LOCAL CONSERVATION LAWS IN QUANTUM INTEGRABLE SYSTEMS

by

MICHAEL STUART HAWKINS

A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Physics and Astronomy
The University of Birmingham

October 2008

Abstract

One of the phenomena associated with quantum integrable systems is the possibility of persistent currents, *i.e.* currents which do not decay away entirely, but have some portion that continues to flow undiminished and indefinitely. These residual currents are shown to be the conserved part of the current operator, and calculable from the conservation laws of the system.

In a particular system, previous attempts to calculate a known residual current from the conservation laws have failed. A numerical investigation is undertaken, and this disparity with the formal results is resolved by the inclusion of a previously overlooked conservation law. An important corollary to these results is that requiring the mutual commutativity of the conservation laws of a quantum integrable system, previously assumed by analogy with the classical case, is an unnecessary and potentially disastrous restriction.

Methods of generating the local conservation laws of a quantum integrable system are investigated, and the current method of using a Boost operator is shown to be subtly flawed. The method is discovered to implicitly require additional knowledge in the form of Hamiltonian identities in order to avoid otherwise unphysical terms.

A new method is proposed based on the idea that the logarithm of the Transfer matrix of a system generates these local conservation laws. The method is applicable to a wide class of systems whose Lax operator obeys a certain condition, and the majority of the work required to generate the local conservation laws is entirely general and thus only needs to be done once.

This new method is then applied to two quite different spin-chain Hamiltonians, the XXZ and Hubbard models, and shown to successfully generate all of the known local conservation laws of these models and some new ones.

Acknowledgements

As is traditional, the first thanks go to my supervisor Dr. Martin Long: without his essentially constant availability and patience in explaining things at length (and often repeatedly), I would not have achieved half as much, and understood even less.

The largest and most deserved thank you however, goes to Jim, who has been there throughout when I needed help or advice, or when I just wanted to vent and moan (which was often). That and for joining me in Monday quizzes, payday meals, Fridays in Staff House, the occasional gin Wednesday and all the other events too numerous to mention that have kept me entertained and, frankly, sane for the last four years.

I would also like to thank my family for their constant (occasionally financial) support: my parents for their irrational and unshakable belief in me, and my brother for keeping me grounded when I believed the hype.

Finally, I gratefully acknowledge my funding by the Engineering and Physical Sciences Research Council. Money after all, makes the world go round, and theirs took me half way round the world.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Generally when a current is started in a system it is expected to decay away with time and eventually vanish. There is a class of systems however, described as being as integrable, where this is not necessarily the case. In systems such as these it has been shown [1] that some portion of the current may continue flowing indefinitely, even at *finite* temperatures.

It has been shown formally that this long-time residual should be calculable from, and indeed interpreted as one of, the conservation laws of the system [2]. For one particular system however, a certain current has a known, non-vanishing residual [3], but it can be shown by symmetry arguments that *none* of the conservation laws can contribute anything towards this residual [1].

Spurred on by a desire to resolve the above inconsistency, this thesis starts as an investigation into the concept of quantum integrability as it relates to the conservation laws of a system; calculating residual currents from conservation laws (using the ideas of Suzuki [2]); and Mazur's inequality [4], which is a crucial part of Suzuki's arguments.

Following the resolution of this conundrum, the focus of the thesis shifts entirely to the conservation laws of a quantum integrable system. Specifically, the idea of generating the *local* conservation laws, used previously to calculate the residual current via Mazur's inequality, is investigated. The current method of generating these conservation laws using a Boost operator [5] is found wanting, and a new method is developed and tested

on some well known integrable systems.

Firstly however, the introduction will consist of a brief tour of the ideas and methods of solving integrable systems.

Investigations into quantum integrable systems have only really taken off in the last thirty or so years. The ideas of the famous Bethe Ansatz and the classical Inverse Scattering method were combined and adapted to create the Quantum Inverse Scattering method and the Algebraic Bethe Ansatz. When introducing and explaining these methods, it is necessary to decide where should one start and what level of detail is appropriate. Given that none of these methods are actually used, this chapter gives only a brief account of them to provide some context, pausing where necessary to give extra details on the major ideas which serve as the starting point for the work that follows.

1.1 Classical Integrability

The first and most widely used definition of integrability for a classical system was provided by Liouville in the nineteenth century, as one where the equations of motion are "solvable by quadrature" [6]. It is pretty much standard now that when a system is described as integrable, what is actually meant is *Liouville-integrable*, as defined fully below, and henceforth the same meaning will be implied here.

A system with a $2n$-dimensional phase space is integrable in the Liouville sense if it has n independent integrals of motion (implying a conserved quantity, or constant of motion) all in involution. A constant of motion is a quantity whose Poisson bracket with the Hamiltonian vanishes, and to be in involution means that the Poisson brackets of the other $n-1$[1] constants with each other also vanish. More formally, there are a set of n quantities $\{C_\alpha(q_i, p_i)\}$ such that

$$\dot{C}_\alpha \;=\; \big\{H, C_\alpha\big\} \;=\; 0 \quad \text{and} \quad \big\{C_\alpha, C_\beta\big\} \;=\; 0, \tag{1.1.1}$$

[1]Since the Hamiltonian automatically qualifies as a constant of motion.

in terms of the standard Poisson bracket

$$\{F, G\} \;=\; \sum_i \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i}. \tag{1.1.2}$$

Note that it is not necessary to know the form of these constants, integrability requires only that they exist. Furthermore, the problem need not be fully separable, but it should be possible to reduce the equation down to quadratures (*i.e.* integrals).

Each of these constants of motion reduces the dimension of phase space, down to an n-dimensional subspace in the integrable case. Assuming periodic motion, this subspace has the topology of an n-torus[2], with the possible trajectories cycling around the handles.

If one were to perform a canonical transformation from the original physical variables of the system, p_i and q_i, to a new set, I_i and ψ_i, such that the Hamiltonian becomes independent of the canonical coordinates (*i.e.* $H = H(\psi_i)$, a function of the conjugate momenta *only*), then the momenta I_i are constants of motion. These new variables I_i and ψ_i are known as the *action-angle variables*, and it should be clear that the momenta I_i will be linear combinations of the conserved quantities $\{C_\alpha\}$ above. The constant action variables can be found in terms of the original variables p_i and q_i (or indeed any canonical set of variables) as

$$I_k \;=\; \oint_k \sum_i \; p_i \; dq_i, \tag{1.1.3}$$

where the integral is over a single period of the motion [7][3]. In terms of these variables, the time evolution becomes very simple: the coordinate variable ψ_i just increases linearly with time, *i.e.* $\psi_i(t) = \psi_i(0) + \alpha_i t$, for some constant $\alpha_i = \dot{\psi}_i$. Having a time evolution of this form, which does not exhibit chaotic features [8], can be used as an alternate (but equivalent) definition of integrability.

Although integrability can be demonstrated by showing the existence of these n con-

[2]This is due to the wonderfully named "hairy ball" theorem [7].

[3]or equivalently, around the handle of the torus.

stants of motion, there still remains the (generally highly non-trivial) task of determining the explicit forms of these action-angle variables (or equivalently solving the original equations of motion).

1.1.1 The Inverse Scattering Method

A general method for solving *linear* partial differential equations (PDEs) given the initial condition is to use Fourier transforms. This is a three stage process of transforming the initial condition into Fourier space; finding the time evolution and then using the inverse transform to return to the (real) space of the original equations. For example, using the standard inverse and Fourier transform

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{u}(k,t)e^{ikx}dk, \quad \text{and} \quad \hat{u}(k,t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-ikx}dx, \tag{1.1.4}$$

the linear PDE $\partial_t u \;=\; -c\,\partial_x u$ with the initial condition $u(x,0)$ becomes (in Fourier space) the linear ordinary differential equation (ODE)

$$\frac{d\hat{u}}{dt} \;=\; -ikc\hat{u}. \tag{1.1.5}$$

This of course has the general solution

$$\hat{u}(k,t) \;=\; \hat{u}(k,0)\exp\left(-ikct\right), \quad \text{where} \quad \hat{u}(k,0) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,0)e^{-ikx}dx, \tag{1.1.6}$$

and putting this back into the inverse transform gives

$$u(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,0)e^{-ikx}dx\exp\left(ik[x-ct]\right)dk. \tag{1.1.7}$$

Unfortunately, there are many physical problems which are modelled by *nonlinear* PDEs, and the Fourier transform method is no longer of any use. In 1967 however, Gardner, Greene, Kruskal and Miura [9] developed a method of solving one particular

nonlinear PDE, the well known Korteweg-de Vries equation

$$\partial_t u + 6u\partial_x u + \partial_x^3 u \;\; = \;\; 0, \tag{1.1.8}$$

by relating it to the time-independent Schrödinger scattering problem

$$Lv \;\; = \;\; \partial_x^2 v + u(x,t)v \;\; = \;\; \lambda v. \tag{1.1.9}$$

The idea is that the field $u(x,t)$ of equation (1.1.8) plays the role of the potential in equation (1.1.9), and t is just treated as a parameter. Usually, the scattering data $S(\lambda,t)$, which depends on the potential $u(x,t)$, can be determined from the eigenvalues and eigenfunctions of equation (1.1.9). This mapping of the potential into the scattering data is known as the direct scattering problem. The *inverse* scattering problem therefore is to obtain the potential given the scattering data.

This Inverse Scattering Method is essentially analogous to the above use of Fourier transforms on linear PDEs [10]: a nonlinear PDE of $u(x,t)$ is rewritten in the form of a scattering problem, where u becomes the potential. The direct scattering problem is solved (the scattering data being the equivalent to the Fourier transform in this analogy), and then given $S(\lambda,0)$ (the scattering data at initial time, from $u(x,0)$) the time evolution can be determined to obtain $S(\lambda,t)$. This is where the analogy ends as (unlike simply applying the inverse Fourier transform) the final step, solving the inverse scattering problem by reconstructing $u(x,t)$ from $S(\lambda,t)$, is *extremely* difficult.

The crucial idea of the Inverse Scattering method is that the nonlinear evolution equations that it solves are integrable Hamiltonian systems[4] [10] (the Hamiltonian form of equation (1.1.8), for example, can be found in [11], among others). In this form, the process of inverse scattering can be thought of as a canonical transform from the original physical variables, q_i, p_i, to the action-angle variables of the system.

[4]with infinite dimension.

1.1.2 The Lax Formalism

Shortly after the work of Gardner *et al*, Lax [12] produced a general method of associating nonlinear evolution equations with a pair of linear operators in such a manner that the eigenvalues of these linear operators are constants of motion of the nonlinear equation.

To do this, Lax formulated the problem in the following manner [10]: first define two matrices L and M, which have some dependence (to be defined shortly) on the canonical coordinates of the system (the q_i and p_i of the Hamiltonian form). The matrix M is assumed to govern the time evolution of the eigenfunctions of the eigenvalue problem involving L, like so

$$Lv = \lambda v, \tag{1.1.10a}$$

$$\frac{dv}{dt} = Mv. \tag{1.1.10b}$$

Differentiating equation (1.1.10a) with respect to time and then substituting in equation (1.1.10b) gives [5]

$$\begin{aligned} L_t v + L v_t &= \lambda_t + \lambda v_t \\ L_t v + LMv &= \lambda_t + \lambda M v \\ &= \lambda_t + MLv, \end{aligned}$$

and requiring that the eigenvalues be independent of time, *i.e.* $\lambda_t = 0$, means that (for nontrivial eigenfunctions v), L and M must satisfy

$$\frac{dL}{dt} = [M, L], \tag{1.1.11}$$

the Lax evolution equation. To link these matrices to the desired system, the q_i, p_i dependence of the matrices L and M must be such that the above equation for $\dot{L}$ recreates the equations of motion of said system (or alternatively, L and M are differential operators

[5] where the subscript t denotes differentiation with respect to time.

such that equation (1.1.11) is the original nonlinear evolution equation).

A set of matrices which satisfy all the above conditions for a given system are known as a *Lax pair*. Note that this pair is not unique; a new matrix $\tilde{L} = ULU^{-1}$, for an arbitrary invertible U and appropriately chosen M, will also satisfy equation (1.1.11) and recreate the same equations of motion[6] [8]. Irrespective of which L is used, the eigenvalues will be functions of the canonical variables q_i, p_i, and (by definition) are independent of time, and are therefore equivalent to the constants of motion of the original system. This hugely important result means that *any* system whose equations of motion can be written in this Lax form is therefore *necessarily* an integrable system. The downside however, is that there is no general way of showing if a system has a Lax representation, or if it is known to be integrable, what the explicit form of L and M might be.

In addition to using the eigenvalues, there are two other methods which can be used to generate alternate (but equivalent) sets of constants of motion from L [8]. The first is to take the traces of increasing powers of L, the results of which can be shown to be conserved as follows: from equation (1.1.11), it can be proved by induction that, for any power k

$$\frac{dL^k}{dt} = \left[M, L^k\right], \tag{1.1.12}$$

and it is then simple to show that the trace of any power of L is a conserved quantity [13]

$$\begin{aligned} \frac{d}{dt} tr\left(L^k\right) &= tr\left(\frac{dL^k}{dt}\right) \\ &= tr\left(\left[M, L^k\right]\right) \\ &= tr\left(ML^k\right) - tr\left(L^k M\right) \\ &= 0, \end{aligned} \tag{1.1.13}$$

since $tr(XY) = tr(YX)$ for two arbitrary n x n matrices X and Y. This result is briefly revisited in Chapter 6, where it is seen that the conserved quantities generated in this

[6] This is shown in detail for the case where U is a unitary transform in Chapter 6.

fashion from the classical Lax matrix of the Toda lattice are also the local conservation laws of the quantum system. The second method involves evaluating the following determinant, which generates a polynomial in the eigenvalue λ, the coefficients of which are another equivalent set of constants of motion

$$\det\left[\lambda - L\right] \;=\; \lambda^N + \sum_{n=1}^{N} \lambda^{N-n} J_n. \tag{1.1.14}$$

This determinant will also be seen again in Chapter 6, when it is used to link the classical and quantum Toda lattices, and its logarithm is seen to provide the local conservation laws of that system.

1.2 Quantum Integrable Systems

Unlike the classical case, the definition of a *Quantum* integrable system is much less clear cut. When making the transition from a classical to a quantum system, the standard idea is that Poisson brackets become commutators. The equivalent of the classical integral of motion whose Poisson bracket with the Hamiltonian vanishes therefore, is the (quantum mechanical) conserved quantity which commutes with the Hamiltonian, *i.e.*

$$\left\{H, C_\alpha\right\} \;=\; 0 \quad \rightarrow \quad \left[\hat{H}, \hat{C}_\alpha\right] \;=\; \hat{H}\;\hat{C}_\alpha - \hat{C}_\alpha\;\hat{H} \;=\; 0. \tag{1.2.1}$$

Similarly the requirement that the Poisson bracket between two integrals of motion vanishes would become the restriction that any two of the conserved quantities $\hat{C}_\alpha$ must commute. The quantum analogue of Liouville's definition of classical integrability would therefore be a system with n mutually commuting conserved quantities (which includes the Hamiltonian), where n is now the number of states of the system.

This is the most generally accepted description of quantum integrability, but it is by no means ideal. The reduction of phase space by the constants of motion does not have any equivalent in the above definition of quantum integrability. Furthermore, as is discussed

in detail in Chapter 2, any power of a conserved quantity (including the Hamiltonian) is also a (distinct) conserved quantity, and so the entire required set of mutually commuting operators could in theory be generated from a single conservation law, or indeed just from the Hamiltonian. Obviously, just having the Hamiltonian and its powers is not enough, or every system would be integrable! The solution to this, again described fully in Chapter 2, is to use the projection operator equivalent to a conserved quantity, since a projection operator to any power just returns the same operator.

The idea of using projection operators, which are a restriction to a particular eigenvalue subspace of the system in question, suggests an analogous idea to that of the reduced phase space in classical integrable systems. Consider the matrix representing the Hamiltonian: when attempting to diagonalise, the result is in general a series of blocks along the leading diagonal, each representing a different eigenvalue subspace. Since a conservation law commutes with the Hamiltonian, it is simultaneously diagonalisable and can potentially be used to reduce the degeneracy in a given subspace. The analogue to reducing classical phase space down to an N-torus is therefore to use conservation laws to *fully* diagonalise the Hamiltonian. Note that unlike the classical case this does not necessarily require *all* of the conservation laws of the system, nor does it mean the conservation laws themselves will be fully diagonalised: it all depends on the *degeneracies* of the system.

1.2.1 The Bethe Ansatz

In 1931, Hans Bethe [14] submitted a paper in which he determined the energy eigenstates of the one-dimensional Heisenberg model. It outlines a method (nowadays referred to as the "Coordinate Bethe Ansatz", to avoid confusion with the variants developed later), for constructing the many-particle wavefunction which diagonalises the Hamiltonian [15].

The German word "ansatz" roughly translates as "beginning", which is appropriate as the method used by Bethe marks the starting point for the solutions to quantum integrable systems. In the mathematical sense that it is used here however, it describes the proposal of (the form of) a solution, which is then proved to be correct. In this case,

Bethe posited that the wavefunctions of the model were superpositions of plane waves [16], as is described below, and then used linear combinations of these wavefunctions as the eigenstates of the system. Using these states in the eigenvalue equation $\hat{H}|\psi\rangle = E|\psi\rangle$ provides a series of equations (the "Bethe Ansatz equations"), each solution of which represents an eigenvector of the Hamiltonian (with the originally chosen form) and it's associated eigen-energy [17].

The one-dimensional Heisenberg magnet Bethe considered consists of an array of quantum mechanical spin-$\frac{1}{2}$ particles at fixed, equidistant points on a chain (1D lattice) [16]. The Hamiltonian for this model can be written as [18]

$$\begin{aligned} \hat{H} &= \frac{J}{2}\sum_{j=1}^{N}(\hat{\boldsymbol{\sigma}}_{\boldsymbol{j}}\cdot\hat{\boldsymbol{\sigma}}_{\boldsymbol{j+1}}+1) \\ &= J\sum_{j=1}^{N}\left(\hat{\sigma}_j^+\hat{\sigma}_{j+1}^- + \hat{\sigma}_j^-\hat{\sigma}_{j+1}^+ + \frac{1}{2}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z + \frac{1}{2}\right), \end{aligned} \tag{1.2.2}$$

where $\boldsymbol{\sigma_j} = (\hat{\sigma}_j^x, \hat{\sigma}_j^y, \hat{\sigma}_j^z)$ are the Pauli pin matrices at site j, and $\hat{\sigma}_j^{\pm} = \hat{\sigma}_j^x \pm i\hat{\sigma}_j^y$, with the periodic boundary condition $j + N = j$. The constant J can be used to control the style of magnetism: when J is negative (positive) the model is (anti)ferromagnetic, *i.e.* (anti)parallel spins are energetically preferred [16].

The above Hamiltonian remains unchanged under a rotation about the z-axis, which implies that the total spin $\hat{S}_{Total}^z = \frac{1}{2}\sum_{n=1}^{N}\hat{\sigma}_n^z$ is conserved [17]. This suggest the grouping of basis states by their (increasing) z-component of spin, which will block diagonalise the Hamiltonian matrix. The first block will then consist solely of the state with all spins pointing up, the ground state of the ferromagnetic model consider by Bethe [17].

The remaining blocks consist of C_r^N states, r being the number of down spins which defines the group of states in that block. Each block can then be fully diagonalised by writing the eigenstates as a linear combination of its basis states

$$|\psi_r\rangle = \sum a(x_1,\cdots,x_r)|x_1,\cdots,x_r\rangle, \tag{1.2.3}$$

where $|x_1, \cdots, x_r\rangle$ is the state where the r spins $x_1 > \cdots > x_r$ are pointing down, and the sum is over all C_r^N states in the block. Bethe's ansatz was to assume the coefficients in the above linear combination took the form

$$a(x_1, \cdots, x_r) \;=\; \sum_P A_P \exp\left(\sum_{j=1}^{r} i k_{P_j} x_j\right), \tag{1.2.4}$$

where the P being summed over are the $r!$ permutations of the labels $\{1, 2, \cdots, r\}$ [16, 17, 18]. Substituting these hypothesised eigenvectors into the eigenvalue equation $\hat{H}|\psi_r\rangle = E|\psi_r\rangle$, Bethe was able to calculate the final part of the above coefficients, A_P, in terms of a two-body interaction

$$A_P \;=\; \epsilon_P \prod_{1 \le i < j \le n} \boldsymbol{s}_{P_i P_j}, \quad \text{where} \quad \boldsymbol{s}_{ij} \;=\; 1 - 2\exp(ik_j) + \exp(ik_i + ik_j), \tag{1.2.5}$$

and ϵ_P is the signature of the permutation. The wavenumbers k_j must also satisfy the r conditions

$$\exp(ik_j N) \;=\; (-1)^{r-1} \prod_{i \neq j} \frac{\boldsymbol{s}_{ji}}{\boldsymbol{s}_{ij}}, \quad \text{for} \quad j \;=\; 1, \cdots, r, \tag{1.2.6}$$

which are known as the *Bethe Ansatz equations*. The final task is then to solve these equations for the wavenumbers k_j, which (from the eigenvalue equation) give the energy eigenvalues in the form

$$E_r \;=\; JN + J\sum_{j=1}^{r} (2\cos k_j - 2). \tag{1.2.7}$$

with the associated eigenstates (1.2.3), whose coefficients come from substituting the k_j back into (1.2.4). There are $N+1$ classes of eigenvalues E_r (including the ground state) with $C_r^N (r = 0, \cdots, N)$ associated eigenvalues, which gives a total of 2^N eigenstates[7]. Since 2^N is the total number of eigenstates of the N-site Heisenberg chain, this represents

[7] As, from the binomial expansion, $\sum_{r=0}^{N} C_r^N \;=\; \sum_{r=0}^{N} \frac{N!}{(N-r)!r!} \;=\; (1+1)^N$.

the complete eigenspectrum of the problem [16].

1.2.2 Quantum Inverse Scattering

To *completely* solve any given quantum-mechanical problem, two things are needed. Firstly, it is necessary to solve the eigenvalue problem $\hat{H}|\psi\rangle = E|\psi\rangle$ and obtain the complete eigenspectrum of the Hamiltonian. The second requirement is the ability to calculate the correlation functions of the various operators $\hat{O}$, corresponding to the physical observables of the problem, *i.e.* to be able to (numerically or analytically) compute the thermal average

$$\langle \hat{O} \rangle = \frac{tr\left(\hat{O}e^{-\beta\hat{H}}\right)}{tre^{-\beta\hat{H}}}. \tag{1.2.8}$$

In the years following Bethe's publication, his ansatz for the Heisenberg chain was adapted and used to obtain the eigenspectrum of several (1+1)-dimensional[8] models, such as the one-dimensional Bose gas and the massive Thirring and Hubbard models [19]. The method was not generally applicable however, and not much use for calculating correlation functions due to the complicated form of the eigenvectors it produced [20].

After the introduction of the Classical Inverse Scattering Method however, investigations began into a quantum equivalent. At the same time, the Bethe Ansatz was also being used on 2D classical statistical models such as the six-vertex, or ice model. The discovery of a relationship between the 2D classical models and 1D quantum spin chains [21, 22] was used in conjunction with the classical methods and the Bethe Ansatz, and the ideas of Quantum Inverse Scattering theory were born [23].

While continuous models can also be considered [19], the following ideas were developed from quantum Heisenberg-like and classical vertex models, and are in a way tailor made for discrete systems [15]. Since all systems considered later on will be lattice models, the methods that follow will relate purely to models defined on a periodic chain of sites.

[8] *i.e.* one spatial dimension and time.

The construction of the main tools of the quantum inverse problem begins with the quantisation of the classical L matrix to obtain the *quantum Lax operator* $\hat{L}$, whose elements are naturally now quantum mechanical operators, which do not necessarily commute. The classical L matrix comes from a more generalised (and discrete) version of equation (1.1.10a) [10, 23]

$$\Psi_{n+1} = L_n(\lambda)\Psi_n, \tag{1.2.9}$$

where n is a site index signifying position on the lattice and λ is known as the *spectral parameter*. Following suit, the equivalent Lax operator $\hat{L}$ can therefore be thought of as representing transport along the lattice, suggesting the introduction of an operator which describes transport along the entire length of the lattice[9] [24]

$$\begin{aligned} \hat{M}(\lambda) &= \left(\tilde{L}_N(\lambda)\cdots\tilde{L}_2(\lambda)\tilde{L}_1(\lambda)\right) \\ &= \prod_{n}^{\curvearrowleft} \tilde{L}_n(\lambda). \end{aligned} \tag{1.2.10}$$

This is known as the Monodromy matrix, and from it one can construct the Transfer matrix, which is defined as its trace

$$\hat{T}(\lambda) = tr\ \hat{M}(\lambda). \tag{1.2.11}$$

This matrix will later be seen to be the generator of the conservation laws of a system, and in Chapter 3 its construction from the Lax operator will be central to the development of a new method for calculating a set of these conservation laws.

These objects form the core of Quantum Inverse Scattering theory, alongside one final operator and the identity it is required to satisfy: the R-matrix and the celebrated Yang-Baxter equations, which are used to define the commutation relations of the above operators.

[9] Or equivalently given the periodic boundary conditions, once around the chain.

The Yang-Baxter Equation

The Transfer matrices just defined are of course the same as those used in calculating the partition functions of the two-dimensional classical statistical models. These matrices have been shown to generate, and commute with, the Hamiltonians of quantum one-dimensional lattice systems [21, 22, 25].

Since it commutes with the Hamiltonian, $\hat{T}$ must represent a conservation law of the associated system. The Transfer matrix is, in fact, a *generator* of the conserved quantities of the related Hamiltonian [26]. Furthermore it can be shown that Transfer matrices with different values of λ commute, as will now be demonstrated.

The Fundamental Commutation Relations [24] below define the commutation relations between the various (operator) elements of the Lax operator $\hat{L}_n$

$$R(\lambda,\mu)\left(\hat{L}_n(\lambda)\otimes\hat{L}_n(\mu)\right) = \left(\hat{L}_n(\mu)\otimes\hat{L}_n(\lambda)\right)R(\lambda,\mu), \tag{1.2.12}$$

where R is some invertible complex matrix. If the Lax operators for different sites commute, a similar relation can be constructed for the Monodromy matrix

$$R(\lambda,\mu)\left(\hat{M}(\lambda)\otimes\hat{M}(\mu)\right) = \left(\hat{M}(\mu)\otimes\hat{M}(\lambda)\right)R(\lambda,\mu). \tag{1.2.13}$$

Alternatively, this can be written as

$$R_{12}(\lambda,\mu)\hat{M}_1(\lambda)\hat{M}_2(\mu) = \hat{M}_2(\mu)\hat{M}_1(\lambda)R_{12}(\lambda,\mu), \tag{1.2.14}$$

where the subscripts denote the subspaces the operators act in, with

$$\hat{M}_1(\lambda) = \hat{M}(\lambda)\otimes\hat{I}_N, \quad \text{and} \quad \hat{M}_2(\mu) = \hat{I}_N\otimes\hat{M}(\mu), \tag{1.2.15}$$

$\hat{I}_N$ being the N x N identity matrix. Taking the trace of both sides of equation (1.2.13), it is then easy to show that relations of the above form for the Lax operators and Mon-

odromy matrix imply that any two Transfer matrices with different values of the spectral parameter commute, *i.e.*

$$\left[\hat{T}(\lambda), \hat{T}(\lambda')\right] = 0. \tag{1.2.16}$$

For this commutation to hold however, an appropriate R-matrix is required. It was shown by Baxter [27] that, for equation (1.2.12) to be satisfied, the R-matrix must also satisfy the relation

$$\hat{R}_{12}(\mu)\hat{R}_{13}(\lambda)\hat{R}_{23}(\lambda,\mu) \;=\; \hat{R}_{23}(\lambda,\mu)\hat{R}_{13}(\lambda)\hat{R}_{12}(\mu), \tag{1.2.17}$$

the famous *Yang-Baxter equation*.

This equation is central to the inverse scattering problem, and indeed to integrable systems as a whole. For any given Hamiltonian, an R-matrix which satisfies the Yang-Baxter equation can be used to generate a Lax pair, and is sufficient proof of integrability of the associated system [26]. It also leads to equation (1.2.16), which proves that the Transfer matrix gives a mutually commuting set of conserved quantities. Moreover, *any* of those conserved quantities could be considered a Hamiltonian, and so in fact one has an entire family of quantum integrable systems.

The solution to the Quantum Inverse Scattering problem however, revolves around the Transfer matrix. As well as using it to generate the commuting conserved quantities, it is also needed to find the eigenvectors of the Hamiltonian (related to its own since commuting matrices are simultaneously diagonalisable). These conserved quantities and eigenvectors can be considered as the quantum analogues of the action and angle variables respectively [23].

Algebraic Bethe Ansatz

Often used synonymously with the Quantum Inverse Scattering *Method*, the Algebraic Bethe Anstaz is a generalisation of the original (coordinate) version, centered around the

idea of creating the eigenfunctions of the Hamiltonian using creation and annihilation operators on a pseudovacuum [19]. These operators stem from the matrix elements of the Monodromy matrix, while the eigenfunctions of the Transfer matrix determine the eigenfunctions of the Hamiltonian, since the two commute[10].

These ideas will be briefly described by once again considering the one-dimensional Heisenberg magnet. The Monodromy matrix for this system can be written as the 2 x 2 matrix [15, 19, 20, 23, 24, 28]

$$\hat{M}(\lambda) = \begin{pmatrix} \hat{A}(\lambda) & \hat{B}(\lambda) \\ \hat{C}(\lambda) & \hat{D}(\lambda) \end{pmatrix}. \tag{1.2.18}$$

The four elements of the above matrix are the operators that will be used to construct the eigenfunctions of the Hamiltonian. The pseudovacuum (also known as the generating vector) $|0\rangle$ in this case is the same as the reference state used for the Coordinate Bethe Ansatz, *i.e.* the ferromagnetic state with all spins pointing up[11]. It is an important requirement that $|0\rangle$ obeys the relations

$$\hat{A}(\lambda)|0\rangle = a(\lambda)|0\rangle; \tag{1.2.19a}$$

$$\hat{D}(\lambda)|0\rangle = d(\lambda)|0\rangle; \tag{1.2.19b}$$

$$\hat{C}(\lambda)|0\rangle = 0. \tag{1.2.19c}$$

Here $\hat{C}(\lambda)$ acts like an annihilation operator ($\hat{B}(\lambda)$ will later be used as a kind of creation operator), while the pseudovacuum is an eigenvector of $\hat{A}(\lambda)$ and $\hat{D}(\lambda)$ with "vacuum eigenvalues" $a(\lambda)$ and $d(\lambda)$ respectively. With the Monodromy matrix in the above form, the Transfer matrix becomes the sum of the two operators on the diagonal

$$\hat{T}(\lambda) = \hat{A}(\lambda) + \hat{D}(\lambda), \tag{1.2.20}$$

[10] Not just the Hamiltonian, of course, but all the mutually commuting conservation laws too.

[11] Although the state with all spins pointing down could also be used [19], in which case the operators $\hat{B}(\lambda)$ and $\hat{C}(\lambda)$ swap roles.

and so $|0\rangle$ is clearly also an eigenvector of $\hat{T}(\lambda)$, and therefore also of the Hamiltonian. The commutation relations for the four operator elements of $\hat{M}(\lambda)$ are of course given by equation (1.2.13) where the R-matrix in this particular case takes the form

$$\hat{R}(\lambda,\mu) \;=\; \begin{pmatrix} r(\lambda,\mu) & 0 & 0 & 0 \\ 0 & s(\lambda,\mu) & 1 & 0 \\ 0 & 1 & s(\lambda,\mu) & 0 \\ 0 & 0 & 0 & r(\lambda,\mu) \end{pmatrix}, \tag{1.2.21}$$

where the diagonal elements are

$$r(\lambda,\mu) \;=\; 1+\frac{ic}{\mu-\lambda}, \quad \text{and} \quad s(\lambda,\mu) \;=\; \frac{ic}{\mu-\lambda}. \tag{1.2.22}$$

Without going into the full details (see *e.g.* [19, 23]), the sixteen sets of commutation relations provided by the Yang-Baxter equation (1.2.13) are used to find the effect of the operators $\hat{A}(\mu)$ and $\hat{D}(\mu)$ on the state $|\Psi_M\rangle = \prod_i^n \hat{B}(\lambda_i)|0\rangle$. Taking the resulting equations and requiring that $|\Psi_M\rangle$ be an eigenvector of $\hat{A}(\lambda)$ and $\hat{D}(\lambda)$, leads to the set of equations

$$\frac{a(\lambda_n)}{d(\lambda_n)} \prod_{i\neq n}^{M} \frac{r(\lambda_i,\lambda_n)}{r(\lambda_n,\lambda_i)} \;=\; 1, \qquad n \;=\; 1,\cdots,M. \tag{1.2.23}$$

If these equations are satisfied then the $|\Psi_M\rangle$ must be eigenvectors of the Transfer matrix $\hat{T}(\lambda)$, and so from the eigenvalue equation $\hat{T}(\mu)|\Psi_M\rangle = E(\mu,\{\lambda_j\})|\Psi_M\rangle$, their corresponding eigenvalues are

$$E(\mu,\{\lambda_j\}) \;=\; a(\mu)\prod_{j=1}^{M} r(\lambda_j,\mu) + d(\mu)\prod_{j=1}^{M} r(\mu,\lambda_j). \tag{1.2.24}$$

Finally, the vacuum eigenvalues of the one-dimensional Heisenberg model are

$$a(\lambda_n) \;=\; \Big(\lambda - i\frac{c}{2}\Big)^M, \quad \text{and} \quad d(\lambda_n) \;=\; \Big(\lambda + i\frac{c}{2}\Big)^M, \tag{1.2.25}$$

and substituting these into equation (1.2.23) and using the reparameterisation

$$\exp ik_j = \frac{2\lambda_j + ic}{2\lambda_j - ic}, \tag{1.2.26}$$

one finds that these equations are in fact the original Bethe Ansatz equations (1.2.6).

There are many subtleties, even in this simple example, which will not be covered here[12]. The aim of this section was only to show roughly how the new formulation of quantum inverse scattering was used, as it will not appear or be mentioned again. Hopefully this whistle-stop tour of quantum integrability will provide a suitable context, as we now consider the key ideas which form the starting point from which the investigations of the following chapters began.

1.3 Local Conservation Laws and Correlation Functions

Earlier it was shown how different values of the spectral parameter could be used in the Transfer matrix to provide a set of conserved quantities. In practise however, actually using this method to generate conservation laws is a subtle issue as there is not a one-to-one correspondence between values of the spectral parameter and conservation laws. Furthermore, the conserved quantities produced in this way are in the form of matrices which commute with the Hamiltonian, and are generally analytically intractable. It would be highly preferable therefore, to have a method of *analytically* generating the conserved quantities of the system.

Fortunately such a method exists, although it has it's own subtleties which will be considered in detail in Chapter 3. It was shown by Tetelman [29] that the conservation laws of a one-dimensional (integrable) spin system could be constructed iteratively by

[12]The actual eigenvectors of the *Hamiltonian*, for example, still need to be found. This is achieved by using trace identities which can be found in [19], among others.

using the relation

$$\hat{C}_{n+1} \;=\; \left[\hat{B}, \hat{C}_n\right]. \tag{1.3.1}$$

The series starts with the Hamiltonian of the system, *i.e.* $\hat{C}_0 = \hat{H}$, and then commutation with the Boost operator[13] $\hat{B}$ generates the first conserved quantity, which is commuted with the Boost operator to provide the second conserved quantity, and so on. These conservation laws form a mutually commuting set, and a Boost operator for any integrable system can be derived from the Yang-Baxter equation (1.2.17) [5, 30]. As an example, for fundamental models (*i.e.* models whose R-matrix and Lax operator coincide [19]) such as the Heisenberg chain, this Boost operator has the form

$$\hat{B} \;=\; \sum_n n\; \hat{h}_{n,n+1}, \quad \text{where} \quad \hat{H} \;=\; \sum_n \hat{h}_{n,n+1}. \tag{1.3.2}$$

An important feature of the conservation laws generated by the Boost operator is their *locality*: an interaction involving some number of sites vanishes as the distance between them increases [5]. In the case of the conservation laws produced above, $\hat{C}_m$ represents an interaction which spans no more than $m+2$ sites.

It should also be noted that the above example uses a nearest-neighbour Hamiltonian, *i.e.* one that operates only on two sites, which are next to each other on the lattice. This property is necessary for the functioning of the Boost operator method [31], and for the new method detailed in Chapter 3, and so henceforth only Hamiltonians of this type will be considered.

Given their aforementioned role as generators of the conserved quantities of the associated Hamiltonian, one might hope to more directly link the Transfer matrix to these local conservation laws. As well as using different values of the spectral parameter, one could also expand out the Transfer matrix as a power series in the spectral parameter. The coefficients of each power of lambda in this series form a family of mutually commuting

[13] Also known as a ladder operator.

operators [20]. These are conservation laws of the system since the second (the coefficient of the λ term) is related to the Hamiltonian via [32]

$$\hat{H} \;=\; -\,\hat{T}(0)^{-1}\,\frac{d}{d\lambda}\,\hat{T}(\lambda)\Big|_{\lambda=0}. \tag{1.3.3}$$

The Hamiltonian is a local conservation law, but the other coefficients are in general not. Up to a sign, the left hand side of equation (1.3.3) corresponds to the first derivative of the logarithm of the Transfer matrix (with λ then set to 0) [32]. It can then be shown that the higher derivatives of this logarithm all commute, and so form a set of conserved quantities [5, 32]. These quantities are the same local conservation laws described above, formally generated via

$$\ln\left(\hat{T}(\lambda)\hat{T}(0)^{-1}\right) = \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\tilde{C}_{n-1}. \tag{1.3.4}$$

It is not at all obvious *why* taking the logarithm should return *local* conservation laws, but the locality of these $\hat{C}_m$ has been shown [33], and since $\hat{C}_0$ is again the Hamiltonian, the coefficients of the above expansion form a mutually commuting set of conservation laws.

This idea of taking the logarithm of the Transfer matrix forms the basis of the method developed in Chapter 3. A new representation of the Lax operator will be used to allow for a straightforward evaluation of this logarithm which iteratively returns the local conservation laws.

1.3.1 Persistent Currents

One of the more actively studied properties of integrable systems is the non-standard decay of current operators: once started, a current will usually decay away with time and eventually vanish, but in integrable systems this is not necessarily the case. In these systems a current will decay some extent, but it is possible to have a non-trivial long-time residual current, *i.e.* a portion of the current which will continue to flow undiminished and indefinitely [3].

The time average of a correlation function has been shown to be expressible in terms of canonical averages involving the conserved quantities of the system [2]. As described in Chapter 2, the long-time residual current is calculable in this manner, and there is an implied suggestion that it should be thought of as the conserved part of the current operator.

An important corollary to the work by Suzuki [2] is the rederivation of Mazur's inequality [4], which can be used to obtain a lower bound for the time average of a correlation function of hermitian operators

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left\langle \hat{X}(t)\hat{X}(0) \right\rangle dt \;\; \geq \;\; \frac{\left\langle \hat{X}(0)\hat{C}_a \right\rangle \left\langle \hat{C}_a^\dagger \hat{X}(0) \right\rangle}{\left\langle \hat{C}_a^\dagger \hat{C}_a \right\rangle}. \tag{1.3.5}$$

The important point, as Mazur himself notes, is that this lower bound may be found without solving the dynamics of the system. In the case of the long-time residual current, the above inequality can be used as a test: if any one conservation law ($\hat{C}_a$) returns a non-trivial value, that is sufficient to prove the existence of a persistent current (although obviously if it does return a vanishing contribution, this is not enough to show the residual current does not exist).

In combining Suzuki's idea with Mazur's inequality one arrives at the following relationship between conservation laws and persistent currents: given all of the conservation laws of an integrable system, the sum of the result of using each one in the above inequality should be equal to the long-time residual current. This should be obviously and trivially true since this residual current can be shown to be one of these conservation laws, and yet in the following example this result appears to fail.

A Residual Current in the XXZ Model

In the process of showing the relationship between correlation functions and the conserved quantities of a system, it was assumed by Suzuki [2] that these conservation laws should form a complete mutually commuting set. This obviously matches nicely with the ideas of

quantum integrability made in analogy to the classical case earlier. However, this idea of mutual commutativity of the conserved quantities has been the cause of some confusion: the XXZ, or anisotropic Heisenberg, model has a known and non-vanishing long-time residual current [3]. It can also be shown however, that *all* of the conservation laws must contribute nothing when used in Mazur's inequality [1].

This lack of agreement between the formal results and a particular example needs to be resolved: is the restriction to a complete mutually commuting set of conservation laws an unnecessary extra condition which causes some, and in this case the vital, conservation laws to be missed? Has a mistake been made in the treatment of the XXZ model, whereby the crucial conservation laws have somehow been missed out, or is there some other more subtle effect at work?

1.4 Overview of Thesis

This thesis is an attempt to investigate the nature of conservation laws in quantum integrable systems, and is split into two distinct projects.

In Chapter 2 the ideas of persistent currents and how they relate to conservation laws are considered. This starts with a general (*i.e.* independent of model and current) analysis of the conserved part of the current operator, showing that it must itself be a conservation law, and how thinking of it as a sum of an arbitrary set of conservation laws lead to Mazur's inequality. Projection operators are then introduced as a natural way to represent conservation laws in a quantum system, and their use with Mazur's inequality to obtain estimates for the residual current in terms of the (still general) current operator is described. This section concludes with a short discussion of the ideas of Suzuki, the Mazur inequality and calculating a persistent current given a set of conservation laws. The chapter is then completed by a numerical investigation of a specific example. The system considered is the XXZ model, where the long-time residual of a particular current has been calculated, but all Mazur based estimates of this residual vanish as it can be shown

that *none* of the local conservation laws contribute anything to the conserved current.

The final four chapters then cover the main focus of the thesis: generating the local conservation laws of an integrable system. Chapter 3 starts with a discussion and critique of the Boost operator method of generating the local conservation laws of a system, using the XXZ model as an example to show how detailed knowledge of the model in question is necessary for the method to work. Following on from the construction of the Transfer matrix just presented, a new and far more general approach is described. An alternate representation of the quantum Lax operator leads to the logarithm of the Transfer matrix taking the form of a well known mathematical formula, that of Campbell, Baker and Hausdorff. Use of the Campbell-Baker-Hausdorff expansion is then described and general expressions for the local conservation laws in terms of commutators of the (model specific) operator $\hat{V}_{ij}$ are obtained.

Chapters 4 and 5 are then used to demonstrate the new method of Chapter 3 for the XXZ and Hubbard models respectively. In each case, a material is first introduced and shown to be described by the model in question, to provide a physical context for its study. The Lax operator is rewritten into its preferred form, and then the operator $\hat{V}_{ij}$ calculated for the model concerned. This is then substituted into the expressions at the end of Chapter3 to obtain the local conservation laws of the model, which are in both cases seen to coincide with their previous calculation. Chapter 4 then concludes with a discussion of Reshetikhin's condition, which is vital when using the Boost method, and is shown to be a natural consequence of the Yang-Baxter equation, with the Lax operator written in the preferred form. Chapter 5 meanwhile, finishes with an attempt to extract the local conservation laws of the Heisenberg model from those of the Hubbard model, but finds it to be more calculationally taxing than just using the method of Chapter 3.

As a continuation of Chapter 4, Appendix B contains what is believed to be the first attempt to write down the explicit form of the sixth local conservation law of the XXZ model.

Chapter 6 then looks at a markedly different system: the Toda Lattice. Despite

the approach of Chapter 3 being inapplicable, due to the nature of the quantum Lax operator, a set of conservation laws are found which match those of the classical system. Investigating the classical system via the Lax pair formalism, a new classical Lax matrix is constructed which has a simple relation to the trace of the Monodromy matrix in the quantum problem. This matrix is then used to generate the local conservation laws of the (quantum and classical) system as the trace of it to increasing powers, as seen earlier.

Finally, Chapter 7 summarises and briefly discusses the conclusions and important ideas that should be taken should be taken from this work, and suggests avenues of further study.

Chapter 2

Residual Current in the XXZ Model

In this chapter the relationship between persistent currents and the conservation laws of a system are investigated. This starts with the derivation of an expression for the residual current in terms of the elements of the matrix representing the current operator. The conserved part of a general operator is then considered, and correlation functions of an operator split into conserved and non-conserved parts are compared with the expression for the long-time residual current. By considering the conserved part of the current operator as a linear combination of conservation laws and some non-conserved part, Mazur's inequality [4] is rederived. This inequality is then used to generate expressions for the contribution to the residual current from the Hamiltonian, and from some unspecified set of conservation laws. These quantities and the idea that a quantum integrable systems conservation laws should form a mutually commuting set are considered and discussed, and the mutual commutativity requirement shown to be unnecessary and potentially catastrophic.

The final section then highlights these issues by considering the specific example where the formal ideas appear to fail. The anisotropic Heisenberg or XXZ model has already received a lot of attention [1, 3, 34, 35, 36, 37], and has been the source of some confusion. At half-filling the long-time residual of the standard current is a known, non-trivial quantity, but the contribution to this residual from the conserved quantities has been shown

to vanish.

Utilising Mazur's identity [4] as a test here fails since *all* of the conservation laws have been shown to contribute nothing to the conserved current, however Suzuki's ideas [2] imply that the conservation laws should be sufficient to calculate the entire long-time residual. This section looks at this model in terms of the ideas of the first part of the chapter, and then resolves the issue with a numerical investigation into whether or not the conserved quantities are indeed sufficient to calculate the residual, or if there is something more subtle going on.

2.1 Calculating the Long-time Residual

As described in the previous chapter, these persistent or residual currents are a feature associated with integrable systems. One would normally expect a current to decay over time and eventually vanish, but in integrable systems this is not necessarily the case, and some residual part of the current may continue flowing indefinitely. The long-time limit of a (decaying) current in an integrable system can be calculated using the following thermal average

$$I_\infty \quad = \quad \langle\, \hat{\jmath}(\infty)\, \hat{\jmath}(0)\, \rangle \quad \equiv \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \langle\, \hat{\jmath}(t)\, \hat{\jmath}(0)\, \rangle\, . \tag{2.1.1}$$

Note that this quantity is trivially related to the charge stiffness (also known as the Drude weight) by an additional coefficient of $\left(\frac{\beta}{2L}\right)$, and if non-zero it implies an ideally conducting system [1]. Writing the current operator in the Heisenberg representation, and working in a basis which diagonalises the Hamiltonian, *i.e.*

$$\hat{H}\ \ |n,m\rangle \quad = \quad \epsilon_n\ \ |n,m\rangle\, , \tag{2.1.2}$$

where m is an arbitrary degeneracy label, the integrand of equation (2.1.1) becomes

$$\begin{aligned}
\langle\, \hat{\jmath}(t)\, \hat{\jmath}(0)\, \rangle &= \frac{tr\left(e^{-\beta\hat{H}}\hat{\jmath}(t)\hat{\jmath}(0)\right)}{tr\left(e^{-\beta\hat{H}}\right)} \\
&= \frac{1}{Z}\sum_n e^{-\beta\epsilon_n}\sum_m \langle n,m|\,\hat{\jmath}(t)\hat{\jmath}(0)\,|n,m\rangle \\
&= \frac{1}{Z}\sum_n e^{-\beta\epsilon_n}\sum_m \langle n,m|\,e^{i\hat{H}t}\hat{\jmath}e^{-i\hat{H}t}\hat{\jmath}\,|n,m\rangle \\
&= \frac{1}{Z}\sum_{nm} e^{-\beta\epsilon_n}\sum_{n'm'} e^{i\epsilon_n t}\,\langle n,m|\,\hat{\jmath}e^{-i\hat{H}t}\,|n',m'\rangle\,\langle n',m'|\,\hat{\jmath}\,|n,m\rangle \\
&= \frac{1}{Z}\sum_{nm} e^{-\beta\epsilon_n}\sum_{n'm'} e^{i(\epsilon_n-\epsilon_{n'})t}\,\langle n,m|\,\hat{\jmath}\,|n',m'\rangle\,\langle n',m'|\,\hat{\jmath}\,|n,m\rangle\,, \qquad (2.1.3)
\end{aligned}$$

where Z is the partition function

$$Z = \sum_n M_n\, e^{-\beta\epsilon_n}, \qquad (2.1.4)$$

and M_n is the degeneracy in n. First setting $t = 0$ to find an expression for what will be referred to throughout as the initial current [1] (though current susceptibility would be more accurate)

$$I_0 = \frac{1}{Z}\sum_{nm} e^{-\beta\epsilon_n}\sum_{n'm'} \langle n,m|\,\hat{\jmath}\,|n',m'\rangle\,\langle n',m'|\,\hat{\jmath}\,|n,m\rangle\,, \qquad (2.1.5)$$

and then looking at just the time dependent part of equation (2.1.3) in equation (2.1.1) finds

$$\begin{aligned}
\frac{1}{T}\int_0^T e^{i(\epsilon_n-\epsilon_{n'})t}dt &= \frac{1}{T}\frac{1}{i(\epsilon_n-\epsilon_{n'})}\left(e^{i(\epsilon_n-\epsilon_{n'})T}-1\right) \\
&\to\ 0 \text{ as } T\to\infty,
\end{aligned}$$

[1] In equilibrium of course, one would not generally expect there to be a current flowing (*i.e.* $\langle j\rangle = 0$). The quantities considered here are the response to some impulse which has started a current flowing in the system, I_0 being the initial response of the system and I_∞ the response the system relaxes to a long time later.

except in the case where $\epsilon_n = \epsilon_{n'}$, in which case the time dependent exponent vanishes, leaving

$$I_\infty \;=\; \frac{1}{Z}\sum_{nm} e^{-\beta\epsilon_n} \sum_{m'} \langle n,m|\,\hat{j}\,|n,m'\rangle\,\langle n,m'|\,\hat{j}\,|n,m\rangle\,. \tag{2.1.6}$$

Clearly for there to be a long-time residual the current operator must have a non zero part which is diagonal in energy. This diagonal part of the current operator must also be large in comparison to the rest of the operator, since this component only represents N out of the N^2 elements (where N itself scales up as 2^n, n being the number of sites), *i.e.* it is expected to be one N^{th} of the total operator, which clearly vanishes in the thermodynamic limit.

2.2 Conservation Laws

In this section the infinite-time correlation functions of an operator split into its conserved and non-conserved parts are considered. The conserved part is then thought of in terms of a sum of conservation laws, and projection operators are introduced as the natural way to represent those conservation laws. The amount of the long-time residual current calculable from various sets of conservation laws is considered, and the idea of the mutual commutativity of a set of conservation laws is discussed.

2.2.1 The Conserved Part of a General Operator

Conservation laws stem from the symmetries of a system, and have the defining property that the operator representing a conserved quantity commutes with the Hamiltonian, *i.e.*

$$\left[\hat{H},\hat{C}\right] \;=\; 0.$$

A general operator can be represented as

$$\hat{O} = \sum_{nn'} \sum_{mm'} |n, m\rangle \, O_{nn'}^{mm'} \, \langle n', m'| \,, \tag{2.2.1}$$

in which form the Hamiltonian becomes

$$\hat{H} = \sum_{nm} |n, m\rangle \, \epsilon_n \, \langle n, m| \,. \tag{2.2.2}$$

The commutator of the two is then

$$\begin{aligned}
\left[\hat{O}, \hat{H}\right] &= \hat{O}\hat{H} - \hat{H}\hat{O} \\
&= \sum_{nm} \sum_{n'm'} \sum_{\tilde{n}\tilde{m}} |n, m\rangle \, O_{n\tilde{n}}^{m\tilde{m}} \, \langle \tilde{n}, \tilde{m} \, |n', m'\rangle \, \epsilon_{n'} \, \langle n', m'| \\
&\quad - \sum_{nm} \sum_{n'm'} \sum_{\tilde{n}\tilde{m}} |n, m\rangle \, \epsilon_n \, \langle n, m \, |\tilde{n}, \tilde{m}\rangle \, O_{\tilde{n}n'}^{\tilde{n}m'} \, \langle n', m'| \\
&= \sum_{nm} \sum_{n'm'} \Big(|n, m\rangle \, O_{nn'}^{mm'} \epsilon_{n'} \, \langle n', m'| - |n, m\rangle \, \epsilon_n O_{nn'}^{mm'} \, \langle n', m'| \Big) \\
&= \sum_{nm} \sum_{n'm'} |n, m\rangle \, O_{nn'}^{mm'} \, (\epsilon_{n'} - \epsilon_n) \, \langle n', m'| \,,
\end{aligned} \tag{2.2.3}$$

and so for the operator $\hat{O}$ to commute with the Hamiltonian we (again) have the condition that $\epsilon_{n'} = \epsilon_n$. This suggests the splitting of an operator into conserved and non-conserved parts, like so

$$\begin{aligned}
\hat{O} &= \hat{O}_c + \hat{O}_{\bar{c}} \\
&= \sum_{n} \sum_{mm'} |n, m\rangle \, O_{nn}^{mm'} \, \langle n, m'| + \sum_{\substack{nn' \\ n' \neq n}} \sum_{mm'} |n, m\rangle \, O_{nn'}^{mm'} \, \langle n', m'| \,.
\end{aligned} \tag{2.2.4}$$

Splitting the current operator in the original thermal average (equation (2.1.1)) in this manner

$$\begin{aligned}
\langle \, \hat{\jmath}(\infty) \, \hat{\jmath}(0) \, \rangle &= \Big\langle \Big(\hat{\jmath}_c(\infty) + \hat{\jmath}_{\bar{c}}(\infty) \Big) \Big(\hat{\jmath}_c(0) + \hat{\jmath}_{\bar{c}}(0) \Big) \Big\rangle \\
&= \langle \, \hat{\jmath}_c \, \hat{\jmath}_c \, \rangle + \langle \, \hat{\jmath}_c \, \hat{\jmath}_{\bar{c}}(0) \, \rangle + \langle \, \hat{\jmath}_{\bar{c}}(\infty) \, \hat{\jmath}_c \, \rangle + \langle \, \hat{\jmath}_{\bar{c}}(\infty) \, \hat{\jmath}_{\bar{c}}(0) \, \rangle \,,
\end{aligned}$$

using the idea that $\hat{j}_c(\infty) = \hat{j}_c(0) = \hat{j}_c(t) = \hat{j}_c$, since the conserved part of the operator is independent of time. The second and third terms will vanish, because they involve taking the trace of something that is off diagonal, *i.e.*

$$\begin{aligned} \langle\, \hat{j}_c\, \hat{j}_{\bar{c}}(0)\, \rangle &= \frac{1}{Z} \sum_n \sum_{mm'} e^{-\beta\epsilon_n} (J_c)^{mm'}_{nn} (J_{\bar{c}})^{m'm}_{nn} = 0, \\ \langle\, \hat{j}_{\bar{c}}(\infty)\, \hat{j}_c\, \rangle &= \frac{1}{Z} \sum_n \sum_{mm'} e^{-\beta\epsilon_n} (J_{\bar{c}})^{mm'}_{nn} (J_c)^{m'm}_{nn} = 0, \end{aligned}$$

since all the diagonal elements of $j_{\bar{c}}$ are zero by definition. The fourth term is

$$\langle\, \hat{j}_{\bar{c}}(t)\, \hat{j}_{\bar{c}}\, \rangle = \frac{1}{Z} \sum_{n \neq n'} \sum_{mm'} e^{-\beta\epsilon_n} e^{it[\epsilon_n - \epsilon_{n'}]} J^{mm'}_{nn'} J^{m'm}_{n'n},$$

and since $n' \neq n$, this vanishes in the long-time limit, as was shown previously. This leaves just the first term

$$\begin{aligned} \langle\, \hat{j}(\infty)\, \hat{j}(0)\, \rangle &= \langle\, \hat{j}_c\, \hat{j}_c\, \rangle \\ &= \frac{1}{Z} \sum_n e^{-\beta\epsilon_n} \sum_{mm'} J^{mm'}_{nn} J^{m'm}_{nn}, \end{aligned}$$

which is the same as the previous result for the long-time residual, equation (2.1.6), being diagonal in energy and therefore due to the conserved part of the current operator. It is clear therefore that, as suggested by Suzuki, the long-time residual *is* a conservation law, and that conservation law can be represented as the above operator $\hat{j}_c$.

One should note at this point that, thus far, everything has been kept entirely general, *i.e.* no knowledge of model, current or *basis* has been used or assumed. Since Mazur's inequality [4] involves finding the overlap between some observable operator and the conserved quantities of the system, it would seems sensible to try to describe that operator in terms of some set of those conserved quantities.

With this in mind, the conserved part of the operator is represented in terms of some *arbitrary*, linearly independent collection of conservation laws, $\{\hat{C}_a\}$, in the following

manner

$$\hat{j}_c \equiv \sum_a \alpha_a \hat{C}_a + \Delta\hat{j}_c, \tag{2.2.5}$$

where $\Delta\hat{j}_c$ is a catch-all term containing all of $\hat{j}_c$ that cannot be written as a linear combination of the conservation laws considered, and the coefficients α_a need to be determined. Then, by considering the $\hat{C}_a$s to be an orthogonal set of conservation laws, *i.e.*

$$\left\langle \hat{C}_a^\dagger \hat{C}_{a'} \right\rangle = \delta_{aa'} \left\langle \hat{C}_a^\dagger \hat{C}_a \right\rangle, \tag{2.2.6}$$

and since (by definition) there is no overlap between these conservation laws and $\Delta\hat{j}_c$, we have

$$\begin{aligned} \left\langle \hat{C}_a^\dagger \Delta\hat{j}_c \right\rangle = 0 \Rightarrow \left\langle \hat{C}_a^\dagger \hat{j}_c \right\rangle &= \left\langle \hat{C}_a^\dagger \Big(\sum_{a'} \alpha_{a'} \hat{C}_{a'} + \Delta\hat{j}_c \Big) \right\rangle \\ &= \sum_{a'} \alpha_{a'} \left\langle \hat{C}_a^\dagger \hat{C}_{a'} \right\rangle + \left\langle \hat{C}_a^\dagger \Delta\hat{j}_c \right\rangle = \alpha_a \left\langle \hat{C}_a^\dagger \hat{C}_a \right\rangle \\ \Rightarrow \quad \alpha_a &= \frac{\left\langle \hat{C}_a^\dagger \hat{j}_c \right\rangle}{\left\langle \hat{C}_a^\dagger \hat{C}_a \right\rangle} = \frac{\left\langle \hat{C}_a^\dagger \hat{j}(0) \right\rangle}{\left\langle \hat{C}_a^\dagger \hat{C}_a \right\rangle}. \end{aligned} \tag{2.2.7}$$

Alternatively, considering the quantity

$$\left\langle \Delta\hat{j}_c^\dagger \Delta\hat{j}_c \right\rangle = \langle \hat{j}_c \hat{j}_c \rangle - \sum_a \alpha_a^\dagger \left\langle \hat{C}_a^\dagger \hat{j}_c \right\rangle - \sum_b \left\langle \hat{j}_c^\dagger \hat{C}_b \right\rangle \alpha_b + \sum_{ab} \alpha_a^\dagger \left\langle \hat{C}_a^\dagger \hat{C}_b \right\rangle \alpha_b, \tag{2.2.8}$$

and minimising the right hand side with respect to $\alpha_a^\dagger$ or α_b gives

$$\left\langle \hat{C}_a^\dagger \hat{j}_c \right\rangle = \sum_b \left\langle \hat{C}_a^\dagger \hat{C}_b \right\rangle \alpha_b, \tag{2.2.9a}$$

$$\text{or} \quad \left\langle \hat{j}_c^\dagger \hat{C}_b \right\rangle = \sum_a \alpha_a^\dagger \left\langle \hat{C}_a^\dagger \hat{C}_b \right\rangle, \tag{2.2.9b}$$

respectively. These are consistent and find the coefficients of equation (2.2.5) to be

$$\alpha_d \;=\; \sum_a \Big[\,\Big\langle \hat{C}^\dagger \hat{C}\Big\rangle\Big]^{-1}_{da} \Big\langle \hat{C}^\dagger_a \hat{\jmath}_c\Big\rangle\,, \tag{2.2.10}$$

where $[\ ]^{-1}$ signifies a *matrix inverse*. Note that this second method does *not* require mutual commutativity of the $\hat{C}_a$s. Using this expression for the coefficients yields

$$\big\langle \Delta\hat{\jmath}^\dagger_c \Delta\hat{\jmath}_c\big\rangle \;=\; \langle\, \hat{\jmath}_c\hat{\jmath}_c\rangle - \sum_{ab} \Big\langle \hat{\jmath}^\dagger_c \hat{C}_a\Big\rangle \Big[\,\Big\langle \hat{C}^\dagger_a \hat{C}_b\Big\rangle\Big]^{-1}_{ab} \Big\langle \hat{C}^\dagger_a \hat{\jmath}_c\Big\rangle\,. \tag{2.2.11}$$

As one might expect, the coefficients (2.2.10) are the same as those from equation (2.2.7) if the conservation laws under consideration are indeed orthogonal. In this case the above equation leads to the form of Mazur's inequality [4] used by Suzuki [2], namely

$$\langle \hat{\jmath}(\infty)\hat{\jmath}(0)\rangle \;\geq\; \sum_a \frac{\Big\langle \hat{\jmath}(0)\hat{C}_a\Big\rangle \Big\langle \hat{C}^\dagger_a \hat{\jmath}(0)\Big\rangle}{\Big\langle \hat{C}^\dagger_a \hat{C}_a\Big\rangle}, \tag{2.2.12}$$

where the right hand side is the total contribution to the residual current from a linear combination of the set of conservation laws $\{\hat{C}_a\}$, with the two sides being equal when

$$\langle \hat{\jmath}(0)\Delta\hat{\jmath}_c\rangle \;\equiv\; \big\langle \Delta\hat{\jmath}^\dagger_c \Delta\hat{\jmath}_c\big\rangle \;=\; 0. \tag{2.2.13}$$

Since the above is positive definite, this requires $\Delta\hat{\jmath}_c = 0$, and so the two sides of equation (2.2.12) are equal when the conserved part of the current operator can be written *entirely* as a linear combination of the conservation laws (though in general this is not something we should expect). This quantity $\Delta\hat{\jmath}_c$ is the crux of the entire matter: if it is indeed zero then Suzuki is correct and the Mazur estimate will generate the full long-time residual, but if $\Delta\hat{\jmath}_c$ is non-trivial then the set of conservation laws will only partially describe the conserved part of the current, if at all.

As was mentioned previously, Suzuki required the set of conservation laws, which

include the Hamiltonian, to be mutually commuting, *i.e.*

$$\left[\hat{C}^a, \hat{C}^b\right] \;=\; 0 \quad \text{where} \quad \hat{C}^0 = \hat{H},$$

but we have not yet given thought as to how these conservation laws will be generated. It is advisable, therefore, to proceed with the assumption that we will only have a *limited* set of the available laws, and see what effect this has on our calculation of the long-time residual.

2.2.2 Conservation from the Hamiltonian

The first step is to consider the set of conservation laws consisting of the Hamiltonian and its powers (since powers of the Hamiltonian clearly commute with the Hamiltonian and each other, and are also distinct conservation laws). The best way to examine conservation laws of this form is via projection operators, mathematically defined as

$$\hat{P}^2 \;=\; \hat{P} \;\Leftrightarrow\; \hat{P}\left(\hat{P}-1\right) \;=\; 0, \tag{2.2.14}$$

which have the eigenvalues one or zero. From this, $1-\hat{P}$ can also be shown to be a projection operator, with the same eigenvalues. The operator $\hat{P}$ will project onto the states for which its eigenvalue is one, while $1-\hat{P}$ will project away from these states. These ideas can be extended to a *complete, orthogonal* set of projection operators, *i.e.*

$$\hat{P}_n\hat{P}_m \;=\; \delta_{nm}\hat{P}_n \quad \text{with} \quad \sum_n \hat{P}_n \;=\; 1. \tag{2.2.15}$$

These projection operators can then be used to enforce a restriction to a certain subset of states. For example, the operator

$$\hat{P}_n \;\equiv\; \prod_{n'\neq n} \frac{\left[\hat{H}-\epsilon_{n'}\right]}{\left[\epsilon_n-\epsilon_{n'}\right]}, \tag{2.2.16}$$

will project states onto the subspace for which $\hat{H}$ has the eigenvalue ϵ_n. Writing this in the notation previously used for operators, $\hat{P}_n$ becomes

$$\hat{P}_n \equiv \sum_m |n,m\rangle \langle n,m| , \tag{2.2.17}$$

and so equation (2.2.2) can be rewritten as

$$\hat{H} \equiv \sum_n \epsilon_n \hat{P}_n. \tag{2.2.18}$$

As the projection operators can be created from the Hamiltonian and vice versa, the two are essentially equivalent, and since any power of a particular projection operator will clearly just return the same projection operator again, these operators give *all* of the conservation due to the Hamiltonian.

If these projection operators are now used as the set of conservation laws $\hat{C}_a$ in equation (2.2.12), then the numerator and denominator respectively become

$$\begin{aligned}
\left\langle \hat{P}_n \hat{j}_c \right\rangle &= \frac{1}{Z} \sum_{n'm'} e^{-\beta\epsilon_{n'}} \langle n',m'| \sum_m |n,m\rangle \langle n,m| \sum_{\bar{n}} \sum_{\bar{m}\dot{m}} |\bar{n},\bar{m}\rangle J^{\bar{m}\dot{m}}_{\bar{n}\bar{n}} \langle \bar{n},\dot{m} |n',m'\rangle \\
&= \frac{1}{Z} \sum_{n'm'} e^{-\beta\epsilon_{n'}} \sum_{n\bar{n}} \sum_{\bar{m}\dot{m}} \delta_{n'n}\delta_{m'm}\delta_{n\bar{n}}\delta_{m\bar{m}} J^{\bar{m}\dot{m}}_{\bar{n}\bar{n}} \delta_{\bar{n}n'}\delta_{\dot{m}m'} \\
&= \frac{1}{Z} e^{-\beta\epsilon_n} \sum_{m'} J^{m'm'}_{nn}, \\
\left\langle \hat{P}_n \hat{P}_n \right\rangle &= \left\langle \hat{P}_n \right\rangle = \frac{1}{Z} \sum_{n'm} e^{-\beta\epsilon_{n'}} \langle n',m| \sum_{m'} |n,m'\rangle \langle n,m' |n',m\rangle \\
&= \frac{1}{Z} \sum_{n'} e^{-\beta\epsilon_{n'}} \sum_{mm'} \delta_{n'n}\delta_{mm'}\delta_{nn'}\delta_{m'm} \\
&= \frac{1}{Z} e^{-\beta\epsilon_n} \sum_m 1 \\
&= \frac{1}{Z} e^{-\beta\epsilon_n} M_n.
\end{aligned}$$

Substituting these results back into equation (2.2.12), the contribution to the long-time

residual due to the conservation laws generated from the Hamiltonian is

$$\begin{aligned} I_H &= \sum_n \frac{\left\langle \hat{P}_n \hat{j}_c \right\rangle \left\langle \hat{j}_c \hat{P}_n \right\rangle}{\left\langle \hat{P}_n \right\rangle} \\ &= \sum_n \frac{1}{\frac{1}{Z} e^{-\beta \epsilon_n} M_n} \left(\frac{1}{Z} e^{-\beta \epsilon_n} \sum_{m'} J_{nn}^{m'm'} \right)^2 \\ &= \frac{1}{Z} \sum_n e^{-\beta \epsilon_n} \frac{1}{M_n} \sum_{mm'} J_{nn}^{mm} J_{nn}^{m'm'}. \end{aligned}$$

At this stage the importance of degeneracy to the problem is clear: in the completely non-degenerate case (*i.e.* ignoring degeneracy labels) the above quantity is equal to the long-time residual current, *i.e.* $I_H = I_\infty$, and *only* the Hamiltonian is required to find the long-time residual. Not only is the conserved part of the current related solely to the Hamiltonian, but in fact *all* conservation laws can be generated by it, and as described in the previous chapter, the system should not be thought of as integrable in the classical sense.

2.2.3 A General Set of Conservation Laws

Next consider some general set of mutually commuting conservation laws which, as with any group of commuting Hermitian operators, may be simultaneously diagonalised. This prompts us to use the basis $|n, l, m\rangle$, in which case

$$\hat{C}_a |n, l, m\rangle = \epsilon_{nl}^a |n, l, m\rangle , \qquad \text{with} \qquad \epsilon_{nl}^0 \equiv \epsilon_n,$$

and for every pair of basis states with the same n but different ls there is an a such that $\epsilon_{nl}^a \neq \epsilon_{nl'}^a$ *i.e.* given n, for any two values of l there is a conservation law where the states are non-degenerate. Writing the conservation laws in terms of projection operators, as

with the purely Hamiltonian case, gives

$$\hat{P}^a_{nl} \equiv \prod_{\substack{n',l' \\ \epsilon^a_{n'l'} \neq \epsilon^a_{nl}}} \frac{[\hat{C}^a - \epsilon^a_{n'l'}]}{[\epsilon^a_{nl} - \epsilon^a_{n'l'}]}, \quad \text{with} \quad \hat{C}^a \equiv \sum_{nlm} |n,l,m\rangle \, \epsilon^a_{nl} \, \langle n,l,m| \,, \tag{2.2.19}$$

and so the following operator projects onto the subspace with eigenvalues ϵ^a_{nl}

$$\hat{P}_{nl} = \prod_a \hat{P}^a_{nl} = \sum_m |n,l,m\rangle \, \langle n,l,m| \,. \tag{2.2.20}$$

These can be used as a set of conservation laws, in exactly the same way as before, in the elements of the left hand side of equation (2.2.12)

$$\begin{aligned}
\left\langle \hat{P}_{nl} \hat{\jmath}_c \right\rangle &= \frac{1}{Z} \sum_{n'} e^{-\beta \epsilon_{n'}} \sum_{ml} \langle n',l,m| \sum_{m'} |n,l',m'\rangle \, \langle n,l',m'| \\
&\qquad \times \sum_{\bar{n}\bar{l}} \sum_{\bar{m}\dot{m}} \left|\bar{n},\bar{l},\bar{m}\right\rangle J^{\bar{m}\dot{m}}_{\bar{n}\bar{l}\bar{n}\bar{l}} \left\langle \bar{n},\bar{l},\dot{m} \right. |n',l,m\rangle \\
&= \frac{1}{Z} \sum_{n'\bar{n}} e^{-\beta \epsilon_{n'}} \sum_{l\bar{l}} \sum_{mm'} \sum_{\bar{m}\dot{m}} \delta_{n'n} \delta_{ll'} \delta_{mm'} \delta_{n\bar{n}} \delta_{l'\bar{l}} \delta_{m'\bar{m}} J^{\bar{m}\dot{m}}_{\bar{n}\bar{l}\bar{n}\bar{l}} \delta_{\bar{n}n'} \delta_{\bar{l}l} \delta_{\dot{m}m} \\
&= \frac{1}{Z} \, e^{-\beta \epsilon_n} \sum_m J^{mm}_{nlnl}, \\
\left\langle \hat{P}_{nl} \hat{P}_{nl} \right\rangle &= \left\langle \hat{P}_{nl} \right\rangle = \frac{1}{Z} \sum_{n'} e^{-\beta \epsilon_{n'}} \sum_{mm'} \sum_{l} \langle n',l,m \, |n,l',m'\rangle \\
&\qquad \times \langle n,l',m'| \sum_{\bar{m}} \left|\bar{n},\bar{l},m'\right\rangle \left\langle \bar{n},\bar{l},m' \right. |n',l,m\rangle \\
&= \frac{1}{Z} \sum_{n'} e^{-\beta \epsilon_{n'}} \sum_{ml} \sum_{m'\bar{m}} \delta_{n'n} \delta_{ll'} \delta_{m\bar{m}} \delta_{n\bar{n}} \delta_{l'\bar{l}} \delta_{\bar{m}m'} \delta_{\bar{n}n'} \delta_{\bar{l}l} \delta_{m'm} \\
&= \frac{1}{Z} \, e^{-\beta \epsilon_n} \sum_m 1 \\
&= \frac{1}{Z} \, e^{-\beta \epsilon_n} M_{nl}.
\end{aligned}$$

Once again, substituting these into equation (2.2.12) leads to the following equation for the contribution to the long-time residual due to a general set of commuting conservation

laws

$$\begin{aligned} I_M &= \sum_{nl} \frac{\left\langle \hat{P}_{nl}\hat{j}_c \right\rangle \left\langle \hat{j}_c \hat{P}_{nl} \right\rangle}{\left\langle \hat{P}_{nl} \right\rangle} \\ &= \sum_{nl} \frac{1}{\frac{1}{Z}e^{-\beta\epsilon_n} M_{nl}} \left(\frac{1}{Z} e^{-\beta\epsilon_n} \sum_{m'} J^{m'm'}_{nlnl} \right)^2 \\ &= \frac{1}{Z} \sum_{nl} e^{-\beta\epsilon_n} \frac{1}{M_{nl}} \sum_{mm'} J^{mm}_{nlnl} J^{m'm'}_{nlnl}. \end{aligned}$$

Comparing these two Mazur contributions (I_M and I_H) with the expressions for the initial and residual current (grouped together in equation (2.2.21) for reference), we see that they involve only the diagonal elements of the current operator. The contribution from many conservation laws, I_M, involves the average over the degeneracy label m, while the Hamiltonian based Mazur estimate, I_H, is a sum of the averages of all the elements in each energy subspace. These two quantities have lost the detailed information provided by the degeneracy, and so can only be equal to the residual current I_∞ if the current operator is proportional to the identity in each energy subspace(n), or a basis can be found in which it has this proportionality..

2.2.4 Discussion

Thus far the following four quantities have been calculated, corresponding respectively to the initial current, the residual current (after infinite time), and the Mazur (general set of conservation law) and Hamiltonian contributions to this residual

$$I_0 = \frac{1}{Z} \sum_{nm} e^{-\beta\epsilon_n} \sum_{n'm'} \sum_{ll'} J^{mm'}_{nln'l'} J^{m'm}_{n'l'nl}, \tag{2.2.21a}$$

$$I_\infty = \frac{1}{Z} \sum_{n} e^{-\beta\epsilon_n} \sum_{mm'} \sum_{ll'} J^{mm'}_{nlnl'} J^{m'm}_{nl'nl}, \tag{2.2.21b}$$

$$I_M = \frac{1}{Z} \sum_{nl} e^{-\beta\epsilon_n} \frac{1}{M_{nl}} \sum_{mm'} J^{mm}_{nlnl} J^{m'm'}_{nlnl}, \tag{2.2.21c}$$

$$I_H = \frac{1}{Z} \sum_{n} e^{-\beta\epsilon_n} \frac{1}{M_{n}} \sum_{mm'} \sum_{ll'} J^{mm}_{nlnl} J^{m'm'}_{nl'nl'}. \tag{2.2.21d}$$

Using the following identity for degenerate Hermitian matrices

$$\begin{aligned} & \sum_{xx'} \left[H_{xx'} - \delta_{xx'} \frac{1}{N_x} \sum_y H_{yy} \right] \left[H_{x'x} - \delta_{x'x} \frac{1}{N_x} \sum_y H_{yy} \right] \\ = & \sum_{xx'} H_{xx'} H_{x'x} - \sum_{xx'} \frac{H_{xx} H_{x'x'}}{N_x} \quad \geq \quad 0, \end{aligned} \tag{2.2.22}$$

the difference between, for example, equations (2.2.21b) and (2.2.21c) can be found

$$\begin{aligned} I_\infty - I_M &= \frac{1}{Z} \sum_n e^{-\beta \epsilon_n} \sum_{mm'} \sum_{ll'} \left(J^{mm'}_{nlnl'} J^{m'm}_{nl'nl} - \delta_{ll'} \frac{1}{M_{nl}} J^{mm}_{nlnl} J^{m'm'}_{nlnl} \right) \\ &= \frac{1}{Z} \sum_n e^{-\beta \epsilon_n} \sum_{mm'} \sum_{ll'} \left[J^{mm'}_{nlnl'} - \delta_{ll'} \delta_{mm'} \sum_{ok} \frac{J^{o\ o}_{nknk}}{M_{nl}} \right] \left[J^{m'm}_{nl'nl} - \delta_{ll'} \delta_{mm'} \sum_{ok} \frac{J^{o\ o}_{nknk}}{M_{nl}} \right] \\ &\geq 0, \end{aligned}$$

and it can therefore be shown that the above four quantities are ordered as follows

$$I_0 \quad \geq \quad I_\infty \quad \geq \quad I_M \quad \geq \quad I_H. \tag{2.2.23}$$

Clearly the initial current, I_0, will be the largest of these quantities, with the others being equal to it only if the current does not decay away at all. Comparing I_0 and I_∞ it is obvious that, for the two to be equal it is necessary for the current operator to be diagonal in energy, which is the same as saying that $\hat{j} = \hat{j}_c$, *i.e.* the current operator has only a conserved part. The real issue however, is how much of I_∞ can be found from calculating I_M: are Suzuki's ideas correct, and a complete mutually commuting set of conservation laws used in Mazur's inequality will give the full residual current, *i.e.* $I_M = I_\infty$, or is it possible to have such a set of conservation laws and still not be able to obtain the full residual ($I_M < I_\infty$)?

I_M is different to the other three quantities in that it depends upon the basis being used, which in turn depends upon the particular conservation laws in our mutually commuting set (since the basis which simultaneously diagonalises all these laws is being used), whereas I_0 depends only on the current, while I_∞ and I_H are purely Hamiltonian

dependent. The most interesting cases for this quantity are those where both the contribution from the Hamiltonian is zero, *and* some (but not all) of the current decays away, *i.e* $I_0 > I_\infty > I_H = 0$.

As mentioned above, I_M depends entirely on the set of conservation laws used to calculate it. If, for example, the set contains $\hat{j}_c$ itself, then it will clearly be possible to obtain the full residual. The other extreme, of course, is the possibility that there could be a set of mutually commuting conservation laws (including the Hamiltonian) which are all orthogonal to $\hat{j}_c$. In this instance it should be clear that it will not be possible to find *any* of the long-time residual, as our conservation laws contain none of the conserved part of the current operator. Therefore, given some set of mutually commuting conservation laws, it is not possible to say what fraction of the long-time residual can be obtained from I_M without knowing what those conservation laws are, even if the set of conservation laws is complete (that is, there are no more conservation laws we could add and still have a mutually commuting set).

This ties in with our understanding of a quantum integrable system. In the classical case there are $2N$ degrees of freedom, and each conservation law removes a degree of freedom. An integrable system therefore being one where there are N conservation laws which reduce the problem down to N independent degrees of freedom (an N-torus in phase space, with trajectories cycling around the handles). The quantum analogue to this is a set of eigenstates with degenerate eigenvalue subspaces, where the degeneracy represents the residual freedom of the system. Having fixed the eigenvalue and restricted the system to its associated subspace, additional conservation laws can be used to further reduce the subspace we've restricted ourselves to. The equivalent of a classical integrable system is then one where it is possible to simultaneously diagonalise *all* of the conservation laws to obtain a single, unique basis. Clearly it is this unique basis which would ideally be used when calculating I_M, though any basis which diagonalises the set of conservation laws provided will be sufficient.

It is important to consider the possibility that the conserved part of the current op-

erator consists solely of terms off-diagonal in the basis which diagonalises the available conservation laws. Since the Hamiltonian is diagonal in an eigenvalue subspace, and all elements have the same value (the eigenvalue for that subspace), it acts much like the identity. The conservation laws must also be given as the diagonal elements, in which case they will contribute nothing to the long-time residual. Here it is the requirement that the conservation laws are mutually commuting that has denied access to the conserved part of the current; with the *full* set of conservation laws there would be no problem calculating the long-time residual. This can be shown with a simple example of just four conservation laws

$$|1\rangle\langle 1| \qquad |2\rangle\langle 2| \qquad |1\rangle\langle 2| \qquad |2\rangle\langle 1|\,.$$

Here the first two form a 'complete' mutually commuting set, but they will miss any contribution from the other two conservation laws. Being a complete mutually commuting set is not necessarily enough to give the full contribution to the long-time residual from all conservation laws; here it is imposing the condition of mutual commutativity that causes the problem, as simply including *all* the conservation laws would give everything. If the current operator were to connect states at the same energy but with different values of the other conservation laws, then using a mutually commuting set of conservation laws could cause us to miss some of this contribution to the long-time residual.

These ideas are now investigated in the context of the XXZ model, where all of the conservation laws are known (and are mutually commuting), but have been shown to generate none of the (non-trivial) residual current, *i.e.* $I_\infty > I_M = 0$. The dangers of restricting to a mutually commuting set are seen when considering their compatibility with the current operator, but this is shown not to be the problem in this case: a single, vital conservation law had been overlooked, and its inclusion resolves the issue and leads to a Mazur estimate equal to the full long-time residual.

2.3 The XXZ Model

Having looked at this problem formally, the specific example of the XXZ model is now considered. The Hamiltonian for this model can be written as

$$\hat{H} \;=\; J\sum_i \left[\hat{S}_i^x\hat{S}_{i+1}^x + \hat{S}_i^y\hat{S}_{i+1}^y + \Delta\hat{S}_i^z\hat{S}_{i+1}^z\right], \tag{2.3.1}$$

where Δ is some measure of anisotropy. Using the following (spinless) Jordan-Wigner transformation to write the spin operators in terms of fermionic creation and annihilation operators

$$\hat{S}_l^+ \;=\; \hat{f}_l^+\exp\left(i\pi\sum_{j<l}\hat{n}_j\right); \quad \hat{S}_l^- \;=\; \exp\left(-i\pi\sum_{j<l}\hat{n}_j\right)\hat{f}_l; \quad \hat{S}_l^z \;=\; \hat{n}_l \;-\frac{1}{2}; \tag{2.3.2}$$

with $\hat{n}_l \;=\; \hat{f}_l^+\hat{f}_l$ the usual number operator; the Hamiltonian (2.3.1) becomes

$$\hat{H} \;=\; t\sum_i\left(\hat{f}_i^+\hat{f}_{i+1} + \hat{f}_{i+1}^+\hat{f}_i\right) + V\sum_i\left(\hat{f}_i^+\hat{f}_i - \frac{1}{2}\right)\left(\hat{f}_{i+1}^+\hat{f}_{i+1} - \frac{1}{2}\right), \tag{2.3.3}$$

where $t = \frac{J}{2}$ and $V = J\Delta$. This is the fermionic form of the one-dimensional Heisenberg model, known as the '$t-V$ model' [1]. The Heisenberg states of spin up or down are now the presence or absence of a fermion, of which there can of course only be one per site. In this context the meaning of the previously used phrase 'half-filling' becomes clear: the system has half as many fermions as there are sites. The equivalent Heisenberg subspace is then clearly that of $\hat{S}_{total}^z = 0$, *i.e.* an equal number of up and down spins, which includes the antiferromagnetic groundstate.

The aforementioned 'standard' current operator of the Heisenberg model is

$$\begin{aligned}\hat{\jmath} \;&=\; 2\sum_n \hat{S}_n^y\hat{S}_{n+1}^x - \hat{S}_n^x\hat{S}_{n+1}^y \\ &=\; (-i)\sum_n \hat{S}_n^+\hat{S}_{n+1}^- - \hat{S}_n^-\hat{S}_{n+1}^+,\end{aligned} \tag{2.3.4}$$

which represents the spin current due to the conservation of $\hat{S}^z$.[2] Using the Jordan-Wigner transformation, the equivalent current for the $t-V$ model is

$$\hat{j} \;=\; (-i)\sum_n \left[\hat{f}_n^+ \hat{f}_{n+1} - \hat{f}_{n+1}^+ \hat{f}_n\right], \tag{2.3.5}$$

which then clearly corresponds to the current produced by electrons hopping one site along the chain. Using the matrix form of this current operator the initial and residual current can be calculated using the expressions for I_0 and I_∞ from equations (2.2.21a) and (2.2.21b), and the target inconsistency can now be investigated.

For small, finite systems, it is possible to generate *all* of the conservation laws, using which it should be possible to calculate the *full* long-time residual. Indeed, any one giving a non-zero Mazur estimate of this residual would be enough to show its existence. For this particular model however, it can be shown that *none* of the conservation laws give a non-zero contribution to the long-time residual.

Consider a simple rotation of π about the x-axis, causing $\hat{S}^y$ and $\hat{S}^z$ to become $-\hat{S}^y$ and $-\hat{S}^z$ respectively. Under this rotation, the Hamiltonian (2.3.1) remains unchanged, while the current operator (2.3.4) has the opposite sign. This rotation is equivalent to the operator

$$\hat{U} = \exp\left(i\pi \sum_j \hat{S}_j^x\right), \tag{2.3.6}$$

and in the basis where the matrices representing $\hat{H}$ and $\hat{U}$ are both diagonal, the matrix representing $\hat{j}$ is off-diagonal, and so there can be no contribution to the residual current from the Hamiltonian. This is true not just for the Hamiltonian, but for *all* of the conservation laws generated using the Boost operator, so there can be no Mazur contribution since the expectation value of the current and any combination of these conservation laws will be zero.

[2] All conservation laws have an associated current, which can be found as the solution to putting that particular conservation law into the continuity equation.

This is all true of the XXZ model at 'half-filling' (again: the subspace where $\hat{S}^z = 0$), so an extra term is added to the Hamiltonian, relating to an external magnetic field

$$\hat{H} = \hat{H}_0 - B \sum_i \hat{S}_i^z. \tag{2.3.7}$$

Note that the extra term commutes with both the Hamiltonian and all of the conservation laws, but not with $\hat{U}$. The purpose of this natural extension to the model is that it grants control over $\hat{S}^z$; by varying the value of B we can move away from half filling to get a non-zero contribution to long-time residual from the conservation laws, and analyse whether or not half filling is a special case. This extension may seem slightly artificial, but under the previously described transformation to the t-V model it becomes clear that this addition corresponds to a term in the chemical potential, and that varying B allows one to change the doping of the system (and so move away from half filling).

Before continuing we should first address the issue of the operator $\hat{U}$ and its effect on the *compatibility* of the current operator $\hat{\jmath}$ with our conservation laws. As long as the Hamiltonian and conservation laws commute with $\hat{U}$, they will remain diagonal and have zero overlap with the off-diagonal current operator. This means that if, for example, we had chosen to study the XXX model ($\Delta = 1$) and added on a term involving the x-component of spin instead of the z-component, $\hat{U}$ *would* still commute with the conservation laws, which would therefore contribute nothing to the Mazur estimate of the long-time residual. This is an extreme version of the example at the end of the previous section. There is a 'complete' mutually commuting set of conservation laws, which misses the contribution to the long-time residual (all of it!) from other conservation laws, which do not form part of the 'complete' set. The current operator is not compatible with our 'complete' set of mutually commuting conservation laws. As noted above, $\hat{U}$ does not commute with $\hat{S}^z$ and the current operator *is* compatible with the conservation laws in the case we are investigating, though the contribution from the Hamiltonian is still zero, which can be shown using spatial inversion symmetry.

Mentioned briefly in the Introduction, the Boost operator and Transfer matrix are properly introduced in the following chapter as methods of generating the conservation laws of an integrable system. The Boost operator is of no use here since finite systems are being considered and the associated periodic boundary conditions will not allow for its use. Instead, the Transfer matrix is used, as described in the next chapter, to generate the necessary number of conservation laws.

For a system comprising a chain of length N sites, N linearly independent conservation laws are generated. Unfortunately, the required number of conservation laws scales as the number of states, and the state-space increases exponentially with system size; an N-site system will have 2^N states, a number which grows inconveniently large all too quickly. To investigate the problem at hand, the initial N conservation laws are used to calculate the first (henceforth: linear) Mazur estimate of the residual current. Since a product of conservation laws is itself a conserved quantity, the $N(N+1)/2$ possible combinations of products of two of the original set of conservation laws are then used, *combined with* the original ones, to form a new set of conserved quantities, from which a second (quadratic) Mazur estimate is obtained. A third and final (cubic) Mazur estimate of the residual is then found from a set of conservation laws comprising of all those created so far and a further $N(N+1)(N+2)/6$ conservation laws, which are the product of any three of the original N. For larger systems this process can be carried on to a quartic estimate or higher as necessary, to ensure the final set has more than the 2^N conservation laws required.

The results of these calculations for two values of Δ (corresponding to a metal and an insulator) are shown in figure 2.1, which demonstrate clearly the conundrum under consideration: for a finite system *all* of the available conservation laws have been generated and used, and yet the Mazur estimate of the residual current falls and smoothly vanishes at zero field.

Fortunately the solution to this problem is trivial: not all of the conservation laws were generated! The z-component of spin, $\hat{S}^z$, is clearly a conserved quantity, but was missed

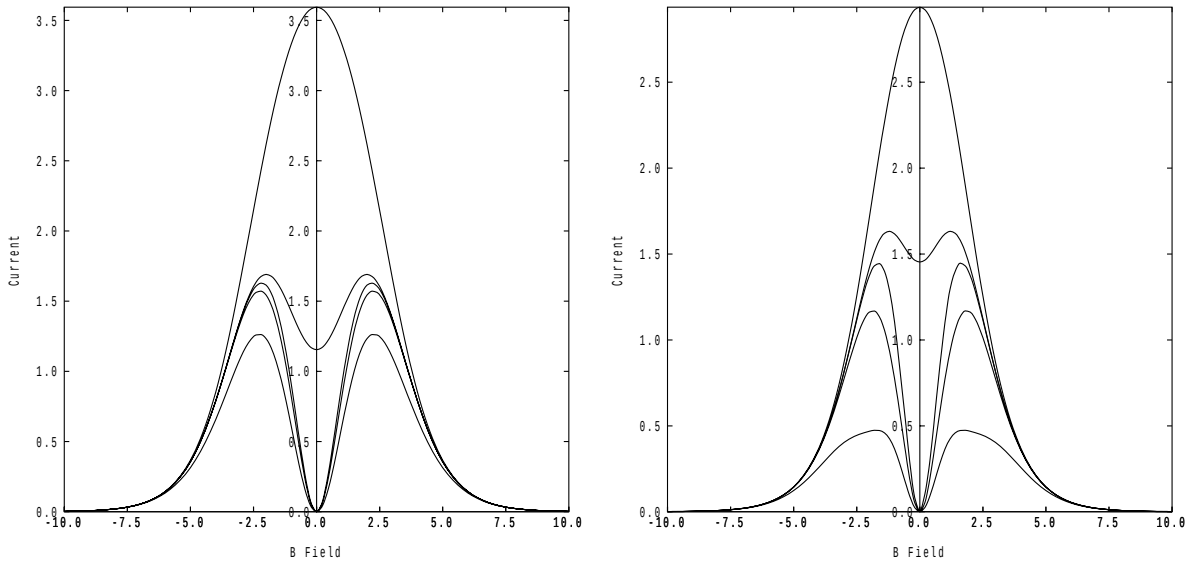


Figure 2.1: The Initial current, the long-time residual and three Mazur estimates using the conservation laws as described, for a system with $N = 6$, $\beta = 1$ and $\Delta > 1$ on the left, $\Delta < 1$ on the right.

out because the methods of generating conservation laws start with a two-site operator (the Hamiltonian) and then produce more laws of increasing complexity. Furthermore, $\hat{S}^z$ represents a very different style of conservation law to the local ones generated. As well as being non-local, it is what should be considered a *fundamental* conservation law that exists separately to the integrability of the model: if a second nearest neighbour interaction were added, the integrability and local conservation laws would be lost, but $\hat{S}^z$ will still be a conservation law.

This extra conservation law can easily be incorporated into the calculations, by using the product of $\hat{S}^z$ with each of the N original conserved quantities as a new starting set from which a linear Mazur estimate can be obtained, and then creating more conservation laws and quadratic and cubic estimates as before. Figure 2.2 shows the Mazur estimates for the original $\hat{S}^z$-less sets of conservation laws, and the new estimates obtained when $\hat{S}^z$ is included in the calculations. The problems at half-filling are gone and so in these finite systems, using all the conservation laws does indeed give a Mazur estimate equal to the long-time residual current.

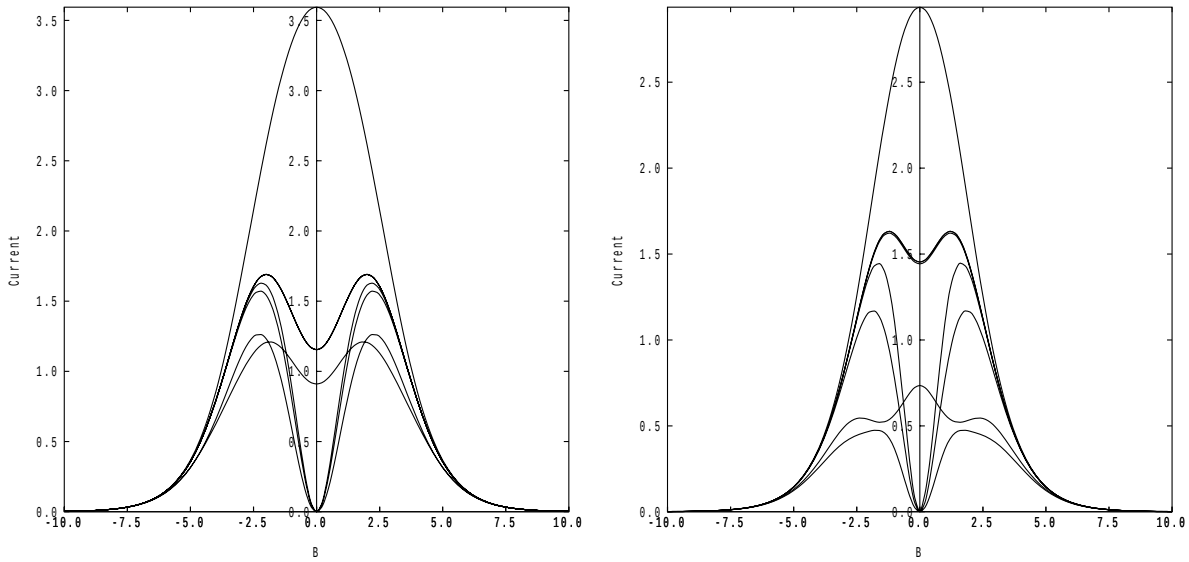


Figure 2.2: The Initial current, the long-time residual and three Mazur estimates using the conservation laws as described, both with and without $\hat{S}^z$, for a system with $N = 6$, $\beta = 1$ and $\Delta > 1$ on the left, $\Delta < 1$ on the right. The second and third $\hat{S}^z$ calculations generate the full long-time residual for all values of Field.

When moving from the finite systems studied thus far to consider the thermodynamic limit, the long-time residual is seen to behave in an odd manner [3, 36] (see figure 2.3). In the metallic regime ($\Delta < 1$) there is a non-trivial long-time residual which falls smoothly to zero as Δ approaches 1. There is a transition when we reach the Heisenberg case ($\Delta = 1$) and there is then no residual in the insulating case ($\Delta > 1$).

The non-analytic nature of the residual is a problem, as one of the local conservation laws will struggle to show this kind of behaviour. Some combination / function of conservation laws is required whose Taylor expansion exists for $0 < \Delta < 1$ but vanishes for all $\Delta > 1$, a complicated dependence on the anisotropy parameter that will require conservation laws which are non-analytic in nature.

2.4 Summary

The idea that integrable systems allow for currents which do not decay away entirely but have a finite residual which continues to flow indefinitely has been investigated. Specifi-

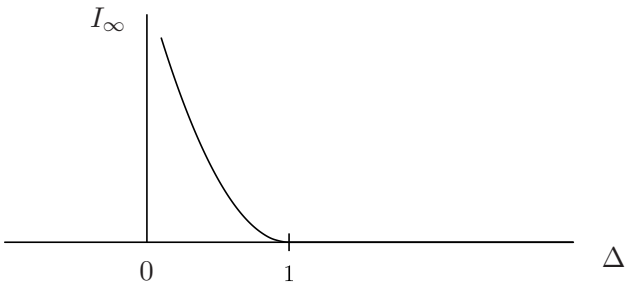


Figure 2.3: Behaviour of the residual current against the anisotropy of the XXZ model, in the thermodynamic limit.

cally, the relationship between these undiminishing currents and subsets of the conservation laws of the system are considered. It has been shown that this residual current is the conserved part of the current operator, and that it is possible to calculate it given *all* the conservation laws of a system by summing their contributions from Mazur's inequality. As a part of this result, it has also been shown that restricting the set of conservation laws to only those which mutually commute can cause some of these Mazur contributions to be missed, and in extreme cases generate none of the residual current.

With these ideas in mind, attention was turned to the XXZ model where the residual current had a known non-trivial value, but a complete set of mutually commuting conservation laws gave nothing. The problem with the mutual commutativity of the conservation laws was demonstrated by considering their compatibility with the current operator. This was seen not to be the cause of the issues with this model however, as there was a previously overlooked conservation law, the product of which with the already known mutually commuting operators were used with Mazur's inequality, and shown to generate the full residual current.

Finally, thought was briefly given to the thermodynamic limit, where the residual current displays some rather exotic behaviour as the anisotropy parameter is varied. It is apparent that the local operators used in the finite calculations will not suffice and that more complicated, non-local conservation laws are required. The combination of $\hat{S}^z$ with the local conservation laws, which resolved the problems at half-filling on the finite chain,

are non-local however. Further investigation is required to see if these combinations will have the structure required to provide a Mazur contribution in the thermodynamic limit and prove the existence of a long-time residual there.

In the following chapter, the methods of generating the analytic forms of the local conservation laws of a system are considered. The Boost operator method is analysed in detail, using the XXZ model to demonstrate how it works and show that there are hidden requirements, making it more subtle than it might first appear. A second method is then proposed which allows for a relatively simple evaluation of the logarithm of the Transfer matrix, and the iterative generation of the local conservation laws.

Chapter 3

Analytic Generation of Local Conservation Laws

In this chapter methods of deriving the exact form of the local conservation laws are investigated. The first method, involving the Boost operator introduced in Chapter 1, is used to generate the first three non-trivial local conservation laws of the XXZ model. During the course of these calculations, this Boost operator method is shown to have hidden requirements of knowledge about the structure of the Hamiltonian of the system being considered. Without this knowledge the unphysical terms produced by the Boost method cannot be shown to cancel, and so the local conservation laws cannot be found.

Having detailed the failings of the Boost operator method, we return to the idea of generating the local conservation laws by taking the logarithm of the Transfer matrix. As described in the first chapter, the result of taking this logarithm can be expressed as an infinite polynomial in the spectral parameter, the coefficients of which are the local conservation laws of the system. First removing the permutation part, and noting how this relates to the statistics of a model, the Lax operator is then written as an exponential. Following the definitions of the Monodromy and Transfer matrices, the problem becomes that of taking the logarithm of a product of exponentials, a well known mathematical problem solved in terms of a Campbell-Baker-Hausdorff expansion. This results in a

series of recurrence relations, which are increasing long-winded and arduous to solve. Once solved however, they can be applied to any system whose Lax operator can be written in the permutationless form which initiated the method. The Lax operators can then be written as the exponential of a second operator, which is substituted into the solutions to the recurrence relations and the local conservation laws obtained. This process will be described in detail in Chapters 4 and 5 for the XXZ and Hubbard models respectively, but the XXZ results are included at the end of this chapter for comparison with the Boost result.

3.1 The Boost Operator Method

In the previous chapters the use of a ladder or 'Boost' operator was described as a way of iteratively generating the local conservation laws of the XXZ model. This appears to be the only viable method for generating the analytic form of these local conservation laws, but it is an increasingly cumbersome process: after relatively few iterations the calculation by its very nature becomes impossibly lengthy, even with the aid of algebraic manipulation software [5]. This Boost method and the reason for the growing difficulty in the calculation of each term is easily demonstrated. The Hamiltonian (equation (2.3.1)) is written in the form

$$\hat{H} \;=\; \sum_n \hat{\Sigma}_{n,n+1} + (\Delta - 1)\hat{P}_{n,n+1} \;=\; \sum_n \hat{h}_{n,n+1}, \tag{3.1.1}$$

where

$$\hat{\Sigma}_{n,n+1} \;=\; \frac{1}{2} + 2\ \hat{S}_n.\hat{S}_{n+1}, \tag{3.1.2}$$

is the permutation operator, switching the spins on two neighbouring sites, and

$$\hat{P}_{n,n+1} \;=\; \frac{1}{2} + 2\ \hat{S}^z_n\hat{S}^z_{n+1}, \tag{3.1.3}$$

is the spin-projection operator, projecting onto the basis for $\hat{S}^z_{total}$. The Boost operator can be derived from the Yang-Baxter equation (equation (1.2.17)) [30] and, assuming periodic boundary conditions, the Boost operator corresponding to models such as this has been found to be [5, 30]

$$\hat{B} \;=\; \sum_m m\hat{h}_{m,m+1}. \tag{3.1.4}$$

A series of conservation laws can be constructed by repeatedly commuting each new conservation law with the above operator, *i.e.*

$$\hat{C}_{n+1} \;=\; \left[\hat{B}, \hat{C}_n\right], \tag{3.1.5}$$

and each new conservation laws range of interaction is increased by one site (with the choice of $\hat{C}_0$ below, $\hat{C}_m$ represents the interactions of up to $m+2$ neighbouring sites).

Starting with $\hat{C}_0 \equiv \hat{H}$, the first conservation law can now be calculated trivially

$$\begin{aligned}
\hat{C}_1 &= \left[\sum_m m\hat{h}_{m,m+1}, \sum_n \hat{h}_{n,n+1}\right] \\
&= \sum_n n\Big(\left[\hat{h}_{n,n+1}, \hat{h}_{n-1,n}\right] + \left[\hat{h}_{n,n+1}, \hat{h}_{n+1,n+2}\right] \Big) \\
&= -\sum_n \left[\hat{h}_{n,n+1}, \hat{h}_{n+1,n+2}\right],
\end{aligned} \tag{3.1.6}$$

where the first step relies upon the fact that operators at long range (*i.e.* which do not share a site index) commute, and the second involves shifting the summation index $n \to n+1$ for the first term, and then cancelling terms with an n-dependent coefficient.

Adopting the notation $\hat{h}_n \equiv \hat{h}_{n,n+1}$ for the sake of brevity, the second conservation law requires only a little more effort than the first

$$\begin{aligned}
\hat{C}_2 &= \left[\sum_m m\hat{h}_m, -\sum_n \left[\hat{h}_n, \hat{h}_{n+1}\right]\right] \\
&= -\sum_n n\Big(\left[\hat{h}_n, \left[\hat{h}_{n-2}, \hat{h}_{n-1}\right]\right] + \left[\hat{h}_n, \left[\hat{h}_{n-1}, \hat{h}_n\right]\right]
\end{aligned}$$

$$+\left[\hat{h}_n,\left[\hat{h}_n,\hat{h}_{n+1}\right]\right]+\left[\hat{h}_n,\left[\hat{h}_{n+1},\hat{h}_{n+2}\right]\right]\Big)$$
$$=\sum_n 2\left[\hat{h}_n,\left[\hat{h}_{n+1},\hat{h}_{n+2}\right]\right]+n\Big(\left[\hat{h}_n,\left[\hat{h}_n,\hat{h}_{n-1}\right]\right]-\left[\hat{h}_n,\left[\hat{h}_n,\hat{h}_{n+1}\right]\right]\Big). \quad (3.1.7)$$

However, it has now become clear that the Boost operator is creating terms whose coefficients depend on the site index being summed over. This is highly unphysical as it leads to terms which increase with distance along the chain and therefore fail to satisfy the (periodic) boundary conditions. Since it can be shown [5] that the Boost operator does indeed create a series of conservation laws we would expect that (as in the first case) these terms can be expanded, the summations shifted and the n-dependent parts will cancel. This is indeed possible, and in fact necessary for every conservation law, but the process of cancelling these terms and gathering up the remainder becomes extremely time-consuming after just a few iterations.

In the case of this second conservation law, the final step requires that the following reduction identities be used

$$\left[\hat{h}_n,\left[\hat{h}_n,\hat{h}_{n+1}\right]\right]=(1+\Delta^2)\hat{h}_{n+1}+\Delta(1-\Delta^2)\hat{P}_{n+1,n+2}-2\Delta\hat{\Sigma}_{n,n+2}, \quad (3.1.8a)$$
$$\left[\left[\hat{h}_n,\hat{h}_{n+1}\right],\hat{h}_{n+1}\right]=(1+\Delta^2)\hat{h}_n+\Delta(1-\Delta^2)\hat{P}_{n,n+1}-2\Delta\hat{\Sigma}_{n,n+2}. \quad (3.1.8b)$$

Shifting the summation index on the second term of equation (3.1.7) and substituting in the above identities, the terms with n-dependent coefficients cancel trivially, leaving the second local conservation law

$$\hat{C}_2=2\sum_n\left[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\right]+(1+\Delta^2)\hat{h}_n+\Delta(1-\Delta^2)\hat{P}_{n,n+1}-\Delta\hat{\Sigma}_{n,n+2}. \quad (3.1.9)$$

Note that for the first term of this new conservation law a new notation has been adopted. A set of N operators written as such implies an (N-1)-tuple nested commutator of those operators (in the written order), where all non-trivial orderings (*i.e.* orderings where the commutator vanishes) of the internal commutators lead to the same result, in

this case

$$\left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}\right] = \left[\hat{h}_n, \left[\hat{h}_{n+1}, \hat{h}_{n+2}\right]\right] = \left[\left[\hat{h}_n, \hat{h}_{n+1}\right], \hat{h}_{n+2}\right]. \tag{3.1.10}$$

As is shown at the end of the next chapter, the difference of the identities (3.1.8a) and (3.1.8b) yields the Reshetikhin condition for this model [30, 31, 38], which in turn stems from the Yang-Baxter equation. They could of course just be calculated directly given the Hamiltonian, but either way the Boost operator alone is not enough to find the conservation laws, as this additional knowledge of the Hamiltonian is also required for cancelling the n in the coefficients.

Calculating the third conservation law, for example, is already much more challenging

$$\begin{aligned}
\hat{C}_3 &= 2\left[\sum_m m\hat{h}_m, \sum_n \left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}\right]\right] + 2\left(1+\Delta^2\right)\left[\sum_m m\hat{h}_m, \sum_n \hat{h}_n\right] \\
&\quad +2\Delta(1-\Delta^2)\left[\sum_m m\hat{h}_m, \sum_n \hat{P}_{n+1,n+2}\right] - 2\Delta\left[\sum_m m\hat{h}_m, \sum_n \hat{\Sigma}_{n,n+2}\right] \\
&= 2\sum_n \Big\{ -3\left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+2}\right] + n\left[\hat{h}_n, \left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}\right]\right] \\
&\quad +(n+1)\left[\hat{h}_{n+1}, \left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}\right]\right] + (n+2)\left[\hat{h}_{n+2}, \left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}\right]\right] \\
&\quad +n\Delta(1-\Delta^2)\Big(\left[\hat{h}_n, \hat{P}_{n+1,n+2}\right] + \left[\hat{h}_n, \hat{P}_{n-1,n}\right]\Big) + \Delta\left[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+1}\right] \\
&\quad -n\Delta\Big(\left[\hat{h}_n, \hat{\Sigma}_{n+1,n+3}\right] + \left[\hat{h}_n, \hat{\Sigma}_{n,n+2}\right]\Big)\Big\} + 2(1+\Delta^2)\hat{C}_1.
\end{aligned}$$

To remove the factors of n from the coefficients of the above terms it is necessary to first reorder the triple commutators, so that the reduction identities (3.1.8a) and (3.1.8b) can be used to rewrite them as several single commutators, which then cancel with the other n-dependent terms, leaving

$$\begin{aligned}
\hat{C}_3 &= 2\sum_n \Big\{ -3\left[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+3}\right] + \Delta\left[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+1}\right] \\
&\qquad -2\Delta(1-\Delta^2)\Big(\left[\hat{h}_n, \hat{P}_{n+1,n+2}\right] + \left[\hat{P}_{n+1,n+2}, \hat{h}_{n+2}\right]\Big) \\
&\qquad +3\Delta\Big(\left[\hat{h}_n, \hat{\Sigma}_{n+1,n+3}\right] + \left[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\right]\Big)\Big\} + 8(1+\Delta^2)\hat{C}_1
\end{aligned}$$

$$= -6\sum_n \Big\{ \Big[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+3}\Big] - \Delta\Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+1}\Big] \\ -\Delta\Big(\Big[\hat{h}_n, \hat{\Sigma}_{n+1,n+3}\Big] + \Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\Big] \Big)\Big\} + (8+4\Delta^2)\hat{C}_1, \quad (3.1.11)$$

where the final step employs the relation

$$\sum_n (1-\Delta^2)\Big(\Big[\hat{h}_n, \hat{P}_{n+1,n+2}\Big] + \Big[\hat{P}_{n,n+1}, \hat{h}_{n+1}\Big] \Big) + \Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+1}\Big] = \Delta\hat{C}_1. \quad (3.1.12)$$

The calculation of $\hat{C}_4$ involves quadruple commutators, and though the identities (3.1.8a) and (3.1.8b) can be used to reduce them down to double commutators, these then involve not just the Hamiltonian but also the permutation and spin-projection operators, and so further identities and relations are required (see Appendix B), and the effort required to eradicate site dependency from the coefficients is quite considerable.

Clearly using the method of the Boost operator to generate the local conservation laws of a system entails a lot more work than one might initially suspect. Worse still, the same level of work will be needed again and again, for each new system considered. Consider, for example, the most productive (in terms of conservation laws produced) investigation into integrable spin chains [5]. Rather than repetitive commutation with the Boost operator, the XYZ Hamiltonian is studied to determine the structure of the terms that result from such a commutation. The unphysical coefficients are then circumvented as the Hamiltonian structure is used to detemine the terms that make up each successive conservation law, and recurrence relations found that will generate the correct coefficients. While this does eventually provide the conservation laws of the system[1], it proves to be of little use when a new model (the Hubbard model) is investigated.

In the following sections a more general method will be formulated which gives the local conservation laws as the coefficients of a power expansion. These coefficients are commutators of an operator which, for a given Hamiltonian, can be found with relative ease. Unlike the Boost method therefore, the majority of the work is done once in gen-

[1]And in fairness, some useful insight into the structure of the higher order conservation laws of the XYZ and related models.

erating general expressions for the coefficients / conservation laws, and then finding the specific expressions given a particular system requires only a little further work.

The development of this new method starts with a reintroduction to some of the main ideas and operators of quantum inverse scattering.

3.2 Lax, Monodromy and Transfer Matrices

The Lax operator comes from an early method of proving integrability, based on the ideas of classical integrable systems, as described in Chapter 1. Restating the formulation in terms of a quantum spin chain: if $\hat{S}_i$ is a spin-like variable defined at site i on the quantum chain, then the Lax operators $\tilde{L}_{ij}$ and $\tilde{K}_i$ define the linear system

$$\hat{S}_j \quad = \quad \tilde{L}_{ij}\hat{S}_i, \qquad\qquad \frac{d\hat{S}_i}{dt} \quad = \quad \tilde{K}_i\hat{S}_i, \tag{3.2.1}$$

i.e. $\tilde{L}_{ij}$ controls transport along the chain and $\tilde{K}_i$ governs the time evolution of the system. Differentiating the first equation with respect to time and substituting in the second generates the following compatibility condition for the above system, known as the Lax equation

$$\frac{d\tilde{L}_{ij}}{dt} \quad = \quad \tilde{K}_j\tilde{L}_{ij} - \tilde{L}_{ij}\tilde{K}_i, \tag{3.2.2}$$

and a system is considered completely integrable if this equation is equivalent to the equations of motion of that system. It has since been shown that the Lax operator can be obtained from the Yang-Baxter equation [26], and that a models R-matrix can be normalised such that

$$\hat{R}_{ij}(\lambda, \mu = 0) \quad = \quad \tilde{L}_{ij}(\lambda). \tag{3.2.3}$$

This is automatically true for fundamental models such as the Heisenberg model [19] as their R-matrices obey the difference property

$$\hat{R}_{ij}(\lambda,\mu) \;=\; \hat{R}_{ij}(\lambda-\mu), \tag{3.2.4}$$

and furthermore

$$\hat{R}_{ij}(\lambda,\mu=0) \;=\; \hat{R}_{ij}(\lambda) \;=\; \tilde{L}_{ij}(\lambda), \tag{3.2.5}$$

i.e. for these models the R-matrix and Lax operator are one and the same. In either case, the Yang-Baxter equation can be written as

$$\tilde{L}_{ki}(\mu)\tilde{L}_{kj}(\lambda)\hat{R}_{ij}(\lambda,\mu) \;=\; \hat{R}_{ij}(\lambda,\mu)\tilde{L}_{kj}(\lambda)\tilde{L}_{ki}(\mu), \tag{3.2.6}$$

which is equivalent to the Fundamental Commutation Relations (1.2.12).

If the system is extended to include a zeroth site with a dummy spin, $\hat{S}_0$, the Monodromy matrix is constructed by taking the product of the Lax matrices of this zeroth site with each of the other sites in sequence, like so

$$\begin{aligned}\hat{M}(\lambda) \;&=\; \Big(\tilde{L}_{01}(\lambda)\tilde{L}_{02}(\lambda)\cdots\tilde{L}_{0N}(\lambda)\Big) \\ &=\; \prod_n \tilde{L}_{0n}(\lambda),\end{aligned} \tag{3.2.7}$$

and the Transfer matrix, $\hat{T}$ is just the trace of the Monodromy matrix, over the zeroth site [5], *i.e.*

$$\hat{T}(\lambda) = tr_0\, \hat{M}(\lambda). \tag{3.2.8}$$

This zeroth site is a mysterious addition, and the reasons for its inclusion have never been made entirely clear. It will be shown later in this chapter however, that its function is purely to maintain the periodic boundary conditions imposed upon the system. As described in Chapter 1, this Transfer matrix is central to the generation of the conservation

laws of a system. Following on from an alteration to the above Lax operator, it will be the dominant object of the method develpoed in the next section.

It can be inferred from equation (3.2.6) that any two of these Transfer matrices with different values of the spectral parameter λ will commute [27], *i.e.*

$$\left[\hat{T}[\lambda], \hat{T}[\lambda']\right] = 0. \tag{3.2.9}$$

It is this result which tells us that operators generated by this matrix will be linearly independent, and so using a variety of values will give a mutually commuting set of conserved quantities (indeed, this is how the conservation laws were created for the previous chapter's numerical work). These conserved quantities are rather intractable however, due to their non-locality, and a more useful set of operators to consider are the *local* conservation laws of the system, which as per Chapter 1 can be found as the coefficients of the expansion

$$\ln\left(\hat{T}(\lambda)\hat{T}(0)^{-1}\right) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \tilde{C}_{n-1}, \tag{3.2.10}$$

and as with the Boost operator method $\tilde{C}_0 \equiv \hat{H}$. Taking the logarithm ensures locality [33] as $\tilde{C}_n$ does not involve interactions between spins more than $(n+1)$ sites apart. Why this logarithm should create purely local operators is not clear, but the fact it does allows for the following method of generating the above expansion and calculating the local conservation laws.

3.3 An Alternative Approach

In this section the major result of the thesis is derived: a new and potentially much simpler method of calculating the local conservation laws of a wide range of integrable systems. The permutation operator is separated out from the Lax operator, and therefore from the Monodromy and Transfer matrices, and eventually shown to vanish from the problem entirely, along with the associated issue of statistics. The Lax operator meanwhile, is

rewritten as the exponential of a new operator, $\hat{V}_{ij}$. Taking the logarithm of the resultant Transfer matrix leads to a series of recurrence relations which need to be solved order by order to generate expressions for the local conservation laws in terms of $\hat{V}_{ij}$. Finding this operator for a specific system and substituting it into these expressions will then return the local conservation laws for that system.

3.3.1 Rewriting the Lax Operator

An important feature of the Lax operator is that, when a value of zero is used for the spectral parameter, it becomes the permutation operator for whichever model is being considered, *i.e.*

$$\tilde{L}_{ij}[0] \;=\; \hat{\Sigma}_{ij}, \tag{3.3.1}$$

with

$$\hat{\Sigma}_{ij}\tilde{L}_{jk}[\lambda]\hat{\Sigma}_{ij} \;=\; \tilde{L}_{ik}[\lambda]. \tag{3.3.2}$$

Another property which will be useful later is that, if one has the permutation operator between the zeroth site and any other, the trace of this operator over the site-zero subspace is unity

$$tr_0\left(\hat{\Sigma}_{0\alpha}\right) \;=\; 1. \tag{3.3.3}$$

The first step towards evaluating the logarithm in equation (3.2.10) is to extract out the permutation component of the Lax operator to give a new 'permutationless' Lax operator

$$\hat{L}_{ij}[\lambda] = \tilde{L}_{ij}[\lambda]\hat{\Sigma}_{ij} \quad \Leftrightarrow \quad \tilde{L}_{ij}[\lambda] = \hat{L}_{ij}[\lambda]\hat{\Sigma}_{ij}, \tag{3.3.4}$$

where this new operator becomes the identity when λ is set to zero

$$\hat{L}_{ij}[0] = \hat{I}. \tag{3.3.5}$$

This separating out of the permutation aspect of the Lax operator represents a subtle issue which needs to be addressed: that of the statistics of the models being considered. The 'loss' of statistics when transforming from the (fermionic) Hubbard model (equation (5.2.1)) to the (bosonic) coupled-spin model (equation (5.2.4)), for example, has long been a concern [39, 40]. As the following steps show, rewriting the Lax operator in the above manner removes almost all of the permutation from the argument of the left hand side of equation (3.2.10), leaving only that between the ends of the lattice / chain. This final permutation term is then seen to care only about preserving the boundary conditions (and thus be irrelevant in the infinite lattice / chain). One might therefore suspect that the local conservation laws are independent of the statistics, a suspicion which is confirmed by the results of fermionic calculations matching the original bosonic ones (under the Jordan-Wigner transformation) [41].

In terms of this new Lax operator the Transfer matrix becomes

$$\hat{T}[\lambda] \;=\; tr_0\left(\hat{L}_{01}[\lambda]\hat{\Sigma}_{01}\hat{L}_{02}[\lambda]\hat{\Sigma}_{02}\cdots\hat{L}_{0N}[\lambda]\hat{\Sigma}_{0N}\right). \tag{3.3.6}$$

Taking the first (leftmost) permutation operator and permuting it with each other operator, all the way to the right, gives

$$\hat{T}[\lambda] \;=\; tr_0\left(\hat{L}_{01}[\lambda]\hat{L}_{12}[\lambda]\hat{\Sigma}_{12}\cdots\hat{L}_{1N}[\lambda]\hat{\Sigma}^{\star}_{1N}\hat{\Sigma}_{01}\right), \tag{3.3.7}$$

and then repeating the process with each successive permutation operator, permuting the leftmost one right past each other operator, but stopping after the $\hat{\Sigma}_{\alpha,N}$ permutation operator, marked with a $\star$ above (*i.e.* leaving the permutation operators in reverse order),

leads to

$$\hat{T}[\lambda] \;=\; tr_0\left(\hat{L}_{01}[\lambda]\hat{L}_{12}[\lambda]\cdots\hat{L}_{N-1,N}[\lambda]\hat{\Sigma}_{N-1,N}\cdots\hat{\Sigma}_{23}\hat{\Sigma}_{12}\hat{\Sigma}_{01}\right). \tag{3.3.8}$$

The final (rightmost) permutation operator can now be carefully permuted back through all the others, which can then be taken outside the trace

$$\hat{T}[\lambda] \;=\; tr_0\left(\hat{L}_{01}[\lambda]\hat{L}_{12}[\lambda]\cdots\hat{L}_{N-1,N}[\lambda]\hat{\Sigma}_{0N}\right)\hat{\Sigma}_{N-1,N}\cdots\hat{\Sigma}_{23}\hat{\Sigma}_{12}. \tag{3.3.9}$$

Using the properties (3.3.3) and (3.3.5) it is trivial to show that

$$\hat{T}[0] \;=\; \hat{\Sigma}_{N-1,N}\cdots\hat{\Sigma}_{23}\hat{\Sigma}_{12}, \tag{3.3.10}$$

and so the expression we are taking the logarithm of in equation (3.2.10) becomes

$$\hat{T}[\lambda]\hat{T}[0]^{-1} \;=\; tr_0\left(\hat{L}_{01}[\lambda]\hat{L}_{12}[\lambda]\cdots\hat{L}_{N-1,N}[\lambda]\hat{\Sigma}_{0N}\right). \tag{3.3.11}$$

The key choice, and the only (but for complicated models very) mathematically taxing step when applying the method, is to now write this permutationless Lax operator as

$$\hat{L}_{ij}[\lambda] = e^{\,\mu\hat{V}_{ij}[\mu]}, \tag{3.3.12}$$

where μ and the original spectral parameter λ are related by some (analytic) functions

$$\lambda = \mu f(\mu) \quad \Leftrightarrow \quad \mu = \lambda F(\lambda).$$

In the following chapter on the XXZ model for example, the two parameters are chosen to be related as

$$\mu \;=\; \frac{1}{2}\ln\left(\frac{\sin\delta + \sin\frac{\lambda\delta}{2}}{\sin\delta - \sin\frac{\lambda\delta}{2}}\right)$$

$$= \left[\frac{\delta}{2\sin\delta}\right]\lambda + \left[\frac{\delta}{2\sin\delta}\right]^3\left[\frac{1}{3} - \frac{\sin^2\delta}{3!}\right]\lambda^3 + \left[\frac{\delta}{2\sin\delta}\right]^5\left[\frac{\sin^4\delta}{5!} - \frac{\sin^2\delta}{3!} + \frac{1}{5}\right]\lambda^5 + \cdots .$$

Note that this choice is by no means arbitrary, but is made to ensure that to first order in μ, $\hat{V}_{ij}[\mu]$ is equal to the Hamiltonian of the system[2]. This choice ensures that the first conservation law generated will be the Hamiltonian.

If an expansion in one parameter is written in terms of the other, the coefficients of the new expansion will be a linear combination of the coefficients of the original one, and so equation (3.2.10) becomes

$$\ln\left(\hat{T}[\mu]\hat{T}[0]^{-1}\right) = \sum_{n=1}^{\infty}\frac{\mu^n}{n!}\hat{L}_{n-1} + g(\mu). \tag{3.3.13}$$

Discarding the irrelevant $g(\mu)$, this will give a transformed set of conservation laws, equivalent to the originals but each containing a linear superposition of the lower order conservation laws. Substituting in this exponential form of the Lax operator, the Transfer Matrix becomes

$$\hat{T}[\mu]\hat{T}[0]^{-1} = tr_0\left(e^{\mu\hat{V}_{01}[\mu]}e^{\mu\hat{V}_{12}[\mu]}\cdots e^{\mu\hat{V}_{N-1,N}[\mu]}\hat{\Sigma}_{0N}\right). \tag{3.3.14}$$

The next issues to address is that of the dummy site, which, though vital for ensuring order with finite periodic boundary conditions, can be shown to be unimportant in the limit of the infinite chain. When considering a conservation law whose order is n lower than the length of the chain, any combination of n of the Lax operators may be replaced by unity (this is the same as calculating the final conservation law of a chain with n fewer sites). If we now employ the restriction that only 'neighbouring' operators do not commute

$$\left[\hat{V}_{m,m+1}, \hat{V}_{n,n+1}\right] = 0 \quad \text{unless} \quad |m-n| = 1, \tag{3.3.15}$$

[2]And for the isotropic case (XXX or pure Heisenberg model), $\hat{V}_{ij}[\mu]$ is *exactly* the Hamiltonian.

and look for the conservation law of order one lower than that of the chain then, removing the $\hat{V}_{r,r+1}$ term from equation (3.3.14), the block of operators that were to the left of that term can be permuted to the right of the permutation operator (maintaining the order of the block). At this point all of the exponentials can be taken outside of the trace which can then be performed trivially, leaving an ordered product of exponentials

$$\begin{aligned}\hat{T}[\mu]\hat{T}[0]^{-1} &= tr_0\left(e^{\mu\hat{V}_{01}[\mu]}e^{\mu\hat{V}_{12}[\mu]}\cdots e^{\mu\hat{V}_{r-1,r}[\mu]}e^{\mu\hat{V}_{r+1,r+2}[\mu]}\cdots e^{\mu\hat{V}_{N-1,N}[\mu]}\hat{\Sigma}_{0N}\right)\\ &= e^{\mu\hat{V}_{r+1,r+2}[\mu]}\cdots e^{\mu\hat{V}_{N-1,N}[\mu]}e^{\mu\hat{V}_{N1}[\mu]}e^{\mu\hat{V}_{12}[\mu]}\cdots e^{\mu\hat{V}_{r-1,r}[\mu]}. \qquad (3.3.16)\end{aligned}$$

It is clear at this point that the only purpose of the dummy site is to enforce the periodic boundary conditions for the ordering of terms in the conservation laws, and can therefore be safely ignored in the infinite system: the N^{th} conservation law can be found by considering an N site section of a longer chain with all the other $\hat{V}$ operators removed, and then the trace is handled simply in the above manner, leaving just the exponential terms

$$\hat{T}[\mu]\hat{T}[0]^{-1} = e^{\mu\hat{V}_{12}[\mu]}e^{\mu\hat{V}_{23}[\mu]}\cdots e^{\mu\hat{V}_{N-1,N}[\mu]}. \qquad (3.3.17)$$

While the above expression is sufficient for the infinite chain, it is not properly correct for the finite periodic chain: the result *must* be periodised. This is necessary for the generation of expressions which run past the end of the chain and back to the start, *i.e.* which involve the term $e^{\mu\hat{V}_{N1}[\mu]}$, as in equation (3.3.16). The dummy site therefore (in the form of the $\hat{\Sigma}_{0N}$ term) is absolutely necessary for the periodisation of the above result for finite chains.

3.3.2 Expanding the Logarithm

The final step is to take the logarithm of equation (3.3.17) and write it as an expansion in powers of μ, and as in equation (3.3.13) the coefficient of each term will be a new local conservation law plus a linear combination of the previous ones. The above product of

exponentials can be written as

$$e^{\hat{X}_{n+1}} = \prod_{m=1}^{n+1} e^{\mu \hat{V}_m}, \tag{3.3.18}$$

again using the simplified notation $\hat{V}_m \equiv \hat{V}_{m,m+1}[\mu]$ and setting $\hat{X}_0 = 0$. The local conservation laws can then be generated order by order by using the Campbell-Baker-Hausdorff identity (see Appendix A) to iteratively solve

$$\hat{X}_{n+1} = \ln\left(e^{\hat{X}_n} e^{\mu \hat{V}_{n+1}}\right), \tag{3.3.19}$$

to find the coefficients of the expansion

$$\hat{X}_n = \sum_{p=1}^{\infty} \frac{\mu^p}{p!}\, \hat{X}_n^{(p)}[\mu] \qquad \text{where} \qquad \hat{X}^{(p)} = O(\hat{V}^p).$$

The final subtlety is that these coefficients are also functions of the parameter μ, and so each will also contribute terms to both its own and higher order conservation laws.

Employing the Campbell-Baker-Hausdorff expansion to expand the right hand side of equation (3.3.19) in terms of commutators of $\hat{X}_n$ and $\hat{V}_n$ gives

$$\begin{aligned}
\hat{X}_{n+1} \;=\; & \hat{X}_n + \mu\hat{V}_{n+1} + \frac{1}{2!}\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big] + \frac{1}{3!}\frac{1}{2}\Big(\Big[\hat{X}_n, \Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big]\Big] + \Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big) \\
& + \frac{1}{4!}\Big[\hat{X}_n, \Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big] + \frac{1}{5!}\Big\{\Big[\hat{X}_n, \Big[\hat{X}_n, \Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big]\Big] + \\
& \Big[\Big[\hat{X}_n, \Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big], \mu\hat{V}_{n+1}\Big] - \frac{1}{3}\Big(\Big[\Big[\hat{X}_n, \Big[\hat{X}_n, \Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big]\Big]\Big], \mu\hat{V}_{n+1}\Big] \\
& + \Big[\hat{X}_n, \Big[\Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big]\Big) - \frac{1}{6}\Big(\Big[\hat{X}_n, \Big[\hat{X}_n, \Big[\hat{X}_n, \Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big]\Big]\Big]\Big] \\
& + \Big[\Big[\Big[\Big[\hat{X}_n, \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big], \mu\hat{V}_{n+1}\Big]\Big)\Big\} + \cdots
\end{aligned} \tag{3.3.20}$$

Writing each instance of $\hat{X}_\alpha$ as its expansion in powers of μ and equating the terms on each side of this expression with the same order in $\hat{V}_n$ gives a series of recurrence relations

$$\hat{X}_{n+1}^{(1)} \;=\; \hat{X}_n^{(1)} + \hat{V}_{n+1},$$

$$
\begin{aligned}
\hat{X}^{(2)}_{n+1} &= \hat{X}^{(2)}_{n} + \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \\
\hat{X}^{(3)}_{n+1} &= \hat{X}^{(3)}_{n} + \frac{3}{2}\Big[\hat{X}^{(2)}_{n}, \hat{V}_{n+1}\Big] + \frac{1}{2}\Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big]\Big] + \frac{1}{2}\Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big], \\
\hat{X}^{(4)}_{n+1} &= \hat{X}^{(4)}_{n} + 2\Big[\hat{X}^{(3)}_{n}, \hat{V}_{n+1}\Big] + \Big[\hat{X}^{(2)}_{n}, \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big]\Big] + \Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(2)}_{n}, \hat{V}_{n+1}\Big]\Big] \\
&\quad + \Big[\Big[\hat{X}^{(2)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big] + \Big[\hat{X}^{(1)}_{n}, \Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big], \\
\hat{X}^{(5)}_{n+1} &= \hat{X}^{(5)}_{n} + \frac{5}{2}\Big[\hat{X}^{(4)}_{n}, \hat{V}_{n+1}\Big] + \frac{5}{3}\Big[\hat{X}^{(3)}_{n}, \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big]\Big] + \frac{5}{3}\Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(3)}_{n}, \hat{V}_{n+1}\Big]\Big] \\
&\quad + \frac{5}{2}\Big[\hat{X}^{(2)}_{n}, \Big[\hat{X}^{(2)}_{n}, \hat{V}_{n+1}\Big]\Big] + \frac{5}{3}\Big[\Big[\hat{X}^{(3)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big] + \frac{5}{2}\Big[\hat{X}^{(2)}_{n}, \Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big] \\
&\quad + \frac{5}{2}\Big[\hat{X}^{(1)}_{n}, \Big[\Big[\hat{X}^{(2)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big] + \Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big]\Big] \\
&\quad + \Big[\Big[\hat{X}^{(1)}_{n}, \Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big], \hat{V}n+1\Big] - \frac{1}{3}\Big[\Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big]\Big]\Big], \hat{V}_{n+1}\Big] \\
&\quad - \frac{1}{3}\Big[\hat{X}^{(1)}_{n}, \Big[\Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big]\Big] - \frac{1}{6}\Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big]\Big]\Big]\Big] \\
&\quad - \frac{1}{6}\Big[\Big[\Big[\Big[\hat{X}^{(1)}_{n}, \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big], \hat{V}_{n+1}\Big].
\end{aligned}
$$

These equations need to be solved order by order to give the terms which make up the local conservation laws. For example the equation for the first coefficient, involving terms $O(\hat{V})$, can be solved trivially to give

$$\hat{X}^{(1)}_{M} = \sum_{n=0}^{M} \hat{V}_{n}. \tag{3.3.21}$$

The solution to the next equation (terms $O(\hat{V}^2)$) requires the previous result, as well as the previously imposed restriction on the commutation relations of $\hat{V}_m$ (equation(3.3.15))

$$
\begin{aligned}
\hat{X}^{(2)}_{n+1} - \hat{X}^{(2)}_{n} &= \Big[\hat{X}^{(1)}_{n}, \hat{V}_{n}\Big] \\
&= \sum_{m=0}^{n-1}\Big[\hat{V}_{m}, \hat{V}_{n}\Big] \quad = \quad \Big[\hat{V}_{n-1}, \hat{V}_{n}\Big],
\end{aligned}
$$

which finds the second coefficient to be

$$\hat{X}^{(2)}_{M} = \sum_{n=0}^{M}\Big[\hat{V}_{n-1}, \hat{V}_{n}\Big]. \tag{3.3.22}$$

Both of these are now needed to solve at the next order, and so on, each equation needing the solutions for all previous coefficients and giving a series of increasingly lengthy expressions in commutators of $\hat{V}_m$

$$\hat{X}_M^{(3)} = \sum_{n=0}^{M} 2\Big[\hat{V}_{n-2},\hat{V}_{n-1},\hat{V}_n\Big] + \frac{1}{2}\Big[\hat{V}_{n-1},\Big[\hat{V}_{n-1},\hat{V}_n\Big]\Big] + \frac{1}{2}\Big[\Big[\hat{V}_{n-1},\hat{V}_n\Big],\hat{V}_n\Big] \tag{3.3.23}$$

$$\begin{aligned}\hat{X}_M^{(4)} = \sum_{n=0}^{M} & 6\Big[\hat{V}_{n-3},\hat{V}_{n-2},\hat{V}_{n-1},\hat{V}_n\Big] + 2\Big(\Big[\Big[\hat{V}_{n-2},\hat{V}_{n-1}\Big],\Big[\hat{V}_{n-1},\hat{V}_n\Big]\Big] \\ & + \Big[\hat{V}_{n-2},\hat{V}_{n-1},\hat{V}_n,\hat{V}_n\Big] + \Big[\hat{V}_{n-2},\hat{V}_{n-2},\hat{V}_{n-1},\hat{V}_n\Big]\Big) + \Big[\hat{V}_{n-1},\hat{V}_{n-1},\hat{V}_n,\hat{V}_n\Big]\end{aligned} \tag{3.3.24}$$

$$\begin{aligned}\hat{X}_M^{(5)} = \sum_{n=0}^{M} & 24\Big[\hat{V}_{n-4},\hat{V}_{n-3},\hat{V}_{n-2},\hat{V}_{n-1},\hat{V}_n\Big] + 9\Big[\hat{W}_{n-2},\hat{V}_{n-1},\hat{V}_n\Big] + 9\Big[\hat{V}_{n-3},\hat{V}_{n-2},\hat{U}_n\Big] \\ & + 6\Big[\Big[\hat{V}_{n-3},\hat{W}_{n-1}\Big],\hat{V}_n\Big] + 3\Big[\Big[\hat{U}_{n-2},\hat{V}_{n-1}\Big],\hat{V}_n\Big] + 6\Big[\hat{V}_{n-3},\Big[\hat{U}_{n-1},\hat{V}_n\Big]\Big] \\ & + 3\Big[\hat{V}_{n-3},\Big[\hat{V}_{n-2},\hat{W}_n\Big]\Big] + 4\Big[\hat{V}_{n-2},\Big[\hat{V}_{n-2},\hat{U}_n\Big]\Big] + 4\Big[\hat{V}_{n-2},\Big[\hat{U}_{n-1},\hat{V}_n\Big]\Big] \\ & + \Big[\hat{V}_{n-2},\Big[\hat{V}_{n-2},\hat{W}_n\Big]\Big] - \Big[\hat{W}_{n-1},\Big[\hat{V}_{n-1},\hat{V}_n\Big]\Big] + 4\Big[\Big[\hat{V}_{n-2},\hat{W}_n\Big],\hat{V}_n\Big] \\ & + \Big[\Big[\hat{U}_{n-1},\hat{V}_n\Big],\hat{V}_n\Big] - \Big[\Big[\hat{V}_{n-2},\hat{V}_{n-1}\Big],\hat{U}_n\Big] + \Big[\Big[\hat{V}_{n-2},\hat{V}_{n-1}\Big],\hat{W}_n\Big] \\ & + \Big[\hat{U}_{n-1},\Big[\hat{V}_{n-1},\hat{V}_n\Big]\Big] - \frac{2}{3}\Big[\Big[\hat{U}_{n-1},\hat{V}_{n-1}\Big],\hat{V}_n\Big] - \frac{2}{3}\Big[\hat{V}_{n-2},\Big[\hat{V}_{n-1},\hat{W}_n\Big]\Big] \\ & + \Big[\hat{V}_{n-1},\Big[\hat{V}_{n-1},\hat{U}_n\Big]\Big] - \frac{1}{3}\Big[\Big[\hat{V}_{n-1},\hat{W}_n\Big],\hat{V}_n\Big] + \Big[\Big[\hat{V}_{n-1},\hat{U}_n\Big],\hat{V}_n\Big] \\ & - \frac{1}{3}\Big[\hat{V}_{n-1},\Big[\hat{U}_n,\hat{V}_n\Big]\Big] + \frac{2}{3}\Big[\hat{V}_{n-2},\Big[\hat{U}_n,\hat{V}_n\Big]\Big] + \frac{2}{3}\Big[\hat{V}_{n-2},\Big[\hat{W}_{n-1},\hat{V}_n\Big]\Big] \\ & - \frac{1}{6}\Big[\hat{V}_{n-1},\Big[\hat{V}_{n-1},\hat{W}_n\Big]\Big] - \frac{1}{6}\Big[\Big[\hat{U}_n,\hat{V}_n\Big],\hat{V}_n\Big],\end{aligned} \tag{3.3.25}$$

where

$$\hat{U}_n = \Big[\Big[\hat{V}_{n-1},\hat{V}_n\Big],\hat{V}_n\Big] \quad \text{and} \quad \hat{W}_n = \Big[\hat{V}_{n-1},\Big[\hat{V}_{n-1},\hat{V}_n\Big]\Big]. \tag{3.3.26}$$

Thus far the steps taken have been entirely general and apply to any Lax integrable system[3], not just fundamental ones such as the Heisenberg-like models. Though the calculations are again becoming incredibly long winded (the sixth coefficient is still achievable in a reasonable time, but higher orders will require computational aid), this generality

[3]where $\hat{L}$ can be written in the permutationless form.

means that the coefficients need only be found once, and can then be used for any suitable model.

As was said earlier, for a given model the only non-trivial task is finding the operator $\hat{V}_n[\mu]$, the log of the Lax operator for the system in question (and which obeys the commutations in equation (3.3.15)). Assuming this can be done, then this operator can be substituted into the above coefficients of $\hat{X}_n$ and then terms which are the same order in μ can be grouped together and the coefficients of these groups are the local conservation laws and can simply be read off.

The following chapter demonstrates this method in full by finding the appropriate $\hat{V}_n[\mu]$ for the XXZ model to be

$$\hat{V}_n[\mu] \;=\; \hat{h}_n + \Delta(1-\Delta^2)\hat{P}_{n,n+1}\left(\frac{\mu^2}{3!}+\frac{\mu^4}{5!}(1-9\Delta^2)+\cdots\right)+\mathcal{O}(\mu^6), \quad (3.3.27)$$

and calculating and simplifying the first few local conservation laws. For comparison purposes the first four conservation laws generated are

$$\hat{L}_1 \;=\; \sum_n \hat{h}_n \;=\; \hat{C}_0, \quad (3.3.28a)$$

$$\hat{L}_2 \;=\; \sum_n \Big[\hat{h}_n, \hat{h}_{n+1}\Big] \;=\; -\hat{C}_1, \quad (3.3.28b)$$

$$\begin{aligned}\hat{L}_3 \;&=\; \sum_n 2\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]+\frac{1}{2}\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{h}_{n+1}\Big]\Big]+\frac{1}{2}\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]\\ &\qquad +\Delta(1-\Delta^2)\hat{P}_{n,n+1}\\ &=\; \hat{C}_2-(1+\Delta^2)\,\hat{C}_0, \end{aligned} \quad (3.3.28c)$$

$$\begin{aligned}\hat{L}_4 \;&=\; \sum_n 6\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3}\Big]+2\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\Big[\hat{h}_{n+2},\hat{h}_{n+3}\Big]\Big]\\ &\quad +2\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+2}\Big]+2\Big[\hat{h}_n,\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]\\ &\quad +\Big[\hat{h}_n,\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+1}\Big]+\frac{1}{2}\Delta(1-\Delta^2)\Big(\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]+\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big]\Big)\\ &=\; -\hat{C}_3+4(1+\Delta^2)\,\hat{C}_1. \end{aligned} \quad (3.3.28d)$$

In each case the local conservation law ($\hat{L}$) generated by the new method has also been

written as a linear combination of those generated by using the Boost operator method ($\hat{C}$), as calculated earlier in the chapter and in [5, 33, 42, 43]. Up to an irrelevant minus sign, and the predicted additional linear combination of lower order laws, the new methods generates exactly the same the same conservation laws as the Boost method.

3.4 Summary

This chapter started with an analysis of the Boost operator method of generating the local conservation laws, by attempting to use it for the XXZ model. It has been shown that, while the local conservation laws *can* be found using this method, there are subtleties. Specifically, additional knowledge of the Hamiltonian in the form of identities such as equations (3.1.8a) and (3.1.8b) is necessary to show the cancellation of unphysical terms. Without this extra information the terms with the site dependent coefficients remain and the local conservation laws cannot be found. Furthermore, this knowledge and the cancellation process is necessary for every model one wishes to find the local conservation laws of.

Using the idea that the local conservation laws of a system can also be generated from the logarithm of the Transfer matrix of that system, a new method has been developed. The starting point is to remove the permutation aspect from the Lax, Monodromy and Transfer operators. As a side effect, this ends up showing the local conservation laws produced to be independent of the statistics of a model. The logarithm of the Lax operator is then taken to give the new operator $\hat{V}_{ij}[\mu]$, and the Transfer matrix can then be written as a product of exponentials. Taking the logarithm of such a product is a well known problem, whose iterative solution leads to a series of recurrence relations, which in turn give the coefficients of the expansion of the logarithm as commutators in $\hat{V}_{ij}[\mu]$. Substituting this operator into the coefficients and collecting terms of the same order in μ then gives the local conservation laws.

This new method has two major advantages over the Boost operator. Firstly, though

the local conservation laws produced can be simplified (increasingly so at higher orders), there is no requirement to cancel any terms or otherwise fix something first. The second advantage is that essentially, all of the hard work of the method has already been done. Although, finding the $\hat{V}_{ij}[\mu]$ operator for a model is not always trivial, it should not involve any major calculations. So, for any model where this operator can be found, the local conservation laws can be generated almost effortlessly by substituting it into the $\hat{X}$ coefficients. The only difficulty now is to continue the method and find the higher order coefficients, though this will require a truly Herculean effort. It would be no more work than finding the next order conservation law of the XXZ model though, and drastically less work if one also wished to then calculate conservation laws of the same order for the Hubbard and other models.

This will now be demonstrated in the following two chapters, where the local conservation laws of the XXZ and Hubbard models are calculated.

Chapter 4

Local Conservation Laws of the XXZ Model

In this chapter, the previously outlined method is demonstrated fully via the full calculation of the first five local conservation laws of the XXZ model. The preferred version of the Lax operator is found and its logarithm calculated to give the operator $\hat{V}_{ij}$, as described in the last chapter. This is then substituted into equations (3.3.21) to (3.3.25) and the resulting terms are collected together at each order in μ and used to generate the first five local conservation laws. First, however, a well known one-dimensional antiferromagnet is considered and shown to be described by the Heisenberg model.

4.1 $KCuF_3$

Potassium Copper Fluoride has received a lot of attention in recent years [44, 45, 46, 47, 48] and is an excellent realisation of a one-dimensional system. The compound has a pseudocubic perovskite structure as shown in figure 4.1, which is distorted due to a cooperative Jahn-Teller effect [48]. The Cu ions form a simple cubic lattice with a F ion at the midpoint of each nearest neighbour pair, forming a series of corner sharing octahedra with a Cu ion at the centre of each octahedron. The large remaining spaces are occupied by K ions such that they also form a simple cubic lattice with each K at the

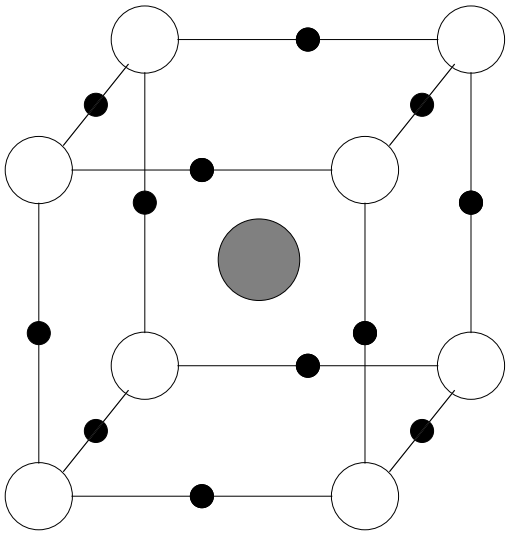


Figure 4.1: Atomic arrangement of $KCuF_3$: Potassium is shown in grey, Copper in white and Fluorine in black.

centre of a cube of F ions.

Each K donates an electron, while each F gains one, leaving both with a filled outer shell. Since there are three F ions for each K, there is a shortfall in electrons which is made up by the Cu, each of which donates two electrons to become Cu^{2+}. These Cu ions now have nine electrons in the $3d$ orbitals, which is one short of full. It is equivalent (and conventional) to think of this as there being a single hole in the $3d$ orbitals, and the orbital and magnetic ordering of the compound are determined by the dynamics of this hole. The d-shell has five orbitals; the threefold t_{2g} states are all filled, leaving the hole to reside in the twofold e_g group [45]. This final orbital degeneracy is broken by the Jahn-Teller effect, leading to alternate orbital occupation of the sites on the ab plane. Along the c-axis, planes are stacked such that they are either the same or alternate, corresponding to the experimentally observed a and d polytypes respectively [44]. The hopping of the hole between Cu ions is thus restricted in plane and occurs only along the c-axis, which is why the material can be modelled as a one-dimensional system.

The archetypal model for studying strong correlations in electron systems is the Hub-

bard model

$$\hat{H} = -t \sum_{<i,j>} \sum_{\sigma} \hat{C}^{+}_{i\sigma} \hat{C}_{j\sigma} + U \sum_{i} \hat{C}^{+}_{i\uparrow} \hat{C}_{i\uparrow} \hat{C}^{+}_{i\downarrow} \hat{C}_{i\downarrow}, \tag{4.1.1}$$

over some lattice of sites i, where $< i, j >$ denotes a nearest neighbour pair. In this particular case the operators represent the creation and annihilation of holes rather than electrons, and since $KCuF_3$ has exactly one hole for each Cu ion, it is said to be *half-filled*. The cost of having a doubly occupied site is very much greater than the hopping energy, *i.e.* $U >> t$, and so the ground state requires this penalty be avoided by only ever having one spin (hole) per site. This leads to a highly degenerate ground state however, as *all* configurations with one spin per site are equally viable. Going to second order in perturbation theory, which represents a virtual hopping onto neighbouring sites[1], lifts the degeneracy and provides an effective Hamiltonian

$$\begin{aligned}
\hat{H}_{eff} &= \sum_{<i,j>} \left(-t \sum_{\sigma} \hat{C}^{+}_{i\sigma} \hat{C}_{j\sigma} \right) \frac{-1}{U} \left(-t \sum_{\tau} \hat{C}^{+}_{j\tau} \hat{C}_{i\tau} \right) \\
&= \frac{-t^2}{U} \sum_{<i,j>} \sum_{\sigma\tau} \hat{C}^{+}_{i\sigma} \hat{C}_{j\sigma} \hat{C}^{+}_{j\tau} \hat{C}_{i\tau} \\
&= \frac{-t^2}{U} \sum_{<i,j>} \sum_{\sigma\tau} \hat{C}^{+}_{i\sigma} \hat{C}_{i\tau} \left(\delta_{\sigma\tau} - \hat{C}^{+}_{j\tau} \hat{C}_{j\sigma} \right) \\
&= \frac{-t^2}{U} \sum_{<i,j>} \left(1 - \sum_{\sigma\tau} \hat{C}^{+}_{i\sigma} \hat{C}_{i\tau} \hat{C}^{+}_{j\tau} \hat{C}_{j\sigma} \right).
\end{aligned}$$

By using the following associations to switch from creation and annihilation to spin operators

$$\begin{aligned}
\hat{C}^{+}_{l\uparrow} \hat{C}_{l\uparrow} &= \frac{1}{2} + \hat{S}^{z}_{l}, & \hat{C}^{+}_{l\uparrow} \hat{C}_{l\downarrow} &= \hat{S}^{+}_{l} = \hat{S}^{x}_{l} + i\hat{S}^{y}_{l}, \\
\hat{C}^{+}_{l\downarrow} \hat{C}_{l\downarrow} &= \frac{1}{2} - \hat{S}^{z}_{l}, & \hat{C}^{+}_{l\downarrow} \hat{C}_{l\uparrow} &= \hat{S}^{-}_{l} = \hat{S}^{x}_{l} - i\hat{S}^{y}_{l},
\end{aligned} \tag{4.1.2}$$

[1]Where a hop takes the system out of a ground state, an energy penalty is paid for the double occupancy, and a second hop returns the system to a ground state.

the effective Hamiltonian becomes

$$\begin{aligned}\hat{H}_{\mathit{eff}} &= \frac{t^2}{U}\sum_{<i,j>}\left(-1+\left[\frac{1}{2}+\hat{S}_i^z\right]\left[\frac{1}{2}+\hat{S}_j^z\right]+\left[\frac{1}{2}-\hat{S}_i^z\right]\left[\frac{1}{2}-\hat{S}_j^z\right]+\hat{S}_i^+\hat{S}_j^-+\hat{S}_i^-\hat{S}_j^+\right)\\ &= \frac{t^2}{U}\sum_{<i,j>}\left(\hat{S}_i^+\hat{S}_j^-+\hat{S}_i^-\hat{S}_j^++2\hat{S}_i^z\hat{S}_j^z-\frac{1}{2}\right).\end{aligned}$$

The derivation thus far has been independent of lattice or dimension. Since the problem under consideration has been reduced to a one-dimensional chain of sites, the sum over nearest neighbour pairs becomes the sum over the site index i, and $j \to i+1$. Including a factor of two to avoid double counting, this gives

$$\begin{aligned}\hat{H}_{\mathit{eff}} &= \frac{2t^2}{U}\sum_i\left(\hat{S}_i^+\hat{S}_{i+1}^-+\hat{S}_i^-\hat{S}_{i+1}^++2\hat{S}_i^z\hat{S}_{i+1}^z-\frac{1}{2}\right)\\ &= \frac{4t^2}{U}\sum_i\left(\hat{S}_i^x\hat{S}_{i+1}^x+\hat{S}_i^y\hat{S}_{i+1}^y+\hat{S}_i^z\hat{S}_{i+1}^z-\frac{1}{4}\right). \qquad (4.1.3)\end{aligned}$$

Ignoring the constant term, this is equation (2.3.1) with exchange constant $J = \frac{4t^2}{U}$ and $\Delta = 1$, *i.e.* the one-dimensional Heisenberg model, theoretical solutions of which fit very well with experimental data [49, 50].

4.2 Rewriting the Lax Operator

Over the next few sections the method developed in the last chapter will be applied to the XXZ model. This process starts by determining a 'new' Lax operator of the model, which is the product of the original Lax operator and the Permutation operator.

As in the previous chapter the Hamiltonian can be rewritten as

$$\begin{aligned}\hat{H} &= 2\sum_i\left(\hat{S}_i^x\hat{S}_{i+1}^x+\hat{S}_i^y\hat{S}_{i+1}^y+\Delta\hat{S}_i^z\hat{S}_{i+1}^z-\frac{\Delta}{4}\right)\\ &= \sum_i\hat{\Sigma}_{i,i+1}+(\Delta-1)\hat{P}_{i,i+1}.\end{aligned}$$

This is a fundamental model [19] and so, as stated in the previous chapter, the Lax

operator is the same as its R-matrix, which has been shown to be [34, 35]

$$\begin{aligned}\tilde{L}_{ij}[\lambda] &= \frac{S(\lambda)}{2} + 2\left(\hat{S}_i^x\hat{S}_j^x + \hat{S}_i^y\hat{S}_j^y + C(\lambda)\hat{S}_i^z\hat{S}_j^z\right) \\ &= \hat{\Sigma}_{ij} + \Big(C(\lambda)-1\Big)\hat{P}_{ij} + \frac{1}{2}\Big(S(\lambda)-C(\lambda)\Big), \end{aligned} \tag{4.2.1}$$

where the functions $S(\lambda)$ and $C(\lambda)$ depend upon the anisotropy parameter Δ (itself parameterised by δ) as follows

$$\begin{aligned} \Delta = \cos\delta \leq 1 \qquad & S(\lambda) = \frac{\sin\frac{\delta}{2}(1+\lambda)}{\sin\frac{\delta}{2}} \qquad & C(\lambda) = \frac{\cos\frac{\delta}{2}(1+\lambda)}{\cos\frac{\delta}{2}} \\ \Delta = 1 \qquad & S(\lambda) = 1+\lambda \qquad & C(\lambda) = 1 \\ \Delta = \cosh\delta \geq 1 \qquad & S(\lambda) = \frac{\sinh\frac{\delta}{2}(1+\lambda)}{\sinh\frac{\delta}{2}} \qquad & C(\lambda) = \frac{\cosh\frac{\delta}{2}(1+\lambda)}{\cosh\frac{\delta}{2}}. \end{aligned}$$

The following calculations will assume the trigonometric representation, although this will eventually be irrelevant as the final result will only involve Δ (see equation (4.3.15)).

The first task is to take the above Lax operator and extract from it the permutation aspect *i.e.* to find the operator suggested in equation (3.3.4) which, when producted with the Permutation operator, returns the original Lax operator

$$\begin{aligned}\hat{L}_{ij}[\lambda] &= \tilde{L}_{ij}[\lambda]\hat{\Sigma}_{ij} \\ &= 1 + \Big(\frac{\cos\frac{\delta}{2}(1+\lambda)}{\cos\frac{\delta}{2}} - 1\Big)\hat{P}_{ij} + \frac{1}{2}\Big(\frac{\sin\frac{\delta}{2}(1+\lambda)}{\sin\frac{\delta}{2}} - \frac{\cos\frac{\delta}{2}(1+\lambda)}{\cos\frac{\delta}{2}}\Big)\hat{\Sigma}_{ij} \\ &= 1 + \Big(\cos\frac{\lambda\delta}{2} - \frac{\sin\frac{\delta}{2}\sin\frac{\lambda\delta}{2}}{\cos\frac{\delta}{2}} - 1\Big)\hat{P}_{ij} + \frac{1}{2}\Big(\frac{\sin\frac{\lambda\delta}{2}\cos\frac{\delta}{2}}{\sin\frac{\delta}{2}} + \frac{\sin\frac{\lambda\delta}{2}\sin\frac{\delta}{2}}{\cos\frac{\delta}{2}}\Big)\hat{\Sigma}_{ij} \\ &= 1 + \Big(\cos\frac{\lambda\delta}{2} - 1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}(\cos\delta - 1)\Big)\hat{P}_{ij} + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\hat{\Sigma}_{ij}, \end{aligned} \tag{4.2.2}$$

where the idea that the Permutation operator squared is unity has been used, along with the fact that

$$\hat{P}_{ij}\,\hat{\Sigma}_{ij} = \hat{P}_{ij}. \tag{4.2.3}$$

Writing $\hat{L}_{ij}[\lambda]$ in terms of the Hamiltonian instead of the permutation operator then gives

$$\hat{L}_{ij}[\lambda] \;=\; 1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\hat{h}_{ij} + \Big(\cos\frac{\lambda\delta}{2} - 1\Big)\hat{P}_{ij}. \tag{4.2.4}$$

4.3 Calculating $\hat{V}_{ij}$

The next step is to write this new version of the Lax operator as an exponential *i.e.* to find the operator $\hat{V}_{ij}$ such that

$$\mu\hat{V}_{ij}[\mu] = \ln(\hat{L}_{ij}). \tag{4.3.1}$$

This can be done by writing the logarithm of $\hat{L}_{ij}$ as an expansion in powers of the Hamiltonian, and employing the identities

$$\hat{h}_{ij}^2 \;=\; 1 + (\Delta^2 - 1)\hat{P}_{ij}, \tag{4.3.2}$$

which follows directly from the definitions of the Hamiltonian (3.1.1), the projection and permutation operators, and the property (4.2.3); and

$$\Big(\hat{h}_{ij} + 1\Big)\Big(\hat{h}_{ij} - 1\Big)\Big(\hat{h}_{ij} - \Delta\Big) \;=\; 0, \tag{4.3.3}$$

which can be deduced from the characteristic polynomial of $\hat{h}_{ij}$. Equation (4.3.3) implies that the expansion can be truncated at the $\hat{h}_{ij}^2$ term, as it can be rearranged to write terms cubic in the Hamiltonian as a quadratic (and thus repeated use will reduce higher order terms down to a quadratic as well). Using equation (4.3.2), the $\hat{h}_{ij}^2$ term can instead be written using $\hat{P}_{ij}$ and so the logarithm of the Lax operator can be written as

$$\ln\left(1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\hat{h}_{ij} + \Big(\cos\frac{\lambda\delta}{2} - 1\Big)\hat{P}_{ij}\right) \;=\; \alpha + \beta\hat{h}_{ij} + \gamma\hat{P}_{ij} \;=\; \mu\hat{V}_{ij}[\mu]. \tag{4.3.4}$$

The three coefficients α, β and γ are functions of λ which need to be found and then rewritten in terms of μ, once the relation between the two parameters has been decided upon. Setting $\hat{h}_{ij} = \pm 1$ in equation (4.3.2) gives $\hat{P}_{ij} = 0$ and so equation (4.3.4) becomes

$$\ln\left(1 \pm \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\right) = \alpha \pm \beta, \tag{4.3.5}$$

and the sum and difference of these two equations give α and β respectively, *i.e.*

$$2\alpha = \ln\left(1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\right) + \ln\left(1 - \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\right), \tag{4.3.6}$$

$$2\beta = \ln\left(1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\right) - \ln\left(1 - \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\right). \tag{4.3.7}$$

Similarly finding γ requires setting $\hat{h}_{ij} = \Delta$, in which case $\hat{P}_{ij} = \hat{I}$ and so equation (4.3.4) becomes

$$\ln\left(1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\Delta + \left[\cos\frac{\lambda\delta}{2} - 1\right]\right) = \alpha + \beta\Delta + \gamma. \tag{4.3.8}$$

Substituting in the above results for α and β and rearranging slightly, γ is found to be

$$\gamma = \ln\Big(\cos\frac{\lambda\delta}{2} + \Delta\frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\Big) - \frac{1}{2}\ln\Big(1 - \frac{\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta}\Big) - \frac{\Delta}{2}\ln\Big(\frac{\sin\delta + \sin\frac{\lambda\delta}{2}}{\sin\delta - \sin\frac{\lambda\delta}{2}}\Big), \tag{4.3.9}$$

and putting all three coefficients back into equation (4.3.4) gives

$$\begin{aligned}\mu\hat{V}_{ij}[\mu] = {} & \frac{1}{2}\ln\left(1 - \frac{\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta}\right) + \frac{1}{2}\ln\left(\frac{\sin\delta + \sin\frac{\lambda\delta}{2}}{\sin\delta - \sin\frac{\lambda\delta}{2}}\right)\hat{h}_{ij} \\ & + \left[\ln\Big(\cos\frac{\lambda\delta}{2} + \Delta\frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\Big) - \frac{1}{2}\ln\Big(\Big[1 + \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\Big]^{1+\Delta}\Big[1 - \frac{\sin\frac{\lambda\delta}{2}}{\sin\delta}\Big]^{1-\Delta}\Big)\right]\hat{P}_{ij}.\end{aligned} \tag{4.3.10}$$

The final step is to define μ in terms of λ, with the obvious choice being to set it to

be the coefficient of the Hamiltonian, *i.e.*

$$\mu \;=\; \frac{1}{2}\ln\left(\frac{\sin\delta+\sin\frac{\lambda\delta}{2}}{\sin\delta-\sin\frac{\lambda\delta}{2}}\right). \tag{4.3.11}$$

This choice is convenient as it will guarantee that the first conservation law generated by putting this $\hat{V}_{ij}[\mu]$ into the coefficients at the end of the previous chapter will be *exactly* the Hamiltonian. The constant and coefficient of $\hat{P}_{ij}$ now need to be rearranged so that they are functions of μ instead of λ. From equation (4.3.11) it can be shown that

$$\cosh\mu=\left[\frac{\sin^2\delta}{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}\right]^{\frac{1}{2}}, \qquad \sinh\mu=\left[\frac{\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}\right]^{\frac{1}{2}}, \tag{4.3.12}$$

and then the constant α trivially becomes

$$\alpha \;=\; \frac{1}{2}\ln\left(1-\frac{\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta}\right) \;=\; \ln\left(\frac{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta}\right)^{\frac{1}{2}} \;=\; -\ln\left(\cosh\mu\right). \tag{4.3.13}$$

Working from equation (4.3.9) and remembering that $\cos\delta=\Delta$, the coefficient of $\hat{P}_{ij}$ can be rewritten as

$$\begin{aligned}
\gamma &= \ln\left(\Big[\frac{\cos^2\frac{\lambda\delta}{2}\sin^2\delta}{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}\Big]^{\frac{1}{2}}+\Delta\Big[\frac{\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}\Big]^{\frac{1}{2}}\right)-\Delta\mu\\
&= \ln\left(\Delta\sinh\mu+\Big[\frac{(1-\sin^2\delta)\sin^2\frac{\lambda\delta}{2}}{\sin^2\delta-\sin^2\frac{\lambda\delta}{2}}+1\Big]\right)-\Delta\mu\\
&= \ln\left(\Delta\sinh\mu+\Big[\Delta^2\sinh^2\mu+1\Big]^{\frac{1}{2}}\right)-\Delta\mu\\
&= \sinh^{-1}\left(\Delta\sinh\mu\right)-\Delta\mu,
\end{aligned} \tag{4.3.14}$$

and the complicated δ dependence has now vanished, leaving just the original anisotropy parameter. Substituting these coefficients back into equation (4.3.11) gives

$$\mu\hat{V}_{ij}[\mu] \;=\; -\ln(\cosh\mu)+\mu\hat{h}_{ij}+\left(\sinh^{-1}\Big[\Delta\sinh\mu\Big]-\Delta\mu\right)\hat{P}_{ij}, \tag{4.3.15}$$

and writing the above as an expansion in powers of μ and ignoring the irrelevant constant

$$\hat{V}_{ij}[\mu] \;=\; \hat{h}_{ij} + \Delta(1-\Delta^2)\hat{P}_{ij}\left(\frac{\mu^2}{3!} + \frac{\mu^4}{5!}(1-9\Delta^2) + \mathcal{O}(\mu^6)\right). \tag{4.3.16}$$

Note that removing the anisotropy by setting $\Delta = 1$ (the pure Heisenberg model) returns the surprisingly simple result $\hat{V}_{ij} = \hat{h}_{ij}$, and the coefficients of equation (3.3.2) are already the local conservation laws.

At this stage the relative ease of this method becomes clear, since all of the necessary calculations have already been done, and all that remains is to substitute the above form of $\hat{V}_{ij}$ into the $\hat{X}^{(i)}$ from the previous chapter. The local conservation laws are then easily read off, as is shown in the next section.

4.4 Generating the Local Conservation Laws

Putting equation (4.3.16) into the coefficients (3.3.21) to (3.3.25) and looking at $\hat{X}[\mu]$, at each order in μ a new local conservation law is found. Once again adopting $\hat{h}_n \equiv \hat{h}_{n,n+1}$, the first coefficient is just the operator $\hat{V}_{ij}$ summed over all sites

$$\hat{X}^{(1)} \;=\; \sum_n \hat{h}_n + \hat{P}_{n,n+1}\left(\Delta(1-\Delta^2)\frac{\mu^2}{3!} + \Delta(1-\Delta^2)(1-9\Delta^2)\frac{\mu^4}{5!}\right) + \mathcal{O}(\mu^6), \tag{4.4.1}$$

whilst the next three coefficients, up to the order necessary to calculate the fourth local conservation law, are

$$\begin{aligned}
\hat{X}^{(2)} \;=\; & \sum_n \left[\hat{h}_{n-1}, \hat{h}_n\right] + \left[\hat{P}_{n-1,n}\left(\Delta(1-\Delta^2)\frac{\mu^2}{3!} + \Delta(1-\Delta^2)(1-9\Delta^2)\frac{\mu^4}{5!}\right), \hat{h}_n\right] \\
& + \left[\hat{h}_{n-1}, \hat{P}_{n,n+1}\left(\Delta(1-\Delta^2)\frac{\mu^2}{3!} + \Delta(1-\Delta^2)(1-9\Delta^2)\frac{\mu^4}{5!}\right)\right] \\
& + \left[\hat{P}_{n-1,n}\Delta(1-\Delta^2)\frac{\mu^2}{3!}, \hat{P}_{n,n+1}\Delta(1-\Delta^2)\frac{\mu^2}{3!}\right] + \mathcal{O}(\mu^6), \qquad (4.4.2) \\
\hat{X}^{(3)} \;=\; & \sum_n 2\left[\hat{h}_{n-2}, \hat{h}_{n-1}, \hat{h}_n\right] + \frac{1}{2}\left[\hat{h}_{n-1}, \left[\hat{h}_{n-1}, \hat{h}_n\right]\right] + \frac{1}{2}\left[\left[\hat{h}_{n-1}, \hat{h}_n\right], \hat{h}_n\right] + \Delta(1-\Delta^2)\cdot \\
& \cdot\frac{\mu^2}{3!}\left(2\left[\hat{P}_{n-2,n-1}, \hat{h}_{n-1}, \hat{h}_n\right] + 2\left[\hat{h}_{n-2}, \hat{P}_{n-1,n}, \hat{h}_n\right] + 2\left[\hat{h}_{n-2}, \hat{h}_{n-1}, \hat{P}_{n,n+1}\right]\right.
\end{aligned}$$

$$+\frac{1}{2}\Big[\hat{P}_{n-1,n},\Big[\hat{h}_{n-1},\hat{h}_n\Big]\Big]+\frac{1}{2}\Big[\hat{h}_{n-1},\Big[\hat{P}_{n-1,n},\hat{h}_n\Big]\Big]+\frac{1}{2}\Big[\hat{h}_{n-1},\Big[\hat{h}_{n-1},\hat{P}_{n,n+1}\Big]\Big]$$
$$+\frac{1}{2}\Big[\Big[\hat{P}_{n-1,n},\hat{h}_n,\Big]\hat{h}_n\Big]+\frac{1}{2}\Big[\Big[\hat{h}_{n-1},\hat{P}_{n,n+1},\Big]\hat{h}_n\Big]+\frac{1}{2}\Big[\Big[\hat{h}_{n-1},\hat{h}_n,\Big]\hat{P}_{n,n+1}\Big]\Big)$$
$$+\mathcal{O}(\mu^4), \tag{4.4.3}$$

and

$$\hat{X}^{(4)} = \sum_n 6\Big[\hat{h}_{n-3},\hat{h}_{n-2},\hat{h}_{n-1},\hat{h}_n\Big]+2\Big(\Big[\Big[\hat{h}_{n-2},\hat{h}_{n-1}\Big],\Big[\hat{h}_{n-1},\hat{h}_n\Big]\Big]+\Big[\hat{h}_{n-2},\hat{h}_{n-1},\hat{h}_n,\hat{h}_n\Big]$$
$$+\Big[\hat{h}_{n-2},\hat{h}_{n-2},\hat{h}_{n-1},\hat{h}_n\Big]\Big)+\Big[\hat{h}_{n-1},\hat{h}_{n-1},\hat{h}_n,\hat{h}_n\Big]+\mathcal{O}(\mu^2). \tag{4.4.4}$$

Using the above coefficients in equation (3.3.2), there is only one term at $\mathcal{O}(\mu)$, from the first coefficient, and so the first local conservation law is just

$$\hat{L}_1 = \sum_n \hat{h}_n, \tag{4.4.5}$$

i.e. the Hamiltonian, which is not surprising given the choice of μ. Similarly, there is only one term of $\mathcal{O}(\mu^2)$, stemming from the second coefficient, and so the second local conservation law is simply

$$\hat{L}_2 = \sum_n \Big[\hat{h}_n,\hat{h}_{n+1}\Big]. \tag{4.4.6}$$

Expanding the commutator and writing out the terms explicitly

$$\begin{aligned}
\hat{L}_2 &= i\sum_n \epsilon^{\alpha\beta\gamma}\Delta^\alpha\Delta^\beta\hat{S}^\beta_n\hat{S}^\gamma_{n+1}\hat{S}^\alpha_{n+2}\\
&= i\sum_n\Big\{\hat{S}^x_n\hat{S}^z_{n+1}\hat{S}^y_{n+2}-\hat{S}^y_n\hat{S}^z_{n+1}\hat{S}^x_{n+2}\\
&\qquad +\Delta\Big(\hat{S}^y_n\hat{S}^x_{n+1}\hat{S}^z_{n+2}+\hat{S}^z_n\hat{S}^y_{n+1}\hat{S}^x_{n+2}-\hat{S}^x_n\hat{S}^y_{n+1}\hat{S}^z_{n+2}-\hat{S}^z_n\hat{S}^x_{n+1}\hat{S}^y_{n+2}\Big)\Big\}, \qquad (4.4.7)\\
&= 2i\sum_n\Big\{\hat{S}^+_n\hat{S}^z_{n+1}\hat{S}^-_{n+2}-\hat{S}^-_n\hat{S}^z_{n+1}\hat{S}^+_{n+2}
\end{aligned}$$

$$-\Delta\Big(\hat{S}^z_n\Big[\hat{S}^+_{n+1}\hat{S}^-_{n+2}-\hat{S}^-_{n+1}\hat{S}^+_{n+2}\Big]+\Big[\hat{S}^+_n\hat{S}^-_{n+1}-\hat{S}^-_n\hat{S}^+_{n+1}\Big]\hat{S}^z_{n+2}\Big)\Big\}, \quad (4.4.8)$$

which matches all the previous attempts to calculate this first non-trivial conservation law [5, 33, 42, 43]. Clearly, with almost as many different ways of representing these conservation laws as there have been attempts to calculate them, a decision needs to be made as to which one should be used. Whilst the final of the above forms is the clearest in terms of the what the conservation laws actually do (to a state), and that used in [5] looks to be the most concise, the preference here is to leave the conservation laws in the commutator form that falls naturally from this method, since it is reasonably compact while still giving some insight into its effects.

For the third local conservation law there are contributions from the third and first coefficients. In fact, given that $\hat{V}_{ij}$ is an expansion only in even powers of μ, the coefficients will only have terms that are *either* odd *or* even in μ, and will therefore only contribute terms to the even or odd numbered conservation laws (respectively). Using the reduction identities (3.1.8a) and (3.1.8b), this third conservation law becomes

$$\begin{aligned}\hat{L}_3 &= \sum_n 2\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]+\frac{1}{2}\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{h}_{n+1}\Big]\Big]+\frac{1}{2}\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]\\ &\qquad +\Delta(1-\Delta^2)\hat{P}_{n,n+1}\\ &= \sum_n 2\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]+(1+\Delta^2)\hat{h}_n+2\Delta(1-\Delta^2)\hat{P}_{n,n+1}-2\Delta\hat{\Sigma}_{n,n+2}. \quad (4.4.9)\end{aligned}$$

Using these reduction identities again on the contribution from the fourth coefficient to the next local conservation law, plus the contribution from the second coefficient, yields

$$\begin{aligned}\hat{L}_4 &= \sum_n 6\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3}\Big]+2\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\Big[\hat{h}_{n+2},\hat{h}_{n+3}\Big]\Big]\\ &\quad +2\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+2}\Big]+2\Big[\hat{h}_n,\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]\\ &\quad +\Big[\hat{h}_n,\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+1}\Big]+\frac{1}{2}\Delta(1-\Delta^2)\Big(\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]+\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big]\Big)\\ &= \sum_n 6\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3}\Big]+6\Delta\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big]+6\Delta\Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big]\end{aligned}$$

$$+2\Delta\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+1}\Big]-4\Delta(1-\Delta^2)\Big(\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]+\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big]\Big)$$
$$-4(1+\Delta^2)\Big[\hat{h}_n,\hat{h}_{n+1}\Big], \tag{4.4.10}$$

and then using the relation (3.1.12) to simplify this gives

$$\begin{aligned}\hat{L}_4 \;=\; \sum_n\Big\{&6\left[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3}\right]+6\Delta\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big]+6\Delta\Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big]\\ &+6\Delta\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+1}\Big]\Big\}-4\hat{L}_2.\end{aligned} \tag{4.4.11}$$

Calculating the fifth local conservation law requires the lowest order contribution from equation (3.3.25)

$$\begin{aligned}\hat{X}^{(5)} \;=\; &\sum_{n=0}24\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3},\hat{h}_{n+4}\Big]+9\Big[\hat{g}_n,\hat{h}_{n+2},\hat{h}_{n+3}\Big]+9\Big[\hat{h}_n,\hat{h}_{n+1},\hat{k}_{n+2}\Big]\\
&+6\Big[\Big[\hat{h}_n,\hat{g}_{n+1}\Big],\hat{h}_{n+3}\Big]+3\Big[\Big[\hat{k}_n,\hat{h}_{n+2}\Big],\hat{h}_{n+3}\Big]+6\Big[\hat{h}_n,\Big[\hat{k}_{n+1},\hat{h}_{n+3}\Big]\Big]\\
&+3\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{g}_{n+2}\Big]\Big]+4\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{k}_{n+1}\Big]\Big]+4\Big[\hat{h}_n,\Big[\hat{k}_n,\hat{h}_{n+2}\Big]\Big]\\
&+\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{g}_{n+1}\Big]\Big]-\Big[\hat{g}_n,\Big[\hat{h}_{n+1},\hat{h}_{n+2}\Big]\Big]+4\Big[\Big[\hat{h}_n,\hat{g}_{n+1}\Big],\hat{h}_{n+2}\Big]\\
&+\Big[\Big[\hat{k}_n,\hat{h}_{n+2}\Big],\hat{h}_{n+2}\Big]-\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{k}_{n+1}\Big]+\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{g}_{n+1}\Big]\\
&+\Big[\hat{k}_n,\Big[\hat{h}_{n+1},\hat{h}_{n+2}\Big]\Big]-\frac{2}{3}\Big[\Big[\hat{k}_n,\hat{h}_{n+1}\Big],\hat{h}_{n+2}\Big]-\frac{2}{3}\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{g}_{n+1}\Big]\Big]\\
&+\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{k}_n\Big]\Big]-\frac{1}{3}\Big[\Big[\hat{h}_n,\hat{g}_n\Big],\hat{h}_{n+1}\Big]+\Big[\Big[\hat{h}_n,\hat{k}_n\Big],\hat{h}_{n+1}\Big]\\
&-\frac{1}{3}\Big[\hat{h}_n,\Big[\hat{k}_n,\hat{h}_{n+1}\Big]\Big]+\frac{2}{3}\Big[\hat{h}_n,\Big[\hat{k}_{n+1},\hat{h}_{n+2}\Big]\Big]+\frac{2}{3}\Big[\hat{h}_n,\Big[\hat{g}_n,\hat{h}_{n+2}\Big]\Big]\\
&-\frac{1}{6}\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{g}_n\Big]\Big]-\frac{1}{6}\Big[\Big[\hat{k}_n,\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]+\mathcal{O}(\mu^2),\end{aligned} \tag{4.4.12}$$

using the simplifications

$$\hat{k}_n=\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big] \quad\text{and}\quad \hat{g}_n=\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{h}_{n+1}\Big]\Big], \tag{4.4.13}$$

plus the following contributions from the third and first coefficients respectively

$$\begin{aligned}\frac{10}{3}\sum_{n=0} \; & 2\Delta(1-\Delta^2)\Big(\Big[\hat{h}_n,\hat{h}_{n+1},\hat{P}_{n+2,n+3}\Big]+\Big[\hat{h}_n,\hat{P}_{n+1,n+2},\hat{h}_{n+2}\Big]+\Big[\hat{P}_{n,n+1},\hat{h}_{n+1},\hat{h}_{n+2}\Big]\Big)\\ & +\frac{1}{2}\Delta(1-\Delta^2)\Big(\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]\Big]+\Big[\hat{h}_n,\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big]\Big]+\Big[\hat{P}_{n,n+1},\Big[\hat{h}_n,\hat{h}_{n+1}\Big]\Big]\\ & +\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{P}_{n+1,n+2}\Big]+\Big[\Big[\hat{h}_n,\hat{P}_{n+1,n+1}\Big],\hat{h}_{n+1}\Big]+\Big[\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]\Big)\\ +\sum_{n=0} \; & \Delta(1-\Delta^2)(1-9\Delta^2)\hat{P}_{n,n+1}. \end{aligned} \tag{4.4.14}$$

The double commutators $\hat{g}_n$ and $\hat{k}_n$ are just the left hand sides of the reduction identities (3.1.8a) and (3.1.8b), and replacing them with their corresponding right hand sides leaves in each case another double commutator. Combining the above three contributions then finds

$$\begin{aligned}\hat{L}_5 = \sum_n \; & 24\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2},\hat{h}_{n+3},\hat{h}_{n+4}\Big]+20\Big[\hat{h}_n,\hat{h}_{n+1},\hat{h}_{n+2}\Big]-24\Delta\Big[\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big],\hat{h}_{n+3}\Big]\\ & +\frac{25}{3}\Delta(1-\Delta^2)\Big[\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big],\hat{h}_{n+2}\Big]+15\Delta(1-\Delta^2)\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{P}_{n+2,n+3}\Big]\Big]\\ & -\frac{2}{3}\Delta\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{\Sigma}_{n,n+2}\Big]\Big]-12\Delta\Big[\Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big],\hat{h}_{n+3}\Big]-10\Delta\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big]\Big]\\ & -24\Delta\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{\Sigma}_{n+2,n+4}\Big]\Big]-\frac{28}{3}\Delta\Big[\Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big],\hat{h}_{n+2}\Big]-2\Delta\Big[\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big],\hat{h}_{n+2}\Big]\\ & +\frac{4}{3}\Delta\Big[\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+1}\Big],\hat{h}_{n+2}\Big]+\frac{4}{3}\Delta\Big[\hat{h}_n,\Big[\hat{h}_{n+1},\hat{\Sigma}_{n+1,n+3}\Big]\Big]-\frac{28}{3}\Delta\Big[\hat{h}_n,\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big]\Big]\\ & -\frac{5}{3}\Delta\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{\Sigma}_{n,n+2}\Big]\Big]-\frac{4}{3}\Delta\Big[\Big[\hat{h}_n,\hat{\Sigma}_{n,n+2}\Big],\hat{h}_{n+1}\Big]+\frac{1}{3}\Delta\Big[\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]\\ & +8\Delta(1-\Delta^2)\Big[\hat{h}_n,\Big[\hat{P}_{n+1,n+2},\hat{h}_{n+2}\Big]\Big]+\frac{23}{3}\Delta(1-\Delta^2)\Big[\hat{P}_{n,n+1},\Big[\hat{h}_{n+1},\hat{h}_{n+2}\Big]\Big]\\ & +\frac{33}{6}\Delta(1-\Delta^2)\Big[\hat{h}_n,\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]\Big]+\frac{4}{3}\Delta(1-\Delta^2)\Big[\hat{h}_n,\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big]\Big]+\frac{5}{2}(1+\Delta^2)\hat{g}_n\\ & +\Delta(1-\Delta^2)\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{P}_{n+2,n+3}\Big]+\frac{2}{3}\Delta(1-\Delta^2)\Big[\hat{P}_{n,n+1},\Big[\hat{h}_n,\hat{h}_{n+1}\Big]\Big]-\frac{3}{2}(1+\Delta^2)\hat{k}_n\\ & +\frac{2}{3}\Delta(1-\Delta^2)\Big[\Big[\hat{h}_n,\hat{h}_{n+1}\Big],\hat{P}_{n+1,n+2}\Big]+\frac{4}{3}\Delta(1-\Delta^2)\Big[\Big[\hat{h}_n,\hat{P}_{n+1,n+2}\Big]\hat{h}_{n+1}\Big]\\ & +\frac{3}{2}\Delta(1-\Delta^2)\Big[\Big[\hat{P}_{n,n+1},\hat{h}_{n+1}\Big],\hat{h}_{n+1}\Big]+\Delta(1-\Delta^2)(1-9\Delta^2)\hat{P}_{n,n+1}. \end{aligned} \tag{4.4.15}$$

Some of these new double commutators reduce further, while the rest can be rearranged

and combined[2] to eventually give

$$\begin{aligned}\hat{L}_5 &= 24\sum_n \Big\{ \Big[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+3}, \hat{h}_{n+4}\Big] - \Delta\left[\Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\Big], \hat{h}_{n+3}\right] \\ &\quad -\Delta\left[\hat{h}_{n+1}, \hat{\Sigma}_{n+1,n+3}, \hat{h}_{n+3}\right] - \Delta\left[\hat{h}_n, \left[\hat{h}_{n+1}, \hat{\Sigma}_{n+2,n+4}\right]\right] \\ &\quad +\Delta(1-\Delta^2)\Big(\Big[\Big[\hat{P}_{n,n+1}, \hat{h}_{n+1}\Big], \hat{h}_{n+2}\Big] + \Big[\hat{h}_n, \Big[\hat{h}_{n+1}, \hat{P}_{n+2,n+3}\Big]\Big] \\ &\quad +\Big[\hat{h}_n, \Big[\hat{P}_{n+1,n+2}, \hat{h}_{n+2}\Big]\Big]\Big) + \Delta^2\hat{\Sigma}_{n,n+3} - \Delta(\Delta-1)\hat{P}_{n,n+3} + \Delta^3\hat{\Sigma}_{n,n+2} \\ &\quad +\Delta(\Delta^2-1)\hat{P}_{n,n+2} - \Delta^3(\Delta+1)\hat{\Sigma}_{n,n+1} + \Delta^3(\Delta^2-1)\hat{P}_{n,n+1}\Big\} \\ &\quad +(10+18\Delta^2)\,\hat{L}_3 + (-9-18\Delta^2+24\Delta^3+3\Delta^4)\,\hat{L}_1. \end{aligned} \tag{4.4.16}$$

All of these results are in agreement with those previously calculated [5, 43], but have been found without the aid of a computer, without major difficulty and in a reasonable time. Calculation of the sixth local conservation law is also possible[3] but beyond that further progress will require algebraic manipulation software or some other form of computational aid, or some other means of keeping the ever growing (in both size and number of) terms under control.

In fairness it should be stated that the identities used to simplify the form of the local conservation laws in this section are the same as would be used to cancel the unphysical terms from the Boost operator method. The difference of course, is that here they just make the conservation laws produced more manageable, as opposed to being necessary for the method to work at all. An unexpected byproduct of this method is that, for fundamental models such as the one considered here, it can be used with the Yang-Baxter equation to generate (at the very least) the first and most well known of these identities.

[2]See Appendix B for a full list of the identities and relations used.
[3]See Appendix B.

4.5 The Reshetikhin Condition

Having an R-matrix which satisfies the Yang-Baxter equation implies severe constraints on the Hamiltonian [38]. One such constraint is the previously described condition derived by Reshetikhin. Essentially, the Yang-Baxter equation is expanded in powers of λ and μ, and a power series form of the R-matrix is then used to show that double commutators of the Hamiltonian reduce to some linear combination of two site operators. It has since been shown that the sum of these commutators over all site indices vanishing provides a sufficient test for the integrability of a model [31], and a generalised form of the condition has been found which applies to non-fundamental models too [30]. The original condition is now rederived from the Yang-Baxter equation using the ideas of the previous chapter, including the exact form of the two-site operators, for the XXZ model.

As was stated in the previous chapter, for fundamental models such as the XXZ the R-matrix and the Lax operator are one and the same, and so the Yang-Baxter equation (3.2.6) becomes

$$\tilde{L}_{ki}(u)\tilde{L}_{kj}(v)\tilde{L}_{ij}(v-u) = \tilde{L}_{ij}(v-u)\tilde{L}_{kj}(v)\tilde{L}_{ki}(u), \tag{4.5.1}$$

which, in terms of 'new' Lax operators from equation (3.3.4), is

$$\hat{L}_{ki}(u)\hat{\Sigma}_{ki}\hat{L}_{kj}(v)\hat{\Sigma}_{kj}\hat{L}_{ij}(v-u)\hat{\Sigma}_{ij} = \hat{L}_{ij}(v-u)\hat{\Sigma}_{ij}\hat{L}_{kj}(v)\hat{\Sigma}_{kj}\hat{L}_{ki}(u)\hat{\Sigma}_{ki}. \tag{4.5.2}$$

Moving each permutation operator to the right and then reordering them on one side to match and cancel with the other leaves

$$\hat{L}_{ki}(u)\hat{L}_{ij}(v)\hat{L}_{ki}(v-u) = \hat{L}_{ij}(v-u)\hat{L}_{ki}(v)\hat{L}_{ij}(u), \tag{4.5.3}$$

and writing these in the exponential form suggested in the previous chapter (equation

(3.3.12)) yields

$$\begin{aligned} & \exp\Big\{\mu_u\hat{V}_{ki}(\mu_u)\Big\}\exp\Big\{\mu_v\hat{V}_{ij}(\mu_v)\Big\}\exp\Big\{\mu_{v-u}\hat{V}_{ki}(\mu_{v-u})\Big\} \\ = \; & \exp\Big\{\mu_{v-u}\hat{V}_{ij}(\mu_{v-u})\Big\}\exp\Big\{\mu_v\hat{V}_{ki}(\mu_v)\Big\}\exp\Big\{\mu_u\hat{V}_{ij}(\mu_u)\Big\}, \end{aligned} \tag{4.5.4}$$

where there are three styles of the new spectral parameter μ, defined earlier in the chapter, but they are no longer trivially related as u, v and $v-u$ were.

Once again the Campbell-Baker-Hausdorff expansion can be used to write each side as a sum of the exponents and increasingly complicated commutators of them (Appendix A), and up to the cubic terms, this is

$$\begin{aligned} & \mu_u\hat{V}_{ki}(\mu_u)+\mu_v\hat{V}_{ij}(\mu_v)+\mu_{v-u}\hat{V}_{ki}(\mu_{v-u}) \\ & +\frac{1}{2}\Big(\mu_u\mu_v\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ij}(\mu_v)\Big]+\mu_v\mu_{v-u}\Big[\hat{V}_{ij}(\mu_v),\hat{V}_{ki}(\mu_{v-u})\Big]+\mu_u\mu_{v-u}\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ki}(\mu_{v-u})\Big]\Big) \\ & +\frac{1}{6}\mu_u\mu_v\mu_{v-u}\Big(\Big[\hat{V}_{ki}(\mu_u),\Big[\hat{V}_{ij}(\mu_v),\hat{V}_{ki}(\mu_{v-u})\Big]\Big]+\Big[\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ij}(\mu_v)\Big],\hat{V}_{ki}(\mu_{v-u})\Big]\Big) \\ & +\frac{1}{12}\Big(\mu_u^2\mu_v\Big[\hat{V}_{ki}(\mu_u),\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ij}(\mu_v)\Big]\Big]+\mu_u^2\mu_{v-u}\Big[\hat{V}_{ki}(\mu_u),\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ki}(\mu_{v-u})\Big]\Big] \\ & \qquad +\mu_v^2\mu_{v-u}\Big[\hat{V}_{ij}(\mu_u),\Big[\hat{V}_{ij}(\mu_u),\hat{V}_{ki}(\mu_{v-u})\Big]\Big]+\mu_u\mu_v^2\Big[\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ij}(\mu_v)\Big],\hat{V}_{ij}(\mu_v)\Big] \\ & +\mu_u\mu_{v-u}^2\Big[\Big[\hat{V}_{ki}(\mu_u),\hat{V}_{ki}(\mu_{v-u})\Big],\hat{V}_{ki}(\mu_{v-u})\Big]+\mu_v\mu_{v-u}^2\Big[\Big[\hat{V}_{ij}(\mu_v),\hat{V}_{ki}(\mu_{v-u})\Big],\hat{V}_{ki}(\mu_{v-u})\Big]\Big) \\ & =\mu_{v-u}\hat{V}_{ij}(\mu_{v-u})+\mu_v\hat{V}_{ki}(\mu_v)+\mu_u\hat{V}_{ij}(\mu_u) \\ & +\frac{1}{2}\Big(\mu_{v-u}\mu_v\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ki}(\mu_v)\Big]+\mu_v\mu_u\Big[\hat{V}_{ki}(\mu_v),\hat{V}_{ij}(\mu_u)\Big]+\mu_{v-u}\mu_u\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ij}(\mu_u)\Big]\Big) \\ & +\frac{1}{6}\mu_u\mu_v\mu_{v-u}\Big(\Big[\hat{V}_{ij}(\mu_{v-u}),\Big[\hat{V}_{ki}(\mu_v),\hat{V}_{ij}(\mu_U)\Big]\Big]+\Big[\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ki}(\mu_v)\Big],\hat{V}_{ij}(\mu_u)\Big]\Big) \\ & +\frac{1}{12}\Big(\mu_v\mu_{v-u}^2\Big[\hat{V}_{ij}(\mu_{v-u}),\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ki}(\mu_u)\Big]\Big]+\mu_u\mu_v^2\Big[\hat{V}_{ki}(\mu_v),\Big[\hat{V}_{ki}(\mu_v),\hat{V}_{ij}(\mu_u)\Big]\Big] \\ & \qquad +\mu_u\mu_{v-u}^2\Big[\hat{V}_{ij}(\mu_{v-u}),\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ij}(\mu_u)\Big]\Big]+\mu_v^2\mu_{v-u}\Big[\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ki}(\mu_v)\Big],\hat{V}_{ki}(\mu_v)\Big] \\ & +\mu_u^2\mu_{v-u}\Big[\Big[\hat{V}_{ij}(\mu_{v-u}),\hat{V}_{ij}(\mu_u)\Big],\hat{V}_{ij}(\mu_u)\Big]+\mu_u^2\mu_v\Big[\Big[\hat{V}_{ki}(\mu_v),\hat{V}_{ij}(\mu_u)\Big],\hat{V}_{ij}(\mu_u)\Big]\Big). \end{aligned} \tag{4.5.5}$$

Expanding out the right hand side of equation (4.3.11) to find

$$\mu_\lambda \;=\; \Big[\frac{\delta}{2\sin\delta}\Big]\lambda+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big]\lambda^3+\Big[\frac{\delta}{2\sin\delta}\Big]^5\Big[\frac{\sin^4\delta}{5!}-\frac{\sin^2\delta}{3!}+\frac{1}{5}\Big]\lambda^5+\cdots.$$

and substituting this and equation (4.3.16) back into equation (4.5.5), each side of the Yang-Baxter equation can be written as an expansion in the original spectral parameters u and v

$$
\begin{aligned}
&\left(\Big[\frac{\delta}{2\sin\delta}\Big]u+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big]u^3\right)\hat{h}_{ki}+\left(\Big[\frac{\delta}{2\sin\delta}\Big]v+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big]v^3\right)\hat{h}_{ij}\\
&+\left(\Big[\frac{\delta}{2\sin\delta}\Big](v-u)+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big](v-u)^3\right)\hat{h}_{ki}\\
&+\Delta(1-\Delta^2)\frac{1}{3!}\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big(u^3\hat{P}_{ki}+v^3\hat{P}_{ij}+(v-u)^3\hat{P}_{ki}\Big)\\
&+\frac{1}{2}\Big[\frac{\delta}{2\sin\delta}\Big]^2\Big(uv\Big[\hat{h}_{ki},\hat{h}_{ij}\Big]+v(v-u)\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]+u(v-u)\Big[\hat{h}_{ki},\hat{h}_{ki}\Big]\Big)\\
&+\frac{1}{6}uv(v-u)\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big(\Big[\hat{h}_{ki},\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]\Big]+\Big[\Big[\hat{h}_{ki},\hat{h}_{ij}\Big],\hat{h}_{ki}\Big]\Big)\\
&+\frac{1}{12}\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big(u^2v\Big[\hat{h}_{ki},\Big[\hat{h}_{ki},\hat{h}_{ij}\Big]\Big]+u^2(v-u)\Big[\hat{h}_{ki},\Big[\hat{h}_{ki},\hat{h}_{ki}\Big]\Big]\\
&\qquad\qquad+v^2(v-u)\Big[\hat{h}_{ij},\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]\Big]+uv^2\Big[\Big[\hat{h}_{ki},\hat{h}_{ij}\Big],\hat{h}_{ij}\Big]\\
&\qquad\qquad+u(v-u)^2\Big[\Big[\hat{h}_{ki},\hat{h}_{ki}\Big],\hat{h}_{ki}\Big]+v(v-u)^2\Big[\Big[\hat{h}_{ij},\hat{h}_{ki}\Big],\hat{h}_{ki}\Big]\Big)+\mathcal{O}(u^4)\\
&=\left(\Big[\frac{\delta}{2\sin\delta}\Big](v-u)+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big](v-u)^3\right)\hat{h}_{ij}\\
&+\left(\Big[\frac{\delta}{2\sin\delta}\Big]v+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big]v^3\right)\hat{h}_{ki}+\left(\Big[\frac{\delta}{2\sin\delta}\Big]u+\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big]u^3\right)\hat{h}_{ij}\\
&+\Delta(1-\Delta^2)\frac{1}{3!}\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big((v-u)^3\hat{P}_{ij}+v^3\hat{P}_{ki}+u^3\hat{P}_{ij}\Big)\\
&+\frac{1}{2}\Big[\frac{\delta}{2\sin\delta}\Big]^2\Big(v(v-u)\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]+vu\Big[\hat{h}_{ki},\hat{h}_{ij}\Big]+u(v-u)\Big[\hat{h}_{ij},\hat{h}_{ij}\Big]\Big)\\
&+\frac{1}{6}uv(v-u)\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big(\Big[\hat{h}_{ij},\Big[\hat{h}_{ki},\hat{h}_{ij}\Big]\Big]+\Big[\Big[\hat{h}_{ij},\hat{h}_{ki}\Big],\hat{h}_{ij}\Big]\Big)\\
&+\frac{1}{12}\Big[\frac{\delta}{2\sin\delta}\Big]^3\Big(v(v-u)^2\Big[\hat{h}_{ij},\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]\Big]+u(v-u)^2\Big[\hat{h}_{ij},\Big[\hat{h}_{ij},\hat{h}_{ij}\Big]\Big]\\
&\qquad\qquad+uv^2\Big[\hat{h}_{ki},\Big[\hat{h}_{ki},\hat{h}_{ij}\Big]\Big]+v^2(v-u)\Big[\Big[\hat{h}_{ij},\hat{h}_{ki}\Big],\hat{h}_{ki}\Big]\\
&\qquad\qquad+u^2(v-u)\Big[\Big[\hat{h}_{ij},\hat{h}_{ij}\Big],\hat{h}_{ij}\Big]+u^2v\Big[\Big[\hat{h}_{ki},\hat{h}_{ij}\Big],\hat{h}_{ij}\Big]\Big)+\mathcal{O}(u^4). \qquad (4.5.6)
\end{aligned}
$$

The first and second order terms cancel trivially, as do the u^3 and v^3 terms, and noting that

$$
\Big[\frac{1}{3}-\frac{\sin^2\delta}{3!}\Big] \;=\; \Big[\frac{2}{3!}-\frac{(1-\cos^2\delta)}{3!}\Big] \;=\; \frac{1}{3!}\left[1+\Delta^2\right], \qquad (4.5.7)
$$

then the remaining cubic terms of equation (4.5.6) can be rearranged to give

$$3(u^2v - uv^2)\Big(\left[1+\Delta^2\right]\left(\hat{h}_{ki}-\hat{h}_{ij}\right)+\Delta(1-\Delta^2)\big(\hat{P}_{ki}-\hat{P}_{ij}\big) \\ -\Big[\hat{h}_{ij},\Big[\hat{h}_{ij},\hat{h}_{ki}\Big]\Big]+\Big[\Big[\hat{h}_{ij},\hat{h}_{ki}\Big],\hat{h}_{ki}\Big]\Big) \;=\; 0. \qquad (4.5.8)$$

Assuming $\hat{u} \neq \hat{v}$ (the trivial solution to the Yang-Baxter equations), this leaves an expression that the $\hat{h}$ and $\hat{P}$ operators must satisfy: the Reshetikhin condition for this model. Note that with the choice $i = n+1,\ j = n+2,\ k = n$, this expression becomes the difference of the identities (3.1.8a) and (3.1.8b) from the previous chapter.

Whilst it is useful to show how the suggested new form for the Lax operator can be used to generate this well known identity, it is the necessity of such identities to the Boost method that is important. Though expansion of the Yang-Baxter equations in the above manner or knowledge of the Hamiltonian and it's structure have been used to find identities which *simplify* the form of the conservation laws found in this chapter, these same identities are *essential* for removing the unphysical terms in the results created using the Boost operator.

In the following chapter a non-fundamental model, the Hubbard model, is considered. The method of Chapter 3 is shown to generate the local conservation laws of the model in a manner identical to the treatment of the XXZ model in this chapter. The non-abelian nature of the terms which make up the Hamiltonian however, mean the calculations and the final form of the conservation laws are much more complicated. Since the Heisenberg model can be derived from the Hubbard model, one might expect that the Heisenberg conservation laws could be obtained as a limit of the Hubbard ones. This possibility is investigated in the final section.

Chapter 5

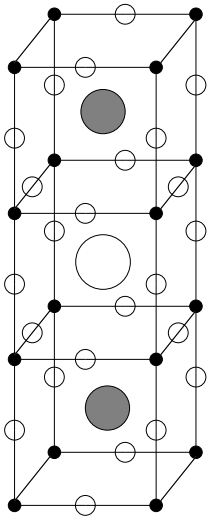
Local Conservation Laws of the Hubbard model

The work of the previous chapter is now repeated for the Hubbard model, and the first four local conservation laws are calculated. Unlike the XXZ model the Hubbard model is not fundamental, and although it is still possible to construct a Boost operator [30], it is non-trivial and the subsequent calculation of the conservation laws is much more taxing than the XXZ case. As before, an example of a material which can be described by this model is presented and then the method of Chapter 3 is followed in an identical manner.

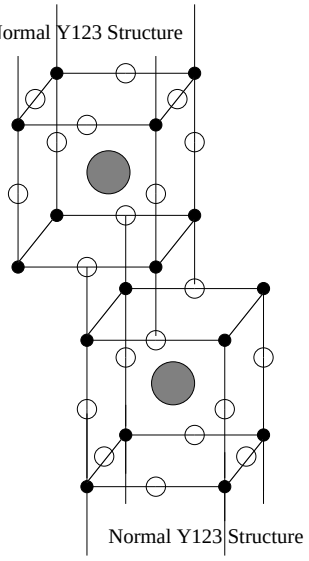
5.1 $YBa_2Cu_3O_7$ and $YBa_2Cu_4O_8$

These materials are among the most famous and well studied high-temperature superconductors. It is generally agreed that superconductivity stems from the CuO_2 planes, and the Hubbard model has been shown to reproduce observed properties [51, 52]. The CuO layers meanwhile, have been shown to be highly one-dimensional [53], and again the Hubbard model has been shown to reproduce important features of both materials [51, 54].

$YBa_2Cu_3O_7$(Y123) and $YBa_2Cu_4O_8$(Y124) have the layered structures depicted in figure 5.1 [55, 56, 57]. The two have an identical basic structure, but Y124 has a double



(a) $YBa_2Cu_3O_7$: The 1D CuO layers are at the top and bottom.



(b) $YBa_2Cu_4O_8$: The 1D CuO layers are directly above and below the offset.

Figure 5.1: Layered structure of $YBa_2Cu_3O_7$ and $YBa_2Cu_4O_8$. Large white circles denote Yttrium, grey are Barium, black are Copper and small white circles represent Oxygen.

(rather than single) layer of CuO chains. Rather than lining up with the first however, this second layer is shifted along one site in the direction of the chain, as shown in figure 5.1b. The normal Y123 structure then continues from there until the next pair of CuO layers, where there is another shift of one site along the chains putting layers back in phase with the first set, and so on, with a shift at each pair of chains so that the subsequent layers are aligned with either the first or second set.

In both materials, each Y donates five electrons and the Ba each donate two electrons (a total of nine electrons donated), while every O gains two electrons. This means that the average charge of the Cu ions in Y123 and Y124 is $\frac{5}{3}$ and $\frac{7}{4}$ respectively. There is a non-stochiometric aspect to these materials that was not an issue with $KCuF_3$, each site has on average between one and two particles on it, *i.e.* a site can now be either singly or doubly occupied (with Y124 having slightly more doubly occupied sites than Y123), and so the Hubbard model is the appropriate model for these materials.

Measurements of the properties of the CuO chains must be taken indirectly by looking

at the difference between results along the a and b axes, so as to eliminate contributions from the CuO_2 planes [58, 59], but even so, it should be possible to experimentally test the transport properties of a one-dimensional Hubbard system using these materials.

5.2 Rewriting the Lax Operator

The next few sections now follow the same path as those of the last chapter, as the method of Chapter 3 is now applied to the Hubbard model. As before the Lax operator is written in its permutationless form, then in the next section its logarithm is taken and the resultant operator $\hat{V}_{ij}$ found. This new operator is the used in the penultimate section of this chapter to create the local conservation laws of the Hubbard model by substituting it into the coefficients from the end of Chapter 3.

The one-dimensional Hubbard Hamiltonian can be written in the following form [60]

$$\hat{H} = -\sum_{i,\sigma}\left(\hat{C}^{+}_{i+1,\sigma}\hat{C}_{i,\sigma}+\hat{C}^{+}_{i,\sigma}\hat{C}_{i+1,\sigma}\right)+U\sum_{i}\left(\hat{n}_{i,\uparrow}-\frac{1}{2}\right)\left(\hat{n}_{i,\downarrow}-\frac{1}{2}\right), \quad (5.2.1)$$

in terms of the standard fermionic creation and annihilation, and number operators. Using a Jordan-Wigner transformation

$$\hat{C}_{j,\uparrow} = \exp\left(i\pi\sum_{m=1}^{j-1}\hat{S}^{+}_{m}\hat{S}^{-}_{m}\right)\hat{S}^{-}_{j}, \quad (5.2.2)$$

$$\hat{C}_{j,\downarrow} = \exp\left(i\pi\sum_{m=1}^{N}\hat{S}^{+}_{m}\hat{S}^{-}_{m}\right)\exp\left(i\pi\sum_{m=1}^{j-1}\hat{T}^{+}_{m}\hat{T}^{-}_{m}\right)\hat{T}^{-}_{j}, \quad (5.2.3)$$

to re-write the problem in terms of the usual spin raising and lowering operators gives the associated coupled-spin model [40]

$$\hat{H} = \sum_{<ij>}\hat{H}_{ij} = \sum_{<ij>}T^{H}_{ij}+g\left(\hat{U}_{i}+\hat{U}_{j}\right), \quad (5.2.4)$$

where $< ij >$ are nearest neighbours; $g = \frac{1}{4}U$ parameterises the interaction;

$$\hat{U}_i = 2\ \hat{S}_i^z\ \hat{T}_i^z \tag{5.2.5}$$

controls it, and the kinetic / hopping term is

$$T_{ij}^H = \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ + \hat{T}_i^+ \hat{T}_j^- + \hat{T}_i^- \hat{T}_j^+. \tag{5.2.6}$$

Note the use of *two* spin operators per site, one for each flavour of spin, where spin up(down) indicates the (non)existence of that style of spin on a particular site. The Lax operator for this model has been found to be [60]

$$\tilde{L}_{ij}[\lambda] = e^{\lambda \hat{U}_i}\ \tilde{S}_{ij}\ \tilde{T}_{ij}\ e^{\lambda \hat{U}_i}, \tag{5.2.7}$$

where the central two operators have the form

$$\tilde{O}_{ij} = \frac{1}{2}\left[\frac{1+\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] + 2\left[\hat{O}_i^x\hat{O}_j^x + \hat{O}_i^y\hat{O}_j^y + \left(\frac{1-\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right)\hat{O}_i^z\hat{O}_j^z\right]. \tag{5.2.8}$$

which is very similar to the Lax operator of the Heisenberg-like models of the previous chapter. The interaction strength can be written in the form [60]

$$g \quad = \quad \frac{U}{4} \quad = \quad \frac{\sinh 2\lambda}{2}\frac{\cosh^2\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}\right)}, \tag{5.2.9}$$

and so as $\lambda \to 0$, this reduces to $2\lambda = g\kappa$, which implies that $\kappa \propto \lambda$ and therefore when $\lambda = \kappa = 0$, equation (5.2.7) becomes

$$\begin{aligned}\tilde{L}_{ij}[0] &= \left(\frac{1}{2} + 2\hat{S}_i.\hat{S}_i\right)\left(\frac{1}{2} + 2\hat{T}_i.\hat{T}_i\right) \\ &= \hat{\Sigma}_{ij}^{xxz}\ \hat{\Sigma}_{ij}^{xxz} \\ &= \hat{\Sigma}_{ij}^{hub},\end{aligned} \tag{5.2.10}$$

i.e. the permutation operator for this representation of the Hubbard model.

As before, the first step is to extract this permutation operator from the Lax operator. This is a slightly more complicated process than with the XXZ model, where the permutation operator already appeared explicitly in the Lax operator, but the similarities in form suggest starting by splitting the $\tilde{O}_{ij}$ into permutation and spin projection parts

$$\begin{aligned}
\tilde{O}_{ij} &= \frac{1}{2}\left[\frac{1+\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] + 2\left[\hat{O}_i^x\hat{O}_j^x + \hat{O}_i^y\hat{O}_j^y + \left(\frac{1-\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right)\hat{O}_i^z\hat{O}_j^z\right] \\
&= \frac{1}{2}\left[\frac{1+\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] + 2\hat{O}_i.\hat{O}_j + 2\left[\frac{1-\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}} - 1\right]\hat{O}_i^z\hat{O}_j^z \\
&= \left(\frac{1}{2} + 2\hat{O}_i.\hat{O}_j\right) - \frac{1}{2} + \frac{1}{2}\left[\frac{1+\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] - 2\left[\frac{\cosh\frac{\kappa}{2}+\sinh\frac{\kappa}{2}-1}{\cosh\frac{\kappa}{2}}\right]\hat{O}_i^z\hat{O}_j^z \\
&= \hat{\Sigma}_{ij(o)}^{xxz} - \left[\frac{\cosh\frac{\kappa}{2}+\sinh\frac{\kappa}{2}-1}{\cosh\frac{\kappa}{2}}\right]\left(\frac{1}{2} + 2\hat{O}_i^z\hat{O}_j^z\right) - \frac{1}{2} \\
&\quad + \frac{1}{2}\left[\frac{1+\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] + \frac{1}{2}\left[\frac{\cosh\frac{\kappa}{2}+\sinh\frac{\kappa}{2}-1}{\cosh\frac{\kappa}{2}}\right] \\
&= \hat{\Sigma}_{ij(o)}^{xxz} - \left[\frac{\cosh\frac{\kappa}{2}+\sinh\frac{\kappa}{2}-1}{\cosh\frac{\kappa}{2}}\right]\hat{P}_{ij(o)}^{xxz} + \frac{\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}} \\
&= \left(1 - \left[\frac{\cosh\frac{\kappa}{2}+\sinh\frac{\kappa}{2}-1}{\cosh\frac{\kappa}{2}}\right]\hat{P}_{ij(o)}^{xxz} + \frac{\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\hat{\Sigma}_{ij(o)}^{xxz}\right)\hat{\Sigma}_{ij(o)}^{xxz}.
\end{aligned} \tag{5.2.11}$$

Along with the fact that

$$\hat{\Sigma}_{ij}^{hub}\ \hat{U}_i = \hat{U}_j\ \hat{\Sigma}_{ij}^{hub}, \tag{5.2.12}$$

this allows for a simple rewriting of the Lax operator

$$\begin{aligned}
\tilde{L}_{ij}[\lambda] &= e^{\lambda\hat{U}_i}\ \tilde{S}_{ij}\ \tilde{T}_{ij}\ e^{\lambda\hat{U}_i} \\
&= e^{\lambda\hat{U}_i}\ \hat{S}_{ij}\ \hat{\Sigma}_{ij\uparrow}^{xxz}\ \hat{T}_{ij}\ \hat{\Sigma}_{ij\downarrow}^{xxz}\ e^{\lambda\hat{U}_i} \\
&= e^{\lambda\hat{U}_i}\ \hat{S}_{ij}\ \hat{T}_{ij}\ \hat{\Sigma}_{ij}^{hub}\ e^{\lambda\hat{U}_i} \\
&= e^{\lambda\hat{U}_i}\ \hat{S}_{ij}\ \hat{T}_{ij}\ e^{\lambda\hat{U}_j}\ \hat{\Sigma}_{ij}^{hub},
\end{aligned} \tag{5.2.13}$$

and so the desired operator is just

$$\hat{L}_{ij}[\lambda] \;=\; e^{\lambda \hat{U}_i}\; \hat{S}_{ij}\; \hat{T}_{ij}\; e^{\lambda \hat{U}_j}, \tag{5.2.14}$$

in terms of the new operators $\hat{O}_{ij}$ which, from equation (5.2.11), are

$$\hat{O}_{ij} \;=\; 1 - \left[\frac{\cosh\frac{\kappa}{2} + \sinh\frac{\kappa}{2} - 1}{\cosh\frac{\kappa}{2}}\right]\; \hat{P}^{xxz}_{ij(o)} + \frac{\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\; \hat{\Sigma}^{xxz}_{ij(o)}. \tag{5.2.15}$$

5.3 Calculating $\hat{V}_{ij}$

Finding the exponential form of the Lax operator requires first obtaining the exponential form of the operators $\hat{O}_{ij}$. This process closely mirrors that of taking the logarithm of the Lax operator for the XXZ model, which is not surprising given their very similar forms, with the equivalent of equation (4.3.4) being

$$\ln\left(1 - \left[\frac{\cosh\frac{\kappa}{2} + \sinh\frac{\kappa}{2} - 1}{\cosh\frac{\kappa}{2}}\right]\; \hat{P}^{xxz}_{ij(o)} + \frac{\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\; \hat{\Sigma}^{xxz}_{ij(o)}\right) \;=\; \alpha + \beta \hat{P}^{xxz}_{ij(o)}. + \gamma \hat{\Sigma}^{xxz}_{ij(o)}. \tag{5.3.1}$$

The coefficients on the left hand side of the above equation can then again be found from the three cases $\hat{P}^{xxz}_{ij(o)} = 0$, $\hat{\Sigma}^{xxz}_{ij(o)} = \pm\, \hat{I}$ and $\hat{P}^{xxz}_{ij(o)} = \hat{I}$, $\hat{\Sigma}^{xxz}_{ij(o)} = \hat{I}$. The first of these reduces equation (5.3.1) down to

$$\begin{aligned} \alpha \pm \gamma \;&=\; \ln\left[1 \pm \frac{\sinh\frac{\kappa}{2}}{\cosh\frac{\kappa}{2}}\right] \;=\; e^{\pm\frac{\kappa}{2}} \frac{1}{\cosh\frac{\kappa}{2}} \\ \Rightarrow\; e^{\alpha} \;&=\; \frac{1}{\cosh\frac{\kappa}{2}} \quad \text{and} \quad \gamma \;=\; \frac{\kappa}{2}, \end{aligned} \tag{5.3.2}$$

and using these and the second case, equation (5.3.1) returns

$$\begin{aligned} \ln\frac{1}{\cosh\frac{\kappa}{2}} \;&=\; \alpha + \beta + \gamma \;=\; \ln\left(\frac{1}{\cosh\frac{\kappa}{2}}\right) + \beta + \frac{\kappa}{2} \\ \Rightarrow\; \beta \;&=\; \frac{\kappa}{2}, \end{aligned} \tag{5.3.3}$$

and then putting these three coefficients into the left hand side of equation (5.3.1) gives

$$\hat{O}_{ij} = \frac{1}{\cosh\frac{\kappa}{2}} e^{\frac{\kappa}{2}\left(\hat{\Sigma}^{xxz}_{ij(o)} - \hat{P}^{xxz}_{ij(o)}\right)} = \frac{1}{\cosh\frac{\kappa}{2}} e^{\frac{\kappa}{2}\left(\hat{o}^+_i\hat{o}^-_j + \hat{o}^-_i\hat{o}^+_j\right)}. \tag{5.3.4}$$

Putting the exponential forms of $\hat{S}_{ij}$ and $\hat{T}_{ij}$ into equation (5.2.14) and combining them so that the exponent becomes the hopping operator T^H_{ij}

$$\begin{aligned}\hat{L}_{ij}[\lambda] &= e^{\lambda\hat{U}_i} \frac{1}{\cosh\frac{\kappa}{2}} e^{\frac{\kappa}{2}\left(\hat{S}^+_i\hat{S}^-_j + \hat{S}^-_i\hat{S}^+_j\right)} \frac{1}{\cosh\frac{\kappa}{2}} e^{\frac{\kappa}{2}\left(\hat{T}^+_i\hat{T}^-_j + \hat{T}^-_i\hat{T}^+_j\right)} e^{\lambda\hat{U}_j} \\ &= \frac{1}{\cosh^2\frac{\kappa}{2}} e^{\lambda\hat{U}_i} e^{\frac{\kappa}{2}\hat{T}^H_{ij}} e^{\lambda\hat{U}_j},\end{aligned} \tag{5.3.5}$$

which now needs to be found as the exponent of a single operator.

Ignoring the coefficient, which merely contributes a constant, the operator $\hat{V}_{ij}$ can be found as a Campbell-Baker-Hausdorff expansion, specifically, by using the identity (A.0.10) it becomes

$$\begin{aligned}\mu\hat{V}_{ij}[\mu] &= \ln e^{\lambda\hat{U}_i} e^{\frac{\kappa}{2}\hat{T}^H_{ij}} e^{\lambda\hat{U}_j} \\ &= \lambda\hat{U}_i + \frac{\kappa}{2}\hat{T}^H_{ij} + \lambda\hat{U}_j + \frac{1}{2}\left(\left[\lambda\hat{U}_i, \frac{\kappa}{2}\hat{T}^H_{ij}\right] + \left[\lambda\hat{U}_i, \lambda\hat{U}_j\right] + \left[\frac{\kappa}{2}\hat{T}^H_{ij}, \lambda\hat{U}_j\right]\right) \\ &\quad + \frac{1}{6}\left(\left[\lambda\hat{U}_i, \left[\frac{\kappa}{2}\hat{T}^H_{ij}, \lambda\hat{U}_j\right]\right] + \left[\left[\lambda\hat{U}_i, \frac{\kappa}{2}\hat{T}^H_{ij}\right], \lambda\hat{U}_j\right]\right) \\ &\quad + \frac{1}{12}\left(\left[\lambda\hat{U}_i, \left[\lambda\hat{U}_i, \frac{\kappa}{2}\hat{T}^H_{ij}\right]\right] + \left[\lambda\hat{U}_i, \left[\lambda\hat{U}_i, \lambda\hat{U}_j\right]\right] + \left[\frac{\kappa}{2}\hat{T}^H_{ij}, \left[\frac{\kappa}{2}\hat{T}^H_{ij}, \lambda\hat{U}_j\right]\right]\right. \\ &\quad \left. + \left[\left[\lambda\hat{U}_i, \frac{\kappa}{2}\hat{T}^H_{ij}\right], \frac{\kappa}{2}\hat{T}^H_{ij}\right] + \left[\left[\lambda\hat{U}_i, \lambda\hat{U}_j\right]\lambda\hat{U}_j\right] + \left[\left[\frac{\kappa}{2}\hat{T}^H_{ij}, \lambda\hat{U}_j\right], \lambda\hat{U}_i\right]\right) + \cdots\end{aligned} \tag{5.3.6}$$

Whilst this method is perfectly reasonable for finding the lower order coefficients (of $\hat{V}_{ij}$ as an expansion in μ), it becomes increasingly difficult to find the higher order terms, and further reduction identities are necessary to control the ever lengthening commutators that appear. It is better therefore, to seek the exact solution for $\hat{V}_{ij}$. This can be achieved by listing all the states for a two site system and calculating the matrix that represents $\hat{L}_{ij}$ in this basis, splitting it up into the subspaces of differing particle number and calculating the logarithm of each of these subspace-matrices. Constructing the matrices represent-

ing the hopping and potential operators ($\hat{T}^H_{ij}$, $\hat{U}_i$ and $\hat{U}_j$) in the same basis and then finding their various commutators by simple matrix multiplication, until the logarithm of $\hat{L}_{ij}$ can be written as a linear combination of these operators and commutators, which completes the calculation. The full details can be found in Appendix C, and, using the reparameterisation

$$\cosh\nu \;=\; \cosh\lambda\;\cosh\kappa, \tag{5.3.7}$$

finds the operator $\hat{V}_{ij}$ to be

$$\begin{aligned}
\mu\hat{V}_{ij}[\mu] =\; & \lambda\left(\hat{U}_i+\hat{U}_j\right)+\frac{\kappa}{4}\sinh\lambda\left(\left[\hat{U}_i,\hat{T}^H_{ij}\right]+\left[\hat{T}^H_{ij},\hat{U}_i\right]\right)+\frac{\kappa}{4}\cosh\lambda\left\{\hat{T}^H_{ij}+\left[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\right]\right\}\\
&+\frac{1}{16}\left\{\frac{\nu}{\sinh\nu}\sinh\lambda\cosh\kappa-\lambda\right\}\left(\left[\hat{U}_i,\hat{T}^H_{ij},\hat{T}^H_{ij}\right]+\left[\hat{T}^H_{ij},\hat{T}^H_{ij},\hat{U}_j\right]\right)\\
&+\frac{1}{4}\frac{\nu\sinh\kappa}{\sinh\nu}\left(\hat{T}^H_{ij}+\left[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\right]\right)\\
=\; & \lambda\left(\hat{U}_i+\hat{U}_j\right)+\frac{1}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}+\frac{\nu\sinh\kappa}{\sinh\nu}\right\}\hat{T}^H_{ij}+\frac{\kappa}{4}\sinh\lambda\left(\left[\hat{U}_i,\hat{T}^H_{ij}\right]+\left[\hat{T}^H_{ij},\hat{U}_i\right]\right)\\
&+\frac{1}{16}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}\left(\left[\hat{U}_i,\hat{T}^H_{ij},\hat{T}^H_{ij}\right]+\left[\hat{T}^H_{ij},\hat{T}^H_{ij},\hat{U}_j\right]\right)\\
&+\frac{1}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}-\frac{\nu\sinh\kappa}{\sinh\nu}\right\}\left[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\right].
\end{aligned}\tag{5.3.8}$$

Rearranging this to be in terms of the Hamiltonian rather than the hopping operator finds

$$\begin{aligned}
\mu\hat{V}_{ij}[\mu] \;=\; & \hat{H}_{ij}\left(\frac{1}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}+\frac{\nu\sinh\kappa}{\sinh\nu}\right\}+\frac{g}{8}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}\right)\\
&+\left(\hat{U}_i+\hat{U}_j\right)\left(\lambda-\frac{g}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}+\frac{\nu\sinh\kappa}{\sinh\nu}\right\}-\frac{g^2}{8}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}\right)\\
&+\frac{\kappa}{4}\sinh\lambda\left(\left[\hat{U}_i,\hat{H}_{ij}\right]+\left[\hat{H}_{ij},\hat{U}_i\right]\right)\\
&+\frac{1}{16}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}\left(\left[\hat{U}_i,\hat{H}_{ij},\hat{H}_{ij}\right]+\left[\hat{H}_{ij},\hat{H}_{ij},\hat{U}_j\right]\right)\\
&+\left[\hat{U}_i,\hat{H}_{ij},\hat{U}_j\right]\left(\frac{1}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}-\frac{\nu\sinh\kappa}{\sinh\nu}\right\}-\frac{g}{8}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}\right)
\end{aligned}\tag{5.3.9}$$

Once again μ is chosen to be the coefficient of the Hamiltonian, and the other coef-

ficients need to be rearranged as functions of it. This is a more challenging task than before though, as they are functions of three parameters, the relationships between which are known and must be used in order to obtain each as a function of a single parameter, which is then written in terms of μ. Firstly, λ can be found as an expansion in κ using equation (5.2.9)

$$\begin{aligned} g &= \frac{\sinh 2\lambda}{2}\frac{\cosh^2\left(\frac{\kappa}{2}\right)}{\sinh\left(\frac{\kappa}{2}\right)} \approx \frac{1}{2}\left(2\lambda+\frac{8\lambda^3}{6}\right)\frac{\left(1+\frac{\kappa^2}{8}\right)^2}{\left(\frac{\kappa}{2}+\frac{\kappa^3}{48}\right)} = \left(\lambda+\frac{2\lambda^3}{3}\right)\frac{\left(1+\frac{\kappa^2}{4}\right)}{\frac{\kappa}{2}\left(1+\frac{\kappa^2}{24}\right)} \\ \Rightarrow\ \lambda+\frac{2\lambda^3}{3} &\approx \frac{g\kappa}{2}\left(1+\frac{\kappa^2}{24}\right)\left(1+\frac{\kappa^2}{4}\right)^{-1} \\ &= \frac{g\kappa}{2}\left(1+\frac{\kappa^2}{24}-\frac{\kappa}{4}\right) \\ &= \frac{g\kappa}{2}\left(1-\frac{5\kappa^2}{24}\right), \end{aligned} \tag{5.3.10}$$

and then by writing λ as an expansion in powers of κ, putting it into the left hand side of the above equation and matching the coefficients

$$\lambda = \frac{\kappa g}{2}-\frac{\kappa^3}{12}\left(\frac{5g}{4}+g^3\right)+\mathcal{O}(\kappa^5). \tag{5.3.11}$$

Equation (5.3.7) can be used to obtain ν as a power expansion in κ in the same way

$$\begin{aligned} \cosh\nu &= \cosh\lambda\cosh\kappa, \\ \Rightarrow\ 1+\frac{\nu^2}{2} &\approx \left(1+\frac{\lambda^2}{2}\right)\left(1+\frac{\kappa^2}{2}\right) = 1+\frac{\kappa^2}{2}\left(1+\frac{g^2}{4}\right), \\ \Rightarrow\ \nu^2 &\approx \kappa^2\left(1+\frac{g^2}{4}\right), \end{aligned} \tag{5.3.12}$$

and finally the above choice of μ (as the coefficient of $\hat{H}$ in equation (5.3.9)) can be used so that it too is written as an expansion in powers of κ

$$\mu = \frac{1}{4}\left\{\frac{\kappa\cosh\nu}{\cosh\kappa}+\frac{\nu\sinh\kappa}{\sinh\nu}\right\} + \frac{g}{8}\left\{\frac{\nu\sinh\lambda\cosh\nu}{\sinh\nu\cosh\lambda}-\lambda\right\}$$

$$
\begin{aligned}
&\approx \frac{1}{4}\left\{\kappa\left(1+\frac{\lambda^2}{2}\right)+\nu\frac{\kappa+\frac{\kappa^3}{6}}{\nu+\frac{\nu^3}{6}}\right\} + \frac{g}{8}\left\{\left(\lambda+\frac{\lambda^3}{6}\right)\left(1-\frac{\nu^2}{6}\right)\left(1+\frac{\kappa^2}{2}\right)-\lambda\right\}\\
&= \frac{1}{4}\left\{\kappa\left(1+\frac{g^2\kappa^2}{8}\right)+\left(\kappa+\frac{\kappa^3}{6}\right)\left(1+\frac{\kappa^2(1+\frac{g^2}{4})}{6}\right)^{-1}\right\}\\
&\quad + \frac{g}{8}\left\{\lambda\left(1+\frac{\lambda^2}{6}-\frac{\nu^2}{6}+\frac{\kappa^2}{2}\right)-\lambda\right\}\\
&= \frac{1}{4}\left\{2\kappa+\frac{\kappa^3 g^2}{12}\right\} + \frac{g}{8}\left\{\frac{\kappa g}{2}\left(\frac{\kappa^2 g^2}{24}-\frac{\kappa^2(1+\frac{g^2}{4})}{6}+\frac{\kappa^2}{2}\right)\right\}\\
&= \frac{1}{4}\left\{2\kappa+\frac{\kappa^3 g^2}{6}\right\}+\mathcal{O}(\kappa^5). \qquad (5.3.13)
\end{aligned}
$$

Rearranging this to have κ (and hence λ and ν) as an expansion in μ

$$
\kappa = 2\mu+\frac{2}{3}g^2\mu^3+\mathcal{O}(\mu^5), \qquad (5.3.14)
$$

and then rewriting each of the coefficients in equation (5.3.9) solely in terms of μ gives

$$
\begin{aligned}
\hat{V}_n = {} & \hat{H}_n-\frac{g}{6}\left(5+6g^2\right)\mu^2\left(\hat{U}_n+\hat{U}_{n+1}\right)\\
&+\left\{\frac{g\mu}{2}-\frac{g}{12}\left(5+7g^2\right)\mu^3\right\}\left(\left[\hat{U}_n,\hat{H}_n\right]+\left[\hat{H}_n,\hat{U}_{n+1}\right]\right)\\
&+\frac{g\mu^2}{12}\left(2g\left[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\right]+\left[\hat{U}_n,\hat{H}_n,\hat{H}_n\right]+\left[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\right]\right)+\mathcal{O}(\mu^4), \qquad (5.3.15)
\end{aligned}
$$

with the usual notation $\hat{O}_n=\hat{O}_{n,n+1}$ for $\hat{H}$ and $\hat{V}$.

5.4 Generating the Local Conservation Laws

As before, finding the local conservation laws is merely a matter of substituting the above operator in to equations (3.3.21) to (3.3.25), and then using those in equation (3.3.2) and collecting together the terms at each order in μ. Writing this out explicitly up to $\mathcal{O}(\mu^3)$ gives

$$
\hat{X}=\sum_{n=0}^{\infty}\mu\Bigg(\hat{H}_n-\frac{g}{6}\left(5+6g^2\right)\mu^2\left(\hat{U}_n+\hat{U}_{n+1}\right)+\frac{g\mu}{2}\left(\left[\hat{U}_n,\hat{H}_n\right]+\left[\hat{H}_n,\hat{U}_{n+1}\right]\right)
$$

$$
\begin{aligned}
&+\frac{g\mu^2}{12}\left(2g\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\hat{U}_n,\hat{H}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\Big]\right)+\cdots\Bigg)\\
&+\frac{\mu^2}{2!}\Bigg(\Big[\hat{H}_n,\hat{H}_{n+1}\Big]+\Big[\frac{g\mu}{2}\left(\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\right),\hat{H}_{n+1}\Big]\\
&\quad+\Big[\hat{H}_n,\frac{g\mu}{2}\left(\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]+\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\right)\Big]\Big]+\cdots\Bigg)\\
&+\frac{\mu^3}{3!}\Bigg(2\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2}\Big]+\frac{1}{2}\Big[\hat{H}_n,\Big[\hat{H}_n,\hat{H}_{n+1}\Big]\Big]+\frac{1}{2}\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big]+\cdots\Bigg)\\
&+\mathcal{O}(\mu^4). \qquad (5.4.1)
\end{aligned}
$$

Clearly there is again only one term that contributes at order μ, the lowest order term of $\hat{X}^1$, and so the first local conservation law is trivially

$$\hat{L}_1 \;=\; \sum_n \hat{H}_n, \qquad (5.4.2)$$

i.e. the Hamiltonian, as would be expected. At order μ^2 there is a contribution from both $\hat{X}^2$ (it's lowest order term) and also from the first order terms of $\hat{X}^1$

$$
\begin{aligned}
\hat{L}_2 &= \frac{1}{2!}\sum_n\Big[\hat{H}_n,\hat{H}_{n+1}\Big]+\frac{g}{2}\sum_n\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\\
&= \sum_n\Big[\hat{H}_n,\hat{H}_{n+1}\Big]+g\left(\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\right). \qquad (5.4.3)
\end{aligned}
$$

Writing out the operators explicitly and performing the commutations, this second local conservation law becomes

$$
\begin{aligned}
\hat{L}_2 &= -2\sum_n\left\{\hat{S}^z_{n+1}\Big(\hat{S}^+_n\hat{S}^-_{n+2}-\hat{S}^-_n\hat{S}^+_{n+2}\Big)+\hat{T}^z_{n+1}\Big(\hat{T}^+_n\hat{T}^-_{n+2}-\hat{T}^-_n\hat{T}^+_{n+2}\Big)\right\}\\
&\quad+U\sum_n\left\{\Big(\hat{S}^+_n\hat{S}^-_{n+1}-\hat{S}^-_n\hat{S}^+_{n+1}\Big)\Big(\hat{T}^z_n+\hat{T}^z_{n+1}\Big)+\Big(\hat{T}^+_n\hat{T}^-_{n+1}-\hat{T}^-_n\hat{T}^+_{n+1}\Big)\Big(\hat{S}^z_n+\hat{S}^z_{n+1}\Big)\right\}\\
&= 4i\sum_n\Bigg\{\hat{S}^z_{n+1}\Big(\hat{S}^x_n\hat{S}^y_{n+2}-\hat{S}^y_n\hat{S}^x_{n+2}\Big)\\
&\qquad-\frac{1}{2}U\Big(\hat{S}^x_n\hat{S}^y_{n+1}-\hat{S}^y_n\hat{S}^x_{n+1}\Big)\Big(\hat{T}^z_n+\hat{T}^z_{n+1}\Big)+\Big(\hat{S}\leftrightarrow\hat{T}\Big)\Bigg\}, \qquad (5.4.4)
\end{aligned}
$$

and inverting the Jordan-Wigner transformation gives the fermionic form

$$\begin{aligned}\hat{L}_2 &= -2\sum_{n,\sigma}\Big\{\Big(\hat{C}^+_{n+2,\sigma}\hat{C}_{n,\sigma}-\hat{C}^+_{n,\sigma}\hat{C}_{n+2,\sigma}\Big)\\ &\qquad -U\Big(\hat{C}^+_{n+1,\sigma}\hat{C}_{n,\sigma}-\hat{C}^+_{n,\sigma}\hat{C}_{n+1,\sigma}\Big)\Big(\hat{n}_{n+1,\sigma'}+\hat{n}_{n,\sigma'}-1\Big)\Big\}, \qquad (5.4.5)\end{aligned}$$

and these are consistent with the expressions obtained by Shastry [60] and others [5, 30, 41, 42].

There are contributions at order μ^3 from each of the first three $\hat{X}^n$'s, which combine to give

$$\begin{aligned}\hat{L}_3 &= \frac{1}{3!}\sum_n 2\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2}\Big]+\frac{1}{2}\Big[\hat{H}_n,\hat{H}_n,\hat{H}_{n+1}\Big]+\frac{1}{2}\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+1}\Big]\\ &\quad +\frac{1}{2!}\sum_n\frac{g}{2}\Big(\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]+\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]+\\ &\quad \Big[\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]\Big)-\sum_n\frac{g}{6}\left(5+6g^2\right)\Big(\hat{U}_n+\hat{U}_{n+1}\Big)\\ &\quad -\frac{g}{12}\Big(2g\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\hat{U}_n,\hat{H}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\Big]\Big)\\ &= \sum_n 2\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2}\Big]+\frac{1}{2}\Big[\hat{H}_n,\hat{H}_n,\hat{H}_{n+1}\Big]+\frac{1}{2}\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+1}\Big]\\ &\quad +\frac{3g}{2}\Big(\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big]+\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]+\Big[\Big[\hat{U}_n,\hat{H}_n\Big]\hat{H}_{n+1}\Big]\\ &\quad +\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]\Big)+\frac{g}{2}\Big(\Big[\hat{U}_n,\hat{H}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\Big]\Big)\\ &\quad +g^2\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big]-g\left(5+6g^2\right)\Big(\hat{U}_n+\hat{U}_{n+1}\Big), \qquad (5.4.6)\end{aligned}$$

and using the relations

$$\begin{aligned}&\sum_n\Big(\Big[\hat{H}_{n-1},\hat{H}_{n-1},\hat{H}_n\Big]+g^2\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big]-\hat{H}_{n,n+1}+g\Big(\hat{U}_n+\hat{U}_{n+1}\Big)\Big)\\ &=\sum_n g\Big(\Big[\hat{H}_{n-1},\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\Big[\hat{U}_{n-1},\hat{H}_{n-1}\Big],\hat{H}_n\Big]\Big), \qquad (5.4.7)\\ &\sum_n\Big(\Big[\hat{H}_{n-1},\hat{H}_n,\hat{H}_n\Big]+g^2\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big]-\hat{H}_{n,n+1}+g\Big(\hat{U}_n+\hat{U}_{n+1}\Big)\Big)\\ &=\sum_n g\Big(\Big[\hat{H}_{n-1},\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{U}_n,\hat{H}_n,\hat{H}_n\Big]+\Big[\hat{H}_{n-1}\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\Big]\Big), \qquad (5.4.8)\end{aligned}$$

$$\sum_n \Big(\Big[\hat{H}_n, \Big[\hat{U}_{n+1}, \hat{H}_{n+1} \Big] \Big] - \Big[\Big[\hat{H}_n, \hat{U}_{n+1} \Big], \hat{H}_{n+1} \Big] \Big) = \sum_n \Big[\hat{U}_{n+1} \Big[\hat{H}_n, \hat{H}_{n+1} \Big] \Big] = 0, \quad (5.4.9)$$

and

$$\begin{aligned} \Big[\hat{H}_n, \Big[\hat{H}_n, \hat{U}_{n+1} \Big] \Big] &= \Big[\Big[\hat{U}_n, \hat{H}_n \Big], \hat{H}_n \Big] - 4\,\hat{U}_n + 4\,\hat{U}_{n+1} \qquad (5.4.10) \\ \Rightarrow \sum_n \Big(\Big[\hat{U}_n, \hat{H}_n, \hat{H}_n \Big] &- \Big[\hat{H}_n, \hat{H}_n, \hat{U}_{n+1} \Big] \Big) = 0, \end{aligned}$$

equation (5.4.6) reduces to

$$\begin{aligned} \hat{L}_3 = \sum_n & 2\Big[\hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2} \Big] + 2\Big[\hat{H}_n, \hat{H}_n, \hat{H}_{n+1} \Big] + 2\Big[\hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+1} \Big] \\ & -2g\Big[\hat{U}_n, \hat{H}_n, \hat{H}_n \Big] + 4g^2 \Big[\hat{U}_n, \hat{H}_n, \hat{U}_{n+1} \Big] - 3\hat{H}_n - 2g(2+6g^2)\hat{U}_n. \quad (5.4.11) \end{aligned}$$

At order μ^4 there are significantly many more terms to deal with. This is due to more complicated nature of $\hat{V}_{ij}$, which means that, unlike the XXZ model, there are contributions to the n^{th} conservation law from all of the coefficients up to and including $\hat{X}^n$. Not only that, but the first $n-1$ coefficients contribute more terms than in the previous case too, and so the effort required in calculating these conservation laws is increasing much more quickly than with the XXZ model. Thus collecting together the contributions to the fourth local conservation law gives the somewhat ponderous

$$\begin{aligned} \hat{L}_4 = & \frac{1}{4!} \sum_n 6\Big[\hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2}, \hat{H}_{n+3} \Big] + 2\Big[\Big[\hat{H}_n, \hat{H}_{n+1} \Big], \Big[\hat{H}_{n+1}, \hat{H}_{n+2} \Big] \Big] \\ & +2\Big[\hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2}, \hat{H}_{n+2} \Big] + 2\Big[\hat{H}_n, \hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+2} \Big] \\ & +\Big[\hat{H}_n, \hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+1} \Big] + \frac{1}{3!} \sum_n g\Big(\Big[\Big[\hat{U}_n, \hat{H}_n \Big], \Big[\hat{H}_{n+1}, \hat{H}_{n+2} \Big] \Big] \\ & +\Big[\hat{H}_n, \Big[\Big[\hat{U}_{n+1}, \hat{H}_{n+1} \Big], \hat{H}_{n+2} \Big] \Big] + \Big[\hat{H}_n, \Big[\hat{H}_{n+1}, \Big[\hat{U}_{n+2}, \hat{H}_{n+2} \Big] \Big] \Big] \\ & +\Big[\Big[\hat{H}_n, \hat{U}_{n+1} \Big], \Big[\hat{H}_{n+1}, \hat{H}_{n+2} \Big] \Big] + \Big[\hat{H}_n, \Big[\Big[\hat{H}_{n+1}, \hat{U}_{n+2} \Big], \hat{H}_{n+2} \Big] \Big] \\ & +\Big[\hat{H}_n, \Big[\hat{H}_{n+1}, \Big[\hat{H}_{n+2}, \hat{U}_{n++3} \Big] \Big] \Big] \Big) + \frac{g}{4} \Big(\Big[\Big[\hat{U}_n, \hat{H}_n \Big], \Big[\hat{H}_n, \hat{H}_{n+1} \Big] \Big] \\ & +\Big[\hat{H}_n, \Big[\Big[\hat{U}_n, \hat{H}_n \Big], \hat{H}_{n+1} \Big] \Big] + \Big[\hat{H}_n, \Big[\hat{H}_n, \Big[\hat{U}_{n+1}, \hat{H}_{n+1} \Big] \Big] \Big] \end{aligned}$$

$$
\begin{aligned}
&+\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\Big[\hat{H}_n,\hat{H}_{n+1}\Big]\Big]+\Big[\hat{H}_n,\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]\Big]\\
&+\Big[\hat{H}_n,\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big]+\Big[\Big[\Big[\hat{U}_n,\hat{H}_n\Big],\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big]\\
&+\Big[\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big],\hat{H}_{n+1}\Big]+\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big]\\
&+\Big[\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big]+\Big[\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big],\hat{H}_{n+1}\Big]\\
&+\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big)+\frac{1}{2!}\sum_n\frac{-g}{6}(5+6g^2)\\
&\Big(\Big[\hat{U}_n,\hat{H}_{n+1}\Big]+\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\hat{H}_n,\hat{U}_{n+2}\Big]\Big)\\
&+\frac{g^2}{6}\Big(\Big[\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]+\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big)\\
&+\frac{g}{12}\Big(\Big[\Big[\hat{U}_n,\hat{H}_n,\hat{H}_n\Big],\hat{H}_{n+1}\Big]+\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1},\hat{H}_{n+1}\Big]\Big]\\
&+\Big[\Big[\hat{H}_n,\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]+\Big[\hat{H}_n\Big[\hat{H}_{n+1},\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big)\\
&+\sum_n\frac{-g}{12}(5+7g^2)\Big(\Big[\hat{U}_n,\hat{H}_n\Big]+\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\Big).
\end{aligned}
$$

Using the previous identities and those listed at the end of Appendix C, these terms can be reduced and recombined to give the following expression for the fourth local conservation law

$$
\begin{aligned}
\hat{L}_4 \;=\;& \sum_n 6\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2},\hat{H}_{n+3}\Big]+6\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\Big[\hat{H}_{n+1},\hat{H}_{n+2}\Big]\Big]\\
&+6\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2},\hat{H}_{n+2}\Big]+6\Big[\hat{H}_n,\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2}\Big]\\
&+3\Big[\hat{H}_n,\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+1}\Big]+6g^2\Big(\Big[\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]\\
&+\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big)+(-8+4g^2)\hat{L}_2\\
&-6g(1+6g^2)\Big(\Big[\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\hat{U}_n,\hat{H}_n\Big]\Big).
\end{aligned}
\tag{5.4.12}
$$

The above form of the conservation law was initially chosen since it was the most compact, but it might be more interesting to consider the form below *i.e.* to have the conservation law written like an expansion in the interaction parameter, making it much easier to see what happens to the conservation law when various limits of this parameter

are taken

$$\begin{aligned}\hat{L}_4 &= \sum_n 6\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2},\hat{H}_{n+3}\Big]+6g\Big(\Big[\Big[\Big[\hat{U}_n,\hat{H}_n\Big],\hat{H}_{n+1}\Big],\hat{H}_{n+2}\Big]\\ &+\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\Big[\hat{H}_{n+2},\hat{U}_{n+3}\Big]\Big]\Big]+2\Big[\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big],\hat{H}_{n+2}\Big]\\ &+2\Big[\hat{H}_n,\Big[\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big],\hat{H}_{n+2}\Big]\Big]+\Big[\hat{H}_n,\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big]\Big]\\ &+\Big[\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big],\hat{H}_{n+1}\Big]\Big)+3g^2\Big(\Big[\Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big]\Big],\hat{U}_{n+2}\Big]\\ &+\Big[\hat{U}_n,\Big[\Big[\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_n\Big]\Big]+\Big[\Big[\hat{U}_n,\hat{H}_n\Big],\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big)\\ &+(4+11g^2)\hat{L}_2-(30g+9g^3)\left(\Big[\hat{H}_n,\hat{U}_{n+1}\Big]+\Big[\hat{U}_n,\hat{H}_n\Big]\right). \qquad (5.4.13)\end{aligned}$$

As in the previous chapter, all of the conservation laws calculated here are in agreement with those found by others [5, 42]. Note that this includes those found using a purely fermionic formulation [41], which, in line with the expectations in Chapter 3, are the same as the above bosonic versions converted back to fermionic form via the Jordan-Wigner transformation.

5.5 Heisenberg Conservation Laws as a Limit of Hubbard Ones

In the previous chapter, the Heisenberg model was derived as a limit of the Hubbard model. Following on from this, it should therefore be possible to generate the local conservation laws of the Heisenberg model (*i.e.* those found in the previous chapter, with $\Delta = 1$) as a limit of the Hubbard ones found above.

Starting as before with the Hubbard model at half-filling and $U >> t$, so that there is exactly one particle per site, it is clear that all terms which do not conserve particle number on each site must be discarded. Unfortunately this covers *all* of the terms in the first two local conservation laws, and it is only at the level of the third conservation law

(rewritten here in terms of the hopping operator rather than the Hamiltonian)

$$\begin{aligned}\hat{L}_3 &= 2\sum_n \left[\hat{T}_n^H, \hat{T}_{n+1}^H, \hat{T}_{n+2}^H\right] + g\Big(\left[\hat{T}_n^H, \left[\hat{T}_n^H, \hat{U}_{n+1}\right]\right] + 2\left[\left[\hat{T}_n^H, \hat{T}_{n+1}^H\right], \hat{U}_{n+2}\right] \\ &\quad +2\left[\left[\hat{U}_{n+1}, \hat{T}_{n+1}^H\right], \hat{T}_{n+2}^H\right] + 2\left[\hat{T}_n^H, \left[\hat{U}_{n+1}, \hat{T}_{n+1}^H\right]\right]\Big) + 4g^2\left[\hat{U}_n, \hat{T}_n^H, \hat{U}_{n+1}\right] \\ &\quad +\alpha\hat{L}_1 + \beta\hat{U}_n, \end{aligned} \tag{5.5.1}$$

that suitable terms appear. The terms in question stem from the second commutator, the only quantity thus far to involve the hopping operator for the same site twice, which is necessary to maintain the one particle per site restriction. Expanding out this commutator gives

$$\begin{aligned}\left[\hat{T}_n^H, \left[\hat{T}_n^H, \hat{U}_{n+1}\right]\right] &= 4\left(\hat{T}_{n+1}^z\left(\hat{S}_{n+1}^z - \hat{S}_n^z\right) + \hat{S}_{n+1}^z\left(\hat{T}_{n+1}^z - \hat{T}_n^z\right)\right. \\ &= \left. +\left(\hat{T}_n^+\hat{T}_{n+1}^- - \hat{T}_n^-\hat{T}_{n+1}^+\right)\left(\hat{S}_n^+\hat{S}_{n+1}^- - \hat{S}_n^-\hat{S}_{n+1}^+\right)\right). \end{aligned} \tag{5.5.2}$$

Ignoring the two terms which project away from single occupancy, and using

$$-\hat{T}_{n+1}^z\hat{S}_n^z - \hat{S}_{n+1}^z\hat{T}_n^z = \frac{1}{2}\left(\left[\hat{S}_n^z - \hat{T}_n^z\right]\left[\hat{S}_{n+1}^z - \hat{T}_{n+1}^z\right] - \left[\hat{S}_n^z + \hat{T}_n^z\right]\left[\hat{S}_{n+1}^z + \hat{T}_{n+1}^z\right]\right), \tag{5.5.3}$$

the relevant terms become

$$\begin{aligned}4\Bigg(&- \hat{T}_n^+\hat{T}_{n+1}^-\hat{S}_n^-\hat{S}_{n+1}^+ - \hat{T}_n^-\hat{T}_{n+1}^+\hat{S}_n^+\hat{S}_{n+1}^- \\ &+ \frac{1}{2}\left(\left[\hat{S}_n^z - \hat{T}_n^z\right]\left[\hat{S}_{n+1}^z - \hat{T}_{n+1}^z\right] - \left[\hat{S}_n^z + \hat{T}_n^z\right]\left[\hat{S}_{n+1}^z + \hat{T}_{n+1}^z\right]\right) - \frac{1}{2}\Bigg) \end{aligned} \tag{5.5.4}$$

where the idea that $\hat{S}_\alpha^z\hat{T}_\alpha^z = -\frac{1}{4}$ has also been used. This final identity stems from the fact that, since the single occupancy of sites has been enforced so that there must be either an up spin or a down spin on each site, $\hat{S}_\alpha^z$ and $\hat{T}_\alpha^z$ must have the opposite sign. The rest of the terms in equation (5.5.4) can be considered in a similar manner.

First, a basis is needed of the possible states on the two sites (n and $n+1$) on which

the operators of equation (5.5.4) act. There are four such states

$$|\uparrow\downarrow\,;\,\uparrow\downarrow\rangle, \tag{5.5.5a}$$

$$|\uparrow\downarrow\,;\,\downarrow\uparrow\rangle, \tag{5.5.5b}$$

$$|\downarrow\uparrow\,;\,\uparrow\downarrow\rangle, \tag{5.5.5c}$$

$$|\downarrow\uparrow\,;\,\downarrow\uparrow\rangle, \tag{5.5.5d}$$

where the first set of spins are for site n and the second for site $n+1$. The first and second spins in these sets then correspond to the flavours of spin acted upon by the $\hat{S}$ and $\hat{T}$ operators respectively. These are equivalent to the four states that make up the two-site basis for the Heisenberg model

$$|\uparrow\,;\,\uparrow\rangle, \quad |\uparrow\,;\,\downarrow\rangle, \quad |\downarrow\,;\,\uparrow\rangle, \quad |\downarrow\,;\,\downarrow\rangle, \tag{5.5.6}$$

where the arrows now represent the spin of the actual particle on site n, then site $n+1$. We will denote the operators which act upon these states with a tilde, *i.e.* $\tilde{S}_n^+$, $\tilde{S}_n^-$ and $\tilde{S}_n^Z$. Note that, in each case the actual spin on the site is identical to that of the flavour acted upon by the $\hat{S}$ operators in the previous representation. As was noted above, $\hat{T}_\alpha^z$ will have the opposite sign to $\hat{S}_\alpha^z$, and so their sum will vanish, while their difference will be twice the value of $\hat{S}_\alpha^z$. It is clear, therefore, that $\hat{S}_n^z - \hat{T}_n^z$ will have the same effect on the states of equation (5.5.5) as $2\tilde{S}_n^Z$ will have on the states in (5.5.6). Finally we come to the terms $\hat{T}_n^+\hat{T}_{n+1}^-\hat{S}_n^-\hat{S}_{n+1}^+$ and $\hat{T}_n^-\hat{T}_{n+1}^+\hat{S}_n^+\hat{S}_{n+1}^-$. The first turns state (5.5.5b) in to state (5.5.5c), and returns zero when acted upon the other three states, exactly the same relationship $\tilde{S}_n^+$ has with the states (5.5.6). Similarly the second term acted upon state (5.5.5c) returns (5.5.5b), and zero when it acts upon the other three, which is equivalent to $\tilde{S}_n^-$ acting the states (5.5.6).

In terms of $\tilde{S}_n^+$, $\tilde{S}_n^-$ and $\tilde{S}_n^Z$ acting on the equivalent Heisenberg states (5.5.6) therefore,

(5.5.4) becomes

$$4\left(-\tilde{S}_n^+\tilde{S}_{n+1}^- - \tilde{S}_n^-\tilde{S}_{n+1}^+ + 2\tilde{S}_n^z\tilde{S}_{n+1}^z - \frac{1}{2}\right). \tag{5.5.7}$$

Summed over n, this is the Heisenberg model, *i.e.* the XXZ model as defined previously in Chapters 3 and 4, with a negative coefficient and $\Delta = -1$. While it may not be surprising that the Heisenberg conservation laws can be found in this way, it is important to check to show that the method and the conservation laws generated are consistent.

Repeating this procedure for higher order conservation laws of the Hubbard model should yield higher order Hubbard ones. Unfortunately the only term from $\hat{L}_4$ which contains an even number of each hopping operator (*i.e.* doesn't project away from single occupancy) vanishes, and so to proceed further would first require the calculation of the fifth Hubbard conservation law. Evidently, attempting to calculate the Heisenberg conservation laws in this manner is impractical and should be done as in the previous chapter.

In the next chapter the local conservation laws of a third model, the Toda chain, are generated. The model is not like the spin chains considered so far, and the method of Chapter 3 is shown to be incompatible with it. Instead an alternative method of evaluating the Transfer matrix is used to generate non-local conservation laws, which are then shown to coincide with the conserved quantities of the classical system. After a slight reformulation of the classical Lax operator, a direct link to the quantum Transfer matrix is established.

Chapter 6

Local Conservation Laws of the Toda Chain

In this chapter a very different style of Hamiltonian is considered: the one-dimensional Toda lattice, or Toda chain [61]. The method previously applied to the XXZ and Hubbard models cannot be used here, and attempts to use the Monodromy matrix provide only the well known non-local conserved quantities originally found by Hénon [62]. Calculation of the integrals of motion via the classical Lax matrix is described since it can be shown [63] that they are equivalent to the conserved quantities of the quantum model. Furthermore, this matrix can be used to generate the local conservation laws of the system, in a way that naturally incorporates the awkward combinatorics of their original construction [64]. A minor simplification to this process is made, and the relationship between the classical L-matrix and the quantum Monodromy matrix is then shown.

In stark contrast to the previous spin models, the one-dimensional Toda lattice consists of a chain of particles (of equal masses) which interact with their nearest neighbours via an exponential potential. The Hamiltonian for an N-site system with periodic boundary conditions (*i.e.* $\hat{q}_{N+1} = \hat{q}_1$) is

$$\hat{H} \;=\; \frac{1}{2}\sum_{j=1}^{N} \hat{p}_j^2 + \sum_{j=1}^{N} \exp\Big(\hat{q}_j - \hat{q}_{j+1}\Big), \tag{6.0.1}$$

$$= \sum_{j=1}^{N} \frac{\partial^2}{\partial \hat{q}_j^2} + \sum_{j=1}^{N} \exp\Big(\hat{q}_j - \hat{q}_{j+1}\Big), \tag{6.0.2}$$

where $\hat{p}_j$ is the momentum of the j^{th} particle and $\hat{q}_j$ is its displacement from its equilibrium position. This model is somewhat of a rarity as both the classical (6.0.1) and quantum mechanical (6.0.2) versions have been shown to be integrable [62, 65], and the conserved quantities of each case can be shown to be the same.

Unlike the previous two models, there are no materials whose physical behaviour suggests they can be modelled by the Toda Hamiltonian (6.0.2). This is not surprising given the curious nature of the potential. This is more easily seen by considering a particles absolute position (with respect to some origin, *e.g.* the first particle in the chain), *i.e.* $\hat{q}_i = \hat{x}_i - \hat{x}_i^{(e)}$, where $\hat{x}_i^{(e)}$ is the i^{th} particles equilibrium position. The potential then obviously vanishes for $\hat{x}_{j+1} \gg \hat{x}_j$, *i.e.* when the $j+1^{th}$ particle is far to the right of the j^{th}, and increases rapidly when $\hat{x}_j \gg \hat{x}_{j+1}$, *i.e.* when the $j+1^{th}$ particle is far to the left of the j^{th}. The potential therefore abhors particles being positioned out of order, and is minimised by having each particle as far to the right of the preceding one as possible. Indeed it is only the periodic boundary conditions that keeps the particles together, not a situation likely to be found in nature.[1] This has not stopped the model from being studied extensively [67, 68] however, and so the investigation of this model begins by considering the quantum mechanical case.

6.1 The Quantum Lax Operator

In Chapters 4 and 5 the Lax operators of the XXZ and Hubbard models were rewritten to make taking the logarithm of the transfer matrix easier, using the approach outlined in Chapter 3. Despite having a markedly different style of Hamiltonian, as an integrable system the Toda chain must still have a Lax operator which satisfies the Yang-Baxter equation, from which the conserved quantities of the system can be constructed. The Lax

[1] Although the classical model is another story, see *e.g.* [66].

operator in this case has been found to be [65]

$$\hat{L}_j[u] \;=\; \begin{bmatrix} u-\hat{p}_j & \exp(\hat{q}_j) \\ -\exp(-\hat{q}_j) & 0 \end{bmatrix}. \tag{6.1.1}$$

Unlike the models dealt with in earlier chapters however, setting the spectral parameter to zero in this case doesn't return a permutation operator. Without this ability to re-express the Lax operator as in the previous cases (equation (3.3.1)), it is impossible to proceed along the same path towards writing the Monodromy matrix as a product of exponential operators. As with the previous models, the Monodromy matrix for an N-site system is still the product of the N Lax operators

$$\hat{M}_N[u] \;=\; \hat{L}_1\hat{L}_2\cdots\hat{L}_N, \tag{6.1.2}$$

but rather than a matrix, the trace of this returns a polynomial in u, the coefficients of which form a mutually commuting set of conserved operators. These operators are no more or less than the original (non-local) conservation laws discovered by Hénon [62], as will now be demonstrated.

The Lax operator (6.1.1) can be split up and rewritten in the following manner

$$\hat{L}_j[u] \;=\; \hat{Q}_{-j}\Big\{\hat{P}_j+\hat{\Sigma}\Big\}\hat{Q}_j, \tag{6.1.3}$$

in terms of the operators

$$\hat{Q}_{\pm j} \;=\; \begin{bmatrix} 1 & 0 \\ 0 & \pm e^{\pm\hat{q}_j} \end{bmatrix}, \quad \hat{P}_j \;=\; \begin{bmatrix} u-\hat{P}_j & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \hat{\Sigma} \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{6.1.4}$$

Products of the operators $\hat{Q}_{\pm j}$ and $\hat{P}_i$ can be reduced since

$$\hat{P}_j\;\hat{Q}_{\pm i} \;=\; \hat{P}_j \;=\; \hat{Q}_{\pm i}\;\hat{P}_j, \tag{6.1.5}$$

while $\hat{\Sigma}$ acts much like a permutation operator when either side of another operator

$$\hat{\Sigma} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \hat{\Sigma} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}. \tag{6.1.6}$$

Employing the notation $\hat{Q}_j \hat{Q}_{-(j+1)} = \hat{Q}_{j,j+1}$, the product of two Lax matrices for neighbouring sites becomes

$$\begin{aligned} \hat{L}_j[u]\hat{L}_{j+1}[u] &= \hat{Q}_{-j}\Big\{\hat{P}_j + \hat{\Sigma}\Big\}\, \hat{Q}_{j,j+1}\Big\{\hat{P}_{j+1} + \hat{\Sigma}\Big\}\, \hat{Q}_{j+1} \\ &= \hat{Q}_{-j}\, \hat{P}_j\, \hat{Q}_{j,j+1}\, \hat{P}_{j+1}\, \hat{Q}_{j+1} + \hat{Q}_{-j}\, \hat{P}_j\, \hat{Q}_{j,j+1}\, \hat{\Sigma}\, \hat{Q}_{j+1} \\ &\quad + \hat{Q}_{-j}\, \hat{\Sigma}\, \hat{Q}_{j,j+1}\, \hat{P}_{j+1}\, \hat{Q}_{j+1} + \hat{Q}_{-j}\, \hat{\Sigma}\, \hat{Q}_{j,j+1}\, \hat{\Sigma}\, \hat{Q}_{j+1}. \end{aligned} \tag{6.1.7}$$

The $\hat{Q}$'s in the first term will cancel out, leaving $\hat{P}_j \hat{P}_{j+1}$, and since both the $\hat{P}$ and $\hat{Q}$ are diagonal, the second and third terms will have only off-diagonal elements, which will be irrelevant when the trace is taken.

Now consider the trace of the Monodromy matrix

$$\begin{aligned} \hat{T}_N[u] &= tr \prod_{i=1}^{N} \hat{L}_i[u] \\ &= tr\Big(\hat{Q}_{-1}\Big\{\hat{P}_1 + \hat{\Sigma}\Big\}\, \hat{Q}_{1,2}\Big\{\hat{P}_2 + \hat{\Sigma}\Big\} \cdots \hat{Q}_{l-1,l}\Big\{\hat{P}_l + \hat{\Sigma}\Big\}\, \hat{Q}_{l,l+1} \\ &\quad \cdots \hat{Q}_{N-1,N}\Big\{\hat{P}_N + \hat{\Sigma}\Big\}\, \hat{Q}_N\Big). \end{aligned} \tag{6.1.8}$$

As with the two Lax matrices example above, any term which contains an *odd* number of $\hat{\Sigma}$ operators will be off-diagonal and therefore contribute nothing when the trace is taken. The remaining terms can be understood by considering the sequence

$$\cdots \hat{P}_j\, \hat{Q}_{j,j+1}\Big\{\hat{P}_{j+1} + \hat{\Sigma}\Big\}\, \hat{Q}_{j+1,j+2}\Big\{\hat{P}_{j+2} + \hat{\Sigma}\Big\}\, \hat{Q}_{j+2,j+3} \hat{P}_{j+3} \cdots . \tag{6.1.9}$$

Assuming the rest of the sequence includes an even number of $\hat{\Sigma}$'s, when expanded this

will yield two off-diagonal terms which are ignored, and the following two terms

$$\cdots \hat{P}_j \; \hat{Q}_{j,j+1}\hat{\Sigma}\hat{Q}_{j+1,j+2}\hat{\Sigma}\hat{Q}_{j+2,j+3}\hat{P}_{j+3}\cdots$$
$$= \cdots \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u-\hat{p}_j)(-\exp\left(\hat{q}_{j+1}-\hat{q}_{j+2}\right))(u-\hat{p}_{j+3})\cdots, \quad (6.1.10)$$

and the term with no $\hat{\Sigma}$'s

$$\cdots \hat{P}_j \; \hat{Q}_{j,j+1}\hat{P}_{j+1}\hat{Q}_{j+1,j+2}\hat{P}_{j+2}\hat{Q}_{j+2,j+3}\hat{P}_{j+3}\cdots$$
$$= \cdots \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u-\hat{p}_j)(u-\hat{p}_{j+1})(u-\hat{p}_{j+2})(u-\hat{p}_{j+3})\cdots. \quad (6.1.11)$$

Note that each site index only appears once, and in both cases the only non-zero element is the first one. In fact, comparing the right hand sides of the two equations, the sequence of terms (6.1.10) can be obtained from (6.1.11) by left multiplying by the operator

$$-\exp\left(\hat{q}_{j+1}-\hat{q}_{j+2}\right)\frac{\partial^2}{\partial\hat{p}_{j+1}\partial\hat{p}_{j+2}}. \quad (6.1.12)$$

Using this operator the sequence (6.1.9) can be rewritten as

$$\cdots \hat{P}_j \; \hat{Q}_{j,j+1}\left\{\hat{P}_{j+1}+\hat{\Sigma}\right\} \hat{Q}_{j+1,j+2}\left\{\hat{P}_{j+2}+\hat{\Sigma}\right\} \hat{Q}_{j+2,j+3}\hat{P}_{j+3}\cdots$$
$$= \cdots \hat{P}_j \; \hat{Q}_j \left[1-\exp\left(\hat{q}_{j+1}-\hat{q}_{j+2}\right)\frac{\partial^2}{\partial\hat{p}_{i+1}\partial\hat{p}_{i+2}}\right] \hat{P}_{j+1}\hat{P}_{j+2} \; \hat{Q}_{-(j+3)}\hat{P}_{j+3}\cdots, \quad (6.1.13)$$

without affecting the trace. Due to the reduction property (6.1.5), each $\hat{Q}$ must either have a $\hat{\Sigma}$ on each side or none neighbouring it, as in the above example, or the sequence vanishes.

Going back to the Monodromy matrix and ordering the terms in (6.1.8) by increasing

number of Σ operators, the first term will be the product

$$\hat{P}_i \cdots \hat{P}_N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u - \hat{p}_1) \cdots (u - \hat{p}_N). \tag{6.1.14}$$

The next set of terms to contribute to the trace are the N combinations of this product left multiplied by the operator (6.1.12); one term for each pair of neighbouring sites. Subsequent sets of terms involve left multiplying by (6.1.12) again for each pair of neighbouring sites, which did not appear in the first instance of its use, and then repeating this process until there are one or no $\hat{p}$ terms remaining (depending on whether N is odd or even). In terms of this operator, the trace of the Monodromy matrix can therefore be written as

$$\hat{T}_N[u] = \prod_{i=1} \left[1 - \exp\left(\hat{q}_i - \hat{q}_{i+1}\right) \frac{\partial^2}{\partial \hat{p}_{i+1} \partial \hat{p}_{i+2}} \right] \prod_{j=1}^{N} \left(u - \hat{p}_j\right), \tag{6.1.15}$$

where the index of leftmost product increases in steps of two, up to $N-2$ or $N-1$ for an odd or even number of sites respectively. This can then be expanded order by order to provide a set of conserved quantities

$$\hat{T}_N[u] = \sum_{m=0}^{N} \hat{C}_m u^{N-m}, \tag{6.1.16}$$

the first few of which can easily be extracted

$$\hat{C}_0 = 1, \tag{6.1.17a}$$

$$\hat{C}_1 = -\sum_{j}^{N} \hat{p}_j, \tag{6.1.17b}$$

$$\hat{C}_2 = \sum_{i>j}^{N} \hat{p}_i \hat{p}_j - \sum_{j}^{N} \exp(\hat{q}_j - \hat{q}_{j+1}), \tag{6.1.17c}$$

$$\hat{C}_3 = -\sum_{i>j>k}^{N} \hat{p}_i \hat{p}_j \hat{p}_k + \sum_{i \neq j, j+1}^{N} \hat{p}_i \exp(\hat{q}_j - \hat{q}_{j+1}), \tag{6.1.17d}$$

and correspond exactly with the classical integrals of motion found by Hénon [62]. Given that the quantum and classical conserved quantities are the same, the next logical step is to consider a much more elegant method of generating them from the classical model.

6.2 The Classical Integrals of Motion

This section very briefly deals with Flaschka's method of generating the integrals of motion of the Toda chain via a Lax pair formalism [64]. The first step is to introduce a new set of variables

$$\hat{a}_n \;=\; e^{-\frac{1}{2}\left(\hat{q}_n - \hat{q}_{n-1}\right)}, \qquad \hat{b}_n \;=\; -\hat{p}_{n-1}, \tag{6.2.1}$$

where the periodic boundary conditions are still in effect, so that $\hat{q}_0 = \hat{q}_N$. Using the time derivative of the Hamiltonian (6.0.1), it is then trivial to show that $\hat{a}_n$ and $\hat{b}_n$ satisfy the equations

$$\frac{d\hat{a}_n}{dt} \;=\; \hat{a}_n(\hat{b}_{n+1} - \hat{b}_n), \quad \text{and} \quad \frac{d\hat{b}_n}{dt} \;=\; \hat{a}_n^2 - \hat{a}_{n-1}^2. \tag{6.2.2}$$

In terms of these variables, a pair of matrices can be defined

$$L \;=\; \begin{pmatrix} \hat{b}_1 & \hat{a}_1 & 0 & \dots & \hat{a}_N \\ \hat{a}_1 & \hat{b}_2 & \hat{a}_2 & & \\ 0 & \hat{a}_2 & \hat{b}_3 & & \\ \vdots & & & \ddots & \hat{a}_{N-1} \\ \hat{a}_N & & & \hat{a}_{N-1} & \hat{b}_N \end{pmatrix}, \; B \;=\; \begin{pmatrix} 0 & \hat{a}_1 & 0 & \dots & -\hat{a}_N \\ -\hat{a}_1 & 0 & \hat{a}_2 & & \\ 0 & -\hat{a}_2 & 0 & & \\ \vdots & & & \ddots & \hat{a}_{N-1} \\ \hat{a}_N & & & \hat{a}_{N-1} & 0 \end{pmatrix}, \tag{6.2.3}$$

where the commutator of the two matrices is equal to the time derivative of the first, *i.e.*

$$\dot{L} \;=\; \big[B, L\big]. \tag{6.2.4}$$

The above matrix relation recreates the equations (6.2.2), and so the matrices L and B form a *Lax pair* for the system. As was seen in Chapter 1, this not only proves the integrable nature of the model, but also provides several methods for generating the conservation laws of the system.

Since the eigenvalues of the above Lax matrix L are conserved quantities (see Chapter 1), the coefficients of the characteristic polynomial

$$P(\lambda) \;=\; \lambda^N + I_1\lambda^{N-1} + I_2\lambda^{N-2} + \cdots + I_N, \tag{6.2.5}$$

are also conserved. In fact, these coefficients I_n are the integrals of motion constructed by Hénon [62, 64]

$$I_n \;=\; \sum \hat{p}_{i_1}\hat{p}_{i_2}\cdots\hat{p}_{i_k}\big(-\hat{X}_{j_1}\big)\big(-\hat{X}_{j_2}\big)\cdots\big(-\hat{X}_{j_l}\big), \quad \text{where} \quad \hat{X}_j \;=\; \hat{a}_j^2. \tag{6.2.6}$$

The sum is performed such that each of the site indices 1 through n appears exactly once (either explicitly or implicitly in a factor of $\hat{X}$) and terms which differ only in the order of factors are considered the same, and so only one of them appears in the sum. Setting $n = 0, 1, 2, 3$ and applying these rules for the summation, it is then simple to verify that the first four integrals of motion are indeed the same as the conservation laws (6.1.17a) to (6.1.17d) generated in the previous section.

When considering the infinite lattice, Hénon constructed a second set of integrals, similar to the above I_n but with an additional sum involving some complicated combinatorics which restrict the eventual integrals to the sum over a single site index [62]. These new conserved quantities are the *local* conservation laws of the model which have a very simple relationship with the classical L-matrix [64]

$$n\hat{J}_n \;=\; (-2)^n\; tr\big\{L^n\big\} \quad \text{for} \quad m < N, \tag{6.2.7a}$$

$$N\hat{J}_N \;=\; (-2)^N\; tr\big\{L^n\big\} - 2N(-1)^N. \tag{6.2.7b}$$

Whilst the above method is very elegant and efficient at generating the quantities I_n and $\hat{J}_n$, it unfortunately offers no insight as to the nature of these conserved quantities, and the above relation between the matrix L and the local conservation laws is almost as mysterious as the conservation laws themselves.

6.3 A Minor Improvement

In this final section, the previous method is made mildly easier by the introduction of a slightly simpler initial matrix. Related to Flaschka's L-matrix, this new matrix $\tilde{L}_N$ is shown to generate the local conservation laws in an identical manner, and is then used to create the trace of the Monodromy matrix (6.1.8).

Taking an identical form to L but with simpler off-diagonal terms, this new matrix is

$$\tilde{L}_N = \begin{pmatrix} \hat{p}_1 & e^{\hat{q}_1} & 0 & \dots & e^{-\hat{q}_1} \\ e^{-\hat{q}_2} & \hat{p}_2 & e^{\hat{q}_2} & & \\ 0 & e^{-\hat{q}_3} & \hat{p}_3 & & \\ \vdots & & & \ddots & e^{\hat{q}_{N-1}} \\ e^{\hat{q}_N} & & & e^{-\hat{q}_N} & \hat{p}_N \end{pmatrix}, \tag{6.3.1}$$

and is closely related to the classical L-matrix (6.2.3). To see this one needs first to consider the effect of changing the sign of the variable $\hat{b}_n$, *i.e.* $\hat{b}_n = \hat{p}_{n-1}$. Keeping $\hat{a}_n$ the same, it then follows that the time derivatives of these variables, and by extension $\dot{L}$, also change sign. At this point, one should be concerned about the status of this new L as part of a Lax pair, since $\dot{L}$ has changed sign entirely while L has only changed down the leading diagonal. It is elementary however, to show that satisfying the Lax relation (6.2.4) with this new definition of L requires only that $B \to -B$ [68]. The proposed new

Lax matrix L_N is then related to this sign altered L by the transformation

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & \alpha_1^{-1} \\ \alpha_2^{-1} & 0 & 0 & & \\ 0 & \alpha_3^{-1} & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & \alpha_N^{-1} & 0 \end{bmatrix} \tilde{L}_N \begin{bmatrix} 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_3 & & \\ 0 & 0 & 0 & & \\ \vdots & & & \ddots & \alpha_N \\ \alpha_1 & & & & 0 \end{bmatrix} = L \tag{6.3.2}$$

where the non-zero elements of the (unitary) transformation operators are given by

$$\alpha_i = e^{\frac{1}{2}\hat{q}_{i-1}} \prod_{j=1}^{i-1} e^{-\hat{q}_j} \qquad \text{for } i \geq 2, \quad \alpha_1 = e^{\frac{1}{2}\hat{q}_N} \prod_{j=1}^{N} e^{-\hat{q}_j}. \tag{6.3.3}$$

Inverting the above transformation and writing it as $\tilde{L}_N = ULU^{-1}$, and then taking the time derivative of both sides finds

$$\begin{aligned} \partial_t \tilde{L}_N &= \dot{U}LU^{-1} + U\dot{L}U^{-1} - ULU^{-2}\dot{U} \\ &= \dot{U}U^{-1}\tilde{L}_N + U\{BL - LB\}U^{-1} - \tilde{L}_N U^{-1}\dot{U} \\ &= \left[\dot{U}U^{-1}, \tilde{L}_N\right] + UBU^{-1}\tilde{L}_N - \tilde{L}_N UBU^{-1} \\ &= \left[\dot{U}U^{-1} + UBU^{-1}, \tilde{L}_N\right], \end{aligned} \tag{6.3.4}$$

and so this new matrix $\tilde{L}_N$ forms a Lax pair with $\tilde{B}_N = \dot{U}U^{-1} + UBU^{-1}$. Note that this U need not be unitary, the above treatment works equally well for any invertible matrix. Since it is also a Lax matrix for the model, $\tilde{L}_N$ will generate the same (or linear combinations of) the conserved quantities calculated from L.

The trace of the Monodromy matrix $\hat{T}_N[u]$ (equation (6.1.8)) can be calculated using the matrix $\tilde{L}_N$ in the following manner

$$\hat{T}_N[u] \equiv \det\left[u - \tilde{L}_N\right] + \left(e^{\hat{q}_1 + \cdots + \hat{q}_N} + e^{-\hat{q}_1 - \cdots - \hat{q}_N}\right), \tag{6.3.5}$$

where the first term should be recognisable as the final method of generating the classical constants of motion given in Chapter 1. From this the local conservation laws of the system can be found using the generating function

$$\begin{aligned} G[u] &= \ln\left(\hat{T}_\infty[u]\right) \\ &= tr\left(\ln\left[u-\tilde{L}_\infty\right]\right) \\ &= \sum_{m=0}^{\infty}\frac{-1}{m}u^{-m}tr\left[\tilde{L}_\infty^m\right], \end{aligned} \tag{6.3.6}$$

where the logarithm has been taken to ensure locality in an analogous way to the previous models. The coefficients of the above expansion correspond directly to Flaschka's $\hat{J}_n$ at the end of the last section. Note the limit $N \to \infty$ is not necessary as the m^{th} conservation law needs only $N > m$ to be exact, *i.e.* there is a correction to the N^{th} law, in a similar manner to $\hat{J}_N$, stemming from the second term of equation (6.3.5), which is dropped in the infinite limit.

Having shown how the local conservation laws stem from the matrix $\tilde{L}_N$, this section is rounded out with a proof of equation (6.3.5), which demonstrates the relationship between the trace of the Monodromy matrix and the matrix $\tilde{L}_N$. Central to establishing this result is the determinant of the tridiagonal $(n-m+1)$-square matrix

$$\hat{D}_{m,n} \equiv \det\begin{bmatrix} u-\hat{p}_m & -e^{\hat{q}_m} & & & \\ -e^{-\hat{q}_{m+1}} & u-\hat{p}_{m+1} & & & \\ & & \ddots & & \\ & & & u-\hat{p}_{n-1} & -e^{\hat{q}_{n-1}} \\ & & & -e^{-\hat{q}_n} & u-\hat{p}_n \end{bmatrix}. \tag{6.3.7}$$

Expanding the first term in equation (6.3.5) in terms of this determinant finds

$$\begin{aligned}
\det\left[u-\hat{L}_N\right] &= \left(u-\hat{P}_1\right)\hat{D}_{2,N} - e^{\hat{q}_1}\Big(e^{-\hat{q}_2}\hat{D}_{3,N} + (-1)^{N-2}e^{\hat{q}_N}\det\begin{bmatrix} -e^{\hat{q}_2} & 0 & \cdots \\ u-\hat{p}_3 & -e^{\hat{q}_3} & \\ \vdots & & \ddots \end{bmatrix}\Big) \\
&\quad + (-1)^{N-1}e^{-\hat{q}_1}\Big(e^{-\hat{q}_2}\det\begin{bmatrix} -e^{-\hat{q}_3} & u-\hat{p}_3 & \cdots \\ 0 & -e^{-\hat{q}_4} & \\ \vdots & & \ddots \end{bmatrix} + (-1)^{N-2}e^{\hat{q}_N}\hat{D}_{2,N-1}\Big) \\
&= \left(u-\hat{P}_1\right)\hat{D}_{2,N} - e^{\hat{q}_1-\hat{q}_2}\hat{D}_{3,N} \;-\; (-1)^{N-2}e^{\hat{q}_1+\hat{q}_N}\prod_{j=2}^{N-1}-e^{\hat{q}_j} \\
&\qquad + (-1)^{N-1}e^{-\hat{q}_1-\hat{q}_2}\prod_{j=3}^{N}-e^{-\hat{q}_j} \;+\; (-1)^{N-1}(-1)^{N-2}e^{-\hat{q}_1+\hat{q}_N}\hat{D}_{2,N-1} \\
&= \hat{D}_{1,N} - e^{\hat{q}_N-\hat{q}_1}\hat{D}_{2,N-1} - \prod_{j=1}^{N}e^{\hat{q}_j} - \prod_{j=1}^{N}e^{-\hat{q}_j}, \qquad (6.3.8)
\end{aligned}$$

and so trace of the Monodromy matrix may be written as

$$\hat{T}_N \;=\; \hat{D}_{1,N} - e^{\hat{q}_N-\hat{q}_1}\hat{D}_{2,N-1}. \qquad (6.3.9)$$

Expanding the determinant $\hat{D}_{m,n}$ about its top row, so that it can be written in terms of lower order determinants, yields the recurrence relation

$$\hat{D}_{m,n} \;=\; \left(u-\hat{p}_m\right)\hat{D}_{m+1,n} - e^{\hat{q}_m-\hat{q}_{m+1}}\hat{D}_{m+2,n}, \qquad (6.3.10)$$

which can then be reformulated as the following matrix equation

$$\begin{bmatrix} \hat{D}_{m,n} \\ \hat{D}_{m+1,n} \end{bmatrix} \;=\; \begin{bmatrix} u-\hat{p}_m & -e^{\hat{q}_m-\hat{q}_{m+1}} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{D}_{m+1,n} \\ \hat{D}_{m+2,n} \end{bmatrix} \;=\; \hat{K}_m\begin{bmatrix} \hat{D}_{m+1,n} \\ \hat{D}_{m+2,n} \end{bmatrix}, \qquad (6.3.11)$$

where the resultant matrix $\hat{K}_m$ looks remarkably similar to the Lax operator $\hat{L}_m$ defined at the start of the chapter.

Expanding out the right-hand side of this matrix equation as a product of the $\hat{K}$ matrices up to the final column vector, which can be thought of as

$$\begin{bmatrix} \hat{D}_{n-1,n} \\ \hat{D}_{n,n} \end{bmatrix} = \hat{K}_{n-1} \begin{bmatrix} u - \hat{p}_n \\ 1 \end{bmatrix} = \hat{K}_{n-1}\hat{K}_n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{6.3.12}$$

and remembering that $\hat{q}_{N+1} = \hat{q}_1$, the recurrence relation can then be solved to find

$$\hat{D}_{m,n} \equiv \begin{bmatrix} 1, 0 \end{bmatrix} \prod_{r=m}^{n} \hat{K}_r \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{6.3.13}$$

Substituting this product form of $\hat{D}_{m,n}$ back into equation (6.3.9), the trace of the Monodromy matrix becomes

$$\begin{aligned} \hat{T}_N[u] &= \begin{bmatrix} 1, 0 \end{bmatrix} \prod_{r=1}^{N} \hat{K}_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{\hat{q}_N - \hat{q}_1} \begin{bmatrix} 1, 0 \end{bmatrix} \prod_{r=2}^{N-1} \hat{K}_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1, 0 \end{bmatrix} \prod_{r=1}^{N} \hat{K}_r \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0, 1 \end{bmatrix} \hat{K}_1 \prod_{r=2}^{N-1} \hat{K}_r \hat{K}_N \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= tr \prod_{r=1}^{N} \hat{K}_r, \end{aligned} \tag{6.3.14}$$

where rewriting the second term requires the use of the identities

$$\begin{bmatrix} 0, 1 \end{bmatrix} \hat{K}_1 = \begin{bmatrix} 0, 1 \end{bmatrix} \begin{bmatrix} u - \hat{p}_1 & -e^{\hat{q}_1 - \hat{q}_2} \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1, 0 \end{bmatrix}, \tag{6.3.15}$$

at the front, and

$$\hat{K}_N \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} u - \hat{p}_N & -e^{\hat{q}_N - \hat{q}_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -e^{\hat{q}_N - \hat{q}_1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.3.16}$$

at the end. It is now a trivial matter to relate the matrix $\hat{K}_m$ from equation (6.3.11) to

the original Lax operator (6.1.1)

$$\hat{L}_m \equiv \begin{bmatrix} u-\hat{p}_m & e^{\hat{q}_m} \\ -e^{-\hat{q}_m} & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & -e^{-\hat{q}_m} \end{bmatrix} \hat{K}_m \begin{bmatrix} 1 & 0 \\ 0 & -e^{\hat{q}_{m+1}} \end{bmatrix}. \tag{6.3.17}$$

Due to the cyclic nature of the trace, rearranging the above relation for $\hat{K}_m$ and substituting it back into equation (6.3.14) returns

$$\hat{T}_N[u] = tr \prod_{r=1}^{N} \hat{K}_r = tr \prod_{r=1}^{N} \hat{L}_r, \tag{6.3.18}$$

i.e. the trace of the Monodromy matrix as defined in the previous chapter, meaning that equation (6.3.5) and equation (6.1.8) are equivalent.

6.4 Summary

The Toda chain presents a very different type of model from those previously considered, not just in its style of Hamiltonian, but in the methods used to construct its conserved quantities and the relation of the quantum and classical systems. Since the quantum Lax operator (6.1.1) does not have the property (3.3.1), the method used in Chapters 4 and 5 is no longer applicable. Instead a different treatment is used which involves splitting the Lax operator into the sum of its diagonal (momentum operator) and off-diagonal (position operator) components. It is shown that for the product of many such operators, the trace is the same as the sum of the product of the diagonal components, with from zero up to a half-N applications of the operator (6.1.12), where each application involves different site indices. The coefficients of the resultant polynomial are the same as the well known non-local conservation laws of the classical model. The classical model is then considered and the previously calculated conservation laws are found from the characteristic polynomial of the Lax matrix L. The *local* conservation laws are then noted as being equivalent to the trace of different powers of L. Finally, an alternate Lax pair is formed and a relation

given between the new Lax matrix $\tilde{L}_N$ and the trace of the Monodromy matrix which is used to show the equivalence of the logarithm of this trace with the trace of the different powers of $\tilde{L}_N$ (and by extension, L), thus establishing the locality of the conservation laws produced.

Chapter 7

Summary and Conclusions

Classically, an integrable system is defined as one which has an integral of motion for each coordinate-momentum pair of degrees of freedom, all in involution. These integrals of motion restrict the allowed trajectories in phase space down to the handles of an N-torus (N being the number of integrals / half the number of degrees of freedom).

For a quantum integrable system, the analogue is to have a number of *families* of conservation laws (related by their projection operator) equal to the number of states of the system. The analogue to the classical restriction of phase space is then to simultaneously diagonalise *enough* of the conservation laws of the system, so that the Hamiltonian is *fully* diagonalised.

The requirement that the conservation laws of the quantum system form a complete mutually commuting set should *not* be included in the definition of integrability. This fact was demonstrated during the investigation of persistent currents in Chapter 2. These persistent (or long-time residual) currents are a phenomena associated with systems which are integrable or close to integrable (*e.g.* could be transformed into an integrable system by the tuning of some parameter): generally, if a current is started in a system, it would be expected to decay over time until it eventually vanishes. In integrable systems however, part of the current may indeed decay away, but it is possible to have a finite proportion that will continue flowing *indefinitely*[1] [1].

[1]In the case of systems close to integrability in parameter space, an anomalously slow decaying of

It was shown by Suzuki [2] that the time average of a correlation function could be expressed in terms of the canonical averages involving the conserved quantities of a system. This implies that the long-time residual of a current can be thought of as the conserved part of the current operator, which was shown to be true in Chapter 2. A major element of Suzuki's work was Mazur's inequality [4], which provides a lower bound for the time average of a correlation function. Since the long-time residual can be calculated in such a manner, Mazur's inequality can be used as a test: a single conservation law giving a non-trivial minimum value is sufficient to prove the existence of a persistent current[2]. Suzuki argues that it should be possible to calculate the long-time residual current in terms of the conservation laws of the system. There is an assumption however, that these conservation laws should form a complete mutually commuting set, and this assumption has caused the conundrum considered in the second half of Chapter 2.

In the particular case of the anisotropic Heisenberg model with an external magnetic field (or XXZh model), there is a current which has a calculable, non-trivial long-time residual [3]. It can be shown however, that at zero field *all* of the conservation laws yield a vanishing contribution to this residual current [1]. Whilst considering the ability of a given set of conservation laws to generate the full residual current, it was discovered that it was entirely feasible that the Hamiltonian and a set of mutually commuting conservation laws could be simultaneously diagonalised by a basis in which the conserved part of the current operator is off-diagonal. Essentially, it is possible for the relevant conservation laws to be non-abelian, and thus for some portion, or indeed all, of the conserved part of the current operator to be inaccessible to a mutually commuting set of conservation laws.

This was not the case in the system in question however, as it was then shown that simply including the z-component of spin in combination with the existing conservation laws was sufficient to generate the full long-time residual current. The above ideas could be seen in action when considering the pure Heisenberg (XXX) model however, specifically

currents has been observed.

[2]Any one conservation law giving zero as a lower bound however, is not enough to show the residual current does not exist.

the *compatibility* of the current with the set of conservation laws. As with the anisotropic case, including the z-component of spin will give the full conserved current, but if the x-component were to be used instead then it becomes impossible to generate any of the long-time residual. In this case the current operator is not *compatible* with the mutually commuting set of conservation laws and they can generate *none* of the residual current, *i.e.* the conservation laws corresponding to the persistent current are not part of the mutually commuting set.

Having considered the relationship between conservation laws and the definition of integrability, and their use in evaluating the time average of correlation functions, the natural next step was to consider how to generate the analytic forms of these conservation laws. First, the existing method of generating the local conservation laws of an integrable system was described and evaluated, using the XXZ model as an example.

The Boost operator method constructs the local conservation laws iteratively by commuting the eponymous operator with the Hamiltonian and using the result as the first conservation law. The Boost operator is then commuted with this new conservation law and the result used as the second conservation law, and so on, the commutator of each new conservation law and the Boost operator providing the next conservation law [5]. This appears to be a straightforward and simple process, but it is shown in Chapter 3 that knowledge of the structure of the Hamiltonian used to initiate the process is required for the method to work.

The problem with the method is this: when commuted with each new conservation law, the Boost method produces terms whose coefficient is the site index. As was describe previously, these terms are extremely *unphysical* and do not form part of the desired conservation laws. For each new conservation law, it is possible to show that these unphysical terms cancel out, but to do so it necessary either to expand out commutators in the Hamiltonian explicitly, or to have prior knowledge of identities involving these commutators (the Reshetikhin condition for the model being the major one). An attempt to use the Boost method for some general Hamiltonian would therefore fail before even

producing the second conservation law, since one would not have sufficient knowledge to cancel the unphysical terms. It would therefore be necessary to investigate a particular model to discover its commutator properties and identities, and then go through the increasingly long-winded process of using these identities to cancel these terms and find each new conservation law. This would all then need repeating should one wish to find the conservation laws of a different model.

In response to the failings of the Boost operator, a new method of generating the local conservation laws of integrable systems has been developed. The method centres on the idea of the Transfer matrix, which is found as the trace of a product of the Lax operators which are associated with these systems. The logarithm of this matrix is known to provide the local conservation laws of a model as the coefficients of a polynomial in the spectral parameter, and the new method essentially involves a simple way to analytically evaluate this logarithm.

The method works by removing the permutation aspect of the Lax operator and writing the remainder as an exponential of some function of the spectral parameter (essentially the Hamiltonian plus some corrections). A side effect of separating off the permutation part of the operators is that it effectively removes statistics from the problem, showing that they do not affect the local conservation laws. The problem is reduced to taking the logarithm of a product of exponentials, which can be handled in a straightforward manner with the Campbell-Baker-Hausdorff identity. Though technically simple, this step is where the majority of the work for this method is required. Separating out and solving the various recurrence relations seen in Chapter 3 is simple for the first few orders, but rapidly becomes arduous in the extreme, and to tackle orders higher than those included here will required some clever and powerful computer programming. Whilst the effort required at this stage cannot be overstressed, neither can the upside: the work up to this stage was entirely general and thus only needs to be done *once*. When considering a specific model, it is necessary only calculate the exact form of the function the Lax operator is written as an exponential of, and then substitute this back into the coefficients already

found by solving the recurrence relations.

This is shown for two examples in Chapters 4 and 5 for the XXZ and Hubbard models respectively. Each time, the Lax operator is rewritten and the model specific operator ($\hat{V}_{ij}$) found. This operator and the coefficients from Chapter 3 are then combined to almost trivially produce the first few conservation laws of each system, as far as (and in agreement with) those previously calculated, but without having to deal with any unphysical terms or worry about statistics. As a side note it is shown that with this new exponential form of the Lax operator, the Reshetikhin condition appears as a natural consequence of the Yang-Baxter equations (for fundamental models at least).

Finally, in an attempt to move away from spin systems and prove the methods total generality, the Toda lattice was considered. This turned out to be a poor choice since the form of the quantum Lax operator for model meant it was not compatible with the treatment used in the previous chapters. This may have something to do with the highly unusual fact that the conservation laws of the quantum system are the same as the classical integrals of motion. This has been shown explicitly and in some sense explained by demonstrating that the trace of the quantum Monodromy matrix can be generated from the classical Lax matrix.

There are two natural, if somewhat uninspired, avenues for the continuation of this investigation. Firstly, it may be of interest to continue the generating of the conservation laws of the Heisenberg and Hubbard models. Calculation of the next two for the Hubbard model will be a painfully long-winded process, but technically simple as the necessary recurrence relations have already been solved and so all that is required is to substitute in the operator $\hat{V}_{ij}$ found in Chapter 5. Beyond this, further work (on either model) necessitates looking into the possibility of using computer programs to generate and solve the higher order recurrence relations, and to then deal with the multitude of commutators which result from substituting in $\hat{V}_{ij}$. One would also have to consider the issue of how to represent these new conservation laws as the number of terms involved is growing quite rapidly, see for example the sixth conservation law of the XXZ model in Appendix B.

The hope here is that creating more conservation laws will aid in identifying any patterns that exist, either patterns that are common to all conservation laws or ones the emerge later as the range of interaction of these conserved operators increases.

In a similar vein, the second logical extension is to widen the scope of the investigation by considering more integrable models, and attempting to calculate their local conservation laws with this new method. The aim here is not just to go from model to model generating conservation laws, but to compare the models at every step: initial and permutationless forms of their Lax operators; the form of the operator $\hat{V}_{ij}$ in each case, and of course the eventual conservation laws. The most interesting cases may well end up being those where, as with the Toda lattice, the new method cannot be applied. It may even be possible that by studying these systems one can construct an alternative treatment which can be used when the method of Chapter 3 fails.

Appendix A

The Campbell-Baker-Hausdorff Expansion

A key element of this work is how to deal with the product of exponentials when the exponents do not commute, *i.e.* to find $\hat{Z}$ such that

$$\hat{Z} \;=\; \ln\left[e^{\hat{A}}e^{\hat{B}}\right] \quad \text{when} \quad \left[\hat{A},\hat{B}\right] \;\neq\; 0. \tag{A.0.1}$$

The solution of $\hat{Z}$ as an infinite series of terms of increasingly higher orders of commutations of $\hat{A}$ and $\hat{B}$ is known as the Campbell-Baker-Hausdorff expansion, and this section outlines one[1] of the methods of generating it.

Introducing a scaling parameter, t (which will eventually be set to unity), so that

$$e^{\hat{Z}(t,\hat{A},\hat{B})} \;=\; e^{t\hat{A}}\; e^{t\hat{B}}, \tag{A.0.2}$$

then the following three Lemmata are used to to write $\hat{Z}$ as an expansion in powers of t, where the coefficients are functions of commutators of $\hat{A}$ and $\hat{B}$.

[1] the second algorithm described in [69]

Lemma 1

$$\delta e^{\hat{Z}} \;=\; \int_0^1 ds\; e^{(1-s)\hat{Z}}\;\delta\hat{Z}\;e^{s\hat{Z}},$$

for any non-commutative $\hat{Z}$, and some operator δ.

Lemma 2

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} \;=\; e^{\;[\hat{A}}\hat{B},$$

is a well known combinatoric identity, and by right multiplying both sides of lemma 1 by $e^{-\hat{Z}}$ and using lemma 2

$$\begin{aligned}\left(\delta e^{\hat{Z}}\right)e^{-\hat{Z}} &= \left(\int_0^1 ds\; e^{(1-s)\hat{Z}}\;\delta\hat{Z}\;e^{s\hat{Z}}\right)e^{-\hat{Z}} = e^{\hat{Z}}\left(\int_0^1 ds\; e^{-s\hat{Z}}\;\delta\hat{Z}\;e^{s\hat{Z}}\right)e^{-\hat{Z}} \\ &= e^{\hat{Z}}\left(\int_0^1 ds\; e^{-s[\hat{Z}}\;\delta\hat{Z}\;\right)e^{-\hat{Z}} \quad = e^{\hat{Z}}\left(\frac{-1}{[\hat{Z}}e^{-[\hat{Z}}\;\delta\hat{Z}+\frac{1}{[\hat{Z}}\;\delta\hat{Z}\right)e^{-\hat{Z}} \\ &= e^{\hat{Z}}\left(\frac{1-e^{-[\hat{Z}}}{[\hat{Z}}\;\delta\hat{Z}\right)e^{-\hat{Z}} \qquad = e^{\;[\hat{Z}}\frac{1-e^{-[\hat{Z}}}{[\hat{Z}}\;\delta\hat{Z},\end{aligned}$$

which leads to

Lemma 3

$$\left(\delta e^{\hat{Z}}\right)e^{-\hat{Z}} \;=\; \frac{e^{[\hat{Z}}-1}{[\hat{Z}}\;\delta\hat{Z}.$$

To generate the Campbell-Baker-Hausdorff expansion, use equation (A.0.2) and the choice of operator $\delta=\partial_t$ in the left hand side of lemma 3

$$\begin{aligned}\left(\partial_t e^{t\hat{A}}\;e^{t\hat{B}}\right)\left(^{t\hat{A}}\;e^{t\hat{B}}\right)^{-1} &= \hat{A}e^{t\hat{A}}\;e^{t\hat{B}}\left(e^{t\hat{A}}\;e^{t\hat{B}}\right)^{-1}+e^{t\hat{A}}\hat{B}e^{t\hat{B}}\left(e^{t\hat{A}}\;e^{t\hat{B}}\right)^{-1} \\ &= \hat{A}+e^{t\hat{A}}\hat{B}e^{-t\hat{A}},\end{aligned}$$

and then using lemma 2 gives

$$\hat{A} + e^{t\ [\hat{A}}\hat{B} \;=\; \frac{e^{\ [\hat{Z}} - 1}{[\hat{Z}}\ \delta\hat{Z}. \tag{A.0.3}$$

Writing $\hat{Z}$ as an expansion in powers of the scaling parameter

$$\hat{Z}(t) \;\equiv\; \sum_{n=1}^{\infty} t^n Z_n, \tag{A.0.4}$$

the Z_n can be found by evaluating the coefficients of the powers of t on each side of equation (A.0.3). Writing everything in power expansion form

$$\hat{A} + \hat{B} + \sum_{n=1}^{\infty} \frac{t^n}{n!} \Big(\Big[\hat{A}, \Big)^n \hat{B} \ \Big]^n \;=\; \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \left(\Big[\sum_{p=1}^{\infty} t^p Z_p, \right)^m \sum_{q=1}^{\infty} q t^{q-1} Z_q \ \Big]^m \tag{A.0.5}$$

which needs to be solved order by order as each new Z_n will be a function of all the previous ones. Note that solving at order t^n will give the Z^{n+1}th coefficient. Writing out each side explicitly up to the quadratic terms

$$\begin{aligned}
& \hat{A} + \hat{B} + \Big[\hat{A}, \hat{B}\Big] t + \frac{1}{2}\Big[\hat{A}, \Big[\hat{A}, \hat{B}\Big]\Big] t^2 \\
= \;& Z_1 + 2Z_2 t + 3Z_3 t^2 + \cdots + \frac{1}{2}\Big(\Big[Z_1 t + Z_2 t^2 + \cdots, Z_1 + 2Z_2 t + \cdots \Big] \Big) \\
& + \frac{1}{3!}\Big(\Big[Z_1 t + \cdots, \Big[Z_1 t + \cdots, Z_1 + \cdots \Big] \Big) + O(t^3) \\
= \;& Z_1 + t\left(2Z_2 + \frac{1}{2}\Big[Z_1, Z_1\Big] \right) + t^2 \left(3Z_3 + \frac{1}{2}\Big(\Big[Z_1, 2Z_2\Big] + \Big[Z_2, Z_1\Big] \Big) + \frac{1}{3!}\Big[Z_1, Z_1, Z_1\Big] \right).
\end{aligned}$$

The first three coefficients can be read off easily and so substituting them back into equation (A.0.4)

$$\hat{Z}(t) \;=\; \hat{A} + \hat{B} + \frac{1}{2}\Big[\hat{A}, \hat{B}\Big] t + \frac{1}{12}\Big(\Big[\hat{A}, \Big[\hat{A}, \hat{B}\Big]\Big] + \Big[\Big[\hat{A}, \hat{B}\Big], \hat{B}\Big] \Big) t^2 + \mathcal{O}(t^3) \tag{A.0.6}$$

and then setting $t = 1$, these are the first few terms of the Campbell-Baker-Hausdorff expansion. For reference, the full expansion up to and including the quintuple commutator

terms (the highest order used here) is

$$\begin{aligned}
\hat{Z} \;=\; & \hat{A}+\hat{B}+\frac{1}{2}\Big[\hat{A},\hat{B}\Big]+\frac{1}{12}\Big(\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]+\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big)+\frac{1}{24}\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big] \\
& +\frac{1}{120}\Bigg(\Big[\hat{A},\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big]\Big]+\Big[\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big],\hat{B}\Big] \\
& \quad -\frac{1}{3}\Bigg\{\Big[\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]\Big],\hat{B}\Big]+\Big[\hat{A},\Big[\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big],\hat{B}\Big]\Big]\Bigg\} \\
& \quad -\frac{1}{6}\Bigg\{\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]\Big]\Big]+\Big[\Big[\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big],\hat{B}\Big],\hat{B}\Big]\Bigg\}\Bigg) \\
& +\frac{1}{480}\Bigg(\Big[\hat{A},\Big[\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big],\hat{B}\Big]\Big]+\Big[\Big[\hat{A},\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big]\Big],\hat{B}\Big] \\
& \quad -\frac{1}{3}\Bigg\{\Big[\hat{A}\Big[\hat{A},\Big[\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big],\hat{B}\Big]\Big]\Big]+\Big[\Big[\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]\Big],\hat{B}\Big],\hat{B}\Big]\Bigg\} \\
& \quad -\frac{1}{3}\Bigg\{\Big[\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]\Big]\Big],\hat{B}\Big]+\Big[\Big[\hat{A},\Big[\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big],\hat{B}\Big]\Big],\hat{B}\Big]\Bigg\}\Bigg) \\
& +\cdots . \qquad\qquad (A.0.7)
\end{aligned}$$

An obvious extension to the above result, which is used briefly in Chapter 5, is to consider the product of three exponential terms; *i.e.* to find $\hat{Z}$ when

$$e^{\hat{Z}} \;=\; e^{\hat{A}}e^{\hat{B}}e^{\hat{C}}. \qquad\qquad (A.0.8)$$

The solution to this can be swiftly constructed from the above result for two exponents, by considering the problem as

$$e^{\hat{Z}} \;=\; e^{\hat{A}}e^{\hat{B}'} \qquad \text{with} \qquad e^{\hat{B}'} \;=\; e^{\hat{B}}e^{\hat{C}}, \qquad\qquad (A.0.9)$$

and then $\hat{Z}$ can be written as the standard Campbell-Baker-Hausdorff expansion, but with every instance of $\hat{B}'$ being replaced with it's own expansion. Grouping the terms by the total number of commutators / operators, and rewriting and combining terms where possible, finds

$$\hat{Z} \;=\; \hat{A}+\hat{B}+\hat{C}+\frac{1}{2}\Big(\Big[\hat{A},\hat{B}\Big]+\Big[\hat{A},\hat{C}\Big]+\Big[\hat{B},\hat{C}\Big]\Big)+\frac{1}{6}\Big(\Big[\hat{A},\Big[\hat{B},\hat{C}\Big]\Big]+\Big[\Big[\hat{A},\hat{B}\Big],\hat{C}\Big]\Big)$$

$$+\frac{1}{12}\Big(\Big[\hat{A},\Big[\hat{A},\hat{B}\Big]\Big]+\Big[\hat{A},\Big[\hat{A},\hat{C}\Big]\Big]+\Big[\hat{B},\Big[\hat{B},\hat{C}\Big]\Big]+\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]+\Big[\Big[\hat{A},\hat{C}\Big],\hat{C}\Big]$$
$$+\Big[\Big[\hat{B},\hat{C}\Big],\hat{C}\Big]\Big)+\frac{1}{24}\Big(\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{B}\Big]\Big]+\Big[\hat{A},\Big[\Big[\hat{A},\hat{C}\Big],\hat{C}\Big]\Big]+\Big[\hat{B},\Big[\Big[\hat{B},\hat{C}\Big],\hat{C}\Big]\Big]\Big)$$
$$+\frac{1}{12}\Big(\Big[\hat{A},\Big[\Big[\hat{A},\hat{B}\Big],\hat{C}\Big]\Big]+\Big[\Big[\hat{A},\hat{B}\Big],\Big[\hat{B},\hat{C}\Big]\Big]+\Big[\Big[\hat{A},\Big[\hat{B},\hat{C}\Big]\Big],\hat{C}\Big]\Big)+\cdots. \quad \text{(A.0.10)}$$

Several of the above terms will vanish in the case considered in Chapter 5, however, since it also has the property that $\hat{A}$ and $\hat{C}$ commute.

Appendix B

Sixth Conservation Law of the XXZ Model

Continuing the process of expanding equation (3.3.19) and solving the sixth order recurrence relation gives

$$
\begin{aligned}
\hat{X}_M^{(6)} \;=\; & \sum_{n=0} 120\Big[\hat{V}_n,\hat{V}_{n+1},\hat{V}_{n+2},\hat{V}_{n+3},\hat{V}_{n+4},\hat{V}_{n+5}\Big] \\
& +48\Big[\Big[\Big[\hat{U}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big]+48\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{V}_{n+2},\hat{W}_{n+3}\Big]\Big]\Big] \\
& +36\Big[\Big[\Big[\hat{V}_n,\hat{U}_{n+2}\Big],\hat{V}_{n+3}\Big],\hat{V}_{n+4}\Big]+12\Big[\Big[\Big[\hat{W}_n,\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big],\hat{V}_{n+4}\Big] \\
& +36\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{W}_{n+2},\hat{V}_{n+4}\Big]\Big]\Big]+12\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{V}_{n+2},\hat{U}_{n+4}\Big]\Big]\Big] \\
& +24\Big[\hat{V}_n,\Big[\Big[\hat{W}_{n+1},\hat{V}_{n+3}\Big],\hat{V}_{n+4}\Big]\Big]+24\Big[\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{U}_{n+3}\Big]\Big],\hat{V}_{n+4}\Big] \\
& +15\Big[\Big[\Big[\hat{V}_n,\hat{W}_n\Big],\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big]+6\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{U}_{n+2},\hat{V}_{n+3}\Big]\Big]\Big] \\
& +15\Big[\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{U}_{n+2}\Big]\Big],\hat{V}_{n+2}\Big]+6\Big[\Big[\Big[\hat{V}_n,\hat{W}_{n+1}\Big],\hat{V}_{n+3}\Big],\hat{V}_{n+3}\Big] \\
& +18\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{W}_{n+1},\hat{V}_{n+3}\Big]\Big]\Big]+3\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{U}_{n+3}\Big]\Big]\Big] \\
& +18\Big[\Big[\Big[\hat{V}_n,\hat{U}_{n+2}\Big],\hat{V}_{n+3}\Big],\hat{V}_{n+3}\Big]+3\Big[\Big[\Big[\hat{W}_n,\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big],\hat{V}_{n+3}\Big] \\
& +21\Big[\hat{U}_n,\hat{W}_{n+1}\Big]+24\Big[\Big[\hat{V}_n,\Big[\hat{U}_{n+2},\hat{V}_{n+2}\Big]\Big],\hat{V}_{n+3}\Big] \\
& -6\Big[\Big[\Big[\hat{V}_n,\hat{V}_{n+1}\Big],\hat{W}_{n+1}\Big],\hat{V}_{n+3}\Big]-6\Big[\hat{V}_n,\Big[\hat{U}_{n+2},\Big[\hat{V}_{n+2},\hat{V}_{n+3}\Big]\Big]\Big] \\
& +6\Big[\Big[\hat{W}_n,\hat{V}_{n+2}\Big],\Big[\hat{V}_{n+2},\hat{V}_{n+3}\Big]\Big]+6\Big[\Big[\hat{V}_n,\hat{V}_{n+1}\Big],\Big[\hat{V}_{n+1},\hat{U}_{n+3}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-3\Big[\hat{W}_n,\hat{U}_{n+3}\Big]+6\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{W}_{n+1}\Big]\Big]\Big]\\
&+6\Big[\Big[\Big[\hat{U}_n,\hat{V}_n\Big],\hat{V}_{n+1}\Big],\hat{V}_{n+1}\Big]-3\Big[\hat{U}_n,\hat{W}_n\Big] \qquad \text{(B.0.1)}\\
&+6\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{W}_{n+2},\hat{V}_{n+3}\Big]\Big]\Big]+6\Big[\Big[\Big[\hat{V}_n,\hat{U}_{n+1}\Big],\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big]\\
&+12\Big[\Big[\hat{W}_n,\Big[\hat{V}_{n+1},\hat{V}_{n+2}\Big]\Big],\hat{V}_{n+3}\Big]-6\Big[\Big[\Big[\hat{W}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big],\hat{V}_{n+3}\Big]\\
&+12\Big[\hat{V}_n,\Big[\Big[\hat{V}_{n+1},\hat{V}_{n+2}\Big],\hat{U}_{n+3}\Big]\Big]-6\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{V}_{n+2},\hat{U}_{n+3}\Big]\Big]\Big]\\
&+6\Big[\hat{V}_n,\Big[\Big[\hat{U}_{n+1},\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big]\Big]-2\Big[\Big[\hat{V}_n,\hat{U}_{n+1}\Big],\Big[\hat{V}_{n+1},\hat{V}_{n+1}\Big]\Big]\\
&+6\Big[\hat{V}_n,\Big[\Big[\hat{V}_{n+1},\hat{W}_{n+1}\Big],\hat{V}_{n+2}\Big]\Big]-2\Big[\Big[\hat{V}_n,\hat{V}_{n+1}\Big],\Big[\hat{W}_{n+1},\hat{V}_{n+2}\Big]\Big]\\
&+4\Big[\Big[\hat{V}_n,\Big[\hat{U}_{n+1},\hat{V}_{n+2}\Big]\Big],\hat{V}_{n+2}\Big]+4\Big[\hat{V}_n,\Big[\Big[\hat{V}_n,\hat{W}_{n+1}\Big],\hat{V}_{n+2}\Big]\Big]\\
&+3\Big[\Big[\Big[\hat{V}_n,\hat{W}_n\Big],\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big]+6\Big[\hat{V}_n,\Big[\Big[\hat{V}_n,\hat{V}_{n+1}\Big],\hat{U}_{n+2}\Big]\Big]\\
&-3\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big]-2\Big[\hat{V}_n,\Big[\Big[\hat{W}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big]\Big]\\
&+3\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{U}_{n+2},\hat{V}_{n+2}\Big]\Big]\Big]+6\Big[\Big[\hat{W}_n,\Big[\hat{V}_{n+1},\hat{V}_{n+2}\Big]\Big],\hat{V}_{n+2}\Big]\\
&-3\Big[\Big[\Big[\hat{W}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big],\hat{V}_{n+2}\Big]-2\Big[\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\hat{U}_{n+2}\Big]\Big],\hat{V}_{n+2}\Big]\\
&+3\Big[\hat{V}_n,\Big[\Big[\hat{V}_n,\hat{W}_n\Big],\hat{V}_{n+1}\Big]\Big]-\Big[\hat{V}_n,\Big[\hat{V}_n,\Big[\hat{W}_n,\hat{V}_{n+1}\Big]\Big]\Big]\\
&-\Big[\Big[\hat{V}_n,\Big[\hat{V}_n,\hat{U}_{n+1}\Big]\Big],\hat{V}_{n+2}\Big]-\Big[\hat{V}_n,\Big[\Big[\hat{W}_{n+1},\hat{V}_{n+2}\Big],\hat{V}_{n+2}\Big]\Big]\\
&-\frac{1}{2}\Big[\Big[\Big[\hat{W}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+1}\Big],\hat{V}_{n+2}\Big]-\frac{1}{2}\Big[\hat{V}_n,\Big[\hat{V}_{n+1},\Big[\hat{V}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big]\\
&-\frac{1}{2}\Big[\Big[\hat{V}_n,\Big[\hat{V}_n,\hat{U}_{n+1}\Big]\Big],\hat{V}_{n+1}\Big]-\frac{1}{2}\Big[\hat{V}_n,\Big[\Big[\hat{W}_n,\hat{V}_{n+1}\Big],\hat{V}_{n+1}\Big]\Big], \qquad \text{(B.0.2)}
\end{aligned}
$$

with $\hat{U}$ and $\hat{W}$ as defined in Chapter 3. Only the lowest order of the above coefficient is needed, in which case $\hat{V}$ is just $\hat{h}$, and $\hat{U}$ and $\hat{W}$ become $\hat{k}$ and $\hat{g}$ respectively. As before, there are also contributions from earlier coefficients, namely the second and fourth: the latter contributing triple commutators of three $\hat{h}$s and a single projection operator (*i.e.* every possible option of replacing one of the $\hat{h}$s in equation (4.4.4) with a $\hat{P}$); the former a solitary single commutator of two projection operators (the final term in equation (4.4.2)).

B.1 Identities and Relations

As well as the the reduction identities (3.1.8a) and (3.1.8b) and the relation (3.1.12), the following were also employed in the final step of the calculation of the fifth conservation law, and are required again here.

These identities were found whilst calculating the local conservation laws with the aim of simplifying them as much as possible, and comparing them to previous results. Their existence is to be expected due to the structure of the problem: any two-site operator can be written as a linear combination of the permutation and spin-projection operators, and since all three site operators are products of two two-site operators[1], it is possible to form a basis of three-site operators from these products of two-site ones. Multiplying by further two-site operators will give expressions which still only span three sites, and must thus be expressible as a linear combination of the two- and three-site operators already obtained. Although unproven, it is believed that the identities in Chapter 3 and those below represent *all* the identities that exist for three- and four-site operators.[2]

The first set of identities reduce double commutator terms down to a linear combination of operators

$$\left[\hat{h}_{n+1},\left[\hat{h}_{n+1},\hat{P}_{n+2,n+3}\right]\right] = 2\hat{P}_{n+2,n+3} - 2\hat{P}_{n,n+2}, \tag{B.1.1}$$

$$\left[\left[\hat{P}_{n,n+1},\hat{h}_{n+1}\right],\hat{h}_{n+1}\right] = 2\hat{P}_{n,n+1} - 2\hat{P}_{n,n+2}, \tag{B.1.2}$$

$$\begin{aligned}\left[\hat{P}_{n,n+1},\left[\hat{h}_{n},\hat{h}_{n+1}\right]\right] &= \Delta\hat{h}_{n+1} - \Delta^2\hat{P}_{n,n+1} + \hat{P}_{n,n+2} - \hat{\Sigma}_{n,n+2} \\ &= \left[\hat{h}_{n},\left[\hat{P}_{n},\hat{h}_{n+1}\right]\right],\end{aligned} \tag{B.1.3}$$

$$\begin{aligned}\left[\left[\hat{h}_{n},\hat{h}_{n+1}\right],\hat{P}_{n+1,n+2}\right] &= \Delta\hat{h}_{n} - \Delta^2\hat{P}_{n,n+1} + \hat{P}_{n,n+2} - \hat{\Sigma}_{n,n+2} \\ &= \left[\left[\hat{h}_{n},\hat{P}_{n+1,n+2}\right],\hat{h}_{n+1}\right],\end{aligned} \tag{B.1.4}$$

$$\left[\hat{h}_{n},\left[\hat{h}_{n},\hat{\Sigma}_{n+1,n+3}\right]\right] = (1+\Delta^2)\hat{\Sigma}_{n+1,n+3} + (1-\Delta^2)\hat{P}_{n+1,n+3} - 2\hat{J}_{n+1}, \tag{B.1.5}$$

$$\left[\left[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\right],\hat{h}_{n+2},\right] = (1+\Delta^2)\hat{\Sigma}_{n,n+2} + (1-\Delta^2)\hat{P}_{n,n+2} - 2\hat{J}_{n+1}, \tag{B.1.6}$$

[1] which share a common site *e.g.* $\hat{P}_{1,2}$ and $\hat{\Sigma}_{2,3}$.

[2] but note that the summation over the site index can cause unexpected complications.

where the latter two involve the four-site operator

$$\hat{J}_{n+1} = \Delta\hat{h}_{n,n+3} - (1-\Delta^2)\hat{P}_{n,n+3}. \tag{B.1.7}$$

The second group allows for rearrangement and different representations of the results

$$\begin{aligned}
\Big[\hat{h}_n, \hat{\Sigma}_{n,n+2}\Big] &= -\Delta\Big[\hat{h}_n, \hat{h}_{n+1}\Big] - (1-\Delta^2)\Big(\Big[\hat{h}_n, \hat{P}_{n+1,n+2}\Big] + \Big[\hat{P}_{n,n+1}, \hat{h}_{n+1}\Big]\Big) \\
&= \Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+1}\Big], \qquad \text{(B.1.8)} \\
\Big[\hat{P}_{n,n+1}, \hat{\Sigma}_{n,n+2}\Big] &= -\Big[\hat{h}_n, \hat{h}_{n+1}\Big] + \Delta\Big(\Big[\hat{h}_n, \hat{P}_{n+1,n+2}\Big] + \Big[\hat{P}_{n,n+1}, \hat{h}_{n+1}\Big]\Big) \\
&= \Big[\hat{\Sigma}_{n,n+2}, \hat{P}_{n+1,n+2}\Big], \qquad \text{(B.1.9)} \\
\Big[\hat{J}_{n+1}, \hat{h}_{n+2}\Big] &= -\Delta^2\Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\Big] - (1-\Delta^2)\Big[\hat{P}_{n,n+2}, \hat{h}_{n+2}\Big] \\
&\quad -\Delta(1-\Delta^2)\Big[\hat{\Sigma}_{n,n+2}, \hat{P}_{n+2,n+3}\Big]. \qquad \text{(B.1.10)}
\end{aligned}$$

B.2 The Sixth Local Conservation Law

Collecting together the aforementioned contributions and with repeated application of the various commutator identities and relations mentioned above and in Chapter 3, the sixth local conservation law may written as

$$\begin{aligned}
\hat{L}_6 &= 120\sum_n \Big[\hat{h}_n, \hat{h}_{n+1}, \hat{h}_{n+2}, \hat{h}_{n+3}, \hat{h}_{n+4}, \hat{h}_{n+5}\Big] \\
&\quad -\Delta\Big(\Big[\Big[\Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\Big], \hat{h}_{n+3}\Big], \hat{h}_{n+4}\Big] + \Big[\hat{h}_n, \Big[\hat{h}_{n+1}, \Big[\hat{h}_{n+2}, \hat{\Sigma}_{n+3,n+5}\Big]\Big]\Big] \\
&\quad +\Big[\hat{h}_n, \Big[\Big[\hat{\Sigma}_{n+1,n+3}, \hat{h}_{n+3}\Big], \hat{h}_{n+4}\Big]\Big] + \Big[\Big[\hat{h}_n, \Big[\hat{h}_{n+1}, \hat{\Sigma}_{n+2,n+4}\Big]\Big], \hat{h}_{n+4}\Big]\Big) \\
&\quad +\Delta(1-\Delta^2)\Big(\Big[\hat{P}_n, \Big[\hat{h}_{n+1}, \Big[\hat{h}_{n+2}, \hat{h}_{n+3}\Big]\Big]\Big] + \Big[\Big[\Big[\hat{h}_n, \hat{h}_{n+1}\Big], \hat{h}_{n+2}\Big], \hat{P}_{n+3}\Big] \\
&\quad +\Big[\hat{h}_n, \Big[\hat{P}_{n+1}, \Big[\hat{h}_{n+2}, \hat{h}_{n+3}\Big]\Big]\Big] + \Big[\Big[\Big[\hat{h}_n, \hat{h}_{n+1}\Big], \hat{P}_{n+2}\Big], \hat{h}_{n+3}\Big]\Big) \\
&\quad +\Delta^2\Big[\hat{\Sigma}_{n,n+2}, \hat{\Sigma}_{n+2,n+4}\Big] + \Big[\hat{J}_n, \hat{h}_{n+2}\Big] + \Big[\hat{h}_n, \hat{J}_{n+2}\Big] \\
&\quad +2\Delta^3\Big(\Big[\hat{\Sigma}_{n,n+2}, \hat{h}_{n+2}\Big] + \Big[\hat{h}_n, \hat{\Sigma}_{n+1,n+3}\Big]\Big) + \Delta\Big(\Big[\hat{J}_n, \hat{h}_{n+1}\Big] + \Big[\hat{h}_n, \hat{J}_{n+1}\Big]\Big) \\
&\quad -2\Delta^3(1-\Delta^2)\Big(\Big[\hat{P}_n, \hat{h}_{n+1}\Big] + \Big[\hat{h}_n, \hat{P}_{n+1}\Big]\Big)
\end{aligned}$$

$$
\begin{aligned}
&+20(1+2\Delta)\hat{L}_4 - 80(1+2\Delta^2+3\Delta^4)\hat{L}_2 \\
&+6\sum_n \Delta\Big(\Big[\Big[\hat{\Sigma}_{n,n+2},\Big[\hat{h}_{n+1},\hat{h}_{n+2}\Big]\Big],\hat{h}_{n+3}\Big] \\
&+\Big[\hat{h}_n,\Big[\Big[\hat{h}_{n+1},\hat{h}_{n+2}\Big],\hat{\Sigma}_{n+2,n+4}\Big]\Big] + \Big[\Big[\hat{h}_{n+1},\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big]\Big],\hat{h}_{n+3}\Big] \\
&+\Big[\hat{h}_n,\Big[\Big[\hat{h}_{n+1},\hat{\Sigma}_{n+2,n+4}\Big],\hat{h}_{n+2}\Big]\Big]\Big) - 12\Big(\Big[\hat{J}_n,\hat{h}_{n+2}\Big] + \Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big]\Big) \\
&-12\Delta\Big(\Big[\hat{\Sigma}_{n,n+2},\hat{h}_{n+2}\Big] + \Big[\hat{h}_n,\hat{\Sigma}_{n+1,n+3}\Big]\Big) \\
&+12\Delta(1-\Delta^2)\Big(\Big[\hat{P}_{n,n+2},\hat{h}_{n+2}\Big] + \Big[\hat{h}_n,\hat{P}_{n+1,n+3}\Big]\Big) \\
&+12\Delta(1-\Delta^2)\Big(\Big[\hat{P}_n,\hat{h}_{n+1}\Big] + \Big[\hat{h}_n,\hat{P}_{n+1}\Big]\Big) + 16(1+3\Delta^2)\hat{L}_2. \qquad \text{(B.2.1)}
\end{aligned}
$$

There is a striking (and unexpected) difference between the form of this conservation law and the previous ones. Until now every term of a conservation law had the same coefficient; $(n-1)!$ for the n^{th} conservation law (ignoring linear combinations of lower order conservation laws). This new conservation law bucks the trend and contains several terms which do not fit this pattern, and all attempts at rearrangement into this form have failed. This begs the question what, if anything, do these anomalous terms mean? Whether they are significant and correspond to new effects which become important with increasing lattice length cannot be determined and requires further study and the calculation of higher order conservation laws.

Appendix C

Exact form of $\hat{V}_{ij}$ for the Hubbard Model

In Chapter 5 the Campbell-Baker-Hausdorff expansion was used to find $\hat{V}_{ij}$, where

$$\begin{aligned} \mu\hat{V}_{ij} &= \ln\ \hat{L}_{ij} \\ &= \ln\left(e^{\lambda\hat{U}_i}e^{\frac{\kappa}{2}\hat{T}^H_{ij}}e^{\lambda\hat{U}_j}\right), \end{aligned} \tag{C.0.1}$$

as an expansion in the parameters. It is desirable however, to find the exact form, which can be obtained using the method outlined in the same chapter. Table C.1 lists the sixteen two site states, split up by the number of (fermionic) particles in that state. It also contains the bosonic form of these states (where the two spins on a site represent the presence (down) or not (up) of a spin up and spin down fermion respectively), and the effect of the hopping and potential operators defined at the beginning of Chapter 5. The states have been grouped by (fermionic) particle number, as none of the operators (or combinations of them) will change this and so the calculation of $\hat{V}_{ij}$ can be broken down and considered one particle number subspace at a time.

Tables C.2 and C.3 contain the matrix form of the hopping and potential operators and their various commutators, again divided into the subspaces for differing particle number (note that the subspaces with zero and four particles look exactly the same). Higher order

Table C.1: Two site states of the Hubbard model and the effect of the Hopping and Potential operators on them.

State	fermionic		bosonic		Effect of operator		
	site i	site j	site i	site j	$\hat{T}_{ij}^H$	$\hat{U}_i$	$\hat{U}_j$
$\|1\rangle$	-	-	$\uparrow\uparrow$	$\uparrow\uparrow$	0	$\frac{1}{2}\|1\rangle$	$\frac{1}{2}\|1\rangle$
$\|2\rangle$	$\uparrow$	-	$\downarrow\uparrow$	$\uparrow\uparrow$	$\|3\rangle$	$-\frac{1}{2}\|2\rangle$	$\frac{1}{2}\|2\rangle$
$\|3\rangle$	-	$\uparrow$	$\uparrow\uparrow$	$\downarrow\uparrow$	$\|2\rangle$	$\frac{1}{2}\|3\rangle$	$-\frac{1}{2}\|3\rangle$
$\|4\rangle$	$\downarrow$	-	$\uparrow\downarrow$	$\uparrow\uparrow$	$\|5\rangle$	$-\frac{1}{2}\|4\rangle$	$\frac{1}{2}\|4\rangle$
$\|5\rangle$	-	$\downarrow$	$\uparrow\uparrow$	$\uparrow\downarrow$	$\|4\rangle$	$\frac{1}{2}\|5\rangle$	$-\frac{1}{2}\|5\rangle$
$\|6\rangle$	$\uparrow$	$\uparrow$	$\downarrow\uparrow$	$\downarrow\uparrow$	0	$-\frac{1}{2}\|6\rangle$	$-\frac{1}{2}\|6\rangle$
$\|7\rangle$	$\frac{1}{\sqrt{2}}[(\uparrow\downarrow)_i + (\uparrow\downarrow)_j]$		$\frac{1}{\sqrt{2}}[(\downarrow\downarrow)_i(\uparrow\uparrow)_j + (\uparrow\uparrow)_i(\downarrow\downarrow)_j]$		$2\|8\rangle$	$\frac{1}{2}\|7\rangle$	$\frac{1}{2}\|7\rangle$
$\|8\rangle$	$\frac{1}{\sqrt{2}}[\uparrow_i\downarrow_j + \downarrow_i\uparrow_j]$		$\frac{1}{\sqrt{2}}[(\downarrow\uparrow)_i(\uparrow\downarrow)_j + (\uparrow\downarrow)_i(\downarrow\uparrow)_j]$		$2\|7\rangle$	$-\frac{1}{2}\|8\rangle$	$-\frac{1}{2}\|8\rangle$
$\|9\rangle$	$\frac{1}{\sqrt{2}}[\uparrow_i\downarrow_j - \downarrow_i\uparrow_j]$		$\frac{1}{\sqrt{2}}[(\uparrow\downarrow)_i(\downarrow\uparrow)_j - (\uparrow\downarrow)_i(\downarrow\uparrow)_j]$		0	$-\frac{1}{2}\|9\rangle$	$-\frac{1}{2}\|9\rangle$
$\|10\rangle$	$\frac{1}{\sqrt{2}}[(\uparrow\downarrow)_i - (\uparrow\downarrow)_j]$		$\frac{1}{\sqrt{2}}[(\downarrow\downarrow)_i(\uparrow\uparrow)_j - (\uparrow\uparrow)_i(\downarrow\downarrow)_j]$		0	$\frac{1}{2}\|10\rangle$	$\frac{1}{2}\|10\rangle$
$\|11\rangle$	$\downarrow$	$\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	0	$-\frac{1}{2}\|11\rangle$	$-\frac{1}{2}\|11\rangle$
$\|12\rangle$	$\uparrow\downarrow$	$\uparrow$	$\downarrow\downarrow$	$\downarrow\uparrow$	$\|13\rangle$	$\frac{1}{2}\|12\rangle$	$-\frac{1}{2}\|12\rangle$
$\|13\rangle$	$\uparrow$	$\uparrow\downarrow$	$\downarrow\uparrow$	$\downarrow\downarrow$	$\|12\rangle$	$-\frac{1}{2}\|13\rangle$	$\frac{1}{2}\|13\rangle$
$\|14\rangle$	$\uparrow\downarrow$	$\downarrow$	$\downarrow\downarrow$	$\uparrow\downarrow$	$\|15\rangle$	$\frac{1}{2}\|14\rangle$	$-\frac{1}{2}\|14\rangle$
$\|15\rangle$	$\downarrow$	$\uparrow\downarrow$	$\uparrow\downarrow$	$\downarrow\downarrow$	$\|14\rangle$	$-\frac{1}{2}\|15\rangle$	$\frac{1}{2}\|15\rangle$
$\|16\rangle$	$\uparrow\downarrow$	$\uparrow\downarrow$	$\downarrow\downarrow$	$\downarrow\downarrow$	0	$\frac{1}{2}\|16\rangle$	$-\frac{1}{2}\|16\rangle$

commutators have not been included because they all either vanish or can be written as a linear combination of those already there. It must therefore also be possible to write the right hand side of equation (C.0.1) as a linear combination of the operators in these tables.

Starting with the simplest subspaces, the effect of the Lax operator on the zero and four particle states is

$$\begin{aligned} \hat{L}_{ij}|1\rangle &= \exp\left\{\lambda\hat{U}_i\right\}\exp\left\{\frac{\kappa}{2}\hat{T}_{ij}^H\right\}\exp\left\{\lambda\hat{U}_j\right\}|1\rangle \\ &= e^{\lambda}|1\rangle, \qquad \text{(C.0.2)} \\ \hat{L}_{ij}|16\rangle &= \exp\left\{\lambda\hat{U}_i\right\}\exp\left\{\frac{\kappa}{2}\hat{T}_{ij}^H\right\}\exp\left\{\lambda\hat{U}_j\right\}|16\rangle \\ &= e^{\lambda}|16\rangle. \qquad \text{(C.0.3)} \end{aligned}$$

The only operators to be non-zero in these subspaces are the $\hat{U}_\alpha$, and so trivially

$$\ln\Big[e^{\lambda}\Big] \;=\; \Big[\lambda\Big] \;=\; \lambda\Big(\hat{U}_i+\hat{U}_j\Big),$$

but note that this combination also contributes something to the two particle subspace.

In the subspace of states with a single particle, the Lax operator acts on each state as follows

$$\begin{aligned}
\hat{L}_{ij}|2\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|2\rangle \\
&= \cosh\Big(\frac{\kappa}{2}\Big)|2\rangle + e^{\lambda}\sinh\Big(\frac{\kappa}{2}\Big)|3\rangle, \qquad \text{(C.0.4)}\\
\hat{L}_{ij}|3\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|3\rangle \\
&= \cosh\Big(\frac{\kappa}{2}\Big)|3\rangle + e^{-\lambda}\sinh\Big(\frac{\kappa}{2}\Big)|2\rangle, \qquad \text{(C.0.5)}\\
\hat{L}_{ij}|4\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|4\rangle \\
&= \cosh\Big(\frac{\kappa}{2}\Big)|4\rangle + e^{\lambda}\sinh\Big(\frac{\kappa}{2}\Big)|5\rangle, \qquad \text{(C.0.6)}\\
\hat{L}_{ij}|5\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|5\rangle \\
&= \cosh\Big(\frac{\kappa}{2}\Big)|5\rangle + e^{-\lambda}\sinh\Big(\frac{\kappa}{2}\Big)|4\rangle. \qquad \text{(C.0.7)}
\end{aligned}$$

Here the problem is already block diagonalised, with both blocks the same, and so it is only necessary to calculate the logarithm of the two-by-two matrix

$$\ln\begin{bmatrix} \cosh\left(\frac{\kappa}{2}\right) & e^{-\lambda}\sinh\left(\frac{\kappa}{2}\right) \\ e^{\lambda}\sinh\left(\frac{\kappa}{2}\right) & \cosh\left(\frac{\kappa}{2}\right)\end{bmatrix} \;=\; \frac{1}{2}\begin{bmatrix} 0 & \kappa e^{-\lambda} \\ \kappa e^{\lambda} & 0\end{bmatrix}, \qquad \text{(C.0.8)}$$

and noting from tables C.2 and C.3 that in this subspace, for the same two-by-two blocks, the off-diagonal elements can be written as the following combination of commutators

$$\begin{bmatrix} 0 & 4 \\ 0 & 0\end{bmatrix} \;=\; -\Big\{\Big[\hat{U}_i,\hat{T}^H_{ij}\Big]+\Big[\hat{T}^H_{ij},\hat{U}_j\Big]\Big\}+\hat{T}^H_{ij}+\Big[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\Big], \qquad \text{(C.0.9)}$$

$$\begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} = \Big[\hat{U}_i, \hat{T}^H_{ij}\Big] + \Big[\hat{T}^H_{ij}, \hat{U}_j\Big] + \hat{T}^H_{ij} + \Big[\hat{U}_i, \hat{T}^H_{ij}, \hat{U}_j\Big], \tag{C.0.10}$$

and so in this subspace the operator $\mu\hat{V}_{ij}$ can be written as

$$\begin{aligned} \frac{1}{2}\begin{bmatrix} 0 & \kappa e^{-\lambda} \\ \kappa e^{\lambda} & 0 \end{bmatrix} &= \frac{1}{8}\Big\{\kappa e^{-\lambda}(C.0.9) + \kappa e^{\lambda}(C.0.10)\Big\} \qquad \text{(C.0.11)} \\ &= \frac{\kappa}{4}\sinh\lambda\Big\{\Big[\hat{U}_i, \hat{T}^H_{ij}\Big] + \Big[\hat{T}^H_{ij}, \hat{U}_j\Big]\Big\} + \frac{\kappa}{4}\cosh\lambda\Big\{\hat{T}^H_{ij} + \Big[\hat{U}_i, \hat{T}^H_{ij}, \hat{U}_j\Big]\Big\}. \end{aligned}$$

Note that this combination of operators returns zero in the two particle subspace. In the three particle subspace however, $\hat{U}_\alpha \to -\hat{U}_\alpha$, and so the λs in equation (C.0.8) change sign, as do the single commutators in equations (C.0.9) and(C.0.10). These changes in sign then cancel out when constructing the final relation, and so the above combination of commutators also gives the correct form of $\mu\hat{V}_{ij}$ in the three particle subspace.

This just leaves the two particle space to be dealt with. The states in table C.1 were chosen to ensure $\hat{T}^H_{ij}$ was already almost fully diagonal, so that applying $\hat{L}_{ij}$ to the six states

$$\begin{aligned} \hat{L}_{ij}|6\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|6\rangle \\ &= e^{-\lambda}|6\rangle, \qquad \text{(C.0.12)} \\ \hat{L}_{ij}|7\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|7\rangle \\ &= e^{\lambda}\cosh\Big(\kappa\Big)|7\rangle + \sinh\Big(\kappa\Big)|8\rangle, \qquad \text{(C.0.13)} \\ \hat{L}_{ij}|8\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|8\rangle \\ &= e^{-\lambda}\cosh\Big(\kappa\Big)|8\rangle + \sinh\Big(\kappa\Big)|7\rangle, \qquad \text{(C.0.14)} \\ \hat{L}_{ij}|9\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|9\rangle \\ &= e^{-\lambda}|9\rangle, \qquad \text{(C.0.15)} \\ \hat{L}_{ij}|10\rangle &= \exp\Big\{\lambda\hat{U}_i\Big\}\exp\Big\{\frac{\kappa}{2}\hat{T}^H_{ij}\Big\}\exp\Big\{\lambda\hat{U}_j\Big\}|10\rangle \\ &= e^{-\lambda}|10\rangle, \qquad \text{(C.0.16)} \end{aligned}$$

$$\begin{aligned}\hat{L}_{ij}|11\rangle &= \exp\left\{\lambda\hat{U}_i\right\}\exp\left\{\frac{\kappa}{2}\hat{T}^H_{ij}\right\}\exp\left\{\lambda\hat{U}_j\right\}|11\rangle \\ &= e^{-\lambda}|11\rangle, \end{aligned} \tag{C.0.17}$$

leads to the relatively simple block diagonal matrix

$$\begin{bmatrix} e^{-\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda}\cosh\kappa & \sinh\kappa & 0 & 0 & 0 \\ 0 & \sinh\kappa & e^{-\lambda}\cosh\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\lambda} \end{bmatrix}. \tag{C.0.18}$$

After taking the logarithm, the four pure diagonal terms can be seen to stem from the same combination as was used for the zero and four particle subspaces, which just leaves the two-by-two block to be dealt with

$$\ln\begin{bmatrix} e^{\lambda}\cosh\kappa & \sinh\kappa \\ \sinh\kappa & e^{-\lambda}\cosh\kappa \end{bmatrix} = \frac{\nu}{\sinh\nu}\begin{bmatrix} \sinh\lambda\cosh\kappa & \sinh\kappa \\ \sinh\kappa & -\sinh\lambda\cosh\kappa \end{bmatrix}, \tag{C.0.19}$$

in terms of the new parameter $\cosh\nu = \cosh\lambda\cosh\kappa$. Again referring to tables C.2 and C.3, the elements of this two-by-two block can be related by the equations

$$\begin{bmatrix} 16 & 0 \\ 0 & -16 \end{bmatrix} = \left[\hat{U}_i, \hat{T}^H_{ij}, \hat{T}^H_{ij}\right] + \left[\hat{T}^H_{ij}, \hat{T}^H_{ij}, \hat{U}_j\right], \tag{C.0.20}$$

$$\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} = \hat{T}^H_{ij} - \left[\hat{U}_i, \hat{T}^H_{ij}, \hat{U}_j\right], \tag{C.0.21}$$

and so, including the combination from the zero particle subspace, $\mu\hat{V}_{ij}$ in this subspace

can be written in the form

$$\lambda\Big(\hat{U}_i+\hat{U}_j\Big)+\frac{1}{16}\Big\{\frac{\nu}{\sinh\nu}\sinh\lambda\cosh\kappa-\lambda\Big\}(C.0.20)+\frac{1}{4}\frac{\nu\sinh\kappa}{\sinh\nu}(C.0.21). \quad \text{(C.0.22)}$$

Finally, the two combinations of operators (C.0.12) and (C.0.22) give $\mu\hat{V}_{ij}$ in the odd and even particle number subspaces respectively, and so, summing the two gives the full and exact form:

$$\begin{aligned}\mu\hat{V}_{ij}[\mu] \;=\; & \lambda\left(\hat{U}_i+\hat{U}_j\right)+\frac{\kappa}{4}\sinh\lambda\left(\big[\hat{U}_i,\hat{T}^H_{ij}\big]+\big[\hat{T}^H_{ij},\hat{U}_i\big]\right)+\frac{\kappa}{4}\cosh\lambda\Big\{\hat{T}^H_{ij}+\big[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\big]\Big\}\\ &+\frac{1}{16}\Big\{\frac{\nu}{\sinh\nu}\sinh\lambda\cosh\kappa-\lambda\Big\}\left(\big[\hat{U}_i,\hat{T}^H_{ij},\hat{T}^H_{ij}\big]+\big[\hat{T}^H_{ij},\hat{T}^H_{ij},\hat{U}_j\big]\right)\\ &+\frac{1}{4}\frac{\nu\sinh\kappa}{\sinh\nu}\left(\hat{T}^H_{ij}+\big[\hat{U}_i,\hat{T}^H_{ij},\hat{U}_j\big]\right). \qquad \text{(C.0.23)}\end{aligned}$$

C.1 Identities and Relations

Unlike the XXZ model where the permutation and spin-projection operators commute, the hopping and potential operators which form the Hubbard Hamiltonian do not. This non-abelian aspect makes life considerably more complicated, as it is the eight operators in Tables C.2 and C.3 which form a basis for all two-site operators, as opposed to just the two in the previous case. As before, three-site operators are constructed as products of two-site ones, but the number of combinations has risen dramatically, and any hope of finding *all* possible relations and reduction identities at higher orders is lost. Bearing this in mind, the following complete the group of identities discovered so far, which have been used for the simplification of the conservation laws in Chapter 5

$$\begin{aligned}\sum_n\Big\{&\Big[\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2},\hat{H}_{n+2}\Big]-(1+g)\left(\Big[\hat{H}_n,\hat{H}_{n+1}\Big]-g\Big[\hat{H}_n,\hat{U}_{n+1}\Big]\right)\\ &-g\left(\Big[\hat{H}_n,\Big[\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big],\hat{H}_{n+2}\Big]\Big]+\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\Big[\hat{H}_{n+2},\hat{U}_{n+3}\Big]\Big]\Big]\right)\Big\}=0,\text{(C.1.1)}\end{aligned}$$

Table C.2: Hopping and Potential operators and their Commutators.

Operator	Particle Number subspace			
	0/4	1	2	3
$\hat{T}_{ij}^{H}$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&2&0&0&0\\0&2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$
$\hat{U}_i$	$\begin{bmatrix}\frac{1}{2}\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$
$\hat{U}_j$	$\begin{bmatrix}\frac{1}{2}\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}-1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{bmatrix}$	$\frac{1}{2}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$
$\left[\hat{U}_i, \hat{T}_{ij}^{H}\right]$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&2&0&0&0\\0&-2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$
$\left[\hat{T}_{ij}^{H}, \hat{U}_j\right]$	$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&-2&0&0&0\\0&-2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$

Table C.3: Double Commutators of the Hopping and Potential operators.

Operator	Particle Number subspace			
	0/4	1	2	3
$\Big[\Big[\hat{U}_i,\hat{T}^H_{ij}\Big],\hat{T}^H_{ij}\Big]$	$[0]$	$\begin{bmatrix}-2&0&0&0\\0&2&0&0\\0&0&-2&0\\0&0&0&2\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&8&0&0&0&0\\0&0&-8&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}2&0&0&0\\0&-2&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$
$\Big[\hat{T}^H_{ij},\Big[\hat{T}^H_{ij},\hat{U}_j\Big]\Big]$	$[0]$	$\begin{bmatrix}2&0&0&0\\0&-2&0&0\\0&0&2&0\\0&0&0&-2\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&8&0&0&0&0\\0&0&-8&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}-2&0&0&0\\0&2&0&0\\0&0&-2&0\\0&0&0&2\end{bmatrix}$
$\Big[\hat{U}_i,\Big[\hat{T}^H_{ij},\hat{U}_j\Big]\Big]$	$[0]$	$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0&0&0\\0&0&-2&0&0&0\\0&-2&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$

$$\sum_n \Big\{ \Big[\hat{H}_n,\hat{H}_n,\hat{H}_{n+1},\hat{H}_{n+2}\Big] - (1+g)\Big(\Big[\hat{H}_n,\hat{H}_{n+1}\Big] - g\Big[\hat{U}_n,\hat{H}_n\Big]\Big) \\ -g\Big(\Big[\hat{H}_n,\Big[\Big[\hat{U}_{n+1},\hat{H}_{n+1}\Big],\hat{H}_{n+2}\Big]\Big] + \Big[\hat{U}_n,\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big]\Big)\Big\} = 0, \quad \text{(C.1.2)}$$

$$\sum_n \Big[\hat{U}_n,\Big[\hat{H}_n,\Big[\hat{H}_n,\hat{H}_{n+1}\Big]\Big]\Big] + \Big[\hat{H}_n,\Big[\hat{H}_n,\Big[\hat{H}_{n+1},\hat{U}_{n+2}\Big]\Big]\Big] - (1+3g^2)\Big[\hat{H}_n,\hat{U}_{n+1}\Big] = \\ \sum_n \Big[\Big[\Big[\Big[\hat{U}_n,\hat{H}_n\Big],\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big] + \Big[\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big],\hat{U}_{n+2}\Big] - (1+3g^2)\Big[\hat{U}_n,\hat{H}_n\Big], \quad \text{(C.1.3)}$$

$$\sum_n \Big\{ \Big[\hat{U}_n,\Big[\hat{H}_n,\Big[\hat{H}_n,\hat{H}_{n+1}\Big]\Big]\Big] + \Big[\Big[\Big[\hat{H}_n,\hat{H}_{n+1}\Big],\hat{H}_{n+1}\Big],\hat{U}_{n+2}\Big] \\ -g\Big(\Big[\Big[\hat{U}_n,\hat{H}_n,\hat{U}_{n+1}\Big],\hat{H}_{n+1}\Big] + \Big[\hat{H}_n,\Big[\hat{U}_{n+1},\hat{}H_{n+1},\hat{U}_{n+2}\Big]\Big]\Big) \\ -2g\Big[\hat{H}_n,\hat{H}_{n+1}\Big] + 3g^2\Big(\Big[\hat{U}_n,\hat{H}_n\Big] + \Big[\hat{H}_n,\hat{U}_{n+1}\Big]\Big)\Big\} = 0, \quad \text{(C.1.4)}$$

$$\sum_n \Big\{ 2\Big[\hat{H}_n, \Big[\hat{H}_n, \Big[\hat{U}_{n+1}, \hat{H}_{n+1}\Big]\Big]\Big] - \Big[\hat{U}_{n+1}, \Big[\hat{H}_n, \Big[\hat{H}_n, \hat{H}_{n+1}\Big]\Big]\Big] \\ -\Big[\Big[\hat{U}_n, \hat{H}_n, \hat{H}_n\Big], \hat{H}_{n+1}\Big] - \Big[\Big[\hat{H}_n, \hat{H}_n, \hat{U}_{n+1}\Big], \hat{H}_{n+1}\Big] \\ -\Big[\hat{U}_n, \hat{H}_n\Big] - 4g\Big[\hat{H}_n, \hat{H}_{n+1}\Big] + g^2\left(4\Big[\hat{U}_n, \hat{H}_n\Big] + 5\Big[\hat{H}_n, \hat{U}_{n+1}\Big]\right)\Big\} = 0, \quad \text{(C.1.5)}$$

$$\sum_n \Big\{ 2\Big[\Big[\Big[\hat{H}_n, \hat{U}_{n+1}\Big], \hat{H}_{n+1}\Big], \hat{H}_{n+1}\Big] - \Big[\Big[\Big[\hat{H}_n, \hat{H}_{n+1}\Big], \hat{H}_{n+1}\Big], \hat{H}_{n+1}\Big] \\ -\Big[\hat{H}_n, \Big[\hat{U}_{n+1}, \hat{H}_{n+1}, \hat{H}_{n+1}\Big]\Big] - \Big[\hat{H}_n, \Big[\hat{H}_{n+1}, \hat{H}_{n+1}, \hat{U}_{n+2}\Big]\Big] \\ -\Big[\hat{H}_n, \hat{U}_{n+1}\Big] - 4g\Big[\hat{H}_n, \hat{H}_{n+1}\Big] + g^2\left(5\Big[\hat{U}_n, \hat{H}_n\Big] + 4\Big[\hat{H}_n, \hat{U}_{n+1}\Big]\right)\Big\} = 0, \quad \text{(C.1.6)}$$

$$\begin{aligned} & g\sum_n \Big\{ \Big[\hat{H}_n, \Big[\hat{H}_n, \Big[\hat{H}_{n+1}, \hat{U}_{n+2}\Big]\Big]\Big] + \Big[\Big[\Big[\hat{U}_n, \hat{H}_n\Big], \hat{H}_{n+1}\Big], \hat{H}_{n+1}\Big] \\ & \qquad +2\,\Big[\Big[\hat{U}_n, \hat{H}_n, \hat{H}_n\Big], \hat{H}_{n+1}\Big] + 2\,\Big[\hat{H}_n, \Big[\hat{H}_{n+1}, \hat{H}_{n+1}, \hat{U}_{n+2}\Big]\Big]\Big\} \\ = & \sum_n \Big\{ 2\,\Big[\hat{H}_n, \hat{H}_n, \hat{H}_{n+1}, \hat{H}_{n+1}\Big] \\ & \qquad +g^2\Big(\Big[\Big[\hat{U}_n, \hat{H}_n, \hat{U}_{n+1}\Big], \hat{H}_{n+1}\Big] + \Big[\hat{H}_n, \Big[\hat{U}_{n+1}, \hat{H}_{n+1}, \hat{U}_{n+2}\Big]\Big]\Big) \\ & \qquad -2g^2\Big[\hat{H}_n, \hat{H}_{n+1}\Big] - g(7+2g^2)\left(\Big[\hat{U}_n, \hat{H}_n\Big] + \Big[\hat{H}_n, \hat{U}_{n+1}\Big]\right)\Big\}. \end{aligned} \quad \text{(C.1.7)}$$

List of references

[1] X. Zotos, F. Naef, and P. Prelovsek, Phys. Rev. B **55**, 11029 (1997).

[2] M. Suzuki, Physica **51**, 277 (1971).

[3] X. Zotos, Phys. Rev. Lett **82**, 1764 (1999).

[4] P. Mazur, Physica **43**, 533 (1969).

[5] M. P. Grabowski and P. Mathieu, Ann. Phys. **243**, 299 (1995).

[6] O. Babelon, D. Bernard, and M. Talon, *Introduction to classical integrable systems* (Cambridge University Press, Cambridge, 2003).

[7] L. N. Hand and J. D. Finch, *Analytical Mechanics* (Cambridge University Press, Cambridge, 1998).

[8] F. Calogero, *Classical many-body problems amenable to exact treatments* (Springer, Berlin, 2001).

[9] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. **19**, 1095 (1967).

[10] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

[11] Y. Nutku, J. Math. Phys. **25**, 2007 (1984).

[12] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).

[13] W.-H. Steeb, *Problems in Theoretical Physics vol. II* (Bibliographisches Institut, Mannheim, 1990).

[14] H. Bethe, Z. Phys. **71**, 205 (1931).

[15] A. R. Chowdhury and A. G. Choudhury, *Quantum Integrable Systems* (CRC Press LLC, Boca Raton, Florida, 2004).

[16] M. T. Batchelor, Physics Today **60**, 436 (2007).

[17] M. Karbach and G. Muller, Computers in Physics **11**, 36 (1997).

[18] B. Sutherland, *Beautiful Models* (World Scientific, Singapore, 2004).

[19] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, 1993).

[20] N. A. Slavnov, Russian Mathematical Surveys **62**, 727 (2007).

[21] E. H. Lieb, Phys. Rev. **162**, 162 (1967).

[22] R. Baxter, *Exactly solved models in statistical mechanics* (Academis Press, London, 1982).

[23] L. A. Takhtadzhan and L. D. Faddeev, Russ. Math. Surveys **34**, 11 (1979).

[24] L. D. Faddeev, Int. J. Mod. Phys. **A10**, 1845 (1995).

[25] B. Sutherland, J. Math. Phys. **11**, 3183 (1970).

[26] M. Q. Zhang, Commun. Math. Phys. **141**, 523 (1991).

[27] R. J. Baxter, Ann. Phys. **70**, 193 (1972).

[28] J.-M. Maillet, Bethe Ansatz: 75 years later PoS(Solvay)008 (2006).

[29] M. G. Tetelman, Sov. Phys. JETP **55**, 306 (1982).

[30] J. Links, H. Zhou, R. H. McKenzie, and M. D. Gould, Phys. Rev. Lett. **86**, 5096 (2001).

[31] M. P. Grabowski and P. Mathieu, J. Phys. A **28**, 4777 (1995).

[32] M. Wadati, T. Nagao, and K. Hikami, Physica D **68**, 162 (1993).

[33] M. Lüscher, Nucl. Phys. **B117**, 475 (1976).

[34] K. Sakai and A. Klümper, J. Phys. A: Math. Gen **36**, 11617 (2003).

[35] A. Klümper and K. Sakai, J. Phys. A: Math. Gen **35**, 2173 (2002).

[36] K. Fabricius and B. M. McCoy, Phys. Rev. B **57**, 8340 (1998).

[37] P. Jung and A. Rosch, Spin conductivity in almost integrable spin chains, 2007.

[38] T. Kennedy, J. Phys. A: Math. Gen **25**, 2809 (1992).

[39] E. Olmedilla and M. Wadati, Phys. Rev. Lett. **60**, 1595 (1988).

[40] Y. Umeno, M. Shiroishi, and M. Wadati, J. Phys. Soc. Jpn. **67**, 2242 (1998).

[41] M. Shiroishi, H. Ujino, and M. Wadati, J. Phys. A: Math. Gen **31**, 2341 (1998).

[42] H. Zhou, L. Jiang, and J. Tang, J. Phys. A: Math. Gen **23**, 213 (1990).

[43] B. Fuchssteiner, in *Symmetries and Nonlinear Phenomena* (World Scientific Publishers, Singapore, 1988), pp. 22–50.

[44] N. Binggeli and M. Altarelli, Phys. Rev. B **70**, 085117 (2004).

[45] R. Caciuffo, L. Paolasini, A. Sollier, P. Ghigna, E. Pavarini, J. van den Brink, and M. Altarelli, Phys. Rev. B **65**, 174425 (2002).

[46] M. Takahashi, M. Usuda, and J. Igarashi, Phys. Rev. B **67**, 064425 (2003).

[47] A. M. Oleś, G. Khaliullin, P. Horsch, and L. F. Feiner, Phys. Rev. B **72**, 214431 (2005).

[48] C. Oliva, M. Scavini, S. Cappelli, C. Bottalo, C. Mazzoli, and P. Ghigna, J. Phys. Chem. B **111**, 5976 (2007).

[49] S. K. Satija, J. D. Axe, G. Shirane, H. Yoshizawa, and K. Hirakawa, Phys. Rev. B **21**, 2001 (1980).

[50] M. T. Hutchings, H. Ikeda, and J. M. Milne, J. Phys. C **12**, L739 (1979).

[51] K. Ishii *et al.*, Phys. Rev. Lett. **94**, 187002 (2005).

[52] N. M. Bogoliubov and V. E. Korepin, Int. J. Mod. Phys. **3**, 427 (1989).

[53] D. J. Derro, E. W. Hudson, K. M. Lang, S. H. Pan, J. C. Davis, J. T. Markert, and A. L. de Lozanne, Phys. Rev. Lett. **88**, 097002 (2002).

[54] A. Y. Zavidonov and D. Brinkmann, Phys. Rev. B **61**, 3282 (2000).

[55] P. Marsh, R. M. Fleming, M. L. Mandich, A. M. DeSantolo, J. Kwo, M. Hong, and L. J. Martinez-Miranda, Nature **334**, 141 (1988).

[56] H. W. Zandbergen, R. Gronsky, K. Wang, and G. Thomas, Nature **331**, 596 (1988).

[57] J. Karplinski, S. Rusiecki, E. Kaldis, B. Bucher, and E. Jilek, Physica C **331**, 449 (1989).

[58] R. Gagnon, C. Lupien, and L. Taillefer, Phys. Rev. B **50**, 3458 (1994).

[59] B. Bucher and P. Wachter, Phys. Rev. B **51**, 3309 (1995).

[60] B. S. Shastry, Phys. Rev. Lett. **56**, 1529 (1986).

[61] M. Toda, J. Phys. Soc. Jpn. **23**, 501 (1967).

[62] M. Henon, Phys. Rev. B **9**, 1921 (1974).

[63] R. Siddharthan and B. S. Shastry, 1996, arXiv:cond-mat/9611221v2.

[64] H. Flaschka, Phys. Rev. B **9**, 1924 (1974).

[65] V. Pasquier and M. Gaudin, J. Phys. A: Math. Gen **25**, 5243 (1992).

[66] W. Ebeling, 2002, arXiv:cond-mat/0210197v1.

[67] K. Hikami, J. Phys. Soc. Jpn. **65**, 1213 (1996).

[68] A. Matsuyama, J. Phys. A: Math. Gen **29**, 4837 (1996).

[69] C. Zachos, Crib notes on Campbell-Baker-Hausdorff expansions, 1999.