Certification of a Class of Industrial Predictive Controllers without Terminal Conditions*

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***Abstract*— Three decades have passed encompassing a flurry of research and commercial activities in model predictive control (MPC). However, the massive strides made by the academic community in guaranteeing stability through a state-space framework have not always been directly applicable in an industrial setting. This paper is concerned with a priori and/or a posteriori certification of persistent feasibility, boundedness of industrial MPC controllers (i) based on input-output formulation (ii) using shorter control than prediction horizon (iii) and without terminal conditions.**

I. INTRODUCTION

MPC is a form of control in which the current control action is obtained by solving, at each sampling instant, a finite horizon open-loop constrained optimal control problem, using the current state (i.e. past inputs, outputs) of the plant as the initial state; the optimization yields an optimal control sequence and the first control in this sequence is applied to the plant. Given this systematic means of handling constraints, MPC has had a tremendous impact on industrial control practice [1].

The vast majority of research in stabilizing MPC and guaranteeing infinite time feasibility has invariably enforced one or all of the three ingredients: terminal penalty, terminal constraints, terminal control law [2]. The desirable features of using terminal conditions in MPC are [3]:

1) Nominal stability follows easily from the properties of stabilizing ingredients.
2) The problem is infinite time feasible (i.e. a solution exists that satisfies the constraints every time).

The objections raised to this method of stabilization include:

1) The stabilizing ingredients may be difficult to compute.
2) Adding a fictitious terms may compromise performance.
3) The region where the problem is feasible may shrink.
4) Most of the stabilizing ingredients are not used in the process industry.

However, the theoretical framework is not necessarily implemented for practical systems, given that industrial implementations largely continue to use input-output formulations of MPC and avoided using the mentioned stabilizing ingredients, thus appearing to defy mathematical analysis. It is not always easy to construct an MPC controller which has an a-priori guarantee of infinite feasibility and stability, either due to theoretical complications, or pragmatic decisions in practice [4]. Instead, we might have a situation where we are given an MPC controller, and the goal is to deduce a region where the problem is infinite time feasible/stable. The principal contribution of this paper is to deduce such a region of inputs and outputs where the industrial controllers without stabilizing ingredients are certifiable with respect to stability through infinite time feasibility.

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II. PROBLEM FORMULATION

The research literature on MPC has adopted the following formulation as standard (which is a regulation problem in the nominal case) [5]:

Problem 1:

$$V^*(x(t)) = \min_{u(.|.)} \{ \sum_{k=0}^{N_2-1} L(x(t+k|t), u(t+k|t)) + T(x(t+N_2|t))\}, \text{ subj. to: } x(t+N_2) \in \mathbb{T}$$
$$u(t+k) \in \mathbb{U}, \quad x(t+k) \in \mathbb{X}, \ \forall k \in [0, N_2) \tag{1}$$

under the state-space model $x(t+1) = f(x(t), u(t))$, with prediction horizon between $[N_1, N_2]$.

Theorem 1: The MPC problem 1 is infinite time feasible and asymptotically stable if the stage cost $L(.)$ and terminal cost $T(.)$ are positive definite; $T(.)$ is a control Lyapunov function (CLF) for the closed-loop system under a terminal control law $h(x(t+k)), \forall k \geq N_2$; and terminal constraint set $\mathbb{T}$ is invariant under $h(.)$.

Proof: Since, $T(.)$ is a CLF, it implies that:

$$T(f(x(t), h(x(t))) - T(x(t)) \leq -L(x(t), h(x(t))), \quad \forall x(t) \in \mathbb{T} \tag{2}$$

From this, it follows (with an optimally computed sequence $U^* = \{u^*(t), \ldots, u^*(t+N_2-1)\}$):

$$V^*(f(x(t), U^*(x(t))) - V^*(x(t)) \leq -L(x(t), u^*(t)) \tag{3}$$

and therefore the optimal cost is a Lyapunov function for the closed-loop system, and convergence to the origin is guaranteed. The invariance of $\mathbb{T}$ guarantees the feasibility of an N_2-step control sequence that enters $\mathbb{T}$ in $N_2 - 1$ steps from $x(t+1|t)$. Thus, feasibility of Problem 1 at a given time instant implies feasibility also at the next time instant. ■

However in many industrial MPC formulations, apart from using a transfer function representation and shorter control horizon, one or more of the three mentioned stabilizing ingredients are not considered in the design phase. Some of the advantages of the transfer function models for plant and disturbance are that these can be more intuitive, grounded with system identification techniques and compact representation of time delays. One can immediately write down the differences between the academic MPC setup to the industrial one as follows:

- D1 The predictions are made over input-output models together with disturbance filter.
- D2 The control and prediction horizons are not equal.
- D3 There is no terminal constraint $\mathbb{T}$.
- D4 There is no terminal cost $T(.)$.
- D5 There is no terminal control law $h(.)$.

In this case neither stability nor infinite time feasibility can be guaranteed (as the optimal cost is no longer a Lyapunov function for the closed loop, and there is no enforced invariance). Some authors have indeed considered omission of terminal constraint [D3], in which the control horizon (=prediction horizon) is made sufficiently large to ensure that the terminal constraint is automatically satisfied [6], [7]. A recent book [8] deals extensively in solving [D3-5] via the choice of an appropriate horizon employing the assumption that the system is asymptotically controllable. Finally, some tools for showing infinite-feasibility have been developed in [9], [10] using invariance and in-feasibility of a MPC controller with [D3-5] has been proposed in [4] by using bi-level programming.

To our knowledge, there has been no prior-work in dealing with all the five mentioned differences [D1-5] together and delivering to the industry a region of attraction for which the controller (either optimal or not) is certified infinite time feasible/stable. Thus, we build upon the necessary state transformation matrices and set-theoretic tools to be able to develop a procedure to certify stability through infinite time feasibility in discrete-time.

III. Input-Output to State-Space MPC

The state-space based MPC matured through the efforts of many researchers [11], [12] and now rests on a firm theoretical foundation. Next to these, the process industry and the adaptive control community saw a rise of its own version of MPC, some of the representative techniques developed in parallel were Model Heuristic Predictive Control (MHCP) [13], Dynamic Matrix Control (DMC), Extended Prediction Self-Adaptive Control (EPSAC) [14] and Generalised Predictive Control (GPC) [15], which predominantly employed transfer function representation. We now briefly describe the EPSAC-MPC formulation which is based on input-output modelling and filtering techniques. Due to its simplicity of implementation, this algorithm has been used extensively in industrial applications.

The process is modeled with $y(t), \hat{y}(t), n(t)$ as process output, model output, disturbance respectively as [16]

$$y(t) = \hat{y}(t) + n(t) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(t) + \frac{C(q^{-1})}{D(q^{-1})}e(t) \quad (4)$$

where B/A represents the model dynamics with $d \geq 0$ samples delay and C/D is chosen to form the disturbance filter in the backward shift operator q^{-1}, with e as white noise. Let the system polynomials be defined as (without loss of generality):

$$\begin{aligned} A &= 1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a} \\ B &= b_1 q^{-1} + \ldots + b_{n_b} q^{-n_b} \\ C &= 1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c} \\ D &= 1 + d_1 q^{-1} + \ldots + d_{n_d} q^{-n_d} \end{aligned} \quad (5)$$

Note that, even if a SISO system is considered for the exposé, the development applies to MIMO systems by considering a transfer function matrix instead. The fundamental step is based on the prediction using the basic process model given by

$$y(t+k|t) = \hat{y}(t+k|t) + n(t+k|t) \quad (6)$$

where $y(t+k|t)$ is the prediction of process output k steps in future computed at time t, over prediction horizon, based on prior measurements and postulated values of inputs. Prediction of model output $\hat{y}(t+k|t)$ and of colored noise process $n(t+k|t)$ can be obtained by the recursion of process model and filtering techniques respectively. The optimal control is then obtained by minimizing the following cost function with respect to the vector $u(.|.)$

$$\begin{aligned} &V = \Sigma_{k=N_1}^{N_2}[r(t+k|t) - y(t+k|t)]^2 + \lambda\Sigma_{k=0}^{N_u-1}[u(t+k|t)]^2 \\ &subj.\ to\ \ \Delta u(t+k|t) = 0,\ \ \forall k \in [N_u, N_2) \\ &u(t+k-N_1|t) \in \mathbb{U},\ \ y(t+k|t) \in \mathbb{Y},\ \ k \in [N_1, N_2] \end{aligned} \quad (7)$$

where $r(t+k|t)$ is the desired reference trajectory and N_u is the control horizon.

A. The Mappings

Consider the disturbance augmented state-space description of the form

$$\begin{aligned} \begin{bmatrix} \hat{x}(t+1) \\ n(t+1) \end{bmatrix} &= \begin{bmatrix} \hat{A} & \mathbf{0} \\ \mathbf{0} & \alpha \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ n(t) \end{bmatrix} + \begin{bmatrix} \hat{B} \\ \mathbf{0} \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ \beta \end{bmatrix} e(t) \\ y(t) &= [\hat{C}\ \ \kappa] \cdot \begin{bmatrix} \hat{x}(t) \\ n(t) \end{bmatrix} \end{aligned} \quad (8)$$

where the first state vector $\hat{x}(t)$ and the associated terms represent the dynamics of the nominal model and the vector $n(t)$ and its associated terms are used for disturbance evolution with the output $y(t)$ summing up the respective effects. A concise representation of (8) is given below in (9) with the respective terms having direct correspondence:

$$x(t+1) = \mathcal{A} \cdot x(t) + \mathcal{B} \cdot u(t) + \mathcal{E} \cdot e(t) \quad (9a)$$

$$y(t) = \mathcal{C} \cdot x(t) \quad (9b)$$

The disturbances are bounded by:

$$n(t) \in \mathbb{W},\ \ w(t) = \mathcal{E}.e(t) \in \mathbb{E}. \quad (10)$$

The system is subject to pointwise-in-time constraints on the control input and/or the states:

$$u(t) \in \mathbb{U}, \quad x(t) \in \mathbb{X}. \quad (11)$$

The set $\mathbb{U}$ is compact, while $\mathbb{X}, \mathbb{W}, \mathbb{E}$ are closed. It is assumed that the system and constraints are time invariant.

Theorem 2: The input-output process model of (4) (after absorbing the delay in B) can be transformed to state-space form of (9) by choosing the state vector as:
$x(t) = [\hat{y}(t), \dots \hat{y}(t-n_a+1), u(t-1), \dots u(t-n_b+1), n(t), \dots n(t-n_d+1), e(t-1), \dots e(t-n_c+1)]^T$.

Proof: The time-series model of (4) as explained before has two additive parts i.e. the model $\hat{y}(t) = B/A \cdot u(t)$ and the disturbance $n(t) = C/D \cdot e(t)$. Therefore, the time history of all these four variables constitute the state vector and a final addition gives the output, through the following transformation matrices:

$$\mathcal{A} = \left(\begin{array}{c|c|c|c} \mathcal{A}_a & \mathcal{A}_b & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{A}_1 & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathcal{A}_d & \mathcal{A}_c \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{A}_1 \end{array}\right)$$

where,

$$\mathcal{A}_a = \begin{pmatrix} -a_1 & \dots & -a_{na} \\ 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots 1 & 0 \end{pmatrix} \mathcal{A}_b = \begin{pmatrix} b_2 & \dots & b_{nb} \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}$$

$$\mathcal{A}_c = \begin{pmatrix} -d_1 & \dots & -d_{nd} \\ 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots 1 & 0 \end{pmatrix} \mathcal{A}_d = \begin{pmatrix} c_2 & \dots & c_{nc} \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots 1 & 0 \end{pmatrix} \mathbf{0} = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{B} &= \begin{bmatrix} b_1 \ 0 \dots \ 0 \ |1 \ 0 \dots \ 0 \ |0 \ 0 \dots \ 0 \ |0 \ 0 \dots \ 0 \end{bmatrix} \\ \mathcal{E} &= \begin{bmatrix} 0 \ 0 \dots \ 0 \ |0 \ 0 \dots \ 0 \ |c_1 \ 0 \dots \ 0 \ |1 \ 0 \dots \ 0 \end{bmatrix} \\ \mathcal{C} &= \begin{bmatrix} 1 \ 0 \dots \ 0 \ |0 \ 0 \dots \ 0 \ |1 \ 0 \dots \ 0 \ |0 \ 0 \dots \ 0 \end{bmatrix} \end{aligned} \quad (12)$$

This formulation, though non-minimal is crucial for deriving set theoretic properties starting from input-output models. It is now trivial to see that, substitution of the derived $\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{C}$ matrices in the state-space model of (9) gives exactly the same output as obtained starting from the input-output equations of (4). ■

Lemma 1: The constraints from the input-output model i.e $\hat{y}(t) \in \mathbb{Y}, u(t) \in \mathbb{U}, n(t) \in \mathbb{W}, e(t) \in \mathbb{E}$ are mapped to the constraints on the state as follows:
$x(t) \in \mathbb{X} = [\mathbb{Y} \times \dots \times \mathbb{Y} \times \mathbb{U} \times \dots \times \mathbb{U} \times \mathbb{W} \times \dots \times \mathbb{W} \times \mathbb{E} \times \dots \times \mathbb{E}]^T$.

Proof: Directly follows from the transformed state-vector representation in theorem 2. ■

Note that, for auto-regressive disturbances with $C = I$, $\mathcal{E} = 0$ and the state vector does not contain white noise terms. The associated non-minimal representation has the advantage that disturbances now appear exclusively in the state vector.

Theorem 3: The finite horizon constrained optimal control problem in the input-output formulation of (7) translates exactly to the following minimization problem in the state-space framework:

Problem 2:

$$\begin{aligned} &V^*(x(t)) = \min_U \ \{\sum_{k=N_1}^{N_2} [r(t+k|t) - \mathcal{C} \cdot x(t+k|t)]^2 + \lambda \cdot \\ &\sum_{k=0}^{N_u-1} u(t+k|t)^2\}, \text{subj. to:} \Delta u(t+k|t) = 0, \forall k \in [N_u, N_2) \\ &u(t+k-N_1|t) \in \mathbb{U}, \ \ x(t+k|t) \in \mathbb{X}, \ \ \forall k \in [N_1, N_2] \end{aligned} \quad (13)$$

where $\Delta u(t+k|t) = u(t+k|t) - u(t+k-1|t)$, rest of the terms have exactly the same meaning as before. The predictions are made by using the state-space maps of (9). Note that, the formulation is also referred to as independent model with augmented disturbance dynamics which has similar interpretation as realigned model for open-loop stable linear systems.

Proof: Since the state-space model of (9) has been proven to be equivalent to the input-output model of (4) in theorem 2, the cost function of (13) exactly matches that of (7) through the transformation vector $\mathcal{C}$. Further, the state constraints in (13) are obtained from the input-output constraints by using lemma 1. ■

IV. Invariant Set Theory

Invariant set theory has been shown to be crucial in understanding the behavior of constrained systems, since constraints can be satisfied if and only if the initial state is bounded in a set which is invariant (i.e. trajectories do not exit this set). Consider the following discrete-time system:

$$\begin{aligned} x(t+1) = f(x(t), u(t), n(t)) \ &= \mathcal{A} \cdot x(t) + \mathcal{B} \cdot u(t) \\ y(t) = g(x(t), n(t)) \ &= \mathcal{C} \cdot x(t), \ \text{for LTI} \\ u(t) \in \mathbb{U}, \ x(t) \in \mathbb{X}, \ & n(t) \in \mathbb{W} \end{aligned} \quad (14)$$

Definition 1: [17] The set $\mathbb{X} \subset \mathbb{R}^n$ is **robust control invariant** for the system $x(t+1) = f(x(t), u(t), n(t))$ iff there exists a feedback control law $u(t) = h(x(t))$ such that $\mathbb{X}$, is robust positively invariant set for the closed-loop system $x(t+1) = f(x(t), h(x(t)), n(t))$ and $u(t) \in \mathbb{U}, \forall x(t) \in \mathbb{X}$.

Definition 2: The **robust output admissible set** $\tilde{X}^g$ is the set of states for which the output constraints are satisfied for all allowable disturbances, i.e.

$$\tilde{X}^g \triangleq \{x(t) \in \mathbb{X} | y(t) \in \mathbb{Y}, \forall n(t) \in \mathbb{W}\} \quad (15)$$

Definition 3: The **robust reach set** $\tilde{R}(\mathbb{X})$ is the set of states to which the system will evolve at the next time step given any $x(t)$, admissible control input and allowable disturbance, i.e.

$$\begin{aligned} \tilde{R}(\mathbb{X}) \triangleq &\{x(t+1) \in \mathbb{R}^n | \exists x(t) \in \mathbb{X}, u(t) \in \mathbb{U}, n(t) \in \mathbb{W} : \\ &x(t+1) = f(x(t), u(t), n(t))\} \end{aligned} \quad (16)$$

Definition 4: The i-step **robust controllable set** $\tilde{K}_i(\mathbb{X}, \mathbb{T})$ is the set of states in $\mathbb{X}$ which can be driven by an admissible input sequence of length i to an arbitrary target set $\mathbb{T}$ in exactly i steps, while keeping the evolution of the state inside $\mathbb{X}$ for the first $i-1$ steps, for all allowable disturbances i.e.

$$\begin{aligned} \tilde{K}_i(\mathbb{X}, \mathbb{T}) \triangleq &\{x(t) \in \mathbb{R}^n | \exists \{u(t+k) \in \mathbb{U}\}_0^{i-1} : \{x(t+k) \\ &\in \mathbb{X}\}_{k=1}^{i-1} \in \mathbb{X}, x(t+i) \in \mathbb{T}, \forall \{n(t+k) \in \mathbb{W}\}_{k=0}^{i-1}\} \end{aligned} \quad (17)$$

Remark 1: If $\tilde{K}_i(\mathbb{X}, \mathbb{T}) = \tilde{K}_{i+1}(\mathbb{X}, \mathbb{T})$, then $\tilde{K}_\infty(\mathbb{X}, \mathbb{T}) = \tilde{K}_i(\mathbb{X}, \mathbb{T})$ is the **robust infinite-time controllable set** with determinedness index $i^*_K = i$.

Definition 5: The i-step **robust admissible set** $\tilde{C}_i(\mathbb{X})$ contained in $\mathbb{X}$ is the set of states for which an admissible control sequence of length i exists, while keeping the evolution of the state inside $\mathbb{X}$ for i steps, for all allowable disturbances i.e.

$$\tilde{C}_i(\mathbb{X}) \triangleq \{x(t) \in \mathbb{R}^n | \exists \{u(t+k) \in \mathbb{U}\}_{k=0}^{i-1} : \\ \{x(t+k) \in \mathbb{X}\}_{k=1}^{i} \in \mathbb{X}, \forall \{n(t+k) \in \mathbb{W}\}_{k=0}^{i-1}\} \quad (18)$$

Now, we introduce a new set tailored towards solving problems with control horizons shorter then prediction horizons.

Definition 6: An i-step **robust tunnel set** $\tilde{L}_i(\mathbb{X})$ is an i-step robust admissible set $\tilde{C}_i(\mathbb{X})$ subject to the constraint that the admissible control sequence remains constant i.e.

$$\tilde{L}_i(\mathbb{X}) \triangleq \{\tilde{C}_i(\mathbb{X}) | \{\Delta u(t+k) = 0\}_{k=1}^{i-1}\} \quad (19)$$

Remark 2: If $\tilde{L}_{i+1}(\mathbb{X}) = \tilde{L}_i(\mathbb{X})$, then $\tilde{L}_\infty(\mathbb{X}) = \tilde{L}_i(\mathbb{X})$ is the **maximal robust tunnel set** with determinedness index $i^*_L = i$.

Note that, all the tools developed for robust sets remain perfectly valid for nominal systems, in which case $n(t) = 0$ and the invariant sets are represented devoid of the ‘tilde’.

V. Persistent Feasibility

In practice, especially when the system is nonlinear, one cannot guarantee that the solution is unique nor that the solver will return the optimal solution to problem 2. It would therefore be useful if a result could be derived which allowed one to guarantee that the MPC controller is feasible for all time and for all disturbance sequences, even if a suboptimal control input is computed each time. In this section, we derive the region of attraction for the given MPC problem 2:

1) That satisfies conditions $[D2 - D5]$
2) And without requiring optimality of the solution.

to be infinite-time (persistent) feasible/stable (Recollect that, the requirement $D1$ is already satisfied by the transformation matrices introduced before). Now, we introduce new definitions and characterizations for the feasible region of our MPC problem 2 with short control horizon and no terminal conditions.

Definition 7: The **robust feasible set** $\tilde{X}_F$ is the set of states for which an admissible control sequence exists that satisfies the state/output constraints i.e. a robust feasible control sequence to the MPC problem 2, for all disturbance sequences.

Theorem 4: The robust feasible set $\tilde{X}_F(\mathbb{X}, N_u, N_2)$ of the MPC regulation problem 2 is given by:

$$\tilde{X}_F(\mathbb{X}, N_u, N_2) = \tilde{K}_{N_u}(\mathbb{X}, \tilde{L}_{N_2 - N_u}(\mathbb{X})) \quad (20)$$

Proof: The construction of the robust feasible set can be divided into two parts in time by approaching it from the end.

Consider the second part of the MPC problem 2 i.e. between control and prediction horizon $N_2 - N_u$, where the requirement is to keep the control moves constant to an admissible set $\mathbb{U}$ and satisfy the state constraints $\mathbb{X}$, for disturbance set $\mathbb{W}$. This set by definition 6 is the robust tunnel set $\tilde{L}_{N_2 - N_u}(\mathbb{X})$.

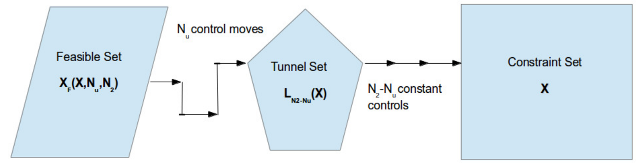

Fig. 1. Relationship among the various sets leading to the feasibility set

Next, we consider the first part of the MPC problem 2 i.e. the window over the control horizon N_u, now the requirement is again to be able to find a control sequence in $\mathbb{U}$ that satisfies the state constraints $\mathbb{X}$ and lies in the target set $\tilde{L}_{N_2 - N_u}(\mathbb{X})$ after N_u moves, for all disturbances. This by definition 4, is the robust controllable set $\tilde{K}_{N_u}(\mathbb{X}, \tilde{L}_{N_2 - N_u}(\mathbb{X}))$, which is the overall robust feasible set and completes the proof. ■

The entire process is depicted conceptually in Fig. 1 for the nominal case.

Definition 8: The MPC problem is **robust persistently feasible** iff the initial state and future evolutions belong to the robust feasible set i.e. $x(t+k) \in \tilde{X}_F, \forall k \in \mathbb{N}$.

If the dynamics are linear and the constraints are compact, convex polyhedra, then $\tilde{X}_F$ is also a compact convex polyhedron.

A. Guidelines for stabilizing horizons

Proposition 1: If the robust feasible set $\tilde{X}_F$ is bounded and the MPC problem is robust persistently feasible, then the system is **robust stable** in a practical Lyapunov sense i.e. trajectories remain bounded.

Now, for the MPC problem to be robust stabilizing, we must ensure robust persistently feasibility through the parameters which are the control and the prediction horizon. Note that, the robust persistently feasible set $\tilde{X}_F$ is independent of the cost function and optimality of the solution.

Theorem 5: The MPC problem is robust persistently feasible if the difference between the prediction and control horizons is larger than the determinedness index of the maximal i^*_L robust tunnel set $\tilde{L}_\infty(\mathbb{X})$ i.e. $N_2 - N_u \geq i^*_L$.

Proof: Since, $\tilde{L}_\infty(\mathbb{X})$ is control invariant under constant control, any $i-$step controllable set to it i.e $\tilde{K}_i(\mathbb{X}, \tilde{L}_\infty(\mathbb{X})), \forall i \geq 1$ is also control invariant, which is in fact also the robust feasible set, and by the nesting property of invariant sets $\tilde{X}_F(\mathbb{X}, N_{u1}, i^*_L) \subseteq \tilde{X}_F(\mathbb{X}, N_{u2}, i^*_L), N_{u1} < N_{u2}$, which is necessary for robust strong feasibility. ■

Theorem 6: If $N_2 \geq N_u + i^*_L$, then the size of the robust feasible set $\tilde{X}_F$ increases with the control horizon N_u until it exceeds the determinedness index $i^*_K$of the infinite robust controllable set $\tilde{K}_\infty(\mathbb{X}, \tilde{L}_{N_2 - N_u}(\mathbb{X}))$.

Proof: The maximal robust tunnel set $\tilde{L}_{i^*_L}$ being control invariant, induces control invariance on $\tilde{K}_{N_u}(\mathbb{X}, \tilde{L}_{i^*_L}(\mathbb{X})), \forall N_u \in \mathbb{N}^+$. The sets being enclosed inside the other with increasing N_u is a property of robust control invariant sets. The size increase stops beyond the determinedness index i^*_K of controllable sets, as by definition the set sizes remain exactly the same. ■

Corollary 7: For a fixed control horizon N_u, the size of the feasibility set decreases with increasing prediction horizon until the determinedness index of the tunnel set i.e. $\tilde{X}_F(\mathbb{X}, N_u, N_{2a}) \subseteq \tilde{X}_F(\mathbb{X}, N_u, N_{2b}), N_{2a} > N_{2b}$ for all $N_2 \leq i_L^*$.

Theorem 8: If $\tilde{X}_F(\mathbb{X}, N_u, N_2)$ is robust control invariant, then the MPC problem
$\tilde{X}_F(\mathbb{X}, N_u+n, N_2+n), n \geq 1$ is robust persistently feasible.

Proof: The robust control invariance of $\tilde{X}_F(\mathbb{X}, N_u, N_2)$ by definition implies that
$\tilde{X}_F(\mathbb{X}, N_u, N_2) \subseteq \tilde{K}_1(\mathbb{X}, \tilde{X}_F(\mathbb{X}, N_u, N_2)$. However,
$\tilde{X}_F(\mathbb{X}, N_u+1, N_2+1) = \tilde{K}_1(\mathbb{X}, \tilde{X}_F(\mathbb{X}, N_u, N_2)$ by application of the MPC control law. This implies that $\tilde{X}_F(\mathbb{X}, N_u + n, N_2 + n) \subseteq \tilde{X}_F(\mathbb{X}, N_u, N_2)$, for $n = 1$ and is true $\forall n > 1$ by induction, which concludes robust persistent feasibility. ■

Based on the above three theorems, the following algorithms for tuning the horizons are proposed to stabilize the MPC problem without terminal conditions.

Algorithm 1: Control horizon-1 tuning procedure (important for efficiency of computation):

1) Compute determinedness of the robust tunnel set i_L^*, if finitely determined.
2) The stabilizing horizons are $N_u = 1, N_2 = i_L^* + 1$.

Now, if an additional requirement is to obtain the largest possible robust feasible region $\tilde{X}_F$, then:

Algorithm 2: Maximal robust feasible region $\tilde{X}_F$ tuning procedure:

1) Compute determinedness of the robust tunnel set i_L^*, if finitely determined.
2) Compute determinedness of the robust controllable set i_K^*, if finitely determined.
3) The stabilizing horizons are $N_u = i_K^*, N_2 = i_L^* + i_K^*$.

If, the robust tunnel set is not finitely determined, then:

Algorithm 3: Heuristic robust persistent $\tilde{X}_F$ tuning procedure:

1) Iterate over different values of the horizons with $N_u < N_2$ until $\tilde{X}_F(\mathbb{X}, N_u, N_2)$ is robust control invariant for N_u^*, N_2^*.
2) The stabilizing horizons are $N_u = N_u^* + 1, N_2 = N_2^* + 1$.

B. A posteriori certification of stability

In order to guarantee robust constraint satisfaction in safety-critical applications it is desirable that infeasibility of the MPC optimization problem is avoided at all costs. In other words, once inside the feasible set the system evolution should remain inside the feasible set for all time and for all disturbance sequences. Here, we develop test for checking robust persistently feasibility, given the MPC problem 2 without terminal conditions and shorter control horizon has already been implemented.

Theorem 9: The MPC regulator that solves problem 2 is robust persistently feasible iff:

$$\tilde{R}(\tilde{X}_F(\mathbb{X}, N_u, N_2)) \cap \tilde{X}_F(\mathbb{X}, N_u - 1, N_2 - 1) \subseteq \tilde{X}_F(\mathbb{X}, N_u, N_2) \tag{21}$$

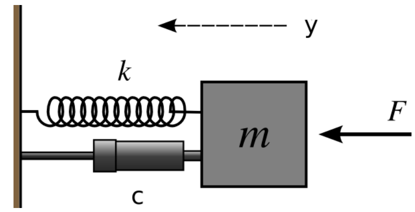

Fig. 2. A schematic representation of the mass-spring-damper system.

Proof: $\tilde{R}(\tilde{X}_F(\mathbb{X}, N_u, N_2))$ is the set of states reachable from the robust feasible set $\tilde{X}_F(\mathbb{X}, N_u, N_2)$ using admissible inputs, while the set $\tilde{R}(\tilde{X}_F(\mathbb{X}, N_u, N_2)) \cap \tilde{X}_F(\mathbb{X}, N_u - 1, N_2 - 1)$ is the subset which is reachable using feasible control inputs which obey the state constraints. Therefore, after applying the feasible control computed by the MPC regulator, the next state $\hat{x}(t+1|t) \in \tilde{R}(\tilde{X}_F(\mathbb{X}, N_u, N_2)) \cap \tilde{X}_F(\mathbb{X}, N_u - 1, N_2 - 1)$. Now, if $\hat{x}(t+1|t) \in \tilde{X}_F(\mathbb{X}, N_u, N_2)$, then by mathematical induction all future evolutions of the system remain within the robust feasible set, which completes the proof. ■

Corollary 10: In the special case of control horizon $N_u = 1$, the robust persistent feasiblity test reduces to:

$$\tilde{R}(\tilde{X}_F(\mathbb{X}, 1, N_2)) \cap \tilde{L}_{N_2-1}(\mathbb{X}) \subseteq \tilde{X}_F(\mathbb{X}, 1, N_2) \tag{22}$$

Proof: In the case with $N_u = 1$,
$\tilde{X}_F(\mathbb{X}, 0, N_2 - 1) = L_{N_2-1}(\mathbb{X})$. ■

Lemma 2: In case of output disturbance, the state constraints $\mathbb{X}$ must be replaced by the robust output admissible set $\tilde{X}^g$, refer definition 2. Note that, input disturbances can be moved to the output by filtering it through the plant denominator A.

This result in practice can be conservative, as optimality of the solution is not considered. Note that all the derived results, as usual, hold for the nominal case with zero disturbance i.e. $n(t) = 0$ and the corresponding sets are represented without the 'tilde'.

VI. Example: Mass-Spring-Damper

In this section, we demonstrate the procedure of obtaining and testing persistent feasibility in the nominal and perturbed cases, starting from transfer function model and using horizon $N_u \ll N_2$ without any terminal conditions, over a mass-spring-damper (MSD) setup of Fig. 2. The continuous time input-output model of the MSD is given by

$$m \cdot \ddot{y}(t) + c \cdot \dot{y}(t) + k \cdot y(t) = F(t) \tag{23}$$

where $y(t), u(t) = F(t)$ are the measured output displacement and input force respectively with parameters m, c, k being the mass, damping and spring constant respectively. Now, this is discretized with sampling time $T_s = 10\text{ms}$ with the parameter values $m = 1.7\text{kg},\ c = 9\text{N/m/s},\ k = 450\text{N/m}$ to obtain the following system:

$$y(t)[\text{mm}] = \frac{0.0545}{1 - 1.902q^{-1} + 0.9264q^{-2}} u(t)[\text{N}] + n(t)$$
$$||y(t)||_\infty \leq 2\text{mm}, \quad ||u(t)||_\infty \leq 1\text{N} \quad ||n(t)||_\infty \leq \gamma \tag{24}$$

An input-output MPC regulator is designed with control

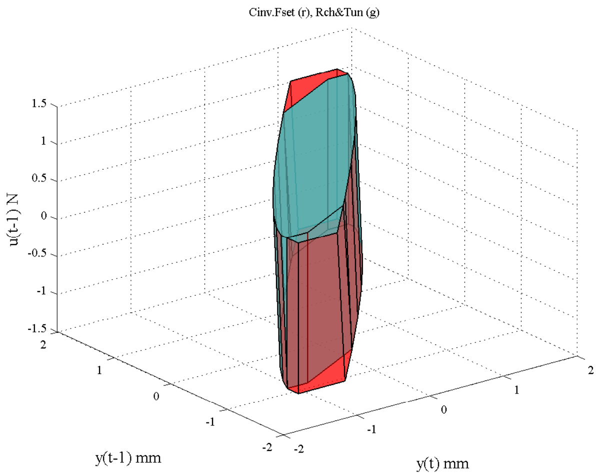

Fig. 3. Robust Persistent feasibility test: $\tilde{R}(\tilde{X}_F(\mathbb{X},1,8)) \cap \tilde{L}_7(\mathbb{X})\ \ (green) \subseteq \tilde{X}_F(\mathbb{X},1,8)\ \ (red)$

horizon $N_u = 1$ and prediction horizon $N_2 = 8$. In the nominal case $\gamma = 0$ and in the robust case $\gamma > 0$ is the upper bound on the output disturbance. Note that, no other information is required as the persistent feasibility technique is independent of the cost function. The first-step is to deduce the state-space representation. The state vector is $x(t) = [y(t), y(t-1), u(t-1)]^T$ and the transformation matrices:

$$x(t+1) = \begin{bmatrix} 1.902 & -0.9264 & 0.0545 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)[\text{N}]$$
$$y(t)[\text{mm}] = [1\ \ 0\ \ 0] \cdot x(t) + n(t) \tag{25}$$
$$||x(t)||_\infty \leq [2 \times 2 \times 1]^T \ominus [\gamma \times 0 \times 0]^T, \quad ||u(t)||_\infty \leq 1\text{N}$$

where $\ominus$ is the Pontryagin difference. Recollect that, in case of output disturbance, the state constraints are mapped to the output admissible set, which explains the state constraint mapping above. A perturbation in the form of 10% additive output disturbance is considered i.e. $|\gamma| = 0.2$. In this case, the robust feasibility set is computed as per theorem 4 to:

$$\tilde{X}_F(\mathbb{X},1,8) = \tilde{K}_1(\mathbb{X},\tilde{L}_7(\mathbb{X})) \tag{26}$$

after the computation of the 7-steps robust tunnel set $\tilde{L}_7(\mathbb{X})$ and the 1-step robust controllable set to it i.e. $K_1(\mathbb{X}, L_7(\mathbb{X}))$. Next, the robust reach set of the robust feasibility set $\tilde{R}(\tilde{X}_F(\mathbb{X},1,8))$ is computed and to check robust persistent feasibility of the MPC problem, we make use of corollary 10:

$$\tilde{R}(\tilde{X}_F(\mathbb{X},1,8)) \cap \tilde{L}_7(\mathbb{X}) \subseteq \tilde{X}_F(\mathbb{X},1,8) \tag{27}$$

Indeed, this test is fulfilled as can be seen graphically in Fig. 3 and thus a certificate of practical robust stability can now be issued to this MPC controller.

Alternately, one may design a persistently feasible and stabilizing controller by following the guidelines of V-A for the nominal MSD system of (VI). Algorithm 1 suggests $N_u = 1, N_2 = 32$, algorithm 2 gives $N_u = 9, N_2 = 40$, and algorithm 3 requires $N_u = 3, N_2 = 4$. All the three designs satisfy the persistent feasibility subset test of theorem 9.

VII. Conclusions

A theoretical framework to certify industrial MPC controllers formulated in input-output domain with shorter control than prediction horizons and without any of the stabilizing terminal conditions has been developed. First, a way to transform the input-output MPC to state-space MPC has been formulated. Then, a new robust tunnel set is introduced to explicitly handle shorter control than prediction horizon. Based on these, the feasible set of the MPC problem has been characterized. Next, on one hand new guidelines are given to choose stabilizing horizons without terminal conditions, and on the other a mechanism for certifying existing industrial MPC controllers without terminal conditions are given through the notion of robust persistent feasibility.

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