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The relationship between a strip Wiener–Hopf problem and a line Riemann–Hilbert problem

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In this paper, the Wiener–Hopf factorization problem is presented in a unified framework with the Riemann–Hilbert factorization. This allows to establish the exact relationship between the two types of factorization. In particular, in the Wiener–Hopf problem one assumes more regularity than for the Riemann–Hilbert problem. It is shown that Wiener–Hopf factorization can be obtained using Riemann–Hilbert factorization on certain lines.

Keywords: Wiener–Hopf; Riemann-Hilbert; integral equations.

1. Introduction

The Wiener–Hopf and the Riemann–Hilbert problems are a subject of many books and articles (Câmara & dos Santos, 1999; Ehrhardt & Speck, 2002; Èrkhardt & Spitkovskiĭ, 2001; Gohberg *et al.*, 2003; Rogosin & Mishuris). The similarities of the two techniques are easily visible and have been noted in many places. Nevertheless, to the author's knowledge there was no systematic study of the exact relationship of the two methods. To fill this gap is the purpose of this article.

It has been suggested in Noble (1958, Chapter 4.2) that the Wiener–Hopf equation are a special case of a Riemann–Hilbert equation. Specifically, the Riemann–Hilbert problem connects boundary values of two analytic functions on a contour and the Wiener–Hopf equation is defined on the strip of common analyticity of two functions. In the simplest case, both methods use the key concept of functions analytic in half-planes. The additional regularity for the Wiener–Hopf equation allows to express the solution in more simple terms than the Riemann–Hilbert equation. To be well defined the Riemann–Hilbert problem requires some additional regularity, e.g. the coefficients need to be Hölder continuous on the contour.

In a different book (Gakhov & Čerskiĭ, 1978, Chapter 14.4), it has been stated that the Wiener–Hopf equations results from a bad choice of functions spaces and instead a Riemann–Hilbert equations should be considered. Confusingly, those Riemann–Hilbert equations are sometimes referred to as a Wiener–Hopf equations. Historically, there has been insufficient interaction between the communities using the Wiener–Hopf and the Riemann–Hilbert methods, this results in obscuring disagreements in terminology and notations. Such differences if not reconciled properly have a tendency to widen.

We consider the following problem as an illustration. Given a function $F(t)$ on the real axis:

$$F(t) = \sqrt{\frac{t^2 - (\frac{1}{2}i + 1)t - \frac{1}{4}i + \frac{3}{4}}{t^2 - \frac{3}{2}it + 1} \frac{(t - i + 2)}{(t + i + \frac{3}{2})}} \tag{1.1}$$

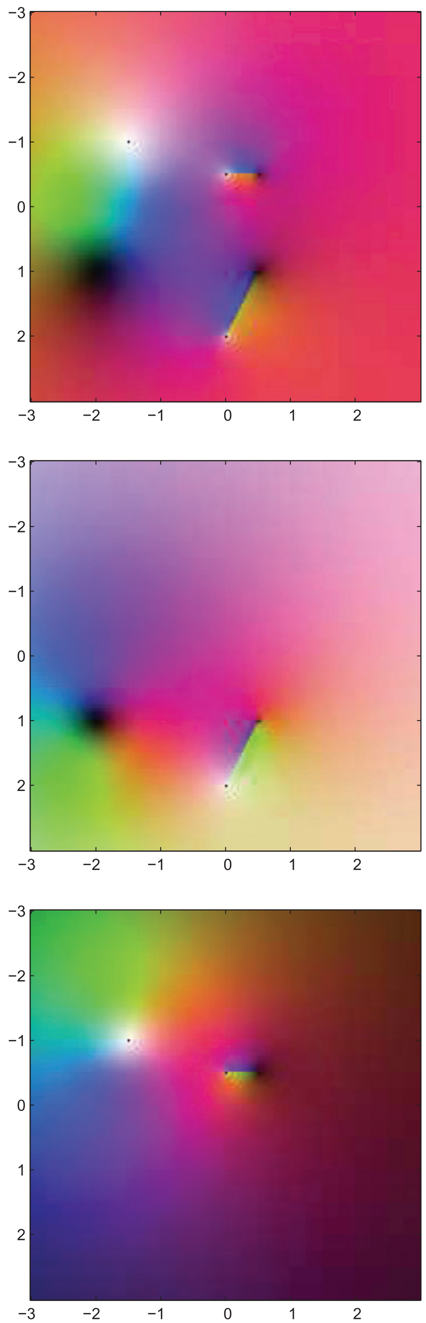

FIG. 1. An image of functions (1.2). The top picture shows $F(z)$ followed by $F^{+}(z)$ and $F^{-}(z)$. We use a scheme developed by John Richardson, black is small magnitude and white is large magnitude. Branch cuts appear as discontinuities. Produced using MATLAB package `zviz.m`.

find two factors $F^+(t)$ and $F^-(t)$, which have analytic extensions in the upper and lower half-planes respectively. In this rare case, the factorization can be obtained by inspection

$$\begin{aligned} F(t) &= \left(\sqrt{\frac{(t + \frac{1}{2}i - \frac{1}{2})}{(t + \frac{1}{2}i)}} \frac{1}{(t + i + \frac{3}{2})} \right) \times \left(\sqrt{\frac{(t - i - \frac{1}{2})}{(t - 2i)}} (t - i + 2) \right) \\ &= F^+(t) F^-(t). \end{aligned} \tag{1.2}$$

These functions are depicted in Fig. 1. We will comment on this example in both (the Riemann–Hilbert and Wiener–Hopf) frameworks at the end of this paper.

The structure of the paper is as follows. Section 2 collects some preliminaries. In Section 3, the usual theory of the Wiener–Hopf equation is recalled. It is presented to highlight the differences with the theory of the Riemann–Hilbert equations, Section 4. Section 5 describes at the relationship between the two methods in the context of integral equations. In Section 6, the solution to Wiener–Hopf and Riemann–Hilbert factorization is re-expressed in terms of the Fourier transforms instead of the Cauchy type integrals. This allows to prove theorems about the exact relationship of Wiener–Hopf and Riemann–Hilbert factorization.

2. Preliminaries

This section will review important properties of the Fourier transform, the Cauchy type integral and the relationship between the two. We also states the generalized version of Liouville's theorem together with analytic continuation, in the form which will be used later.

The *Fourier transform* of a function $f(t)$ is defined as

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t, \quad -\infty < x < \infty. \tag{2.1}$$

Throughout this paper the Fourier image and original function will be denoted by the same letter but they will be upper and lower case respectively. The inverse Fourier transform is given:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, \mathrm{e}^{-\mathrm{i}xt}\, \mathrm{d}t, \quad -\infty < t < \infty.$$

It is an important fact that if $f(t) \in L_2(\mathbb{R})$ then $F(t) \in L_2(\mathbb{R})$. This makes the space of square integrable functions very convenient to work in. Moreover, Fourier transform is an isometry of $L_2(\mathbb{R})$ due to Plancherel's theorem.

The Fourier transform need not be confined to the real axis, as long as the integral (2.1) is absolutely convergent for a complex x. On an open domain consisting of such parameters x, the Fourier transform is an analytic function. The Paley–Wiener theorem (see (2.6–2.8)) gives the conditions on the decay at infinity of $f(t)$ to ensure such analyticity in different strips and half-planes.

Another key concept is the Cauchy type integral. Let $F(\tau)$ be integrable on a simple Jordan curve L, the integral

$$\frac{1}{2\pi\mathrm{i}} \int_L \frac{F(\tau)}{\tau - z}\, \mathrm{d}\tau$$

is called a *Cauchy type integral* and defines an analytic function on the complement of L. If L divides the complex plane into two disjoint open components then it makes sense to consider two functions

F^- and F^+ on the respective domains. In particular, for the real line we use the notation:

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, \mathrm{d}\tau = \begin{cases} F^+(z) & \text{if } \operatorname{Im} z > 0, \\ F^-(z) & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{2.2}$$

The relationship of the Cauchy type integral to the Fourier transform is outlined below.

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, \mathrm{d}\tau = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(t) \, \mathrm{e}^{\mathrm{i}zt} \, \mathrm{d}t, \quad \text{if } \operatorname{Im} z > 0, \tag{2.3}$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - z} \, \mathrm{d}\tau = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(t) \, \mathrm{e}^{\mathrm{i}zt} \, \mathrm{d}t, \quad \text{if } \operatorname{Im} z < 0. \tag{2.4}$$

The above integrals are the two ways of arriving at functions analytic in half-planes.

It was already mentioned that $L_2(\mathbb{R})$ is a very convenient with regards to the Fourier transform. The Hölder continuous functions produce well-defined boundary values of the Cauchy integral. Thus, their intersection Gakhov & Čerskiĭ (1978), § 1.2:

$$\{\{0\}\} = L_2(\mathbb{R}) \cap \text{Hölder},$$

turns out to be very useful for both the Wiener–Hopf and the Riemann–Hilbert problems. The pre-image of $\{\{0\}\}$ under the Fourier transform is denoted $\{0\}$.

Formulae (2.3–2.4) show the significance of functions which are zero on the positive or the negative half lines. Given a function on the real axis, we can define the splitting

$$f_+(t) = \begin{cases} f(t) & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases} \quad f_-(t) = \begin{cases} 0 & \text{if } t > 0, \\ -f(t) & \text{if } t < 0. \end{cases} \tag{2.5}$$

We will say that if $f \in \{0\}$ then $f_+(t) \in \{0, \infty\}$ and $f_-(t) \in \{-\infty, 0\}$.

In the rest of the paper, we will need to refer to functions that are analytic on strips or (shifted) half-planes. Following Gakhov & Čerskiĭ (1978), § 13, we define $f \in \{a\}$ if $\mathrm{e}^{-ax} f \in \{0\}$, that is a shift in the Fourier space. Finally, $f = f_+ + f_- \in \{a, b\}$ if $f_+ \in \{a\}$ and $f_- \in \{b\}$. From the definition of f_+ and f_- it is clear that also $f_+ \in \{a, \infty\}$ and $f_- \in \{-\infty, b\}$.

The Fourier transform of functions in the class $\{a, b\}$ is denoted $\{\{a, b\}\}$. The celebrated Paley–Wiener theorem states that the following is equivalent:

$$F_+(z) \text{ analytic in } \operatorname{Im} z > a \iff F_+(z) \in \{\{a, \infty\}\}, \tag{2.6}$$

$$F_-(z) \text{ analytic in } \operatorname{Im} z < b \iff F_-(z) \in \{\{-\infty, b\}\}, \tag{2.7}$$

$$F(z) \text{ analytic in } a < \operatorname{Im} z < b \iff F(z) \in \{\{a, b\}\}. \tag{2.8}$$

Recall, a convolution of two functions on the real line is given by

$$h(t) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} f(t - s) g(s) \, \mathrm{d}s. \tag{2.9}$$

One of the key properties, which make Fourier techniques useful in integral equations, is that the Fourier transform of the convolution is the product of the Fourier transform of the functions $f(t)$ and $g(t)$, i.e. $H(x) = F(x)G(x)$.

If $g \in \{a, b\}$ and $f \in \{\alpha, \beta\}$ $(a < b,\ \alpha < \beta)$ then $h(x)$ will be in the class $\{\max(a, \alpha), \min(b, \beta)\}$ (Gakhov & Čerskiĭ, 1978, § 1.3) provided

$$\max(a, \alpha) < \min(b, \beta).$$

After taking the Fourier transform this will become:

$$H(z) = F(z)G(z)$$

and $H(z) \in \{\{\max(a, \alpha), \min(b, \beta)\}\}$ i.e analytic in the strip

$$\max(a, \alpha) < \mathrm{Im}(z) < \min(b, \beta). \tag{2.10}$$

We will need a generalized version of Liouville's theorem together with analytic continuation.

THEOREM 2.1 (Gakhov & Čerskiĭ, 1978, § 3.1) If functions $F_1(z)$, $F_2(z)$ are analytic in the upper and lower half-planes, respectively, with exception of $z_0 = \infty$, z_k $(k = 1, 2, \ldots, n)$ where they have poles with principal parts:

$$G_0(z) = c_1^0 z + \cdots + c_{m_0}^0 z_0^m, \tag{2.11}$$

$$G_k\left(\frac{1}{z - z_k}\right) = \frac{c_1^k}{z - z_k} + \cdots + \frac{c_{m_k}^k}{(z - z_k)^{m_k}} \tag{2.12}$$

with $F_1(z)$ and $F_2(z)$ equal on the real axis, then they define a rational function on the whole plane:

$$F(z) = c + G_0(z) + \sum_1^n G_k\left(\frac{1}{z - z_k}\right),$$

where c is an arbitrary constant. Poles z_k can lie anywhere on the half-planes or on the real axis.

3. The Wiener–Hopf method

In this section, the Wiener–Hopf method is presented in the way it appears in most papers. This will be revisited to establish the relationship with the Riemann–Hilbert method. For some applications of Wiener-Hopf in areas like elasticity, crack propagation and acoustics see Mishuris & Rogosin (2014), Jaworski & Peake (2013), Abrahams *et al.* (2008) and Veitch & Peake (2008).

The next theorem provides a constructive existence theorem for the additive Wiener–Hopf decomposition (in terms of a Cauchy type integral).

THEOREM 3.1 (Noble, 1958, Ch. 1.3) Let $f(\alpha)$ be a function of variable $\alpha = \sigma + \mathrm{i}\tau$, analytic in the strip $\tau_- < \tau < \tau_+$, such that $f(\sigma + \mathrm{i}\tau) < C|\sigma|^{-p}$, $p > 0$ as $|\sigma| \to \infty$, the inequality holding uniformly for all τ in the strip $\tau_- + \epsilon < \tau < \tau_+ - \epsilon$, $\epsilon > 0$. Then for $\tau_- < c < \tau < d < \tau_+$,

$$f(\alpha) = f_-(\alpha) + f_+(\alpha),$$

$$f_-(\alpha) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty + id}^{\infty + id} \frac{f(\zeta)}{\zeta - \alpha}\, \mathrm{d}\zeta; \quad f_+(\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty + \mathrm{i}c}^{\infty + \mathrm{i}c} \frac{f(\zeta)}{\zeta - \alpha}\, \mathrm{d}\zeta, \tag{3.1}$$

where f_- is analytic in $\tau < \tau_+$ and f_+ is analytic in $\tau > \tau_-$.

We wish to highlight the simplicity of demonstration of this result. Indeed, to prove (3.1) one applies Cauchy's integral theorem to the rectangle with vertices $\pm a + \mathrm{i}c$, $\pm a + \mathrm{i}d$. From the assumption as regards to the behaviour of $f(\alpha)$ as $|\sigma| \to \infty$ in the strip, the integral on $\sigma = \pm a$ tends to zero as $a \to \infty$ and the result follows.

The next theorem is a useful variation of the previous theorem, obtained by taking logarithms. This provides a way to achieve the multiplicative Wiener–Hopf factorization.

THEOREM 3.2 (Noble, 1958, Ch. 1.3) If $\log K$ satisfies the conditions for Theorem 3.1 in particular that $K(\alpha)$ is analytic and non-zero in the strip and $K(\alpha) \to 1$ as $|\sigma| \to \infty$ then $K(\alpha) = K_+(\alpha)K_-(\alpha)$ and K_+, K_- are analytic, bounded and non-zero when $\tau > \tau_-$, $\tau < \tau_+$ respectively.

The above factors are unique up to a constant (Kranzer, 1967). In other words, if there are two such factorizations $K = K_+K_-$ and $K = P_+P_-$, then

$$K_+ = cP_+ \quad \text{and} \quad K_- = c^{-1}P_-,$$

where c is some complex constant. This can be seen by applying analytic continuation to K_+/P_+ and P_-/K_- and then using the extended Liouville Theorem 2.1.

4. The Riemann–Hilbert method

In this section, the main differences from the Wiener–Hopf method are presented. They are rooted in the conditions imposed on the functions and the resulting formula. For applications of the Riemann–Hilbert problem the interested reader is refereed to Fokas *et al.* (2006) and Gakhov (1966).

The Riemann–Hilbert problem can be stated as follows:

PROBLEM 4.1 (Riemann–Hilbert) On the real line two functions are given, $H(t)$ and non-zero $D(t)$ with

$$D(t) - 1 \in \{\{0\}\} \quad \text{and} \quad H(t) \in \{\{0\}\}.$$

It is required to find two functions $F^{\pm}(z)$ analytic in the upper and lower half-planes such that their boundary values $F^{\pm}(t)$ on the real line satisfy two conditions:

(1) $F^{\pm}(t)$ belong to the classes $\{\{0, \infty\}\}$ and $\{\{-\infty, 0\}\}$;

(2) The identity holds:

$$D(t)F^-(t) + H(t) = F^+(t), \quad \text{for all real } t. \tag{4.1}$$

As it has been seen for the Wiener–Hopf method the key steps in the solution is the additive splitting or jump problem (Theorem 3.1) and the factorization (Theorem 3.2). First splitting is addressed as before.

PROBLEM 4.2 (Jump problem) On the real line a function $F(t) \in \{\{0\}\}$ is given. It is required to find two functions $F^{\pm}(z)$ analytic in the upper and lower half-planes with boundary functions on the real line belonging to the classes $\{\{0, \infty\}\}$ and $\{\{-\infty, 0\}\}$ and satisfying:

$$F(t) = F^+(t) + F^-(t),$$

on the real line.

The solution is offered by the Sokhotskyi–Plemelj formula:

$$F^+(x) - F^-(x) = F(x), \quad F^+(x) + F^-(x) = \frac{1}{\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau - x}\, \mathrm{d}\tau. \tag{4.2}$$

We shall note that the derivation of (4.2) is more involved than the proof of Theorem 3.1.

Next the factorization problem is examined. The *index* of a continuous non-zero function $K(t)$ on the real line is

$$\mathrm{ind}(K(t)) = \frac{1}{2\pi}\left(\lim_{t\to+\infty} \arg K(t) - \lim_{t\to-\infty} \arg K(t)\right). \tag{4.3}$$

In other words, the index is the winding number of the curve $(\operatorname{Re} K(t), \operatorname{Im} K(t))$ $t \in \mathbb{R}$. Note that $\mathrm{ind}(t-i)/(t+i) = 1$. Given a function $K(t)$ with index κ one can reduce it to zero index by considering a new function

$$K_0(t) = K(t)\left(\frac{t-i}{t+1}\right)^{-\kappa}.$$

The reason it is important to consider functions $K(t)$ with zero index is to ensure $\ln K(t)$ is single valued. For the rest of this paper, we will assume that all functions have zero index. Taking logarithms and applying the Sokhotskyi–Plemelj formula, we get a solution to the factorization problem Gakhov & Čerskiĭ (1978).

Problem 4.3 (Riemann–Hilbert factorization) Let a non-zero function $K(t)$, such that $K(t) - 1 \in \{\{0\}\}$ and $\mathrm{ind}\, K(t) = 0$, be given. It is required to find two functions $K^{\pm}(z)$ analytic in the upper and lower half-planes with boundary functions $K^{\pm}(t)$ on the real line belonging to the classes $\{\{0,\infty\}\}$ and $\{\{-\infty, 0\}\}$ and satisfying

$$K(t) = K^+(t)K^-(t),$$

on the real line.

5. Relationship between Wiener-Hopf and Riemann-Hilbert via integral equations

This section demonstrates how one type of integral equation can either lead to the Wiener–Hopf problem or the Riemann–Hilbert problem, depending on the class of function where the solution is sought. Integral equations have historically motivated the introduction of the Wiener–Hopf equation (Lawrie & Abrahams, 2007).

In applications, a time-invariant process can be modelled by an integral equations with convolution on the half-line:

$$\int_0^{\infty} k(x-y) f(y)\, \mathrm{d}y = g(x), \quad 0 < x < \infty. \tag{5.1}$$

Here the kernel $k(x-y) \in \{a, b\}$ represents the process, $g(x)$ is a given output and $f(y)$ is an input to be determined. To solve the equation (5.1), we complement the domain of x:

$$\int_0^{\infty} k(x-y) f(y)\, \mathrm{d}y = h(x), \quad -\infty < x < 0, \tag{5.2}$$

where $h(x)$ is unknown. Then, by applying the Fourier transform, we get the equation:

$$F_+(\alpha)K(\alpha) - G_+(\alpha) = H_-(\alpha). \tag{5.3}$$

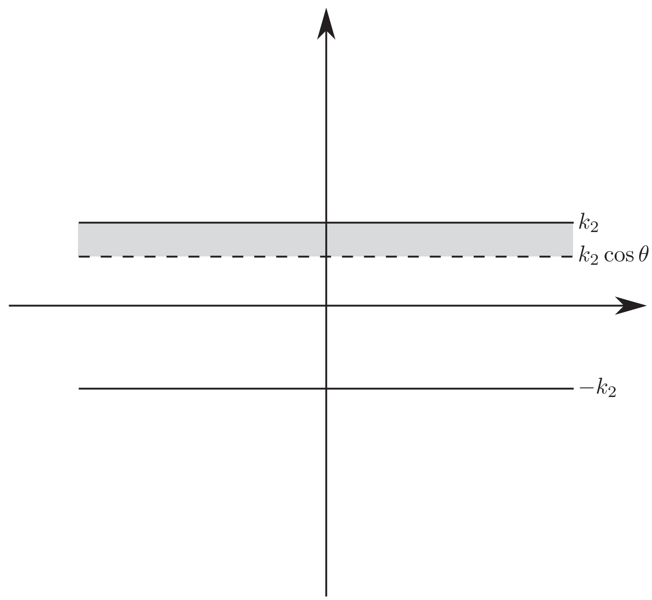

FIG. 2. Figure showing the strip of analyticity resulting in Sommerfeld's half-plane problem.

Now it is time to examine the above equation more carefully and in particular clarify the analyticity regions. There will be two different cases considered: the first one will be a typical example from applications and the second the most general solution. We will see that the former will lead to a Wiener–Hopf equation and the latter to a Riemann–Hilbert equation.

(1) In equation (5.3) $K(\alpha)$ and $G_+(\alpha)$ are known, thus their region of regularity can be determined. From the maximal growth rate of $f(x)$ as $x \to +\infty$ and $h(x)$ as $x \to -\infty$ the analyticity half-planes are determined as in (2.6–2.8). For example, in the integral equation for Sommerfeld's half-plane problem (Noble, 1958, Ch. 2.5), the known functions are in the following classes:

$$G_+(\alpha) \in \{\{a\cos\theta, \infty\}\}, \quad K(\alpha) \in \{\{-a, a\}\}$$

and the unknown is in

$$F_+(\alpha) \in \{\{a\cos\theta, \infty\}\}, \quad H_-(\alpha) \in \{\{-\infty, a\}\}.$$

Here a and θ are some constant. From (2.10), $K(\alpha)F_+(\alpha) \in \{\{a\cos\theta, a\}\}$ and Equation (5.3) holds in the strip $a\cos\theta < \operatorname{Im}\alpha < a$, Fig. 2. We obtained a Wiener–Hopf equation.

(2) We considered $K(\alpha) \in \{\{-a, a\}\}$ and the other functions will be assumed to belong to the largest possible class for (5.3) to be solvable. If $F_+(\alpha) \in \{\{b, \infty\}\}$ the convolution will exist for $b \leqslant a$, so we will take equality as the minimal condition on regularity. Then, the maximal class (Gakhov & Čerskiĭ, 1978), in which the integral equation has a solution, is:

$$G_+(\alpha) \in \{\{a, \infty\}\}, \quad F_+(\alpha) \in \{\{a, \infty\}\}, \quad H_-(\alpha) \in \{\{-\infty, a\}\}.$$

Furthermore, from (2.10) we have $K(\alpha)F_+(\alpha) \in \{\{a, a\}\}$. Hence, (5.3) is only valid on a line $\operatorname{Im}\alpha = a$ and it is a Riemann–Hilbert equation.

The above shows that the Wiener–Hopf equation is the result of a better regularity of the function at infinity than is minimally needed for a solution to exist.

6. Relationship between the Wiener–Hopf and the Riemann-Hilbert equations

To examine the relationship between the Wiener–Hopf and the Riemann–Hilbert equations, we will restate problems for the same class of functions and re-express the solution in terms of the Fourier integrals instead of Cauchy type integrals.

We begin with the jump problem in the Riemann–Hilbert case:

THEOREM 6.1 The solution to Jump Problem 4.2 can be expressed as:

$$F^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t \quad F^-(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t, \tag{6.1}$$

where $f(t)$ is the inverse Fourier transform of $F(t)$.

Proof. The solution to the problem is given by the Sokhotskyi–Plemelj formula (4.2). We re-express them in terms of the Fourier integrals using (2.3):

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t, \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t, \\ &= F^+(x) - F^-(x) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\pi \mathrm{i}} \int_{-\infty}^\infty \frac{F(\tau)}{\tau - x}\, \mathrm{d}\tau = F^+(x) + F^-(x), \\ = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(t)\, \mathrm{e}^{\mathrm{i}xt}, \\ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \operatorname{sgn}(t) f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t. \end{aligned}$$

In other words, the Sokhotskyi–Plemelj formula for the real line in terms of the Fourier transform is

$$F^+(x) - F^-(x) = F(x), \quad F^+(x) + F^-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \operatorname{sgn}(t) f(t)\, \mathrm{e}^{\mathrm{i}xt}\, \mathrm{d}t. \tag{6.2}$$

From this the result as follows. □

REMARK 6.2 The problem does not change if it is shifted by real a to the classes $F(t) \in \{\{a\}\}$ with $\{\{a, \infty\}\}$ and $\{\{-\infty, a\}\}$.

To clarify the relation between two problems, we re-state the Wiener–Hopf jump problem from Theorem 3.1 as follows.

PROBLEM 6.3 (Wiener–Hopf jump problem) On a strip $S = \{z : a < \mathrm{Im}(z) < b\}$ a function $F(z) \in \{\{a, b\}\}$ is given. It is required to find two functions $F^{\pm}(z)$ analytic in the half-planes $\{z : a < \mathrm{Im}(z)\}$ and $\{z : \mathrm{Im}(z) < b\}$, respectively, which belong to the classes $\{\{a, \infty\}\}$ and $\{\{-\infty, b\}\}$ and satisfying:

$$F(z) = F^{+}(z) + F^{-}(z),$$

on the strip S.

Similarly to Theorem 6.1 we find the following theorem.

THEOREM 6.4 The solution of Problem 6.3 can be expressed as

$$F^{+}(z) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t \quad F^{-}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t,$$

where $f(t)$ is the inverse Fourier transform of $F(t)$.

Proof. To prove the theorem, we use the solution (6.1) of the Riemann–Hilbert problem first for the line $\mathrm{Im}(z) = a$ and then for $\mathrm{Im}(z) = b$. On the line $\mathrm{Im}(z) = a$ we obtain

$$F(x + \mathrm{i}a) = F^{+}(x + \mathrm{i}a) + F^{-}(x + \mathrm{i}a), \tag{6.3}$$

with the factors given by

$$F^{+}(z) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t \quad \text{and} \quad F^{-}(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t. \tag{6.4}$$

Note that

$$F(x + \mathrm{i}a) - F^{-}(x + \mathrm{i}a) = F^{+}(x + \mathrm{i}a), \tag{6.5}$$

where the left-hand side has continuous analytic extension into the strip S, thus the same will be true for $F^{+}(x + ia)$. This allows to move to the other side of the strip:

$$F(x + \mathrm{i}b) = F^{+}(x + \mathrm{i}b) + F^{-}(x + \mathrm{i}b). \tag{6.6}$$

Now, an application of the Sokhotskyi–Plemelj formula gives

$$F^{+}(z) = \frac{1}{\sqrt{2\pi}} \int_{b}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}zt}\, \mathrm{d}t. \tag{6.7}$$

In other words, the analyticity of $F^{+}(z)$ was extended to the strip S and the result follows. □

REMARK 6.5 Note, that in the above proof one could have chosen initially the $\mathrm{Im}(z) = b$ and then extended function $F^{-}(x + \mathrm{i}b)$ to the line $\mathrm{Im}(z) = a$. In fact, any line in between $\mathrm{Im}(z) = c$ with $a < c < b$ could have been taken and a solution of the Riemann–Hilbert problem obtained. Then, the existance of analytical extension of both functions F^{+} and F^{-} can be shown in a similar manner.

Similarly, we describe the relationship between the factorization in the Wiener–Hopf and Riemann–Hilbert setting.

THEOREM 6.6 The solution or Problem 4.3 can be expressed as

$$K^{+}(z)=\exp\left(\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right)\quad\text{and}\quad K^{-}(z)=\exp\left(-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right),$$

where $\kappa(t)$ is the inverse Fourier transform of $\ln K(t)$.

Proof. From the $\operatorname{ind} K(t)=0$ it follows that $\ln K(t)$ is single valued. An application of Theorem 6.1 to $\ln K(t)$ yields:

$$\ln K(t)=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t+\left(-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right).$$

The result follows from taking exponents of both sides of the last identity. □

To express relations between two factorization problems, we formulate the Wiener–Hopf factorization in a suitable form.

PROBLEM 6.7 (Wiener–Hopf factorization) A function $K(z)$ is non-zero on the whole strip $S=\{z: a<\operatorname{Im}(z)<b\}$, furthermore $K(z)-1\in\{\{a,b\}\}$ and $\operatorname{ind} K(x+\mathrm{i}a)=0$. It is required to find two functions $K^{\pm}(z)$ analytic in the half-planes $\{z: a<\operatorname{Im}(z)\}$ and $\{z:\operatorname{Im}(z)<b\}$ belonging to the classes $\{\{a,\infty\}\}$ and $\{\{-\infty,b\}\}$, respectively, such that

$$K(z)=K^{+}(z)K^{-}(z)$$

on the strip $a<\operatorname{Im}(z)<b$.

Similarly to Theorem 6.6 we find the following theorem.

THEOREM 6.8 The solution of Problem 6.7 can be expressed as:

$$K^{+}(z)=\exp\left(\frac{1}{\sqrt{2\pi}}\int_{b}^{\infty}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right)\quad\text{and}\quad K^{-}(z)=\exp\left(-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right),$$

where $\kappa(t)$ is the inverse Fourier transform of $\ln K(t)$.

Proof. The function $K(z)$ on the line $\operatorname{Im} z=a$ satisfies all the assumptions of Theorem 6.6, thus we obtain

$$K(x+\mathrm{i}a)=K^{+}(x+\mathrm{i}a)K^{-}(x+\mathrm{i}a)$$

with

$$K^{+}(z)=\exp\left(\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right)\quad\text{and}\quad K^{-}(z)=\exp\left(-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right).$$

Since $\operatorname{ind} K(x+\mathrm{i}a)=0$ and $K(z)$ zero free on the strip, it follows that $\operatorname{ind} K(x+\mathrm{i}s)=0$ for all $a<s<b$. This is again expressed as

$$K(x+\mathrm{i}a)K^{-}(x+\mathrm{i}a)=K^{+}(x+\mathrm{i}a).$$

Because the left hand side has continuous analytic extension in the strip S, the function $K^{+}(x+\mathrm{i}a)$ has the extension as well. This gives meaning to the expression

$$K(x+\mathrm{i}b)=K^{+}(x+\mathrm{i}b)K^{-}(x+\mathrm{i}b).$$

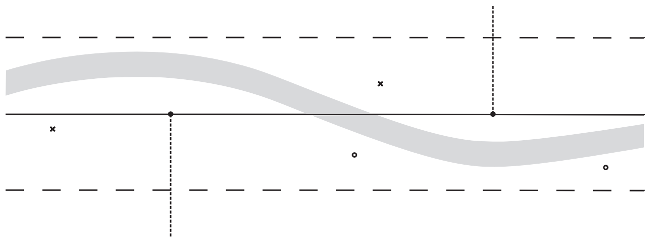

FIG. 3. The strip deformation (shaded) around the singularities of a given function.

Now, $K(x+\mathrm{i}b)$ satisfies all the assumptions of Theorem 6.6, thus

$$K^{+}(z)=\exp\left(\frac{1}{\sqrt{2\pi}}\int_{b}^{\infty}\kappa(t)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\right).$$

This provides the required factorization. □

Remark 6.5 also holds here. There are some differences in the formulation of Theorems 3.2 and 6.8. The most significant difference[1] is the assumption that $\operatorname{ind} K(x+\mathrm{i}a)=0$. This is because Theorem 3.2 assumes the existence of single valued $\ln K(t)$.

The above derivations show that the Wiener–Hopf equations is characterised by extra regularity, namely a domain of analyticity. Noteworthy, there are further applications of this feature, for example, a strip deformation. Consider a Wiener–Hopf equation:

$$A(\alpha)\Phi_{+}(\alpha)+\Psi_{-}(\alpha)+C(\alpha)=0.$$

Assume as before that $\Phi_{+}(\alpha)$ and $\Psi_{-}(\alpha)$ are analytic in the upper or lower half-planes and the strip, respectively. However, assume that this time $A(\alpha)$ and $C(\alpha)$ have singularities in the strip , see Fig. 3 for an illustration . Then, by taking a subset of the strip as shown on Fig. 3 the Wiener–Hopf equations can still be solved (Veitch & Abrahams, 2007).

Finally, we are going to revisit the factorization of $F(t)$ (1.1). Instead of spotting the factors by inspection the Riemann–Hilbert formula can be used, yielding factors analytic in the half-planes. By inspection of singularities in the complex plane, $F(z)$ has a strip of analyticity $-1+\epsilon<\operatorname{Im}(z)<1/2-\epsilon$. Hence, it is also possible to apply the Wiener–Hopf formula to obtain the factorization. Due to the uniqueness of the factors and analytic continuation it follows, that both methods shall produce results coinciding with (1.2) and illustrated by Fig. 1.

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[1] Remarkably, the excellent book Noble (1958) does not mention the index of functions at all.

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