yielding explicit constraints on the system's long-time behavior without requiring the calculation of any system trajectory. It is part of ongoing research to determine to what extent the technical conditions stated here can be relaxed while still yielding incremental asymptotic stability, and whether they can also provide necessary conditions for delay-independent convergence.

[7] To be precise, the example shows that incremental asymptotic stability is not guaranteed unless the set of initial conditions is bounded away from the coordinate axes. It is straightforward to see that incremental asymptotic stability also does not necessarily hold when the initial condition set is not bounded from above by considering the simple system $\dot{x} = \max\{1 - x, -1\}$ in $\mathbb{R}$. This equation has an equilibrium at $x = 1$ and satisfies assumption (A) with $\phi = 1$. Furthermore, the difference between solutions through initial conditions k and $k + 1$ for any $k \in \mathbb{N} \setminus \{0, 1\}$ remains equal to 1 for all $t \leq k - 2$. Hence, by letting $k \to \infty$, the rate at which these solutions converge to one another can be made arbitrarily slow. This is incompatible with Definition 2, demonstrating the necessity of the upper boundedness requirement in Theorems 7 and 14.

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