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# A COMBINATORIAL PROOF OF A PLETHYSTIC MURNAGHAN–NAKAYAMA RULE

## 1. Introduction

The purpose of this note is to give a combinatorial proof of a plethystic generalization of the Murnaghan–Nakayama rule, first stated in [1]. The key step in the proof uses a sign-reversing pairing on sequences of bead moves on James' abacus (see [3, page 78]), inspired by the theme of [4] 'when beads bump, objects cancel'. The only prerequisites are the Murnaghan–Nakayama rule and basic facts about plethysms of symmetric functions. The necessary combinatorial background on border-strips and James' abacus is recalled in Section 2 below.

Let $s_{\lambda/\nu}$ denote the Schur function corresponding to the skew-partition $\lambda/\nu$ and let $p_r$ denote the power-sum symmetric function of degree $r \in \mathbf{N}$. Let $\mathrm{sgn}(\lambda/\nu) = (-1)^\ell$ if $\lambda/\nu$ is a border-strip of height $\ell \in \mathbf{N}_0$, and let $\mathrm{sgn}(\lambda/\nu) = 0$ otherwise. The Murnaghan–Nakayama rule (see, for instance, [6, Theorem 7.17.1]) states that if $\nu$ is a partition and $r \in \mathbf{N}$ then

$$s_\nu p_r = \sum_{\lambda \vdash r+|\nu|} \mathrm{sgn}(\lambda/\nu) s_\lambda. \tag{1}$$

To generalize (1) we need some further definitions. Let $\lambda/\nu$ be a skew-partition and let $d$ be minimal such that $\lambda_d > \nu_d$. We say that an $r$-border-strip $\lambda/\mu$ is the *final* $r$-border-strip in $\lambda/\nu$ if $\mu/\nu$ is a skew-partition and $\lambda_d > \mu_d$. Thus $\lambda/\nu$ has a (necessarily unique) final $r$-border-strip if and only if the $r$ boxes at the top-right of the rim of the Young diagram of $\lambda/\nu$ can be removed to leave a skew-partition. We say that $\lambda/\nu$ is $r$-*decomposable* if there exist partitions $\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(m)}$ such that

$$\lambda = \mu^{(0)} \supset \mu^{(1)} \supset \ldots \supset \mu^{(m)} = \nu$$

and $\mu^{(i)}/\mu^{(i+1)}$ is the final $r$-border-strip in $\mu^{(i)}/\nu$ for each $i$. In this case we define $\mathrm{sgn}_r(\lambda/\nu) = \mathrm{sgn}(\mu^{(0)}/\mu^{(1)}) \ldots \mathrm{sgn}(\mu^{(m-1)}/\mu^{(m)})$. If $\lambda/\nu$ is not $r$-decomposable we define $\mathrm{sgn}_r(\lambda/\mu) = 0$. Let $f \circ g$ denote the plethysm of symmetric functions $f$ and $g$, as defined in [5, I.8] or [6, Appendix 2]. Finally let $h_m = s_{(m)}$ denote the complete symmetric function of degree $m \in \mathbf{N}_0$.

We shall prove that if $\nu$ is a partition and $r$, $m \in \mathbf{N}$ then

$$s_\nu(p_r \circ h_m) = \sum_{\lambda \vdash rm+|\nu|} \mathrm{sgn}_r(\lambda/\nu) s_\lambda. \tag{2}$$

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Taking $m = 1$ recovers (1). The formula for $s_\mu(p_r \circ h_{m_1} \dots h_{m_d})$ given in [1, page 29] follows by repeated applications of (2). This formula is proved in [1] using Muir's rule. Similarly (2) implies combinatorial formulae for $s_\mu(p_{r_1} \dots p_{r_c} \circ h_m)$, and, more generally, for $s_\mu(p_{r_1} \dots p_{r_c} \circ h_{m_1} \dots h_{m_d})$. An alternative proof of (2) using the character theory of the symmetric group was given in [2, Proposition 4.3]. The special case $\nu = \varnothing$ of (2) follows from [5, I.8, Example 8].

## 2. Background on border-strips, signs and James' abacus

Let $\lambda$ be a partition of $n$ with $p$ parts. The *Young diagram* of $\lambda$ is the set

$$[\lambda] = \{(i, j) : 1 \leq i \leq p,\ 1 \leq j \leq \lambda_i\}.$$

The elements of a Young diagram are called *boxes*. The *rim* of $\lambda$ consists of all boxes $(i, j)$ such that $(i+1, j+1) \notin [\lambda]$. A *border-strip* of *length* $s$ (also called an $s$*-border-strip*) in $\lambda$ consists of $s$ adjacent boxes in the rim of $\lambda$ whose removal from $[\lambda]$ leaves the Young diagram of a partition. The *top-right* and *bottom-left* boxes of a border-strip are defined with respect to the 'English' convention for drawing Young diagrams, shown in Figure 1 below. The *height* of a border-strip with bottom-left box $(i, j)$ and top-right box $(i', j')$ is $i - i'$.

Take a coordinate system in which $(i, j)$ is the bottom-right corner of the box $(i, j) \in [\lambda]$. Clearly $\lambda$ is determined by the sequence of right and up steps that starts at $(p, 0)$, visits the bottom-right corner of each box in the rim of $\lambda$, and finishes at $(0, \lambda_1)$. Encoding each right step by a *gap*, denoted

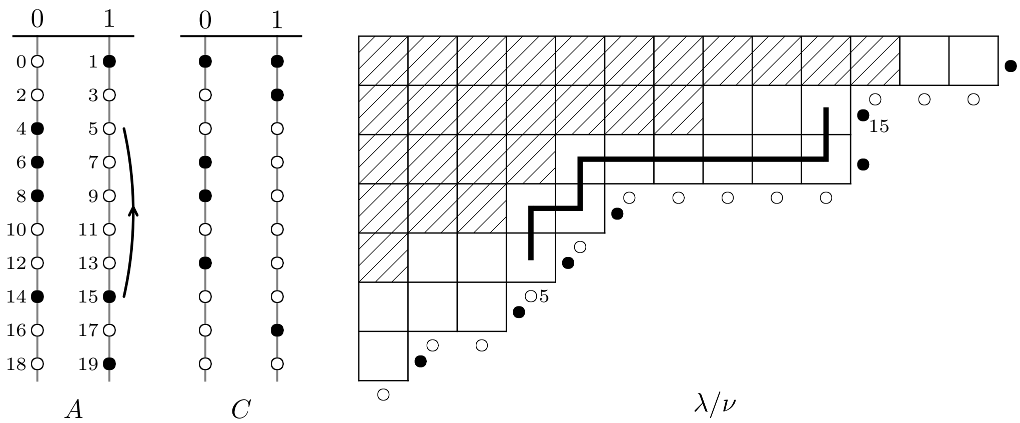


FIGURE 1. The 2-decomposable skew-partition $\lambda/\nu$ where $\lambda = (13, 10, 10, 5, 4, 3, 1)$ and $\nu = (11, 7, 4, 3, 1)$. The normalized abacus $A$ for $\lambda$ and an abacus $C$ for $\nu$ are shown on two runners. The marked 10-border-strip of height 3 has top-right box $(2, 10)$ and bottom-left box $(5, 4)$; it corresponds to the bead in position 15. Swapping this bead with the gap in position 5 removes this border-strip, giving $\mu = (13, 9, 4, 3, 3, 3, 1)$. In the walk along the rim of $\mu$, step 5 is up $\bullet$, rather that right $\circ$; the walk then agrees with that for $\lambda$ until step 15, which is right $\circ$, rather than up $\bullet$.

$\circ$, and each up step by a *bead*, denoted $\bullet$, we obtain the *normalized abacus* for $\lambda$. (This term is not entirely standard, but is convenient here.) More generally, an *abacus* for $\lambda$ is a sequence consisting of any number of beads, followed by the normalized abacus for $\lambda$, followed by any number of gaps. We number the positions in such a sequence from 0.

In the proof of (2) we shall only be concerned with border-strips whose length is a multiple of a fixed $r \in \mathbf{N}$. In this case it is useful to represent the gaps and beads on $r$ *runners*, so that for each $t \in \{0, 1, \ldots, r-1\}$ the positions on runner $t$ are $t, t+r, t+2r, \ldots$. We say that position $t + jr$ is *above* position $t + j'r$ if $j < j'$. Define a *single-step* bead move to be a move of a bead from a position $\beta$ to the position $\beta - r$ immediately above it. These definitions are illustrated in Figure 1 on the previous page.

The following two lemmas record some basic results on the abacus.

**Lemma 1.** *Let $A$ be an abacus for the partition $\lambda$. Let $s \in \mathbf{N}$. Let $\mathcal{A}_s$ be the set of bead positions $\beta$ of $A$ such that $\beta - s \geq 0$ and $A$ has a gap in position $\beta - s$.*

(i) *The map sending $\beta \in \mathcal{A}_s$ to the corresponding box in $[\lambda]$ is a bijection between $\mathcal{A}_s$ and the top-right boxes in the $s$-border-strips in $\lambda$.*
(ii) *Let $\beta \in \mathcal{A}_s$ and let $B$ be the abacus obtained from $A$ by swapping the bead in position $\beta$ with the gap in position $\beta - s$. Then $B$ is an abacus for the partition obtained by removing the $s$-border-strip from $\lambda$ corresponding to $\beta$.*
(iii) *The height of the $s$-border-strip in $\lambda$ corresponding to the bead in position $\beta \in \mathcal{A}_s$ is the number of beads in positions $\beta - s + 1, \ldots, \beta - 1$ of $A$.*

*Proof.* Parts (i) and (ii) follow from Lemma 2.7.13 in [3], using that the set of bead positions in $A$ is a set of $\beta$-numbers for $\lambda$. (An alternative proof, avoiding $\beta$-numbers, is indicated in the caption to Figure 1.) For (iii), observe that the beads in positions $\beta - s + 1, \ldots, \beta$ of $A$ encode the steps up made when walking the $s$-border-strip in $\lambda$ corresponding to $\beta$. $\square$

**Lemma 2.** *Let $\lambda = \mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(c)} = \nu$ be a sequence of partitions such that $\mu^{(i)}/\mu^{(i+1)}$ is an $s_i$-border-strip in $\mu^{(i)}$ for each $i \in \{0, 1, \ldots, c-1\}$. Let $A$ be an abacus for $\lambda$. Let $\mathcal{J}$ be the set of pairs of positions $\{\beta, \beta'\}$ such that*

(i) *$\beta < \beta'$;*
(ii) *$A$ has beads $b$ and $b'$ in positions $\beta$ and $\beta'$, respectively;*
(iii) *after the sequence of bead moves that removes the border-strips $\mu^{(0)}/\mu^{(1)}$, $\ldots$, $\mu^{(c-1)}/\mu^{(c)}$, bead $b$ finishes in a greater numbered position than bead $b'$.*

*Then* $\operatorname{sgn}(\mu^{(0)}/\mu^{(1)}) \ldots \operatorname{sgn}(\mu^{(c-1)}/\mu^{(c)}) = (-1)^{|\mathcal{J}|}$.

*Proof.* We work by induction on $c$. The base case $c = 0$ is trivial. Let $\mathcal{I}$ be the set defined in the same way as $\mathcal{J}$ for the sequence $\lambda = \mu^{(0)}$, $\mu^{(1)}$, $\ldots$, $\mu^{(c-1)} = \mu$. Let $B$ be the abacus for $\mu$ obtained from $A$ by the sequence of bead moves specified in (iii), stopping at $\mu$. Suppose that the border-strip $\mu/\nu$ corresponds to the bead in position $\gamma$ of $B$, and that this bead was in position $\beta$ of $A$. Let $\ell$ be the height of $\mu/\nu$. By Lemma 1(iii) there are $\ell$ beads in positions $\gamma - s_c + 1, \ldots, \gamma - 1$ of $B$. Suppose that exactly $j$ of these beads were originally in a position $\beta' > \beta$ of $A$. These $j$ beads correspond to pairs $\{\beta, \beta'\} \in \mathcal{I}\backslash\mathcal{J}$ and the remaining $\ell - j$ beads correspond to pairs $\{\beta, \beta'\} \in \mathcal{J}\backslash\mathcal{I}$. Apart from these pairs, the sets $\mathcal{I}$ and $\mathcal{J}$ agree. Thus $|\mathcal{J}| = |\mathcal{I}| - j + (\ell - j) = |\mathcal{I}| + \ell - 2j$. Hence, by induction,

$$\begin{aligned}(-1)^{|\mathcal{J}|} &= (-1)^{|\mathcal{I}|}(-1)^{\ell} \\ &= \mathrm{sgn}(\mu^{(0)}/\mu^{(1)}) \ldots \mathrm{sgn}(\mu^{(c-2)}/\mu^{(c-1)})\, \mathrm{sgn}(\mu^{(c-1)}/\mu^{(c)})\end{aligned}$$

as required. □

For the remainder of this section, fix $r$, $m \in \mathbf{N}$ and let $\lambda/\nu$ be a skew-partition of $rm$ such that $\nu$ can be obtained from $\lambda$ by repeatedly removing $r$-border-strips. Let $A$ be an $r$-runner abacus for $\lambda$ and let $C$ be the abacus for $\nu$ obtained from $A$ by a sequence of bead moves that removes these $r$-border-strips.

Using Lemma 2 we obtain the following proposition, which is equivalent to [3, 2.7.26].

**Proposition 3.** *Let $r$, $m \in \mathbf{N}$, $\lambda/\nu$ and $A$ be as just defined. There exists $\sigma \in \{1, -1\}$ such that if $\mu^{(0)}$, $\mu^{(1)}$, …, $\mu^{(m)}$ is any sequence of partitions such that $\mu^{(0)} = \lambda$, $\mu^{(m)} = \nu$ and $\mu^{(i)}/\mu^{(i+1)}$ is an $r$-border-strip in $\mu^{(i)}$ for each $i \in \{0, 1, \ldots, m-1\}$, then*

$$\sigma = \mathrm{sgn}(\mu^{(0)}/\mu^{(1)}) \ldots \mathrm{sgn}(\mu^{(m-1)}/\mu^{(m)}).$$

*Proof.* The sequence $\mu^{(0)}$, $\mu^{(1)}$, …, $\mu^{(m)}$ corresponds to a sequence of single-step bead moves on $A$ leading to the abacus $C$. Since the final positions of the beads moved on $A$ are independent of the order of moves, the result follows from Lemma 2. □

An immediate corollary of Proposition 3 is that $\mathrm{sgn}_r(\lambda/\nu)$ can be computed by removing $r$-border-strips in any way, provided that $\lambda/\nu$ is $r$-decomposable. We use this corollary in the proof of Proposition 6 below.

We end this section with a characterization of $r$-decomposable partitions using the abacus.

**Definition 4.** Let the abaci $A$ and $C$ be as defined above. Let $t \in \{0, \ldots, r-1\}$. We say that runner $t$ of $A$ is *$r$-decomposable* if it has positions $\alpha_1 < \beta_1 < \cdots < \alpha_c < \beta_c$ such that, for each $k \in \{1, \ldots, c\}$, position $\beta_k$ has a bead, positions $\alpha_k, \alpha_k + r, \ldots, \beta_k - r$ have gaps, and runner $t$ of $C$ is obtained by moving the bead in each position $\beta_k$ to the gap in position $\alpha_k$.

**Lemma 5.** *Let the skew-partition $\lambda/\nu$ and the abacus $A$ be as defined above.*

(i) *Let $\beta$ be the greatest numbered position of $A$ that has a bead moved in a sequence of bead moves leading to $\nu$. Then $\lambda/\nu$ has a final $r$-border-strip if and only if $A$ has a gap in position $\beta - r$.*
(ii) *The skew-partition $\lambda/\nu$ is $r$-decomposable if and only if runner $t$ of $A$ is $r$-decomposable for each $t \in \{0, \ldots, r-1\}$.*

*Proof.* Let $d$ be minimal such that $\lambda_d > \nu_d$. The bead in position $\beta$ corresponds to the box in position $(d, \lambda_d)$ of $[\lambda]$. By Lemma 1(i), this box is the top-right box in an $r$-border-strip in $\lambda$ if and only if there is a gap in position $\beta - r$ of $A$. This proves (i).

Let $t \in \{0, \ldots, r-1\}$. By Definition 4, runner $t$ of $A$ is $r$-decomposable if and only if it is possible to obtain runner $t$ of $C$ by a sequence of single-step bead moves in which beads are moved in decreasing order of their position. Part (ii) follows from this remark by repeated applications of (i). $\square$

The skew-partition $\lambda/\nu$ shown in Figure 1 is 2-decomposable. It may be used to give an example of Proposition 3 and Lemma 5.

## 3. Proof of Equation (2)

The proof is by induction on $m$. We begin with the identity

$$mh_m = \sum_{\ell=1}^{m} p_\ell h_{m-\ell}, \tag{3}$$

which may be proved in a few lines working from the generating functions $\sum_{m=0}^{\infty} h_m t^m = \prod_{i=1}^{\infty}(1 - x_i t)^{-1}$ and $\sum_{\ell=1}^{\infty} p_\ell t^\ell = \sum_{i=1}^{\infty} x_i t(1 - x_i t)^{-1}$, or found in [5, I, Equation (2.11)]. The map $f \mapsto p_r \circ f$ is an endomorphism of the ring of symmetric functions (see [5, I.8.6]), so (3) implies that

$$p_r \circ mh_m = p_r \circ \sum_{\ell=1}^{m} p_\ell h_{m-\ell} = \sum_{\ell=1}^{m} (p_r \circ p_\ell)(p_r \circ h_{m-\ell}) = \sum_{\ell=1}^{m} p_{r\ell}(p_r \circ h_{m-\ell}).$$

Since $p_r \circ mh_m = mp_r \circ h_m$, it follows that

$$ms_\nu(p_r \circ h_m) = \sum_{\ell=1}^{m} s_\nu(p_r \circ h_{m-\ell})p_{r\ell}.$$

By (1) and induction we get

$$ms_\nu(p_r \circ h_m) = \sum_{\ell=1}^{m} \sum_{\mu \vdash r(m-\ell)+|\nu|} \sum_{\lambda \vdash r\ell + |\mu|} \operatorname{sgn}_r(\mu/\nu)\operatorname{sgn}(\lambda/\mu)s_\lambda.$$

It is therefore sufficient to prove that if $\lambda/\nu$ is a skew-partition of $rm$ then

$$m \operatorname{sgn}_r(\lambda/\nu) = \sum_{\mu} \operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_r(\mu/\nu) \tag{4}$$

where the sum is over all partitions $\mu$ such that $\lambda/\mu$ is a border-strip of length divisible by $r$ and $\mu/\nu$ is a skew-partition.

Fix an $r$-runner abacus $A$ for $\lambda$. We may assume that one side of (4) is non-zero, and so an abacus $C$ for $\nu$ can be obtained by a sequence of bead moves on the runners of $A$. We say that a runner of $A$ is of *type*

(I) if it is $r$-decomposable (see Definition 4);
(II) if it is not $r$-decomposable but an $r$-decomposable runner can be obtained by swapping a bead on this runner with a gap one or more positions above it;
(III) if it is neither of type (I) nor of type (II).

For example, in Figure 2 after the proof of Proposition 7, runner 0 of $A$ has type (II) and runner 1 has type (I).

By Lemma 1(ii), swapping a bead and a gap as described in (II) corresponds to removing a border-strip of length divisible by $r$ from $\lambda$ to leave a partition $\mu$. The corresponding contribution of $\operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_r(\mu/\nu)$ to the right-hand side of (4) is non-zero if and only if $\mu/\nu$ is an $r$-decomposable skew-partition. Hence if $A$ has a runner of type (III) or two or more runners of type (II), then both sides of (4) are zero. The following two propositions deal with the remaining cases. In both cases an example is given following the proof.

**Proposition 6.** *If all runners of $A$ have type* (I) *then* (4) *holds.*

*Proof.* Let $\mu$ be a partition such that $\lambda/\mu$ is a border-strip of length divisible by $r$ and $\mu/\nu$ is a skew-partition. Suppose that $\mu$ is obtained by moving a bead $b$ on runner $t$ of $\lambda$. By Lemma 5(ii) there are positions $\alpha_1 < \beta_1 < \cdots < \alpha_c < \beta_c$ on runner $t$, such that for each $k \in \{1, \ldots, c\}$, position $\beta_k$ has a bead, positions $\alpha_k, \alpha_k + r, \ldots, \beta_k - r$ have gaps, and $\nu$ is obtained by moving the bead in position $\beta_k$ to the gap in position $\alpha_k$. Since $\mu/\nu$ is a skew-partition, the bead $b$ must be in one of the positions $\beta_k$ and, after the move giving $\mu$, it must be in position $\beta_k - rq$ for some $q$ such that $1 \le q \le (\beta_k - \alpha_k)/r$. Since there are gaps in $A$ in positions $\beta_k - rq, \ldots, \beta_k - r$, this move can also be achieved by a sequence of $q$ single-step bead moves of $b$. Since there are gaps in $A$ in positions $\alpha_k, \ldots, \beta_k - (r+1)q$, the runner obtained after moving bead $b$ is still $r$-decomposable. Hence, by Lemma 5, $\mu/\nu$ is $r$-decomposable. By Proposition 3, noting that bead $b$ can be moved from position $\beta_k - rq$ to position $\alpha_k$ by single-step bead moves, we have $\operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_r(\mu/\nu) = \operatorname{sgn}_r(\lambda/\nu)$.

It follows that all the non-zero summands on the right-hand side of (4) are equal to $\operatorname{sgn}_r(\lambda/\nu)$. The number of partitions $\mu$ obtained by moving a bead on runner $t$ that give a non-zero summand is $(\beta_1 - \alpha_1)/r + \cdots + (\beta_c - \alpha_c)/r$. Summing over all runners, and using that $\lambda/\nu$ is a skew-partition of $rm$, we see that there are exactly $m$ non-zero summands. This completes the proof. $\square$

For an example consider the skew-partition $\lambda/\nu$ and the border-strip $\lambda/\mu$ shown in Figure 1. The partition $\mu$ is obtained by moving the bead in position 15 to position 5: we denote this move by $(15,5)$. The final 2-border-strips removed from $\mu$ to obtain $\nu$ correspond to the bead moves $(19,17),(14,12),(5,3),(4,2),(2,0)$ and have heights $0,0,1,1,1$. Since $\lambda/\mu$ has height 3, we have $\operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_2(\mu/\nu) = (-1)^6 = 1$. The final 2-border-strips removed from $\lambda$ to obtain $\nu$ correspond to the bead moves $(19,17)$, $(15,13)$, $(14,12)$, $(13,11)$, $(11,9)$, $(9,7)$, $(7,5)$, $(5,3)$, $(4,2)$, $(2,0)$ and have heights $0,1,1,1,0,1,1,\ 1,\ 1,\ 1$, so $\operatorname{sgn}_2(\lambda/\nu) = (-1)^8 = 1$. In this case $m = 10$ and so there are 10 choices for $\mu$, corresponding to the bead moves $(19,17),(15,13),(15,11),(15,9),(15,7),(15,5),(15,3),(14,12),(4,2),(4,0)$.

**Proposition 7.** *If there is a unique runner of $A$ of type* (II) *and all other runners have type* (I) *then both sides of* (4) *are zero.*

*Proof.* By Lemma 5(ii), $\lambda/\nu$ is not $r$-decomposable. Hence the left-hand side of (4) is zero. Let runner $t$ be the unique runner of $A$ of type (II). Since runner $t$ is not $r$-decomposable, there are beads $d$ and $d^\star$ on this runner, in positions $\delta$ and $\delta^\star$ respectively, such that $\delta > \delta^\star$ and in any sequence of single-step bead moves leading from $A$ to $C$, bead $d$ finishes weakly above position $\delta^\star$. Choose $\delta$ maximal with this property. Then there are positions

$$\alpha_0 < \alpha_1 \leq \beta_1 < \alpha_2 \leq \beta_2 < \cdots < \alpha_b \leq \beta_b < \beta_{b+1}$$

on runner $t$ such that (a) $\beta_b = \delta^\star$ and $\beta_{b+1} = \delta$, (b) the beads between positions $\alpha_0$ and $\beta_{b+1}$ on runner $t$ are exactly those in positions $\beta_1,\ldots,\beta_{b+1}$, (c) in any sequence of single-step bead moves leading from $A$ to $C$, for each $k \in \{1,\ldots,b+1\}$, the bead in position $\beta_k$ finishes in the gap in position $\alpha_{k-1}$, and (d) swapping bead $d$ with the gap in position $\alpha_0$ gives an $r$-decomposable runner.

Let $P$ be the set of pairs $(\varepsilon,\gamma)$ such that $\varepsilon$ and $\gamma$ are positions on runner $t$ and the runner obtained by swapping the bead in position $\varepsilon$ with the gap in position $\gamma$ is $r$-decomposable. It follows from the choice of $\delta$ that if $(\varepsilon,\gamma) \in P$ then $\varepsilon \in \{\delta^\star,\delta\}$ and that $(\delta,\gamma) \in P$ if and only if $(\delta^\star,\gamma) \in P$. Hence

$$P = \big\{(\varepsilon,\gamma) : \varepsilon \in \{\delta,\delta^\star\},\ \gamma \in \{\alpha_0,\ldots,\alpha_1 - 1\}\big\}.$$

Let $(\delta,\gamma) \in P$, let $B$ be the abacus obtained by swapping bead $d$ with the gap in position $\gamma$, and let $\mu$ be the partition represented by $B$. Define $B^\star$ and $\mu^\star$ analogously, replacing $d$ with $d^\star$. It suffices to show that

$$\operatorname{sgn}(\lambda/\mu^\star)\operatorname{sgn}_r(\mu^\star/\nu) = -\operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_r(\mu/\nu) \tag{5}$$

so the contributions from $\mu$ and $\mu^\star$ to (4) cancel. We do this using Lemma 2 and a sign reversing pairing on sequences of bead moves from $A$ to $C$. This pairing is illustrated in Figure 2 and in the example following this proof.

Fix a sequence of bead moves that first swaps bead $d$ with the gap in position $\gamma$ (giving $B$) then makes single-step bead moves to go from $B$ to $C$. This sequence is paired with the sequence that first swaps bead $d^\star$ with the gap in position $\gamma$ (giving $B^\star$), then moves bead $d$ to position $\delta^\star$ by single-step moves (giving $B$, with beads $d$ and $d^\star$ swapped compared to the first sequence) and then makes the same sequence of single-step moves to go from $B$ to $C$. Let $\mathcal{J}$ and $\mathcal{J}^\star$ be the set of pairs $\{\beta,\beta'\}$ defined, as in Lemma 2, for these two sequences of bead moves. It is clear that $\mathcal{J}$ and $\mathcal{J}^\star$ agree except for pairs involving the positions $\delta$ and $\delta^\star$. Moreover $\{\delta,\delta^\star\} \in \mathcal{J}\backslash\mathcal{J}^\star$.

The final positions of beads $d$ and $d^\star$ in $C$ after the sequence of moves from $A$ to $B$ to $C$ are $\alpha = \alpha_0$ and $\alpha^\star = \alpha_b$, respectively. (Equivalently, $\alpha$ and $\alpha^\star$ are, respectively, the final positions of beads $d^\star$ and $d$ in $C$, after the sequence of moves from $A$ to $B^\star$ to $B$ to $C$.) Let $\mathcal{A}$ be the set of positions of $A$ that have a bead, excluding positions $\delta$ and $\delta^\star$. For $\beta \in \mathcal{A}$, let $\bar\beta$ be the final position in $C$, after either sequence of moves, of the bead starting in position $\beta$ of $A$.

The following four claims are routine to check:

$$\begin{aligned}
\{\beta,\delta\} \in \mathcal{J} \text{ and } \{\beta,\delta^\star\} \notin \mathcal{J}^\star &\iff \delta^\star < \beta < \delta \text{ and } \alpha < \bar\beta,\\
\{\beta,\delta^\star\} \in \mathcal{J} \text{ and } \{\beta,\delta\} \notin \mathcal{J}^\star &\iff \delta^\star < \beta < \delta \text{ and } \bar\beta < \alpha^\star,\\
\{\beta,\delta\} \notin \mathcal{J} \text{ and } \{\beta,\delta^\star\} \in \mathcal{J}^\star &\iff \delta^\star < \beta < \delta \text{ and } \bar\beta < \alpha,\\
\{\beta,\delta^\star\} \notin \mathcal{J} \text{ and } \{\beta,\delta\} \in \mathcal{J}^\star &\iff \delta^\star < \beta < \delta \text{ and } \alpha^\star < \bar\beta.
\end{aligned}$$

Let $X_{\mathcal{J}}$, $Y_{\mathcal{J}}$, $X_{\mathcal{J}^\star}$, $Y_{\mathcal{J}^\star}$ be the sets of $\beta \in \mathcal{A}$ satisfying each of these conditions, respectively. These sets are obstacles to a bijection $\mathcal{J}\backslash\big\{\{\delta,\delta^\star\}\big\} \longleftrightarrow \mathcal{J}^\star$ defined by $\{\beta,\delta\} \longleftrightarrow \{\beta,\delta^\star\}$. Observe that

$$\begin{aligned}
X_{\mathcal{J}} &= \{\beta \in \mathcal{A} : \delta^\star < \beta < \delta,\ \alpha < \bar\beta < \alpha^\star\} \cup Y_{\mathcal{J}^\star},\\
Y_{\mathcal{J}} &= \{\beta \in \mathcal{A} : \delta^\star < \beta < \delta,\ \alpha < \bar\beta < \alpha^\star\} \cup X_{\mathcal{J}^\star}.
\end{aligned}$$

It follows that

$$|\mathcal{J}| = 2\big|\{\beta \in \mathcal{A} : \delta^\star < \beta < \delta,\ \alpha < \bar\beta < \alpha^\star\}\big| + |\mathcal{J}^\star| + 1$$

where the final summand comes from $\{\delta,\delta^\star\}$. Hence $|\mathcal{J}|$ and $|\mathcal{J}^\star|$ have opposite parities. Equation (5) now follows from Lemma 2. This completes the proof. $\square$

In the example shown in Figure 2 overleaf with $r = 2$, we have $\delta = 18$, $\delta^\star = 14$, $\alpha^\star = 12$, $\alpha = 2$ and $\gamma = 4$. The set $P$ is $\{(\delta,2),(\delta^\star,2),(\delta,4),(\delta^\star,4)\}$. The sets $\mathcal{J}$ and $\mathcal{J}^\star$ are

$$\begin{aligned}
\mathcal{J} &= \big\{\{10,18\},\{9,18\},\{8,18\},\{3,18\}\big\} \cup \big\{\{17,18\},\{14,17\}\big\} \cup \big\{\{14,18\}\big\},\\
\mathcal{J}^\star &= \big\{\{10,14\},\{9,14\},\{8,14\},\{3,14\}\big\}.
\end{aligned}$$

The second set in the union for $\mathcal{J}$ gives the pairs coming from $X_{\mathcal{J}} = Y_{\mathcal{J}} = \{\beta \in \mathcal{A} : \delta^\star < \beta < \delta,\ \alpha < \bar\beta < \alpha^\star\} = \{17\}$. In this example $X_{\mathcal{J}^\star} = Y_{\mathcal{J}^\star} = \varnothing$. We have $\operatorname{sgn}(\lambda/\mu^\star)\operatorname{sgn}_2(\mu^\star/\nu) = 1 = -\operatorname{sgn}(\lambda/\mu)\operatorname{sgn}_2(\mu/\nu)$ as predicted by (4).

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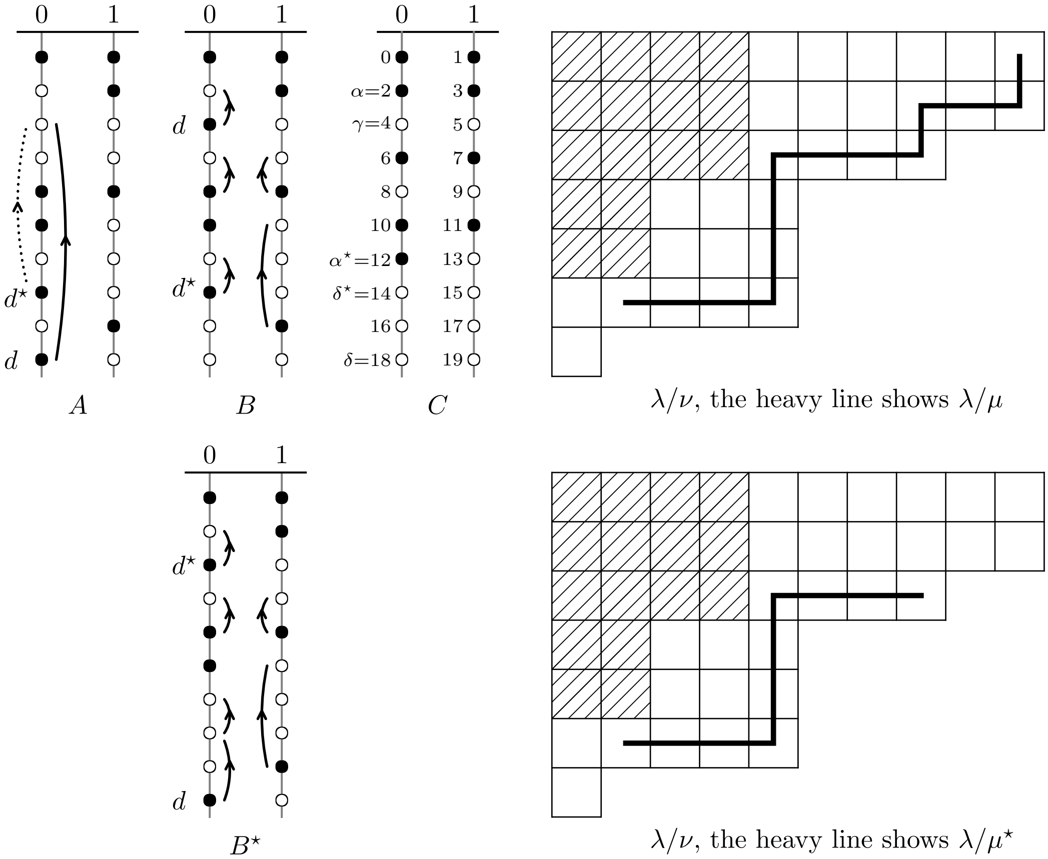


FIGURE 2. Example to illustrate Proposition 7 when $r = 2$ and $\lambda/\nu = (10, 10, 8, 5, 5, 5, 1)/(4, 4, 4, 2, 2)$. The partitions $\mu$ and $\mu^\star$ are $(9, 7, 4, 4, 4, 1, 1)$ and $(10, 10, 4, 4, 4, 1, 1)$. Abaci $A$, $B$, $B^\star$, $C$ for $\lambda$, $\mu$, $\mu^\star$, $\nu$, respectively, are shown. The bead moves between these abaci are indicated by arrows: $B$ is obtained from $A$ by the move $(\delta, \gamma) = (18, 4)$ shown by a solid arrow and $B^\star$ is obtained from $A$ by the move $(\delta^\star, \gamma) = (14, 4)$ shown by a dotted arrow.

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