Nominal Presentation of Cubical Sets Models of Type Theory

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Abstract

The cubical sets model of Homotopy Type Theory introduced by Bezem, Coquand and Huber [2] uses a particular category of presheaves. We show that this presheaf category is equivalent to a category of sets equipped with an action of a monoid of name substitutions for which a finite support property holds. That category is in turn isomorphic to a category of nominal sets [15] equipped with operations for substituting constants 0 and 1 for names. This formulation of cubical sets brings out the potentially useful connection that exists between the homotopical notion of *path* and the nominal sets notion of *name abstraction*. The formulation in terms of actions of monoids of name substitutions also encompasses a variant category of cubical sets with diagonals, equivalent to presheaves on Grothendieck's "smallest test category" [8, pp. 47–48]. We show that this category has the pleasant property that path objects given by name abstraction are exponentials with respect to an interval object.

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1 Introduction

We begin by reviewing the notion of *cubical set* introduced by Bezem, Coquand and Huber [2, Section 2] and motivate the move to a nominal presentation of it.

▶ **Definition 1.1** (Directions). Throughout the paper $\mathbb{D}$ is a fixed, infinite set whose elements we write as $\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots$ and call *directions*. We assume $\mathbb{D}$ is disjoint from the two-element set $2 = \{0, 1\}$.

Let $\mathbf{C}$ be the small category whose objects $X, Y, \ldots$ are finite subsets of $\mathbb{D}$ and whose hom-sets $\mathbf{C}(X, Y)$ consist of all functions $s : X \to Y \cup 2$ with the property

$$(\forall \mathsf{x}, \mathsf{x}' \in X)\ s\,\mathsf{x} = s\,\mathsf{x}' \notin 2 \ \Rightarrow\ \mathsf{x} = \mathsf{x}' \tag{1}$$

Such a function is extended to one defined on the whole of $X \cup 2$ by taking $s\,0 = 0$ and $s\,1 = 1$. Then composition in $\mathbf{C}$ is given by composition of functions, with identity morphisms given by inclusions $X \hookrightarrow X \cup 2$.

▶ **Definition 1.2** (Cubical sets). A *cubical set* is a functor $\mathbf{C} \to \mathbf{Set}$ and a morphism of cubical sets is a natural transformation between such functors. Thus the category of cubical sets is the category $[\mathbf{C}, \mathbf{Set}]$ of set-valued presheaves on the category $\mathbf{C}^{\mathrm{op}}$.

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We refer the reader to Bezem, Coquand and Huber [2] for the geometric intuition behind the terminology 'cubical set' and the connection with the notion of the same name in homotopy theory. Like any presheaf category, $[\mathbf{C}, \mathbf{Set}]$ gives rise to a model of extensional Martin-Löf Type Theory, organised as a *category with families* (CwF) in the sense of Dybjer [5]. See Hofmann [9, Section 4] for an account of presheaf CwFs. In order to model Homotopy Type Theory and in particular Voevodsky's Univalence Axiom [21], Bezem, Coquand and Huber consider families of presheaves equipped with operations for filling open boxes – a more uniform version of the classic Kan filling condition in combinatorial homotopy theory. The resulting families of *Kan cubical sets* support an interpretation of identity types and [2] contains a sketch of why there is a universe satisfying the Univalence Axiom with respect to these identity types.

Motivation for a nominal approach

Presheaf models of type theory in general, and in particular the cubical sets model of Homotopy Type Theory mentioned above, inevitably involve quantifications over Kripke possible-worlds (which are finite sets of directions in the cubical case) that tend to obscure the simple intuition behind these models, because of the need to write explicit weakening functions from a world to future worlds. Furthermore, cubical sets of paths and the Kan filling condition make use of constructions involving a choice of directions $\mathsf{x} \in \mathbb{D}$ that are suitably fresh, but whose properties are independent of which particular fresh direction is chosen. This is precisely the situation for which the theory of *nominal sets* [6, 15] was created. In particular it admits a rich theory of *freshness* that makes implicit the dependence upon possible worlds of directions. According to the authors of the experimental implementation of Kan cubical sets [4], "it was convenient to use the alternative presentation of cubical sets as nominal sets". That alternative presentation was announced in [14]. Here we take an alternative approach based on monoids of name substitutions, leading to the equivalences of Theorems 2.9 and 2.13 below. This facilitates the description of Π-types (Section 3.2) and universes (Section 3.3); but more importantly, it allows path objects to be described in terms of the well-developed nominal sets theory of *name abstraction* (Section 2.2). The presentation in terms of monoids of substitutions also encompasses a variant of cubical sets with diagonals (Section 4), equivalent to presheaves on Grothendieck's "smallest test category" [8, pp. 47–48] and referred to in [2]. We show that this category has the pleasant property that path objects given by name abstraction are exponentials with respect to an interval object (Theorem 4.2).

A note on constructivity

The model of univalence based on simplicial sets [12] uses classical set theory. One of Bezem, Coquand and Huber's motivations for considering cubical sets instead of simplicial sets is that they can be made a model of univalence within constructive logics, which makes a computational version possible. It is therefore of interest whether the results in this paper are constructively valid. Like [15], upon some of whose results it relies, this paper is written using naive classical set theory. In a constructive setting, equality for elements of the set $\mathbb{D}$ of directions should be assumed to be decidable and $\mathbb{D}$ should be 'finitely inexhaustible', in the sense that for each subset $X \subseteq \mathbb{D}$ that is in bijection with a finite ordinal, there exists some $\mathsf{x} \in \mathbb{D}$ with $\mathsf{x} \notin X$. Starting from that basis, it seems likely that much of the theory of nominal sets is constructively valid. However, at the very least one has to replace the use of *smallest* finite support sets in arguments by the existence of *some* finite support set. For if equality is undecidable in some set upon which name permutations act, then the existence of some finite support for an element of the set does not necessarily mean there is a smallest one; see [20, Section 1.2.1]. We leave for future work the questions of whether use of smallest supports can always be avoided and whether the results of this paper are constructively valid.

2 Monoids of Substitutions

In this section we reformulate cubical sets in terms of monoids of substitutions, where the crucial property of 'finite support' gives a well-behaved theory of degeneracy via freshness.

▶ **Definition 2.1** (Substitutions). As far as this paper is concerned, a *finite substitution* is a function $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$ for which $\mathrm{Dom}\,\sigma \triangleq \{\mathsf{x} \in \mathbb{D} \mid \sigma\,\mathsf{x} \neq \mathsf{x}\}$ is finite. Let $\mathbf{Sb}$ denote the monoid whose elements are finite substitutions, with the monoid operation given by composition: $\sigma\sigma' \triangleq \hat{\sigma} \circ \sigma'$, where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ is the function

$$\begin{aligned} \hat{\sigma}\,\mathsf{b} &\triangleq \mathsf{b} && \text{if } \mathsf{b} \in 2, \\ \hat{\sigma}\,\mathsf{x} &\triangleq \sigma\,\mathsf{x} && \text{if } \mathsf{x} \in \mathbb{D}. \end{aligned} \tag{2}$$

(Note that $\mathrm{Dom}\,\sigma\sigma'$ is indeed finite, since it is contained in $\mathrm{Dom}\,\sigma \cup \mathrm{Dom}\,\sigma'$.) The identity element $\iota \in \mathbf{Sb}$ is given by the inclusion $\mathbb{D} \hookrightarrow \mathbb{D} \cup 2$. If $\mathsf{x} \in \mathbb{D}$ and $i \in \mathbb{D} \cup 2$, we write

$$(i/\mathsf{x}) \in \mathbf{M} \tag{3}$$

for the finite substitution mapping x to i and otherwise acting like the identity; and if $\mathsf{x}, \mathsf{x}' \in \mathbb{D}$, then we write

$$(\mathsf{x}\ \mathsf{x}') \in \mathbf{M} \tag{4}$$

for the finite substitution that transposes x and x' and otherwise acts like the identity. By a *monoid of substitutions* $\mathbf{M}$ we mean any submonoid of $\mathbf{Sb}$ containing $(\mathsf{x}\ \mathsf{x}')$ and (b/x) for all $\mathsf{b} \in 2$ and all $\mathsf{x}, \mathsf{x}' \in \mathbb{D}$.

The notion of finite support is most often applied to actions of permutations, for example in the theory of nominal sets [15, Chapter 2]. However, it generalizes well to actions of more general forms of substitution; see [7, Definition 7], for example.

▶ **Definition 2.2** (Finitely supported $\mathbf{M}$-sets). For any monoid $\mathbf{M}$ we write write $\mathbf{Set}^{\mathbf{M}}$ for the category whose objects are sets Γ equipped with a (left) $\mathbf{M}$-action $\_ \cdot \_ : \mathbf{M} \times \Gamma \to \Gamma$

$$\iota \cdot d = d \qquad \sigma' \cdot (\sigma \cdot d) = \sigma'\sigma \cdot d \qquad (d \in \Gamma, \sigma, \sigma' \in \mathbf{M}) \tag{5}$$

and whose morphisms are functions $\gamma : \Gamma \to \Gamma'$ preserving the action

$$\gamma(\sigma \cdot d) = \sigma \cdot (\gamma\, d) \qquad (\sigma \in \mathbf{M}, d \in \Gamma) \tag{6}$$

When $\mathbf{M}$ is a monoid of substitutions (Definition 2.1) and $\Gamma \in \mathbf{Set}^{\mathbf{M}}$, we say that a finite subset $X \subseteq_{\mathrm{fin}} \mathbb{D}$ *supports* an element $d \in \Gamma$ if

$$(\forall \sigma, \sigma' \in \mathbf{M})\ ((\forall \mathsf{x} \in X)\ \sigma\,\mathsf{x} = \sigma'\mathsf{x})\ \Rightarrow \sigma \cdot d = \sigma' \cdot d \tag{7}$$

We write $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ for the full subcategory of $\mathbf{Set}^{\mathbf{M}}$ consisting of those Γ such that for all $d \in \Gamma$ there exists a finite subset $X \subseteq_{\mathrm{fin}} \mathbb{D}$ that supports d. We call $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ the *category of finitely supported* $\mathbf{M}$*-sets*.

▶ **Example 2.3** (The interval). Let $\mathbf{M}$ be a monoid of substitutions. We make $\mathrm{I} \triangleq \mathbb{D} \cup 2$ into an object of $\mathbf{Set}^{\mathbf{M}}$ via the action given by function application: $\sigma \cdot i \triangleq \hat{\sigma}\, i$, for all $i \in \mathrm{I}$. With respect to this action, an element of $\mathsf{x} \in \mathbb{D} \subseteq \mathrm{I}$ is supported by $\{\mathsf{x}\}$ and the two elements of $2 \subseteq \mathrm{I}$ are supported by $\emptyset$. We call I the *interval* in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$.

▶ **Lemma 2.4.** *Let $\mathbf{M}$ be a monoid of substitutions and $\mathsf{b} \in 2$ some fixed Boolean value. For each $d \in \Gamma \in \mathbf{Set}^{\mathbf{M}}$, a finite subset $X \subseteq_{\mathrm{fin}} \mathbb{D}$ supports d iff*

$$(\forall \mathsf{x} \in \mathbb{D})\ \mathsf{x} \notin X \ \Rightarrow\ (\mathsf{b}/\mathsf{x}) \cdot d = d \tag{8}$$

Proof. Taking $\sigma = (\mathsf{b}/\mathsf{x})$ and $\sigma' = \iota$ in (7), we get that it implies (8). To prove (8) implies (7), we proceed by induction on the size of the finite set

$$\mathrm{ds}(\sigma, \sigma') \ \triangleq\ \{\mathsf{x} \in \mathbb{D} \mid \sigma\,\mathsf{x} \neq \sigma'\mathsf{x}\} \tag{9}$$

(It is finite, because it is contained in $\mathrm{Dom}\,\sigma \cup \mathrm{Dom}\,\sigma'$.) The base case is trivial. For the induction step, suppose

$$(\forall \mathsf{x} \in X)\ \sigma\,\mathsf{x} = \sigma'\mathsf{x} \tag{10}$$

and that $\mathsf{y} \in \mathrm{ds}(\sigma, \sigma')$. We have to prove that $\sigma \cdot d = \sigma' \cdot d$. Since $\sigma\,\mathsf{y} \neq \sigma'\mathsf{y}$, from (10) we must have $\mathsf{y} \notin X$ and hence $(\forall \mathsf{x} \in X)\ \sigma(\mathsf{b}/\mathsf{y})\,\mathsf{x} = \sigma'(\mathsf{b}/\mathsf{y})\,\mathsf{x}$. Since $\mathrm{ds}(\sigma(\mathsf{b}/\mathsf{y}), \sigma'(\mathsf{b}/\mathsf{y})) = \mathrm{ds}(\sigma, \sigma') - \{\mathsf{y}\}$, by induction hypothesis $\sigma(\mathsf{b}/\mathsf{y}) \cdot d = \sigma'(\mathsf{b}/\mathsf{y}) \cdot d$. But since X satisfies (8) and $\mathsf{y} \notin X$, it follows that $(\mathsf{b}/\mathsf{y}) \cdot d = d$. Therefore $\sigma \cdot d = \sigma \cdot ((\mathsf{b}/\mathsf{y}) \cdot d) = \sigma(\mathsf{b}/\mathsf{y}) \cdot d = \sigma'(\mathsf{b}/\mathsf{y}) \cdot d = \sigma' \cdot ((\mathsf{b}/\mathsf{y}) \cdot d) = \sigma' \cdot d$, as required. ◀

▶ **Corollary 2.5.** *Suppose $\Gamma \in \mathbf{Set}^{\mathbf{M}}$ and $d \in \Gamma$ is supported by $X \subseteq_{\mathrm{fin}} \mathbb{D}$.*
1. *For any morphism $\gamma : \Gamma \to \Gamma'$ in $\mathbf{Set}^{\mathbf{M}}$, $\gamma\, d \in \Gamma'$ is also supported by X.*
2. *For any $\sigma \in \mathbf{M}$, $\sigma \cdot d \in \Gamma$ is supported by the finite subset $\sigma X \cap \mathbb{D} = \{\sigma\,\mathsf{x} \mid \mathsf{x} \in X \wedge \sigma\,\mathsf{x} \notin 2\}$.*

Proof. Fix some $\mathsf{b} \in 2$. For part 1, if $\mathsf{x} \in \mathbb{D}$ satisfies $\mathsf{x} \notin X$, then

$$\begin{aligned} (\mathsf{b}/\mathsf{x}) \cdot (\gamma\, d) &= \gamma((\mathsf{b}/\mathsf{x}) \cdot d) && \text{by (6)} \\ &= \gamma\, d && \text{by Lemma 2.4.} \end{aligned}$$

So by Lemma 2.4 again, X supports $\gamma\, d$.

For part 2, if $\mathsf{y} \notin \sigma X \cap \mathbb{D}$, then $(\forall \mathsf{x} \in X)\ (\mathsf{b}/\mathsf{y})\sigma\,\mathsf{x} = \widehat{(\mathsf{b}/\mathsf{y})}(\sigma\,\mathsf{x}) = \sigma\,\mathsf{x}$; so because X supports d we have $(\mathsf{b}/\mathsf{y}) \cdot (\sigma \cdot d) = (\mathsf{b}/\mathsf{y})\sigma \cdot d = \sigma \cdot d$. So Lemma 2.4 implies that $\sigma X \cap \mathbb{D}$ supports $\sigma \cdot d$. ◀

▶ **Definition 2.6** (Least supports). Let $\mathbf{M}$ be a monoid of substitutions and $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$. By Lemma 2.4, for each $d \in \Gamma$

$$\mathrm{supp}\, d \triangleq \{\mathsf{x} \in \mathbb{D} \mid (0/\mathsf{x}) \cdot d \neq d\} \tag{11}$$

is finite and is the least finite supporting set of directions for d. Note that $\mathrm{supp}\, d = \{\mathsf{x} \in \mathbb{D} \mid (1/\mathsf{x}) \cdot d \neq d\}$.

▶ **Definition 2.7** (The monoid $\mathbf{Cb}$). Let $\mathbf{Cb} \subseteq \mathbf{Sb}$ be the subset consisting of finite substitutions σ satisfying an injectivity condition like (1):

$$(\forall \mathsf{x}, \mathsf{x}' \in \mathbb{D})\ \sigma\,\mathsf{x} = \sigma\,\mathsf{x}' \notin 2 \ \Rightarrow\ \mathsf{x} = \mathsf{x}'$$

Cb is a monoid of substitutions in the sense of Definition 2.1. It enjoys the following homogeneity property with respect to the small category **C** from Section 1.

▶ **Lemma 2.8** (Homogeneity). *For all morphism $s \in \mathbf{C}(X, Y)$ there is a finite substitution $\sigma \in \mathbf{Cb}$ satisfying $(\forall \mathsf{x} \in X)\ s\,\mathsf{x} = \sigma\,\mathsf{x}$.*

Proof. Given $s \in \mathbf{C}(X, Y)$, let $X_1 \triangleq \{\mathsf{x} \in X \mid s\,\mathsf{x} \notin 2\}$ and $X_2 \triangleq \{\mathsf{x} \in X \mid s\,\mathsf{x} \in 2\}$. Thus $X = X_1 \uplus X_2$ and s restricts to a bijection between X_1 and $Y_1 \triangleq \{s\,\mathsf{x} \mid \mathsf{x} \in X_1\}$. Pick a finite permutation π of $\mathbb{D}$ that agrees with s on X_1 and is the identity outside the finite set $X_1 \cup Y_1$ (it is always possible to do so – see for example [15, Lemma 1.14]). Then

$$\sigma\,\mathsf{x} \triangleq \begin{cases} s\,\mathsf{x} & \text{if } \mathsf{x} \in X_2 \\ \pi\,\mathsf{x} & \text{otherwise} \end{cases}$$

is a suitable element of **Cb**. ◀

▶ **Theorem 2.9.** *The category $[\mathbf{C}, \mathbf{Set}]$ of cubical sets is equivalent to the category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ of finitely supported $\mathbf{Cb}$-sets.*

Proof. We define a functor $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ as follows. Each inclusion $X \subseteq Y$ between finite subsets of $\mathbb{D}$ yields a morphism $X \hookrightarrow Y$ in **C**. So given $C \in [\mathbf{C}, \mathbf{Set}]$ we can take the colimit of C restricted to the poset $(\mathrm{P_{fin}}\,\mathbb{D}, \subseteq)$ of finite subsets of $\mathbb{D}$: $I^*C \triangleq \mathrm{colim}_{X \in \mathrm{P_{fin}}\,\mathbb{D}}\, C\,X$. Concretely, I^*C consists of equivalence classes $[X, x]$ of pairs $(X, x) \in \sum_{X \in \mathbf{C}} C\,X$ for the equivalence relation that relates (X, x) and (X', x') when there is some $Y \supseteq X \cup X'$ with $C(X \hookrightarrow Y)x = C(X' \hookrightarrow Y)x'$. Note that by definition of the monoid **Cb**, for each $\sigma \in \mathbf{Cb}$ and $X \in \mathbf{C}$ the restricted function $\sigma|_X : X \to \sigma X$ is a morphism in $\mathbf{C}(X, \sigma X \cap \mathbb{D})$. Then

$$\sigma \cdot [X, x] \triangleq [\sigma X \cap \mathbb{D}, C(\sigma|_X)x] \tag{12}$$

gives a well-defined **Cb**-action on I^*C. Furthermore, with respect to this action an element $[X, x] \in I^*C$ is supported by X; for if σ and σ' agree on X, then $C(\sigma|_X) = C(\sigma'|_X)$ and hence $\sigma \cdot [X, x] = \sigma' \cdot [X, x]$. So $I^*C \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$.

The assignment $C \in [\mathbf{C}, \mathbf{Set}] \mapsto I^*C \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ extends to a functor as follows. Given a natural transformation $\varphi : C \to C'$ in $[\mathbf{C}, \mathbf{Set}]$ we get a well-defined function $I^*\varphi : I^*C \to I^*C'$ by defining

$$I^*\varphi\,[X, x] \triangleq [X, \varphi_X x] \tag{13}$$

The naturality of φ_X in $X \in \mathbf{C}$ ensures not only that this definition is independent of the choice of representative (X, x) for the element $[X, x]$, but also that $I^*\varphi$ preserves the **Cb**-action (12).

We complete the proof of the theorem by showing that $I^* : [\mathbf{C}, \mathbf{Set}] \to \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ is faithful, full and essentially surjective.

- I^* *is faithful*: Note that any inclusion $X \hookrightarrow Y$ in **C** is split, for example by the morphism $p \in \mathbf{C}(Y, X)$ where

  $$p\,\mathsf{y} \triangleq \begin{cases} \mathsf{y} & \text{if } \mathsf{y} \in X \\ 0 & \text{otherwise} \end{cases} \qquad (\mathsf{y} \in Y)$$

  Therefore $C(X \hookrightarrow Y) : C\,X \to C\,Y$ is an injective function in **Set** with left inverse $C\,p$. Thus if $\varphi, \varphi' \in [\mathbf{C}, \mathbf{Set}](C, C')$ and $I^*\varphi = I^*\varphi'$, then for any $X \in \mathbf{C}$ and $x \in C\,X$ we have $[X, \varphi_X x] = I^*\varphi\,[X, x] = I^*\varphi'[X, x] = [X, \varphi'_X x]$, so that for some $Y \supseteq X$, $C(X \hookrightarrow Y)(\varphi_X x) = C(X \hookrightarrow Y)(\varphi'_X x)$; and since $C(X \hookrightarrow Y)$ is injective this gives $\varphi_X x = \varphi'_X x$. Therefore $\varphi = \varphi'$.

- I^* *is full*: Suppose $C, C' \in [\mathbf{C}, \mathbf{Set}]$ and $\gamma \in \mathbf{Set}_{\text{fs}}^{\mathbf{Cb}}(I^*C, I^*C')$. It is not hard to see that

$$(\forall d \in I^*C')\ \operatorname{supp} d \subseteq X \ \Rightarrow\ (\exists! x \in C'\,X)\ d = [X, x] \tag{14}$$

Indeed if $d = [Y, y]$ and $\operatorname{supp} d \subseteq X$, then d is supported by $X \cap Y$ and hence the substitution $\sigma \in \mathbf{Cb}$ mapping each $\mathsf{y} \in Y - X$ to 0 and otherwise acting like the identity satisfies $\sigma \cdot d = d$; therefore $d = \sigma \cdot [Y, y] = [X \cap Y, C'(\sigma|_Y)y] = [X, C'(X \cap Y \hookrightarrow X)C'(\sigma|_Y)y]$. The uniqueness part of (14) follows from the injectivity of each $C'(X \hookrightarrow Y)$, noted above. For each $X \in \mathbf{C}$ and $x \in C\,X$, by part 1 of Corollary 2.5 we have that $\operatorname{supp}(\gamma[X, x]) \subseteq \operatorname{supp}[X, x]$ and hence that $\operatorname{supp}(\gamma[X, x]) \subseteq X$. Therefore from (14) we have $(\forall x \in C\,X)(\exists! x' \in C'X)\ \gamma[X, x] = [X, x']$. So for each $X \in \mathbf{C}$ there is a function $\varphi_X : C\,X \to C'X$ satisfying

$$(\forall x \in C\,X)\ \gamma[X, x] = [X, \varphi_X x] \tag{15}$$

It suffices to show that φ_X is natural in X, since then by combining (13) with (15) we have that $\varphi \in [\mathbf{C}, \mathbf{Set}](C, C')$ satisfies $I^*\varphi = \gamma$. For naturality, given $s \in \mathbf{C}(X, Y)$ to prove $\varphi_Y(C\,s\,x) = C's\,(\varphi_X x)$ it suffices to show $[Y, \varphi_Y(C\,s\,x)] = [Y, C's\,(\varphi_X x)]$, because of the injectivity of the functions $C(Y \hookrightarrow Z) : C\,Y \to C\,Z$ (see above). Now we use the homogeneity property in Lemma 2.8: picking a substitution $\sigma \in \mathbf{Cb}$ that agrees with s on X, we have $[Y, \varphi_Y(C\,s\,x)] = \gamma[Y, C\,s\,x] = \gamma[\sigma X \cap \mathbb{D}, C(\sigma|_X)x] = \gamma(\sigma \cdot [X, x]) = \sigma \cdot (\gamma[X, x]) = \sigma \cdot [X, \varphi_X x] = [\sigma X \cap \mathbb{D}, C'(\sigma|_X)(\varphi_X x)] = [Y, C's\,(\varphi_X x)]$, as required.
- I^* *is essentially surjective*: Given $\Gamma \in \mathbf{Set}_{\text{fs}}^{\mathbf{Cb}}$, for each $X \in \mathbf{C}$ consider the subset of Γ consisting of the elements supported by the finite subset $X \subseteq_{\text{fin}} \mathbb{D}$:

$$I_*\Gamma\,X \triangleq \{d \in \Gamma \mid \operatorname{supp} d \subseteq X\} \tag{16}$$

For each $s \in \mathbf{C}(X, Y)$ there is a well-defined function $I_*\Gamma\,s : I_*\Gamma\,X \to I_*\Gamma\,Y$ satisfying

$$I_*\Gamma\,s\,d = \sigma \cdot d \quad \text{where } \sigma \in \mathbf{Cb} \text{ is any substitution satisfying } \sigma|_X = s \tag{17}$$

(There is such a σ by Lemma 2.8; $I_*\Gamma\,s\,d$ is independent of the choice of σ because X supports d; and $I_*\Gamma\,s\,d \in I_*\Gamma\,Y$ by part 2 of Corollary 2.5.) Since $\iota|_X = \mathrm{id}_X$ we get $I_*\Gamma\,\mathrm{id}_x\,d = \iota \cdot d = d$; and since $(\sigma'\sigma)|_X = s' \circ s$ when $s = \sigma|_X$ and $s' = \sigma'|_Y$, we get $I_*\Gamma\,(s' \circ s)\,d = \sigma'\sigma \cdot d = \sigma' \cdot (\sigma \cdot d) = I_*\Gamma\,s'(I_*\Gamma\,s\,d)$. So $I_*\Gamma \in [\mathbf{C}, \mathbf{Set}]$. To complete the proof we will construct an isomorphism $\varepsilon_\Gamma : I^*(I_*\Gamma) \cong \Gamma$ in $\mathbf{Set}_{\text{fs}}^{\mathbf{Cb}}$.
First note that in (17), if s is an inclusion $X \hookrightarrow Y$, then we can take $\sigma = \iota$ and therefore $I_*\Gamma\,(X \hookrightarrow Y)d = \iota \cdot d = d$. It follows that if (X, d) and (X', d') both represent the same equivalence class in $I^*(I_*\Gamma)$, then $d = I_*\Gamma\,(X \hookrightarrow Y)d = I_*\Gamma\,(X' \hookrightarrow Y)d' = d'$ (where $Y \supseteq X \cup X'$). So we get a well-defined function $\varepsilon_\Gamma : I^*(I_*\Gamma) \to \Gamma$ satisfying

$$\varepsilon_\Gamma[X, d] = d \tag{18}$$

This preserves the $\mathbf{Cb}$-action because

$$\begin{aligned}
\varepsilon_\Gamma(\sigma \cdot [X, d]) &= \varepsilon_\Gamma[\sigma X \cap \mathbb{D}, I_*\Gamma(\sigma|_X)d] && \text{by (12)}\\
&= I_*\Gamma(\sigma|_X)d && \text{by (18)}\\
&= \sigma \cdot d && \text{by (17)}\\
&= \sigma \cdot (\varepsilon_\Gamma[X, d]) && \text{by (18) again.}
\end{aligned}$$

It is an injective function, because if $[X, d], [X', d'] \in I^*(I_*\Gamma)$ satisfy $d = d'$, then $\operatorname{supp} d = \operatorname{supp} d' \subseteq X \cap X'$ (by Lemma 2.4) and as above we have $I_*\Gamma(X \cap X' \hookrightarrow X)d = d = d' = I_*\Gamma(X \cap X' \hookrightarrow X')d'$; hence $[X, d] = [X', d']$. It is a surjective function, because each $d \in \Gamma$ is finitely supported by some $X \subseteq_{\text{fin}} \mathbb{D}$ and hence $d = \varepsilon_\Gamma[X, d]$. So altogether, ε_Γ is an isomorphism in $\mathbf{Set}_{\text{fs}}^{\mathbf{Cb}}$. ◀

2.1 Nominal sets with 01-substitution

In this section we deduce from Theorem 2.9 the equivalence announced in [14]. We assume some familiarity with the theory of nominal sets; see for example [15].

Let **Nom** denote the category of nominal sets and equivariant functions over the set of atoms $\mathbb{D}$. If $\mathbf{M}$ is any monoid of substitutions (Definition 2.1), then the group $\mathrm{Perm}\,\mathbb{D}$ of finite permutations of $\mathbb{D}$ is a submonoid of $\mathbf{M}$, because every finite permutation is the composition of finitely many transpositions. Thus the $\mathbf{M}$-action on each $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ restricts to a $\mathrm{Perm}\,\mathbb{D}$-action. If $X \subseteq \mathbb{D}$ supports $d \in \Gamma$ in the sense of Definition 2.2, then it is in particular a support in the usual sense of nominal sets [15, Section 2.1]:

$$(\forall \pi \in \mathrm{Perm}\,\mathbb{D})\ ((\forall \mathsf{x} \in X)\ \pi\,\mathsf{x} = \mathsf{x}) \ \Rightarrow\ \pi \cdot d = d \tag{19}$$

Hence each $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ is a nominal set and indeed there is a forgetful functor $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}} \to \mathbf{Nom}$, since morphisms in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ are in particular equivariant functions.

▶ **Lemma 2.10** (Freshness). *Suppose* $\mathbf{M}$ *is a monoid of substitutions and that* $d \in \Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$*. Then* $\mathrm{supp}\,d$ *as defined in Definition 2.6 is the least finite support for* $d \in \Gamma$ qua *nominal sets, that is, the least finite subset* $X \subseteq \mathbb{D}$ *satisfying* (19)*. Hence the relation*

$$\mathsf{x} \mathrel{\#} d \ \triangleq\ \mathsf{x} \notin \mathrm{supp}\,d \qquad (\mathsf{x} \in \mathbb{D}, d \in \Gamma)$$

coincides with the nominal notion of freshness [15, Chapter 3]*.*

Proof. First note that since (7) implies (19), $\mathrm{supp}\,d$ is a finite subset of $\mathbb{D}$ satisfying (19). If $X \subseteq_{\mathrm{fin}} \mathbb{D}$ is any other such, we will show that (8) holds and hence that $\mathrm{supp}\,d \subseteq X$, by Lemma 2.4. Indeed, if $\mathsf{b} \in 2$ and $\mathsf{x} \in \mathbb{D} - X$, choose some $\mathsf{y} \in \mathbb{D}$ not in the finite subset $X \cup \{\mathsf{x}\} \cup \mathrm{supp}\,d$. Then

$$\begin{aligned}
(\mathsf{b}/\mathsf{x}) \cdot d &= (\mathsf{x}\ \mathsf{y})(\mathsf{b}/\mathsf{y})(\mathsf{x}\ \mathsf{y}) \cdot d && \text{since } (\mathsf{b}/\mathsf{x}) = (\mathsf{x}\ \mathsf{y})(\mathsf{b}/\mathsf{y})(\mathsf{x}\ \mathsf{y}) \in \mathbf{M}\\
&= (\mathsf{x}\ \mathsf{y})(\mathsf{b}/\mathsf{y}) \cdot d && \text{by (19) with } \pi = (\mathsf{x}\ \mathsf{y}), \text{ since } \mathsf{x}, \mathsf{y} \notin X\\
&= (\mathsf{x}\ \mathsf{y}) \cdot d && \text{by Lemma 2.4, since } \mathsf{y} \notin \mathrm{supp}\,d\\
&= d && \text{by (19) again}
\end{aligned}$$

as required for (8). ◀

▶ Remark 2.11. By contrast with nominal sets in general, the freshness relation for objects of $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ can be characterised in terms of substitution of 0 or 1, as follows:

$$\mathsf{x} \mathrel{\#} d \ \Leftrightarrow\ (0/\mathsf{x}) \cdot d = d \ \Leftrightarrow\ (1/\mathsf{x}) \cdot d = d \qquad (\mathsf{x} \in \mathbb{D}, d \in \Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}) \tag{20}$$

This is an immediate consequence of Definition 2.6, which relies upon the characterisation of support in Lemma 2.4.

▶ **Definition 2.12** (Nominal 01-substitution structures). Let **01-Nom** be the category whose objects are nominal sets Γ equipped with *source* and *target* operations $(\mathsf{x} := 0)\_, (\mathsf{x} := 1)\_ : \Gamma \to \Gamma$ in each direction $\mathsf{x} \in \mathbb{D}$ satisfying for all $\pi \in \mathrm{Perm}\,\mathbb{D}$, $\mathsf{x}, \mathsf{x}' \in \mathbb{D}$, $\mathsf{b}, \mathsf{b}' \in 2$ and $d \in \Gamma$

$$\pi \cdot ((\mathsf{x} := \mathsf{b})d) = (\pi\,\mathsf{x} := \mathsf{b})(\pi \cdot d) \tag{21}$$

$$\mathsf{x} \mathrel{\#} (\mathsf{x} := \mathsf{b})d \tag{22}$$

$$\mathsf{x} \mathrel{\#} d \Rightarrow (\mathsf{x} := \mathsf{b})d = d \tag{23}$$

$$\mathsf{x} \neq \mathsf{x}' \Rightarrow (\mathsf{x} := \mathsf{b})(\mathsf{x}' := \mathsf{b}')d = (\mathsf{x}' := \mathsf{b}')(\mathsf{x} := \mathsf{b})d \tag{24}$$

The morphisms of $\mathbf{01\text{-}Nom}$ are the equivariant functions $\gamma \in \mathbf{Nom}(\Gamma, \Gamma')$ that also commute with the source and target operations in each direction: $\gamma((\mathsf{x} := \mathsf{b})d) = (\mathsf{x} := \mathsf{b})(\gamma\, d)$. Composition and identities are as in $\mathbf{Nom}$.

▶ **Theorem 2.13.** *The category* $\mathbf{01\text{-}Nom}$ *of nominal sets with* 01*-substitution structure is isomorphic to the category* $\mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}$ *of finitely supported* $\mathbf{Cb}$*-sets and hence (by Theorem 2.9) is equivalent to the category* $[\mathbf{C}, \mathbf{Set}]$ *of cubical sets.*

Proof. We noted above that the $\mathbf{Cb}$-action on $\Gamma \in \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}$ restricts to a $\mathrm{Perm}\,\mathbb{D}$-action, making it a nominal set. We get source and target operations in each direction $\mathsf{x} \in \mathbb{D}$ by defining $(\mathsf{x} := \mathsf{b})d \triangleq (\mathsf{b}/\mathsf{x}) \cdot d$. These satisfy (21) because $\pi(\mathsf{b}/\mathsf{x}) = (\mathsf{b}/\pi\,\mathsf{x})\pi \in \mathbf{Cb}$; they satisfy (22) because of part 2 of Corollary 2.5 and Lemma 2.10; they satisfy (23) because of Lemmas 2.4 and 2.10; and they satisfy (24) because $(\mathsf{b}/\mathsf{x})(\mathsf{b}'/\mathsf{x}') = (\mathsf{b}'/\mathsf{x}')(\mathsf{b}/\mathsf{x}) \in \mathbf{Cb}$ when $\mathsf{x} \neq \mathsf{x}'$. Furthermore, since each morphism $\gamma \in \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}(\Gamma, \Gamma')$ commutes with the $\mathbf{Cb}$-action, it is not only an equivariant function, but also preserves the source and target operations defined as above. So we get a functor $\mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}} \to \mathbf{01\text{-}Nom}$ which is the identity on underlying nominal sets.

Conversely, given $\Gamma \in \mathbf{01\text{-}Nom}$, we can combine the $\mathrm{Perm}\,\mathbb{D}$-action with the source and target operations to get a $\mathbf{Cb}$-action on Γ as follows: for each $\sigma \in \mathbf{Cb}$ and $d \in \Gamma$, consider

$$\sigma \cdot d \triangleq \pi \cdot (\mathsf{x}_1 := \mathsf{b}_1) \cdots (\mathsf{x}_n := \mathsf{b}_n)d \tag{25}$$

where $\mathsf{x}_1, \ldots, \mathsf{x}_n$ are the distinct element of $\{\mathsf{x} \in \mathrm{Dom}\,\sigma \mid \sigma\,\mathsf{x} \in 2\}$, where $\mathsf{b}_i = \sigma\,\mathsf{x}_i$ for $i = 1, \ldots, n$, and where $\pi \in \mathrm{Perm}\,\mathbb{D}$ is a finite permutation agreeing with σ on $\{\mathsf{x} \in \mathrm{Dom}\,\sigma \mid \sigma\,\mathsf{x} \notin 2\}$. Note that there is such a permutation, because σ is injective on $\{\mathsf{x} \in \mathrm{Dom}\,\sigma \mid \sigma\,\mathsf{x} \notin 2\}$; and (25) is independent of which π we choose, and independent of the order in which we list the elements of $\{\mathsf{x} \in \mathrm{Dom}\,\sigma \mid \sigma\,\mathsf{x} \in 2\}$ (because of property (24)). In case $n = 0$, by $(\mathsf{x}_1 := \mathsf{b}_1) \cdots (\mathsf{x}_n := \mathsf{b}_n)d$ we mean d. Thus $\iota \cdot d = d$; and it is not hard to see that this definition also satisfies $\sigma' \cdot (\sigma \cdot d) = \sigma'\sigma \cdot d$. So we get a $\mathbf{Cb}$-action on Γ and clearly each $d \in \Gamma$ is supported by $\mathrm{supp}\, d$ with respect to this action. Furthermore, for each morphism $\gamma \in \mathbf{01\text{-}Nom}(\Gamma, \Gamma')$, since $\gamma(\pi \cdot (\mathsf{x}_1 := \mathsf{b}_1) \cdots (\mathsf{x}_n := \mathsf{b}_n)d) = \pi \cdot (\mathsf{x}_1 := \mathsf{b}_1) \cdots (\mathsf{x}_n := \mathsf{b}_n)(\gamma\, d)$ and $\mathrm{supp}(\gamma\, d) \subseteq \mathrm{supp}\, d$, we get $\gamma(\sigma \cdot d) = \sigma \cdot (\gamma\, d)$. So we get a functor $\mathbf{01\text{-}Nom} \to \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}$, which once again is the identity on underlying nominal sets.

It is easy to see that these two functors are mutually inverse, so that $\mathbf{01\text{-}Nom} \cong \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}$. ◀

▶ Remark 2.14. The proof of the equivalence $[\mathbf{C}, \mathbf{Set}] \simeq \mathbf{01\text{-}Nom}$ given in [14] is somewhat different from the above one and was inspired by proofs of equivalences between (pre)sheaf categories and nominal sets equipped with substitution structures studied by Staton [17]; see in particular the proof of Proposition 9 in [18].

2.2 Path objects

One of the advantages of replacing cubical sets by the equivalent notion of nominal sets with 01-substitution (Theorem 2.13) is that the construct used in [2, Section 8.2] to model identity types coincides across the equivalence with a central and widely used notion of nominal set theory, namely that of *name abstraction* [15, Chapter 4].

Given a nominal set $\Gamma \in \mathbf{Nom}$, the nominal set $[\mathbb{D}]\Gamma$ of name-abstractions of elements of Γ has underlying set consisting of equivalence classes of pairs $(\mathsf{x}, d) \in \mathbb{D} \times \Gamma$ for a generalised form of α-equivalence, namely the equivalence relation

$$(\mathsf{x}, d) \approx_\alpha (\mathsf{x}', d') \;\triangleq\; (\exists \mathsf{y} \mathrel{\#} (\mathsf{x}, d, \mathsf{x}', d'))\; (\mathsf{y}\;\mathsf{x}) \cdot d = (\mathsf{y}\;\mathsf{x}') \cdot d' \tag{26}$$

We write $\langle \mathsf{x} \rangle d$ for the $\approx_\alpha$-equivalence class of (x, d). The action of finite permutations $\pi \in \operatorname{Perm} \mathbb{D}$ on such equivalence classes is well-defined by

$$\pi \cdot \langle \mathsf{x} \rangle d \triangleq \langle \pi\, \mathsf{x} \rangle (\pi \cdot d) \tag{27}$$

and one can show that the least support of $\langle \mathsf{x} \rangle d$ with respect to this action is $\operatorname{supp} d - \{\mathsf{x}\}$; see [15, Proposition 4.5].

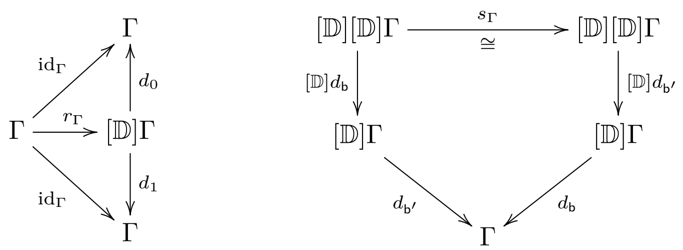
If $\Gamma \in$ **01-Nom**, then because of properties (21) and (22), the source and target operations induce morphisms $d_0, d_1 \in \mathbf{Nom}([\mathbb{D}]\Gamma, \Gamma)$ satisfying

$$d_{\mathsf{b}} \langle \mathsf{x} \rangle d = (\mathsf{x} := \mathsf{b}) d \qquad (\mathsf{b} \in 2, \mathsf{x} \in \mathbb{D}, d \in \Gamma)$$

Then properties (23) and (24) correspond to the commutation of the following diagrams, where $r_\Gamma \in \mathbf{Nom}(\Gamma, [\mathbb{D}]\Gamma)$ and $s_\Gamma \in \mathbf{Nom}([\mathbb{D}][\mathbb{D}]\Gamma, [\mathbb{D}][\mathbb{D}]\Gamma)$ are morphisms satisfying

$$r_\Gamma\, d = \langle \mathsf{x} \rangle d \qquad \text{for some/any } \mathsf{x} \mathrel{\#} d \tag{28}$$

$$s_\Gamma \langle \mathsf{x} \rangle \langle \mathsf{y} \rangle d = \langle \mathsf{y} \rangle \langle \mathsf{x} \rangle d \tag{29}$$



In fact these are diagrams in **01-Nom**, because there is a well-defined nominal 01-substitution structure on each $[\mathbb{D}]\Gamma$ satisfying

$$(\mathsf{x} := \mathsf{b})(\langle \mathsf{y} \rangle d) = \langle \mathsf{y} \rangle ((\mathsf{x} := \mathsf{b}) d) \quad \text{if} \quad \mathsf{x} \neq \mathsf{y} \tag{30}$$

and then d_0, d_1, r_Γ and s_Γ are morphisms in **01-Nom**.

For each $\Gamma \in$ **01-Nom**, one can think of elements $p \in [\mathbb{D}]\Gamma$ as *paths* in Γ from $d_0 p$ to $d_1 p$. For each $d \in \Gamma$, $r_\Gamma\, d \in [\mathbb{D}]\Gamma$ is a degenerate path from d to itself. The object $\langle d_0\,, d_1 \rangle : [\mathbb{D}]\Gamma \to \Gamma \times \Gamma$ of the slice category **01-Nom**$/\Gamma \times \Gamma$ corresponds under the equivalence of Theorem 2.13 to the structure that Bezem, Coquand and Huber use to model identity types, at least in the case that the cubical set corresponding to Γ satisfies a uniform Kan filling condition [2, Section 5].

3 Modelling Type Theory with Families of $\mathbb{M}$-sets

We have reformulated cubical sets in a way that emphasises actions of monoids of substitutions. Since any monoid $\mathbf{M}$ can be regarded as a one-object category, $\mathbf{Set}^{\mathbf{M}}$ is in particular a category of set-valued presheaves and so can be given the standard category-with-families structure for such a category [9, Section 4]. However, in this case the structure is quite simple (if one is familiar with monoid actions): as we will see, one just uses a dependently-typed version of monoid action. We begin by recalling briefly the definition of category-with-families in order to fix notation; see [9] for more details and [1] for a more abstract, category-theoretic perspective.

▶ **Definition 3.1** (Category with families [5]). A *category with families* (*CwF*) is specified by a category $\mathcal{C}$ with a terminal object 1, together with the following structure:

- For each object $\Gamma \in \mathcal{C}$, a collection $\mathcal{C}(\Gamma)$, whose elements are called *families* over Γ.
- For each object $\Gamma \in \mathcal{C}$ and family $A \in \mathcal{C}(\Gamma)$, a collection $\mathcal{C}(\Gamma \vdash A)$ of *elements* of the family A over Γ.
- Operations for re-indexing families and elements along morphisms in $\mathcal{C}$

$$\frac{A \in \mathcal{C}(\Gamma) \qquad \gamma \in \mathcal{C}(\Gamma', \Gamma)}{A[\gamma] \in \mathcal{C}(\Gamma')} \qquad \frac{a \in \mathcal{C}(\Gamma \vdash A) \qquad \gamma \in \mathcal{C}(\Gamma', \Gamma)}{a[\gamma] \in \mathcal{C}(\Gamma' \vdash A[\gamma])}$$

satisfying

$$\begin{aligned} A[\mathrm{id}_\Gamma] &= A && (A \in \mathcal{C}(\Gamma)) \\ A[\gamma \circ \gamma'] &= A[\gamma][\gamma'] && (A \in \mathcal{C}(\Gamma), \gamma \in \mathcal{C}(\Gamma', \Gamma), \gamma' \in \mathcal{C}(\Gamma'', \Gamma')) \\ a[\mathrm{id}_\Gamma] &= a && (a \in \mathcal{C}(\Gamma \vdash A) \\ a[\gamma \circ \gamma'] &= a[\gamma][\gamma'] && (a \in \mathcal{C}(\Gamma \vdash A), \gamma \in \mathcal{C}(\Gamma', \Gamma), \gamma' \in \mathcal{C}(\Gamma'', \Gamma')) \end{aligned}$$

- For each family $A \in \mathcal{C}(\Gamma)$, a *comprehension* object $\Gamma.A \in \mathcal{C}$ equipped with a projection morphism $\mathrm{p} \in \mathcal{C}(\Gamma.A, \Gamma)$, a *generic element* $\mathrm{v} \in \mathcal{C}(\Gamma.A \vdash A[\mathrm{p}])$ and a *pairing operation*

$$\frac{\gamma \in \mathcal{C}(\Gamma', \Gamma) \qquad A \in \mathcal{C}(\Gamma) \qquad a \in \mathcal{C}(\Gamma' \vdash A[\gamma])}{\langle \gamma \, , a \rangle \in \mathcal{C}(\Gamma', \Gamma.A)}$$

satisfying

$$\begin{aligned} \mathrm{p} \circ \langle \gamma \, , a \rangle &= \gamma \\ \mathrm{v}[\langle \gamma \, , a \rangle] &= a \\ \langle \gamma \, , a \rangle \circ \gamma' &= \langle \gamma \circ \gamma' \, , a[\gamma'] \rangle \\ \langle \mathrm{p} \, , \mathrm{v} \rangle &= \mathrm{id}_{\Gamma.A} \end{aligned}$$

For each object $\Gamma \in \mathcal{C}$, one can make $\mathcal{C}(\Gamma)$ into a category by taking, for each $A, B \in \mathcal{C}(\Gamma)$, the set of morphisms $\mathcal{C}(\Gamma)(A, B)$ to be $\mathcal{C}(\Gamma.A \vdash B[\mathrm{p}])$ with identities given by generic elements and composition given by $c \circ b \triangleq c[\langle \mathrm{p} \, , b \rangle]$. Then the mapping $A \in \mathcal{C}(\Gamma) \mapsto \mathrm{p} \in \mathcal{C}(\Gamma.A, \Gamma)$ extends to a full and faithful functor to the slice category

$$\mathcal{C}(\Gamma) \to \mathcal{C}/\Gamma \tag{31}$$

$$A \xrightarrow{b} B \mapsto \begin{array}{ccc} \Gamma.A & \xrightarrow{\langle \mathrm{p}, b \rangle} & \Gamma.B \\ & {\scriptstyle \mathrm{p}} \searrow \quad \swarrow {\scriptstyle \mathrm{p}} & \\ & \Gamma & \end{array}$$

The re-indexing operations are mapped to pullback functors between slices, since for each $A \in \mathcal{C}(\Gamma)$ and $\gamma \in \mathcal{C}(\Gamma', \Gamma)$

$$\begin{array}{ccc} \Gamma'.A[\gamma] & \xrightarrow{\langle \gamma \circ \mathrm{p}, \mathrm{v} \rangle} & \Gamma.A \\ {\scriptstyle \mathrm{p}} \downarrow \lrcorner & & \downarrow {\scriptstyle \mathrm{p}} \\ \Gamma' & \xrightarrow[\gamma]{} & \Gamma \end{array} \tag{32}$$

is a pullback in $\mathcal{C}$; see [9, Proposition 3.9].

The contexts, types-in-context, terms-in-context and term-substitutions of Type Theory are interpreted in a CwF by its objects, families, elements and morphisms respectively; see [9, Section 3.5]. Furthermore, one can translate each type-forming construct to an equivalent structure within CwFs. For example:

▶ **Definition 3.2** (Σ- and Π-types in CwFs). A Cwf $\mathcal{C}$ has

- Σ-*types* if there are operations

$$\frac{A \in \mathcal{C}(\Gamma) \qquad B \in \mathcal{C}(\Gamma.A)}{\Sigma\, A\, B \in \mathcal{C}(\Gamma)} \qquad \frac{a \in \mathcal{C}(\Gamma \vdash A) \qquad B \in \mathcal{C}(\Gamma.A) \qquad b \in \mathcal{C}(\Gamma \vdash B[\langle \mathrm{id}_\Gamma\, , a\rangle])}{\mathrm{pair}\, a\, b \in \mathcal{C}(\Gamma \vdash \Sigma\, A\, B)}$$

$$\frac{c \in \mathcal{C}(\Gamma \vdash \Sigma\, A\, B)}{\mathrm{fst}\, c \in \mathcal{C}(\Gamma \vdash A)} \qquad \frac{c \in \mathcal{C}(\Gamma \vdash \Sigma\, A\, B)}{\mathrm{snd}\, c \in \mathcal{C}(\Gamma \vdash B[\langle \mathrm{id}_\Gamma\, , \mathrm{fst}\, c\rangle])}$$

satisfying

$$\begin{aligned}
(\Sigma\, A\, B)[\gamma] &= \Sigma(A[\gamma])(B[\langle \gamma \circ \mathrm{p}\, , \mathrm{v}\rangle]) \\
(\mathrm{pair}\, a\, b)[\gamma] &= \mathrm{pair}\, (a[\gamma])\, (b[\gamma]) \\
(\mathrm{fst}\, c)[\gamma] &= \mathrm{fst}(c[\gamma]) \\
(\mathrm{snd}\, c)[\gamma] &= \mathrm{snd}(c[\gamma]) \\
\mathrm{fst}(\mathrm{pair}\, a\, b) &= a \\
\mathrm{snd}(\mathrm{pair}\, a\, b) &= b \\
\mathrm{pair}\, (\mathrm{fst}\, c)\, (\mathrm{snd}\, c) &= c
\end{aligned}$$

- Π-*types* if there are operations

$$\frac{A \in \mathcal{C}(\Gamma) \qquad B \in \mathcal{C}(\Gamma.A)}{\Pi\, A\, B \in \mathcal{C}(\Gamma)} \qquad \frac{b \in \mathcal{C}(\Gamma.A \vdash B)}{\mathrm{lam}\, b \in \mathcal{C}(\Gamma \vdash \Pi\, A\, B)} \qquad \frac{\begin{array}{c} c \in \mathcal{C}(\Gamma \vdash \Pi\, A\, B) \\ a \in \mathcal{C}(\Gamma \vdash A) \end{array}}{\mathrm{app}\, c\, a \in \mathcal{C}(\Gamma \vdash B[\langle \mathrm{id}_\Gamma\, , a\rangle])}$$

satisfying

$$\begin{aligned}
(\Pi\, A\, B)[\gamma] &= \Pi(A[\gamma])(B[\langle \gamma \circ \mathrm{p}\, , \mathrm{v}\rangle]) \\
(\mathrm{lam}\, b)[\gamma] &= \mathrm{lam}\, b[\langle \gamma \circ \mathrm{p}\, , \mathrm{v}\rangle] \\
(\mathrm{app}\, c\, a)[\gamma] &= \mathrm{app}\, (c[\gamma])\, (a[\gamma]) \\
\mathrm{app}\, (\mathrm{lam}\, b)\, a &= b[\langle \mathrm{id}_\Gamma\, , a\rangle] \\
\mathrm{lam}(\mathrm{app}\, (c[\mathrm{p}])\, \mathrm{v}) &= c
\end{aligned}$$

▶ Remark 3.3. If $\mathcal{C}$ is a locally cartesian closed category, it is always possible to find a CwF with the same underlying category $\mathcal{C}$ for which the functors in (31) are not only full and faithful, but also essentially surjective; see [13, 1]. In that case each category of families $\mathcal{C}(\Gamma)$ is equivalent to the slice category $\mathcal{C}/\Gamma$ and the CwF structure is just providing an equivalent version of the traditional use of slice categories to model families of objects in category theory [16] – one in which pullback strictly commutes with composition and hence correctly models properties of substitution in type theory. This applies to the categories we consider in this paper, $[\mathbf{C}, \mathbf{Set}]$, $\mathbf{Set}^{\mathbf{M}}$ and $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$, since they are all toposes and hence in particular locally cartesian closed. However, in these cases it not necessary to apply a general construction as in [13, 1], since there are natural and useful notions of 'family of presheaves' and 'family of $\mathbf{M}$-sets' equivalent to the use of slice categories. Such families are used in [2] for the category of cubical sets; and we describe analogues for $\mathbf{Set}^{\mathbf{M}}$ and $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ in the next two sections. Note that the equivalence $I^* : [\mathbf{C}, \mathbf{Set}] \simeq \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}$ from Theorem 2.9 gives an equivalence $[\mathbf{C}, \mathbf{Set}]/C \simeq \mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}/I^*C$ for each cubical set C; and therefore the category of families over C, being equivalent to $[\mathbf{C}, \mathbf{Set}]/C$ and hence to $\mathbf{Set}^{\mathbf{Cb}}_{\mathrm{fs}}/I^*C$, is also equivalent to the category of families for I^*C in the CwF described in Section 3.2 for the case $\mathbf{M} = \mathbf{Cb}$.

3.1 CwF structure of $\mathrm{Set}^{\mathbf{M}}$

Let $\mathbf{M}$ be an arbitrary monoid.

- *Families* $\mathbf{Set}^{\mathbf{M}}(\Gamma)$ over an object $\Gamma \in \mathbf{Set}^{\mathbf{M}}$ consist of Γ-indexed families of sets $A = (A_d \mid d \in \Gamma)$ equipped with a 'dependently-typed $\mathbf{M}$-action', that is, a family of functions

  $$\_ \cdot \_ \in \textstyle\prod_{\sigma\in\mathbf{M}} \prod_{a \in A_d} A_{\sigma\cdot d} \qquad (d \in \Gamma)$$

  satisfying $\iota \cdot a = a \in A_d (= A_{\iota \cdot d})$ and $\sigma' \cdot (\sigma \cdot a) = \sigma'\sigma \cdot a \in A_{\sigma'\sigma\cdot d} (= A_{\sigma'\cdot(\sigma\cdot d)})$.
- *Elements* $\mathbf{Set}^{\mathbf{M}}(\Gamma \vdash A)$ of a family $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ consist of dependently-typed functions $f \in \prod_{d\in\Gamma} A_d$ that preserve the $\mathbf{M}$-action, in the sense that $\sigma \cdot (f\, d) = f(\sigma \cdot d) \in A_{\sigma\cdot d}$.
- *Re-indexing* of a family $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ along $\gamma \in \mathbf{Set}^{\mathbf{M}}(\Gamma', \Gamma)$ is the family $A[\gamma] \triangleq (A_{\gamma\, d'} \mid d' \in \Gamma')$ with dependently-typed $\mathbf{M}$-action: $\sigma \in \mathbf{M}, a \in A[\gamma]_{d'} = A_{\gamma\, d'} \;\mapsto\; \sigma \cdot a \in A_{\sigma\cdot(\gamma\, d')} = A_{\gamma(\sigma\cdot d')} = A[\gamma]_{\sigma\cdot d'}$. The re-indexing of an element $f \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash A)$ along $\gamma \in \mathbf{Set}^{\mathbf{M}}(\Gamma', \Gamma)$ is the element $f[\gamma] \in \mathbf{Set}^{\mathbf{M}}(\Gamma' \vdash A)$, where $f[\gamma]\, d' = f(\gamma\, d')$.
- *Comprehension* for the CwF $\mathbf{Set}^{\mathbf{M}}$ is created by that for $\mathbf{Set}$. Thus given $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$, the comprehension object $\Gamma.A \in \mathbf{Set}^{\mathbf{M}}$ is given by the dependent product of sets

  $$\Gamma.A \triangleq \textstyle\sum_{d\in\Gamma} A_d \quad \text{equipped with the } \mathbf{M}\text{-action} \quad \sigma \cdot (d, a) \triangleq (\sigma \cdot d, \sigma \cdot a) \tag{33}$$

  First projection yields a morphism $\mathrm{p} \in \mathbf{Set}^{\mathbf{M}}(\Gamma.A, \Gamma)$ and the generic element $\mathrm{v} \in \mathbf{Set}^{\mathbf{M}}(\Gamma.A \vdash A[\mathrm{p}])$ is given by second projection: $\mathrm{v}(d, a) \triangleq a \in A_d = A[\mathrm{p}]_{(d,a)}$. The pairing operation is

  $$\frac{\gamma \in \mathbf{Set}^{\mathbf{M}}(\Gamma', \Gamma) \qquad f \in \mathbf{Set}^{\mathbf{M}}(\Gamma' \vdash A[\gamma])}{\langle \gamma\,,f\rangle \in \mathbf{Set}^{\mathbf{M}}(\Gamma', \Gamma.A)} \qquad \langle\gamma\,,f\rangle\, d' \triangleq (\gamma\, d', f\, d') \qquad (d' \in \Gamma')$$

These operations satisfy the equations required for a CwF (Definition 3.1). In this case the functors (31) are equivalences: any object $\gamma : \Gamma' \to \Gamma$ of the slice category $\mathbf{Set}^{\mathbf{M}}/\Gamma$ is isomorphic to $\mathrm{p} : \Gamma.A \to \Gamma$ for some family $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$, namely $A_d \triangleq \{d' \in \Gamma' \mid \gamma\, d' = d\}$ with dependently-typed action given by the $\mathbf{M}$-action of Γ'. Since $\mathbf{Set}^{\mathbf{M}}$ is a topos (being a presheaf category), it is in particular locally cartesian closed. One can use the equivalences $\mathbf{Set}^{\mathbf{M}}(\Gamma) \simeq \mathbf{Set}^{\mathbf{M}}/\Gamma$ to transfer this local cartesian closed structure to operations in the CwF $\mathbf{Set}^{\mathbf{M}}$ for modelling Σ- and Π-types (Definition 3.2). Given families $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ and $B \in \mathbf{Set}^{\mathbf{M}}(\Gamma.A)$, then $\Sigma\, A\, B \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ is given by the dependent product of sets

$$(\Sigma\, A\, B)_d \triangleq \textstyle\sum_{a\in A_d} B_{(d,a)} \quad \text{equipped with } \mathbf{M}\text{-action} \quad \sigma \cdot (a, b) \triangleq (\sigma \cdot a, \sigma \cdot b) \tag{34}$$

with $\mathrm{pair}\, a\, b$, $\mathrm{fst}\, c$ and $\mathrm{snd}\, c$ as for $\mathbf{Set}$. However, $\Pi\, A\, B \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ is more complicated:

$$\begin{aligned}(\Pi\, A\, B)_d \triangleq \{f \in \textstyle\prod_{\sigma'\in\mathbf{M}} \prod_{a\in A_{\sigma'\cdot d}} B_{(\sigma'\cdot d, a)} \mid (\forall \sigma, \sigma' \in \mathbf{M})(\forall a \in A_{\sigma'\cdot d})&\\ \sigma \cdot (f\, \sigma'\, a) = f\, (\sigma\sigma')\, (\sigma \cdot a) \in B_{(\sigma\sigma'\cdot d, \sigma\cdot a)}\}& \qquad (d \in \Gamma)\end{aligned} \tag{35}$$

with $\mathbf{M}$-action:

$$\frac{\sigma \in \mathbf{M} \qquad f \in (\Pi\, A\, B)_d}{\sigma \cdot f \in (\Pi\, A\, B)_{\sigma\cdot d}} \qquad \sigma \cdot f \triangleq \lambda\sigma' \in \mathbf{M}.\lambda a \in A_{\sigma'\sigma\cdot d}.\, f\,(\sigma'\sigma)\, a$$

Application is given by

$$\frac{g \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash \Pi\, A\, B) \qquad h \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash A)}{\mathrm{app}\, g\, h \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash B[\langle \mathrm{id}\,, h\rangle])} \qquad \mathrm{app}\, g\, h\, d \triangleq g\, d\, \iota\, (h\, d) \qquad (d \in \Gamma) \tag{36}$$

and currying by

$$\frac{k \in \mathbf{Set}^{\mathbf{M}}(\Gamma.A \vdash B)}{\mathrm{lam}\, k \in \mathbf{Set}^{\mathbf{M}}(\Gamma \vdash \Pi\, A\, B)} \qquad \mathrm{lam}\, k\, d \triangleq \lambda\sigma' \in \mathbf{M}.\lambda a \in A_{\sigma'\cdot d} k\, (\sigma' \cdot d, a) \qquad (d \in \Gamma) \tag{37}$$

3.2 CwF structure of $\mathrm{Set}^{\mathbf{M}}_{\mathrm{fs}}$

Now let $\mathbf{M}$ be a monoid of substitutions (Definition 2.1)

▶ **Lemma 3.4.** *The full subcategory $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ is closed under taking finite limits in $\mathbf{Set}^{\mathbf{M}}$.*

Proof. Just note that for each element $(d_1, \ldots, d_n)$ of a finite limit, since the action of $\mathbf{M}$ on the finite limit is componentwise, if each component d_i is supported by X_i, the the whole element is supported by $X_1 \cup \cdots \cup X_n$. ◀

▶ **Lemma 3.5.** *Given $\Gamma \in \mathbf{Set}^{\mathbf{M}}$, define*

$$\Gamma_{\mathrm{fs}} \triangleq \{d \in \Gamma \mid d \textit{ is supported by some finite subset } X \subseteq_{\mathrm{fin}} \mathbb{D}\}$$

Then $\Gamma \mapsto \Gamma_{\mathrm{fs}}$ is the object part of a right adjoint to the inclusion functor $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}} \hookrightarrow \mathbf{Set}^{\mathbf{M}}$.

Proof. First note that by part 2 of Corollary 2.5, Γ_{fs} is closed under the $\mathbf{M}$-action on Γ and hence gives an object in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$. For the right adjointness we just have to see that given $\Gamma' \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$, any morphism $\gamma \in \mathbf{Set}^{\mathbf{M}}(\Gamma', \Gamma)$ factors (necessarily uniquely) through the inclusion $\Gamma_{\mathrm{fs}} \hookrightarrow \Gamma$. But if $d' \in \Gamma'$ is supported by $X \subseteq_{\mathrm{fin}} \mathbb{D}$, then by part 1 of Corollary 2.5, $\gamma\, d' \in \Gamma$ is also supported by X. ◀

▶ Remark 3.6. Combining Lemmas 3.4 and 3.5, we have that if $\mathbf{M}$ is a monoid of substitutions, then $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ is a topos and there is a geometric surjection $\mathbf{Set}^{\mathbf{M}} \to \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ whose direct image part is the right adjoint functor $(\_)_{\mathrm{fs}} : \mathbf{Set}^{\mathbf{M}} \to \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ (see [11, Proposition 4.15(ii)], for example).

The CwF structure on $\mathbf{Set}^{\mathbf{M}}$ given above restricts to one for $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ when $\mathbf{M}$ is a monoid of substitutions. For each $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ we define:

- *Families* $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ are families of $\mathbf{M}$-sets $A \in \mathbf{Set}^{\mathbf{M}}(\Gamma)$ for which the comprehension object (33) is in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$. This amounts to requiring that for each $d \in \Gamma$, every $a \in A_d$ possesses a finite support with respect to the dependently-typed $\mathbf{M}$-action, that is, a finite subset $X \subseteq_{\mathrm{fin}} \mathbb{D}$ satisfying $\operatorname{supp} d \subseteq X$ and $(\forall \sigma, \sigma' \in \mathbf{M})((\forall \mathsf{x} \in X)\ \sigma\,\mathsf{x} = \sigma'\,\mathsf{x}) \Rightarrow \sigma \cdot a = \sigma' \cdot a$. (The condition $\operatorname{supp} d \subseteq X$, i.e. X supports d, is important since with it, when $(\forall \mathsf{x} \in X)\ \sigma\,\mathsf{x} = \sigma'\,\mathsf{x}$ holds, it makes sense to compare $\sigma \cdot a$ and $\sigma' \cdot a$ for equality, because we have $\sigma \cdot a \in A_{\sigma \cdot d} = A_{\sigma' \cdot d} \ni \sigma' \cdot a$.) Note that the functor from Lemma 3.5 extends to a fibre-wise version:

$$\begin{aligned} &\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}, A \in \mathbf{Set}^{\mathbf{M}}(\Gamma) \ \mapsto\ A_{\mathrm{fs}} \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma) \\ &(A_{\mathrm{fs}})_d \triangleq \{a \in A_d \mid (d, a) \in (\Gamma.A)_{\mathrm{fs}}\} \qquad (d \in \Gamma) \end{aligned} \tag{38}$$

- *Elements* $f \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma \vdash A)$ are the same as in $\mathbf{Set}^{\mathbf{M}}$, namely dependently-typed functions $f \in \prod_{d \in \Gamma} A_d$ that preserve the $\mathbf{M}$-action.
- *Re-indexing* is the same as in $\mathbf{Set}^{\mathbf{M}}$, since if X supports $(\gamma\, d', a)$ in $\Gamma.A$, then it supports (d', a) in $\Gamma'.A[\gamma]$.
- *Comprehension objects* are as in (33), since by definition $\Gamma.A$ is in the subcategory $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ when $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$.

▶ Remark 3.7. We noted above that the functors (31) give equivalences when $\mathcal{C} = \mathbf{Set}^{\mathbf{M}}$. Because of the definition of families in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ (and the fact that the objects of $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ are closed under isomorphisms in $\mathbf{Set}^{\mathbf{M}}$), it follows that (31) is also an equivalence when $\mathcal{C} = \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$.

Σ-types in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ are as in (34), since $(a,b) \in (\Sigma\, A\, B)_d$ is supported by any $X \supseteq \mathrm{supp}(d,a)$ that supports $b \in B_{(d,a)}$.

Π-types in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ are obtained by applying the functor (38) to (35). Thus for each $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$, $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ and $B \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma.A)$ we define $\Pi_{\mathrm{fs}}\, A\, B \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ by

$$(\Pi_{\mathrm{fs}}\, A\, B)_d \triangleq ((\Pi\, A\, B)_{\mathrm{fs}})_d \qquad (d \in \Gamma) \tag{39}$$

and one can check that the application (36) and currying (37) operations preserve the finite support property. Combining (35) with (39) in the case that $\Gamma = 1$, we recover the following description of exponentials in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ that will be useful later.

▶ **Lemma 3.8** (Exponentials in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$)**.** *Given $\Gamma, \Delta \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$, their exponential Δ^{Γ} is given by the set $(\Gamma \to_{\mathbf{M}} \Delta)_{\mathrm{fs}}$ of finitely supported elements of*

$$\Gamma \to_{\mathbf{M}} \Delta \;\triangleq\; \{f \in \mathbf{Set}(\mathbf{M} \times \Gamma, \Delta) \mid (\forall \sigma, \sigma' \in \mathbf{M})(\forall d \in \Gamma)\; \sigma \cdot f(\sigma', d) = f(\sigma\sigma', \sigma \cdot d)\}$$

where the $\mathbf{M}$-action on $\Gamma \to_{\mathbf{M}} \Delta$ is

$$\sigma \cdot f \;\triangleq\; \lambda(\sigma', d) \in \mathbf{M} \times \Gamma.\, f(\sigma'\sigma, d)$$

The evaluation morphism $\mathrm{ev} \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Delta^{\Gamma} \times \Gamma, \Delta)$ is given by $\mathrm{ev}(f,d) = f(\iota, d)$; and the currying of $\gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma' \times \Gamma, \Delta)$ is $\mathrm{cur}\,\gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma', \Delta^{\Gamma})$, where $\mathrm{cur}\,\gamma\, d' = \lambda(\sigma, d) \in \mathbf{M} \times \Gamma.\, \gamma(\sigma \cdot d', d)$. ◀

3.3 Hofmann-Streicher universes

Hofmann and Streicher [10] describe a way of lifting a Grothendieck universe in $\mathbf{Set}$ to a type-theoretic universe in any presheaf category. This is used by Bezem, Coquand and Huber [2] to construct a universe within the category $[\mathbf{C}, \mathbf{Set}]$ of cubical sets. We give the construction for the case when the presheaf category is $\mathbf{Set}^{\mathbf{M}}$ for a monoid $\mathbf{M}$ and then apply the coreflection $(\_)_{\mathrm{fs}} : \mathbf{Set}^{\mathbf{M}} \to \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ from Lemma 3.5 when $\mathbf{M}$ is a monoid of substitutions.

Let $\mathcal{U}$ be a Grothendieck universe (see [19], for example) containing $\mathbb{D}$ and hence also $\mathbf{M}$. We lift $\mathcal{U}$ to an object U of $\mathbf{Set}^{\mathbf{M}}$ whose underlying set consists of certain pairs (F, act) where F is a function from $\mathbf{M}$ to $\mathcal{U}$ and $\mathit{act} \in \prod_{\sigma \in \mathbf{M}} \prod_{\sigma' \in \mathbf{M}} (F(\sigma'\sigma))^{F\,\sigma}$. Thus F is an $\mathbf{M}$-indexed family of sets $F\,\sigma \in \mathcal{U}$ ($\sigma \in \mathbf{M}$) and act maps $\sigma, \sigma' \in \mathbf{M}$ to a function $\mathit{act}\,\sigma\,\sigma' : F\,\sigma \to F(\sigma'\sigma)$. We use the following notation for act:

$$\sigma' \cdot a \;\triangleq\; \mathit{act}\,\sigma\,\sigma'\,a \in F(\sigma'\sigma) \qquad (\sigma' \in \mathbf{M}, a \in F\,\sigma) \tag{40}$$

and refer to (F, act) via F. For it to be in U we require act to be a dependently typed $\mathbf{M}$-action (cf. Section 3.1), in the sense that if $a \in F\,\sigma$, then

$$\iota \cdot a = a \in F\,\sigma = F(\iota\sigma) \tag{41}$$

$$\sigma'' \cdot (\sigma' \cdot a) = (\sigma''\sigma') \cdot a \in F((\sigma''\sigma')\sigma) = F(\sigma''(\sigma'\sigma)) \tag{42}$$

If $F \in U$ and $\sigma \in \mathbf{M}$, then we get $\sigma \cdot F \in U$ by defining

$$(\sigma \cdot F)\,\sigma' \triangleq F(\sigma'\sigma) \tag{43}$$

with dependently typed $\mathbf{M}$-action on $\sigma \cdot F$ given by the one for F. (This makes sense, since if $a \in (\sigma \cdot F)\,\sigma' = F(\sigma'\sigma)$, then $\sigma'' \cdot a \in F(\sigma''(\sigma'\sigma)) = (\sigma \cdot F)(\sigma''\sigma')$.) These definitions make U into an object in $\mathbf{Set}^{\mathbf{M}}$, since one can easily check that $\iota \cdot F = F$ and $\sigma' \cdot (\sigma \cdot F) = (\sigma'\sigma) \cdot F$.

▶ **Definition 3.9 (Hofmann-Streicher lifting for $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$).** Let $\mathbf{M}$ be a monoid of substitutions and let $U \in \mathbf{Set}^{\mathbf{M}}$ be the $\mathbf{M}$-set derived from a Grothendieck universe $\mathcal{U} \in \mathbf{Set}$ as above. Let $E \in \mathbf{Set}^{\mathbf{M}}(U)$ be the family mapping each $F \in U$ to

$$E_F \triangleq F\,\iota \quad \text{with dependently-typed } \mathbf{M}\text{-action given by (40)}$$

(Note that this makes sense, because if $a \in E_F = F\,\iota$, then $\sigma \cdot a \in F(\sigma\iota) = F(\iota\sigma) = (\sigma \cdot F)\,\iota = E_{\sigma \cdot F}$.) Applying the functor $(\_)_{\mathrm{fs}} : \mathbf{Set}^{\mathbf{M}} \to \mathbf{Sub}$ from Lemma 3.5 to the projection morphism $\mathrm{p} : U.E \to U$ we get a morphism $\mathrm{p} : (U.E)_{\mathrm{fs}} \to U_{\mathrm{fs}}$ in $\mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ and (by Remark 3.7) a corresponding family $E_{\mathrm{fs}} \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(U_{\mathrm{fs}})$, where

$$(E_{\mathrm{fs}})_F \triangleq \{a \in F\,\iota \mid (F, a) \in (U.E)_{\mathrm{fs}}\} \qquad (F \in U_{\mathrm{fs}}) \tag{44}$$

Note that if $F \in U_{\mathrm{fs}}$, then $F\,\iota \in \mathcal{U}$ and hence $(E_{\mathrm{fs}})_F \in \mathcal{U}$. In general we say that a family $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ *has fibres in* $\mathcal{U}$ if $A_d \in \mathcal{U}$ for all $d \in \Gamma$. The family (44) not only has fibres in $\mathcal{U}$, but is weakly universal among such families, in the following sense.

▶ **Theorem 3.10.** *Let $\mathbf{M}$ be a monoid of substitutions and $E_{\mathrm{fs}} \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(U_{\mathrm{fs}})$ be the Hofmann-Streicher universe in the CwF of finitely supported $\mathbf{M}$-sets derived from a Grothendieck universe $\mathcal{U} \in \mathbf{Set}$. Then for each $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$ and family $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ with fibres in $\mathcal{U}$, there is a morphism $\ulcorner A \urcorner \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma, U_{\mathrm{fs}})$ with $A = E_{\mathrm{fs}}[\ulcorner A \urcorner]$.*

Proof. For each $d \in \Gamma$ and $\sigma \in \mathbf{M}$ define

$$\ulcorner A \urcorner\, d\, \sigma \triangleq A_{\sigma \cdot d} \in \mathcal{U} \tag{45}$$

If $\sigma' \in \mathbf{M}$ and $a \in \ulcorner A \urcorner\, d\, \sigma = A_{\sigma \cdot d}$, then the dependently-typed $\mathbf{M}$-action on $A \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma)$ gives us $\sigma' \cdot a \in A_{\sigma' \cdot (\sigma \cdot d)} = \ulcorner A \urcorner\, d\,(\sigma'\sigma)$, satisfying (41) and (42). So for each $d \in \Gamma$, we get $\ulcorner A \urcorner\, d \in U$. Furthermore

$$\begin{aligned}
(\sigma' \cdot (\ulcorner A \urcorner\, d))\,\sigma &= (\ulcorner A \urcorner\, d)(\sigma\sigma') && \text{by (43)}\\
&= A_{\sigma\sigma' \cdot d} && \text{by (45)}\\
&= A_{\sigma \cdot (\sigma' \cdot d)} && \\
&= \ulcorner A \urcorner\,(\sigma' \cdot d)\,\sigma && \text{by (45) again}
\end{aligned}$$

so that $\ulcorner A \urcorner \in \mathbf{Set}^{\mathbf{M}}(\Gamma, U)$. Since $\Gamma \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}$, if follows from Lemma 3.5 that $\ulcorner A \urcorner$ factors through $U_{\mathrm{fs}} \hookrightarrow U$ to give $\ulcorner A \urcorner \in \mathbf{Set}^{\mathbf{M}}_{\mathrm{fs}}(\Gamma, U_{\mathrm{fs}})$. Since it follows from this that $\mathrm{supp}(\ulcorner A \urcorner\, d) \subseteq \mathrm{supp}\, d$, if $a \in A_d$ is supported by $X \supseteq \mathrm{supp}\, d$, then X also supports $(\ulcorner A \urcorner\, d, a)$ in $U.E$. Therefore by (44), for all $d \in \Gamma$ we have

$$A_d = \{a \in \ulcorner A \urcorner\, d\, \iota \mid (\ulcorner A \urcorner\, d, a) \in (U.E)_{\mathrm{fs}}\} = E_{\mathrm{fs}}[\ulcorner A \urcorner]_d$$

so that re-indexing the family E_{fs} along $\ulcorner A \urcorner$ gives $E_{\mathrm{fs}}[\ulcorner A \urcorner] = A$. ◀

4 Cubical sets with diagonals

In footnote 2 of [2] the authors say

> 'In a previous attempt, we have been considering the category of finite sets with maps $I \to J + 2$ (i.e. the Kleisli category for the monad $I + 2$). This category appears on pages 47–48 in Pursuing Stacks [8] as "in a sense, the smallest test category".'

Call this category $\mathbf{S}$. Thus $\mathbf{S}$ is like the category $\mathbf{C}$ from Section 1, but without the injectivity condition (1) on morphisms. In Section 2 we moved from the small category $\mathbf{C}$ to the submonoid $\mathbf{Cb}$ of the monoid $\mathbf{Sb}$ of all substitutions (Definition 2.1) and replaced cubical sets $[\mathbf{C}, \mathbf{Set}]$ by the equivalent category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ of finitely supported $\mathbf{Cb}$-sets. If one starts from $\mathbf{S}$ rather than $\mathbf{C}$, then one gets the whole monoid of substitutions $\mathbf{Sb}$ and can consider the category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ of finitely supported $\mathbf{Sb}$-sets.

▶ **Theorem 4.1.** *The categories* $[\mathbf{S}, \mathbf{Set}]$ *and* $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ *are equivalent.*

Proof. One can check that the proof method of Theorem 2.9 still goes through when one replaces the category $\mathbf{C}$ by $\mathbf{S}$ and the monoid $\mathbf{Cb}$ by $\mathbf{Sb}$. Indeed the proof is easier, because the 'homogeneity' property (the analogue of Lemma 2.8) needed for the fullness and essential surjectivity of the functor $I^* : [\mathbf{S}, \mathbf{Set}] \to \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ is trivial: for each $s \in \mathbf{S}(X, Y)$ we get a substitution $\sigma \in \mathbf{Sb}$ that agrees with s on X simply by defining

$$\sigma\,\mathsf{x} \triangleq \begin{cases} s\,\mathsf{x} & \text{if } \mathsf{x} \in X \\ \mathsf{x} & \text{otherwise.} \end{cases}$$

◀

One advantage of $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ over $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$ stems from the following theorem. Regarding each $\Gamma \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ as a nominal set as in Section 2.1, we can make the nominal set $[\mathbb{D}]\Gamma$ of name abstractions into an object of $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ via an $\mathbf{Sb}$-action satisfying

$$\mathsf{x} \mathrel{\#} \sigma \;\Rightarrow\; \sigma \cdot \langle \mathsf{x}\rangle d = \langle \mathsf{x}\rangle(\sigma \cdot d) \qquad (\mathsf{x} \in \mathbb{D}, \sigma \in \mathbf{Sb}, d \in \Gamma) \tag{46}$$

where we regard $\mathbf{Sb}$ as a nominal set, and hence make sense of the condition $\mathsf{x} \mathrel{\#} \sigma$, via the conjugation action of permutations: $\pi \cdot \sigma \triangleq \pi\sigma\pi^{-1}$. (The support of σ with respect to this action is $\mathrm{Dom}\,\sigma \cup \bigcup_{\mathsf{x} \in \mathrm{Dom}\,\sigma} \mathrm{supp}(\sigma\,\mathsf{x})$.) Thus the action of σ is well-defined by sending an element $\langle \mathsf{x}\rangle d \in [\mathbb{D}]\Gamma$ to $\langle \mathsf{y}\rangle(\sigma(\mathsf{x}\ \mathsf{y}) \cdot d)$, where y is some (or indeed, any) direction satisfying $\mathsf{y} \mathrel{\#} (\mathsf{x}, \sigma, d)$; cf. [15, Theorem 9.18].

▶ **Theorem 4.2.** *$[\mathbb{D}]\Gamma$ is isomorphic in the category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ to the exponential Γ^{I} of Γ by the interval object* I *from Example 2.3.*

Proof. Recall the definition of $[\mathbb{D}]\Gamma$ in terms of the equivalence relation $\approx_\alpha$ (26). If $(\mathsf{x}, d) \approx_\alpha (\mathsf{x}', d')$, then picking any $\mathsf{y} \mathrel{\#} (\mathsf{x}, d, \mathsf{x}', d')$ we have $(\mathsf{y}\ \mathsf{x}) \cdot d = (\mathsf{y}\ \mathsf{x}') \cdot d' \in \Gamma$. Since for any $i \in \mathrm{I}$, the substitutions $(i/\mathsf{y})(\mathsf{y}\ \mathsf{x})$ and (i/x) agree on $\mathrm{supp}\,d$, we have $(i/\mathsf{x}) \cdot d = (i/\mathsf{y})(\mathsf{y}\ \mathsf{x}) \cdot d$; similarly $(i/\mathsf{x}') \cdot d' = (i/\mathsf{y})(\mathsf{y}\ \mathsf{x}') \cdot d'$. Therefore $(i/\mathsf{x}) \cdot d = (i/\mathsf{x}') \cdot d'$. So there is a well-defined function $\mathrm{ev} : [\mathbb{D}]\Gamma \times \mathrm{I} \to \Gamma$ satisfying

$$\mathrm{ev}(\langle \mathsf{x}\rangle d, i) = (i/\mathsf{x}) \cdot d \qquad (\mathsf{x} \in \mathbb{D}, d \in \Gamma, i \in \mathrm{I}) \tag{47}$$

(Note that since $\mathbf{Cb}$ does not contain the substitution (i/x) when $i \in \mathbb{D} - \{\mathsf{x}\}$, it would not be possible to make this definition in the category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$.)

When $\mathsf{x} \mathrel{\#} \sigma$, we have $\sigma(i/\mathsf{x}) = (\sigma\, i/\mathsf{x})\sigma \in \mathbf{Sb}$ and hence $\sigma \cdot \mathrm{ev}(\langle \mathsf{x}\rangle d, i) = \mathrm{ev}(\sigma \cdot \langle \mathsf{x}\rangle d, \sigma \cdot i)$. So ev is a morphism in $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}([\mathbb{D}]\Gamma \times \mathrm{I}, \Gamma)$. We verify that it has the universal property required for the exponential. Given $\gamma \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}(\Gamma' \times \mathrm{I}, \Gamma)$ we get a well-defined function $\mathrm{cur}\,\gamma : \Gamma' \to [\mathbb{D}]\Gamma$

$$\mathrm{cur}\,\gamma\,d' \;\triangleq\; \langle \mathsf{x}\rangle\gamma(d', \mathsf{x}) \quad \text{where } \mathsf{x} \mathrel{\#} d' \tag{48}$$

This satisfies $\sigma \cdot (\operatorname{cur}\gamma\, d') = \operatorname{cur}\gamma\,(\sigma \cdot d')$ and hence gives a morphism $\operatorname{cur}\gamma \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}(\Gamma', [\mathbb{D}]\Gamma)$. Note that

$$
\begin{aligned}
\operatorname{ev}(\operatorname{cur}\gamma\, d', i) &= \operatorname{ev}(\langle \mathsf{x}\rangle\gamma(d',\mathsf{x}), i) && \text{where } \mathsf{x} \mathrel{\#} d' \\
&= (i/\mathsf{x}) \cdot \gamma(d',\mathsf{x}) && \text{by (47)} \\
&= \gamma((i/\mathsf{x}) \cdot d', (i/\mathsf{x}) \cdot \mathsf{x}) && \text{since } \gamma \text{ is a morphisms in } \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}} \\
&= \gamma(d', i) && \text{since } \mathsf{x} \mathrel{\#} d'
\end{aligned}
$$

so that $\operatorname{ev} \circ (\operatorname{cur}\gamma \times \mathrm{id}_{\mathrm{I}}) = \gamma$. The uniqueness of $\operatorname{cur}\gamma$ with this property follows from an η-rule for elements of $[\mathbb{D}]\Gamma$:

$$
(\forall p \in [\mathbb{D}]\Gamma)(\forall \mathsf{x} \in \mathbb{D})\ \mathsf{x} \mathrel{\#} p \ \Rightarrow\ p = \langle \mathsf{x}\rangle \operatorname{ev}(p, \mathsf{x}) \tag{49}
$$

which in turn follows the fact that for any $\langle \mathsf{x}\rangle d \in [\mathbb{D}]\Gamma$ and $\mathsf{y} \mathrel{\#} (\mathsf{x}, d)$ it is the case that $\langle \mathsf{x}\rangle d = \langle \mathsf{y}\rangle((\mathsf{y}\ \mathsf{x}) \cdot d) = \langle \mathsf{y}\rangle((\mathsf{y}/\mathsf{x}) \cdot d)$. ◀

Iterating the theorem, we get that the exponential Γ^{I^n} (the object of n-cubes in Γ) is isomorphic to $[\mathbb{D}]^{(n)}\Gamma$, where

$$
\begin{aligned}
[\mathbb{D}]^{(0)}\Gamma &\triangleq \Gamma \\
[\mathbb{D}]^{(n+1)}\Gamma &\triangleq [\mathbb{D}]([\mathbb{D}]^{(n)}\Gamma)
\end{aligned} \tag{50}
$$

Note that $[\mathbb{D}]^{(n)}\Gamma \in \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ is the nominal set of n-ary name abstractions $\langle \mathsf{x}_1, \ldots, \mathsf{x}_n\rangle d$ (with $\mathsf{x}_1, \ldots, \mathsf{x}_n$ mutually distinct directions) equipped with the **Sb**-action satisfying the evident generalisation of (46) to n-ary name abstractions.

One may think of objects of $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ as cubical sets 'with diagonals', because (unlike the case for $[\mathbf{C}, \mathbf{Set}] \simeq \mathbf{Set}_{\mathrm{fs}}^{\mathbf{Cb}}$) each square $\langle \mathsf{x}, \mathsf{y}\rangle d \in [\mathbb{D}]^{(2)}\Gamma$ contains a diagonal path $\langle \mathsf{z}\rangle(\mathsf{z}/\mathsf{x})(\mathsf{z}/\mathsf{y}) \cdot d \in [\mathbb{D}]\Gamma$. Of course, under the isomorphism in the above theorem, diagonalization $[\mathbb{D}]^{(2)}\Gamma \to [\mathbb{D}]\Gamma$ corresponds to the morphism $\Gamma^{\mathrm{I}^2} \to \Gamma^{\mathrm{I}}$ given by precomposing with the diagonal $\langle \mathrm{id}_{\mathrm{I}}, \mathrm{id}_{\mathrm{I}}\rangle : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$.

▶ Remark. Gabbay and Hofmann [7] prove the analogue of Theorem 4.2 for their category of 'nominal renaming sets'. This category is like **01-Nom** except that it uses nominal sets equipped with name-for-name substitutions, rather than 01-for-name substitutions. They also have an analogue of Theorem 2.13: an equivalence between the category of nominal renaming sets and a sheaf subcategory the presheaf category $[\mathbf{F}, \mathbf{Set}]$, where $\mathbf{F}$ is the small category whose objects are finite subsets of $\mathbb{D}$ and whose morphisms are all functions between such subsets; see [7, Theorem 38].

5 Conclusion

We have shown how to reformulate cubical sets, originally given as presheaves, in terms of sets whose elements are finitely supported with respect to a given action of a monoid of name substitutions. Because of the equivalences we have established (Theorems 2.9, 2.13 and 4.1), there is no difference in the category-theoretic properties of the two formulations. However, the approach using monoids of name substitutions leads to a relatively simple notion of *family* of cubical sets (Section 3) and allows access to the well-developed nominal sets notions of *freshness* to calculate with degeneracy of cubes and *name abstraction* to calculate with paths (proofs of equality); see the implementation of Kan cubical sets [4].

We saw that in the category $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$, paths are arbitrary functions from an interval object (Theorem 4.2). Coquand [3] has noted that this property can enable simpler formulations of the Kan filling condition, simpler proofs of closure of Kan complete families under taking Π-types, and more natural realizers for operations like the elimination rule for the circle. So there may be a sub-CwF of $\mathbf{Set}_{\mathrm{fs}}^{\mathbf{Sb}}$ consisting of families satisfying some Kan-filling condition which yields a technically simpler model of univalent foundations than the one in [2]. Of course, to be computationally useful, such a model has to exist in a constructive meta-theory. We leave this for future investigation.

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